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Global integral formulas for solving the $\bar{\partial}$-equation on Stein manifolds

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Abstract. The Cauchy–Fantappie–Leray formula is generalized to Stein manifolds. For general analytic polyhedra (this is a class of domains, which contains the strictly pseudoconvex domains and the Weil polyhedra) holomorphic kernels and formulas for solving the $\bar{\partial}$-equation are constructed. Theses formulas are especially useful for domains of the Weil polyhedron type, where in distinction to the strictly pseudoconvex case, for many problems, it is impossible to reduce the problem to the case C^n by localization. In particular, these formulas yield solutions of the $\bar{\partial}$-equation with uniform estimates for regular Weil polyhedra on arbitrary Stein manifolds.

0. Preliminaries

Throughout this paper we use the following notation:

M	is a Stein manifold with the orientation defined by the condition: If $z_1, \dots, z_n$ are local holomorphic coordinates, then the form $(-i)^n\, d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \wedge dz_1 \wedge \dots \wedge dz_n$ is positive;
n	is the complex dimension of M;
$T(M)$	is the complex tangent boundle of M;
$T^*(M)$	is the complex cotangent boundle of M;
$\tilde{T}(M \times M)$, $\tilde{T}^*(M \times M)$	are the pull-backs of $T(M)$ and $T^*(M)$, respectively, relative to the projection $M \times M \to M$, $(z, \zeta) \to z$;
D	is a relatively compact domain in M, whose boundary is of class C^1 or at least piecewise of class C^1;
$\Delta(Y)$	is the diagonal in the product $Y \times Y$ when Y is some set;
${}_N\mathcal{O}$	is the sheaf of germs of local holomorphic functions on N, when N is some complex manifold.

Let $v = (v_1, \dots, v_n)$ and $u = (u_1, \dots, u_n)$ be collections of complex C^1-functions on some C^1-manifold X. Then we define

$$\langle v, u\rangle = v_1 u_1 + \ldots + v_n u_n, \qquad \omega(u) = du_1 \wedge \ldots \wedge du_n,$$

$$\omega'(v) = \sum_{j=1}^{n} (-1)^{j-1} v_j \bigwedge_{s \neq j} dv_s, \qquad \Omega(v, u) = \frac{(n-1)!}{(2\pi i)^n} \frac{\omega'(v) \wedge \omega(u)}{\langle v, u\rangle^n}.$$

If ζ denotes the variable in X and if the functions v_j and u_j depend also on other variables, then we write ω_ζ, ω'_ζ and Ω_ζ. Let us list some simple properties of these forms.

The form $\Omega(v, u)$ is closed (see, for example, [12]):

$$d\Omega(v(\zeta), u(\zeta)) = 0 \quad \text{for } \langle v(\zeta), u(\zeta)\rangle \neq 0. \tag{0.1}$$

If X is an open set in M and the functions u_j are holomorphic, then, for every continuous function f on X, we have

$$d(f\Omega(v, u)) = \bar{\partial} f \wedge \Omega(v, u). \tag{0.2}$$

In fact, $\partial f \wedge \omega(u) = 0$ by degree considerations. Consequently, $\partial f \wedge \Omega(v, u) = 0$. In view of (0.1) this implies (0.2).

For every complex C^1-function ψ on X, one has the relations

$$\omega'(\psi v) = \psi^n \omega'(v) \tag{0.3}$$

and

$$\Omega(\psi v, u) = \Omega(v, u), \tag{0.4}$$

which follow by computation or from the representation (0.7) below.

The following lemma shows that the forms $\omega'(v) \wedge \omega(u)$ and $\Omega(v, u)$ have an invariant meaning also in the case when, for some point $z \in M$, u is a map with values in the tangent space $T_z(M)$ and v is a map with values in the corresponding cotangent space $T_z^*(M)$.

LEMMA. *Let $\Gamma = (\gamma_{ij})$ be a complex $n \times n$ matrix not depending on $\zeta \in X$ such that* $\det \Gamma \neq 0$, *and let $\Gamma_t = (\gamma_{ji})$. Then*

$$\omega'(\Gamma_t^{-1} v) \wedge \omega(\Gamma u) = \omega'(v) \wedge \omega(u) \tag{0.5}$$

and, consequently,

$$\Omega(\Gamma_t^{-1} v, \Gamma u) = \Omega(v, u). \tag{0.6}$$

Proof. We recall that, for an $n \times n$ matrix $A = (a_{\alpha\beta})$ with entries in a non-commutative C-algebra, the determinant can be defined as follows:

$$\det A = \sum_{\sigma} \operatorname{sgn}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(n),n},$$

where the summation is spread over all permutations σ of $\{1, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. The usual relations between row operations and determinants hold also in this non-commutative case. And so, as usually,

one can prove that $\det(\Gamma A) = \det\Gamma \cdot \det A$ (however, in general, $\det(A\Gamma) \neq \det A \cdot \det\Gamma$). Since the form $\omega'(v)$ can be written as

$$\omega'(v) = \frac{1}{(n-1)!} \det \begin{pmatrix} v_1 & dv_1 & dv_1 \\ \vdots & \vdots & \vdots \\ v_n & dv_n & dv_n \end{pmatrix}, \tag{0.7}$$

it follows that $\omega'(\Gamma_t^{-1} v) = (\det\Gamma)^{-1}\omega'(v)$. This and the obvious relation $\omega(\Gamma u) = \det\Gamma\omega(u)$ imply (0.6).

1. Introduction

1.1. Let f be a continuous function on $\bar{D}$ which is holomorphic in D. For $M = \boldsymbol{C}^n$ we have the Cauchy–Leray formula [12]

$$f(z) = \int_{\zeta\in\partial D} f(\zeta)\,\Omega_\zeta(u^*(z,\zeta), \zeta - z), \qquad z \in D,$$

where u^*: $D \times \partial D \to \boldsymbol{C}^n$ is a smooth map satisfying $\langle u^*(z,\zeta), z-\zeta\rangle \neq 0$ for all $z \in D$ and $\zeta \in \partial D$. Important special cases are integral formulas with holomorphic kernels as the Bergman–Weil formula for regular analytic polyhedra (see [1], [4], [14], [25]) and formulas for strictly pseudoconvex and piecewise strictly pseudoconvex domains in $\boldsymbol{C}^n$ (see [6], [18], [19], [21]).

It is an interesting task to generalize the Cauchy–Leray formula to the case of Stein manifolds M. First generalizations have been obtained in [16], [23], however, only for the special case where M is a complete intersection in some $\boldsymbol{C}^m$.

In the present paper we give a generalization of the Cauchy–Leray formula and formulas for solving the $\bar{\partial}$-equation on arbitrary Stein manifolds M. These formulas are especially useful for domains of the Weil polyhedron type, where, contrary to the strictly pseudoconvex case, for many problems, it is impossible to reduce the problem to the case $M = \boldsymbol{C}^n$ by localization. In particular, these formulas yield solutions of the $\bar{\partial}$-equation with uniform estimates for regular Weil polyhedra on arbitrary Stein manifolds. As a consequence one obtains, for example, a uniform approximation theorem for such domains. For piecewise strictly pseudoconvex domains in $\boldsymbol{C}^n$ as well as in general Stein manifolds, such results were obtained in [5], [7], [10], [13], [15], [18], [21]. For regular Weil polyhedra in $\boldsymbol{C}^n$, solutions of the $\bar{\partial}$-equation with uniform estimates are contained in [8] (see also [3]).

In the problem of extending the Cauchy–Leray formula to Stein manifolds, the point of gravity is contained in the following question: By what map should $\zeta - z$ be replaced? If there exist a collection $u = (u_1, \ldots, u_n)$ of holomorphic functions on $M \times M$ generating the ideal of the diagonal $\Delta(M)$

in the sheaf ${}_{M\times M}\mathcal{O}$ (that is, if $\Delta(M)$ is a complete intersection in $M\times M$), then as a Cauchy–Leray formula one can use the formula

$$f(z) = \int_{\zeta\in\partial D} f(\zeta)\,\Omega_\zeta(u^*(z,\zeta), u(z,\zeta)), \qquad z\in D,$$

where $u^*\colon D\times\partial D\to \mathbf{C}^n$ is a smooth map such that $\langle u^*(z,\zeta), u(z,\zeta)\rangle \neq 0$ for all $z\in D$ and $\zeta\in\partial D$. However, such functions do not exist, in general, even for parallelizable Stein manifolds M. Namely, it was shown by M. Schneider [22] that $\Delta(M)$ is a complete intersection in $M\times M$ if and only if M is parallelizable and, moreover, $\Delta(M)$ is zero in $H^{2n}(M\times M, \mathbf{Z})$.

1.2. In order to avoid this difficulty, in 1974 A. Dynin proposed to use some cross section in $\tilde{T}(M\times M)$ instead of a collection of functions. We will explain this.

Suppose that there is a holomorphic section $s(z,\zeta)\colon M\times M\to \tilde{T}(M\times M)$ which generates the ideal of $\Delta(M)$ in ${}_{M\times M}\mathcal{O}$, that is:

(i) *For every* $z\in M$

$$s(z,z) = 0$$

and the map

$$s(z,\cdot)\colon M\to T_z(M)$$

is biholomorphic in some neighbourhood of z,

(ii) $$s(z,\zeta)\neq 0 \quad \textit{for all } z\neq\zeta.$$

Now let

$$s^*(z,\zeta)\colon D\times\partial D\to \tilde{T}^*(M\times M)$$

be a smooth section such that

$$\langle s^*(z,\zeta), s(z,\zeta)\rangle \neq 0 \quad \text{for all } z\in D \text{ and } \zeta\in\partial D,$$

where $\langle s^*, s\rangle$ denotes the value of the covector s^* at s.

Then, for every fixed $z\in D$, we can define the form

$$\Omega^0(1, s^*, s; z) := \Omega_\zeta(s^*(z,\zeta), s(z,\zeta)), \qquad \zeta\in\partial D,$$

as follows: Let (U,h) be holomorphic coordinates in a neighbourhood of z, and let $u^*(\zeta), u(\zeta)\colon \partial D\to \mathbf{C}^n$ express $s^*(z,\zeta)$ and $s(z,\zeta)$ in the induced coordinates in $T_z^*(M)$ and $T_z(M)$, respectively. Then we define

$$\Omega^0(1, s^*, s; z) := \Omega(u^*, u).$$

In view of (0.6) this definition does not depend on the choice of local coordinates. Indeed, if $(\hat{U},\hat{h})$ are other coordinates, then $\hat{u}^* = (\partial\hat{h}/\partial h)_t^{-1}u^*$ and $\hat{u} = (\partial\hat{h}/\partial h)u$, where the Jacobi matrix $\partial\hat{h}/\partial h$ depends only on the fixed point z.

The form $\Omega^0(1, s^*, s; z)$ is continuous on ∂D and we get the following generalization of the Cauchy–Leray formula

$$f(z) = \int_{\partial D} f\Omega^0(1, s^*, s; z), \quad z \in D.$$

This formula is a partial case of Theorem 2.2.1 proved below.

1.3. Unfortunatly, this elegant approach does not work in general, because such a section $s(z, \zeta)$ does not exist, as a rule. As it was remarked by A. Dynin, it is not hard to construct a holomorphic section s in a neighbourhood of $\Delta(M)$ with property (i). To get a global section A. Dynin proposed to proceed as follows:

(1) extend this section to a continuous section on $M \times M$ with properties (i) and (ii);

(2) prove Oka's principle for this situation, i.e. prove the following proposition: If there is a continuous section s: $M \times M \to \tilde{T}(M \times M)$ holomorphic in a neighbourhood of $\Delta(M)$ and satisfies conditions (i) and (ii), then there is also a holomorphic section with these properties.

It turned out that Oka's principle is true for this situation [11]. However, in general, there are topological obstructions to realize the first step. In fact, by the result of M. Schneider cited above, there are parallelizable Stein manifolds M such that the diagonal $\Delta(M)$ is not a complete intersection in $M \times M$, but for a parallelizable manifold M the existence of a holomorphic section s with properties (i) and (ii) means that $\Delta(M)$ is a complete intersection in $M \times M$.

1.4. The Cauchy–Leray formula for arbitrary Stein manifolds given in the present paper is obtained by some combination of A. Dynin's idea with an idea of E. Bishop [2], Theorem 7. At first we construct a holomorphic section s: $M \times M \to \tilde{T}(M \times M)$ satisfying (i), whereas condition (ii) can be violated (see Lemma 2.1.1 below). Then we choose a holomorphic function φ on $M \times M$ such that $\varphi(z, z) = 1$ for all $z \in M$ and, for a sufficiently large integer $\varkappa$, the function $\varphi^{\varkappa}|s|^{-2}$ is C^1 on $(M \times M) \setminus \Delta(M)$ (see Lemmas 2.1.2 and 2.1.3 below). Now let

$$s^*(z, \zeta)\colon\ D \times \partial D \to T^*(M \times M)$$

be a smooth section such that for some integer $\varkappa^* \geqslant 0$ and every point $z \in D$ the function

$$\frac{\varphi^{\varkappa^*}(z, \zeta)}{\langle s^*(z, \zeta), s(z, \zeta)\rangle}, \quad \zeta \in \partial D,$$

is continuously differentiable([1]). Then, for every integer $\nu \geqslant \max\{\varkappa^*, \varkappa\}$ and each fixed $z \in D$, the form

$$\Omega^0(\varphi^\nu, s^*, s; z) := \varphi^{\nu n}(z, \cdot)\Omega^0(1, s^*, s; z)$$

is continuous on ∂D and we obtain the following Cauchy–Leray formula:

$$f(z) = \int_{\partial D} f\Omega^0(\varphi^\nu, s^*, s; z), \quad z \in D$$

(see Theorem 2.2.1 below).

1.5. As it was pointed out above, the reason for replacing n functions by a section was the non-existence of n functions generating the ideal of $\Delta(M)$ in ${}_{M\times M}\mathcal{O}$. However, also in the case of a section, it was impossible to fulfil condition (ii). Thus the question arises whether there exist, in general, n holomorphic functions on $M \times M$ which generate the ideal of $\Delta(M)$ in ${}_{M\times M}\mathcal{O}$, at least in some neighbourhood of $\Delta(M)$. The answer is negative, because such functions exist if and only if M is parallelizable (see [22]).

1.6. The further contents of the paper is as follows:

2. Integral formulas an arbitrary domains with piecewise smooth boundaries.

2.1. Construction of the section $s(z, \zeta)$ and the function $\varphi(z, \zeta)$.

2.2 Integral representations with one support function.

2.3. Integral representations with several support functions.

2.4. Generalization to $(0, q)$-forms.

3. Integral formulas on general analytic polyhedra.

3.1. The function $\Phi(z, \zeta)$ and solution of the equation $\langle s^*, s\rangle = \varphi\Phi$ on strictly pseudoconvex domains.

3.2. Formula for solving the $\bar\partial$-equation.

4. Uniform estimates for the $\bar\partial$-equation and a uniform approximation theorem (without proofs).

2. Integral formulas on arbitrary domains with piecewise smooth boundaries

2.1. Construction of the section $s(z, \zeta)$ and the function $\varphi(z, \zeta)$.

2.1.1. LEMMA. *There exists a holomorphic section* $s\colon M \times M \to \tilde T(M \times M)$ *satisfying condition* (i) *from Section* 1.3.

Proof. Let $f\colon M \to \boldsymbol{C}^m$ be a holomorphic imbedding and $E(z) = df(z)\,T_z(M)$ the image of the differential of f at z. Then the family

([1]) Using the properties of the section s and a partition of unity, there is no problem in finding such a section s^*, as long as we do not require additional properties as, for example, holomorphic dependence on z or growth conditions.

$\{E(z)\}_{z\in M}$ forms a holomorphic vector boundle. As was discovered by M. A. Shubin [24], it follows from Theorem B that there is a holomorphic $m\times m$ matrix function P on M such that $P^2 = P$ and $\operatorname{Im} P(z) = E(z)$ for all $z\in M$. Setting

$$s(z,\zeta) := (df(z))^{-1} P(z)\big(f(\zeta)-f(z)\big),$$

we conclude the proof.

Let s be the section from Lemma 2.1.1. We denote by $\mathscr{F}_s$ the analytic subsheaf of ${}_{M\times M}\mathscr{O}$ generated by s, i.e. by its components with respect to local holomorphic coordinates. From Theorem B we get

2.1.2. LEMMA. *There is a holomorphic function φ on $M\times M$ such that $\varphi(z,z) = 1$ for all $z\in M$, and the restriction of φ to $(M\times M)\setminus\Delta(M)$ belongs to $\mathscr{F}_s\big((M\times M)\setminus\Delta(M)\big)$.*

We also have the following lemma, which is easily verified.

2.1.3. LEMMA. *Let φ be the function from Lemma 2.1.2, and let $|s|$ be the modulus of s with respect to some smooth Hermitian metric in $T(M)$. Then there is an integer $\varkappa\geqslant 0$ such that the function $\varphi^{\varkappa}|s|^{-2}$ is C^1 on $(M\times M)\setminus\Delta(M)$. (It is enough to choose $\varkappa\geqslant 5$.)*

2.2. Integral representations with one support function.

Throughout this section $s(z,\zeta)$ is the section from Lemma 2.1.1, $\varphi(z,\zeta)$ is the function from Lemma 2.1.2 and $\varkappa$ is the integer from Lemma 2.1.3.

Suppose that

$$s^*(z,\zeta)\colon\ D\times\partial D\to \tilde T^*(M\times M)$$

is a section with the following property:

(i*) *There exists an integer $\varkappa^*\geqslant 0$ such that, for every fixed $z\in D$, the maps $s^*(z,\cdot)$ and $\varphi^{\varkappa^*}(z,\zeta)/\langle s^*(z,\zeta), s(z,\zeta)\rangle$ are C^1 on ∂D.*

Then, for every point $z\in D$, we can define the form $\Omega^0(1, s^*, s; z)$ on ∂D (cf. Section 1.2). This form is continuous for all points $\zeta\in D$ satisfying $\langle s^*(z,\zeta), s(z,\zeta)\rangle\neq 0$. At other points singularities can appear. However, in view of (i*), for each integer $\nu\geqslant\varkappa^*$, the form

$$\Omega^0(\varphi^\nu, s^*, s; z) := \varphi^{\nu-n}(z,\zeta)\,\Omega^0(1, s^*, s; z)$$

is continuous everywhere on ∂D.

2.2.1. THEOREM (Cauchy–Leray formula). *Let $\nu\geqslant\varkappa+\varkappa^*$ be an integer. Then we have*

$$f(z) = \int_{\partial D} f\,\Omega^0(\varphi^\nu, s^*, s; z),\qquad z\in D,$$

for every continuous function f on $\bar D$ holomorphic in D.

Proof. Fix $z \in D$. Let $u = \{u_j\}$ and $u^* = \{u_j^*\}$ stand for the maps $s(z, \cdot)$ and $s^*(z, \cdot)$ written in some holomorphic coordinates in a neighbourhood of z. Then by definition

$$\Omega^0(\varphi^\nu, s^*, s; z) = \varphi^{\nu n}(z, \cdot)\,\Omega(u^*, u). \tag{2.2.2}$$

We set

$$v^*(\zeta, \lambda) = \varphi^\varkappa(z, \zeta)\left[\lambda u^*(\zeta) + (1-\lambda)\frac{\langle u^*(\zeta), u(\zeta)\rangle\,\overline{u(\zeta)}}{|u(\zeta)|^2}\right]$$

for $\zeta \in \partial D$ and $0 \leqslant \lambda \leqslant 1$ and consider the form $\varphi^{\nu n}(z, \cdot)\,\Omega(v^*, u)$ defined on the bordered manifold $\partial D \times [0, 1]$. By Lemma 2.1.3, the map v^* is C^1 on $\partial D \times [0, 1]$. Since $\nu \geqslant \varkappa + \varkappa^*$ and $\langle v^*, u\rangle = \varphi^\varkappa \langle u^*, u\rangle$, it follows by (i*) that the form $\varphi^{\nu n}(z, \cdot)\,\Omega(v^*, u)$ is continuous on $\partial D \times [0, 1]$. It follows from (0.2) that

$$d_{\zeta,\lambda}\, f\varphi^{\nu n}(z, \cdot)\,\Omega(v^*, u) = 0. \tag{2.2.3}$$

Therefore, by Stokes' formula,

$$\int\limits_{\partial D \times 1} f\varphi^{\nu n}(z, \cdot)\,\Omega(v^*, u) = \int\limits_{\partial D \times 0} f\varphi^{\nu n}(z, \cdot)\,\Omega(v^*, u). \tag{2.2.4}$$

Taking into account that $v^*(\cdot, 0) = [\varphi^\varkappa(z, \cdot)\langle u^*, u\rangle/|u|^2]\,\bar{u}$ and $v^*(\cdot, 1) = \varphi^\varkappa(z, \cdot)u^*$, we get from (0.4) $\Omega(v^*(\cdot, 0), u) = \Omega(\bar{u}, u)$ and $\Omega(v^*(\cdot, 1), u) = \Omega(u^*, u)$. Thus it follows from (2.2.4) that

$$\int\limits_{\partial D} f\varphi^{\nu n}(z, \cdot)\,\Omega(u^*, u) = \int\limits_{\partial D} f\varphi^{\nu n}(z, \cdot)\,\Omega(\bar{u}, u). \tag{2.2.5}$$

In view of property (i) of section s, for some $\varepsilon > 0$, the vector function u maps some neighbourhood of z biholomorphically onto a neighbourhood of the ball $B_\varepsilon = \{\xi\colon |\xi| \leqslant \varepsilon\}$ in C^n.

It follows from (0.2) and Lemma 2.1.3 that the form $f\varphi^{\nu n}(z, \cdot)\,\Omega(\bar{u}, u)$ is continuous on $\bar{D}$ and closed in D. Consequently, using Stokes' formula, we get from (2.2.5) the relation

$$\begin{aligned}\int\limits_{\partial D} f\varphi^{\nu n}(z, \cdot)\,\Omega(u^*, u) &= \int\limits_{u^{-1}(\partial B_\varepsilon)} f\varphi^{\nu n}(z, \cdot)\,\Omega(\bar{u}, u) \\ &= \int\limits_{|\xi|=\varepsilon} f\big(u^{-1}(\xi)\big)\,\varphi^{\nu n}\big(z, u^{-1}(\xi)\big)\,\Omega_\xi(\xi, \xi).\end{aligned} \tag{2.2.6}$$

Now, by the Martinelli–Bochner formula, the right-hand side of this relation is equal to $f(u^{-1}(0))\,\varphi^{\nu n}(z, u^{-1}(0)) = f(z)$ (recall that $\varphi(z, z) = 1$). And so, (2.2.1) follows from (2.2.2) and (2.2.6). The theorem is proved.

Now we will generalize the Martinelli–Bochner formula for smooth functions f on $\bar{D}$. We recall that for $M = C^n$ this formula has the form

$$f(z) = \int\limits_{\zeta \in \partial D} f(\zeta)\,\Omega_\zeta(\bar{\zeta} - \bar{z}, \zeta - z) - \int\limits_{\zeta \in D} \bar{\partial} f(\zeta) \wedge \Omega_\zeta(\bar{\zeta} - \bar{z}, \zeta - z). \tag{2.2.7}$$

As in the case of the Cauchy–Leray formula, the map $\zeta - z$ will be replaced by the section $s(z,\zeta)$ from Lemma 2.1.1, and the map $\bar\zeta - \bar z$ will be replaced by some section in $\tilde T^*(M\times M)$. Contrary to the case of the Cauchy–Leray formula, the section in $T^*(M\times M)$ now must be defined for all $\zeta\in\bar D$. For this reason we introduce a map σ: $\tilde T(M\times M)\to \tilde T^*(M\times M)$, which corresponds to the map

$$z\to\bar z \tag{2.2.8}$$

in C^n: First, by (2.2.8), we can define some local mappings

$$\sigma_j\colon \tilde T(M\times M)|\,U_j\times M\to \tilde T^*(M\times M)|\,U_j\times M,$$

where $\{U_j\}$ is a locally finite open covering of M. Then we set

$$\sigma=\sum\chi_j\sigma_j,$$

where $\{\chi_j\}$ is a C^∞-partition of unity for $\{U_j\}$. The map σ so defined is C^∞, anti-linear and bijective in each fiber. Further, $\langle\sigma x, x\rangle > 0$ for all $x\neq 0$. Consequently, the map

$$x\to |x|_\sigma := \sqrt{\langle\sigma x, x\rangle}$$

is a smooth Hermitian metric on $\tilde T(M\times M)$, and the map

$$x^*\to |x^*|_\sigma := \sqrt{\langle x^*, \sigma^{-1}x^*\rangle}$$

is a smooth Hermitian metric on $\tilde T^*(M\times M)$.

We define

$$\bar s(z,\zeta)=\sigma s(z,\zeta). \tag{2.2.9}$$

Then, in view of Lemma 2.1.3, the function

$$\frac{\varphi^\varkappa(z,\zeta)}{\langle s(z,\zeta), s(z,\zeta)\rangle}=\frac{\varphi^\varkappa(z,\zeta)}{|s(z,\zeta)|^2_\sigma}$$

is C^1 for $z\neq\zeta$. (In particular, the section $\bar s$ has property (i*) for $\varkappa^*=\varkappa$.) Consequently, for each integer $\nu\geqslant\varkappa$ and for every point $z\in M$, the equation

$$\Omega^0(\varphi^\nu,\bar s, s; z) := \varphi^{\nu n}(z,\cdot)\,\Omega^0(1,\bar s, s; z)$$

defines a continuous form on $M\backslash z$ (the form $\Omega^0(1,\bar s, s; z)$ was defined in Section 1.3).

2.2.2. THEOREM (Martinelli–Bochner formula). *Let $\nu\geqslant 2\varkappa$ be an integer and let f be a continuous function on $\bar D$ such that the form $\bar\partial f$ is also continuous on $\bar D$. Then*

$$f(z)=\int_{\partial D} f\Omega^0(z)-\int_D \bar\partial f\wedge\Omega^0(z),\qquad z\in D, \tag{2.2.10}$$

where $\Omega^0(z)=\Omega^0(\varphi^\nu,\bar s, s; z)$.

Proof. First we remark that the second integral on the right of (2.2.10) converges, because the singularity of $\Omega^0(z)$ at $\zeta = z$ is of order $2n-1$.

Now we fix a point $z \in D$. For sufficiently small $\varepsilon > 0$, we denote by B_ε the open ball of radius ε centered at z (with respect to some fixed local coordinates). Since, by (0.2), $d(f\Omega^0(z)) = \bar\partial f \Omega^0(z)$, it follows from Stokes' formula

$$\int\limits_{D\setminus B_\varepsilon} \bar\partial f \wedge \Omega^0(z) = \int\limits_{\partial D} f\Omega^0(z) - \int\limits_{\partial B_\varepsilon} f\Omega^0(z). \tag{2.2.11}$$

In view of the Cauchy–Leray formula (2.2.1) we have

$$\int\limits_{\partial B_\varepsilon} f\Omega^0(z) = f(z) + \int\limits_{\partial B_\varepsilon} (f - f(z))\Omega^0(z). \tag{2.2.12}$$

Since the form $\Omega^0(z)$ has a singularity of order $2n-1$ at z, it follows that the integral on the right-hand side tends to zero as $\varepsilon \to 0$. Thus, (2.2.10) follows from (2.2.11) and (2.2.12) as $\varepsilon \to 0$. The theorem is proved.

Finally we will generalize a third formula. This formula was obtained in [7] in order to get a means for solving the $\bar\partial$-equation on strongly pseudoconvex domains in $\boldsymbol{C}^n$. In [7] this formula was called the Leray–Stokes formula.

Consider again the section $\bar s$ defined by (2.2.9). Moreover, assume that we have an additional section

$$s^*(z,\zeta)\colon \dot D \times \partial D \to \dot T^*(M \times M)$$

satisfying condition (i*) for some integer $\varkappa^* \geqslant 0$. For every integer $\nu \geqslant \max\{\varkappa, \varkappa^*\}$ and each point $z \in D$, we define

$$t^*(z,\zeta,\lambda) = \varphi^\nu(z,\zeta)\left[\lambda \frac{\bar s(z,\zeta)}{\langle \bar s(z,\zeta), s(z,\zeta)\rangle} + (1-\lambda)\frac{s^*(z,\zeta)}{\langle s^*(z,\zeta), s(z,\zeta)\rangle}\right]. \tag{2.2.13}$$

The map $t^*(z,\cdot,\cdot)$ is defined and is C^1 for all points $(\zeta,\lambda) \in (\partial D \times [0,1]) \cup \cup ((M\setminus z)\times 1)$. This follows from the relation $\varphi^\nu/\langle \bar s, s\rangle = \varphi^\nu/|s|^2_\sigma$, Lemma 2.1.3 and property (i*) of the section s^*.

As in Section 1.2, for every point $z \in D$ we can define a continuous form

$$\Omega^1(\varphi^\nu, \bar s, s^*, s; z) := \frac{(n-1)!}{(2i)^n}\,\omega'_{\zeta,\lambda}(t^*(z,\zeta,\lambda)) \wedge \omega_\zeta(s(z,\zeta))$$

on the set $(\partial D \times [0,1]) \cup ((M\setminus z)\times 1)$.

2.2.3. Lemma. *Let $\nu \geqslant \max\{\varkappa, \varkappa^*\}$ be an integer and f a continuous function on $\bar D$ such that $\bar\partial f$ is also continuous on $\bar D$. Then, for all $z \in D$.*

$$d_{\zeta,\lambda}(f\Omega^1(z)) = \bar\partial f \wedge \Omega^1(z) \quad \text{on } \partial D \times [0,1] \tag{2.2.14}$$

in the sense of distributions, where $\Omega^1(z) = \Omega^1(\varphi^\nu, \bar s, s^* s; z)$.

Proof. Since $\Omega^1(z)$ contains a factor of type $(n, 0)$, it suffices to prove that

$$(\bar\partial_\zeta + d_\lambda)\Omega^1(z) = 0 \quad \text{on } \partial D \times [0, 1]. \tag{2.2.15}$$

Let $u^* = \{u_j^*\}_1^n$ and $u = \{u_j\}_1^n$ express the maps $t^*(z, \cdot)$ and $s(z, \cdot)$, respectively, in some local holomorphic coordinates in a neighbourhood of z. Then

$$\Omega^1(z) = \frac{(n-1)!}{(2\pi i)^n}\, \omega'_{\zeta,\lambda}(u^*) \wedge \omega_\zeta(u).$$

Since $(\bar\partial_\zeta + d_\lambda)\omega'_{\zeta,\lambda}(u^*) = \bigwedge_1^n (\bar\partial_\zeta + d_\lambda) u_j^*$ and $(\bar\partial_\zeta + d_\lambda)\omega_\zeta(u) = 0$, it follows that

$$(\bar\partial_\zeta + d_\lambda)\Omega^1(z) = \frac{n!}{(2\pi i)^n} \bigwedge_{j=1}^n (\bar\partial_\zeta + d_\lambda) u_j^* \wedge \omega_\zeta(u). \tag{2.2.16}$$

From $\sum u_j^* u_j = \langle t^*, s\rangle = \varphi^\nu(z, \cdot)$ we get

$$\sum u_j^* u_j = \varphi^\nu(z, \cdot).$$

This implies the relation

$$\sum u_j (\bar\partial_\zeta + d_\lambda) u_j^* = 0,$$

because the functions u_j and $\varphi(z, \cdot)$ are holomorphic. Consequently,

$$\bigwedge_1^n (\bar\partial_\zeta + d_\lambda) u_j^* = 0,$$

which implies (2.2.15) in view of (2.2.16). The lemma is proved.

2.2.4. THEOREM (Leray–Stokes formula). *Let $\nu \geqslant \max\{2\varkappa, \varkappa^*\}$ be an integer, and let f be a continuous function on $\bar D$ such that $\bar\partial f$ is also continuous on $\bar D$. Then*

$$f(z) = \int_{\partial D \times 0} f\Omega^1(z) - \int_{\partial D \times [0,1]} \bar\partial f \wedge \Omega^1(z) - \int_{D \times 1} \bar\partial f \wedge \Omega^1(z), \quad z \in D, \tag{2.2.17}$$

where $\Omega^1(z) = \Omega^1(\varphi^\nu, \bar s, s^*, s; z)$.

Proof. Since on $(M \setminus z) \times 1$ the form $\Omega^1(z)$ coincides with $\Omega^0(\varphi^\nu, \bar s, s; z)$, we obtain from the Martinelli–Bochner formula (2.2.10) the relation

$$f(z) = \int_{\partial D \times 1} f\Omega^1(z) - \int_{D \times 1} \bar\partial f \wedge \Omega^1(z).$$

From Stokes' formula and Lemma 2.2.3 we get

$$\int_{\partial D \times [0,1]} \bar\partial f \wedge \Omega^1(z) = \int_{\partial D \times 0} f\Omega^1(z) - \int_{\partial D \times 1} f\Omega^1(z).$$

These two relations imply (2.2.17).

2.3. Integral representation with several support functions.

In this section we suppose that the domain D has a piecewise smooth

boundary in the following sense: There are a finite open covering $\{U_j\}_1^k$ of a neighbourhood U of ∂D and C^1-functions ϱ_j: $U_j \to \boldsymbol{R}$ $(j = 1, \ldots, k)$ such that

$$D \cap U = \{z \in U: \text{ for } j = 1, \ldots, k, \text{ either } z \notin U_j \text{ or } \varrho_j(z) < 0\}$$

and, for every collection $1 \leqslant i_1 < \ldots < i_l \leqslant k$, the forms $d\varrho_{i_1}, \ldots, d\varrho_{i_l}$ are linearly independent over $\boldsymbol{R}$ at every point of $U_{i_1} \cap \ldots \cap U_{i_l}$.

We call $\{U_j, \varrho_j\}_1^k$ a frame for D.

We define

$$S_j = \{z \in \partial D \cap U_j: \varrho_j(z) = 0\} \quad (j = 1, \ldots, k),$$

and for every ordered collection $I = (i_1, \ldots, i_l)$ of integers $1 \leqslant i_1, \ldots, i_l \leqslant k$ we set

$$S_I = \begin{cases} S_{i_1} \cap \ldots \cap S_{i_l} & \text{if the integers } i_1, \ldots, i_l \text{ are different in pairs,} \\ 0 & \text{otherwise.} \end{cases}$$

We choose the orientation on S_I inductively requiring that

$$\partial D = \sum_{j=1}^{k} S_j \quad \text{and} \quad \partial S_I = \sum_{j=1}^{k} S_{Ij}. \tag{2.3.1}$$

Let

$$\Delta := \{\lambda = (\lambda_0, \ldots, \lambda_k) \in \boldsymbol{R}^{k+1}: \ \lambda_j \geqslant 0 \text{ and } \sum_{j=1}^{k} \lambda_j = 1\}$$

be the standard k-simplex in $\boldsymbol{R}^{k+1}$ with the canonical orientation. For each collection $J = (j_1, \ldots, j_m)$ of integers $0 \leqslant j_1 < \ldots < j_m \leqslant k$, we define

$$\Delta_J = \{\lambda \in \Delta: \sum_{j \in J} \lambda_j = 1\}.$$

The orientation of each Δ_J is chosen so that

$$\partial \Delta_{j_1, \ldots, j_m} = \sum_{s=1}^{m} (-1)^{s+1} \Delta_{j_1, \ldots, \hat{j}_s, \ldots, j_m},$$

where $\hat{j}_s$ means that j_s is omitted. Then

$$\partial(S_I \times \Delta_J) = \partial S_I \times \Delta_J + (-1)^{|I|} S_I \times \Delta_J,$$

where $|I|$ is the number of components of I.

2.3.1. LEMMA (see [21]).

$$\Big(\sum_I (-1)^{|I|} S_I \times \Delta_{0I}\Big) = \sum_I S_I \times \Delta_I - \partial D \times \Delta_0,$$

where the summation is spread over all collections $I = (i_1, \ldots, i_l)$ *with* $1 \leqslant i_1 < \ldots < i_l \leqslant k$.

Let $s(z,\zeta)$ be the section from Lemma 2.1.1, $\varphi(z,\zeta)$ the function from Lemma 2.1.2, $\varkappa$ the integer from Lemma 2.1.3, and $s(z,\zeta)$ the section defined by (2.2.9).

Suppose, moreover, that there are sections

$$s_j^*(z,\zeta)\colon\ D\times S_j\to \tilde T^*(M\times M)\qquad (j=1,\dots,k)$$

satisfying the following condition:

$(\mathrm{i}^*)_k$ *There is an integer* $\varkappa^*\geqslant 0$ *such that, for every* $z\in D$, *the mappings*

$$s_j^*(z,\cdot)\quad \textit{and}\quad \frac{\varphi^{\varkappa^*}(z,\cdot)}{\langle s_j^*(z,\cdot),s(z,\cdot)\rangle}$$

are C^1 *on* S_j $(j=1,\dots,k)$.

For every $z\in D$ and each integer $\nu\geqslant\max\{\varkappa,\varkappa^*\}$, set

$$t^*(z,\zeta,\lambda)=\varphi^\nu(z,\zeta)\left[\lambda_0\frac{\bar s(z,\zeta)}{\langle\bar s(z,\zeta),s(z,\zeta)\rangle}+\sum_{j=1}^k\lambda_j\frac{s_j^*(z,\zeta)}{\langle s_j^*(z,\zeta),s(z,\zeta)\rangle}\right].$$

For each $z\in D$, this map is defined and is C^1 on the bordered manifold

$$\Big(\bigcup_I S_I\times\Delta_{0I}\Big)\cup\big((M\setminus z)\times\Delta_0\big),\tag{2.3.2}$$

where I runs over all collections $1\leqslant i_1<\dots<i_l\leqslant k$. This follows from Lemma 2.1.3 and property $(\mathrm{i}^*)_k$ of the sections s_j^*. Consequently, for each integer $\nu\geqslant\max\{\varkappa,\varkappa^*\}$ and every point $z\in D$, we can define a continuous form (cf. Section 1.2)

$$\Omega^k(\varphi^\nu,\bar s,s_1^*,\dots,s_k^*,s;z):=\frac{(n-1)!}{(2\pi i)^n}\,\omega'_{\zeta,\lambda}(t^*)\wedge\omega_\zeta(s)$$

on the bordered manifold (2.3.2).

2.3.2. Lemma. *Let* $\nu\geqslant\max\{\varkappa,\varkappa^*\}$ *be an integer and* f *a continuous function on* $\bar D$ *such that the form* $\bar\partial f$ *is also continuous on* $\bar D$. *Then, for every* $z\in D$,

$$d_{\zeta,\lambda}\big(f\Omega^k(z)\big)=\bar\partial f\wedge\Omega^k(z)\qquad \textit{on}\ \bigcup_I S_I\times\Delta_{0I}$$

in the sense of distributions, where $\Omega^k(z)=\Omega^k(\varphi^\nu,\bar s,s_1^*,\dots,s_k^*,s;z)$, *and* I *runs over all collections* $1\leqslant i_1<\dots<i_l\leqslant k$.

Proof. See the proof of Lemma 2.2.3.

We are now ready to give the following generalization of the Leray–Stokes formula (2.2.17).

2.3.3. Theorem (the k-Leray–Stokes formula). *Let* $\nu\geqslant\max\{2\varkappa,\varkappa^*\}$ *be an integer, and let* f *be a continuous function on* D *such that the form* $\bar\partial f$ *is also continuous on* D. *Then*

$$(2.3.3)\quad f(z) = \sum_{|I|\leqslant n} \int_{S_I\times\Delta_I} f\Omega^k(z) - \sum_{|I|\leqslant n-1} (-1)^{|I|} \int_{S_I\times\Delta_{0I}} \bar\partial f\wedge\Omega^k(z) - \int_{D\times\Delta_0} \bar\partial f\wedge\Omega^k(z),\quad z\in D,$$

where $\Omega^k(z) = \Omega^k(\varphi^\nu, \bar s, s_1^*, \dots, s_k^*, s; z)$ *and the summation is spread over all collections* $1\leqslant i_1 < \dots < i_l \leqslant k$ *with* $l\leqslant n$ *and* $l\leqslant n-1$, *respectively.*

In the case $M = \mathbf{C}^n$, $s(z,\zeta) = z-\zeta$ and $\varphi(z,\zeta)\equiv 1$ this formula was obtained in [8], [14], [21].

Proof. Since, on $(M\setminus z)\times\Delta_0$, the form $\Omega^k(z)$ coincides with $\Omega^0(\varphi^\nu, \bar s, s; z)$, by the Martinelle–Bochner formula (2.2.10) we have

$$(2.3.4)\quad f(z) = \int_{\partial D\times\Delta_0} f\Omega^k(z) - \int_{D\times\Delta_0} \bar\partial f\wedge\Omega^k(z).$$

It follows from Stokes' formula, Lemma 2.3.1 and Lemma 2.3.2 that

$$\int_{\partial D\times\Delta_0} f\Omega^k(z) = -\sum_I (-1)^{|I|} \int_{S_I\times\Delta_{0I}} \bar\partial f\wedge\Omega^k(z) + \sum_I \int_{S_I\times\Delta_I} f\Omega^k(z).$$

Together with (2.3.4) this implies (2.3.3) if we take into account that $\Omega^k(z)$ is of degree $\geqslant n$ in ζ and, consequently, the form $f\Omega^k(z)$ vanishes on $S_I\times\Delta_I$ when $\dim_R S_I < n$, and the form $\bar\partial f\wedge\Omega^k(z)$ vanishes on $S_I\times\Delta_{0I}$ if $\dim_R S_I < n-1$.

2.4. Generalization to $(0,q)$-forms.

In this section we give a generalization of the Martinelli–Bochner formula (2.2.10) to $(0,q)$-forms, i.e. a generalization of the Koppelman formula to Stein manifolds. From this one can get also generalizations of the Leray–Stokes and the k-Leray–Stokes formulas (2.2.17) and (2.3.3). We do not pursue this in detail (however, cf. Remark 3.2.3).

Let $v = \{v_j\}_1^n$ be a collection of C^1-functions on an open set $W\subseteq M\times M$. Then we define

$$\bar\omega'_{z,\zeta}(v) = \sum_{j=1}^n (-1)^{j-1} v_j \bigwedge_{s\neq j} \bar\partial_{z,\zeta} v_s \quad \text{on } W.$$

LEMMA 2.4.1. *Let* Γ *be a holomorphic* $n\times n$ *matrix function on* W. *Then*

$$(2.4.1)\quad \bar\omega'_{z,\zeta}(\Gamma v) = (\det\Gamma)\,\bar\omega'_{z,\zeta}(v).$$

Proof. We can just repeat the proof of (0.4), because $\bar\partial(\gamma v) = \gamma\bar\partial v$ for every holomorphic function γ.

Further, for every C^1-function ψ on W, we have

$$(2.4.2)\quad \bar\omega'_{z,\zeta}(\psi v) = \psi^n\,\bar\omega'_{z,\zeta}(v),$$

as is easily verified.

Let $s(z,\zeta)$ be the section from Lemma 2.1.1 and $\bar s(z,\zeta)$ the section defined by (2.2.9). Choose an integer $\varkappa$ so large that the function

$$\frac{\varphi^{\varkappa}(z,\zeta)}{\langle \bar s(z,\zeta), s(z,\zeta)\rangle}$$

is C^2 on $(M\times M)\setminus \Delta(M)$. Let $\nu \geqslant \varkappa$ be an integer.

It follows from Lemma 2.4.1 (cf. Section 1.2) that the form $\bar\omega'_{z,\zeta}(\bar s)\wedge\omega_\zeta(s)$ has an invariant meaning. Consequently, we can define by

$$(\tilde\Omega^0(\varphi^\nu, s, s))(z,\zeta) = \frac{(n-1)!}{(2\pi i)^n}\,\bar\omega'_{z,\zeta}\left(\frac{\varphi^\nu(z,\zeta)\bar s(z,\zeta)}{\langle \bar s(z,\zeta), s(z,\zeta)\rangle}\right)\wedge\omega_\zeta(s(z,\zeta))$$

a C^1-from on $(M\times M)\setminus\Delta(M)$.

We write

$$\tilde\Omega^0(\varphi^\nu,\bar s, s) = \sum_{q=0}^{n-1}\Omega^0_q(\varphi^\nu,\bar s, s),$$

where $\Omega^0_q(\varphi^\nu,\bar s, s)$ is of type $(n, n-q-1)$ in ζ and of type $(0,q)$ in z. Notice that, for fixed z, $\Omega^0_0(\varphi^\nu,\bar s, s)$ coincides with the form $\Omega^0(\varphi^\nu,\bar s, s; z)$ introduced in Section 1.3. For simplicity we write also $\tilde\Omega^0$ and Ω^0_q instead of $\tilde\Omega^0(\varphi^\nu,\bar s, s)$ and $\Omega^0_q(\varphi^\nu,\bar s, s)$ when there is no danger of confusion. We set

$$\Omega^0_{-1} = \Omega^0_n = 0.$$

2.4.2. LEMMA.

$$\bar\partial_{z,\zeta}\tilde\Omega^0 = 0 \tag{2.4.3}$$

and, consequently,

$$\bar\partial_\zeta\Omega^0_q = -\bar\partial_z\Omega^0_{q-1};\qquad q = 0,1,\ldots,n. \tag{2.4.4}$$

Proof. Fix a point $z_0\in M$. Let $\hat u = \{\hat u_j\}$ and $u = \{u_j\}$ express $\bar s$ and s in some local holomorphic coordinates in a neighbourhood $U(z_0)$ of z_0. Then

$$\tilde\Omega^0(z,\zeta) = \frac{(n-1)!}{(2\pi i)^n}\,\bar\omega'_{z,\zeta}\left(\frac{\varphi^\nu\hat u}{\langle\hat u, u\rangle}\right)\wedge\omega_\zeta(u)$$

and so

$$\bar\partial_{z,\zeta}\tilde\Omega^0(z,\zeta) = \frac{(n-1)!}{(2\pi i)^n}\bigwedge_{j=1}^{n}\bar\partial_{z,\zeta}\frac{\varphi^\nu\hat u_j}{\langle\hat u, u\rangle}\wedge\omega_\zeta(u) \tag{2.4.5}$$

on $U(z_0)\times M$. Since

$$\sum\frac{\varphi\hat u_j}{\langle\hat u, u\rangle}u_j = \varphi$$

and the functions u_j, φ are holomorphic, it follows that

$$\sum u_j \bar{\partial}_{z,\zeta} \frac{\varphi \hat{u}_j}{\langle \hat{u}, u \rangle} = 0.$$

Consequently,

$$\bigwedge_{j=1}^{n} \bar{\partial}_{z,\zeta} \frac{\varphi \hat{u}_j}{\langle \hat{u}, u \rangle} = 0.$$

Together with (2.4.5) this implies (2.4.3).

In the proof of the Martinelli–Bochner–Koppelman formula for $M = \boldsymbol{C}^n$, the transformation $(\xi, z) \to (z+\xi, z)$ is used (see, for example, [13], [15]). Under this transformation the map $\zeta - z$ becomes ξ, which does not depend on z. The following lemma gives a local transformation which plays a similar role for $s(z, \zeta)$.

2.4.3. LEMMA. *For every point $z_0 \in M$, there exist a neighbourhood $U(z_0)$ of z_0 and a holomorphic map*

$$\alpha(\xi, z)\colon\ U(z_0) \times U(z_0) \to M$$

satisfying the following conditions:

0° *There exist holomorphic coordinates on $U(z_0)$.*

1° $\alpha(\xi, z_0) = \xi$ *for all* $\xi \in U(z_0)$.

2° $\alpha(z_0, z) = z$ *for all* $z \in U(z_0)$.

3° *For every fixed $z \in U(z_0)$, the map $\alpha(\cdot, z)$ is biholomorphic from $U(z_0)$ onto a neighbourhood $U(z)$ of z.*

4° $s(z, \alpha(\xi, z)) = s(z_0, \xi)$ *and* $\bar{s}(z, \alpha(\xi, z)) = \bar{s}(z_0, \xi)$.

Proof. Let $V(z_0)$ be a neighbourhood of z_0 with holomorphic coordinates. Denote by $u(z, \zeta)\colon V(z_0) \times M \to \boldsymbol{C}^n$ the expression of $s(z, \zeta)$ in these coordinates. If $V(z_0)$ is sufficiently small, then it follows from property (i) of s that, for every $z \in V(z_0)$, the map $u_z := u(z, \cdot)$ is biholomorphic from $V(z_0)$ onto some neighbourhood $W_z(0)$ of $0 \in \boldsymbol{C}^n$. Choose a neighbourhood $U(z_0) \subseteq V(z_0)$ of z_0 such that $u_{z_0}(U(z_0)) \subseteq W_z(0)$ for all $z \in U(z_0)$. Setting

$$\alpha(\xi, z) := u_z^{-1}(u(z_0, \xi)) \quad \text{for } (\xi, z) \in U(z_0) \times U(z_0),$$

we obtain the required map.

2.2.4. THEOREM (Martinelli–Bochner–Koppelman formula). *Let $\nu \geqslant 2\varkappa$ be an integer, and let f be a continuous $(0, q)$-form on D such that $\bar{\partial} f$ is also continuous on $\bar{D}$, $0 \leqslant q \leqslant n$. Then*

$$(2.4.6)\qquad f(z) = (-1)^q \Big[\int_{\zeta \in \partial D} f(\zeta) \wedge \Omega_q^0(z, \zeta) - \int_{\zeta \in D} \bar{\partial}_\zeta f(\zeta) \wedge \Omega_q^0(z, \zeta) + \\ + \bar{\partial}_z \int_{\zeta \in D} f(\zeta) \wedge \Omega_{q-1}^0(z, \zeta) \Big], \qquad z \in D,$$

where $\Omega_q^0(z, \zeta) = \Omega_q^0(\varphi^\nu, \bar{s}, s)(z, \zeta)$.

For $M = \mathbf{C}^n$, $s(z,\zeta) = \zeta - z$, $\bar{s}(z,\zeta) = \bar{\zeta} - \bar{z}$ and $\varphi \equiv 1$, this formula was stated without proof in Koppelman's paper [10a]. Proofs are given in [13], [15], [18]. Theorem 2.4.4 can be proved by the idea from [15], using instead of the transformation $(\xi, z) \to (z+\xi, z)$ the local transformation $(\xi, z) \to$ $\to (\alpha(\xi, z), z)$ from Lemma 2.4.3. We omit this proof here; a complete proof is carried out in [8b].

3. Integral formulas on general analytic polyhedra

3.1. The function $\Phi(z,\zeta)$ and solution of the equation $\langle s^*, s\rangle = \Phi$ on strictly pseudoconvex domains.

In this section we suppose that D is a strictly pseudoconvex C^2-domain, that is, there is a strictly plurisubharmonic C^2-function ϱ on a neighbourhood Z of D such that

$$d\varrho(\zeta) \neq 0 \quad \text{for } \zeta \in \partial D$$

and

$$D = \{z \in Z\colon \varrho(z) < 0\}.$$

3.1.1. Lemma. *There are neighbourhoods $U_{\bar{D}}$ and $U_{\partial D}$ of $\bar{D}$ and ∂D, and a complex C^1-function $\Phi(z,\zeta)$ on $U_{\bar{D}} \times U_{\partial D}$ such that $\Phi(z,z) = 0$ for all $(z,z) \in U_{\bar{D}} \times U_{\partial D}$ and:*

1° *$\Phi(z,\zeta)$ is holomorphic on $U_{\bar{D}}$ for any fixed $\zeta \in U_{\partial D}$;*

2° *there are a neighbourhood $U_{\Delta(\partial D)} \subseteq U_{\bar{D}} \times U_{\partial D}$ of the diagonal $\Delta(\partial D)$ and a constant $C > 0$ such that*

$$|\Phi| > C \quad \text{on } (U_{\bar{D}} \times U_{\partial D}) \setminus U_{\Delta(\partial D)}$$

and

$$\Phi = AB \quad \text{on } U_{\Delta(\partial D)},$$

where $A(z,\zeta)$ and $B(z,\zeta)$ are C^1-functions on $U_{\Delta(\partial D)}$ which depend holomorphically on z and satisfy the following conditions:

$$|B| > C \quad \text{on } U_{\Delta(\partial D)}, \tag{3.1.1}$$

$$d_\zeta A(z,\zeta)|_{\zeta=z} = -d_z A(z,\zeta)|_{z=\zeta} = 2\partial\varrho(z) \quad \text{on } \Delta(\partial D) \tag{3.1.2}$$

and

$$\operatorname{Re} A(z,\zeta) \geqslant [\varrho(\zeta) - \varrho(z)] + C(\operatorname{dist}(z,\zeta))^2 \quad \text{on } U_{\Delta(\partial D)}, \tag{3.1.3}$$

where $\operatorname{dist}(\cdot,\cdot)$ *is a Riemannian distance on M.*

For $M = \mathbf{C}^n$ this lemma was proved in [6], [15], [19]. The case of an arbitrary Stein manifold M does not raise any essential new difficulties.

Let $s(z,\zeta)$ be the section from Lemma 2.1.1 and $\varphi(z,\zeta)$ the function from Lemma 2.1.2.

3.1.2. LEMMA. *Suppose $N \subseteq M$ is a C^1-manifold, $S \subseteq M$ is a Stein domain, and $\Psi(z,\zeta)$ is a complex C^1-function on $S \times N$ which depends holomorphically on $z \in S$. Then, for every compact set $K \subseteq S$, there are a neighbourhood $V_K \subseteq S$ of K and C^1-section*

$$s^*(z,\zeta)\colon\ V_K \times N \to \tilde{T}^*(M \times M)$$

which depends holomorphically on $z \in V_K$ such that

$$\varphi(z,\zeta)\Psi(z,\zeta) = \langle s^*(z,\zeta), s(z,\zeta)\rangle \quad \text{on } V_K \times N.$$

3.1.3. COROLLARY. *Let $\Phi\colon\ U_{\bar{D}} \times U_{\hat{\partial}D} \to \mathbf{C}$ be the function from Lemma 3.1.1. Then there exist a neighbourhood $V_{\bar{D}} \subseteq U_{\bar{D}}$ of $\bar{D}$ and a C^1-section*

$$s^*(z,\zeta)\colon\ V_{\bar{D}} \times U_{\hat{\partial}D} \to \tilde{T}^*(M \times M)$$

which depends holomorphically on $z \in V_{\bar{D}}$ such that

$$\varphi(z,\zeta)\Phi(z,\zeta) = \langle s^*(z,\zeta), s(z,\zeta)\rangle \quad \text{on } V_{\bar{D}} \times U_{\hat{\partial}D}. \tag{3.1.4}$$

For $M = \mathbf{C}^n$, $s(z,\zeta) = z - \zeta$ and $\varphi \equiv 1$, Corollary 3.1.3 was proved in [6], [19] (see also [15]). The proof from [6] (see also [15]) can be generalized to a proof of Lemma 3.1.2.

3.2. Formula for solving the $\bar{\partial}$-equation.

In this section we suppose that D is a general analytic polyhedron in the sense of [9]. That means, D is a domain with piecewise smooth boundary in the sense of Section 2.3, which is defined by a frame $\{U_j, \varrho_j\}_1^k$ of the following type:

$$\varrho_j = \alpha_j \circ \chi_j \quad (j = 1, \ldots, k),$$

where

1° $\chi_j\colon\ U_{\bar{D}} \to M_j$ is a holomorphic mapping from a neighbourhood $U_{\bar{D}}$ of $\bar{D}$ into some Stein manifold M_j with $\dim_{\mathbf{C}} M_j \leqslant \dim_{\mathbf{C}} M$, $j = 1, \ldots, k$;

2° α_j is a strictly plurisubharmonic C^2-function on M_j such that

$$D_j := \{w \in M_j\colon\ \alpha_j(w) < 0\}$$

is a relatively compact domain in M_j and

$$d\alpha_j \neq 0 \quad \text{on } \partial D_j,$$

$j = 1, \ldots, k$;

3° $D = \bigcap_{j=1}^{k} \chi_j^{-1}(D_j)$.

The main examples for general analytic polyhedra are the piecewise strictly pseudoconvex domains and the Weil polyhedra.

Let $s(z,\zeta)$ be the section from Lemma 2.1.1, $\varphi(z,\zeta)$ the function from Lemma 2.1.2, $\varkappa$ the integer from Lemma 2.1.3, and $\bar s(z,\zeta)$ the section defined by (2.2.9). We use the natation of Section 2.3. Additionally we suppose that the number $\varkappa$ is choosen so large that the function

$$\frac{\varphi^{\varkappa}(z,\zeta)}{\langle \bar s(z,\zeta), s(z,\zeta)\rangle} \tag{3.2.1}$$

is C^2 on $(M\times M)\setminus \Delta(M)$.

Applying Lemma 3.1.1 to the domains D_j, we find neighbourhoods $U_{\bar D_j}$ and $U_{\partial D_j}$ of $\bar D_j$ and ∂D_j, as well as functions Φ_j on $U_{\bar D_j}\times U_{\partial D_j}$ with the properties listed in Lemma 3.1.1. Then in some neighbourhood of $\bar D\times \partial D$ we can define the functions

$$\Psi_j(z,\zeta) := \Phi_j(\chi_j(z),\chi_j(\zeta)).$$

From Lemma 3.1.2 we obtain neighbourhoods $V_{\bar D}$, V_{S_j} of $\bar D$, S_j, respectively, and C^1-sections

$$s_j^*(z,\zeta)\colon\ V_{\bar D}\times V_{S_j}\to \tilde T^*(M\times M),$$

which depend holomorphically on $z\in V_{\bar D}$ and are such that

$$\varphi(z,\zeta)\Psi_j(z,\zeta) = \langle s_j^*(z,\zeta), s(z,\zeta)\rangle \quad \text{on } V_{\bar D}\times V_{S_j}.$$

It follows from (3.1.3) that $\Psi_j(z,\zeta)\neq 0$ for $z\in D$ and $\zeta\in S_j$. Since $\varphi/\langle s_j^*, s\rangle = 1/\Psi_j$, this implies that the maps

$$\frac{\varphi(z,\zeta)}{\langle s_j^*(z,\zeta), s(z,\zeta)\rangle} \tag{3.2.2}$$

are C^1 on $D\times S_j$ and depend holomorphically on $z\in D$. Let $\nu\geqslant \max\{1,\varkappa\}$ be an integer. We set

$$l^*(z,\zeta,\lambda) = \varphi^{\nu}(z,\zeta)\left[\lambda_0\frac{\bar s(z,\zeta)}{\langle \bar s(z,\zeta), s(z,\zeta)\rangle} + \sum_{j=1}^{k}\lambda_j\frac{s_j^*(z,\zeta)}{\langle s_j^*(z,\zeta), s(z,\zeta)\rangle}\right].$$

This section is defined over the bordered manifold

$$\Big[\bigcup_I (D\times S_I\times \Delta_{0I})\Big]\cup\big[((M\times M)\setminus\Delta(M))\times\Delta_0\big], \tag{3.2.3}$$

where I runs over all collections $1\leqslant i_1<\ldots<i_l\leqslant k$.

For any collection $v=\{v_j(z,\zeta,\lambda)\}_1^n$ of C^1-functions on (3.2.3), we define the form

$$\bar\omega'_{z,\zeta,\lambda}(v(z,\zeta,\lambda)) = \sum_{j=1}^{n}(-1)^{j-1}v_j(z,\zeta,\lambda)\bigwedge_{s\neq j}(\bar\partial_{z,\zeta}+d_\lambda)v_s(z,\zeta,\lambda).$$

Clearly, Lemma 2.4.1 and its proof hold also for this form. And so, for $\nu \geqslant \max\{1, \varkappa\}$ we can define the form

$$\tilde{\Omega}^k(\varphi^\nu, \bar{s}, s_1^*, \ldots, s_k^*, s)(z, \zeta, \lambda) = \frac{(n-1)!}{(2\pi i)^n}\, \bar{\omega}'_{z,\zeta,\lambda}\big(t^*(z, \zeta, \lambda)\big) \wedge \omega_\zeta\big(s(z, \zeta)\big).$$

We write

$$\tilde{\Omega}^k(\varphi^\nu, \bar{s}, s_1^*, \ldots, s_k^*, s) = \sum_{q=1}^{n-1} \Omega_q^k(\varphi^\nu, \bar{s}, s_1^*, \ldots, s_k^*, s),$$

where $\Omega_q^k(\varphi^\nu, \bar{s}, s_1^*, \ldots, s_k^*, s)$ is of degree $2n-q-1$ in (ζ, λ) and q in z. We set $\Omega_{-1}^k = \Omega_n^k = 0$.

3.2.1. LEMMA.

$$(\bar{\partial}_{\zeta,z} + d_\lambda)\tilde{\Omega}^k = 0$$

and, consequently,

$$(\bar{\partial}_\zeta + d_\lambda)\Omega_q^k = -\bar{\partial}_z \Omega_{q-1}^k; \quad q = 0, 1, \ldots, n.$$

Proof. See the proof of Lemma 2.4.2.

Remark. It follows from the assumption imposed on the functions (3.2.1) and (3.2.2) that the forms $\bar{\partial}_z \Omega_{q-1}^k$ are continuous. And so, Lemma 3.2.1, in particular, implies that the forms $(\bar{\partial}_\zeta + d_\lambda)\Omega_q^k$ are continuous.

3.2.2. THEOREM. *Let $\nu = \max\{1, 2\varkappa\}$ be an integer, $\Omega_q^k = \Omega_q^k(\varphi^\nu, \bar{s}, s_1^*, \ldots, s_k^*, s)$, and $1 \leqslant q \leqslant n$. Then, for every continuous $(0, q)$-form g on $\bar{D}$ such that $\bar{\partial}g$ is also continuous on $\bar{D}$, we have*

$$\begin{aligned}(3.2.4)\quad g(z) = \bar{\partial}_z(-1)^q\Big[\sum_{|I|\leqslant n-q}(-1)^{|I|}\int\limits_{(\zeta,\lambda)\in S_I\times\Delta_{0I}} g(\zeta)\wedge\Omega_{q-1}^k(z,\zeta,\lambda) + \\ + \int\limits_{(\zeta,\lambda)\in D\times\Delta_0} g(\zeta)\wedge\Omega_{q-1}^k(z,\zeta,\lambda)\Big] + \\ + (-1)^{q+1}\Big[\sum_{|I|\leqslant n-q-1}(-1)^{|I|}\int\limits_{(\zeta,\lambda)\in S_I\times\Delta_{0I}}\bar{\partial}_\zeta g(\zeta)\wedge\Omega_q^k(z,\zeta,\lambda) + \\ + \int\limits_{(\zeta,\lambda)\in D\times\Delta_0}\bar{\partial}_\zeta g(\zeta)\wedge\Omega_q^k(z,\zeta,\lambda)\Big], \quad z\in D,\end{aligned}$$

where the summation is spread over all collections $1 \leqslant i_1 < \ldots < i_l \leqslant k$ of length $l \leqslant n-q$ and $l \leqslant n-q-1$, respectively.

Consequently, for $\bar{\partial}g = 0$, the continuous $(0, q-1)$-form

$$\begin{aligned}(3.2.5)\quad f(z) := (-1)^q\Big[\sum_{|I|\leqslant n-q}(-1)^{|I|}\int\limits_{(\zeta,\lambda)\in S_I\times\Delta_{0I}} g(\zeta)\wedge\Omega_{q-1}^k(z,\zeta,\lambda) + \\ + \int\limits_{(\zeta,\lambda)\in D\times\Delta_0} g(\zeta)\wedge\Omega_{q-1}^k(z,\zeta,\lambda)\Big], \quad z\in D,\end{aligned}$$

is a solution of the equation

$$\bar{\partial}f = g \quad \text{on } D.$$

Proof. Since Ω_q^k contains a factor of type $(n,0)$ in ζ, it follows from Lemma 3.2.1 that

$$d_{\zeta,\lambda}(g\wedge\Omega_q^k)=\bar\partial_\zeta g\wedge\Omega_q^k+(-1)^{q+1}g\wedge\bar\partial_z\Omega_{q-1}^k.$$

And so, by Lemma 2.3.1, Stokes' formula gives

$$(3.2.6)\quad \sum_I(-1)^{|I|}\int\limits_{S_I\times\Delta_{0I}}\bar\partial g\wedge\Omega_q^k+(-1)^{q+1}\sum_I(-1)^{|I|}\int\limits_{S_I\times\Delta_{0I}}g\wedge\bar\partial_z\Omega_{q-1}^k$$
$$=\sum_I\int\limits_{S_I\times\Delta_I}g\wedge\Omega_q^k-\int\limits_{\partial D\times\Delta_0}g\wedge\Omega_q^k=-\int\limits_{\partial D\times\Delta_0}g\wedge\Omega_q^k,$$

as $\Omega_q^k=0$ on $S_I\times\Delta_I$, because $q>0$ and the sections s_j^* and s depend holomorphically on z.

For $\lambda\in\Delta_0$, the form Ω_q^k coincides with Ω_q^0. Consequently, it follows from the Martinelli–Bochner–Koppelman formula (2.4.9) that

$$\int\limits_{\partial D\times\Delta_0}g\wedge\Omega_q^k=(-1)^q g(z)+\int\limits_{D\times\Delta_0}\bar\partial g\wedge\Omega_q^k-\bar\partial_z\int\limits_{D\times\Delta_0}g\wedge\Omega_{q-1}^k.$$

Together with (3.2.6) this implies (3.2.4) with the only distinction that summation runs all collections I of arbitrary length. Notice that

$$\int\limits_{S_I\times\Delta_{0I}}f\wedge\Omega_{q-1}^k=0$$

for $|I|>n-q$, because in this case $\dim_R S_I=2n-|I|<n+q$, whereas the form $f\wedge\Omega_{q-1}^k$ contains a factor of degree $q+n$ in ζ. By the same argument,

$$\int\limits_{S_I\times\Delta_{0I}}\bar\partial f\wedge\Omega_q^k=0\quad\text{for } |I|>n-q-1.$$

The theorem is proved.

3.2.3. Remark. The assumption that the sections $s_j^*(z,\zeta)$ depend holomorphically on z was used only for the proof of the relations

$$\sum_I\int\limits_{S_I\times\Delta_I}g\wedge\Omega_q^k=0,\quad q>0.$$

We must add these integrals to the sum on the right of (3.2.4) when the sections $s_j(z,\zeta)$ only smoothly depend on z. For $q=0$, we then obtain the k-Leray–Stokes formula (2.3.3). The formula obtained for $q>0$ can be considered as a generalization of the k-Leray–Stokes formula to $(0,q)$-forms.

4. Uniform estimates for the $\bar\partial$-equation and an approximation theorem

Formula (3.2.5) for solving the $\bar\partial$-equation enables us to get solutions with uniform estimates for several classes of general analytic polyhedra on Stein manifolds. For strongly pseudoconvex domains on Stein manifolds such estimates can be obtained also with use of local formulas only [10] (see also [8a]). However, for Weil polyhedra on Stein manifolds, formula (3.2.5) is the first one to give this possibility.

Suppose that D is a classical Weil analytic polyhedron that is, D is a general analytic polyhedron as defined in Section 3.2, where $\dim_C M_j = 1$ for all j. We set

$$\|g\|_{C(D)} = \sup_{z \in D} |g(z)|_{\wedge^q T^*},$$

for q-forms g on D, where $|g(z)|_{\wedge^q T^*}$ is the modulus of $g(z)$ with respect to some smooth Hermitian metric in $\bigwedge^q T^*(M)$.

4.1. THEOREM. *Suppose that*

$$\bar\partial \varrho_{i_1} \wedge \ldots \wedge \bar\partial \varrho_{i_l} \neq 0 \quad \text{on } S_{i_1,\ldots,i_l} \tag{4.1}$$

for all $1 \leqslant i_1 < \ldots < i_l \leqslant k$ $(l = n-q+1)$. *Then there is a constant* $\gamma = \gamma(D) < \infty$ *such that for every continuous* $(0, q-1)$*-form* g *on* $\bar D$ *with* $\bar\partial g = 0$, *there exists a continuous* $(0, q-1)$*-form* f *on* $\bar D$ *such that*

$$\bar\partial f = g \quad \text{on } D \quad \text{and} \quad \|f\|_{C(D)} \leqslant \gamma \|g\|_{C(D)}.$$

For $M = C^n$ this theorem was proved by G. M. Henkin for $q = 1$ and by P. L. Poljakov for $q > 1$ (see [3], [8]).

For $\dim_C M \leqslant 2$ condition (4.1) can be replaced by a weaker one:

4.2. THEOREM. *Let* $\dim_C M \leqslant 2$ *and*

$$\bar\partial \varrho_i \wedge \bar\partial \varrho_j \not\equiv 0 \quad \text{on } S_{ij}$$

for all $1 \leqslant i < j \leqslant k$. *Then the conclusion of Theorem* 4.1 *is true.*

For polynomial polyhedra in C^2 this theorem was proved in [17].

The proofs of these theorems and their generalization will be given in an other place.

Finally, let us state the following

4.3. COROLLARY. *Let the hypothesis of Theorem* 4.1 *or Theorem* 4.2 *be fulfilled. Then for every* $\varepsilon > 0$ *and every continuous function* f *on* $\bar D$ *holomorphic in* D, *there exists a holomorphic function* f_ε *in a neighbourhood of* $\bar D$ *such that*

$$\|f - f_\varepsilon\|_{C(D)} < \varepsilon.$$

Note added in proof. In paper [23a] uniform estimates for the $\bar\partial$-equation on "non-degenerate" general analytic polyhedra in C^n are obtained. Using this work and the formulas of the present paper one can obtain similar results on Stein manifolds.

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