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On the distribution of the longest success-run in Bernoulli trials

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1. Introduction. The success-run in a sequence of Bernoulli trials was considered in a large number of papers. W. Feller [4] concentrates his attention on the number of runs what find the principal application in the theory of test of randomness and tests of homogeneity. Many papers deal with random variable Z_n introduced by Erdös and Rényi [2] (see also Erdös and Révész [3]), defined as the length of longest head-run during n coin tossings. They give the asymptotic estimations of that random variable if n tends to infinity. The asymptotic estimation of the distribution of Z_n was given by Antónia Földes [5]. Note that in the theory of extremes in random sequences it is proved that in a sequence of random variables geometrically distributed the maximum linearly standarized do not have a limiting distribution (see [7], p. 26). In [1] the multivariate extension of the problem of the largest cube filled up by successes which may be found in a random lattice in a cube of range n. The problem has very much of practical implications.

Our purpose in this paper is to show the recurrent formulas useful in the calculation of the distribution of the random variable Z_n and in the method of calculation of its expected value and also in the test of the accuracy of the limiting estimations.

2. Distribution of Z_n. Let us suppose that the probability p of the success is fixed and $0 < p < 1$. Let us denote $q = 1 - p$,

$$p(n,k) = P(Z_n = k),$$
$$P(n,k) = P(Z_n \leq k),$$

$$R(n,k) = P(Z_n > k),$$
$$\mu(n) = EZ_n,$$

where $n = 0, 1, \ldots, k = 0, 1, \ldots$

In the sequel we may find a sum over an empty set of indices and then it is assumed equal to zero. Note that $p(0,0) = 1$,

$$p(n,k) = 0 \quad \text{for} \quad k < 0 \quad \text{or} \quad k > n,$$
$$P(n,k) = 0 \quad \text{for} \quad k < 0 \quad \text{and} \quad P(n,k) = 1 \quad \text{for} \quad k \geq n,$$
$$R(n,k) = 1 \quad \text{for} \quad k < 0 \quad \text{and} \quad R(n,k) = 0 \quad \text{for} \quad k \geq n.$$

It is evident that

$$P(n,k) = \sum_{i=0}^{k} p(n,i),$$
$$R(n,k) = 1 - P(n,k),$$
$$p(n,k) = P(n,k) - P(n,k-1) = R(n,k-1) - R(n,k).$$

By $[x]$ we denote the integer part of x. The following theorem gives recurrence formulas for the distribution of the longest success-run in Bernoulli trials.

THEOREM 1. *For fixed $k = 1, 2, \ldots$ we have*

(1) $$p(k,k) = p^k,$$

(2) $$p(n,k) = \sum_{j=0}^{k-1} p^j q p(n-j-1,k) + p^k q P(n-k-1,k), \qquad n = k+1,\ k+2, \ldots$$

(3) $$p(n,k) = p^k q P(n-k-1,k) + \sum_{j=0}^{n-k-2} P(j,k-1) p^k q^2 P(n-j-k-2,k) + p^k q P(n-k-1,k-1),$$

(4) $$R(k+1,k) = p^{k+1},$$

(5) $$R(n,k) = \sum_{j=0}^{k} p^j q R(n-j-1,k) + p^{k+1}, \quad n = k+1,\ k+2, \ldots$$

(6) $$R(n+1,k) = R(n,k) + p^{k+1} q \big(1 - R(n-k-1,k)\big), \qquad n = k+1,\ k+2, \ldots$$

Let $N_k, k = 1, 2, \ldots$ denote the number of the trial in which the success-run of length k firstly is realized. Let us denote $p_{nk} = P(N_k = n), n =$

$1, 2, \ldots$, and assume that $p_{00} = 1$. Then

$$P(n, k-1) = P(Z_n < k) = P(N_k > n) = 1 - \sum_{i=1}^{n} p_{ik}, \quad n = 1, 2, \ldots, \; k = 1, 2, \ldots \tag{7}$$

THEOREM 2. *For fixed $k = 1, 2, \ldots$ we have*

$$p_{kk} = p^k, \quad p_{nk} = 0 \quad \text{for } n < k, \tag{8}$$

$$p_{nk} = \sum_{i=0}^{k-1} p^i q p_{n-i-1\,k} \quad \text{for } n = k+1, \; k+2, \ldots \tag{9}$$

Proof of Theorem 1. Clearly (1) and (4) hold. We prove the formulas (2), (3) and (5) by the total probability formula. For proof of (2) we calculate the probability of the event $\{Z_n = k\}$ using as condition the length of the success-run which initiated the considered trials. For the proof of (3) we use as condition the begin of the first success-run of length k in the considered trials. For the proof of (5) we calculate the probability of the event $\{Z_n > k\}$ under the same condition as in the proof of (2).

Now we pass to the proof of (6). Note that the probability of the event $\{Z_n > k\}$ may be interpreted as the reliability of a coherent system, which may be named a structure "$k+1$ in a sequence among n". This system is in the working state if at least $k+1$ elements in sequence from $1, 2, \ldots, n$ are in two working state. The reliability of the structure can be calculated using minimal paths which are

$$\{1, 2, \ldots, k+1\}, \{2, 3, \ldots, k+2\}, \ldots, \{n-k, n-k+1, \ldots, n\}.$$

Here it is convenient to introduce the notation for the considered trials. Let $X_1, X_2, \ldots$ denote the sequence independent 0-1 random variables and let $P(X_i = 1) = p, P(X_i = 0) = q, i = 1, 2, \ldots$ Let us define the random events

$$A_i = \{X_j = 1 \quad \text{for} \quad j = i, i+1, \ldots, i+k\}, \; i = 1, 2, \ldots$$

Then

$$R(n, k) = P\Big(\bigcup_{i=1}^{n-k} A_i\Big).$$

The formula (6) may be easily shown inductively. We have

$$P(A_1) = p^{k+1},$$

$$R(n+1, k) = P\Big(\bigcup_{i=1}^{n-k} A_i \cup A_{n-k+1}\Big)$$

$$= R(n,k) + p^{k+1} - P\Big(\bigcup_{i=k}^{n-k}(A_i \cap A_{n-k+1})\Big)$$

$$= R(n,k) + p^{k+1} - p^{k+1}P\Big(\bigcup_{i=1}^{n-k} B_i\Big),$$

where

$$B_i = \{X_j = 1 \quad \text{for } j = i, i+1, \ldots, \min(i+k, n-k)\},\ i = 1,2,\ldots,n-k.$$

Because

$$\bigcup_{i=1}^{n-k} B_i = \Big(\bigcup_{i=1}^{n-2k-1} A_i\Big) \cup B_{n-k}$$

and since the random events $\bigcup_{i=1}^{n-2k-1} A_i$ and $B_{n-k} = \{X_{n-k} = 1\}$ are independent, thus

$$P\Big(\bigcup_{i=1}^{n-2k-1} A_i\Big) = R(n-k-1,k) + p - R(n-k-1,k)p,$$

gives the result of (6).

Remark.

(a) From (6) in particular we have $R(2n, n-1) = p^n(1+nq)$ which for $p = 1/2$ was given in [3], Lemma 1.

(b) The equivalence of (2) and (5) may be checked immediately

$$\begin{aligned} p(n,k) &= R(n,k-1) - R(n,k) \\ &= \sum_{j=0}^{k-1} p^j q R(n-j-1,k-1) - \sum_{j=0}^{k} p^j q R(n-j-1,k) + p^k - p^{k+1} \\ &= \sum_{j=0}^{k-1} p^j q(R(n-j-1,k-1) - R(n-j-1,k)) \\ &\quad - p^k q R(n-k-1,k) + p^k q \\ &= \sum_{j=0}^{k-1} p^j q p(n-j-1,k) + p^k q(1 - R(n-k-1,k)). \end{aligned}$$

(c) The equivalence of (5) and (6) may be checked immediately

$$\begin{aligned} R(n+1,k) &= \sum_{j=0}^{k} p^j q R(n-j,k) + p^{k+1} \\ &= qR(n,k) + p\sum_{j=0}^{k-1} p^j q R(n-j-1,k) + p^{k+1} \end{aligned}$$

$$= qR(n,k) + p(R(n,k) - R(n-k-1,k)p^k q - p^{k+1}) + p^{k+1}$$
$$= R(n,k) + p^{k+1}q(1 - R(n-k-1,k)).$$

(d) The immediate proof of the equivalence of (3) and others requires more complicated calculations. We prove it as a remark using the generating function of the distribution function.

(e) Theorem 1 is useful in calculations of the moments. We have

$$\mu(n) = \sum_{k=0}^{\infty} P(Z_n > k) = \sum_{k=0}^{n-1} R(n,k).$$

Some reduction of calculations gives the following recurrence formula being a consequence of (6)

$$\mu(1) = p,$$
$$\mu(n+1) = \mu(n) + p - q \sum_{i=1}^{[\frac{n-1}{2}]} R(n-i, i-1)p^i.$$

Proof of Theorem 2. The formula (9) follows from the total probability formula where the condition is the length of the success-run which indicated the considered trials. The formula (8) clearly hold.

3. Generating functions. Let us introduce the generating functions

$$R^*(s,k) = \sum_{n=k+1}^{\infty} s^n R(n,k),$$
$$p_k^*(s) = \sum_{n=k}^{\infty} s^n p_{nk}, k = 0, 1, \ldots,$$
$$R^{**}(s,u) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} s^n u^k R(n,k), \tag{10}$$
$$\mu^*(s) = \sum_{n=1}^{\infty} s^n \mu(n).$$

THEOREM 3. *For* $k = 0, 1, \ldots, |u| < 1, |s| < 1$ *we have*

$$R^*(s,k) = \frac{p^{k+1}s^{k+1}(1-ps)}{(1-s)(1-s+p^{k+1}qs^{k+2})}, \quad k = 0, 1, \ldots, \tag{11}$$

$$P^*(s,k) = \frac{1}{1-s} - R^*(s,k) = \frac{1-p^{k+1}s^{k+1}}{1-s+p^{k+1}qs^{k+1}}, \tag{12}$$

$$R^{**}(s,u) = \frac{ps(1-ps)}{(1-s)^2} \sum_{j=0}^{\infty} \left(\frac{-pqs^2}{1-s}\right)^j \frac{1}{1-(ps)^{j+1}u}, \tag{13}$$

$$\mu^*(s) = R^{**}(s,1), \tag{14}$$

$$p_k^*(s) = \frac{p^k s^k (1-ps)}{1-s+p^k q s^{k+1}}. \tag{15}$$

Proof. Multiplying (4) by s^{k+1} and (6) by s^{n+1} and adding side by side we obtain

$$R^*(s,k) = p^{k+1}s^{k+1} + sR^*(s,k) + p^{k+1}qs^{k+2}\frac{1}{1-s} - p^{k+1}qs^{k+2}R^*(s,k)$$

implying that

$$R^*(s,k)(1-s+p^{k+1}qs^{k+2}) = \frac{p^{k+1}s^{k+1}(1-ps)}{1-s}, \quad k = 0,1,\ldots, \tag{16}$$

which implies (11) and (12). Because from (10) we have

$$\sum_{k=0}^{\infty} u^k R^*(s,k) = \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} u^k s^n R(n,k) = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} s^n u^k R(n,k) = R^{**}(s,u),$$

hence multiplying (16) by u^k and adding side by side we obtain

$$R^{**}(s,u)(1-s) + pqs^2 R^{**}(s,psu) = \frac{ps(1-ps)}{(1-s)(1-psu)}.$$

The solution of this equation is (13). We have

$$\mu^*(s) = \sum_{n=1}^{\infty} s^n \sum_{k=0}^{n-1} R(n,k) = R^{**}(s,1),$$

which proves (14).

Multiplying (8) by s^k and (9) by s^n and adding side by side we obtain the equation

$$p_k^*(s) = p^k s^k + \sum_{i=0}^{k-1} s^{i+1} p^i q p_k^*(s),$$

the solution of which is (15). From (7) we get

$$P^*(s,k-1) = \sum_{n=0}^{\infty} s^n P(n,k) = (1-p_k^*(s))/(1-s) = \frac{1-p^k s^k}{1-s+p^k q s^{k+1}}$$

which is equivalent to (12).

Remark.

(a) Theorem 3 gives the practical possibility of calculation the moments.

We have

$$\mu^*(s) = R^{**}(s,1) = \frac{ps(1-ps)}{(1-s)^2}\sum_{j=0}^{\infty}\left(\frac{-pqs^2}{1-s}\right)^j \frac{1}{1-(ps)^{j+1}}$$

$$= \frac{ps(1-ps)}{1-s}\sum_{k=0}^{\infty}\frac{(ps)^k}{1-s+(ps)^{k+1}qs}.$$

As

$$\sum_{k=0}^{\infty}\frac{(ps)^k}{1-s+(ps)^{k+1}qs} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(ps)^k s^n \sum_{i=0}^{\infty}\binom{n}{i}(-(ps)^{k+1}q)^i$$

$$= \sum_{m=0}^{\infty} s^m \sum_{\substack{k=0\\ k+n+(k+1)i=m}}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}p^k(-p^{k+1}q)^i$$

$$= \sum_{m=0}^{\infty} s^m \sum_{k=0}^{\infty}\sum_{i=0}^{[\frac{m-k}{k+2}]}\binom{m-k-(k+1)i}{i}p^k(-p^{k+1}q)^i = \sum_{m=0}^{\infty} s^m b_m$$

hence

$$\mu^*(s) = \frac{ps(1-ps)}{1-s}\sum_{m=0}^{\infty} s^m b_m,$$

and

$$\frac{1}{p\mu(n+1)} = a_n - pa_{n-1},$$

where

$$a_n = \sum_{m=0}^{n} b_m = \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{[\frac{m-k}{k+2}]}\binom{m-k-(k+1)i}{i}p^k(-p^{k+1}q)^i.$$

But

$$\sum_{m=0}^{n-k-(k+2)i}\binom{m+i}{i} = \binom{n-k-(k+1)i+1}{i+1}$$

(see Feller [4], p.64, formula 12.8), thus

$$a_n = \sum_{k=0}^{n}\sum_{i=0}^{[\frac{n-k}{k+2}]}\binom{n-k-(k+1)i+1}{i+1}p^k(-p^{k+1}q)^i, n = 0,1,\ldots$$

(b) Using the generating function we prove the equivalence of (3) and

(6). The formula (3) in terms of $P(n,k)$ has the form

$$P(n,k) - P(n,k-1) = p^k q(P(n-k-1,k-1) + P(n-k-1,k)) + \sum_{j=0}^{n-k-2} P(j,k-1)p^k qP(n-j-k-1,k-1).$$

Whence and from (1) passing to transforms yields

$$P^*(s,k) - P^*(s,k-1) = p^k s^k + p^k qs^{k+1}(P^*(s,k) + P^*(s,k-1)) + p^k q^2 s^{k+2} P^*(s,k)P^*(s,k-1).$$

The solution of it is (10), which is equivalent to (6).

4. Asymptotic distribution of Z_n. The asymptotic distribution of the longest head-run is given by A.Földes [5]. For the Bernoulli trials the following estimation was proved in [7].

THEOREM 4. *If* $n \to \infty$ *then*

$$P(Z_n - [\log_{1/p} n] \leq k) = \exp(-p^{k-a(n)}) + o(1),$$

where $a(n) = \log_{1/p} n - [\log_{1/p} n]$.

Table 1 contains selected probability distributions $P(Z_n > k)$ for coin tossing and their estimation. It allows to verify the accuracy of estimation for the subsequence $n = 2^m$, $m = 1,2,\ldots$ for which the convergence can be proved as $m \to \infty$. Table 2 contains the expected values of Z_n for coin tossing for some n.

5. Estimation of expected value. Theorem 4 suggests a formula for the estimation of the expected value $\mu(n)$ for large n.

Now we draw this fact from Theorem 3.

THEOREM 5. *If* $n \to \infty$, *then*

$$\mu(n) = \log_{1/p} n + O(1).$$

Proof. Let us introduce the function

$$\mu(t) = \mu([t]),\ t \geq 0,$$

and it Laplace transform

$$\mu^{\otimes}(s) = \int_0^{\infty} \mu(t)e^{-st}\,dt, \qquad Re\ s > 0.$$

It the easy to verify that

$$\mu^{\otimes}(s) = \frac{1-e^{-s}}{s}\mu^{*}(e^{-s}) =$$

$$= \frac{1-e^{-s}}{s}\frac{1-pe^{-s}}{1-e^{-s}}\sum_{k=1}^{\infty}\frac{(pe^{-s})^k}{1-e^{-s}+(pe^{-s})^k qe^{-s}},$$

For real $s > 0$ and $p = e^{-\lambda}, \lambda > 0$ let

$$\phi(x) = \frac{1-e^{-s}}{s}\frac{1-pe^{-s}}{1-e^{-s}}\frac{e^{-(s+\lambda)x}}{1-e^{-s}+e^{-(s+\lambda)x}qe^{-s}}, x \geq 0.$$

Because $\phi(x)$ is decreasing for $x \geq 0$

$$\int_{k}^{k+1}\phi(x)\,dx \leq \phi(k) \leq \int_{k-1}^{k}\phi(x)\,dx,$$

hence

$$\phi_1^{\otimes}(s) \leq \mu^{\otimes}(s) \leq \phi_2^{\otimes}(s),$$

where

$$\phi_2^{\otimes}(s) = \int_{0}^{\infty}\phi(x)\,dx = \frac{1}{s+\lambda}\frac{1-e^{-s}}{s}\frac{1-pe^{-s}}{1-e^{-s}}\frac{1}{qe^{-s}}\log\frac{1-pe^{-s}}{1-e^{-s}}$$

and

$$0 \leq \Delta^{\otimes}(s) = \phi_1^{\otimes}(s) - \phi_2^{\otimes}(s) \leq \phi(0) - \phi(1) = \frac{1}{s}\frac{1-e^{-s}}{1-e^{-s}q}.$$

Let us introduce the function

$$\mu_1^{*}(u) = \frac{1-pu}{1-u}\frac{1}{qu}\log\frac{1-pu}{1-u}.$$

It is easy to verify that

$$\mu_1^{*}(u) = \frac{1-pu}{1-u}\frac{1}{qu}\sum_{k=1}^{\infty}\left(\frac{u^k}{k} - \frac{(pu)^k}{k}\right)$$

$$= \frac{1-pu}{1-u}\sum_{k=1}^{\infty}u^{k-1}\frac{1}{k}(1+p+\ldots+p^{k-1})$$

$$= \frac{1}{1-u}\sum_{k=1}^{\infty}u^{k-1}\left(\frac{1}{k}(1+p+\ldots+p^{k-1}) - \frac{p}{k-1}(1+p+\ldots+p^{k-2})\right)$$

$$= \sum_{k=1}^{\infty}u^{k-1}\left(1-p+\frac{1-p^2}{2}+\ldots+\frac{1-p^{k-1}}{k-1}+\frac{1-p^k}{k(1-p)}\right)$$

$$= \sum_{k=1}^{\infty}u^{k-1}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}-\left(p+\frac{p^2}{2}+\ldots+\frac{p^k}{k}\right)+\frac{1-p^{k+1}}{(k+1)(1-p)}\right).$$

This implies that $\mu_1^*(u)$ is the generating function of the sequence

$$c_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k-1} - \left(p + \frac{p^2}{2} + \ldots + \frac{p^{k-1}}{k-1}\right) + \frac{1-p^k}{k(1-p)}, k = 1, 2, \ldots$$

Hence $\phi_2^{\otimes}(s)$ is the Laplace transform of the convolution of $C(x) = c_{[x]}$ and $F(x) = e^{-\lambda x}, x \geq 0$

$$\phi_2(t) = \int_0^t C(t-x)e^{-\lambda x}\,dx = \frac{1}{\lambda}\int_0^t (1 - e^{-\lambda(t-x)})\,dC(x)$$

$$= \frac{1}{\lambda}C(t) - \frac{1}{\lambda}\int_0^t e^{-\lambda x}\,dC(x) = \log_{1/p} t + O(1), \quad t \to \infty.$$

Since

$$\Delta^{\otimes}(s) = \frac{1-e^{-s}}{s}\frac{1}{1-e^{-s}q} = \frac{1-e^{-s}}{s}(1 + qe^{-s} + q^2e^{-2s} + \ldots)$$

we have

$$\Delta(t) = 1 + q + \ldots + q^{[t]} \to \frac{1}{p}, \quad t \to \infty,$$

Hence $\phi_2(t) - \mu(t) = O(1), t \to \infty$, and it proves of Theorem 5.

Table 1. The distribution function $P(Z_n \geq k)$ for coin tossing

k \ n	4	8	16	32	64	128	approximation for 128
0	0.937	0.996	0.999				
1	.500	.785	.960	.998			
2	.187	.418	.702	.922	.995	.999	.999
3	.062	.187	.395	.664	.897	.990	.982
4		.078	.196	.389	.648	.883	.865
5		.031	.093	.204	.389	.640	.632
6		.012	.043	.102	.211	.390	.393
7		.004	.019	.050	.109	.215	.221
8			.009	.024	.054	.112	.118
9			.004	.011	.028	.057	.061
10			.002	.005	.013	.029	.031
11			.001	.002	.006	.014	.016
12				.001	.003	.007	.008
13					.002	.004	.004
14					.001	.002	.002
15						.001	.001
EZ_n	1.687	2.512	3.425	4.379	5.356	6.345	6.3328

Table 2. Expected values of Z_n for coin tossing

n	$\mu(n)$	n	$\mu(n)$
1	0.5000	15	3.3380
2	1.0000	20	3.7292
3	1.3750	25	4.0364
4	1.6875	30	4.2895
5	1.9375	35	4.5049
6	2.1563	40	4.6923
7	2.3437	45	4.8581
8	2.5117	50	5.0068
9	2.6621	75	5.5817
10	2.7988	100	5.9918

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