ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria III: MATEMATYKA STOSOWANA XXXV (1992)

Ministerstwo Nauki i Szkolnictwa Wyższego

D. VAN BAN

K. MOSZYŃSKI

A. POKRZYWA

Semiiterative methods for linear equations

(Praca wpłynęła do Redakcji 11.12.1989)

Let A denote a linear invertible bounded operator acting in a complex Banach space X. We are interested in solving a linear equation $Ax = f$. We assume that it is only possible to calculate linear combinations of known vectors and to compute vector Az when z is known.

Since at the beggining we know the vector f only we can calculate any linear combination of vectors $f, Af, A^2f, \ldots$ or in other words calculate any vector of the form $p(A)f$ where p is a polynomial. If we could find such a sequence $\{p_n\}$ of polynomials that $p_n(A)f \to A^{-1}f$ we obtain a semiiterative method of solving linear equations. The operator calculus and approximation theory of analytic functions form a theoretical background for methods of this kind. They have been studied by R. S. Varga, L. Reichel and others.

If the space X is finite–dimensional then there exists a polynomial q such that $q(A) = 0, q(0) \neq 0$ — a minimal polynomial of A or its multiple. Let $p(\lambda) = \frac{q(0)-q(\lambda)}{\lambda q(0)}$, p is a polynomial and it follows from its definition that $Aq(0)p(A) = q(0) - q(A)$ and therefore $p(A) = A^{-1}$. If the minimal polynomial q has zero of multiplicity k at λ then $q^{(j)}(\lambda) = 0, (0 \leq j \leq k-1)$ and setting

$$\rho(\lambda) = \lambda^{-1}$$

we have

$$p^j(\lambda) = \frac{1}{q(0)} \sum_{i=0}^{j} \binom{j}{i} (\delta_{0,i} q(0) - q^{(i)}(\lambda)) \rho^{(j-i)}(\lambda) = \rho^{(j)}(\lambda).$$

These simple algebraic considerations lead to the following theorem.

THEOREM 1. *If $0 \notin \Omega \subset C$ then the following conditions are equivalent:*

(*i*) $p_n(A) \to A^{-1}$ *with* $n \to \infty$ *for any invertible finite-dimensional operator* A *whose spectrum is contained in the set* Ω,

(*ii*) $p_n^{(k)}(\lambda) \to \rho^{(k)}(\lambda)$ *for* $k = 0, 1, 2, \ldots$ *and for any* $\lambda \in \Omega$.

Proof. (ii)⇒(i). Any finite-dimensional operator may be represented in the form $A = \sum_j (\lambda_j + N_j)E_j$, where the operators E_j are projections such that $E_iE_j = \delta_{i,j}E_i$, $\sum_j E_j = I$ and $N_j \in L(\mathrm{ran}E_j)$ are nilpotent operators, $N_j^{k_j} = 0$ for some nonnegative integer k_j. This representation is equivalent to the Kronecker canonical form. For any function f analytic in the neighbourhood of the spectrum of A we have

$$f(A) = \sum_j f(\lambda_j + N_j)E_j$$

and using Taylor formula we obtain

$$f(A) = \sum_j \sum_{i=0}^{k_j - 1} \frac{1}{i!} f^{(i)}(\lambda_j) N_j^i E_j.$$

Writting this formula for $f = p_n - \rho$ we obtain the estimation

$$\|p_n(A) - A^{-1}\| \leq \max_{\substack{j \\ 0 \leq i < k_j}} |p_n^{(i)}(\lambda) - \rho^{(i)}(\lambda)| \sum_j \sum_{i=0}^{k_j - 1} \frac{1}{i!} \|N_j^i E_j\|,$$

from which the desired convergence follows.

(i)⇒(ii) Let X be a m-dimensional complex Banach space with basis $e_1, e_2, \ldots, e_m$ and N, A_λ be the operators defined by

$$Ne_k = \begin{cases} e_{k+1}, & \text{for } k = 1, 2, \ldots, m-1, \\ 0, & \text{for } k = m, \end{cases}$$

$A_\lambda = \lambda I + N$. Then $\sigma(A_\lambda) = \{\lambda\}$ and

$$(A_\lambda^{-1} - p_n(A_\lambda))e_1 = \sum_{i=0}^{m-1} \frac{1}{i!} (\rho^{(i)}(\lambda) - p_n^{(i)}(\lambda)) N^i e_1 =$$

$$= \sum_{i=0}^{m-1} \frac{1}{i!} (\rho^{(i)}(\lambda) - p_n^{(i)}(\lambda)) e_{i+1}.$$

In view of the linear independence of the vectors $e_1, e_2, \ldots, e_m$ the convergence $p_n(A_\lambda) \to A_\lambda^{-1}$ implies that $p_n^{(k)}(\lambda) \to \rho^{(k)}(\lambda)$ for $k = 0, 1, 2, \ldots, m-1$.

THEOREM 2. *There exists a sequence of polynomials* $\{p_n\}_{n=0}^{\infty}$ *such that for any invertible finite-dimensional operator* A, $p_n(A) \to A^{-1}$ *with* $n \to \infty$.

Proof. Let $\Omega_n = K(0,n)\backslash K_0(0,\frac{1}{n})\backslash\{z \in C : \frac{1}{n} < \arg z < \frac{2}{n}\}$, where $K(c,r)$ ($K_0(c,r)$) denotes the closed (open) disc of radius r and center c. It follows from the Runge approximation theorem ([3] or [10]) that there exists a polynomial p_n such that $|p_n^{(k)}(\lambda) - \rho^{(k)}(\lambda)| < \frac{1}{n}$ for all $\lambda \in \Omega_n, k = 0,1,2,\ldots,n$. The definition of the sets Ω_n implies that $\sigma(A) \subset \Omega_n$ for sufficiently large n. Now the thesis of the Theorem follows from Theorem 1.

Theorem 2 shows that there exists a universal semiiterative algorithm for all linear finite-dimensional problems. However in practice such an algorithm would be not very efficient — the polynomials p_n used in the proof are of high degrees and hard to construct. Moreover it follows from the maximum principle that for each $n \in N$ there exists z_n such that $|z_n| = 1$ and $|p_n(z_n)| \geq |p_n(\frac{1}{n})| \geq |\rho(\frac{1}{n})| - \frac{1}{n} \geq n - \frac{1}{n}$. Hence if we cannot localize the spectrum of A we cannot forsee for how large n $p_n(A)f$ is sufficiently accurate approximation of $x = A^{-1}f$.

Theorem 3 is an infinite-dimensional version of Theorem 1, Theorem 4 — of Theorem 2. To see that Theorem 2 does not hold for infinite-dimensional operators let us consider the operator S — the bilateral shift defined by $Se_n = e_{n+1}$, where $\{e_j\}_{-\infty}^{\infty}$ is an orthonormal basis of the Hilbert space. The equation $Sx = e_1$ cannot be solved by such methods because for any polynomial p the vector $p(S)e_1$ is orthogonal to the solution $x = e_0$ and hence $\|x - p(S)e_1\|^2 = \|e_0\|^2 + \|p(S)e_1\|^2 \geq 1$.

LEMMA 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic functions such that each one is defined on some neighbourhood of 0 and $f_n(V) \to 0$ (weakly) for any quasinilpotent operator V acting in a complex Hilbert space H. Then there exists a positive number r such that all the functions f_n have analytic extensions to $K(0,r)$ and $f_n(\lambda) \to 0$ uniformly on $K(0,r)$.*

Proof. Let $f_m(z) = \sum_{n=0}^{\infty} a_{m,n} z^n$ be the power series expansion of f_m at zero. Since $\limsup_{n\to\infty} \sqrt[n]{|a_{m,n}|}$, the inverse of the convergence radius of the series expansion of f_m is finite

$$s_m = \sup_{n\in N} \sqrt[n]{|a_{m,n}|} < \infty \qquad \text{for all } m \in N. \tag{1}$$

Let $e_1, e_2, e_3, \ldots$ be an orthonormal basis of H, and S_n the truncated shift

$$S_n e_k = \begin{cases} e_{k+1}, & \text{for } k < n; \\ 0, & \text{for } k \geq n. \end{cases}$$

The convergence $f_m(S_n)e_1 = \sum_{k=0}^{n-1} a_{m,k} e_{k+1} \to 0$ with $m \to \infty$ implies that for all $n \in N$

$$a_{m,n} \to 0 \qquad \text{with } m \to \infty. \tag{2}$$

To show the lemma it suffices to prove that $\sup_{m,n} \sqrt[n]{|a_{m,n}|} < \infty$. Assume the converse

$$\sup_{m,n} \sqrt[n]{|a_{m,n}|} = \infty \tag{3}$$

We shall show by induction that then there exist sequences $\{m_k\}, \{n_k\}$ of natural numbers such that

$$|a_{m_k,n_k}| > \exp(kn_k) \quad \text{and} \quad n_k > 2n_{k-1} \tag{4}$$

for all $k \in N$, where $n_0 = 0$.

Suppose that the numbers m_k, n_k $(k = 1, 2, \ldots, l-1)$ satisfying (4) have been already defined and there exists no pair (n_l, m_l) satisfying (4). This means that

$$|a_{m,n}| \leq \exp(ln) \quad \text{for all} \quad n, m \in N,\ n > 2n_{l-1}. \tag{5}$$

It follows from (2) that there exists $m' > m_{l-1}$ such that

$$|a_{m,n}| < 1 \quad \text{for all} \quad m > m',\ n \leq 2n_{l-1}.$$

This with (5) implies that

$$\sqrt[n]{|a_{m,n}|} \leq \exp(ln) \quad \text{for} \quad m, n \in N \quad m > m'.$$

and this with (1) gives further $\sqrt[n]{|a_{m,n}|} \leq \max\{e^l, s_1, s_2, \ldots, s_{m'}\}$, for all $n, m \in N$ contradicting (3).

This contradiction yields the existence of the sequences $\{n_k\}$, $\{m_k\}$ satisfying (4) for all $k \in N$.

Now we shall construct quasinilpotent weighted shift V such that $f_m(V) \not\to 0$(*weakly*). Let $x_k = \exp\left(-k + \frac{1-n_{k-1}}{n_k - n_{k-1}}\right) = \exp\left(\frac{1-kn_k+(k-1)n_{k-1}}{n_k-n_{k-1}}\right)$ $\alpha_{n_{k-1}+1} = \alpha_{n_{k-1}+2} = \ldots = \alpha_{n_k} = x_k$, $k = 1, 2, \ldots$. It follows from the condition $n_k > 2n_{k-1}$ that $\{x_k\}$ is a decreasing sequence. We define operator V by $Ve_k = \alpha_k e_{k+1}$, $k = 1, 2, 3, \ldots$. Since $\|V^n e_k\| = \alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1} e_{k+n}$ and the sequence $\{\alpha_n\}$ is nonincreasing it is easy to see that $\|V^n\| = \alpha_1 \alpha_2 \ldots \alpha_n$, in particular $\|V^{n_k}\| = x_1^{n_1-n_0} x_2^{n_2-n_1} \ldots x_k^{n_k-n_{k-1}} = \exp(\sum_{l=1}^{k}(1 - ln_l + (l-1)n_{l-1})) = \exp(k - kn_k)$. It now follows from the spectral radius formula that $r(V) = \sup_{\lambda \in \sigma(V)} |\lambda| = \lim_{n\to\infty} \|V^n\|^{1\sigma} = \lim_{k\to\infty} \|V^{n_k}\|^{1\sigma_k} = \lim_{k\to\infty} \exp(k\sigma_k - k) = 0$. Thus V is a quasinilpotent operator.

On the other hand $\langle f_{m_k}(V)e_1, e_{n_k+1}\rangle = \langle a_{m_k,n_k} V^{n_k} e_1, e_{n_k+1}\rangle = a_{m_k,n_k} \alpha_1 \alpha_2 \ldots \alpha_{n_k} = a_{m_k,n_k} \exp(k - kn_k)$ and by (4) $\|f_{m_k}(V)\| \geq |\langle f_{m_k}(V)e_1, e_{n_k+1}\rangle| \geq e^k$. It follows now from the Banach Uniform Principle that $f_m(V) \not\to 0$(*weakly*). This is contradiction with assumption of the Lemma and shows that $M = \sup_{m,n\in N} \sqrt[n]{|a_{m,n}|} < \infty$ i.e. $|a_{m,n}| \leq M^n$ for $m, n \in N$.

Note that if $|\xi|, |\eta| < \frac{1}{2M}$ then $|f_n(\xi)| = |\sum_{n=0}^{\infty} a_{m,n}\xi^n| \leq \sum_{n=1}^{\infty} M^n \frac{1}{2^n M^n} + s_0 = 2 + s_0$, where by (2) $s_0 = \sup_m |a_{m,0}| < \infty$ and $|f_m(\xi) - f_m(\eta)| = |\sum_{n=1}^{\infty} a_{m,n}(\xi - \eta)\sum_{i=0}^{n-1} \xi^i \eta^{n-1-i}| \leq |\xi - \eta| \sum_{n=1}^{\infty} M^n \frac{n}{(2M)^{n-1}} = 4M|\xi - \eta|$. Thus the sequence $\{f_m\}$ is a sequence of uniformly bounded and equicontinous function in the disc $K(0, \frac{1}{2M})$. If f is a limit of a subsequence of this sequence then it follows from (2) that $f = 0$. Therefore $f_n(\lambda) \to 0$ with $n \to \infty$ uniformly in $K(0, \frac{1}{2M})$.

THEOREM 3. *If $0 \notin \Omega \subset C$ then the following conditions are equivalent:*

(i) $\|p_n(A) - A^{-1}\| \to 0$ *for any bounded operator A acting in some Banach space X whose spectrum is contained in the set Ω,*

(ii) $p_n(A)b \to A^{-1}b$ *for any bounded operator A with $\sigma(A) \subset \Omega$ and any vector b,*

(iii) for each compact set $G \subset \Omega$ there exists an open neighbourhood G_0 of G such that $p_n(\lambda) \to \rho(\lambda)$ uniformly in the set G_0.

Proof. (i) $\Rightarrow$(ii) obvious.

(ii)$\Rightarrow$(iii). For any $\lambda \in \Omega$ and any quasinilpotent operator V the convergence $p_n(\lambda I + V) \to (\lambda I + V)^{-1}$ means that $f_{n,\lambda}(V) \to 0$, where $f_{n,\lambda}(\eta) = p_n(\lambda + \eta) - (\lambda + \eta)^{-1}$ is an analytic function in a neighbourhood of 0. It follows from Lemma 1 that there exists an open neighbourhood of 0 on which $f_{n,\lambda} \to 0$ uniformly, or in other words there exists an open neighbourhood G_λ of λ on which $p_n(\eta) \to \rho(\eta)$ uniformly. For any compact subset $G \subset \Omega$ there exists a finite open covering by the sets G_{λ_i}, $G \subset \bigcup_{j=1}^{n(G)} G_{\lambda_j} = G_0$, it is now obvious that $p_n(\lambda) \to \rho(\lambda)$ uniformly in G_0.

(iii)$\Rightarrow$(i). Suppose that A is a bounded operator, $\sigma(A) \subset \Omega$ and G is an open neighbourhood of $\sigma(A)$ such that there exist polynomials p_n such that $p_n(\lambda) \to \rho(\lambda)$ uniformly for $\lambda \in G$. Since $\sigma(A)$ is compact and $C \setminus G$ is closed and disjoint with $\sigma(A)$ $\delta \stackrel{\text{df}}{=} \inf\{|z - \lambda|; z \in C \setminus G, \lambda \in \sigma(A)\} > 0$ there exist a finite number of discs $K_0(x_i, \delta/3)$ with centers in $\sigma' = \bigcup_{z \in \sigma(A)} K(z, \delta/3)$ that cover σ'. It is obvious that $K_0(x_i, \delta/3) \subset G$ and that there exists a closed curve Γ formed from arcs of the circles $S(x_i, \delta/3)$ that contains the set σ' inside. Moreover $\inf\{|z - \lambda|; z \in \Gamma, \lambda \in \sigma(A)\} > \delta/3$ and therefore $\sup_{\lambda \in \Gamma} \|(A - \lambda)^{-1}\| < \infty$. Therefore

$$\|p_n(A) - A^{-1}\| = \left\| -\frac{1}{2\pi i} \int_\Gamma (p_n(\lambda) - \rho(\lambda))(A - \lambda)^{-1} d\lambda \right\|$$

$$\leq \frac{1}{2\pi} |\Gamma| \max_{\lambda \in \Gamma} |p_n(\lambda) - \rho(\lambda)| \max_{\lambda \in \Gamma} \|(A - \lambda)^{-1}\| \to 0 \quad \text{with } n \to \infty .$$

THEOREM 4. *Suppose that Ω is a compact or an open subset of the complex plane such that the set $C \setminus \Omega$ has an unbounded component contai-*

ning zero. Then there exists a sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ *such that* $p_n(A) \to A^{-1}$ *with* $n \to \infty$ *for any bounded operator* A *such that* $\sigma(A) \subset \Omega$.

Proof. (*compact* Ω *case*) If Ω is compact then the unbounded component G of $C \setminus \Omega$ containing the origin is an open and connected subset of the complex plain. This set is also arcwise connected [6]. Let Γ_0 be any curve contained in G that starts at the origin and ends at a point z such that $|z| > \sup_{x\in\Omega} |x|$. Since Ω and Γ_0 are compact and disjoint sets $\delta \stackrel{\text{df}}{=} \inf_{\substack{x\in\Omega \\ z\in\Gamma_0}} |z-x| > 0$.

The set $\Omega_1 = \{z \in C; \inf_{x\in\Omega} |z-x| < \frac{\delta}{3}\}$ is an open neighbourhood of Ω. The complement of its closure has an unbounded component containing the origin and therefore by the Runge approximation theorem [3 or 10] there exists a sequence $\{p_n\}$ of polynomials that converges uniformly to the function $\rho(\lambda) = (\lambda)^{-1}$ on the set $\overline{\Omega}_1$. Applying Theorem 3 we obtain the thesis.

(*open* Ω *case*) Let G be the unbounded component of $C \setminus \Omega$ containing zero and let $\Omega_n = K(0,n) \setminus \bigcup_{z\in G} K_0(z, \frac{1}{n})$ for $n = 1,2,3,\ldots$. Note that Ω_n is compact, $C \setminus \Omega_n \subset G$ and $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots$. Moreover for each compact set $\sigma \subset \Omega$ $\epsilon_\sigma \stackrel{\text{df}}{=} \inf\{|\lambda - z|; \lambda \in \sigma, z \in G\} > 0$ because G is closed, therefore the sets $\bigcup_{z\in G} K_0(z, \epsilon_\sigma/3), \bigcup_{z\in\sigma} K_0(z, \epsilon_\sigma/3)$ are disjoint and for $n > \max\{3/\epsilon_\sigma, \sup_{\lambda\in\sigma} |\lambda|\}$ the set $\bigcup_{z\in\sigma} K_0(z, \epsilon_\sigma/3)$ is an open neighbourhood of σ and is contained in Ω_n. It follows from the Mergelyan approximation theorem [3 or 10] that there exist polynomials p_n such that $\sup_{\lambda\in\Omega_n} |p_n(\lambda) - \rho(\lambda)| < \frac{1}{n}$. Now the thesis of the Theorem follows from Theorem 3.

Let $\Omega \subset C$ be an open domain. Consider the linear space $L^2(\Omega)$ of all functions analytic on Ω and such that

$$\|f\|^2_{L^2(\Omega)} = \int\int_{\Omega} |f(z)|^2 d\Omega < \infty.$$

It is known that $L^2(\Omega)$ with usual scalar product is a Hilbert space [3 or 10]. In this space the norm convergence implies stronger consequences then in usual L^2 spaces.

LEMMA 2. *Let* f *be analytic and* $f \in L^2(\Omega)$, *where* $\Omega \subset C$ *is an open and bounded domain. Then for any* $z_0 \in \Omega$ *and* $j = 0,1,2,\ldots$ *we have*

$$|f^{(j)}(z_0)| \leq \|f\|_{L^2(\Omega)} \frac{j!\sqrt{j+1}}{\sqrt{\pi}} \operatorname{dist}(z_0, \partial\Omega)^{-(j+1)}. \tag{6}$$

Proof. Put $z_0 \in \Omega$ arbitrary and $K = \{z \in C; |z - z_0| < \operatorname{dist}(z_0, \partial\Omega)\}$. For any $z \in K$ we have $z = z_0 + re^{i\phi}$, with $0 \leq r \leq \operatorname{dist}(z_0, \partial\Omega)$, $0 \leq \phi < 2\pi$ and $f(z) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(z_0)}{\nu!}(z-z_0)^\nu$; hence $|f(z)|^2 = \sum_{\nu,\mu=0}^{\infty} \frac{f^{(\nu)}(z_0)\overline{f^{(\mu)}(z_0)}}{\nu!\mu!}(z-$

$z_0)^{\nu}\overline{(z-z_0)^{\mu}}$, which imply

$$\|f\|^2_{L^2(\Omega)} = \int\int_{\Omega} |f(z)|^2 d\Omega \geq \int\int_{K} |f(z)|^2 dK$$

$$= \sum_{\nu,\mu=0}^{\infty} \frac{f^{(\nu)}(z_0)\overline{f^{(\mu)}(z_0)}}{\nu!\mu!} \int_0^{\operatorname{dist}(z_0,\partial\Omega)} r^{\nu+\mu+1} dr \int_0^{2\pi} e^{i(\nu-\mu)\phi} d\phi$$

$$= \pi \sum_{\nu=0}^{\infty} \frac{|f^{(\nu)}(z_0)|^2}{(\nu!)^2(\nu+1)} \operatorname{dist}(z_0,\partial\Omega)^{2(\nu+1)}$$

$$\geq \pi \frac{|f^{(j)}(z_0)|^2}{(j!)^2(j+1)} \operatorname{dist}(z_0,\partial\Omega)^{2(j+1)}$$

and (6) follows.

THEOREM 5. *Let $\Omega \subset C$ be open and bounded, $f, f_n \in L^2(\Omega)$ $n = 0,1,2,\ldots$ analytic functions. Then for $j = 0,1,2,\ldots$ and $z \in \Omega$ arbitrary,*

$$|f^{(j)}(z) - f_n^{(j)}(z)| \leq \frac{j!\sqrt{j+1}}{\sqrt{\pi}} \operatorname{dist}(z,\partial\Omega)^{-(j+1)} \|f - f_n\|_{L^2(\Omega)}$$

holds.

Proof. The thesis follows immediately from Lemma 2.

COROLLARY. *If $0 \notin \overline{\Omega}$ and p_n, $n = 0,1,2,\ldots$ are polynomials such that $\|p_n - \rho\|_{L^2(\Omega)} \to 0$ with $n \to \infty$, where $\rho(z) = \frac{1}{z}$ then*

$$p_n(A) \to A^{-1}, \quad \text{with} \quad n \to \infty$$

for any matrix A such that its spectrum $\sigma(A)$ is contained in Ω.

Proof. The thesis follows immediately from Theorems 1 and 5.

Observe also that general estimate for $\|p_n(A) - A^{-1}\|$ can be obtained from the Dunford integral formula

$$p_n(A) - A^{-1} = \frac{1}{2\pi i} \int_{\partial\Omega} \big(p_n(\lambda) - \rho(\lambda)\big)\big(\lambda I - A\big)^{-1} d\lambda,$$

where the set Ω with sufficiently regular boundary $\partial\Omega$, satisfies assumptions of the Corollary. We get applying Schwartz inequality:

$$\|p_n(A) - A^{-1}\| \leq \frac{|\partial\Omega|}{2\pi} \sup_{z\in\partial\Omega} \|(zI - A)^{-1}\| \int_{\partial\Omega} |p_n(z) - \rho(z)||dz|$$

$$\leq \frac{|\partial\Omega|^2}{2\pi} \sup_{z\in\partial\Omega} \|(zI - A)^{-1}\| \Big(\int_{\partial\Omega} |p_n(z) - \rho(z)|^2 |dz| \Big)^{1/2}$$

$$\leq C \Big(\int_{\partial\Omega} |p_n(z) - \rho(z)|^2 |dz| \Big)^{1/2},$$

where $\|.\|$ stands for any matrix norm, and C is a constant. Moreover

$$\Big(\int\limits_{\partial\Omega} |p_n(z) - \rho(z)|^2 |dz| \Big)^{1/2} = \|p_n - \rho\|_{L^2(\partial\Omega)}$$

holds. This implies the following.

THEOREM 6. *Let Ω be an open domain with sufficiently regular boundary $\partial\Omega$ such that $0 \notin \overline{\Omega}$, $\rho(z) = \frac{1}{z}$ and p_n the sequence of polynomials. If $\|p_n - \rho\|_{L^2(\partial\Omega)} \to 0$ when $n \to \infty$ then $p_n(A) \to A^{-1}$ with $n \to \infty$ for any matrix A such that $\sigma(A) \subset \Omega$.*

Many general ideas of studied methods may be illustrated on the example case $\Omega = K(1, r)$ where $r < 1$. Writting the power series expansion of the function $\rho(\lambda)$ in the point $\lambda_0 = 1$

$$\rho(\lambda) = (1 - (1 - \lambda))^{-1} = \sum_{j=0}^{\infty} (1 - \lambda)^j, \tag{7}$$

we see that the polynomials

$$p_n(\lambda) = \sum_{j=0}^{n} (1 - \lambda)^j$$

converge uniformly to the function ρ on $K(1, r)$ for all $r < 1$. We may also look at the expansion (7) as an expansion into series of orthogonal polynomials — polynomials $q_n(\lambda) = (1 - \lambda)^n$, $n = 0, 1, 2, \ldots$ are orthogonal with respect to the scalar product in $L^2(S(1, r))$ because

$$\int\limits_{S(1,r)} q_n(\lambda)\overline{q_m(\lambda)} |d\lambda| = \int\limits_{S(0,r)} z^n \overline{z}^m |dz|$$

$$= \int\limits_0^{2\pi} r^{n+m+1} e^{itn} e^{-itm} dt = 2\pi r^{2n+1} \delta_{n,m}.$$

The same polynomials are also orthogonal with respect to the scalar product in the space $L^2(K(1, r))$ because

$$\int\int\limits_{K(1,r)} q_n(\lambda)\overline{q_m(\lambda)} dS = \int\int\limits_{K(0,r)} z^n \overline{z}^m dS$$

$$= \int\limits_0^r \rho^{m+n+1} \int\limits_0^{2\pi} e^{it(n-m)} dt d\rho = 2\pi \delta_{n,m} \frac{r^{2n+2}}{2(n+1)}$$

Let us investigate now some computational aspects of general ideas presented above.

The general Theorems 4, 5, 6 can be applied in practice to approximate the solution of big system of linear algebraic equations $Ax = b$, wi-

thout any special assumptions about the matrix A, other then its invertibility.

The only thing we need is the bounded set $\Omega \subset C$ containing the spectrum $\sigma(A)$ of A and such that $0 \notin \overline{\Omega}$. It is clear that this set should be as small as possible. The numerical process based on the above theory will, in general, sharply depend on properties of Ω, such as extremal distance of its points to zero (condition number of A!), the shape of Ω, the qualities of the boundary $\partial\Omega$ (for example its smootness), and also certain general topological properties of Ω as for example its connectedness or simple connectedness. In other words we need some reasonable estimate of the spectrum $\sigma(A)$ of A. Observe that methods of this kind depend on Ω and not on A.

Recall here that it is always possible to multiply the equation $Ax = b$ on the left by the matrix A^* so as to get an equivalent system with symmetric and positive definite matrix. Unfortunately this transformation increases the condition coefficient of the system of equations to the square and may be disastrous in certain situations.

It should be stressed here that numerical process we are interested in, splits in natural way into two parts:

1^0 creation of the polynomial p_n approximating the function $\rho(z) = \frac{1}{z}$ on Ω;

2^0 application of p_n to the system $Ax = b$.

In fact the first part of this process can be accomplished even a long time before the second part. At this stage, the most important thing is to get polynomials p_k, $k = 0, 1, 2, \ldots, n$ which well approximate the function $\frac{1}{z}$ on Ω, the question of computing time being not so important. It could be even desirable to produce the whole library of polynomials for typical domains Ω.

The second part of our process is the strict solver of the problem and on this level time optimization is very important.

There are several possibilities of construction of polynomials p_n. Let us quote some of them:

(i) interpolation of the function $\rho(z) = \frac{1}{z}$ on the complex domain Ω,

(ii) Fourier expansion of $\rho(z)$ in $L^2(\Omega)$, based on orthogonal polynomials,

(iii) Fourier expansion of $\rho(z)$ in $L^2(\partial\Omega)$, based on orthogonal polynomials,

(iv) the best approximation of $\rho(z)$ by complex polynomials of fixed degree n in $C(\Omega)$ (in supremum norm).

(i) Interpolation. For the compact $K \subset \Omega \subset C$ we can take as p_n the Lagrange or Hermite interpolating polynomials. It is known that if $C \setminus K$ the complement of K is simply connected then p_n will converge to $\frac{1}{z}$ uniformly on K with all derivatives, provided that the interpolation nodes are "equidistant" on K. In this case the convergence is optimal. (see [3 or

10]). For large class of such sets, so called Fejer nodes, conformal images of n-th roots of unity, can be applied.

For arbitrary compact K, which is union of finite number of components not degenerating to single points, on which ρ is regular, the choice of so called Fekete's nodes, i.e. points $z_0, z_1, \ldots, z_k$ maximizing the function $F(z_0, z_1, \ldots, z_k) = \prod_{j \neq k} |z_j - z_k|$ assure the uniform convergence on K of p_n to ρ with all derivatives, (for details see [3,Chapter2 or 10])

Let us make here a few remarks concerning the interpolation in the context of general algorithmus solving systems of linear algebraic equations via polynomial approximation of $\rho(z)$. Let $x_n = p_n(A)b$ be the approximate solution of the system $Ax = b$. We are looking for a right choice of the polynomial p_n.

For the residual vector $r_n = b - Ax_n$ we have

$$r_n = b - Ax_n = (I - Ap_n(A))b = V_{n+1}(A)b$$

where $V_{n+1}(z)$ is polynomial of degree $n+1$ such that

$$V_{n+1}(0) = 1. \tag{8}$$

Observe that any polynomial V_{n+1} satisfying (8) can be written in the following form

$$V_{n+1}(z) = \left(1 - \frac{z}{q_0}\right)\left(1 - \frac{z}{q_1}\right)\ldots\left(1 - \frac{z}{q_n}\right),$$

where $q_0, q_1, \ldots, q_n$ are (distinct from zero!) zeros of V_{n+1}. It is easy to prove the following proposition.

PROPOSITION 1. *The polynomial V_{n+1} of degree $n+1$ satisfies condition (8) if and only if there exists polynomial p_n of degree n such that*

$$V_{n+1}(z) = 1 - zp_n(z), \quad z \in C.$$

Put for any $k = 0, 1, 2, \ldots$

$$V_k(z) = \left(1 - \frac{z}{q_{k-1}}\right)\left(1 - \frac{z}{q_{k-2}}\right)\ldots\left(1 - \frac{z}{q_0}\right) \tag{9}$$
$$V_0(z) = 1$$

hence $V_{k+1}(z) = \left(1 - \frac{z}{q_k}\right) V_k(z)$, and by Proposition 1 we get $1 - zp_n(z) = \left(1 - \frac{z}{q_n}\right)(1 - zp_{n-1}(z))$ which gives immediately

$$p_k(z) = p_{k-1}(z) + \frac{1}{q_k} V_k(z).$$

Since $x_k = p_k(A)b$ and $r_k = V_{k+1}(A)b$, the last formula suggests the following algorithm

$$(10) \qquad x_k = x_{k-1} + \frac{r_{k-1}}{q_k}, \quad k = 0, 1, \ldots, n, \quad x_{-1} = 0, \quad r_{-1} = b.$$

We find in (10) the well known old Richardson iterative process. This points out that the zeros $q_0, q_1, \ldots, q_k$ of V_{k+1} play an important role as the coefficients of the Richardson iteration.

PROPOSITION 2. *The following conditions are equivalent:*

(i) *q is a zero of V_{n+1} of order $s > 0$;*

(ii) *q is the node of interpolation of order $s > 0$ of the function $\frac{1}{z}$ by polynomial $p_n(z)$.*

Proof. Since $V_{n+1}(z) = 1 - zp_n(z)$, we have $V_{n+1}^{(j)}(z) = -jp_n^{(j-1)}(z) - zp_n^{(j)}(z)$ for $j = 0, 1, \ldots, s$. This formula proves that $V_{n+1}^{(j)}(q) = 0$ if and only if $p_n^{(j)}(q) = \frac{(-1)^j j!}{q^{j+1}} = \left(\frac{1}{z}\right)^{(j)}\Big|_{z=q}$ $j = 0, 1, \ldots, s$, which ends the proof.

It follows then that p_n is always some interpolating polynomial for $\frac{1}{z}$ and that good choise of p_n means the good choice of interpolation nodes $q_0, q_1, \ldots, q_n$. Moreover those nodes are the coefficients in the Richardson formula (10) defining one of the possible algorithms for solution of our problem.

Observe that (10) do not define uniquely this algorithm if the order of coefficients is not fixed. It is known [7] that numerical properties of Richardson iteration strongly depend on this order.

(ii) Fourier expansion on Ω. Let $\Omega \subset C$ be bounded open domain containing $\sigma(A)$ and such that $0 \notin \overline{\Omega}$. If $\rho \in \overline{\bigcup_{n=0}^{\infty} \operatorname{span}\{1, z^1, \ldots, z^n\}} \subset L^2(\Omega)$ then $\rho(z) = \frac{1}{z}$ can be approximated by polynomial p_n of the best approximation of degree n in the $L^2(\Omega)$ norm. The sequence $\{p_k\}$ of such polynomials will converge uniformly with all derivatives to ρ on any compact $K \subset \Omega$, hence also on $\sigma(A)$. Observe that there is a great variety of domains Ω, also not connected or not simply connected, which satisfy above condition for ρ. On such domains ρ may be approximated by Fourier expansion based on the system $Q_0, Q_1, Q_2, \ldots$ of orthogonal polynomials on Ω: $\langle Q_i, Q_j \rangle = \delta_{i,j}$. For optimal polynomial p_n we have $p_n(z) = \sum_{j=0}^{n} c_j Q_j(z)$ where $c_j = \langle \rho, Q_j \rangle$. Such polynomials often give satisfactory approximation. The system $Q_0, Q_1, Q_2, \ldots$ can readily be generated by orthonormalization procedure of Gram–Schmidt or Lanczos type.

For Gram–Schmidt proces the polynomials $\phi_k(z) = \left(\frac{z - z_0}{d}\right)^k$ $k = 0, 1, 2, \ldots$ are convenient basis for orthogonalization. Complex z_0 and real d have to be choosen so that $\left|\frac{z - z_0}{d}\right| \leq 1$ for $z \in \Omega$. The Gram–Schmidt

procedure can be modified so that all integrations will occure between functions ϕ_k and $\overline{\phi_l}$ only. Hence when computing Fourier coefficient c_k the Green formula for complex analytic functions f, g

$$\iint_{\Omega} f(z)\overline{g'(z)}d\Omega = \frac{1}{2i}\int_{\partial\Omega} f(z)\overline{g(z)}dz$$

(the boundary $\partial\Omega$ has to be positively oriented!) can be easily applied. This hint replaces the double integral on the left hand side by this on the right. As a result, the triangular matrix of Gram–Schmidt formula is produced, as well as the sequence of Fourier coefficients. Any polynomial Q_k as well as p_n can be easily computed.

Computations have been made following this procedure for various domains Ω. Figures below show examples of domains Ω; n is the degree of optimal $L^2(\Omega)$ polynomial $p_n(z)$ and numbers into domains represent residuals $r_n(z) = 1 - zp_n(z)$ for $z \in \Omega$.Computations have been proceeed with 14 decimals. For integration composed Gauss–Legendre quadrature of order 16 has been used.

(iii) Fourier expansions on $\partial\Omega$.

Theorem 6 enables us to apply Fourier expansions of ρ on $\partial\Omega$. First a system of orthogonal polynomials on $\partial\Omega$ should be generated, (as above, Gram–Schmidt proces may be applied) and then Fourier coefficients computed using the scalar product of $L_2(\partial\Omega)$: $\langle f, \rho\rangle_{L_2(\partial\Omega)} = \int_{\partial\Omega} f(z)\overline{\rho(z)}|dz|$; also a priori error bound can be found.

(iv) Best approximation on $C(\Omega)$. Recently a new variant of the Remez algorithm in a complex domain has been published. Authors claim that this algorithm gives effectively a second order convergence. One may hope that this algorithm will enable the effective use of the best approximation by polynomials in supremum norm for our problem. See [8].

Cyclic version. Let us say a few words at the end about the so called cyclic version of polynomial algorithm. Observe that if p_n is chosen so that $\sigma(V_{n+1}(A))$ is inside the unit circle $|z| < 1$, then by the spectral radius formula $\lim_{k\to\infty} \sqrt[k]{\|(V_{n+1}(A))^k\|} < 1$ and therefore $(V_{n+1}(A))^k \to 0$ when $k \to \infty$. This enables us to use the following cyclic algorithm:

$$\begin{aligned} \text{Put} \quad & y_0 = p_n(A)b, \quad s_0 = b - Ay_0 = r_n, \quad \Delta_0 = p_n(A)s_0 \\ & y_{k+1} = y_k + \Delta_k, \quad s_{k+1} = b - Ay_{k+1}, \quad \Delta_{k+1} = p_n(A)s_{k+1}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 0, 1, 2, \ldots \end{aligned}$$

We have then

$$s_{k+1} = b - Ay_k - Ap_n(A)s_k = V_{n+1}(A)s_k$$

and

$$s_k = (V_{n+1}(A))^k r_n \to 0 \quad \text{as} \quad k \to \infty \quad (n \text{ being fixed!}).$$

This version of polynomial algorithm may be useful in case when we can not get good enough approximation of $\rho(z) = \frac{1}{z}$ on Ω to apply it directly. In such a case it is sufficient to find an approximation satisfying $|V_{n+1}(z)| = |1 - zp_n(z)| < 1$, for any $z \in \Omega$. Then the cyclic version may be applied to minimize the residual vector s_k.

Version with starting vector. Infinite–dimensional problems may be approximated by various finite–dimensional ones, problems of smaller dimensions are cheaper to solve but less accurate, their solutions may serve as starting vectors in iteration procedures for higher dimensional systems. Suppose that togeather with the linear system $Ax = f$ we have some approximation x_0 of the solution x. We may seek for better polynomial approximations of x in the form

$$x_n = p_n(A)f + q_n(A)x_0, \tag{11}$$

where p_n, q_n are polynomials. Note that if $x_0 = 0$ then we return to the problem discussed before. It is natural to demand that if x_0 is an exact solution of $Ax = f$ then also $x_n = x$. This restricts our attention to the polynomials p_n, q_n satisfying the identity

$$x = p_n(A)Ax + q_n(A)x \tag{12}$$

for all $x \in X$ and operators A with spectrum contained in the domain Ω. This is equivalent to the condition

$$q_n(\lambda) = 1 - \lambda p_n(\lambda), \tag{13}$$

this is the same condition as that in Proposition 1 for polynomials V_n.

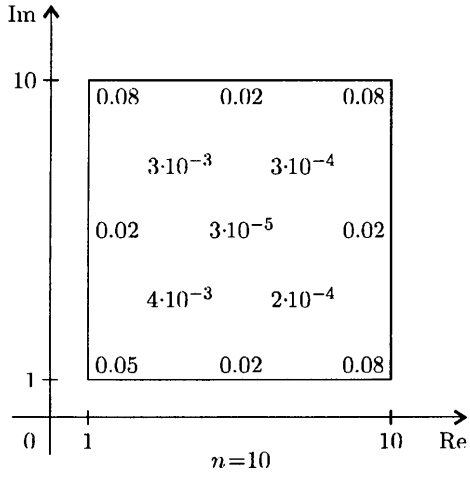
Substracting (12) from (11) we get the expression for the error $x - x_n = q_n(A)(x - x_0)$. Trying to minimize this error we may look for polynomials q_n for which $\sup_{\lambda\in\Omega} |q_n(\lambda)|$ is as small as possible or at least $\sup_{\lambda\in\Omega} |q_n(\lambda)| \to 0$ with $n \to \infty$. It is easy to see that this condition is satisfied by polynomials V_n. Therefore we may construct as before polynomials p_n and then define q_n by (13). We may also try to define polynomials q_n directly, similarily to the ways of construction of polynomials p_n we have the following possibilities:

Interpolation — Interpolate the function identical to 0 at Fejer or Fekete points $\{z_0^{(n)}, z_1^{(n)}, \ldots, z_n^{(n)}\}$ by polynomial q_n of degree at most $n+1$ with additional condition $q_n(0) = 1$. The obtained result is the same as if having defined p_n by interpolation of $\rho(\lambda) = \lambda^{-1}$ at Fejer or Fekete points we put $q_n(\lambda) = 1 - \lambda p_n(\lambda)$.

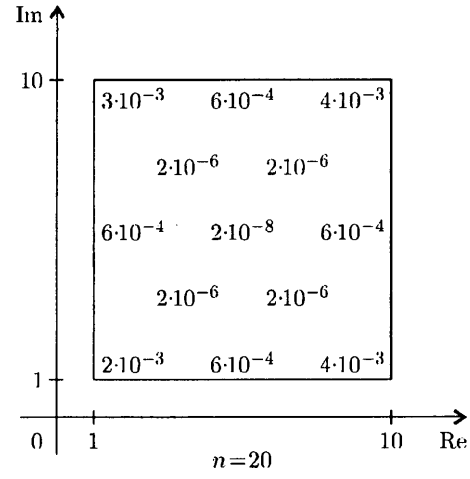
Minimizing $L^2(\Omega)$ or $L^2(\partial\Omega)$ norm — we may define q_n as that polynomial of degree at most $n+1$ satisfying the condition $q_n(0) = 1$ for which the respective norm is the smallest.

In all these cases similar estimations as before may be obtained with nearly the same methods.

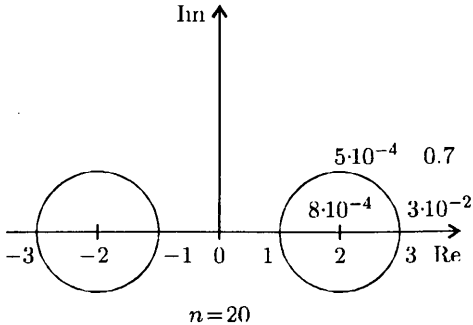
Examples of approximation of $\rho(z) = 1/z$ by optimal $L^2(\Omega)$ polynomial of degree n on various domains Ω



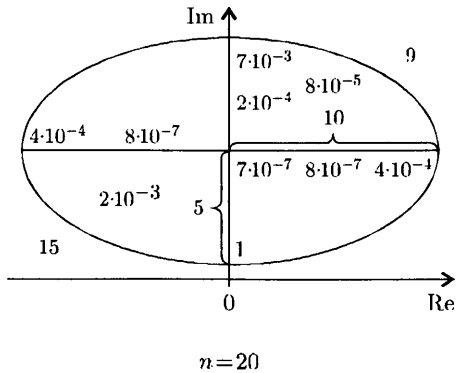
$n=10$



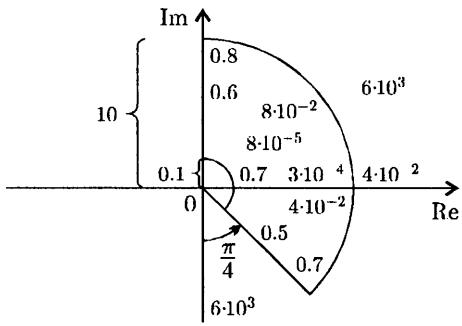
$n=20$



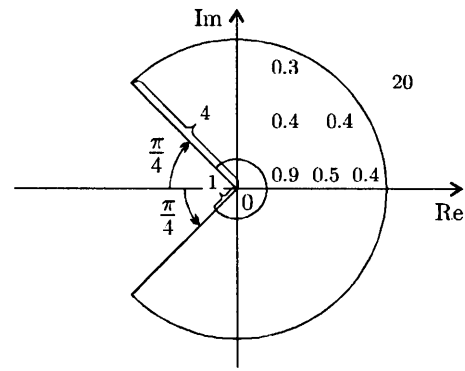
$n=20$



$n=20$



$n=20$



$n=20$

References

[1] Anderssen R.S., Golub G.H., *Richardson's non-stationary matrix iterative procedure*, Report STAN-CS-72-304, Computer Science Department, Stanford University, Stanford, CA, 1972,

[2] Fischer, B., Reichel, Lothar, *A stable Richardson iteration methods for complex linear systems*, Numerische Mathematik

[3] Gaier D., *Lectures on Complex Approximation*, Birkhauser, 1989,

[4] Gragg, W. B.,Reichel, L, *On the applications of orthogonal polynomials to the iterative solution of linear systems of equations with indefinite or non-Hermitian matrices*, Linear Algebra Appl. 88/89, pp. 349-371 (1987),

[5] Kato T., *Perturbation Theory for Linear Operators*, Springer 1966.

[6] Kuratowski K, *Wstęp do Teorii Mnogości i Topologii*, PWN 1972,

[7] Лебедев Б. И.,Финогенов П. А., *О порядке выбора итерационных параметров в Чебышевском циклическом итерационном методе*, Журнал вычислительной математики и математической физики, Vol 11, (1971) No. 2 pp. 425-438,

[8] Ping Tak Peter Tang, *A fast algorithm for linear complex Chebyshev approximation*, Mathematics of Computation 51, pp. 721-739,

[9] Reichel L., *Polynomials by conformal mapping for the Richardson iteration method for complex linear systems*, SIAM Journal Numer. Anal. 25 (1988), pp. 1359-1368,

[10] Walsh, L. J., *Interpolation and Approximation by Rational Functions in the Complex Domain*, 5th ed., American Mathematical Society, Providence, RI, 1969,

[11] Niethammer W.,Varga R. S., *The analysis of k-step methods for linear systems from summability theory*, Numer. Math. 41. (1983) pp. 177-206.