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The Whitehead and the Smale theorems in shape theory

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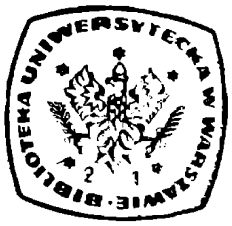

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§1. Introduction

Let H_0 denote the category of pointed connected CW complexes and homotopy classes of maps. Of great importance in homotopy theory is the following classical theorem of J. H. C. Whitehead [Wh]

1.1. THEOREM. *Let $f: X \to Y$ be a morphism of H_0. If $\dim X$, $\dim Y \leqslant n$ and $\pi_k f$ is an isomorphism for $k \leqslant n$ (n finite or infinite), then f is an isomorphism of H_0.*

A sufficient geometric condition for a map to induce isomorphisms of homotopy groups was given by S. Smale [Sm] in the following

1.2. THEOREM. *Let $X \in \mathrm{LC}^n$ be a locally compact connected metrizable space. If $f: X \to Y$ is a proper map onto a metrizable space Y such that $f^{-1}(y) \in \mathrm{LC}^{n-1} \cap \mathrm{C}^{n-1}$ for each $y \in Y$, then $Y \in \mathrm{LC}^n$ and $\pi_k f: \pi_k X \to \pi_k Y$ is an isomorphism for $k \leqslant n-1$ and an epimorphism for $k = n$.*

In the first stage of development of shape theory it was natural to look for shape theoretic analogues of classical theorems of homotopy theory. In this paper (which is based on the author's thesis) we survey some work on the above two theorems – the Whitehead and the Smale theorems.

The first Whitehead theorem in shape theory has been proved by M. Moszyńska [Mos_2] and concerned finite-dimensional continua. The papers of S. Mardešić [$\mathrm{Mar}_{2,3}$], D. A. Edwards and R. Geoghegan [$\mathrm{E\text{-}G}_2$] and K. Morita [Mor_3] arose from an attempt to simplify or generalize [Mos_2]. More recent papers of D. A. Edwards and R. Geoghegan [$\mathrm{E\text{-}G}_3$] and J. Dydak [Dyd_4] concern the infinite-dimensional Whitehead theorem in shape theory.

The classical Smale theorem has been transferred to shape theory by several authors and at various levels of generality [Bog_2], [Kup], [Kod_1], [Dyd_5], [$\mathrm{K\text{-}S}_2$] and [Mor_5].

In the paper we use the notions and methods of category theory which may be found in [ML].

§2. Some classes of objects and morphisms in pro-categories

For any category C, let $\mathrm{Ob}\mathsf{C}$ be the class of its objects and let $\mathsf{C}(X, Y)$ be the set of all morphisms from X to Y in C. By $\mathsf{C}(Y, \cdot)$ (resp. $\mathsf{C}(\cdot, Y)$) we denote the covariant (resp. contravariant) functor from C to the category of sets and functions Ens defined in the following way:

For $X \in \mathrm{Ob}\mathbf{C}$ we set $\mathbf{C}(Y, \cdot)(X) = \mathbf{C}(Y, X)$ (resp. $\mathbf{C}(\cdot, Y)(X) = \mathbf{C}(X, Y)$) and for $f \in \mathbf{C}(X, X')$ we have $\mathbf{C}(Y, \cdot)f(g) = f \cdot g$ for each $g \in \mathbf{C}(Y, X)$ (resp. $\mathbf{C}(\cdot, Y)f(g) = g \cdot f$ for each $g \in \mathbf{C}(X', Y)$).

Consider the class of all inverse systems in $\mathbf{C}$. It is the class of objects of some category which is called the *pro-category* of $\mathbf{C}$ and denoted by pro-$\mathbf{C}$. The sets of morphisms in this category are defined by the equality

2.1. $\text{pro-}\mathbf{C}(X, Y) = \varprojlim_{b} \varinjlim_{a} \mathbf{C}(X_a, Y_b)$, where $X = (X_a, p_a^{a'}, A)$ and $Y = (Y_b, q_b^{b'}, B)$.

Let us explain equality 2.1. It is particularly simple when B is a one-point set, i.e. when Y is an object of $\mathbf{C}$. Then

$$\text{pro-}\mathbf{C}(X, Y) = \varinjlim \left(\mathbf{C}(X_a, Y), \mathbf{C}(\cdot, Y)p_a^{a'}, A\right)$$

and it is quite clear how to define the composition $g \cdot f$, where $f \in \text{pro-}\mathbf{C}(X, Y)$, $g \in \text{pro-}\mathbf{C}(Y, Z)$ and $Y, Z \in \mathrm{Ob}\mathbf{C}$.

Now each morphism $f \in \text{pro-}\mathbf{C}(X, Y)$ may be considered as a family $\{f_b\}_{b \in B}$, where $f_b \in \text{pro-}\mathbf{C}(X, Y_b)$ for each $b \in B$ and $q_b^{b'} \cdot f_{b'} = f_b$ for $b \leqslant b'$. Let $Z = (Z_d, r_d^{d'}, D)$ and $g = \{g_d\}_{d \in D} \in \text{pro-}\mathbf{C}(Y, Z)$. Choose for each $d \in D$ a representative $g_d^{b(d)} : Y_{b(d)} \to Z_d$ of g_d. Then $g \cdot f = \{g_d^{b(d)} \cdot f_{b(d)}\}_{d \in D}$. It remains only to define the *identity morphism* 1_X of $X \in \mathrm{Ob}\,\text{pro-}\mathbf{C}$. It is the family $\{p_a\}_{a \in A}$, where the identity morphism 1_{X_a} is a representative of $p_a : X \to X_a$ for each $a \in A$. The morphism p_a is called a *projection morphism*.

$\mathrm{Ob}\mathbf{C}$ is a subclass of $\mathrm{Ob}\,\text{pro-}\mathbf{C}$ and, in view of 2.1, $\mathbf{C}$ is a full subcategory of pro-$\mathbf{C}$. It is easy to see that each covariant functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ induces a contravariant functor $\text{pro-}F : \text{pro-}\mathbf{C}_1 \to \text{pro-}\mathbf{C}_2$ in a natural way.

Let $X = (X_a, p_a^b, A)$ and $Y = (Y_a, q_a^b, A)$ be two inverse systems over the same directed set $(A, \leqslant)$. Suppose that $\{f_a\}_{a \in A}$ is a family of morphisms such that $f_a \in \mathbf{C}(X_a, Y_a)$ and $q_a^b \cdot f_b = f_a \cdot p_a^b$ for $a \leqslant b$. If for each $a \in A$ we take the morphism $\bar{f}_a \in \text{pro-}\mathbf{C}(X, Y_a)$ whose representative is f_a, then the family $\{\bar{f}_a\}_{a \in A}$ is a morphism from X to Y. Such a morphism is specially simple and we called it the *special morphism* induced by $\{f_a\}_{a \in A}$ (comp. [Mar$_2$]).

Each morphism of pro-$\mathbf{C}$ is equivalent to a special morphism. This is shown in the following theorem (see [Mar$_2$], 2.2 and [A-M], Corollary (3.2) on p. 160).

2.2. THEOREM. *For each morphism $f : X \to Y$ of* pro-$\mathbf{C}$ *there exist inverse systems X', Y' and isomorphisms $i : X' \to X$, $j : Y \to Y'$ such that $j \cdot f \cdot i$ is a special morphism.*

Moreover, each object appearing in X' (resp. Y') can be taken from the objects appearing in X (resp. Y).

If f is a special morphism, then there is a useful criterion for f to be an isomorphism (see [Mor$_3$], Theorem 1.1).

2.3. THEOREM. *A special morphism*

$$f\colon (X_a, p^b_a, A)\to(Y_a, q^b_a, A)$$

induced by a family $\{f_a\}_{a\in A}$ *is an isomorphism of* pro-$\mathbf{C}$ *iff for each* $a\in A$ *there exist a* $b\geqslant a$ *and a* $g^b_a\in \mathbf{C}(Y_b, X_a)$ *such that* $f_a\cdot g^b_a = q^b_a$ *and* $g^b_a\cdot f_b = p^b_a$.

An object $X = (X_a, p^b_a, A)$ of pro-$\mathbf{C}$ is said to be *movable* provided for each $a\in A$ there exists a $b\geqslant a$ such that for all $c\geqslant b$ the bond p^b_a factors as $p^b_a = p^c_a\cdot r$ for some $r\in \mathbf{C}(X_b, X_c)$.

X is *uniformly movable* if for each $a\in A$ there exists a $b\geqslant a$ such that the bond p^b_a factors as $p^b_a = p_a\cdot r$ for some $r\colon X_b\to X$, where $p_a\colon X\to X_a$ is the *projection morphism* of pro-$\mathbf{C}$ (comp. [Mos$_1$]).

If the set A is countable, then the movability of X is equivalent to its uniform movability (see [Sp]).

The notion of movability has been introduced by K. Borsuk [Bor$_4$] in the case where X is a system of neighbourhoods of a compactum lying in the Hilbert cube. S. Mardešić and J. Segal [M-S] extended it to the case of ANR-systems. The most general case was considered by M. Moszyńska [Mos$_1$].

If $X\in \mathrm{Ob}\,\text{pro-}\mathbf{C}$ is isomorphic in pro-$\mathbf{C}$ to an object of $\mathbf{C}$, then we say that X is *stable*.

If $(A, \leqslant)$ is a directed set, then by $\boldsymbol{A}$ we denote the set of all increasing sequences $B = \{a_n\}^\infty_{n=1}$ of A. We define the directed set $(\boldsymbol{A}, \leqslant)$ as follows:

$$B\leqslant B' = \{a'_n\}^\infty_{n=1} \quad \text{iff} \quad a_n\leqslant a'_n \quad \text{for each } n.$$

Let $X = (X_a, p^b_a, A)$ be an inverse system in a category $\mathbf{C}$. If $\{a_n\}^\infty_{n=1} = B\in \boldsymbol{A}$, let $X_B = (X_{a_n}, p^{a_{n+1}}_{a_n})$, and if $\{a'_n\}^\infty_{n=1} = B'\geqslant B$, let $P^{B'}_B\colon X_{B'}\to X_B$ be the special morphism induced by the family $\{p^{a'_n}_{a_n}\}^\infty_{n=1}$. Then it is clear that we obtain an inverse system $\boldsymbol{X} = (X_B, P^{B'}_B, \boldsymbol{A})$ in pro-$\mathbf{C}$ and a morphism $\boldsymbol{P} = \{P_B\}_{B\in\boldsymbol{A}}\colon X\to\boldsymbol{X}$ of pro-pro-$\mathbf{C}$ which is an inverse limit of $\boldsymbol{X}$ (see [A-M], Proposition (4.4) on p. 165 for a description of inverse limits in pro-categories). The morphisms P_B are just equal to $\{p_{a_n}\}^\infty_{n=1}$, where $B = \{a_n\}^\infty_{n=1}$ and $p_{a_n}\colon X\to X_{a_n}$ are projection morphisms.

Now we can pass to two lemmata which allow us to reduce proofs of many theorems to simpler cases.

2.4. LEMMA. *Let* $F_n\colon \mathbf{C}\to\mathbf{D}_n$ *be a sequence of covariant functors. If* $X = (X_a, p^b_a, A)$ *is an object of* pro-$\mathbf{C}$ *such that each* pro-F_nX *is stable, then the set*

$$\boldsymbol{A}' = \{B\in\boldsymbol{A}\colon \text{pro-}F_nP_B \text{ is an isomorphism for each } n\}$$

is a cofinal subset of $\boldsymbol{A}$. *Moreover, if* X *is movable, then* $\boldsymbol{A}'' = \{B\in\boldsymbol{A}'\colon X_B$ *is movable*$\}$ *is a cofinal subset of* $\boldsymbol{A}$ *as well.*

Proof. For each n take an isomorphism

$$q^n = \{q^n_a\}_{a\in A}\colon Y_n\to\text{pro-}F_nX, \quad \text{where} \quad Y_n\in\mathrm{Ob}\,\mathbf{D}_n.$$

Observe that Theorem 2.3 is applicable to such a case because Y_n is canonically isomorphic to $\{Y_n, 1_{Y_n}, A\}$. So if $B = \{b_i\}_{i=1}^{\infty} \in \boldsymbol{A}$, then we can find inductively an increasing sequence $B' = \{a_i\}_{i=1}^{\infty} \in \boldsymbol{A}$ such that the following conditions are fulfilled:

(1) $a_1 = b_1$ and $b_i \leqslant a_i$ for each i,

(2) for $k \leqslant n$ there exists a morphism h_n^k: $F_k X_{a_{n+1}} \to Y_k$ of D_k such that $q_{a_n}^k \cdot h_n^k = F_k p_{a_n}^{a_{n+1}}$ and $h_n^k \cdot q_{a_{n+1}}^k = 1_{Y_k}$,

(3) if X is movable, then for each $m \geqslant n+1$ there exists an r: $X_{a_{n+1}} \to X_{a_m}$ with $p_{a_n}^{a_m} \cdot r = p_{a_n}^{a_{n+1}}$.

Then, by Theorem 2.3, $\{q_{a_n}^k\}_{n=1}^{\infty}$: $Y_k \to$ pro-$F_k X_{B'}$ is an isomorphism for each k. Since $\{q_{a_n}^k\}_{n=1}^{\infty} = (\text{pro-}F_k P_{B'}) \cdot q^k$, we infer that pro-$F_k P_{B'}$ is an isomorphism for each k.

Thus the proof of Lemma 2.4 is concluded.

Let f: $X = (X_a, p_a^b, A) \to Y = (Y_a, q_a^b, A)$ be a special morphism induced by a family $\{f_a\}_{a \in A}$. Then, for each $B = \{a_n\}_{n=1}^{\infty} \in \boldsymbol{A}$, we have the special morphism f_B: $X_B \to Y_B$ induced by the family $\{f_{a_n}\}_{n=1}^{\infty}$. These morphisms induce a special morphism $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ such that $f = \varprojlim \boldsymbol{f}$, i.e. $\boldsymbol{f} \cdot \boldsymbol{P} \underset{\cdot}{=} \boldsymbol{R} \cdot f$, where $\boldsymbol{P}$: $\boldsymbol{X} \to \boldsymbol{X}$ and $\boldsymbol{R}$: $Y \to \boldsymbol{Y}$ are projection morphisms of pro-pro-C.

By similar considerations as in the proof of Lemma 2.4 one can show the following

2.5. LEMMA. *Let F_n: $\mathsf{C} \to \mathsf{D}_n$ be a sequence of covariant functors. If f: $X = (X_a, p_a^b, A) \to Y = (Y_a, q_a^b, A)$ is a special morphism such that each* pro-$F_n f$ *is an isomorphism, then the set $A' = \{B \in \boldsymbol{A}$: pro-$F_n f_B$ is an isomorphism for each $n\}$ is a cofinal subset of $\boldsymbol{A}$.*

Moreover, if Y is movable, then $A'' = \{B \in \boldsymbol{A}'$: Y_B is movable$\}$ is a cofinal subset of $\boldsymbol{A}$ as well.

By Gr we denote the category of groups and homomorphisms.

2.6. THEOREM. *Let $\mathsf{C} = \mathsf{Gr}$ or $\mathsf{C} = \mathsf{Ens}$. A special morphism f: $(X_a, p_a^b, A) \to (Y_a, q_a^b, A)$ of* pro-C *induced by a family $\{f_a\}_{a \in A}$ is an epimorphism of* pro-C *iff for each $a \in A$ there exists a $b \geqslant a$ such that*

$$\operatorname{im} q_a^b \subset \operatorname{im} f_a .$$

Theorem 2.6 has been proved by S. Mardešić [Mar$_3$] and independently by K. Morita [Mor$_3$] (Theorem 1.2) for $\mathsf{C} = \mathsf{Gr}$. The case $\mathsf{C} = \mathsf{Ens}$ may be considered analogously.

A morphism f of pro-C is called a *weak domination* if for each covariant functor F: $\mathsf{C} \to \mathsf{Ens}$ the morphism pro-Ff is an epimorphism of pro-Ens.

It is easy to prove the following

2.7. PROPOSITION. *Let f: $X \to Y$ and g: $Y \to Z$ be two morphisms of*

pro-C. *If f is a domination, then f is a weak domination. If f and g are weak dominations, then $g \cdot f$ is a weak domination. If $g \cdot f$ is a weak domination, then g is a weak domination.*

If f is the special morphism we have a useful criterion for f to be a weak domination.

2.8. THEOREM. *Let $f\colon (X_a, p^b_a, A) \to (Y_a, q^b_a, A)$ be a special morphism induced by a family $\{f_a\}_{a \in A}$. Then f is a weak domination of* pro-C *iff for each $a \in A$ there exist a $b \geqslant a$ and a $g^b_a \in \mathsf{C}(Y_b, X_a)$ with $f_a \cdot g^b_a = q^b_a$.*

Proof. Sufficiency is obvious in view of Theorem 2.6.

Necessity. Let $F\colon \mathsf{C} \to \mathsf{Ens}$ be the restriction of pro-$\mathsf{C}(Y, \cdot)$ to C. Then pro-Ff is the special morphism induced by the family $\{Ff_a\}_{a \in A}$.

Let $a \in A$. By Theorem 2.6 there exists an $a_1 \geqslant a$ such that $\operatorname{im} Fq^{a_1}_a \subset \operatorname{im} Ff_a$. Let $q_a\colon Y \to Y_a$ be the morphism of pro-C whose representative is $q^{a_1}_a$. Then $q_a \in \operatorname{im} Fq^{a_1}_a$ and consequently there exists a morphism $g\colon Y \to X_a$ with $q_a = f_a \cdot g$. Let $g^{a_2}_a\colon Y_{a_2} \to X_a$ be a representative of g. Then $q^{a_1}_a$ and $f_a \cdot g^{a_2}_a$ are representatives of the same morphism from Y to Y_a. So there exists a $b \geqslant a_1, a_2$ such that $q^b_a = f_a \cdot g^{a_2}_a \cdot q^b_{a_2}$. This completes the proof.

2.9. PROPOSITION. *Each weak domination $f\colon X \to Y$ of* pro-C *is an epimorphism of* pro-C.

Proof. Let $g, h\colon Y \to Z$ be two morphisms of pro-C such that $g \cdot f = h \cdot f$. Let $F\colon \mathsf{C} \to \mathsf{Ens}$ be the restriction of the functor pro-$\mathsf{C}(Y, \cdot)\colon$ pro-$\mathsf{C} \to \mathsf{Ens}$. In view of 2.1 we have $\varprojlim \cdot (\text{pro-}F) = \text{pro-}\mathsf{C}(Y, \cdot)$, where

$$\varprojlim\colon \text{pro-}\mathsf{Ens} \to \mathsf{Ens}$$

is the inverse limit functor. Since pro-Ff is an epimorphism of pro-Ens and $g \cdot f = h \cdot f$, we infer that

$$\text{pro-}\mathsf{C}(Y, \cdot)g = \text{pro-}\mathsf{C}(Y, \cdot)h.$$

Hence $g = \text{pro-}\mathsf{C}(Y, \cdot)g(1_Y) = \text{pro-}\mathsf{C}(Y, \cdot)h(1_Y) = h$.

2.10. COROLLARY. *A morphism $f\colon X \to Y$ of* pro-Ens *is a weak domination iff it is an epimorphism of* pro-Ens.

Proof. By 2.9 each weak domination of pro-Ens is an epimorphism of pro-Ens.

Let $f\colon X \to Y$ be an epimorphism of pro-Ens. By Theorem 2.2 we may assume that $X = (X_a, p^b_a, A)$, $Y = (Y_a, q^b_a, A)$ and f is a special morphism induced by a family $\{f_a\}_{a \in A}$. Let $a \in A$. By Theorem 2.6 there exists a $b \geqslant a$ such that $\operatorname{im} q^b_a \subset \operatorname{im} f_a$. Now we can define a function $g\colon Y_b \to X_a$ such that $f_a \cdot g = q^b_a$. By Theorem 2.8, f is a weak domination.

It is well known (see [M-S] and [Mos$_1$]) that dominations of pro-C preserve movability and uniform movability. Here we show a stronger result.

2.11. THEOREM. *Let $f\colon X\to Y$ be a weak domination of* pro-C. *If X is movable (uniformly movable), then Y is movable (uniformly movable) as well.*

Proof. Let us prove only the second part of Theorem 2.11; the proof of the first one is similar.

By Theorem 2.2 together with 3.10 of [Mos$_1$] we may suppose that $X=(X_a, p^b_a, A)$, $Y=(Y_a, q^b_a, A)$ and f is a special morphism induced by a family $\{f_a\}_{a\in A}$. Let $a\in A$. Since X is uniformly movable, there exist a $b\geqslant a$ and an $r\colon X_b\to X$ such that $p_a\cdot r=p^b_a$. By Theorem 2.8 there exist a $c\geqslant b$ and a $g\colon Y_c\to X_b$ with $f_b\cdot g=q^c_b$. Let $s=f\cdot r\cdot g\colon Y_c\to Y$. Then

$$q_a\cdot s=q_a\cdot f\cdot r\cdot g=f_a\cdot p_a\cdot r\cdot g=f_a\cdot p^b_a\cdot g=q^b_a\cdot f_b\cdot g=q^b_a\cdot q^c_b=q^c_a,$$

where $q_a\colon Y\to Y_a$ is the projection morphism.

Thus Y is uniformly movable.

2.12. EXAMPLE. Let us show that Theorem 2.11 does not hold under the weaker assumption that f is an epimorphism of pro-C. In this way we get an epimorphism of pro-Gr which is not a weak domination.

Let $X=(X_n, p^{n+1}_n)$ be the inverse sequence of groups, where $X_n=Z$ is the group of integers for each n and p^{n+1}_n are identities. Let $Y=(Y_n, q^{n+1}_n)$ be the inverse sequence of groups, where $Y_n=Z/2^nZ$ is the group of integers modulo 2^n for each n and q^{n+1}_n are natural projections.

Let $f\colon X\to Y$ be the special morphism induced by the sequence $\{f_n\colon X_n\to Y_n\}^\infty_{n=1}$ of natural projections. By Theorem 2.6, f is an epimorphism of pro-Gr and it is easy to see that Y is not movable. Thus f is not a weak domination of pro-Gr. If we consider f as a morphism of pro-Ens (by ignoring algebraic structures), then f is a weak domination which is not a domination.

If X is a stable object of pro-C, then any isomorphism $p\colon Y\to X$ with $Y\in \mathrm{Ob}C$, is an inverse limit of X in C. Let us describe situations when inverse limits are isomorphisms of pro-C.

2.13. LEMMA. *Let X be an object of* pro-C *which has an inverse limit $p\colon \varprojlim X\to X$ as an inverse system in* C. *p is an isomorphism of* pro-C *iff X is dominated in* pro-C *by an object of* C.

Proof. Necessity is obvious because each isomorphism is a domination.

Let $f\colon X\to Y$ and $g\colon Y\to X$ be morphisms of pro-C such that $g\cdot f=1_X$ and $Y\in \mathrm{Ob}C$. By the property of inverse limits there exists an $r\colon Y\to \varprojlim X$ such that $p\cdot r=g$. Let $s=r\cdot f$. Then $p\cdot s=p\cdot r\cdot f=g\cdot f=1_X$ and $p\cdot s\cdot p=p\cdot 1_{\varprojlim X}$. Hence $s\cdot p=1_{\varprojlim X}$ and the proof is finished.

Lemma 2.13 has been proved by D. A. Edwards and R. Geoghegan [E-G$_5$] for C = Gr.

If X and Y are objects of pro-C which have inverse limits $p\colon \varprojlim X \to X$, $t\colon \varprojlim Y \to Y$ as inverse systems in C, then for each morphism $f\colon X \to Y$ of pro-C there is a morphism

$$\varprojlim f\colon \varprojlim X \to \varprojlim Y$$

such that $t \cdot \varprojlim f = f \cdot p$. Let us verify which properties of $\varprojlim f$ imply the same properties of f for uniformly movable X and Y.

2.14. LEMMA. *If $\varprojlim f$ is an epimorphism (domination) of C, then f is an epimorphism (weak domination) of* pro-C. *In particular, both p and q are weak dominations.*

Proof. Let $Y = (Y_a, q_a^b, A)$ and let $a \in A$. Since Y is uniformly movable, there exist a $b \geqslant a$ and an $r\colon Y_b \to Y$ with $q_a \cdot r = q_a^b$, where $q_a\colon Y \to Y_a$ is a projection morphism. Then there is a morphism $s\colon Y_b \to \varprojlim Y$ with $t \cdot s = r$. Hence

$$q_a^b = q_a \cdot r = q_a \cdot t \cdot s = t_a \cdot s$$

and by Theorem 2.8 the morphism t is a weak domination. Now, if $\varprojlim f$ is an epimorphism (domination) of C, then $f \cdot p = t \cdot \varprojlim f$ is an epimorphism (weak domination) of pro-C. Consequently f is an epimorphism (weak domination) of pro-C.

The first part of Lemma 2.14 has been proved by M. Moszyńska [Mos$_1$].

It is clear that the implication

2.15. $\varprojlim f$ is an isomorphism of $\mathsf{Gr} \Rightarrow f$ is an isomorphism of pro-Gr

in general fails because $\varprojlim p = 1_{\varprojlim X}$ and it is not true that each uniformly movable pro-group is stable.

However, there are some special interesting cases where 2.15 holds. To show them we need a certain preliminary notion.

An object $X = (X_a, p_a^b, A)$ of pro-Gr satisfies the *Mittag-Leffler condition* (M-L) provided for each $a \in A$ there is a $b \geqslant a$ such that $\operatorname{im} p_a^c = \operatorname{im} p_a^b$ for each $c \geqslant b$ (comp. [A-S]).

In other words, X satisfies the Mittag-Leffler condition iff X is movable as an object of pro-Ens.

2.16. PROPOSITION. *If $f\colon X \to Y$ is an epimorphism of* pro-Gr *and X satisfies* M-L, *then Y satisfies* M-L *as well.*

Proof. By Theorem 2.6, f is an epimorphism as a morphism of pro-Ens. Hence, by Corollary 2.10, f is a weak domination as a morphism of pro-Ens. Consequently, Y is movable as an object of pro-Ens by Theorem 2.11 and the proof is concluded.

2.17. LEMMA. *If* $X = (X_a, p_a^b, A)$ *satisfies* M-L, *then there exists an object* $Y = (Y_a, q_a^b, A)$ *of* pro-**Gr** *isomorphic to* X *and such that all bonds* q_a^b *are epimorphisms.*

Proof. For each $a \in A$ choose an element $b(a) \geqslant a$ such that $\operatorname{im} p_a^c = \operatorname{im} p_a^{b(a)}$ for each $c \geqslant b(a)$. Now, if $a_1 \geqslant a$ and $d \geqslant b(a_1)$, $b(a)$, then

$$p_a^{a_1}(\operatorname{im} p_{a_1}^{b(a_1)}) = p_a^{a_1}(\operatorname{im} p_{a_1}^{d}) = \operatorname{im} p_a^{d} = \operatorname{im} p_a^{b(a)} .$$

Thus, $Y = (Y_a, q_a^b, A)$ is an object of pro-**Gr**, where $Y_a = \operatorname{im} p_a^{b(a)}$ and q_a^b is an epimorphism induced by p_a^b. If $i\colon Y \to X$ is the special morphism induced by the family of inclusions $\{i_a\colon Y_a \to X_a\}_{a \in A}$, then, by Theorem 2.3, i is an isomorphism, which completes the proof.

Lemma 2.17 is well known in the case where X is an inverse sequence of groups.

Now we are ready to show a case where implication 2.15 holds.

2.18. LEMMA (J. Keesling [K_1]). *Let*

$$f\colon X = (X_n, p_n^{n+1}) \to Y = (Y_n, q_n^{n+1})$$

be a morphism of pro-**Gr** *such that* $\varprojlim f$ *is an isomorphism and all* X_n, Y_n *are countable. If* X *and* Y *satisfy* M-L, *then* f *is an isomorphism.*

Proof. By Lemma 2.17 and Theorem 2.2 we can suppose that all p_n^{n+1}, q_n^{n+1} are epimorphisms and f is a special morphism induced by a family $\{f_n\colon X_n \to Y_n\}_{n=1}^{\infty}$.

Observe that each f_n is an epimorphism. We topologize $\varprojlim X$ ($\varprojlim Y$) by giving it the inverse limit topology assuming each X_n (Y_n) to have a discrete topology. Then $\varprojlim X$ ($\varprojlim Y$) is a completely metrizable separable topological group and $\varprojlim f\colon \varprojlim X \to \varprojlim Y$ is a continuous epimorphism.

By the well-known theorem of S. Banach [Ban] $\varprojlim f$ is an open map. Consequently, $\varprojlim f$ is a homeomorphism.

Take $n \in N$. Then $p_n^{-1}(0)$ is a neighbourhood of $0 \in \varprojlim X$, where $p_n\colon \varprojlim X \to X_n$ is the projection. Consequently there exists an $m \geqslant n$ such that

$$q_m^{-1}(0) \subset \varprojlim f(p_n^{-1}(0)),$$

where $q_m\colon \varprojlim Y \to Y_m$ is the projection. Hence

$$p_m^{-1} f_m^{-1}(0) = (\varprojlim f)^{-1} q_m^{-1}(0) \subset p_n^{-1}(0)$$

and consequently

$$\ker f_m = f_m^{-1}(0) \subset p_m p_n^{-1}(0) = (p_n^m)^{-1}(0) = \ker p_n^m .$$

Therefore we have a homomorphism

$$h\colon\ Y_m = X_m/\ker f_m \to X_m/\ker p_n^m = X_n$$

such that

$$f_n \cdot h = q_n^m \quad \text{and} \quad h \cdot f_m = p_n^m.$$

By Theorem 2.3, f is an isomorphism and the proof is finished.

As an immediate consequence of Lemma 2.18 we get the following

2.19. COROLLARY. *Let X be an inverse sequence of countable groups. If X satisfies* M-L *and $\varprojlim X$ is countable, then X is stable.*

An elementary proof of 2.19 may be found in [Dyd_{10}].

It follows from Lemma 2.13 that each pro-group dominated in pro-**Gr** by a group is stable. By Lemma 2.14 each uniformly movable pro-group is weakly dominated by a group. So the statement "each pro-group weakly dominated by a group is stable" is not valid in general. However, it is valid in a certain special case.

2.20. LEMMA. *If Y is a pro-group weakly dominated by a finitely generated Abelian group X, then it is stable.*

Proof. Let $f\colon X \to Y$ be a weak domination. By Proposition 2.16, Y satisfies M-L and by Lemma 2.17 we may suppose that $Y = (Y_a, q_a^{a'}, A)$, where all $q_a^{a'}$ are epimorphisms. By Theorem 2.6, each $f_a\colon X \to Y_a$ is an epimorphism. Hence each Y_a is a finitely generated Abelian group.

Let $(B, \leqslant)$ be the directed set where B is the set of all subgroups of X which are of the form $\ker f_a$ for some $a \in A$, and "$\leqslant$" is defined by inclusion.

Then Y is isomorphic to $(X/b, p_b^{b'}, B)$, where $p_b^{b'}\colon X/b' \to X/b$ are quotient homomorphisms. Since X is a finitely generated Abelian group, we infer that B is countable. Consequently we may suppose that $Y = (Y_n, q_n^{n+1})$ is an inverse sequence. By the well-known theorem characterizing finitely generated Abelian groups (see [S], p. 9) we have

$$X = Z^k \oplus \operatorname{Tor} X \quad \text{and} \quad Y_n = Z^{k_n} \oplus \operatorname{Tor} Y_n$$

for each n. Since each Y_n is an epimorphic image of X, we may assume $k_n = l$ for each n. Therefore,

$$\operatorname{rank} \ker q_n^{n+1} = \operatorname{rank} Y_{n+1} - \operatorname{rank} Y_n = l - l = 0$$

and hence $\ker q_n^{n+1} \subset \operatorname{Tor} Y_{n+1}$. Consequently,

$$q_n^{n+1}(Z^l) = Z^l \quad \text{and} \quad q_n^{n+1}(\operatorname{Tor} Y_{n+1}) = \operatorname{Tor} Y_n.$$

Moreover, Tor: **AbGr** $\to$ **AbGr** is a functor, where **AbGr** is the full subcategory of **Gr** whose objects are all Abelian groups. Hence pro-Tor$f$ is a weak domination and this implies $f_n(\operatorname{Tor} X) = \operatorname{Tor} Y_n$ for each n.

But the torsion part of each finitely generated Abelian group is finite and therefore there must be an n such that $\ker q_m^{m+1} = 0$ for each $m \geqslant n$, which completes the proof.

2.21. Remark. Pro-categories have been introduced by A. Grothendieck [Groth] in greater generality than needed here. However, D. A. Edwards and H. M. Hastings [E-H_2] have shown that each object of a pro-category in the sense of Grothendieck is isomorphic to an object of a pro-category described here.

§3. Shape category

Shape category has been introduced by K. Borsuk [Bor_1] for compacta. Next R. H. Fox [Fox] and S. Mardešić [Mar_1] have gradualy extended the class of objects to all topological spaces. Here we give description of shape category, using the results of K. Morita [$Mor_{1,2}$].

The following categories are used throughout:

- T — topological spaces and maps,
- T_0 — pointed topological spaces and maps preserving base points,
- CW — the full subcategory of T whose objects are CW complexes,
- CW_0 — the full subcategory of T_0 whose objects are pointed CW complexes,
- HT — homotopy category of T,
- HT_0 — homotopy category of T_0,
- HCW — homotopy category of CW,
- HCW_0 — homotopy category of CW_0,
- H — the full subcategory of HCW whose objects are connected CW complexes,
- H_0 — the full subcategory of HCW_0 whose objects are pointed connected CW complexes.

We usualy supress base points when describing objects of the above categories of pointed spaces.

Let $(X, x) \in \mathrm{Ob}\mathsf{T}_0$. Let $\{U_a\}_{a \in A}$ be the family of all locally finite normal open covers of X such that each U_a has exactly one member containing x. Then we have an inverse system $\check{C}(X, x) = \big((X_a, x_a), [p_a^b], A\big)$ in HCW_0 by taking the nerves of U_a, by ordering A by means of the refinement of covers, and by taking the homotopy classes of canonical projections. This system is called the *Čech system* of (X, x) (see [Mor_1]). Moreover, we have a morphism

$$p = \{[p_a]\}_{a \in A} \colon (X, x) \to \check{C}(X, x)$$

of pro-HT_0, where $p_a\colon (X, x)\to(X_a, x_a)$ are canonical maps. The main property of the morphism p is expressed in the following

3.1. THEOREM ([Mor$_1$], Theorem 4.3). *If* $Y \in \mathrm{Ob}\,\mathsf{HCW}_0$, *then for each morphism* $f\colon (X, x)\to Y$ *of* HT_0 *there exists a unique morphism* $g\colon \check{C}(X, x)\to Y$ *of* pro-HCW_0 *such that* $g\cdot p = f$.

It is well known that the inclusion functor

$$I\colon \mathsf{HCW}_0\to\mathsf{HT}_0$$

has a right adjoint

$$\mathrm{Sin}\colon \mathsf{HT}_0\to\mathsf{HCW}_0,$$

where $\mathrm{Sin}(X, x)$ is the geometric realization of the singular complex of (X, x). Consequently,

$$\text{pro-}I\colon \text{pro-}\mathsf{HCW}_0\to\text{pro-}\mathsf{HT}_0$$

has the right adjoint

$$\text{pro-Sin}\colon \text{pro-}\mathsf{HT}_0\to\text{pro-}\mathsf{HCW}_0.$$

Let us show that there exists a left adjoint

$$\check{C}\colon \text{pro-}\mathsf{HT}_0\to\text{pro-}\mathsf{HCW}_0$$

of pro-I. First let us observe that Theorem 3.1 can be improved in the following way:

3.2. THEOREM. *If* $Y \in \mathrm{Ob}\,\text{pro-}\mathsf{HCW}_0$, *then for each morphism* $f\colon (X, x)\to Y$ *of* pro-HT_0 *there exists a unique morphism* $g\colon \check{C}(X, x)\to Y$ *of* pro-HCW_0 *such that* $g\cdot p = f$.

Proof. Let

$$f = \{f_d\}_{d\in D} : (X, x)\to Y = (Y_d, q_d^{d'}, D).$$

Then for each $d \in D$ there exists a $g_d\colon \check{C}(X, x)\to Y_d$ such that $g_d\cdot p = f_d$. Since $q_d^{d'}\cdot f_{d'} = f_d$ for $d \leqslant d'$, we infer $q_d^{d'}\cdot g_{d'} = g_d$ for $d \leqslant d'$. Thus $g = \{g_d\}_{d\in D}$ satisfies $g\cdot p = f$. If $g' = \{g'_d\}_{d\in D}$ satisfies $g'\cdot p = f$, then $g_d\cdot p = g'_d\cdot p$ for each $d \in D$. Hence $g'_d = g_d$ for each $d \in D$ and consequently $g' = g$. Thus the proof is concluded.

Now, if $f\colon (X, x)\to(Y, y)$ is a morphism of HT_0, then there exists a unique morphism $\check{C}f\colon \check{C}(X, x)\to\check{C}(Y, y)$ such that $\check{C}f\cdot p = q\cdot f$, where $q\colon (Y, y)\to\check{C}(Y, y)$. Thus we have defined a functor $\check{C}\colon \mathsf{HT}_0\to\text{pro-}\mathsf{HCW}_0$ and we are going to extend it to a functor $\check{C}\colon \text{pro-}\mathsf{HT}_0\to\text{pro-}\mathsf{HCW}_0$.

Recall that pro-categories in the sense of Grothendieck are categories with inverse limits (see [A-M], p. 165) and since each object of a pro-category in the sense of Grothendieck is isomorphic to an object of a pro-category defined here (see [E-H$_2$]), we infer that our pro-categories are categories with inverse limits.

So let us define the functor $\check{C}$: pro-$\mathsf{HT}_0 \to$ pro-HCW_0 as the composition of pro-$\check{C}$: pro-$\mathsf{HT}_0 \to$ pro-pro-HCW_0 and the inverse limit functor

$$\varprojlim: \text{pro-pro-}\mathsf{HCW}_0 \to \text{pro-}\mathsf{CHW}_0.$$

Now, if $X = (X_a, p_a^b, A)$ is an object of pro-HT_0, then the family $\{p_a: X_a \to \check{C}X_a\}_{a \in A}$ induces a morphism $p_X : X \to \check{C}X$ which has (in view of Theorem 3.2) the following property:

If $Y \in \mathrm{Ob}$pro-HCW_0, then for each morphism $f: X \to Y$ of pro-HT_0 there exists a unique morphism $g: \check{C}X \to Y$ with $g \cdot p_X = f$.

Moreover, if $h: X \to Y$ is a morphism of pro-HT_0, then $p_Y \cdot h = \check{C}h \cdot p_X$. Thus $\check{C}$ is a left adjoint to pro-I.

Now we can describe the *shape category* S_0. Its objects are all pointed topological spaces and for $\mathsf{S}_0(X, Y)$ we take pro-$\mathsf{HCW}_0(\check{C}X, \check{C}Y)$, where the composition of morphisms is induced from pro-HCW_0.

The category S_0 will be considered together with the shape functor S: $\mathsf{HT}_0 \to \mathsf{S}_0$ defined as follows:

$$SX = X \text{ for } X \in \mathrm{Ob}\mathsf{HT}_0 \quad \text{and} \quad Sf = \check{C}f \text{ for } f \in \mathsf{S}_0(X, Y).$$

It is easy to see that S_0 may be considered as a full subcategory of pro-HCW_0. Therefore in the sequel we shall identify $X \in \mathrm{Ob}\mathsf{T}_0$ and $\check{C}X$.

One can describe in a similar way the shape category for unpointed topological spaces.

If $X, Y \in \mathrm{Ob}\mathsf{S}_0$, then $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$ means that X and Y are isomorphic in S_0, and $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$ means that X dominates Y in S_0.

Let us emphasize that applications of homotopy theory to general topological spaces are based on the existence of the functor Sin: $\mathsf{HT}_0 \to \mathsf{HCW}_0$ and applications of shape theory to general topological spaces are based on the existence of the functor $\check{C}$: pro-$\mathsf{HT}_0 \to$ pro-HCW_0. In this way shape theory classifies topological spaces from the point of view of their global properties.

§4. Deformation dimension

By the *deformation dimension* $\mathrm{ddim}\,X$ of an object X of pro-HCW_0 (pro-HCW) we mean the least integer $n \geqslant 0$ such that the following condition is satisfied:

For each morphism $f: X \to K \in \mathrm{Ob}\mathsf{HCW}_0$ ($\mathrm{Ob}\mathsf{HCW}$) of pro-HCW_0 (pro-HCW) there exists a morphism $g: X \to K^n$ with $i \cdot g = f$, where i: $K^n \to K$ is the homotopy class of the inclusion of the n-skeleton of K into K.

If $((X_a, x_a), p_a^b, A) \in \mathrm{Ob}pro\mathsf{HCW}_0$, then $(X_a, p_a^b, A) \in \mathrm{Ob}$pro-$\mathsf{HCW}$ and Lemma 1.1 of [S-W] implies

$$\mathrm{ddim}((X_a, x_a), p_a^b, A) = \mathrm{ddim}(X_a, p_a^b, A).$$

4.1. THEOREM. *If* $X = (X_a, p_a^{a'}, A) \in \mathrm{Ob}\,\text{pro-}\mathsf{HCW}_0$, *then* $\operatorname{ddim} X \leqslant n$ *iff there exists an object* $Y = (Y_a, q_a^{a'}, A)$ *of* pro-HCW_0 *isomorphic to* X *such that*

$$Y_a^n = Y_a \quad \textit{for each } a \in A.$$

Proof. Sufficiency is obvious.

Necessity. Let A be the set of all pairs (a, a_0) of elements of A such that $a \leqslant a_0$ and there exists an $r_a^{a_0}\colon X_{a_0} \to X_a^n$ with $i \cdot r_a^{a_0} = p_a^{a_0}$ (i is the homotopy class of the inclusion $X_a^n \hookrightarrow X_a$). Since $\operatorname{ddim} X \leqslant n$, we infer that for each $a \in A$ there exists an $a_0 \in A$ with $(a, a_0) \in A$.

Let us define a directed set $(A, \leqslant)$ as follows: $(a, a_0) \leqslant (a', a_0')$ iff either $a_0 \leqslant a'$ or $(a, a_0) = (a', a_0')$.

Then we define $X' = (X'_a, s_a^{a'}, A)$ and $Y = (Y_a, q_a^{a'}, A)$ as follows: if $a = (a, a_0) \leqslant (a', a_0') = a'$, then

$$X'_a = X_{a_0},\ Y_a = X_a^n, \qquad s_a^{a'} = p_{a_0}^{a_0'}, \qquad q_a^{a'} = r_a^{a_0} \cdot p_{a_0}^{a_0'} \cdot j$$

(j is the homotopy class of the inclusion $X_{a'}^n \hookrightarrow X_{a'}$).

Setting $r_a = r_a^{a_0}\colon X'_a \to Y_a$ for $a = (a, a_0) \in A$, we get for $a' = (a', a_0') \geqslant a$

$$q_a^{a'} \cdot r_{a'} = r_a^{a_0} \cdot p_{a_0}^{a'} \cdot j \cdot r_{a'}^{a_0'} = r_a^{a_0} \cdot p_{a_0}^{a'} \cdot p_{a'}^{a_0'} = r_a^{a_0} \cdot p_{a_0}^{a_0'} = r_a \cdot s_a^{a'}.$$

Thus there exists a special morphism $r\colon X' \to Y$ induced by the family $\{r_a\}_{a \in A}$.

Let $a = (a, a_0) \in A$. Take an element $a' \in A$ such that $a' = (a_0, a') \in A$. Then $i\colon X_{a_0}^n = Y_{a'} \to X'_a = X_{a_0}$ and

$$r_a \cdot i = r_a^{a_0} \cdot i = r_a^{a_0} \cdot p_{a_0}^{a_0} \cdot i = q_a^{a'}, \qquad i \cdot r_{a'} = i \cdot r_{a_0}^{a'} = p_{a_0}^{a'} = s_a^{a'}.$$

By Theorem 2.3 the morphism r is an isomorphism. Since it is clear that X and X' are isomorphic, we infer that X and Y are isomorphic. Thus the proof is concluded.

Theorem 4.1 has been proved by the author [Dyd$_4$] for X being a Čech system of a pointed topological space.

Theorem 4.1 implies that $\operatorname{ddim} X = \operatorname{ddim} Y$ if X and Y are isomorphic. Let us prove a stronger result.

4.2. THEOREM. *Let* $f\colon X \to Y$ *be a weak domination of* pro-HCW_0. *If* $\operatorname{ddim} X \leqslant n$, *then* $\operatorname{ddim} Y \leqslant n$ *as well.*

Proof. By Theorems 2.2 and 4.1 we may suppose that $X = (X_a, [p_a^b], A)$, where $X_a^n = X_a$ for each $a \in A$, $Y = (Y_a, [q_a^b], A)$ and f is a special morphism induced by a family $\{[f_a]\}_{a \in A}$.

Let $[g_a]\colon Y_a \to K \in \mathrm{Ob}\,\mathsf{HCW}_0$ be a representative of a morphism $g\colon Y \to K$ of pro-HCW_0. By Theorem 2.8 there exist a $b \geqslant a$ and a $g_a^b\colon Y_b \to X_a$ with $f_a \cdot g_a^b \simeq q_a^b$. By the cellular approximation theorem (see [L-W],

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p. 72) there exists a map $r: X_a \to K^n$ such that $i \cdot r \simeq g_a \cdot f_a$, where $i: K^n \to K$ is the inclusion map. Now

$$i \cdot r \cdot g_a^b \simeq g_a \cdot f_a \cdot g_a^b \simeq g_a \cdot q_a^b.$$

Hence the morphism $h: Y \to K^n$ whose representative is $[r_a \cdot g_a^b]$ satisfies $[i] \cdot h = g$. Thus the proof is finished.

According to our convention we identify a topological space X and its Čech system $\check{C}X$. Consequently the deformation dimension of a topological space X is defined as $\operatorname{ddim} \check{C}X$.

The deformation dimension has been introduced by the author [Dyd$_4$] for topological spaces as a generalization of the *fundamental dimension* Fd. It satisfies the inequalities

$$\dim X \geqslant \operatorname{Sd} X = \min\{\dim Y\colon \operatorname{Sh} Y \geqslant \operatorname{Sh} X\} \geqslant \operatorname{ddim} X$$

(see [Dyd$_4$]), where dim is the covering dimension considered in [Mor$_1$]. For compacta we have

$$\operatorname{ddim} X = \operatorname{Fd} X$$

by Corollary (1.7) of [Now].

Observe that Theorem E of [Wall] implies for each CW complex X the existence of a CW complex Q of the same homotopy type as X with $\dim Q = \max(3, \operatorname{ddim} X)$.

Notice that D. A. Edwards and R. Geoghegan [E-G$_6$] consider the *homotopy dimension* of an object X of pro-HCW_0 which coincides with the deformation dimension (in view of Theorem 4.1).

§5. Some properties of n-equivalences of pro-$\mathsf{H}_\bullet$.

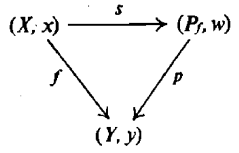
By a *pointed fibration* we mean a morphism

$$p\colon (T, t_0) \to (S, s_0)$$

of T_0 such that for each two maps $g\colon (Z, z_0) \to (T, t_0)$ and $H\colon (Z, z_0) \times I \to (S, s_0)$ satisfying $p \cdot g(z) = H(z, 0)$ for $z \in Z$ there exists a map $G\colon (Z, z_0) \times I \to (T, t_0)$ with $p \cdot G = H$ and $g(z) = G(z, 0)$ for $z \in Z$.

Notice that the proof of Theorem 9 in [S] (p. 99) remains valid in the following more general case:

For each map $f\colon (X, x) \to (Y, y)$ there exists a commutative diagram in T_0 such that $[s]$ is an isomorphism of HT_0 and p is a pointed fibration.



5.1. LEMMA. *Let*

$$\begin{array}{ccccccc}
(K^k \cup K_0, k_0) & \xrightarrow{f} & (X_c, x_c) & \xrightarrow{p_b^c} & (X_b, x_b) & \xrightarrow{p_a^b} & (X_a, x_a) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f_c} & & \downarrow{\scriptstyle f_b} & & \downarrow{\scriptstyle f_a} \\
(K^{k+1}, k_0) & \xrightarrow{g} & (Y_c, y_c) & \xrightarrow{q_b^c} & (Y_b, y_b) & \xrightarrow{q_a^b} & (Y_a, y_a)
\end{array}$$

be a commutative diagram in T_0, *where* K *is a* CW *complex whose each open cell has a vertex in its closure and* $K_0 \subset K^{k+1}$ *is a subcomplex.*

If f_a *is a pointed fibration,*

$$\ker \pi_k[f_c \cdot f] \subset \ker \pi_k[p_b^c \cdot f], \quad \operatorname{im} \pi_{k+1}[q_a^b] \subset \operatorname{im} \pi_{k+1}[f_a],$$

$$\text{and} \quad f(K^0) = \{x_c\},$$

then there is a map $h\colon (K^{k+1}, k_0) \to (X_a, x_a)$ *such that*

$$f_a \cdot h = q_a^b \cdot q_b^c \cdot g \quad \text{and} \quad h/K^k \cup K_0 = p_a^b \cdot p_b^c \cdot f.$$

Proof. Let e be a $(k+1)$-cell of K not contained in K_0 and let $r\colon (B^{k+1}, S^k) \to (e, \dot{e})$ be its characteristic map. Then there is a $v \in S^k$ with $r(v) \in K^0$. Now

$$f_c \cdot f \cdot \big(r/(S^k, v)\big) = g \cdot r/(S^k, v) \quad \text{and} \quad f \cdot r(v) \in f(K^0) = \{x_c\}.$$

Since $\ker \pi_k[f_c \cdot f] \subset \ker \pi_k[p_b^c \cdot f]$ and $g \cdot r/S^k$ extends over B^{k+1}, we infer that $p_b^c \cdot f \cdot \big(r/(S^k, v)\big)$ extends to $u\colon (B^{k+1}, v) \to (X_b, x_b)$.

Let us regard $g \cdot r$ as a map defined on the upper hemisphere B_2^{k+1} of the $(k+1)$-sphere S^{k+1} and u as a map defined on the lower hemisphere B_1^{k+1} of the $(k+1)$-sphere S^{k+1}.

Let us define the map $s\colon (S^{k+1}, v) \to (Y_b, y_b)$ as follows: $s(x) = f_b \cdot u(x)$ for $x \in B_1^{k+1}$ and $s(x) = q_b^c \cdot g \cdot r(x)$ for $x \in B_2^{k+1}$.

Since $\operatorname{im} \pi_{k+1}[q_a^b] \subset \operatorname{im} \pi_{k+1}[f_a]$ and f_a is a pointed fibration, we infer that there exists a map

$$\bar{s}\colon (S^{k+1}, v) \to (X_a, x_a)$$

such that $f_a \cdot \bar{s} = q_a^b \cdot s$.

Let $r_1\colon B_1^{k+1} \times I \to B_1^{k+1} \times \{0,1\} \cup \{v\} \times I$ be a strong deformation retraction. Let us define

$$H_1\colon B_1^{k+1} \times \{0,1\} \cup \{v\} \times I \to X_a$$

as follows:

$$H_1(z) = \begin{cases} \bar{s}(x) & \text{for} \quad z = (x, 0), \\ p_a^b \cdot u(x) & \text{for} \quad z = (x, 1), \\ x_a & \text{for} \quad z = (v, t). \end{cases}$$

Let $p\colon B_1^{k+1}\times I\to B_1^{k+1}$ be the projection. Then the maps $f_a\cdot H_1\cdot r_1$ and $q_a^b\cdot f_b\cdot u\cdot p$ are homotopic rel. $B_1^{k+1}\times\{0,1\}\times\{v\}\times I$. Indeed,

$$f_a\cdot H_1 = q_a^b\cdot f_b\cdot u\cdot p/B_1^{k+1}\times\{0,1\}\cup\{v\}\times I$$

and r_1 is a strong deformation retraction.

Let $B = S^{k+1}\times\{0,1\}\cup B_1^{k+1}\times I$. Take the map $H\colon B\to Y_a$ defined as follows: $H(y,0) = H(y,1) = f_a\cdot\bar{s}(y)$ for $y\in S^{k+1}$ and $H|B_1^{k+1}\times I = f_a\cdot H_1\cdot r_1$. Then H is homotopic to $q_a^b\cdot s\cdot q|B$, where $q\colon S^{k+1}\times I\to S^{k+1}$ is the projection. By the homotopy extension theorem there exists an extension $\bar{H}\colon S^{k+1}\times I\to Y_a$ of H. On the other hand, Theorem 9 in [S] on p. 416 implies the existence of the map $G\colon S^{k+1}\times I\to X_a$, where $f_a\cdot G = \bar{H}$, $G(y,0) = \bar{s}(y)$ for $y\in S^{k+1}$ and $G|B_1^{k+1}\times I = H_1\cdot r_1/B_1^{k+1}\times I$.

Let $g_1(x) = G(x,1)$ for $x\in B_2^{k+1}$. Then

$$f_a\cdot g_1 = q_a^b\cdot q_b^c\cdot g\cdot r \quad\text{and}\quad g_1/S^k = p_a^b\cdot p_b^c\cdot f\cdot(r/S^k).$$

Thus for each e there exists a map $h_e\colon K^k\cup K_0\cup e\to X_a$ such that

$$h_e/K^k\cup K_0 = p_a^b\cdot p_b^c\cdot f \quad\text{and}\quad f_a\cdot h_e = q_a^b\cdot q_b^c\cdot g/K^k\cup K_0\cup e.$$

If we define

$$h(x) = h_e(x) \quad\text{for } x\in e \quad\text{and}\quad h/K^k\cup K_0 = p_a^b\cdot p_b^c\cdot f,$$

then we get the desired map.

If $F\colon \mathsf{C}\to\mathsf{Gr}$ is a covariant functor, then by $\check{F}$ we denote the composition $\varprojlim\cdot(\text{pro-}F)$.

In the sequel we shall need the following

5.2. THEOREM. *Let X be an inverse sequence in H_0. Then there exists a morphism $f\colon Q\to X$ of* pro-H_0 *with $Q\in\mathrm{Ob}\mathsf{H}_0$ such that*

(1) pro-$\mathsf{H}_0(P,\cdot)f$ *is a surjection for each $P\in\mathrm{Ob}\mathsf{H}_0$,*

(2) *if all pro-groups* pro-$\pi_n X$ *satisfy* M-L, *then $\check{\pi}_n f$ is an isomorphism for each n,*

(3) *if X is movable, then f is a weak domination of* pro-H_0.

Proof. If $X = (X_n, [p_n^{n+1}])$, then we can find inductively a sequence of maps $s_n\colon X_n\to Y_n$, $q_n^{n+1}\colon Y_{n+1}\to Y_n$ such that the diagram

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{s_{n+1}} & Y_{n+1} \\ {\scriptstyle p_n^{n+1}}\downarrow & & \downarrow{\scriptstyle q_n^{n+1}} \\ X_n & \xrightarrow{s_n} & Y_n \end{array}$$

is commutative in T_0, $[s_n]$ is an isomorphism of HT_0 for each n and q_n^{n+1} is a pointed fibration for each n.

Let $Y = \varprojlim(Y_n, q_n^{n+1})$ and let $q_n\colon Y \to Y_n$ be the projection for each n. Now, if $g = \{g_n\}_{n=1}^{\infty}\colon Z \to (Y_n, [q_n^{n+1}])$ is a morphism of pro-HT_0 and $Z \in \mathrm{Ob}\mathsf{HT}_0$, then we can find maps $g_n'\colon Z \to Y_n$ such that $[g_n'] = g_n$ and $g' = \{g_n'\}_{n=1}^{\infty}$ is a morphism of pro-T_0. Consequently there is a map $g''\colon Z \to Y$ with $g_n' = q_n \cdot g''$ for each n. Thus pro-$\mathsf{HT}_0(Z, \cdot)q$ is a surjection for each $Z \in \mathrm{Ob}\mathsf{HT}_0$, where $q = \{[q_n]\}_{n=1}^{\infty}\colon Y \to (Y_n, [q_n^{n+1}])$. In particular, $\check{\pi}_k q\colon \pi_k Y \to \check{\pi}_k(Y_n, [q_n^{n+1}])$ is an epimorphism for each k.

Suppose that pro-$\pi_k X$ satisfies M-L for each k. Then pro-$\pi_k(Y_n, [q_n^{n+1}])$ satisfies M-L for each k.

Let $\varphi\colon S^k \to Y$ be a representative of an element of $\ker \check{\pi}_k q$. We may assume that $\mathrm{im}\,\pi_{k+1}[q_n^{n+1}] = \mathrm{im}\,\pi_{k+1}[q_n^m]$ for each $m > n$. Suppose we have maps

$$t_i\colon B^{k+1} \to Y_i \quad \text{for } i \leqslant l$$

such that

$$t_i / S^k = q_i \cdot \varphi \quad \text{and} \quad q_{i-1}^i \cdot t_i = t_{i-1} \quad \text{for} \quad i \leqslant l.$$

Then the diagram

$$\begin{array}{ccccc} S^k & \xrightarrow{q_{l+1}\cdot\varphi} & Y_{l+1} & \xrightarrow{\mathrm{id}} & Y_{l+1} \xrightarrow{\mathrm{id}} Y_{l+1} \\ \downarrow j & & \downarrow q_l^{l+1} & & \downarrow q_l^{l+1} \qquad \downarrow q_{l-1}^{l+1} \\ B^{k+1} & \xrightarrow{t_l} & Y_l & \xrightarrow{\mathrm{id}} & Y_l \xrightarrow{q_{l-1}^{l}} Y_{l-1} \end{array}$$

satisfies the conditions of Lemma 5.1. Consequently there is a map $u\colon B^{k+1} \to Y_{l+1}$ with

$$u \mid S^k = q_{l+1} \cdot \varphi \quad \text{and} \quad q_{l-1}^{l+1} \cdot u = t_{l-1}.$$

Finally we can find a morphism

$$\{u_n\}_{n=1}^{\infty}\colon B^{k+1} \to (Y_n, q_n^{n+1}) \quad \text{with } u_n / S^k = q_n \cdot \varphi.$$

Thus $[\varphi] = 0$ and consequently $\ker \check{\pi}_k q = 0$ for each k.

Now let us observe that

(a) X and $(Y_n, [q_n^{n+1}])$ are isomorphic,

(b) there is a morphism $g\colon \mathrm{Sin}\, Y \to Y$ such that $\check{\pi}_k g$ is an isomorphism for each k and $\mathsf{H}_0(P, \cdot)g$ is a bijection for each $P \in \mathrm{Ob}\mathsf{H}_0$,

which completes the proof of parts (1) and (2) of Theorem 5.2.

Suppose that X is movable. Then for each n there exist an $m \geqslant n$ and an $r\colon X_m \to X$ with $p_n \cdot r = p_n^m$. By condition (1) there is a morphism $g\colon X_m \to Q$ with $f \cdot g = r$. Hence

$$p_n^m = p_n \cdot r = p_n \cdot f \cdot g = f_n \cdot g,$$

and by Theorem 2.8, f is a weak domination.

Thus the proof of Theorem 5.2 is concluded.

Part 2 of Theorem 5.2 has been proved by D. A. Edwards and R. Geoghegan [E-G$_1$] under the additional assumption that X is movable.

5.3. LEMMA. *Let*

$$\begin{array}{ccc} (X_k, x_k) & \xrightarrow{[f_k]} & (Y_k, y_k) \\ \downarrow{\scriptstyle [p^k_{k+1}]} & & \downarrow{\scriptstyle [q^k_{k+1}]} \\ (X_{k+1}, x_{k+1}) & \xrightarrow{f_{k+1}} & (Y_{k+1}, y_{k+1}) \end{array} \qquad 0 \leqslant k \leqslant 2n-1$$

be a collection of $2n$ commutative diagrams in HT_0 such that X_0, Y_0 are arcwise connected and

$$\ker \pi_k[f_{2k}] \subset \ker \pi_k[p^{2k}_{2k+1}] \qquad \text{for} \quad 1 \leqslant k \leqslant n,$$

$$\operatorname{im} \pi_{k+1}[q^{2k+1}_{2k+2}] \subset \operatorname{im} \pi_{k+1}[f_{2k+2}] \qquad \text{for} \quad 0 \leqslant k \leqslant n.$$

Then for each pointed CW *complex (K, k_0) the following conditions are satisfied:*

(1) *for each map $f\colon (K^n, k_0) \to (Y_0, y_0)$ there is a map $g\colon (K^n, k_0) \to (X_{2n}, x_{2n})$ such that*

$$f_{2n} \cdot g \simeq q^{2n-1}_{2n} \cdot \ldots \cdot q^0_1 \cdot f,$$

(2) *if $f, g\colon (K^{n-1}, k_0) \to (X_0, x_0)$ and $f_0 \cdot f \simeq f_0 \cdot g$, then*

$$p^{2n-1}_{2n} \cdot \ldots \cdot p^0_1 \cdot f \simeq p^{2n-1}_{2n} \cdot \ldots \cdot p^0_1 \cdot g.$$

Proof. Since each pointed CW complex has the homotopy type of a pointed simplicial complex (see [Mar$_1$], Theorem 4.1), it is enough to prove Lemma 5.3 in the case where K is a simplicial complex. Moreover, we can suppose that all f_k are pointed fibrations and $q^k_{k+1} \cdot f_k = f_{k+1} \cdot p^k_{k+1}$ for $0 \leqslant k \leqslant 2n-1$.

Let $f, g\colon (K^{n-1}, k_0) \to (X_0, x_0)$ be maps such that K is a simplicial complex, $k_0 \in K^0$ and $f_0 \cdot f \simeq f_0 \cdot g$. Since X_0 is arcwise connected, we may assume that $f(K^0) = g(K^0) = \{x_0\}$. Let $H\colon (K^{n-1}, k_0) \times I \to (Y_0, y_0)$ be a homotopy joining $f_0 \cdot f$ and $f_0 \cdot g$. Let

$$L = K^{n-1} \times I, \qquad L_0 = K^{n-1} \times \{0, 1\} \cup \{k_0\} \times I$$

be CW complexes with the natural structure (see [L-W], p. 58).

Define $t\colon (L_0, (k_0, 0)) \to (X_0, x_0)$ by the formula

$$t(z) = \begin{cases} f(x) & \text{for} \quad z = (x, 0), x \in K^{n-1}, \\ g(x) & \text{for} \quad z = (x, 1), x \in K^{n-1}, \\ x_0 & \text{for} \quad z = (k_0, t), t \in I. \end{cases}$$

We find inductively maps $h_k\colon L^k \cup L_0 \to X_{2k}$, $0 \leqslant k \leqslant n$, such that

$$f_{2k} \cdot h_k = q_{2k}^{2k-1} \cdot \ldots \cdot q_1^0 \cdot H | L^k \cup L_0 \quad \text{and} \quad h_k | L_0 = p_{2k}^{2k-1} \cdot \ldots \cdot p_1^0 \cdot t.$$

Let $h_0 = t$. Suppose we have h_k for a certain $k \leqslant n-1$.

Let $L_1 = L_0^{k+1}$. By Lemma 5.1 there is a map

$$h_{k+1}\colon L^{k+1} \to X_{2k+2}$$

such that

$$f_{2k+2} \cdot h_{k+1} = q_{2k+2}^{2k+1} \cdot \ldots \cdot q_1^0 \cdot H | L^{k+1}$$

and

$$h_{k+1} | L^k \cup L_1 = p_{2k+2}^{2k+1} \cdot \ldots \cdot p_1^0 \cdot h_k | L^k \cup L_1.$$

Let $h_{k+1}(x) = p_{2k+2}^{2k+1} \cdot p_{2k+1}^{2k} \cdot h_k(x)$ for $x \in L_0$. Then we get the map $h_{k+1}\colon L^{k+1} \cup L_0 \to X_{2k+2}$ satisfying the desired conditions.

Since h_n is a homotopy joining

$$p_{2n}^{2n-1} \cdot \ldots \cdot p_1^0 \cdot f \quad \text{and} \quad p_{2n}^{2n-1} \cdot \ldots \cdot p_1^0 \cdot g,$$

condition (2) of Lemma 5.3 is satisfied.

The verification of condition (1) is even simpler.

Let $f\colon (K^n, k_0) \to (Y_0, y_0)$ be a map, where K is a simplicial complex. Since Y_0 is arcwise connected, we may assume that $f(K^0) = \{y_0\}$.

We find inductively maps $g_k\colon K^k \to X_{2k}$, $0 \leqslant k \leqslant n$, such that

$$g_k(K^0) = \{x_{2k}\} \quad \text{and} \quad f_{2k} \cdot g_k = q_{2k}^{2k-1} \cdot \ldots \cdot q_1^0 \cdot f | K^k.$$

Let $g_0\colon K^0 \to X_0$ map K^0 into x_0. Suppose we have $g_k\colon K^k \to X_{2k}$ for a certain $0 \leqslant k \leqslant n-1$.

By Lemma 5.1 there is a $g_{k+1} K^{k+1} \to X_{2k+2}$ such that

$$f_{2k+2} \cdot g_{k+1} = q_{2k+2}^{2k+1} \cdot \ldots \cdot q_1^0 \cdot f | K^{k+1} \quad \text{and} \quad g_{k+1} | K^k = p_{2k+2}^{2k+1} \cdot p_{2k+1}^{2k} \cdot g_k.$$

Setting $g = g_n$, we get the desired map.

Thus the proof of Lemma 5.3 is completed.

We need Lemma 5.3 for the proof of the main lemma in this section. This is Lemma 5.4, which has been proved by the author [Dyd$_4$]; its original proof was based on some results of K. Morita [Mor$_3$]. The additional advantage of Lemma 5.3 is its usefulness for the considerations in §8.

Before stating Lemma 5.4, we introduce some notions.

A morphism $f\colon X \to Y$ of pro-$\mathbf{H}_0$ is called an *n-equivalence* ($n \geqslant 1$) provided pro-$\pi_k f$ is an isomorphism for $k \leqslant n-1$ and an epimorphism for $k = n$. A morphism f is called a *weak equivalence* (comp. [B-K], p. 76) provided it is an n-equivalence for each n.

The notion of n-equivalence corresponds to the notion denoted by the same term in homotopy theory (see [S], p. 404).

5.4. LEMMA. *Let* $f\colon (X_a, p_a^b, A)\to(Y_a, q_a^b, A)$ *be a special morphism of* pro-H_0 *induced by a family* $\{f_a\}_{a\in A}$. *If* f *is an* n*-equivalence, then for each* $a\in A$ *there is a* $b\geqslant a$ *such that the following conditions are fulfilled*:

(1) *if* $g\colon Z\to Y_b$ *is a morphism of* pro-HCW_0 *and* $\operatorname{ddim} Z\leqslant n$, *then there is a morphism* $h\colon Z\to X_a$ *such that* $f_a\cdot h = q_a^b\cdot g$,

(2) *if* h, $h'\colon Z\to X_b$ *are morphisms of* pro-HCW_0 *such that* $\operatorname{ddim} Z\leqslant n-1$ *and* $f_b\cdot h = f_b\cdot h'$, *then*

$$p_a^b\cdot h = p_a^b\cdot h'.$$

Proof. Let $a\in A$. Using Theorems 2.3 and 2.6, we find a sequence $a = a_{2n}\leqslant\ldots\leqslant a_0 = b$ of elements of A such that $\operatorname{im}\pi_{k+1}q_{a_{2k+2}}^{a_{2k+1}}\subset \operatorname{im}\pi_{k+1}f_{a_{2k+2}}$ for $0\leqslant k<n$ and there are homomorphisms

$$h_k\colon \pi_k(Y_{a_{2k}}, y_{a_{2k}})\to\pi_k(X_{a_{2k+1}}, x_{a_{2k+1}})$$

such that $h_k\cdot\pi_k f_{a_{2k}} = \pi_k p_{a_{2k+1}}^{a_{2k}}$ for $1\leqslant k<n$. Hence

$$\ker\pi_k f_{a_{2k}}\subset\ker\pi_k p_{a_{2k+1}}^{a_{2k}}\quad\text{for}\quad 1\leqslant k<n.$$

Thus, if Z is a pointed CW complex, Lemma 5.4 is a consequence of Lemma 5.3. In view of Theorem 4.1 conditions (1) and (2) are satisfied for each $Z\in\operatorname{Ob}$pro-HCW_0.

As an easy consequence of Lemma 5.4 we get the following

5.5. THEOREM. *Let* $f\colon X\to Y$ *be an* n*-equivalence of* pro-H_0. *If* $\operatorname{ddim} Y\leqslant n$, *then* f *is a weak domination.*

One of the main results in this section is the following

5.6. THEOREM. *Let* $f\colon X\to Y$ *be a morphism of* pro-H_0 *such that* pro-$\pi_k f$ *is an isomorphism of* pro-Gr *for* $k\leqslant n$. *Then for each morphism* $g\colon Z\to Y$ *of* pro-HCW_0, *where* $\operatorname{ddim} Z\leqslant n$, *there is a unique morphism* $h\colon Z\to X$ *with* $f\cdot h = g$.

Proof. First let us prove Theorem 5.6 under a stronger assumption, namely that f is an $(n+1)$-equivalence.

Applying Theorem 2.2, it is enough to consider the case where $X = (X_a, p_a^b, A)$, $Y = (Y_a, q_a^b, A)$ and f is a special morphism induced by a family $\{f_a\}_{a\in A}$.

For each $a\in A$, let $b(a)$ be an element of A such that conditions (1) and (2) of Lemma 5.4 are satisfied.

Let $g = \{g_a\}_{a\in A}\colon Z\to Y$ be a morphism of pro-HCW_0, where $\operatorname{ddim} Z\leqslant n$. Hence there exist morphisms $h'_a\colon Z\to X_a$ such that

$$f_a\cdot h'_a = q_a^{b(a)}\cdot g_{b(a)} = g_a.$$

Let $h_a = p_a^{b(a)}\cdot h'_{b(a)}\colon Z\to X_a$. Let us verify that $h = \{h_a\}_{a\in A}$ is a morphism of pro-HCW_0. So let $a_1\leqslant a_2$ and take $a\geqslant b(a_1), b(a_2)$. Then

$$f_{b(a_1)}\cdot p^a_{b(a_1)}\cdot h'_a = q^a_{b(a_1)}\cdot f_a\cdot h'_a = q^a_{b(a_1)}\cdot g_a = g_{b(a_1)} = f_{b(a_1)}\cdot h'_{b(a_1)}$$

and therefore

$$p^a_{a_1}\cdot h'_a = p^{b(a_1)}_{a_1}\cdot p^a_{b(a_1)}\cdot h'_a = p^{b(a_1)}_{a_1}\cdot h'_{b(a_1)} = h_{a_1}.$$

Similarly, $p^a_{a_2}\cdot h'_a = h_{a_2}$ and consequently

$$p^{a_2}_{a_1}\cdot h_{a_2} = p^{a_2}_{a_1}\cdot p^a_{a_2}\cdot h'_a = p^a_{a_1}\cdot h'_a = h_{a_1}.$$

Thus h is a morphism of pro-$\mathbf{HCW}_0$ for which $f\cdot h = g$. If $t = \{t_a\}_{a\in A}: Z\to X$ is another morphism of pro-$\mathbf{HCW}_0$ satisfying $f\cdot t = g$, then

$$f_{b(a)}\cdot t_{b(a)} = g_{b(a)} = f_{b(a)}\cdot h_{b(a)}.$$

Hence $t_a = p^{b(a)}_a\cdot t_{b(a)} = p^{b(a)}_a\cdot h_{b(a)} = h_a$ for each $a\in A$, i.e. $h = t$.

To complete the proof of Theorem 5.6 let us apply a canonical morphism $r: Y\to$pro-$\mathrm{cosk}_{n+1}Y$, where $\mathrm{cosk}_{n+1}: \mathbf{H}_0\to\mathbf{H}_0$ is the functor defined in [A-M] on p. 21 ($\mathrm{cosk}_{n+1}K$ is obtained from K by attaching some p-cells, $p\geqslant n+2$, to kill higher homotopy groups). This morphism is an $(n+1)$-equivalence as well as $r\cdot f$, because pro-π_{k+1}(pro-$\mathrm{cosk}_{n+1}Y$) is a zero object of pro-$\mathbf{Gr}$ for $k\geqslant n+1$. So if $g: Z\to Y$ is a morphism of pro-$\mathbf{HCW}_0$ and $\mathrm{ddim}Z\leqslant n$, then there is an $h: Z\to X$ with $r\cdot f\cdot h = r\cdot g$. Hence $f\cdot h = g$.

If $h, h': Z\to X$, $\mathrm{ddim}Z\leqslant n$ and $f\cdot h = f\cdot h'$, then $r\cdot f\cdot h = r\cdot f\cdot h'$ and consequently $h = h'$.

Theorem 5.6 has been proved by the author [Dyd$_4$] and independently by K. Morita [Mor$_4$] for the case where $X, Y, Z\in\mathrm{Ob}\mathbf{S}_0$ and f is an $(n+1)$-equivalence.

5.7. Corollary. *Let $f: X\to Y$ be a morphism of* pro-$\mathbf{H}_0$. *If* pro-$\pi_k f$ *is an isomorphism for $k\leqslant\mathrm{ddim}\,Y<+\infty$, then f is a domination.*

Corollary 5.7 has been obtained by K. Morita [Mor$_4$] for the case $X, Y\in\mathrm{Ob}\mathbf{S}_0$ and independently by the author [Dyd$_4$] for the case where $X, Y\in\mathrm{Ob}\mathbf{S}_0$ and f is a $(\mathrm{ddim}\,Y+1)$-equivalence.

The next results show that weak equivalences behave in some cases like monomorphisms or epimorphisms.

5.8. Theorem. *Let $f: X\to Y$ be a weak equivalence of* pro-$\mathbf{H}_0$ *and let $Z = (Z_d, r^{d'}_d, D)$ be a movable object of* pro-$\mathbf{HCW}_0$ *such that each Z_d is finite-dimensional.*

If $g, h: Z\to X$ satisfy $f\cdot g = f\cdot h$, then $g = h$.

Proof. It is enough to consider the case where $X = (X_a, p^b_a, A)$, $Y = (Y_a, q^b_a, A)$ and f is a special morphism induced by a family $\{f_a\}_{a\in A}$.

Let $g = \{g_a\}_{a\in A}$ and $h = \{h_a\}_{a\in A}$. Then

$$f_a\cdot g_a = f_a\cdot h_a \quad \text{for each } a\in A.$$

Let $a\in A$. By Lemma 2.5 there exists a sequence $B = \{a_n\}^\infty_{n=1}\in\boldsymbol{A}$ such that $a_1 = a$ and $f_B: X_B\to Y_B$ is a weak equivalence. Now, by simple

considerations, we can find a sequence $\{d_n\}_{n=1}^{\infty} = E \in \boldsymbol{D}$ such that the following conditions are satisfied:

(1) there exist representatives $g_{a_n}^{d_n}\colon Z_{d_n} \to X_{a_n}$ and $h_{a_n}^{d_n}\colon Z_{d_n} \to X_{a_n}$ of g_{a_n} and h_{a_n}, respectively, such that the families $\{g_{a_n}^{d_n}\}_{n=1}^{\infty}$ and $\{h_{a_n}^{d_n}\}_{n=1}^{\infty}$ induce special morphisms $g'\colon Z_E \to X_B$ and $h'\colon Z_E \to X_B$, respectively, and $f_{a_n} \cdot h_{a_n}^{d_n} = f_{a_n} \cdot g_{a_n}^{d_n}$ for each n,

(2) Z_E is a movable object of pro-HCW_0. Hence Z_E is a uniformly movable object of pro-HCW_0 (see [Sp]) and there exists a morphism $s = \{s_n\}_{n=1}^{\infty}\colon Z_{d_k} \to Z_E$ for a certain $k \geqslant 1$ with $s_1 = r_{d_1}^{d_k}$. Now $f_B \cdot g' \cdot s = f_B \cdot h' \cdot s$ and by Theorem 5.6 we have $g' \cdot s = h' \cdot s$. Consequently,

$$g_{a_1}^{d_1} \cdot r_{d_1}^{d_k} = h_{a_1}^{d_1} \cdot r_{d_1}^{d_k},$$

which implies $g_a = h_a$. Thus the proof is completed.

5.9. THEOREM. *Let $f\colon X \to Y$ be a weak equivalence of* pro-H_0. *If Y is a movable object of* pro-H_0, *then f is a weak domination.*

Proof. It is enough to consider the case where $X = (X_a, p_a^b, A)$, $Y = (Y_a, q_a^b, A)$ and f is a special morphism induced by a family $\{f_a\}_{a \in A}$.

Let $a \in A$. By Lemma 2.5 there exists a $\boldsymbol{B} = \{a_n\}_{n=1}^{\infty} \in \boldsymbol{A}$ such that $f_B\colon X_B \to Y_B$ is a weak equivalence, Y_B is movable and $a_1 = a$. Then all pro-groups pro-$\pi_n X_B$, pro-$\pi_n Y_B$ satisfy the Mittag-Leffler condition and by Theorem 5.2 there exist two morphisms

$$g\colon Q \to X_B \quad \text{and} \quad h\colon R \to Y_B,$$

such that $Q, R \in \mathrm{Ob}\mathsf{H}_0$ and all $\check{\pi}_n g, \check{\pi}_n h$ are isomorphisms. Moreover, there exists an $r\colon Q \to R$ with $h \cdot r = f_B \cdot g$. Notice that r is a weak equivalence and by the classical Whitehead theorem r^{-1} exists. By Theorem 5.2, h is a weak domination. Hence $f_B \cdot g = h \cdot r$ is a weak domination and consequently f_B is a weak domination. Therefore there exist a $k \geqslant 1$ and a $t\colon Y_{a_k} \to X_{a_1}$ with $f_{a_1} \cdot t = q_{a_1}^{a_k}$, which completes the proof.

Observe that Theorem 5.9 implies the classical Whitehead theorem. So it can be considered as a convenient generalization of the Whitehead theorem.

§6. Whitehead theorems in shape and pro-homotopy

In this section we summarize the results received up to date concerning sufficient conditions for a morphism of pro-H_0 or S_0 to be an isomorphism.

As an immediate consequence of Theorem 5.6 we get

6.1. THEOREM. *Let $f\colon X \to Y$ be a morphism of* pro-H_0 *such that* pro-$\pi_k f$

is an isomorphism of pro-**Gr** *for* $k \leqslant n$. *If* $\operatorname{ddim} X, \operatorname{ddim} Y \leqslant n$, *then* f *is an isomorphism of* pro-$\mathbf{H}_0$.

In shape theory Theorem 6.1 reads as follows:

6.1′. THEOREM. *Let* $f\colon (X, x) \to (Y, y)$ *be a shape morphism, where* X *and* Y *are connected topological spaces and*

$$n = \max(\operatorname{ddim} X, \operatorname{ddim} Y) < +\infty.$$

If pro-$\pi_k f$ *is an isomorphism of* pro-**Gr** *for* $k \leqslant n$, *then* f *is a shape equivalence.*

Theorem 6.1′ has been proved by K. Morita [Mor$_4$].

As a consequence of Theorems 4.2, 5.5 and 6.1 we get

6.2. THEOREM. *If* $f\colon X \to Y$ *is an* n*-equivalence of* pro-$\mathbf{H}_0$, *where* $n = \max(\operatorname{ddim} X + 1, \operatorname{ddim} Y)$, *then* f *is an isomorphism of* pro-$\mathbf{H}_0$.

In shape theory Theorem 6.2 reads as follows:

6.2′. THEOREM. *Let* $f\colon (X, x) \to (Y, y)$ *be a shape morphism, where* X *and* Y *are connected topological spaces and*

$$n = \max(\operatorname{ddim} X + 1, \operatorname{ddim} Y) < +\infty.$$

If pro-$\pi_k f$ *is an isomorphism of* pro-**Gr** *for* $k \leqslant n-1$ *and an epimorphism for* $k = n$, *then* f *is a shape equivalence.*

Theorem 6.2 has been considered hitherto in the following cases:

(1) $X = (X_k, p_k^{k+1})$ and $Y = (Y_k, q_k^{k+1})$ are inverse sequences in $\mathbf{H}_0$, $n = \max\limits_k \{1 + \dim X_k, \dim Y_k\} + 1$ and pro-$\pi_n f$ is a domination of pro-**Gr** (D. A. Edwards and R. Geoghegan [E-G$_2$]),

(2) $X = (X_a, p_a^{a'}, A)$, $Y = (Y_b, q_b^{b'}, B)$, $n = \max\limits_{a,b} \{1 + \dim X_a, \dim Y_b\} + 1$, f is a morphism induced from pro-$\mathbf{CW}_0$ and pro-$\pi_n f$ is a domination of pro-**Gr** (D. A. Edwards and R. Geoghegan [E-G$_2$]).

Theorem 6.2′ is taken from [Dyd$_4$] and was earlier considered in the following cases:

(1) X and Y are continua and $n = \max(1 + \operatorname{Fd} X, \operatorname{Fd} Y) + 1$ (M. Moszyńska [Mos$_2$]),

(2) X and Y are compact, Y is metrizable and $n = \max(1 + \dim X, \dim Y) + 1$ (S. Mardešić [Mar$_2$]),

(3) f is induced by a map and $n = \max(1 + \dim X, \dim Y) + 1$ (S. Mardešić [Mar$_2$]),

(4) $n = \max(1 + \dim X, \dim Y)$ (K. Morita [Mor$_3$]).

Thus Theorems 6.1′ and 6.2′ are generalizations of the classical finite-dimensional Whitehead theorem. The first idea of proving such a theorem is due to M. Moszyńska [Mos$_2$]. Then arose the question how

to remove the hypotheses of finite-dimensionality and the example of J. Draper and J. Keesling [D-K] indicated that it cannot be done quite simply.

The first attempt to prove the infinite-dimensional Whitehead theorem in shape theory was due to D. A. Edwards and R. Geoghegan [E-G_3]. Their results were later improved by the author [$Dyd_{4,6}$]. Here we give some stronger results.

6.3. THEOREM. *Let $f\colon X\to Y$ be a weak equivalence of* pro-H_0. *If f is a domination and X is movable, then f is an isomorphism of* pro-H_0.

Proof. Let $g\colon Y\to X$ be a morphism of pro-H_0 such that $f\cdot g = 1_Y$. Then g is a weak equivalence and $(g\cdot f)\cdot g = 1_X\cdot g$. By Theorem 5.9 we have $g\cdot f = 1_X$ which completes the proof.

In shape theory Theorem 6.3 reads as follows:

6.3′. THEOREM. *Let (X, x) and (Y, y) be pointed connected topological spaces, where (X, x) is movable. If a shape morphism $f\colon (X, x)\to(Y, y)$ induces isomorphisms of all homotopy pro-groups and is a domination of S_0, then f is a shape equivalence.*

Theorem 6.3′ has been proved by the author [Dyd_4] under the additional assumption of compactness of X.

As an easy consequence of Theorems 4.2, 5.9, and 6.1 we have

6.4. THEOREM. *Let $f\colon X\to Y$ be a weak equivalence of* pro-H_0. *If Y is movable and the deformation dimension of X is finite, then f is an isomorphism of* pro-H_0.

In shape theory the corresponding result is the following

6.4′. THEOREM. *Let (X, x) and (Y, y) be pointed connected topological spaces such that the deformation dimension of X is finite and (Y, y) is movable. If a shape morphism $f\colon (X, x)\to(Y, y)$ induces isomorphisms of all homotopy pro-groups, then it is a shape equivalence.*

Theorem 6.4′ has been proved by the author [Dyd_4] under the additional assumption of the compactness of Y.

6.5. THEOREM. *Let $f\colon X\to Y$ be a weak equivalence of* pro-H_0. *If X is movable and the deformation dimension of Y is finite, then f is an isomorphism of* pro-H_0.

Proof. By Corollary 5.7 there is a morphism $g\colon Y\to X$ such that $f\cdot g = 1_Y$. Then g is a weak equivalence and by Theorem 6.4 g is an isomorphism of pro-H_0. Hence $f = g^{-1}$ is an isomorphism of pro-H_0 as well.

Theorem 6.5 has been proved by D. A. Edwards and R. Geoghegan [E-G_3] for the case where X is an object induced from pro-CW_0 consisting of finite-dimensional CW complexes and Y is a pointed finite-dimensional CW complex.

In shape theory Theorem 6.5 reads as follows:

6.5′. THEOREM. *Let (X, x) and (Y, y) be pointed connected topological spaces, where (X, x) is movable and the deformation dimension of Y is finite. If a shape morphism $f\colon (X, x)\to(Y, y)$ induces isomorphisms of all homotopy pro-groups, then it is a shape equivalence.*

Theorem 6.5′ was obtained by the author [Dyd$_4$] in the case where X is compact. Earlier D. A. Edwards and R. Geoghegan [E-G$_3$] proved it for the case where X is a continuum and (Y, y) is a pointed FANR, i.e. (Y, y) is shape dominated by a pointed CW complex.

Finally let us observe that Lemma 2.18 and Lemma 2.14 together with 6.1′–6.5′ imply the following

6.6. THEOREM. *Let $f\colon (X, x)\to(Y, y)$ be a shape morphism, where (X, x) and (Y, y) are pointed movable continua. If one of the following conditions is satisfied:*

(1) *$\check{\pi}_k f$ is an isomorphism for $k \leqslant \max(\operatorname{ddim} X, \operatorname{ddim} Y)$,*

(2) *$\check{\pi}_k f$ is an isomorphism for $k \leqslant \max(\operatorname{ddim} X+1, \operatorname{ddim} Y)-1$ and an epimorphism for $k = \max(\operatorname{ddim} X+1, \operatorname{ddim} Y)$,*

(3) *$\check{\pi}_k f$ is an isomorphism for each k and either $\operatorname{ddim} X$ or $\operatorname{ddim} Y$ is finite,*

(4) *$\check{\pi}_k f$ is an isomorphism for each k and f is a shape domination, then f is a shape equivalence.*

The first idea of proving the Whitehead theorem in shape theory expressed in terms of shape groups is due to M. Moszyńska [Mos$_2$].

Another sufficient conditions for a morphism of S_0 to be an isomorphism may be found in [E-G$_3$] and [Mos$_{3,4}$].

Some interesting examples are given in [E-H$_1$].

Notice that there are also homological versions of the Whitehead theorem in shape theory (see [Mar$_3$] and [Mor$_3$]). They may be derived from the following result of K. Morita [Mor$_3$];

6.7. THEOREM. *Let $f\colon X\to Y$ be a morphism of* pro-H_0, *where* pro-$\pi_1 X$ = pro-$\pi_1 Y = 0$. *Then for $n \geqslant 2$ the following conditions are equivalent:*

(1) pro-$\pi_k f$ *is an isomorphism for $k < n$ and an epimorphism for $k = n$,*

(2) pro-$H_k f$ *is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

§7. Criteria for stability in shape and pro-homotopy

CW complexes form an important class of spaces in homotopy theory and in shape theory. So it is natural to ask when a topological space has the shape of a CW complex.

Just as in homotopy theory (see [L-W], p. 127), one has a domination criterion for stability in pro-H_0.

7.1. THEOREM. *An object X of* pro-H_0 *is stable iff it is dominated in* pro-H_0 *by an object K of* H_0.

Proof. Take $f\colon X\to K$ and $g\colon K\to X$ with $g\cdot f = 1_X$. Observe that X is isomorphic in pro-H_0 to $\overline{K} = (K_n, q_n^{n+1})$, where $K_n = K$ and $q_n^{n+1} = f\cdot g$ for each n. Indeed, if $f' = \{f_n\}_{n=1}^{\infty}$ is a morphism from X to $\overline{K}$ such that $f_n = f$ for each n and $g'\colon \overline{K}\to X$ is the composition of g and $i\colon \overline{K}\to K$, where i has 1_K as a representative, then

$$f'\cdot g' = 1_{\overline{K}} \quad \text{and} \quad g'\cdot f' = 1_X.$$

By Theorem 5.2 there exist morphisms $h\colon Q\to X$ and $r\colon K\to Q$ such that $Q \in \mathrm{Ob}\mathsf{H}_0$, $\check{\pi}_n h$ is an isomorphism for each n and $h\cdot r = g$. Let $s = r\cdot f$. Then

$$h\cdot s = h\cdot r\cdot f = 1_X \quad \text{and} \quad h\cdot(s\cdot h) = h.$$

Consequently, $\check{\pi}_n(s\cdot h)$ is an isomorphism for each n and by the Whitehead theorem $s\cdot h$ is an isomorphism of H_0. Now, for $t = (s\cdot h)^{-1}$ we have $h = h\cdot t$ and

$$h\cdot(t\cdot s) = h\cdot s = 1_X, \quad (t\cdot s)\cdot h = 1_Q,$$

which completes the proof.

Theorem 7.1 has been proved by D. A. Edwards and R. Geoghegan [E-G$_5$].

In shape theory Theorem 7.1 reads as follows:

7.1′. THEOREM. *A pointed connected topological space X has the shape of a pointed* CW *complex iff it is shape dominated by a pointed* CW *complex.*

Theorem 7.1′ is in fact due to L. Demers [Dem]. Independently D. A. Edwards and R. Geoghegan [E-G$_5$] have obtained the same result.

In the unpointed case Theorem 7.1′ has been announced in [Dem] and [E-G$_4$] (for the compact case) but there are gaps in these papers. Namely the proof of Theorem 2.1 in [E-G$_4$] is valid only in the connected case and Demers, in order to apply Brown's Representability Theorem (see [Br]), uses in an obscure form the following statement:

If $f\colon X\to Y$ is a morphism of H *such that* $\mathsf{H}(Z, \cdot)f$ *is a bijection for each finite* CW *complex Z, then f is an isomorphism.*

This statement is false, as is shown in the following example:

Let A be the group of all bijective functions $f\colon N\to N$, where N is the set of all integers, such that $f(n) = n$ for almost all n. Let B be a sub-

group of A whose elements have 1 as a fixed point. Then these groups have the following properties:

(1) for each finite sequence $f_1, \ldots, f_k$ of elements of A there is an $h \in A$ such that $h \cdot f_i \cdot h^{-1} \in B$ for $i \leqslant k$,

(2) if $f_i, g_i \in B$, $i \leqslant k$, and there is a $g \in A$ such that $f_i = g \cdot g_i \cdot g^{-1}$ for $i \leqslant k$, then there exists an $r \in B$ such that $f_i = r \cdot g_i \cdot r^{-1}$ for $i \leqslant k$.

In fact, if $f_1, \ldots, f_k \in A$, then there is an $n_0 \in N$ such that $f_i(n_0) = n_0$ for $1 \leqslant i \leqslant k$. Then $h \in A$ defined by the formulae

$$h(m) = m \quad \text{for} \quad m \neq 1, n_0, \quad h(1) = n_0, \quad h(n_0) = 1$$

satisfies condition (1).

If $f_i = g \cdot g_i \cdot g^{-1}$ for $1 \leqslant i \leqslant k$, then take $n_0 \in N$ such that $f_i(n_0) = g_i(n_0) = g(n_0) = n_0$ and define h as previously. Let $r = h^{-1} \cdot g \cdot h$. Then

$$r \cdot g_i \cdot r^{-1} = h^{-1} \cdot g \cdot h \cdot g_i \cdot h^{-1} \cdot g^{-1} \cdot h = h^{-1} \cdot g \cdot g_i \cdot g^{-1} \cdot h = h^{-1} \cdot f_i \cdot h = f_i$$

for $i \leqslant k$.

Now, let X be a CW complex of type $(B, 1)$ and let Y be a CW complex of type $(A, 1)$. Take a morphism $f\colon X \to Y$ which corresponds to the inclusion homomorphism $B \hookrightarrow A$ (see [S], Theorem 9 on p. 427). Recall (see [S], Theorem 11 on p. 428) that for any path-connected CW complex Z the set of free homotopy classes of maps from Z to $Y(X)$ is in one-to-one correspondence with the set of conjugacy classes of homomorphisms from $\pi_1(Z, z)$ to $\pi_1(Y, y)$ $(\pi_1(X, x))$. In the case where Z is a finite CW complex the group $\pi_1(Z, z)$ is finitely generated and by conditions (1) and (2) we infer that the inclusion homomorphism $B \hookrightarrow A$ induces a bijection between these sets of conjugacy classes of homomorphisms. Hence $\mathsf{H}(Z, \cdot)f$ is a bijection for each finite CW complex Z and f is not an isomorphism of H in view of $A \neq B$.

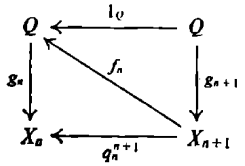
The author is grateful to Dr. P. Minc who found the groups A and B satisfying conditions (1) and (2).

In [Dyd_{11}] there is an example indicating that Theorem 7.1 does not hold in the unpointed case, i.e. *there exists an object of* pro-H *dominated by a* CW *complex but not stable.*

It is well known that a pointed topological space (X, x) (x is non-degenerate) is homotopy equivalent to a pointed CW complex iff X is equivalent to a CW complex Q. A similar result can be proved in pro-homotopy.

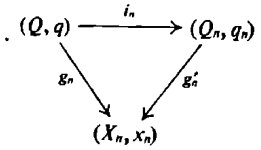
7.2. Lemma (R. Geoghegan, [G]). *Let* $\{(X_n, x_n), [q_n^{n+1}]\}$ *be an inverse sequence in* H_0 *such that* $(X_n, [q_n^{n+1}])$ *is isomorphic in* pro-H *to a* CW *complex* Q. *Then for any choice* $q \in Q^0$ *the object* $\{(X_n, x_n), [q_n^{n+1}]\}$ *is isomorphic in* pro-H_0 *to* (Q, q).

Proof. We may assume that there exist maps $g_n\colon Q\to X_n$, $f_n\colon X_{n+1}\to Q$ such that for each n the diagram



commutes up to homotopy and $g_n(q)=x_n$, $f_n(x_{n+1})=q$ for each n.

For each n let us find a commutative diagram in T_0



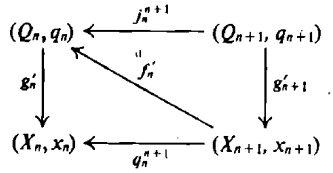
such that g'_n is a pointed fibration and i_n is an isomorphism of HT_0. Hence we can find a map

$$f'_n\colon (X_{n+1}, x_{n+1})\to(Q_n, q_n)$$

such that $g'_n\cdot f'_n = q_n^{n+1}$ and $f_n\simeq i_n\cdot f_n$. Let

$$j_n^{n+1} = f'_n\cdot g'_{n+1}\colon (Q_{n+1}, q_{n+1})\to(Q_n, q_n).$$

Then the diagram



is commutative in T_0. Thus $((X_n, x_n), [q_n^{n+1}])$ and $((Q_n, q_n), [j_n^{n+1}])$ are isomorphic in pro-HT_0. Moreover,

$$j_n^{n+1}\cdot i_{n+1} = f'_n\cdot g'_{n+1}\cdot i_{n+1} = f'_n\cdot g_{n+1}\simeq i_n\cdot f_n\cdot g_{n+1},$$

and since $f_n\cdot g_{n+1}$ is an isomorphism of H_0 (because $f_n\cdot g_{n+1}$ is homotopic to the identity map of Q in the unpointed sense), we infer that j_n^{n+1} is an isomorphism of HT_0 for each n. Hence $((Q_n, q_n), [j_n^{n+1}])$ is isomorphic to (Q, q) in pro-HT_0, which completes the proof.

Lemma 7.2 and its proof were presented by R. Geoghegan at the *Shape Theory and Pro-homotopy Conference*, Dubrovnik, January 1976 (see [G]).

Lemma 7.2 can be improved in the following way:

7.3. THEOREM. *An object* $X = ((X_a, x_a), p_a^b, A)$ *of* pro-HCW_0 *is stable iff* (X_a, p_a^b, A) *is stable in* pro-HCW.

Proof. It is enough to prove Theorem 7.3 in the case where X is an object of pro-H_0.

Let $F\colon \mathsf{H}_0 \to \mathsf{H}$ be the forgetful functor. By Lemma 2.4 the set

$$A' = \{B \in A\colon X_B \text{ is stable in pro-}\mathsf{H} \text{ and } P_B\colon X \to X_B \text{ is an isomorphism of pro-}\mathsf{H}\}$$

is a cofinal subset of A.

Let X and $Q \in \mathrm{Ob}\mathsf{H}$ be isomorphic in pro-H and take $q \in Q^0$. By Lemma 7.2 for each $B \in A'$ there is an isomorphism $h_B\colon X_B \to (Q, q)$ of pro-H_0. Take $B' \geqslant B$, $B' \in A'$. Then $h_B \cdot P_B^{B'} \cdot h_{B'}^{-1}\colon Q \to Q$ is an isomorphism of H. Hence $h_B \cdot P_B^{B'} \cdot h_{B'}^{-1}\colon (Q, q) \to (Q, q)$ is an isomorphism of H_0. Therefore $P_B^{B'}\colon X_{B'} \to X_B$ is an isomorphism of pro-H_0. Since $\boldsymbol{P}\colon X \to (X_B, P_B^{B'}, A')$ is an inverse limit, we infer that $P_B\colon X \to X_B$ is an isomorphism of pro-H_0 for each $B \in A'$, which completes the proof.

Now let us give some algebraic criteria for stability in pro-H_0.

7.4. THEOREM. *An object* X *of* pro-H_0 *with a finite deformation dimension is stable iff each* pro-$\pi_n X$ *is dominated in* pro-Gr *by a group.*

Proof. Necessity is obvious.

Sufficiency. By Theorem 4.1 there is an object $Y = (Y_a, p_a^b, A)$ of pro-H_0 isomorphic to X such that $Y_a^k = Y_a$ for each $a \in A$, where $k = \mathrm{ddim} X$. By Lemma 2.4 there is a $B \in A$ such that $P_B\colon Y \to Y_B$ is a weak equivalence. Hence, by Theorem 6.1, P_B is an isomorphism of pro-H_0. Now, by Theorem 5.2, there is a morphism $f\colon Q \to Y_B$ such that $\check{\pi}_n f$ is an isomorphism for each n and $Q \in \mathrm{Ob}\mathsf{H}_0$. Let $q_n\colon \check{\pi}_n Y_B \to \text{pro-}\check{\pi}_n Y_B$ be projections which are isomorphisms in view of Lemma 2.13. Then

$$\text{pro-}\pi_n f = q_n \cdot \check{\pi}_n f,$$

i.e. f is a weak equivalence. Since $\mathrm{ddim}\, Y_B \leqslant k$, we infer by Corollary 5.7 that there is a morphism

$$g\colon Y_B \to Q$$

with $f \cdot g = 1_{Y_B}$. Then $g \cdot f\colon Q \to Q$ is a weak equivalence and the classical Whitehead theorem implies that $g \cdot f$ is an isomorphism of H_0. Therefore $\mathrm{ddim} Q \leqslant k$, and by Theorem 6.1, f is an isomorphism of pro-H_0, which completes the proof.

Theorem 7.4 has been proved by D. A. Edwards and R. Geoghegan [E-G$_5$] in the case where X is an object induced from pro-CW_0 whose strong homotopy dimension is finite.

Observe that it is not true that an object of pro-H_0 weakly dominated by a pointed CW complex must be stable. As an example take an arbitrary movable inverse sequence in H_0 which is not stable (see Theorem 5.2). However, it can be proved under some additional hypotheses.

7.4. THEOREM. *If an object X of* pro-H_0 *is weakly dominated by a finite* CW *complex Q and* pro-$\pi_1 X = 0$, *then X is stable.*

Proof. Let $f = \{f_a\}_{a\in A}\colon Q\to X = (X_a, p_a^b, A)$ be a weak domination and let $i\colon Q\to Q\cup CQ^1$ be the homotopy class of the inclusion map (CQ^1 is the reduced cone over the 1-skeleton of Q). We are going to show that each morphism pro-$\pi_n f$ factors as

$$\text{pro-}\pi_n f = q_n\cdot\pi_n i$$

for a certain $q_n\colon \operatorname{im}\pi_n i\to\text{pro-}\pi_n X$.

First let us observe that the set

$$A' = \{B\in A\colon B = \{a_i\}_{i=1}^{\infty},\ \{f_{a_i}\}_{i=1}^{\infty}\colon Q\to X_B \text{ is a weak domination and pro-}\pi_1 X_B = 0\}$$

is a cofinal subset of A.

Let $B\in A'$ and take a morphism $g_B\colon Q_B\to X_B$ of pro-H_0 such that $Q_B\in \mathrm{Ob}\mathsf{H}_0$, $\check{\pi}_n g_B$ is an isomorphism for each n and $f_B = g_B\cdot h_B$ for a certain $h_B\colon Q\to Q_B$ (see Theorem 5.2). Since $\pi_1 Q_B = 0$, we infer that h_B factors as

$$h_B = r_B\cdot i$$

for a certain $r_B\colon Q\cup CQ^1\to Q_B$. Consequently, for each n, pro-$\pi_n f_B$ factors as

$$\text{pro-}\pi_n f_B = (q_n)_B\cdot\pi_n i$$

for a certain $(q_n)_B\colon \operatorname{im}\pi_n i\to\text{pro-}\pi_n X_B$. Since pro-$\pi_n X$ is an inverse limit of (pro-$\pi_n X_B$, pro-$\pi_n P_B^{B'}$, A') and $\pi_n i\colon \pi_n Q\to\operatorname{im}\pi_n i$ is an epimorphism, we infer that pro-$\pi_n f = q_n\cdot\pi_n i$ for a certain $q_n\colon \operatorname{im}\pi_n i\to\text{pro-}\pi_n X$. Then q_n is a weak domination of pro-Gr and $\operatorname{im}\pi_n i$ is a finitely generated Abelian group (see [S], p. 509). By Lemma 2.20 each pro-$\pi_n X$ is stable and this completes the proof (see Theorems 4.2 and 7.3).

The following question remains open:

7.5. Question. Let X be an object of pro-H_0 weakly dominated by a finite CW complex. Is it true that X is stable?

Notice that Lemma 2.20 implies in this case that each pro-$H_n X$ is stable and each $\check{H}_n X$ is finitely generated. Moreover, if $\check{\pi}_1 X$ is Abelian, then it is finitely generated as well. Hence a positive answer to Problem (8.2) of [Bor$_6$] would imply a positive answer to Question 7.5 in the case where X is a Čech system of a continuum and $\check{\pi}_1 X$ is Abelian.

For the next criterion we need the functor $\pi_*\colon \mathsf{H}_0\to\mathsf{Gr}$ described on $Q\in\mathrm{Ob}\mathsf{H}_0$ as follows:

$$\pi_* Q = \prod_{i>0}\pi_i Q \qquad \text{(see [E-G}_1\text{])}.$$

7.6. THEOREM. *A movable object X of pro-H_0 is stable iff* pro-$\pi_* X$ *is dominated in* pro-$\mathbf{Gr}$ *by a group.*

Proof. The necessity is obvious, so we prove sufficiency. Notice that each pro-$\pi_n X$ is dominated by pro-$\pi_* X$. By Lemma 2.4 $\boldsymbol{A}' = \{B \in \boldsymbol{A}$: X_B is movable, P_B: $X \to X_B$ is a weak equivalence and pro-$\pi_* P_B$ is an isomorphism$\}$ is cofinal in $\boldsymbol{A}$. Take $B = \{a_i\}_{i=1}^{\infty} \in \boldsymbol{A}'$. By Theorem 5.2 there is a morphism $f = \{f_i\}_{i=1}^{\infty}$: $Q \to X_B$ such that $Q \in \mathrm{Ob}\mathsf{H}_0$ and $\check{\pi}_n f$ are isomorphisms. Hence $\check{\pi}_* f$ is an isomorphism because $\check{\pi}_* Z = \prod_{i>0} \check{\pi}_i Z$ for each $Z \in \mathrm{Ob}\,\text{pro-}\mathsf{H}_0$. Therefore pro-$\pi_* f$ is an isomorphism (see Lemma 2.13). Since f is a weak equivalence, it is a weak domination (see Theorem 5.2) and we can find by induction an increasing sequence $\{k_i\}_{i=1}$ of integers, homomorphisms h_i: $\pi_* X_{b_i} \to \pi_* Q$ $(b_i = a_{k_i})$ being representatives of $(\text{pro-}\pi_* f)^{-1}$ and morphisms r_i: $X_{b_{i+1}} \to Q$ of H_0 such that $f_{k_i} \cdot r_i = p_{b_i}^{b_{i+1}}$. Then $f_{k_i} \cdot r_i \cdot f_{k_{i+1}} = p_{b_i}^{b_{i+1}} \cdot f_{k_{i+1}} = f_{k_i}$ and therefore

$$\pi_*(r_i \cdot f_{k_{i+1}}) = h_i \cdot \pi_* f_{k_i} \cdot \pi_*(r_i f_{k_{i+1}}) = h_i \cdot \pi_* f_{k_i} = 1_{\pi_* Q}.$$

By the Whitehead theorem $r_i f_{k_{i+1}}$ is an isomorphism of H_0 for each i. Therefore (Q_i, q_i^{i+1}) is stable, where $Q_i = Q$ and $q_i^{i+1} = r_i \cdot f_{k_{i+1}}$ for each i. Since $f_{k_i} \cdot r_i = p_{b_i}^{b_{i+1}}$ and $q_i^{i+1} = r_i \cdot f_{k_{i+1}}$, then $(X_{b_i}, p_{b_i}^{b_{i+1}})$ and (Q_i, q_i^{i+1}) are isomorphic. Hence X_B is stable for each $B \in \boldsymbol{A}'$. Since $P_B^{B'}$: $X_{B'} \to X_B$ is a weak equivalence for $B' \geqslant B$, then $P_B^{B'}$ is an isomorphism for B', $B \in \boldsymbol{A}'$. Consequently P_B: $X \to X_B$ is an isomorphism for each $B \in \boldsymbol{A}'$ and X is stable.

Theorem 7.6 has been proved by the author [Dyd$_7$] in the case where X is a Čech system of a compact Hausdorff space as an answer to a conjecture of D. A. Edwards [Edw].

We now give two necessary and sufficient conditions for a topological space X to have the shape of a one-point space (then we say that the space of X is *trivial*).

A connected topological space X is said to be *approximatively k-connected* iff pro-$\pi_k(X, x) = 0$ for a certain $x \in X$.

This notion has been introduced by K. Borsuk [Bor$_5$] for compacta in an equivalent manner. One can prove that it is a shape invariant. If X is a closed subset of $P \in \mathrm{ANR}(\mathfrak{M})$, its approximative k-connectedness is equivalent to the following condition:

For each neighbourhood U of X in P there is a neighbourhood $V \subset U$ of X such that each map h: $S^k \to V$ is null-homotopic in U (see [Bor$_7$], Theorem (2.2) and [Mor$_3$], Lemma 2.3).

If X is approximatively k-connected for each $0 < k \leqslant n$, then we write $X \in \mathrm{AC}^n$. For $n = 0$ this means that X is connected.

As an easy consequence of Theorem 6.1′ we get

7.7. THEOREM. *Let X be a non-empty topological space. The shape of X is trivial iff* $\operatorname{ddim} X \leqslant n$ *and* $X \in \mathrm{AC}^n$ *for a certain* $n \geqslant 0$.

7.8. THEOREM. *Let X be a non-empty topological space. The shape of X is trivial iff X is movable and approximatively k-connected for each k.*

Proof. It is enough to prove the sufficiency.

Just as in [Bor$_5$], one can prove that (X, x) is movable for each $x \in X$. Thus Theorem 7.8 is a consequence of Theorem 6.4′.

7.9. COROLLARY. *If (X, x) is a uniformly movable pointed connected topological space and $\check{\pi}_n(X, x) = 0$ for each n, then the shape of X is trivial.*

Proof. By Lemma 2.14 the projections

$$p_n\colon \check{\pi}_n(X, x) \to \text{pro-}\pi_n(X, x)$$

are weak dominations and by Lemma 2.20 pro-$\pi_n(X, x) = 0$ for each n. Thus Corollary 7.9 is a consequence of Theorem 7.8.

Theorems 7.7, 7.8 and Corollary 7.9 have been proved by K. Borsuk [Bor$_{3,4}$] for compacta. Theorem 7.8 and Corollary 7.9 were obtained independently by S. Bogatyi [Bog$_2$] for compacta. Next G. Kozlowski and J. Segal [K-S$_1$] extended Theorem 7.8 and Corollary 7.9 onto compact Hausdorff spaces.

It is well known that the movability of (X, x) does not imply the uniform movability of (X, x) in general (see [K-S$_1$]). So of some interest is the following

7.10. THEOREM. *Let (X, x) be a pointed connected topological space such that each* pro-$\pi_n(X, x)$ *is dominated in* pro-Gr *by a group. If (X, x) is movable, then it is uniformly movable.*

Proof. By Theorem 5.4 of [E-G$_5$] there exists a weak equivalence $f\colon Q \to (X, x)$ such that $Q \in \mathrm{Ob}\mathsf{H}_0$. By Theorem 5.9, f is a weak domination and Theorem 2.11 implies that (X, x) is uniformly movable.

There is also an algebraic criterion for stability in shape in terms of Čech cohomology, which we formulate without proof.

7.11. THEOREM. *A finite-dimensional approximatively 1-connected continuum X has the shape of a finite* CW *complex iff its Čech cohomology groups are finitely generated.*

Theorem 7.11 was first proved by L. Demers [Dem] for the case where X is locally pathwise connected and simply connected, and then (independently) by R. Geoghegan and R. C. Lacher [G-L].

Observe that Corollary 2.19 and Theorem 7.4 imply the following

7.12. THEOREM. *A pointed continuum* (X, x) *is a pointed* FANR *iff* (X, x) *is movable, the deformation dimension of* X *is finite and all shape groups* $\check{\pi}_n(X, x)$ *are countable.*

Theorem 7.12 has been proved by the author [Dyd$_8$] and independently by T. Watanabe [Wat].

§8. Smale theorem in shape theory

In this section we use the methods of §§ 5 and 6 to generalize the Smale theorem.

Suppose we are given two metrizable locally connected spaces M and N, a map $F\colon M\to N$ and closed subsets $Y\subset N$, $X = F^{-1}(Y)\subset M$ such that $F|X\colon X\to Y$ is closed and surjective.

Let C be the set of all pairs (U, V) such that U is a neighbourhood of X in M whose components intersect X, V is a neighbourhood of Y in N whose components intersect Y and $U\subset F^{-1}(V)$.

A directed set $(C, \leqslant)$ is defined as follows:

$$(U, V)\leqslant (U', V') \quad \text{iff} \quad U'\subset U \text{ and } V'\subset V.$$

If $c = (U, V)\in C$, then we write $U_c = U$, $V_c = V$ and $F_c\colon U_c\to V_c$ is the map defined by F.

Let $\boldsymbol{V}\succ\boldsymbol{U}$ be two open families in M. We say that $\boldsymbol{V}$ is an *n-refinement* of $\boldsymbol{U}$ if each partial realization of any at most $(n+1)$-dimensional simplicial complex K relative to $\boldsymbol{V}$ extends to a full realization of K relative to $\boldsymbol{U}$, i.e. each map $f\colon L\to M$ such that L is a subcomplex of K containing all its vertices and $f(s\cap L)$ is contained in an element of $\boldsymbol{V}$ for each simplex s of K extends to a map $\bar{f}\colon K\to M$ such that $\bar{f}(s)$ is contained in an element of $\boldsymbol{U}$ for each simplex s of K.

An open family $\{U_a\}_{a\in A}$ in M covering X is said to be a *cover of the decomposition* $D = \{F^{-1}(y)\}_{y\in Y}$ provided $U_a\cap X = F^{-1}F(U_a\cap X)$ for each $a\in A$.

8.1. LEMMA. *Each cover* $\boldsymbol{U}$ *of* D *has a star refinement* $\boldsymbol{V}$ *which is a cover of* D.

Proof. Let us define a decomposition D_1 of M as follows: if $x\in X$, then its stratum is $F^{-1}F(x)$, and a stratum of $x\in M-X$ is $\{x\}$. Then the projection $p\colon M\to M/D_1$ from M onto the decomposition space is closed and hence M/D_1 is paracompact. Observe that $p(\boldsymbol{U})$ is an open family in M/D_1 covering $p(X)$. So there exists an open cover $\boldsymbol{W}$ of M/D_1 which is a star refinement of $p(\boldsymbol{U})\cup\{M/D_1 - p(X)\}$. Now

$$\boldsymbol{V} = \{W\in p^{-1}(\boldsymbol{W})\colon W\cap X\neq\emptyset\}$$

is a star refinement of $\boldsymbol{U}$ which is a cover of D.

Lemma 8.1 is due to Y. Kodama [Kod$_2$].

If Z is a closed subset of a topological space T, then $Z \in \mathrm{AC}^n_T$ means that for each neighbourhood U of Z in T there is a neighbourhood $V \subset U$ of Z in T such that each map $h\colon S^k \to V$ $(k \leqslant n)$ is null-homotopic in U (see [Dug], where this notion is denoted by PC^n_T).

8.2. LEMMA. *If $F^{-1}(y) \in \mathrm{AC}^n_M$ for each $y \in Y$, then each cover $\boldsymbol{U}$ of D has an n-refinement $\boldsymbol{V}$ which is a cover of D.*

Proof (by induction). For $n = -1$ we take $\boldsymbol{V} = \boldsymbol{U}$.

Suppose that Lemma 8.2 is verified for $n = k \geqslant -1$ and $F^{-1}(y) \in \mathrm{AC}^{k+1}_M$ for each $y \in Y$. Let $\boldsymbol{U}$ be a cover of D. For each $x \in X$ take a neighbourhood V_x of $F^{-1}F(x)$ such that there is a $U_x \in \boldsymbol{U}$ such that each map $h\colon S^{k+1} \to V_x$ is null-homotopic in U_x. Let $\boldsymbol{U}_1$ be a cover of D which is a star refinement of

$$\{p^{-1}(M/D_1 - p(M - V_x))\}_{x \in X},$$

where $p\colon M \to M/D_1$ is a projection from the proof of Lemma 8.1. By the inductive assumption there exists a k-refinement $\boldsymbol{V}$ of $\boldsymbol{U}_1$ which is a cover of D. Let us show that $\boldsymbol{V}$ is a $(k+1)$-refinement of $\boldsymbol{U}$.

So let $f\colon L \to M$ be a partial realization of an at most $(k+2)$-dimensional simplicial complex K relative to $\boldsymbol{V}$. Then there is an extension $f_1\colon K^{k+1} \to M$ of $f|L \cap K^{k+1}$ which is a full realization of K^{k+1} relative to $\boldsymbol{U}_1$. Now, if $s \notin L$ is a closed $(k+2)$-simplex of K, then $f_1(\partial s)$ is contained in V_x for a certain $x \in X$ because each two $(k+1)$-simplexes from ∂s intersect each other (unless $k = -1$, but $f_1 = f$ in this case) and $\boldsymbol{U}_1$ is a star refinement of $\{V_x\}_{x \in X}$. Therefore we can extend $f_1|\partial s$ over s so that the values of this extension belong to U_x. Thus we get a full realization $\bar{f}\colon K \to M$ of K relative to $\boldsymbol{U}$ which is an extension of f.

Lemma 8.2 summarizes Lemmata 2 and 3 from [Kod$_2$]. See also Theorem 3.1 of [Dug].

Now we are ready to prove the following

8.3. LEMMA. *If $F^{-1}(y) \in \mathrm{AC}^n_M$ for each $y \in Y$, then for each $c' \in C$ and for each open cover $\boldsymbol{U}$ of $V_{c'}$ there exists a $c \geqslant c'$ such that if*

$$g\colon P \to V_c \text{ and } h\colon R \to U_c$$

are two maps, where R is a subpolyhedron of at most $(n+1)$-dimensional polyhedron P and $F_c \cdot h = g|R$, then there exists a map

$$k\colon P \to U_{c'}$$

such that $k|R = h$ and $F_{c'} \cdot k$ and g are $\boldsymbol{U}$-near.

Proof. Let $\boldsymbol{V}$ be a star refinement of $\boldsymbol{U}$.

Let $\boldsymbol{U}_1, \boldsymbol{U}_2 = \{U_a\}_{a \in A}$ be covers of D such that $\boldsymbol{U}_1$ is an n-refinement of $F^{-1}(\boldsymbol{V}) \cap U_{c'}$, $\boldsymbol{U}_2$ is a star refinement of $\boldsymbol{U}_1$ and $U_a \cap X \neq \emptyset$ for each

$a \in A$. By Theorem 9 in [Kur] (p. 226), there exists an open family $\{W_a\}_{a\in A} \succ \boldsymbol{V}$ such that $W_a \cap Y = F(U_a \cap X)$ for each $a \in A$ and $W_{a_1} \cap \ldots \cap W_{a_k} = \emptyset$ iff $F(U_{a_1} \cap X) \cap \ldots \cap F(U_{a_k} \cap X) = \emptyset$.

Choose $c \in C$ such that $V_c \subset \bigcup_{a\in A} W_a$ and $U_c \subset \bigcup_{a\in A} U_a \cap F^{-1}(W_a)$. Let $g\colon P \to V_c$ and $h\colon R \to U_c$ be maps such that $F_c \cdot h = g|R$, P is an at most $(n+1)$-dimensional polyhedron and R is its subpolyhedron. Take a triangulation T of P such that R is a body $|L|$ of some full subcomplex L of T and

$$\{g(s)\}_{s\in T} \succ \{W_a\}_{a\in A} \quad \text{and} \quad \{h(s)\}_{s\in L} \succ \{U_a \cap F^{-1}(W_a)\}_{a\in A}.$$

Choose for each vertex $v \in T^0 - L$ a point $k(v) \in U_a \cap X$, where $g(v) \in W_a$. Let $k(x) = h(x)$ for $x \in R$. Let us verify that $k\colon |T^0 \cup L| \to U_{c'}$ is a partial realization of T relative to $\boldsymbol{U}_1$.

If $s \in T$ and $s \cap L = \emptyset$, then

$$k\big(s \cap (T^0 \cup L)\big) = k(s \cap T^0)$$

is contained in $\bigcup\{U_a\colon W_a \cap W_{a_1} \neq \emptyset\}$, where $g(s) \subset W_{a_1}$.

If $s \cap L = s_1 \neq \emptyset$, then

$$k\big(s \cap (T^0 \cup L)\big) \subset \bigcup\{U_a\colon U_a \cap U_{a_1} \neq \emptyset\},$$

where $g(s) \subset W_{a_1}$, because $h(s_1) \subset U_{a_2} \cap F^{-1}(W_{a_2})$ for a certain a_2 and $W_{a_2} \cap W_{a_1} \neq \emptyset$.

Since $\boldsymbol{U}_2$ is a star refinement of $\boldsymbol{U}_1$, we infer that k is a partial realization of T relative to $\boldsymbol{U}_1$. So we can extend $k\colon |T^0 \cup L| \to U_{c'}$ to a full realization of T, $k\colon P \to U_{c'}$ relative to $F^{-1}(\boldsymbol{V}) \cap U_{c'}$. It is easy to verify that $F_{c'} \cdot k$ and g are $\boldsymbol{U}$-near.

If Z is a closed subset of a topological space P and $z \in Z$, then by $U_P(Z, z)$ we denote the inverse system $\big((U, z), [i_a^{a'}], A\big)$ in HT_0, where A is the set of all open neighbourhoods of Z in P ordered by means of inclusion and $i_a^{a'}$ are inclusion maps.

It is easy to see that the family of maps $\{F_c\}_{c\in C}$ induces a morphism

$$[F]\colon U_M(X, x_0) \to U_N(Y, y_0)$$

of pro-HT_0 for each $x_0 \in X$, $y_0 = F(x_0)$.

8.4. Corollary. *If $F^{-1}(y) \in \mathrm{AC}^n_M$ for each $y \in Y$ and $N \in \mathrm{ANR}(\mathfrak{M})$, then*

$$\text{pro-}\pi_k[F]\colon \text{pro-}\pi_k U_M(X, x_0) \to \text{pro-}\pi_k U_N(Y, y_0)$$

is an isomorphism of pro-Gr *for $k \leqslant n$ and an epimorphism for $k = n+1$.*

Proof. Take $c' \in C$ and choose $c \geqslant c'$ as in Lemma 8.3, where $\boldsymbol{U}$ is an open cover of $V_{c'}$ such that any two $\boldsymbol{U}$-near maps $s_1, s_2\colon (Z, z_0)$

$\to(V_c, y_0)$ are homotopic rel. z_0 (see [Hu], p. 111). Applying Lemma 8.3 first for $P = B^{k+1}$ and $R = S^k$ and then for $P = S^k$ and $R = \{\text{point}\}$, we get

$$\ker \pi_k[F_c] \subset \ker \pi_k[i^c_{c'}] \quad \text{for} \quad k \leqslant n$$

and

$$\operatorname{im} \pi_k[j^c_{c'}] \subset \operatorname{im} \pi_k[F_{c'}] \quad \text{for} \quad k \leqslant n+1.$$

By Theorem 2.6 pro-$\pi_k[F]$ is an epimorphism for $k \leqslant n+1$.

Let $k \leqslant n$ and $c \in C$. Take $c_2 \geqslant c_1 \geqslant c$ such that:

$$\ker \pi_k[F_{c_1}] \subset \ker \pi_k[i^{c_1}_c] \quad \text{and} \quad \operatorname{im} \pi_k[j^{c_2}_{c_1}] \subset \operatorname{im} \pi_k[F_{c_1}].$$

Define $h\colon \pi_k(V_{c_2}, y_0) \to \pi_k(U_c, x_0)$ as follows:

$$h(\alpha) = \pi_k[i^{c_1}_c](\beta), \quad \text{where } \pi_k[F_{c_1}](\beta) = \pi_k[j^{c_2}_{c_1}](\alpha).$$

It is easy to verify that h is a homomorphism and

$$\pi_k[F_c] \cdot h = \pi_k[j^{c_2}_c], \quad h \cdot \pi_k[F_{c_2}] = \pi_k[i^{c_2}_c].$$

Tuhs, by Theorem 2.3, pro-$\pi_k[F]$ is an isomorphism for $k \leqslant n$.

As a consequence of Corollary 8.4 we get

8.5. THEOREM. *If* $f\colon (X, x_0) \to (Y, y_0)$ *is a closed surjective map of metrizable spaces such that* $f^{-1}(y) \in \mathrm{AC}^n$ *for each* $y \in Y$, *then*

$$\text{pro-}\pi_k S[f]\colon \text{pro-}\pi_k(X, x_0) \to \text{pro-}\pi_k(Y, y_0)$$

is an isomorphism of pro-Gr *for* $k \leqslant n$ *and an epimorphism of* pro-Gr *for* $k = n+1$.

Proof. Take two absolute retracts for metrizable spaces M and N' containing X and Y, respectively, as closed subsets (see [Bor_3], p. 79) and let $F'\colon M \to N'$ be an extension of f. Let

$$N = N' \times [0, \infty) \quad \text{and} \quad F(x) = \big(F'(x), \operatorname{dist}(x, X)\big)$$

for $x \in M$. To complete the proof of Theorem 8.5 it is enough to observe that $U_M(X, x_0)$ and $U_N(Y, y_0)$ are naturally isomorphic in pro-HT_0 to $\check{C}(X, x_0)$ and $\check{C}(Y, y_0)$, respectively (see [Mor_2]) and $f^{-1}(y) \in \mathrm{AC}^n$ is equivalent to $f^{-1}(y) \in \mathrm{AC}^n_M$.

Theorem 8.5 has been proved by the author [Dyd_5] (the compact case) and [Dyd_9] (the general case). Independently K. Morita [Mor_5] has obtained the same result.

Theorem 8.5 may be regarded as a natural analogue in shape theory of the Smale theorem because the class of spaces AC^n plays the same role in shape theory as the class C^n does in homotopy theory.

As an immediate consequence of Theorem 8.5 we get

8.6. THEOREM. *If* $f\colon (X, x_0)\to(Y, y_0)$ *is a closed surjective map of metrizable spaces such that* $f^{-1}(y) \in \mathrm{AC}^n$ *for each* $y \in Y$, *then*

$$\check{\pi}_n S[f]\colon \check{\pi}_n(X, x_0)\to\check{\pi}_n(Y, y_0)$$

is an isomorphism of Gr.

This theorem has been outlined by S. Bogatyi [Bog$_2$] and proved by K. Kuperberg [Kup] in the compact case. Next Y. Kodama [Kod$_1$] proved that $\check{\pi}_{n-1}S[f]$ is an isomorphism and G. Kozlowski and J. Segal [K-S$_2$] proved 8.6 for proper maps.

Another consequence of Corollary 8.4 is the following

8.7. THEOREM. *If* $Z \in \mathrm{LC}^n$ *is a metrizable space, then the natural morphism*

$$i_k\colon \pi_k(Z, z)\to\text{pro-}\pi_k(Z, z)$$

is an isomorphism for $k \leqslant n$ *and an epimorphism for* $k = n+1$.

Proof. Take an absolute retract for metrizable spaces N containing Z as a closed subset. Observe that $Z \in \mathrm{LC}^n$ is equivalent to $\{x\} \in \mathrm{AC}^n_Z$ for each $x \in Z$. In view of Corollary 8.4 this completes the proof.

Theorem 8.7 implies at once

8.8. COROLLARY. *If* Z *is a metrizable space, then* $Z \in \mathrm{LC}^n \cap \mathrm{C}^{n+1}$ *implies* $Z \in \mathrm{AC}^{n+1}$ *and* $Z \in \mathrm{AC}^n \cap \mathrm{LC}^n$ *implies* $Z \in \mathrm{C}^n$.

Corollary 8.8 has been obtained by S. Bogatyi [Bog$_2$] for the compact case.

It is well known that if Q is a CW complex, then for each shape morphism $f\colon X\to Q$ there is a unique homotopy class $f'\colon X\to Q$ such that $f = Sf'$. For LC^n metrizable spaces we can prove a similar result.

8.9. THEOREM. *If* $Z \in \mathrm{LC}^n$ *is a metrizable space, then for each shape morphism* $f\colon (X, x)\to(Z, z)$ *such that* $\operatorname{ddim} X \leqslant n$ *there is a unique morphism* $f'\colon (X, x)\to(Z, z)$ *of* HT_0 *such that* $f = Sf'$.

Proof. It is enough to consider the case where Z is connected.

Let $p\colon \mathrm{Sin}(Z, z)\to(Z, z)$ be a canonical morphism of HT_0. Then p induces isomorphisms of all homotopy groups and, by Theorem 8.7, Sp is an $(n+1)$-equivalence. Consequently, by Theorem 5.6, for each morphism $f\colon (X, x)\to(Z, z)$, where $\operatorname{ddim} X \leqslant n$, there is a unique shape morphism $g\colon (X, x)\to\mathrm{Sin}(Z, z)$ with $Sp\cdot g = f$. On the other hand, there is a unique morphism $g'\colon (X, x)\to\mathrm{Sin}(Z, z)$ of HT_0 with $g = Sg'$. Hence $f' = p\cdot g'$ satisfies $Sf' = f$, which completes the proof.

Theorem 8.9 has been announced by S. Bogatyi [Bog$_1$] in the case where Z is compact metrizable and $\dim X \leqslant n$. Later G. Kozlowski and

J. Segal [K-S$_2$] showed it for the case where X is an n-dimensional CW complex.

8.10. EXAMPLE. Let us show that the following statement is not true:

Each shape morphism $f\colon X\to Z$, *where* $Z\in \mathrm{LC}^n$ *and* $\dim X\leqslant n+1$, *is generated by some homotopy class* $f'\colon X\to Z$, *i.e.* $f = Sf'$.

Let

$$B_{mn} = \{(x, y, z)\in R^3\colon (x-m)^2+(y-2^{-n})^2+z^2 = 2^{-2n}\}$$

and let $B = \bigcup_{n=1}^{\infty} B_{0,n}$. We show that $\pi_1(B, a) = 0$, where $a = (0, 0, 0)$.

So let $p\colon (I, \dot{I})\to(B, a)$ be a map and let

$$U_n = \{(x, y, z)\in R^3\colon x^2+(y-2^{-n+1})^2+z^2 < 2^{-2n+2}\}.$$

Then $p^{-1}\left(\bigcup_{n=1}^{\infty} U_n\right)$ is a disjoint union of intervals (a_k, b_k) such that $0 < a_k < b_k < 1$.

Now, for each k, there exists an n such that $p\left((a_k, b_k)\right)\subset U_n$ and we can find a homotopy

$$H_k\colon [a_k, b_k]\times I\to\overline{U}_n \text{ rel. } \{a_k, b_k\}$$

such that

$$H_k(x, 0) = p(x) \quad \text{for} \quad x\in[a_k, b_k],$$

$$\operatorname{diam} H_k(\{x\}\times I) < b_k-a_k \quad \text{for} \quad x\in[a_k, b_k],$$

$$H_k([a_k, b_k]\times\{1\})\subset \overline{U}_n-\{(0, 2^{-n+1}, 0)\}.$$

Define $H\colon (I, \dot{I})\times I\to(B, a)$ by the following formula: $H(x, t) = H_k(x, t)$ if $x\in[a_k, b_k]$ for a certain k and $H(x, t) = p(x)$ otherwise.

Then $H(x, 0) = p(x)$ for $x\in I$ and $H(I\times\{1\})$ does not contain the points $(0, 2^{-n+1}, 0)$, $n = 1, 2, \ldots$ Since the inclusion $B-\bigcup_{n=1}^{\infty}\{(0, 2^{-n+1}, 0)\}\hookrightarrow B$ is null-homotopic, we infer that p represents the trivial element of $\pi_1(B, a)$.

Let $S^1 = \{(x, y, z)\in R^3\colon (x+1)^2+y^2 = 1\}$ and let $Z = B\cup S^1$, $B_n = \bigcup_{k=1}^{n} B_{0,k}$, $Z_n = B_n\cup S^1$. Then Z is locally 1-connected.

Let $L = \{(x, y, z)\in R^3\colon y = z = 0\}$ and $E = L\cup\bigcup_{n=1}^{\infty}\bigcup_{m=-\infty}^{\infty} B_{m,n}$, $E_n = L\cup\bigcup_{m=-\infty}^{\infty}\bigcup_{k=1}^{n} B_{m,k}$.

Let $p\colon E\to Z$ be a map such that

$$p\big((x,0,0)\big) = (\cos 2\pi x - 1, \sin 2\pi x, 0),$$
$$p\big((x,y,z)\big) = (x-m, y, z) \quad \text{if} \quad (x,y,z)\in B_{m,n}.$$

Let $p_n\colon E_n\to Z_n$ be maps induced by p. Then p and all p_n are universal covering projections.

Let $g_{k,l}: S^2\to B_{k,l}$ be a homeomorphism and let $g_n\colon S^2\to E_n$ be a representative of

$$\sum_{l=1}^{n}\sum_{k=-l}^{l}[g_{k,l}]\in\pi_2 E_n.$$

Let $f_n = p_n\cdot g_n\colon S^2\to Z_n$ and let $r_n\colon Z\to Z_n$ be a map such that $r_n|Z_n = \mathrm{id}$ and $r_n(B_{0,m}) = \{a\}$ for $m\geqslant n+1$.

Now, if $r_n^{n+1} = r_n|Z_{n+1}\colon Z_{n+1}\to Z_n$, then $r_n^{n+1}\cdot f_{n+1}\simeq f_n$ for each n, and since $Z = \varprojlim(Z_n, r_n^{n+1})$, we infer that there exists a shape morphism $f\colon S^2\to Z$ such that $S[r_n]\cdot f = S[f_n]$ for each n (see [Mar$_1$]).

Suppose $f = S[g]$ for some map $g\colon S^2\to Z$. We may assume that $g(s_0) = a$ (s_0 is the base point of S^2).

Then there is a map $\bar g\colon S^2\to E$ with $p\cdot\bar g = g$ and $\bar g(s_0) = a$.

Take maps $\bar r_n\colon E\to E_n$ and $\bar r_n^{n+1}\colon E_{n+1}\to E_n$ such that $p_n\cdot\bar r_n = r_n\cdot p$, $p_n\cdot\bar r_n^{n+1} = r_{n+1}\cdot p_{n+1}$, and $\bar r_n^{n+1}(a) = \bar r_n(a) = a$. Then

$$p_n\cdot\bar r_n\cdot\bar g = r_n\cdot p\cdot\bar g = r_n\cdot g\simeq f_n = p_n\cdot g_n,$$

i.e. $p_n\cdot\bar r_n\cdot\bar g$ and $p_n\cdot g_n$ are freely homotopic.

Hence there exist $[\omega_n]\in\pi_1(Z_n, a)$ such that

$$p_n\cdot\bar r_n\cdot\bar g = h_{[\omega_n]}[p_n\cdot g_n]\in\pi_2(Z_n, a)$$

(for a description of the acting of $\pi_1(Z_n, a)$ on $\pi_2(Z_n, a)$ see [S], p. 382).

Notice that $\pi_1(Z_n, a)$ is isomorphic to the group of integers for each n and each element $[\omega]$ of $\pi_1(Z_n, a)$ has a representative $\omega\colon (I, \dot I)\to(Z_n, a)$ of the form $\omega(t) = p_n(m\cdot t, 0, 0)$ for some integer m. Moreover, $\pi_1[r_n^{n+1}]$ can be regarded as the identity.

Now

$$\begin{aligned} h_{[\omega_{n+1}]}[p_n\cdot g_n] &= h_{[\omega_{n+1}]}[r_n^{n+1}\cdot p_{n+1}\cdot g_{n+1}] \\ &= \pi_2[r_n^{n+1}]\,(h_{[\omega_{n+1}]}[p_{n+1}\cdot g_{n+1}]) \\ &= \pi_2[r_n^{n+1}]\,([p_{n+1}\cdot\bar r_{n+1}\cdot\bar g]) = [r_n^{n+1}\cdot p_{n+1}\cdot\bar r_{n+1}\cdot\bar g] \\ &= [p_n\cdot\bar r_n\cdot\bar g] = h_{[\omega_n]}[p_n\cdot g_n] \end{aligned}$$

and hence

$$[\omega_n] = [\omega_{n+1}] \quad \text{for each } n$$

(because $p_n \cdot g_n$ is non-trivial). Thus there exists an integer k such that $[\omega_n] = k$ for each n.

Let h_n be a representative of

$$\sum_{l=1}^{n} \sum_{s=-l}^{l} [g_{s+k,l}], \quad \text{i.e.} \quad [p_n \cdot h_n] = h_{[\omega_n]}[p_n \cdot g_n].$$

Then

$$[p_n \cdot h_n] = h_{[\omega_n]}[p_n \cdot g_n] = [p_n \cdot \bar{r}_n \cdot \bar{g}]$$

and since $\pi_2[p_n]$ is an isomorphism, we infer $[h_n] = [\bar{r}_n \cdot \bar{g}]$. Now, $\bar{g}(S^2)$ is a compact set and there exists an $m > 0$ such that $B_{p,r} \cap \bar{g}(S^2) = \emptyset$ for $|p| \geqslant m$ and each $r \geqslant 1$.

Then $g_{|k|+m+k,\,|k|+m}$ cannot appear in the sum

$$\sum_{l=1}^{k+m} \sum_{s=-l}^{l} [g_{s+k,l}] = [h_{|k|+m}] = [\bar{r}_{|k|+m} \cdot \bar{g}]$$

(because $B_{|k|+m+k,\,|k|+m}$ is not contained in $\bar{r}_{|k|+m} \cdot \bar{g}(S^2)$), a contradiction.

This example shows that Theorem 2 of [Bog$_1$] is false.

For the sake of completeness we prove a slight generalization of the Smale theorem in homotopy (see [Dug] and [Koz]).

8.11. THEOREM. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a closed surjective map of connected metrizable spaces such that $f^{-1}(y) \in \mathrm{LC}^{n-1} \cap \mathrm{C}^{n-1}$ for each $y \in Y$. If $X \in \mathrm{LC}^n$, then $Y \in \mathrm{LC}^n$ and $\pi_k[f]$ is an isomorphism for $k \leqslant n-1$ and an epimorphism for $k = n$.*

Proof. In view of Theorems 8.5, 8.7 and Corollary 8.8 it is enough to prove that $Y \in \mathrm{LC}^n$. This will be proved by induction.

The case $n = 0$ is easy. So suppose $Y \in \mathrm{LC}^{n-1}$. First, let us prove that $\pi_n[f]$ is an epimorphism. Take an absolute retract for metrizable spaces N containing X as a closed subset. Let $F_1\colon X \to N$ be the inclusion map. Then $F_1^{-1}(x) = \{x\} \in \mathrm{AC}^n_X$ for each $x \in X$ and Corollary 8.4 and Theorem 8.7 imply

(1) pro-$\pi_k\, U_X(f^{-1}(y), x) = \pi_k(f^{-1}(y), x) = 0$ for $k \leqslant n-1$ and each $y \in Y$, $x \in f^{-1}(y)$, i.e. $f^{-1}(y) \in \mathrm{AC}^{n-1}_X$,

(2) the natural morphism from $\pi_n(f^{-1}(y), x)$ to pro-$\pi_n\, U_X(f^{-1}(y), x)$ is an epimorphism for each $y \in Y$, $x \in f^{-1}(y)$.

Let P be an at most n-dimensional polyhedron and let $g\colon (P, p_0) \to (Y, y_0)$ be a map. By Lemma 8.3 we can find a sequence of maps $h_n\colon (P, p_0) \to (X, x_0)$ such that $\varrho(f \cdot h_n(p), g(p)) < 1/n$ for each $p \in P$. Let us show that the following condition is satisfied:

(3) For any open cover $\boldsymbol{U}$ of Y there exists a k such that for $m, l > k$ the maps $f \cdot h_l$ and $f \cdot h_m$ are $\boldsymbol{U}$-homotopic rel. p_0.

For each $y \in Y$ take a $U_y \in \boldsymbol{U}$ containing y and find an open neighbourhood $V_y \subset U_y$ of y such that every map $r\colon S^n \to f^{-1}(V_y)$ is homotopic in $f^{-1}(U_y)$ to a map $r'\colon S^n \to f^{-1}(y)$ (see condition (2)). Let $\boldsymbol{V}_1$ be a star refinement of $\{V_y\}_{y\in Y}$ and let $\boldsymbol{V}_2$ be a star refinement of $\boldsymbol{V}_1$. Take an open cover $\boldsymbol{V}_3$ of Y such that if $s_1, s_2\colon (R, r_0) \to (Y, y_0)$ are two $\boldsymbol{V}_3$-near maps and $\dim R \leqslant n-1$, then s_1 and s_2 are $\boldsymbol{V}_2$-homotopic rel. r_0 (see [Hu]). Take a triangulation T of P such that there exists a k such that $f\cdot h_m$ and $f\cdot h_l$ are $\boldsymbol{V}_3$-near for $m, l > k$ and $\{f\cdot h_m(s)\}_{s\in T} \succ \boldsymbol{V}_3$ for $m > k$.

Let $m, l > k$. Then there exists a $\boldsymbol{V}_2$-homotopy $H\colon (|T^{n-1}|, p_0)\times I \to (Y, y_0)$ joining $f\cdot h_m/|T^{n-1}|$ and $f\cdot h_l/|T^{n-1}|$. By Lemma 8.3 we can find an $f^{-1}(\boldsymbol{V}_2)$-homotopy $G\colon (|T^{n-1}|, p_0)\times I \to (X, x_0)$ such that

$$G(x, 0) = h_m(x) \quad \text{and} \quad G(x, 1) = h_l(x)$$

for $x \in |T^{n-1}|$.

Define an extension $G'\colon |T^{n-1}|\times I \cup P\times\{0,1\} \to X$ of G by setting $G'(x, 0) = h_m(x)$ and $G'(x, 1) = h_l(x)$ for $x \in P$. Then for each n-simplex s of T the set $G'/\partial(s\times I)$ is contained in $f^{-1}(V_{y_s})$ for a certain $y_s \in Y$. Hence we can extend $f\cdot G'/\partial(s\times I)$ over $s\times I$ so that the values are in U_{y_s}. Thus we get a $\boldsymbol{U}$-homotopy joining $f\cdot h_m$ and $f\cdot h_l$.

Consequently we may suppose that for each l we have a homotopy $H_l\colon (P, p_0)\times I \to (Y, y_0)$ such that

$$H_l(x, 0) = f\cdot h_{l+1}(x) \quad \text{and} \quad H_l(x, 1) = f\cdot h_l(x)$$

and $\operatorname{diam} H_l(\{x\}\times I) < 1/l$ for each $x \in P$.

Let $H\colon (P, p_0)\times I \to (Y, y_0)$ be defined as follows: $H(x, t) = H_l\big(x, l(l+1)t - l\big)$ if $1/l+1 \leqslant t \leqslant 1/l$ and $H(x, 0) = g(x)$. Thus H is a homotopy joining g and $f\cdot h_1$, i.e. $\pi_n[f]$ is an epimorphism.

Suppose that U is a neighbourhood of $y \in Y$. Take a neighbourhood $V \subset U$ of y such that each map $h\colon S^n \to f^{-1}(V)$ is homotopic to a map $h'\colon S^n \to f^{-1}(y)$ in $f^{-1}(U)$.

Take $g\colon S^n \to V$. Then there is an $h\colon S^n \to f^{-1}(V)$ such that $g \simeq f\cdot h$ (here by f we denote a map from $f^{-1}(V)$ to V induced by f). Hence g is null-homotopic in U, i.e. $Y \in \mathrm{LC}^n$.

See [Dug] and [$\mathrm{La}_{1,2,3}$] for other Smale type theorems in homotopy.

The main lemma in this section is the following

8.12. Lemma. *If $M, N \in \mathrm{ANR}(\mathfrak{M})$ and $F^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$, then for each $c' \in C$ there exists a $c \geqslant c'$ such that for each $x_0 \in X$, $y_0 = F(x_0)$ the following conditions are satisfied:*

(1) *for each map $g\colon (Z, z_0) \to (V_c, y_0)$, where (Z, z_0) is a pointed at most $(n+1)$-dimensional* CW *complex, there exists a map $h\colon (Z, z_0) \to (U_c, x_0)$ such that $F_{c'}\cdot h \simeq g$ in $(V_{c'}, y_0)$,*

(2) *if $g, h\colon (Z, z_0) \to (U_c, x_0)$ are two maps, (Z, z_0) is a pointed at most n-dimensional* CW *complex and $F_c\cdot g \simeq F_c\cdot h$, then $g \simeq h$ in $(U_{c'}, x_0)$.*

Proof. As in the proof of Corollary 8.4, we can find a sequence $c_{2n+2} = c' \leqslant \ldots \leqslant c_0 = c$ of elements of C such that for each $x_1 \in X$ and $y_1 = F(x_1)$ the following conditions are fulfilled:

(3) $\ker \pi_k[F_{c_{2k}}] \subset \ker \pi_k[i^{2k}_{2k+1}]$ for $1 \leqslant k \leqslant n+1$ and
$\operatorname{im} \pi_{k+1}[j^{2k+1}_{2k+2}] \subset \operatorname{im} \pi_{k+1}[F_{c_{2k+2}}]$ for $0 \leqslant k < n+1$, where
$j^k_{k+1}\colon (V_{c_k}, y_1) \to (V_{c_{k+1}}, y_1)$ and
$i^k_{k+1}\colon (U_{c_k}, x_1) \to (U_{c_{k+1}}, x_1)$ are inclusions,

(4) each two components of U_c which are mapped by F_c into some component of V_c are contained in some component of U_{c_0}.

Let $x_0 \in X$ and $y_0 = F(x_0)$. Let $g, h\colon (K, k_0) \to (U_c, x_0)$ be two maps such that (K, k_0) is a pointed at most n-dimensional CW complex and $F_c \cdot g \simeq F_c \cdot h$. Consider an arbitrary component K_0 of K.

Since $F_c \cdot g(K_0) \cup F_c \cdot h(K_0)$ is contained in some component of V_c, there exists a component X_0 of U_{c_0} containing $g(K_0) \cup h(K_0)$. Let Y_0 be a component of V_{c_0} containing $F_{c_0}(X_0)$. Take a pointed CW complex (L, l_0) equal to (K_0, k_0) if $k_0 \in K_0$, otherwise add to K_0 a discrete base point l_0. Take $x_1 \in X_0 \cap X$ equal to x_0 if $x_0 \in X_0$.

Define $r, s\colon (L, l_0) \to (X_0, x_1)$ and $f\colon (X_0, x_1) \to (Y_0, F(x_1))$ as follows: $r(k) = g(k)$ and $s(k) = h(k)$ for $k \in K_0$, $r(l_0) = s(l_0) = x_1$, $f(x) = F(x)$ for $x \in X_0$. Then $f \cdot r \simeq f \cdot s$ and, by Lemma 5.3, $r \simeq s$ in $(U_{c'}, x_1)$. Thus, $g \simeq h$ in $(U_{c'}, x_0)$ and condition (2) is verified. Similarly one can check that condition (1) holds.

Using Lemma 8.12 one can easily prove the following results:

8.13. Theorem. *Let X and Y be metrizable spaces. If $f\colon (X, x_0) \to (Y, y_0)$ is a closed surjective map such that $f^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$, then for each shape morphism $g\colon (Z, z_0) \to (Y, y_0)$, where $\operatorname{ddim} Z \leqslant n$, there is a unique shape morphism $h\colon (Z, z_0) \to (X, x_0)$ with $S[f] \cdot h = g$.*

Theorem 8.13 has been announced by S. Bogatyi [Bog$_1$] for the case where X, Y, Z are compacta and $\dim Z \leqslant n$ (the unpointed version). It is also an improvement of Theorems (3.1) and (3.10) of [Dyd$_2$].

8.14. Theorem. *Let X and Y be metrizable spaces, and let*

$$n+1 = \max(\operatorname{ddim} X + 1, \operatorname{ddim} Y).$$

If $f\colon (X, x_0) \to (Y, y_0)$ is a closed surjective map such that $f^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$, then $S[f]$ is a shape equivalence.

Theorem 8.14 arose as a partial answer to the following question of K. Borsuk [Bor$_2$]:

Let X and Y be compacta. Is it true that $\operatorname{Sh} X = \operatorname{Sh} Y$ if there exists a surjective map $f\colon X \to Y$ such that $f^{-1}(y)$ has the trivial shape for each $y \in Y$?

Theorem 8.14 has hitherto been proved for the following cases (all in the unpointed version):

(1) X and Y are compact finite-dimensional and $f^{-1}(y) \in \mathrm{FAR}$ for each $y \in Y$ (R. B. Sher [Sh] and then S. Bogatyi [Bog$_2$]),

(2) $n = \max(\dim X, \dim Y)$ (Y. Kodama [Kod$_2$]);

(3) $n = \max(\mathrm{Sd}\, X, \mathrm{Sd}\, Y)$ (J. Dydak [Dyd$_2$]);

(4) X and Y are compacta and $n+1 = \max(\mathrm{Fd}\, X+1, \mathrm{Fd}\, Y)$ (J. Dydak [Dyd$_5$]);

(5) $n+1 = \max(\dim X+1, \dim Y)$ (K. Morita [Mor$_5$]) and $n+1 = \max(\mathrm{ddim}\, X+1, \mathrm{ddim}\, Y)$ (J. Dydak [Dyd$_9$]).

8.15. THEOREM. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a surjective map of compact metrizable spaces such that $f^{-1}(y) \in \mathrm{AC}^\infty$ for each $y \in Y$. If one of the following conditions is satisfied:*

(1) *(X, x_0) is movable and $\mathrm{ddim}\, X$ is finite,*

(2) *(Y, y_0) is movable and $\mathrm{ddim}\, Y$ is finite,*

(3) *(X, x_0) is movable and $S[f]$ is a domination, then $S[f]$ is a shape equivalence.*

Theorem 8.15 gives a new partial answer to the question of K. Borsuk and has been proved by the author [Dyd$_3$] (the unpointed version) and independently by S. Bogatyi [Bog$_3$] (without condition 3). Earlier D. A. Edwards and R. Geoghegan [E-G$_3$] obtained it for the case where (Y, y_0) is pointed FANR.

8.16. EXAMPLE. We show an example of a surjective map $f\colon (X, x_0) \to (Y, y_0)$ such that (X, x_0) and (Y, y_0) are pointed movable continua, $f^{-1}(y) \in \mathrm{FAR}$ for each $y \in Y$ and $S[f]$ is not a shape equivalence. This example was given by G. Kozlowski and J. Segal [K-S$_2$] and independently by the author [Dyd$_5$].

Let $g\colon Z \to Q$ be a surjective map from Z onto the Hilbert cube Q such that $g^{-1}(y) \in \mathrm{FAR}$ for each $y \in Q$ and $Z \notin \mathrm{FAR}$ (see [Tay]). We consider Z as a closed subset of a copy Q' of the Hilbert cube. Take $z_0 \in Z$. Choose an extension $F\colon Q' \to Q$ of g and let $\{Z_n\}_{n=1}^\infty$ be a decreasing sequence of ANR's such that $\bigcap_{n=1}^{\infty} Z_n = Z$.

Let (X_n, x_n) be obtained from $\bigcup_{k=1}^{n} Z_k \times \{k\}$ by identifying points $(z_0, 1), \ldots, (z_0, n)$. Define a map $p_n^{n+1}\colon (X_{n+1}, x_{n+1}) \to (X_n, x_n)$ as follows:

$$p_n^{n+1}([z, k]) = [z, k] \text{ for } k \leqslant n \quad \text{and} \quad p_n^{n+1}([z, n+1]) = [z, n]$$

(by $[z, k]$ we denote an equivalence class of (z, k)). Then $(X, x_0) = \varprojlim((X_n, x_n), p_n^{n+1})$ is movable (see [M-S]).

Now let us define (Y, y_0). Let $(Y_1, y_1) = (Q, t)$, where $t = g(z_0)$, and let (Y_{n+1}, y_{n+1}) be the wedge of (Q, t) and (X_n, x_n). The maps

$$q_n^{n+1}\colon (Y_{n+1}, y_{n+1}) \to (Y_n, y_n)$$

are defined as follows:

$$q_n^{n+1}(x) = x \quad \text{for } x \in Q \text{ or } x = [z, k], \text{ where } k \leqslant n-1, \quad \text{and}$$

$$q_n^{n+1}([z, n]) = F(z) \quad \text{for} \quad z \in Z_n.$$

Then $(Y, y_0) = \varprojlim\left((Y_n, y_n), q_n^{n+1}\right)$ is movable.

Let the map $f_n\colon (X_n, x_n) \to (Y_n, y_n)$ be defined by the formulae: $f_n([z, k]) = [z, k]$ for $k \leqslant n-1$ and $f_n([z, n]) = F(z)$ for $z \in Z_n$. Then

$$q_n^{n+1} \cdot f_{n+1} = f_n \cdot p_n^{n+1}$$

and consequently there exists a map $f\colon (X, x_0) \to (Y, y_0)$ induced by f_n, $n \geqslant 1$. Observe that we can regard Z as a subset of X ($z \in Z$ corresponds to $\{[z, n]\}_{n=1}^{\infty} \in X$). Then $Y = X \cup_g Q$ and $f\colon X \to Y$ is the natural projection. Thus f is surjective and $f^{-1}(y) \in \mathrm{FAR}$ for each $y \in Y$.

Suppose that $S[f]$ is a shape equivalence. Let $i\colon Z \to X$ be the inclusion map. Since $S[f \cdot i]$ is the trivial morphism, we infer that $S[i]$ must be trivial. Then all maps $i_n\colon Z \to X_n$ $\left(i_n(z) = [z, n]\right)$ are null-homotopic. So if $r_n\colon X_n \to Z_n$ is the retraction, then the inclusion map $r_n \cdot i_n\colon Z \to Z_n$ is null-homotopic contrary to $Z \notin \mathrm{FAR}$.

The examples of maps $g\colon Z \to Q$ and $f\colon X \to Y$ together with an example of J. Keesling [K_2] show that Theorems 6.3′, 6.4′, 6.5′, and 8.15 cannot be improved by omitting some of the hypotheses.

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