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Translations on ternary algebras

1. Introduction and general observations. In [12] Pócs considered translations on universal algebras generalizing Szász's results given in [13], [14] and in [15]. The purpose of this paper is firstly to present a few observations on translations on universal algebras and thereafter consider translations on ternary algebras. Ternary algebras, in which we are interested, are considered by Nebeský in [4] and [5] and some further results are also given in [6], [7], [8], [9] and in [10].

The concept of an ideal in tree algebras defined by Nebeský [4] suggests us to use a different definition than Pócs'; the concept of a translation is also differently defined because e.g. in simple ternary algebras Pócs' translations are identical mappings.

By $\mathscr{A} = \langle A, \mathscr{F} \rangle$ we denote an algebra with an n-ary operation, $n \geqslant 2$, $Q \in \mathscr{F}$, where $\mathscr{F}$ is the set of the operations on $\mathscr{A}$ and A is the carrier set of $\mathscr{A}$. If $\{a_1, \ldots, a_n\} \subseteq A$, then $Q(a_1, \ldots, a_n) \in A$, too. We assume that for each $x \in A$, $x = Q(a_1, \ldots, a_n)$ for some subset $\{a_1, \ldots, a_n\} \subseteq A$. A subset U of A is said to be an *ideal* of the algebra $\mathscr{A}$, if $x \in U$ when $x = Q(a_1, \ldots, a_n)$ and at least $n-1$ elements of the n-tuple $(a_1, \ldots, a_n)$ belong to U.

A single-valued mapping $\mathscr{S}$ of the set A into A is called a *Q-translation* of the algebra $\mathscr{A} = \langle A, \mathscr{F} \rangle$, $Q \in \mathscr{F}$, if for each n-tuple $(a_1, \ldots, a_n)$ of A and for each i, $1 \leqslant i \leqslant n$, the following formula holds:

$$\mathscr{S}\big(Q(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)\big) = Q\big(\mathscr{S}(a_1), \ldots, \mathscr{S}(a_{i-1}), a_i, \mathscr{S}(a_{i+1}), \ldots, \mathscr{S}(a_n)\big).$$

As we shall consider Q-translations only, we shall call them briefly *translations*. An endomorphism of the algebra $\mathscr{A}$ is a mapping h from A into A such that for each $Q \in \mathscr{F}$

$$h\big(Q(a_1, \ldots, a_n)\big) = Q\big(h(a_1), \ldots, h(a_n)\big)$$

for any n-tuple $(a_1, \ldots, a_n)$ of A. An endomorphism h of $\mathscr{A}$ onto its ideal U which leaves U elementwise fixed is a retraction and U is a retract of $\mathscr{A}$. A function k on $\mathscr{A}$ is called *idempotent*, if $k(k(x)) = k(x)$ for each $x \in A$.

As the following example of Pócs [12] shows, a translation of an algebra need not be idempotent. Let us consider the closed interval $[-1, 1] = A$, where $Q(a_1, \ldots, a_n) = a_1 a_2 \ldots a_{n-1} a_n$, $\{a_1, \ldots, a_n\} \subseteq [-1, 1]$, and let $n = 2m$ for some integer $m \geqslant 1$. Clearly the mapping $f\colon A \to A$ defined as follows: $f(x) = -x$ when $x \in A$, is a translation on $\langle A, Q\rangle$, but $f(f(x)) = x \neq -x = f(x)$. For idempotent translations one can be prove a generalization of [11], Corollary 1.

THEOREM 1. *An idempotent function f on $\mathscr{A} = \langle A, \mathscr{F}\rangle$, $Q \in \mathscr{F}$, is a translation on $\mathscr{A}$ if and only if f is a retraction of $\mathscr{A}$.*

Proof. Let f be a translation and an idempotent function on $\mathscr{A}$. If $y \in f(A)$, then $y = f(x)$ for some $x \in A$. Let $a_1, \ldots, a_{n-1} \in f(A)$ and let us consider an element $Q(a_1, \ldots, a_{n-1}, a_n)$, where $a_n \in A$. As $a_j = f(y_j)$ for some $y_j \in A$ and for each j, $j = 1, \ldots, n-1$, $Q(a_1, \ldots, a_{n-1}, a_n) = Q(f(y_1), \ldots, f(y_{n-1}), a_n) = f(Q(y_1, \ldots, y_{n-1}, a_n)) \in f(A)$. Thus $f(A)$ is an ideal of $\mathscr{A}$ and it remains to show that $f(A)$ is a retract of $\mathscr{A}$ with respect to f.

Let $y \in f(A)$. So $y = Q(f(a_1), \ldots, f(a_{i-1}), a_i, f(a_{i+1}), \ldots, f(a_n))$. Now $f(y) = Q(f(f(a_1)), \ldots, f(f(a_{i-1})), a_i, f(f(a_{i+1})), \ldots, f(f(a_n))) = Q(f(a_1), \ldots, f(a_{i-1}), a_i, f(a_{i+1}), \ldots, f(a_n)) = y$ as f is a translation and idempotent on $\mathscr{A}$. Thus f leaves $f(A)$ elementwise fixed.

Conversely, let f be a retraction on $\mathscr{A}$. Consider the element $y = Q(f(a_1), \ldots, f(a_{i-1}), a_i, f(a_{i+1}), \ldots, f(a_n))$. $y = f(y)$, because $y \in f(A)$ and $f(A)$ is a retract of $\mathscr{A}$. As f is a retraction on $\mathscr{A}$, $y = f(y) = f(f(y))$, and so $y = f(Q(f(a_1), \ldots, f(a_{i-1}), a_i, f(a_{i+1}), \ldots, f(a_n))), Q(f(f(a_1)), \ldots, f(f(a_{i-1})), f(a_i), f(f(a_{i+1})), \ldots, f(f(a_n))) = Q(f(a_1), \ldots, f(a_i), \ldots, f(a_n)) = f(Q(a_1, \ldots, a_n))$, where $f(f(a_j)) = f(a_j)$ as $f(a_j) \in f(A)$ and $f(A)$ is a retract of $\mathscr{A}$, $j = 1, \ldots, n$ and $j \neq i$. Hence f is a translation on $\mathscr{A}$. Clearly f is idempotent. This completes the proof.

The proof above shows that for any translation $\mathscr{S}$ on an algebra $\mathscr{A} = \langle A, \mathscr{F}\rangle$, $Q \in \mathscr{F}$, $\mathscr{S}(A)$ is an ideal of $\mathscr{A}$.

As noted by Kolibiar [3], Theorem 1, each $\vee$-translation λ on a lattice L induces an equivalence relation R_λ on L having the substitution property over the operation $\vee$; briefly R_λ is a $\vee$-congruence on L. The next theorem gives a generalization.

THEOREM 2. *Let $\mathscr{A} = \langle A, \mathscr{F}\rangle$, $Q \in \mathscr{F}$, be an algebra. There is a one-to-one correspondence between idempotent translations $\mathscr{S}$ on $\mathscr{A}$ and Q-congruences R on $\mathscr{A}$ as follows:*

(i) *Any translation $\mathscr{S}$ determines a Q-congruence $R_\mathscr{S}$ having property* (iii) *defined below according to the rule: $xR_\mathscr{S}y \Leftrightarrow \mathscr{S}(x) = \mathscr{S}(y)$.*

(ii) *Any Q-congruence R having property* (iii) *determines a translation* $\mathscr{S}_R$ *as follows:* $\mathscr{S}_R(x) = x'$, *where* xRx' *and* x' *belongs to the ideal* U *given in* (iii).

(iii) *There is an ideal* U *of* $\mathscr{A}$ *such that each congruence class of a Q-congruence* R *contains exactly one element of* U.

Proof. Let $\mathscr{S}$ be an idempotent translation on $\mathscr{A}$. As shown above, $\mathscr{S}(A)$ is an ideal of $\mathscr{A}$. If C is a congruence class of $R_{\mathscr{S}}$, $\mathscr{S}(x) = \mathscr{S}(y)$ for any two elements x, y of C, and so $\mathscr{S}(A) \cap C$ contains one element only. Clearly each class C contains at least one element of $\mathscr{S}(A)$. Thus it remains to show that $R_{\mathscr{S}}$ is a Q-congruence on $\mathscr{A}$. $R_{\mathscr{S}}$ is evidently an equivalence relation on $\mathscr{A}$ according to its definition and so we consider the substitution property only. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two n-tuples of A such that $x_i R_{\mathscr{S}} y_i$ for each value of i, $i = 1, \ldots, n$. Then $\mathscr{S}\big(Q(x_1, \ldots, x_n)\big) = \mathscr{S}\big(\mathscr{S}\big(Q(x_1, \ldots, x_n)\big)\big) = \mathscr{S}\big(Q\big(x_1, \mathscr{S}(x_2), \ldots, \mathscr{S}(x_n)\big)\big) = Q\big(\mathscr{S}(x_1), \mathscr{S}(x_2), \ldots, \mathscr{S}(x_n)\big) = Q\big(\mathscr{S}(y_1), \mathscr{S}(y_2), \ldots, \mathscr{S}(y_n)\big) = \mathscr{S}\big(Q(y_1, \ldots, y_n)\big)$, whence $R_{\mathscr{S}}$ has the substitution property over Q.

Conversely, let R be a Q-congruence on $\mathscr{A}$ with the property given in (iii), and let $\mathscr{S}_R$ be the mapping defined in (ii). We consider the element $Q(a_1, \ldots, a_n) \in A$. According to (iii), for any a_i there is an element $u_i \in U$ such that $a_i R u_i$, $i = 2, \ldots, n$. Moreover, $a_1 R a_1$ and $Q(a_1, u_2, \ldots, u_n) \in U$. According to the definition of $\mathscr{S}_R$, $\mathscr{S}_R\big(Q(a_1, \ldots, a_n)\big) = Q(a_1, u_2, \ldots, u_n) = Q\big(a_1, \mathscr{S}_R(a_2), \ldots, \mathscr{S}_R(a_n)\big)$. We have proved the Q-translation property for the index 1, but the same construction of the proof holds for any indexvalue. Hence $\mathscr{S}_R$ is a translation on $\mathscr{A}$. As $\mathscr{S}_R(a)$ and a belong to one and the same congruence class of R, $\mathscr{S}_R\big(\mathscr{S}_R(a)\big) = \mathscr{S}_R(a)$.

The Q-congruence property allows us now to prove an analogue of Kolibiar's Theorem 2 for idempotent translations on abstract algebras. Let λ and $\mathscr{S}$ be two idempotent translations on $\mathscr{A}$. As usually, $\lambda\mathscr{S}(a) = \lambda\big(\mathscr{S}(a)\big)$, and we say that $\lambda \leqslant \mathscr{S}$ if and only if $R_\lambda \leqslant R_{\mathscr{S}}$.

THEOREM 3. *Let* $\mathscr{S}$ *and* λ *be two idempotent translations on an algebra* $\mathscr{A} = \langle A, \mathscr{F} \rangle$, $Q \in \mathscr{F}$, *and let* $\lambda \leqslant \mathscr{S}$. *Then* $\lambda\mathscr{S} = \mathscr{S}\lambda = \mathscr{S}$ *if and only if for each* z *for which* $\mathscr{S}(z) = z$ *also* $\lambda(z) = z$.

Proof. In other words, $\lambda\mathscr{S} = \mathscr{S}\lambda = \mathscr{S}$ if and only if $\mathscr{S}(A) \subseteq \lambda(A)$. For any $x \in A$, $\lambda\mathscr{S}(x) = \lambda\big(\mathscr{S}(x)\big) = \mathscr{S}(x)$, because $\mathscr{S}\big(\mathscr{S}(x)\big) = \mathscr{S}(x)$.

It follows from the definition $\lambda \leqslant \mathscr{S} \Leftrightarrow R_\lambda \leqslant R_{\mathscr{S}}$ that if $\lambda(a) = \lambda(b)$, then $\mathscr{S}(a) = \mathscr{S}(b)$, because $aR_\lambda b \Rightarrow aR_{\mathscr{S}} b$. Let $x \in A$ be an arbitrary element. Then $\lambda\big(\lambda(x)\big) = \lambda(x)$ implies that $\mathscr{S}\big(\lambda(x)\big) = \mathscr{S}(x)$, whence $\mathscr{S}\lambda(x) = \mathscr{S}\big(\lambda(x)\big) = \mathscr{S}(x)$. Thus $\mathscr{S}\lambda = \lambda\mathscr{S} = \mathscr{S}$.

Conversely, let $\mathscr{S}(z) = z$. Then $\lambda(z) = \lambda\big(\mathscr{S}(z)\big) = \lambda\mathscr{S}(z) = \mathscr{S}(z) = z$, and the desired property holds.

2. Ternary algebras and translations. An ordered pair $\mathscr{A} = (A, Q)$ is called a *ternary algebra*, if A is a non-empty set and Q a ternary operation

on $\mathscr{A}$ such that for each $a, b \in A$, $Q(a, a, b) = a$, and $Q(a, b, c)$ is invariant under all six permutations of $a, b, c \in A$. A ternary algebra is called *normal*, if for each four elements $a, b, c, d \in A$, $Q(Q(a, b, c), c, d) = Q(Q(a, d, c), c, b)$, and it is simple, if for each five elements $a, b, c, d, e \in A$, $Q(Q(a, b, c), d, e) = Q(Q(a, d, e), Q(b, d, e), c)$. A simple ternary algebra is also normal [5]. The lattice structure of finite simple ternary algebras is characterized in [6], and the structure of algebras analogous to normal ternary algebras is studied under the title "Pseudoschar" by Hoehnke in papers [1] and [2].

LEMMA 1. *Each translation $\mathscr{S}$ on a ternary algebra $\mathscr{A} = (A, Q)$ is idempotent.*

Proof. $a = Q(a, a, b)$ for each two elements $a, b \in A$. Then $\mathscr{S}(a) = Q(\mathscr{S}(a), a, \mathscr{S}(b))$ and $\mathscr{S}(\mathscr{S}(a)) = Q(\mathscr{S}(a), \mathscr{S}(a), \mathscr{S}(\mathscr{S}(b))) = \mathscr{S}(a)$.

THEOREM 4. *Let $\mathscr{A} = (A, Q)$ be a ternary algebra and $\mathscr{S}$ and λ two translations on $\mathscr{A}$. $\mathscr{S}$ and λ permute, i.e. $\mathscr{S}\lambda(a) = \lambda\mathscr{S}(a)$ for each $a \in A$, if and only if $\mathscr{S}(A) \cap \lambda(A) \neq \emptyset$.*

Proof. Let $d \in \lambda(A) \cap \mathscr{S}(A)$, and so $\mathscr{S}(d) = d = \lambda(d)$, as $\mathscr{S}(A)$ and $\lambda(A)$ are retracts of $\mathscr{A}$. Now $a = Q(d, a, a)$, and thus $\mathscr{S}\lambda(a) = \mathscr{S}(\lambda(a)) = \mathscr{S}(\lambda(Q(d, a, a))) = \mathscr{S}(Q(\lambda(d), \lambda(a), a)) = \mathscr{S}(Q(d, \lambda(a), a)) = Q(\mathscr{S}(d), \lambda(a), \mathscr{S}(a)) = Q(d, \lambda(a), \mathscr{S}(a)) = Q(\lambda(d), \lambda(a), \mathscr{S}(a)) = \lambda(Q(d, a, \mathscr{S}(a))) = \lambda(Q(\mathscr{S}(d), a, \mathscr{S}(a))) = \lambda(\mathscr{S}(Q(d, a, a))) = \lambda\mathscr{S}(a)$.

Conversely, let $\lambda\mathscr{S}(a) = \mathscr{S}\lambda(a)$ for any $a \in A$ and let $b \in \mathscr{S}(A)$. Then $\lambda\mathscr{S}(b) = \lambda(b) = \mathscr{S}\lambda(b)$, and so $\lambda(b) \in \mathscr{S}(A)$. As λ is idempotent, $\lambda(b) \in \lambda(A)$, whence $\lambda(b) \in \lambda(A) \cap \mathscr{S}(A)$, and the theorem follows.

Now we can prove a uniqueness theorem analogous to [11], Corollary 2, and [14], Theorem 3.

THEOREM 5. *Let $\mathscr{S}$ and λ be two translations on a ternary algebra $\mathscr{A} = (A, Q)$. $\lambda = \mathscr{S}$ if and only if $\mathscr{S}(A) = \lambda(A)$.*

Proof. If $\mathscr{S} = \lambda$, then evidently $\mathscr{S}(A) = \lambda(A)$. So, let $\mathscr{S}(A) = \lambda(A)$. Now for each $a \in A$, $\mathscr{S}\lambda(a) = \mathscr{S}(\lambda(a)) = \lambda(a) = \lambda\mathscr{S}(a) = \lambda(\mathscr{S}(a)) = \mathscr{S}(a)$, as $\mathscr{S}(A) = \lambda(A)$ and $\mathscr{S}\lambda(a) = \lambda\mathscr{S}(a)$.

Let $a, s \in A$ be fixed elements of a ternary algebra $\mathscr{A} = (A, Q)$. The set $P(a, s) = \{y \mid y = Q(x, a, s), x \in A\}$ is called the *principal set* generated by a and s in $\mathscr{A}$.

LEMMA 2. *If $\mathscr{A} = (A, Q)$ is a simple ternary algebra, then $P(a, s)$ is an ideal of $\mathscr{A}$.*

Proof. We should show that when $x, y \in P(a, s)$, then $Q(x, y, z)$, $P(a, s)$ for any $z \in A$. As $x, y \in P(a, s)$, $x = Q(r, a, s)$ and $y = Q(t, a, s)$ for some elements $r, t \in A$. Thus $Q(z, x, y) = Q(z, Q(r, a, s), Q(t, a, s)) = Q(Q(z, r, t), a, s) \in P(a, s)$.

Now we are ready to generalize [11], Theorem 3; we shall do it in two steps. At first a lemma.

LEMMA 3. *If an ideal U of a normal ternary algebra $\mathscr{A}$ is the retract of a translation $\mathscr{S}$ on $\mathscr{A}$, then $U \cap P(a, u)$ is a principal set of $\mathscr{A}$ for any $a \in A$ and any $u \in U$.*

Proof. Let $x \in U \cap P(a, u)$. Then $x = \mathscr{S}(x) = \mathscr{S}(Q(y, a, u)) = Q(\mathscr{S}(y), \mathscr{S}(a), u)$ and so $x \in P(\mathscr{S}(a), u)$. On the other hand, if $x' \in P(\mathscr{S}(a), u)$, then $x' = Q(z, \mathscr{S}(a), u)$. Now $Q(Q(z, \mathscr{S}(a), u), a, u) = Q(Q(\mathscr{S}(a), a, u), z, u)$, and if $Q(\mathscr{S}(a), a, u) = \mathscr{S}(a)$, then $Q(Q(z, \mathscr{S}(a), u), a, u) = Q(z, \mathscr{S}(a), u) = x'$, and so $x' \in P(a, u)$. As $\mathscr{S}(x') = \mathscr{S}(Q(z, \mathscr{S}(a), u)) = Q(z, \mathscr{S}(a), \mathscr{S}(u))$ $= Q(z, \mathscr{S}(a), u) = x'$, $x' \in U$, and consequently, $x' \in U \cap P(a, u)$. Hence, $U \cap P(a, u) = P(\mathscr{S}(a), u)$.

We must further show that $Q(\mathscr{S}(a), a, u) = \mathscr{S}(a)$. As $\mathscr{S}(a), u \in U$, $Q(\mathscr{S}(a), a, u) \in U$ and so it is invariant under $\mathscr{S}$. Thus $Q(\mathscr{S}(a), a, u)$ $= \mathscr{S}(Q(\mathscr{S}(a), a, u)) = Q(\mathscr{S}(a), \mathscr{S}(a), \mathscr{S}(u)) = \mathscr{S}(a)$. This completes the proof.

THEOREM 6. *An ideal U of a simple ternary algebra $\mathscr{A} = (A, Q)$ is a retract of $\mathscr{A}$ if and only if for any $a \in A$ and any $u \in U$ the set $U \cap P(a, u)$ is a principal set of $\mathscr{A}$.*

Proof. According to Lemma 3 it suffices to show the second part of the proof. Let $U \cap P(a, u) = P(r, t)$. Let $x \in P(r, t)$. Then $x = Q(x, r, t)$ and $x = Q(a, u, x)$. Thus $x = Q(Q(x, a, u), r, t) = Q(u, Q(a, r, t), Q(x, r, t))$ $= Q(x, u, Q(a, r, t))$. Hence $P(r, t) = P(k, u)$ for some $k \in U$. We define the mapping $\mathscr{S}$ as follows: $\mathscr{S}(a) = k$. Accordingly, $\mathscr{S}(A) \subseteq U$, and when $a \in U$, then $P(a, u) \subset U$, and so $U \cap P(a, u) = P(a, u)$. Thus $U \subseteq \mathscr{S}(A)$, and $\mathscr{S}(A) = U$ is an ideal of $\mathscr{A}$. As $U \cap P(a, u) = P(a, u)$ for $a \in U$, $\mathscr{S}$ leaves U elementwise fixed. We should further show that $\mathscr{S}$ is an endomorphism on $\mathscr{A}$.

According to the definition of $\mathscr{S}$, we should show that $U \cap P(Q(k, m, n), u)$ $= Q(U \cap P(k, u), U \cap P(m, u), U \cap P(n, u))$ for any three elements $k, m, n \in A$ and $u \in U$. We denote $Q(U \cap P(k, u), U \cap P(m, u), U \cap P(n, u))$ briefly by W_{kmn}. Since $u \in U \cap P(k, u)$, then $U \cap P(k, u)$ is an ideal of $\mathscr{A}$ (see [10], Lemma 2). Moreover, W_{kmn} is an ideal of $\mathscr{A}$ and $W_{kmn} = \{Q(x, y, z) |$ $z \in U \cap P(k, u), y \in U \cap P(m, u)$ and $x \in U \cap P(n, u)\}$ as shown in [10], Theorem 4. By results of Sholander (see [10]), we can transform $\mathscr{A}$ into a meet-semilattice L, where the least element can be choosen freely, $Q(x, y, z) = (x \wedge y) \vee (z \wedge y) \vee (z \wedge x)$, and the join of two elements x and y exists, if there is an element $t \geqslant x, y$ in L. In our situation, we choose u as the least element. Let $d \in W_{kmn}$. Then $d = Q(x, y, z)$, where $x \in U \cap P(k, u)$, and has the representation $x = Q(x, k, u) = x \wedge k$; similarly $y = Q(y, m, u)$ and $z = Q(z, n, u)$. Now $d = Q(x \wedge k, y \wedge m, z \wedge n) = (x \wedge k \wedge y \wedge m) \vee$ $\vee (x \wedge k \wedge z \wedge n) \vee (y \wedge m \wedge z \wedge n) \leqslant (k \wedge m) \vee (k \wedge n) \vee (m \wedge n) = Q(k, m, n)$. Hence $d = Q(d, Q(m, n, k), u)$ and thus $d \in U \cap P(Q(k, m, n), u)$.

Conversely, let $b \in U \cap P(Q(k, m, n), u)$; it means $b = Q(b, Q(k, m, n), u)$ $= Q(Q(b, u, k), Q(b, u, m), n) = (b \wedge k) \vee (b \wedge m) \vee n$. But $b = b \wedge b = (b \wedge k) \vee$

$\vee (b \wedge m) \vee (b \wedge n) = Q(Q(b, u, k), Q(b, u, m), Q(b, u, n))$, where $Q(b, u, k) \in \in U \cap P(k, u)$ as $b, u \in U$ and $u, k \in P(k, u)$. Similarly $Q(b, u, m) \in U \cap P(u, m)$ and $Q(b, u, n) \in U \cap P(n, u)$. Thus $b \in W_{kmn}$ and we can conclude that $W_{kmn} = U \cap P(Q(k, m, n), u)$.

Let $a, c \in A$ be arbitrary elements of $\mathscr{A}$. We define the specified translation generated by a and c as follows: $\mathscr{S}_{ac}(x) = Q(x, a, c)$. As the previous proofs suggests, the translation $\mathscr{S}_{ac}$ characterizes simple ternary algebras.

THEOREM 7. *A ternary algebra $\mathscr{A} = (A, Q)$ is simple if and only if $\mathscr{S}_{ac}$ is a translation on $\mathscr{A}$ for any two elements $a, c \in A$.*

Proof. Let $\mathscr{S}_{ac}$ be a translation for any pair $a, c \in A$. Then $Q(Q(a, b, c), d, e) = \mathscr{S}_{de} Q(a, b, c) = Q(\mathscr{S}_{de}(a), \mathscr{S}_{de}(b), c) = Q(Q(a, d, e), Q(b, d, e), c)$, and thus $\mathscr{A}$ is simple. The converse proof is similar.

A further problem on translations on $\mathscr{A}$ is the characterization of direct sums of $\mathscr{A}$ as done in [9].

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