ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XII (1969)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XII (1969)

Ministerstwo Nauki i Szkolnictwa Wyższego

Z. ROMANOWICZ and K. WOŹNIAK (Opole)

On a property of approaching maps in Euclidean spaces

Let us consider an arbitrary bounded subset G of the real Euclidean space E_n. Let us denote by $r(G)$ the radius of the smallest closed n-ball containing G. The function r is defined for each bounded subset G of E_n.

Let f be an approaching map which maps the set G onto a subset $f(G)$ of E_n, i.e. let f be a map such that for each couple of points $z_1 \in G$ and $z_2 \in G$ there is

$$\|f(z_1)-f(z_2)\| < \|z_1-z_2\|, \tag{1}$$

where $\|u-v\|$ denotes the distance between u and v.

Professor H. Steinhaus raised the problem whether for each finite subset G of E_n and for an arbitrary approaching map f defined on G there is $r\big(f(G)\big) < r(G)$?

A positive answer to this problem in the case of E_2 is given in [3].

The aim of this paper is a generalization of the result of [3] to arbitrary Euclidean space E_n. Namely, we shall prove the following

THEOREM. *If f is an approaching map of a compact subset G of E_n onto the subset $f(G)$ of E_n, then $r\big(f(G)\big) < r(G)$.*

In the proof of the Theorem the following five lemmas will be used.

LEMMA 1. *If each system of $n+1$ points of $G \subset E_n$ is contained in a closed n-ball with the radius R, then the whole set G is contained in a closed n-ball with the radius R.*

This lemma follows immediately from Helly's theorem for convex sets [2].

Denote by $S(z_i, R)$, $i = 1, \ldots, k$, $2 \leqslant k \leqslant n+1$, a family of closed n-balls with centers in z_i and with the radius R. Let Δ be the smallest convex set containing z_i, $i = 1, \ldots, k$.

LEMMA 2. *If a family of n-balls $S(z_i, R)$ satisfies the condition $\bigcap_{i=1}^{k} S(z_i, R) \neq \emptyset$, then there exists at least one point of Δ belonging to each ball of this family.*

Proof. Let $a \in \bigcap_{i=1}^{k} S(z_i, R)$. There exists a (unique) point $z^* \in \Delta$ such that (see [1], Chapter V, § 1, Theorem 1)

$$\|a-z^*\| = \inf_{z\in\Delta}\|a-z\| \quad \text{and} \quad (a-z^*, z-z^*) \leqslant 0$$

for each $z \in \Delta$, where $(a-z^*, z-z^*)$ denotes the scalar product of vectors $a-z^*$ and $z-z^*$. In particular, $(a-z^*, z_i-z^*) \leqslant 0$ for $i = 1, \ldots, k$. Hence,

$$\|z_i-z^*\|^2 \leqslant \|a-z^*\|^2 - 2(a-z^*, z_i-z^*) + \|z_i-z^*\|^2 = \|z_i-a\|^2 \leqslant R^2,$$

which means that $z^* \in \bigcap_{i=1}^{k} S(z_i, R)$.

LEMMA 3. *If an approaching map f maps points $z_i \in E_n$ into points $z_i' = f(z_i)$, $i = 1, \ldots, k$, $2 \leqslant k \leqslant n+1$, and if the condition*

$$\bigcap_{i=1}^{k} S(z_i, R) \neq \emptyset \tag{2}$$

holds, then there exists a positive real number $R' < R$ such that $\bigcap_{i=1}^{k} S(z_i', R') \neq \emptyset$.

Proof. Since f is an approaching map then, as was shown in [4], the condition (2) implies that the set

$$\bigcap_{i=1}^{k} S(z_i', R) \tag{3}$$

is not empty. Thus the lemma will be proved if we show that the set (3) contains more than one point.

Suppose, on the contrary, that set (3) consists of only one point z_0'. Let z_0 be an arbitrary point of $\bigcap_{i=1}^{k} S(z_i, R)$. The existence of z_0 follows from (2).

Let $u_i = z_i - z_0$, $u_i' = z_i' - z_0'$ for $i = 1, \ldots, k$. From the definition of z_0 and from (2) we get

$$\|u_i\| \leqslant R. \tag{4}$$

By the assumption, the set (3) is a one-point set, hence there exist at least two vectors in the family $\{u_1', \ldots, u_k'\}$ of vectors such that the equality

$$\|u_i'\| = R \tag{5}$$

holds.

Let l, $2 \leqslant l \leqslant k$, denote the number of such vectors in $\{u_1', \ldots, u_k'\}$ for which the condition (5) holds. Without loss of generality we may assume that these vectors are $u_1', \ldots, u_l'$. Then, by (4) and (5), we have

$$\|u_i\| \leqslant \|u_i'\| \quad \text{for } i = 1, \ldots, l. \tag{6}$$

Moreover, from (1) we get $\|u_i' - u_j'\| < \|u_i - u_j\|$ for $i, j = 1, \ldots, k$, $i \neq j$. Hence, by (6),

$$(u_i, u_j) < (u_i', u_j') \quad \text{for } i, j = 1, \ldots, l,\ i \neq j. \tag{7}$$

Since z_0' is the only point in the set (3), and since z_0' is an interior point of the set $\bigcap_{i=l+1}^{k} S(z_i', R)$ for $l < k$, then $\bigcap_{i=1}^{l} S(z_i', R) = \{z_0'\}$.

In the case $l = k$ the last equality holds, by the contrary assumption, too.

Let Δ be the smallest convex set containing the points $z_1', \ldots, z_l'$. By Lemma 2, $z_0' \in \Delta$.

By the definition of Δ, there exist real numbers a_i, $i = 1, \ldots, l$, such that $a_i \geqslant 0$ and $\sum_{i=1}^{l} a_i \neq 0$ and such that $\sum_{i=1}^{l} a_i u_i' = 0$.

Multiplying the inequalities (7) by $a_i a_j$, respectively, and then summarizing, we get

$$\Big(\sum_{i=1}^{l} a_i u_i, \sum_{j=1}^{l} a_j u_j\Big) < \Big(\sum_{i=1}^{l} a_i u_i', \sum_{j=1}^{l} a_j u_j'\Big) = 0,$$

a contradiction.

From Lemma 3 we get immediately

LEMMA 4. *If $S = \{z_1, \ldots, z_k\}$, $2 \leqslant k \leqslant n+1$ is a subset of E_n and f is an approaching map defined on S, then $r\big(f(S)\big) < r(S)$.*

Let G^{n+1} denote the Cartesian product of $n+1$ copies of G. If $\mathfrak{z} = (z_1, \ldots, z_{n+1}) \in G^{n+1}$, then $r(\mathfrak{z})$ means $r\big(\bigcup_{i=1}^{n+1} \{z_i\}\big)$.

LEMMA 5. *If G is a compact subset of E_n, then there exists $\mathfrak{z} \in G^{n+1}$ such that $r(G) = r(\mathfrak{z})$.*

Proof. By Lemma 1, $r(G) = \sup\{r(\mathfrak{z}) : \mathfrak{z} \in G^{n+1}\}$. By the definition of the supremum and by the compactnes of G^{n+1}, there exists an infinite sequence $\mathfrak{z}_k \in G^{n+1}$, such that

$$r(\mathfrak{z}_k) \to r(G) \quad \text{and} \quad \mathfrak{z}_k \to \mathfrak{z} \in G^{n+1}. \tag{8}$$

We shall show that $r(\mathfrak{z}) = r(G)$.

Suppose, on the contrary, that $r(\mathfrak{z}) < r(G)$. Let ε be a positive number such that

$$\varepsilon < r(G) - r(\mathfrak{z}). \tag{9}$$

By (8), there exists k_0 such that $r(\mathfrak{z}_{k_0}) > r(G) - \varepsilon/2$ and such that $r(\mathfrak{z}_{k_0}) < r(\mathfrak{z}) + \varepsilon/2$. Hence $r(G) - \varepsilon/2 < r(\mathfrak{z}_{k_0}) < r(\mathfrak{z}) + \varepsilon/2$. Thus, we get $r(G) - r(\mathfrak{z}) < \varepsilon$, contrary to (9).

Proof of the Theorem. Since G is compact and f is, by (1), continuous on G, the set $[f(G)]^{n+1}$ is compact. Hence, by Lemma 5, there exists $\mathfrak{z}' \in [f(G)]^{n+1}$ such that $r(f(G)) = r(\mathfrak{z}')$. Let $\mathfrak{z}' = (q_1', \ldots, q_{n+1}')$ and let $q_i \in G$, $i = 1, \ldots, n+1$, be such that $q_i' = f(q_i)$. Then, by Lemma 4, we get

$$r(f(G)) = r\Big(\bigcup_{i=1}^{n+1} \{q_i'\}\Big) < r\Big(\bigcup_{i=1}^{n+1} \{q_i\}\Big) \leqslant r(G).$$

Thus, the Theorem is proved.

Note that the Theorem remains true in case when G and $f(G)$ lie in arbitrary real vector finite dimensional spaces. The proof remains the same.

References

[1] N. Bourbaki, *Espaces vectoriels topologiques*, Paris 1955.

[2] E. Helly, Jahresber. Deutsch. Math. Verein. 32 (1923), pp. 175-176.

[3] Z. Romanowicz, *On a property of approaching mapping on the plane*, (Polish) Zeszyty Naukowe WSP Opole, Matematyka 4 (1965), pp. 163-167.

[4] F. A. Valentine, *A Lipschitz condition preserving extension for a vector function*, Amer. Journ. Math. 67 (1945), pp. 83-93.