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A VERSION OF NON-HAMILTONIAN LIOUVILLE EQUATION

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Abstract

In this paper we give a version of the theorem on local integral invariants of systems of ordinary differential equations. We give, as an immediate conclusion of this theorem, a condition which guarantees existence of an invariant measure of local dynamical systems. Results of this type lead to the Liouville equation and have been frequently proved under various assumptions. Our method of the proof is simpler and more direct.

Keywords: Liouville equation, invariant measure.

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1. Introduction

The first results containing the Hamiltonian version of the Liouville equation have been derived from the early twentieth century. The Liouville equation describes changes in time of the probability density function of the particle in phase space and it became an essential tool of classical and statistical mechanics. Non-Hamiltonian but the classical version of the Liouville equation appeared in later papers. It was also shown that the function which is the integral invariant of a dynamical system must satisfy this equation. These results play a fundamental role in the theory of certain stochastic differential equations. Various types of problems associated with the Liouville equation are still open issues and still new results concerning this subject appear [1, 2, 3, 4, 6]. Various stochastic versions of the Liouville equation are given in many of the present papers. These results are related to physical systems in which there is a white noise effect. In this paper a generalization of the classical result of the integral invariant of a dynamical

system is given. We consider here systems of differential equations for which the Lipschitz condition is fulfilled only locally. We describe the way we understand the concept of the integral invariant in such case and then we show the equivalent form of the Liouville equation. As an immediate application of our result we obtain a condition under which invariant measures of local dynamical systems exist. Our result can be useful to solve certain type of stochastic differential equations.

2. Preliminaries

We begin with the following notation and definitions. Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{N}$ denote the set of positive integer numbers. Given $a, b \in \mathbb{R}$, $a < b$, the closed interval in $\mathbb{R}$ with ends a and b will be denoted by $[a, b]$. Let $A \times B$ be the cartesian product of sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, $n, m \in \mathbb{N}$. We will denote by L_n, $n \in \mathbb{N}$, the family of Lebesgue measurable sets in $\mathbb{R}^n$ and by μ the Lebesgue measure defined on L_n. Moreover let $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$, $i, k \in \mathbb{N}$.

Let us fix a positive integer n, real numbers a, b such that $a < b$ and an open set $G \subset \mathbb{R}^n$. Let $D = [a, b] \times G$. We will consider the system of ordinary differential equations of the form

$$y_1' = F_1(t, y_1, \ldots, y_n), \ldots, y_n' = F_n(t, y_1, \ldots, y_n), \tag{1}$$

with initial conditions

$$y_1(t_0) = y_{01}, \ldots, y_n(t_0) = y_{0n}, \tag{2}$$

where functions $F_1, \ldots, F_n$ are defined on the set D, $(y_{01}, \ldots, y_{0n}) \in G$ and $t_0 \in [a, b]$.

We will assume that functions $F_1, \ldots, F_n$ are continuous on D and have continuous partial derivatives $\frac{\partial F_i}{\partial y_k}$ on D for $i, k = 1, \ldots, n$.

Let $y = (y_1, \ldots, y_n) \in G$ and $y_0 = (y_{01}, \ldots, y_{0n}) \in G$. Let us fix $t_0 \in [a, b]$ and $y_0 \in G$. We will denote by $y(t, y_0) = (y_1(t, y_0), \ldots, y_n(t, y_0))$, $t \in J_{t_0, y_0}$, the saturated solution of system (1) with the initial condition (2), where J_{t_0, y_0} denotes a maximal interval on which it is defined.

We are going to specify the way we understand the concept of the local integral invariant of system (1).

Definition. Let $A \in L_n$, $A \subset G$ and $t_0 \in [a, b]$. Let $I_{t_0, A} = \bigcap_{y_0 \in A} J_{t_0, y_0}$. Let $\phi_{t_0, t}(y_0) = y(t, y_0)$, for $t \in I_{t_0, A}$ and $y_0 \in A$. Let us assume that $f : D \to \mathbb{R}$ is a function with nonnegative values, with nonpositive values or Lebesgue integrable with respect to the variable $y \in G$. We will call such a function a local integral

invariants of system (1), if the following condition is satisfied

$$\int_A f(t_0, y)\, d\mu = \int_{\phi_{t_0,t(A)}} f(t, y)\, d\mu \tag{3}$$

for all $t_0 \in [a, b]$, $A \in L_n$, $A \subset G$ and for all $t \in I_{t_0,A}$, where $\int_A f(t_0, y)\, d\mu$ and $\int_{\phi_{t_0,t(A)}} f(t, y)\, d\mu$ are integrals with respect to the variable $y \in G$ with respect to the Lebesgue measure in $\mathbb{R}^n$ on sets A and $\phi_{t_0,t(A)}$, respectively.

Remark 1. Let $t_0 \in [a, b]$, $A \in L_n$ and $A \subset G$ are such that $I_{t_0,A} \neq \{t_0\}$. It follows from theorems on dependence of solution $y(t, y_0)$ to the system (1) on the initial condition $y_0 \in G$, that if our assumptions for the system are satisfied then for $t \in I_{t_0,A}$ there exists an open set $U \subset G$ such that

(a) $A \subset U$,
(b) $[t_0, t] \subset I_{t_0,U}$ for $t > t_0$,
(c) $[t, t_0] \subset I_{t_0,U}$ for $t < t_0$,

and $\phi_{t_0,t} : U \to \phi_{t_0,t}(U)$ is a homeomorphism, $\phi_{t_0,t}(A) \in L_n$ for every $t \in I_{t_0,a}$ and our definition is correctly specified.

Moreover, if all saturated solutions of system (1) are defined on whole interval $[a, b]$, then our definition of the local integral invariant of system (1) coincides with the classical definition of the local integral invariant of the dynamical system generated by system (1).

3. Our results

Under the notation of the previous section, we will prove the following theorem.

Theorem 2. *Let an n be a fixed positive integer number. Let $f : D \to \mathbb{R}$ be a function with nonnegative values, with nonpositive values or Lebesgue integrable with respect to a variable $y \in G$ for each fixed $t \in [a, b]$. Moreover let the function f and all its partial derivatives be continuous in D. Then the function f is the local integral invariant of system* (1) *if and only if the equation*

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \frac{\partial (f F_i)}{\partial y_i} = 0 \tag{4}$$

is satisfied for all $(t, y) \in D$.

Equation (4) is known as the Liouville equation. If this equation is satisfied then the function f is the integral invariant of system (1). This fact has been frequently proved under various assumptions. We will show that our version of

this theorem is also valid. Our method of the proof is simpler and more direct then those given in earlier papers concerning this subject.

We will precede the proof of Theorem 2 by the following lemma.

Lemma 3. *Let $K \subset G$ be a bounded closed set, $t_0 \in [a,b]$ and $I_{t_0,K} \neq \{t_0\}$. Let $t = t_0 + \Delta t \in I_{t_0,K}$.*

Let $L \subset \mathbb{R}$ be a closed interval such that $0 \in L$ and 0 is not an endpoint of L if $t_0 \neq a, b$. Moreover, $t \in I_{t_0,K}$ and $t \in I_{t_0,U}$ for $\Delta t \in L$, where $U \subset G$ is some open set satisfying conditions (a), (b), (c) *of Remark* 1.
Then the following conditions hold

(1) $y_i(t, y_0) = y_{0i} + F(t_0, y_0)\Delta t + r_i(\Delta t, y_0)\Delta t$, *for* $y_0 \in K$, $i = 1, \ldots, n$, *where values* $r_i(\Delta t, y_0)$ *are uniformly bounded for* $(\Delta t, y_0) \in L \times K$ *and* $r_i(\Delta t, y_0) \to_{\Delta t \to 0} 0$ *for all* $y_0 \in K$, $i = 1, \ldots, n$.

(2) *The partial derivatives* $\frac{\partial \phi_{t_0,t}(y_0)}{\partial y_{0i}}$, $i = 1, \ldots, n$, *exist and the Jacobian of the function* $y_0 \to \phi_{t_0,t}(y_0)$, $y_0 \in U$, *is given by the formula*

$$Jac\phi_{t_0,t}(y_0) = 1 + \sum_{i=1}^{n} \frac{\partial F_i}{\partial y_i}(t_0, y_0)\Delta t + \gamma(\Delta t, y_0)\Delta t,$$

where values $\gamma(\Delta t, y_0)$ *are uniformly bounded for* $(\Delta t, y_0) \in L \times K$ *and* $\gamma(\Delta t, y_0) \to_{\Delta t \to 0} 0$ *for all* $y_0 \in U$.

Proof. Let $y_0 \in G$, $t_0 \in [a,b]$, and

$$y_i'(t, y_0) = F_i(t, y(t, y_0)), \tag{5}$$

for $i = 1, \ldots, n$, $t \in J_{t_0,y_0}$.

From the Lagrange theorem cf. [5], it follows that

$$y_i(t, y_0) = y_i(t_0, y_0) + y_i'(t_0, y_0)\Delta t + \left(y_i'(\theta, y_0) - y_i'(t_0, y_0)\right)\Delta t \tag{6}$$

with θ between t_0 and t, $t \in J_{t_0,y_0}$, $i = 1, \ldots, n$. Hence we get

$$y_i(t, y_0) = y_{0i} + F_i(t_0, y_0)\Delta t + r_i(\Delta t, y_0)\Delta t, \tag{7}$$

for $y_0 \in K$, $t_0 \in I_{t_0,U}$ and $t = t_0 + \Delta t \in I_{t_0,U}$, where $r_i(\Delta t, y_0) = F_i(\theta, y(\theta, y_0)) - F_i(t_0, y_0)$ for some θ between t_0 and t.

Note that $y(t, y_0)$ is continuous on $I_{t_0,U} \times U$. This fact is a consequence of theorems on dependence of solution of system (1) on the initial condition. Moreover we can assume that (θ, y_0) belongs to some compact set included in $I_{t_0,K} \times K$ for $(\Delta t, y_0) \in L \times K$. Taking into account the continuity of functions F_i on D, $i = 1, \ldots, n$, and formulas of r_i, $i = 1, \ldots, n$, we know that the values

$r_i(\Delta t, y_0)$, $i = 1, \ldots, n$, are bounded as values of a continuous function on a compact set and that $r_i(\Delta t, y_0) \to_{\Delta t \to 0} 0$ for all $y_0 \in K$, $i = 1, \ldots, n$.

We are going to prove the property (2) now.
By hypotheses the partial derivatives $\frac{\partial y_i}{\partial y_{0k}}(t, y_0)$ exist for $i, k = 1, \ldots, n$, $(t, y_0) \in I_{t_0,U} \times U$ and they are continuous on $I_{t_0,U} \times U$. Moreover, we have

$$y_i(t, y_0) = y_{0i} + \int_{t_0}^{t} F_i(s, y(s, y_0))\, ds$$

for $i = 1, \ldots, n$.

Computing a partial derivative with respect to y_{0k} of both sides of the above equality we obtain

$$\begin{aligned}
\frac{\partial y_i}{\partial y_{0k}}(t, y_0) &= \delta_{ik} + \frac{\partial}{\partial y_{0k}} \int_{t_0}^{t} F_i(s, y(s, y_0))\, ds \\
&= \delta_{ik} + \int_{t_0}^{t} \frac{\partial F_i}{\partial y_{0k}}(s, y(s, y_0))\, ds \\
&= \delta_{ik} + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial F_i}{\partial y_j}(s, y(s, y_0)) \frac{\partial y_j}{\partial y_{0k}}(s, y_0)\, ds \qquad (8) \\
&= \delta_{ik} + \sum_{j=1}^{n} \frac{\partial F_i}{\partial y_j}(\theta, y(\theta, y_0)) \frac{\partial y_j}{\partial y_{0k}}(\theta, y_0)\, \Delta t \\
&= \delta_{ik} + \frac{\partial F_i}{\partial y_k}(t_0, y_0)\, \Delta t + h_{ik}(\Delta t, y_0)\, \Delta t
\end{aligned}$$

where θ is between t_0 and $t = t_0 + \Delta t$, and

$$h_{ik}(\Delta t, y_0) = \sum_{j=1}^{n} \frac{\partial F_i}{\partial y_j}(\theta, y(\theta, y_0)) \frac{\partial y_j}{\partial y_{0k}}(\theta, y_0) - \frac{\partial F_i}{\partial y_k}(t_0, y_0). \qquad (9)$$

Taking into account the continuity of functions $y(t, y_0)$ and $\frac{\partial y_i}{\partial y_{0k}}$, $i, k = 1, \ldots, n$, on $I_{t_0,U} \times U$, continuity of partial derivatives $\frac{\partial F_i}{\partial y_k}$ on D and the formula for $h_{ik}(\Delta t, y_0)$, $i, k = 1, \ldots, n$, we obtain that values $h_{i,k}(\Delta t, y_0)$, $i, k = 1, \ldots, n$, as values of a continuous function on a compact set, are uniformly bounded for $(\Delta t, y_0) \in L \times K$. We assume that (θ, y_0) belongs to some compact set. Moreover, we have $\frac{\partial y_j}{\partial y_{0k}}(\theta, y_0) \to_{\Delta t \to 0} \delta jk$, so $h_{i,k}(\Delta t, y_0) \to_{\Delta t \to 0} 0$ for all $y_0 \in K$, $i, k = 1, \ldots, n$.

The property (2) of Lemma 3 is now an immediate consequence of (8). ∎

Now we are in a position to prove Theorem 2.

Proof. We will first prove the equivalence of the condition (3) defining the local integral invariant and the condition (4) from Theorem 1 for a closed and bounded set $K \subset G$.

Let $t_0 \in [a,b]$, $I_{t_0,K} \neq \{t_0\}$ and $t \in I_{t_0,K}$. Let $g_K(t) = \int_{\phi_{t_0,t}(K)} f(t,y)\, d\mu$. Observe that the function g_K is constant on $I_{t_0,K}$ if and only if $g'_K(t) = 0$ for all $t \in I_{t_0,K}$. Remark that if $K \subset G$ is a closed and bounded set, then $\phi_{t_0,t}(K) \subset G$ is also a closed and bounded set, for $t \in I_{t_0,K}$. It is a consequence of the fact that functions $\phi_{t_0,t}$, $t \in I_{t_0,K}$, are homeomorphisms of certain sets, cf. Remark 1.

From the above observation, the definition of function $\phi_{t_0,t}$ for $t \in I_{t_0,K}$ and the definition of the derivative of the function g_K, it follows that the following conditions are equivalent:

(a) $g'_K(t_0) = 0$ for an arbitrary fixed $t_0 \in [a,b]$ and an arbitrary fixed closed bounded set $K \subset G$;

(b) $g'_K(t) = 0$ for $t \in I_{t_0,K}, t_0 \in [a,b]$, for an arbitrary fixed closed bounded set $K \subset G$.

So let us fix a closed and bounded set $K \subset G$, $t_0 \in [a,b]$ such that $I_{t_0,K} \neq \{t_0\}$ and let $t = t_0 + \Delta t \in I_{t_0,K}$. Then, we have

$$g'_K(t_0) = \lim_{\Delta t \longrightarrow 0} \frac{\int_{\phi_{t_0,t}(K)} f(t,y)\, d\mu - \int_K f(t_0,y)\, d\mu}{\Delta t}. \tag{10}$$

Let us make a change of variables $y \to \phi_{t_0,t}(y)$ in the integral $\int_{\phi_{t_0,t}(K)} f(t,y)\, d\mu$ where $t \in I_{t_0,U}$, $y \in U$ and U satisfy conditions (a), (b), (c) in Remark 1. Because of Remark 1 and property (2) of Lemma 3 we obtain for sufficiently small Δt the following equality

$$\begin{aligned} &\int_{\phi_{t_0,t}(K)} f(t,y)\, d\mu = \int_K f(t,\phi_{t_0,t}(y))\, Jac\phi_{t_0,t}(y)\, d\mu \\ &= \int_K f(t,\phi_{t_0,t}(y)) \cdot \left(1 + \left(\sum_{i=1}^{n} \frac{\partial F_i(t_0,y)}{\partial y_i}\right)\Delta t + \gamma(\Delta t, y)\,\Delta t\right) d\mu, \end{aligned} \tag{11}$$

where $\gamma(\Delta t, y)$ is the function such as in the property (2) of Lemma 3.

From from the property (1) of Lemma 3, we infer

$$\phi_{t_0,t}(y) = (y_1 + F_1(t_0,y)\,\Delta t + r_1(\Delta t, y)\,\Delta t, \ldots, y_n + F_n(t_0,y)\,\Delta t + r_n(\Delta t, y)\,\Delta t)$$

for $y = (y_1, \ldots, y_n) \in K$, where $r_i(\Delta t, y)$, $i = 1, \ldots, n$ are such as in the property (1) in Lemma 3. Taking into accout the Taylor formula, we have

$$
\begin{aligned}
&f\left(t,\phi_{t_0,t}\left(y\right)\right)=f\left(t_0,y\right)+\frac{\partial f}{\partial t}\left(t_0,y\right)\Delta t\\
&+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\left(t_0,y\right)\cdot\left(F_i\left(t_0,y\right)+r_i\left(\Delta t,y\right)\right)\Delta t+o\left(\Delta t,y\right)\Delta t\right).
\end{aligned}
\tag{12}
$$

By a reasoning as in the proof of property (1) of Lemma 3 we deduce that $o\left(\Delta t,y\right)\rightarrow_{\Delta t\rightarrow 0}0$ for $y\in K$ and that values of $o\left(\Delta t,y\right)$ are uniformly bounded for sufficiently small Δt and $y\in K$.

From equalities (11) and (12) we have

$$
\int_{\phi_{t_0,t}(K)}f\left(t,y\right)d\mu=
\tag{13}
$$

$$
\int_K\left(f\left(t_0,y\right)+\frac{\partial f}{\partial t}\left(t_0,y\right)\Delta t+\sum_{i=1}^{n}\frac{\partial\left(fF_i\right)}{\partial y_i}\left(t_0,y\right)\Delta t+R\left(\Delta t,y\right)\Delta t\right)d\mu.
$$

Taking into account properties of γ and $r_i\left(\Delta t,y\right)$ for $i=1,\ldots,n$ and properties of $o\left(\Delta t,y\right)$, we obtain that values $R\left(\Delta t,y\right)$ are uniformly bounded for $y\in K$ for sufficiently small Δt and that $R\left(\Delta t,y\right)\rightarrow_{\Delta t\rightarrow 0}0$ for $y\in K$.

From (10) and (13) it follows that

$$
g'_K\left(t_0\right)=\lim_{\Delta t\longrightarrow 0}\int_K\left(\frac{\partial f}{\partial t}\left(t_0,y\right)+\sum_{i=1}^{n}\frac{\partial\left(fF_i\right)}{\partial y_i}\left(t_0,y\right)+R\left(\Delta t,y\right)\right)d\mu.
$$

Because of the continuity of partial derivatives of functions f and F_i, $i=1,\ldots,n$ and properties of $R\left(\Delta t,y\right)$, we can use the Lebegue dominated convergence theorem to obtain

$$
g'_K\left(t_0\right)=\int_K\left(\frac{\partial f}{\partial t}\left(t_0,y\right)+\sum_{i=1}^{n}\frac{\partial\left(fF_i\right)}{\partial y_i}\left(t_0,y\right)\right)d\mu.
$$

Hence we deduce that the condition (3) defining local integral invariants for the closed and bounded set $K\subset G$ is equivalent to the condition (4) from Theorem 2.

Now we are going to verify if the condition (4) in Theorem 2 implies the condition (3) for every bounded set $A\in L_n$, $A\subset G$.

Let $t_0\in[a,b]$ and $t\in I_{t_0,A}$. Let us represent the set A as a sum $A=A_{1+}\cup A_{2+}\cup A_{1-}\cup A_{2-}$, where

$$
A_{1+}=\{y\in A:f\left(t_0,y\right)\geqslant 0,f\left(t,\phi_{t_0,t}\left(y\right)\right)\geqslant 0\},
$$

$$
A_{2+}=\{y\in A:f\left(t_0,y\right)\geqslant 0,f\left(t,\phi_{t_0,t}\left(y\right)\right)<0\},
$$

$$A_{1-} = \{y \in A : f(t_0, y) < 0, f(t, \phi_{t_0,t}(y)) \geqslant 0\},$$
$$A_{2-} = \{y \in A : f(t_0, y) < 0, f(t, \phi_{t_0,t}(y)) < 0\}.$$

Sets A_{1+}, A_{2+}, A_{1-} and A_{2-} are pairwise disjoint and belong to L_n, because the function f is continuous and $\phi_{t_0,t}$ is a homeomorphism. Moreover, these sets are bounded. If we prove that each of the sets A_{1+}, A_{2+}, A_{1-} and A_{2-} satisfy condition (3) then the set A also satisfies condition (3) due to the injectivity of $\phi_{t_0,t}$ and the additivity of the Lebesque integral with respect to a set of integration.

First we will prove that the set A_{1+} satisfies the condition (3). Since $A_{1+} \in L_n$ and A_{1+} is bounded we obtain that for all $\epsilon > 0$ there exists a closed and bounded set $K \subset A_{1+}$ such that $\mu(A_{1+} \setminus K) < \epsilon$. Hence and from the injectivity of $\phi_{t_0,t}$, we get

$$\begin{aligned}
&\int_{\phi_{t_0,t}(A_{1+})} f(t,y)\,d\mu = \int_{\phi_{t_0,t}(K)} f(t,y)\,d\mu + \int_{\phi_{t_0,t}(A_{1+}\setminus K)} f(t,y)\,d\mu \\
&= \int_K f(t_0,y)\,d\mu + \int_{\phi_{t_0,t}(A_{1+}\setminus K)} f(t,y)\,d\mu \\
&= \int_{A_{1+}} f(t_0,y)\,d\mu - \int_{A_{1+}\setminus K} f(t_0,y)\,d\mu + \int_{\phi_{t_0,t}(A_{1+}\setminus K)} f(t,y)\,d\mu \\
&= \int_{A_{1+}} f(t_0,y)\,d\mu - \eta + \int_{\phi_{t_0,t}(A_{1+}\setminus K)} f(t,y)\,d\mu,
\end{aligned}$$

where η is a nonnegative real number which can be sufficiently small while the set K is suitable chosen.

Hence and from the fact that $\int_{\phi_{t_0,t}(A_{1+}\setminus K)} f(t,y)\,d\mu \geqslant 0$ we have $\int_{A_{1+}} f(t_0,y)\,d\mu \leqslant \int_{\phi_{t_0,t}(A_{1+})} f(t,y)\,d\mu$. Because $\phi_{t,t_0}(\phi_{t_0,t}(A_{1+})) = A_{1+}$ similarly we obtain $\int_{A_{1+}} f(t_0,y)\,d\mu \geqslant \int_{\phi_{t_0,t}(A_{1+})} f(t,y)\,d\mu$, so we have $\int_{A_{1+}} f(t_0,y)\,d\mu = \int_{\phi_{t_0,t}(A_{1+})} f(t,y)\,d\mu$ for the set A_{1+}.

Now we are going to show that the set A_{2+} satisfies condition (3). For the closed and bounded set $K \subset \phi_{t_0,t}(A_{2+})$, we get

$$\begin{aligned}
\int_{\phi_{t_0,t}(A_{2+})} f(t,y)\,d\mu &= \int_K f(t,y)\,d\mu + \int_{\phi_{t_0,t}(A_{2+})\setminus K} f(t,y)\,d\mu \\
&= \int_K f(t,y)\,d\mu - \eta = \int_{\phi_{t,t_0}(K)} f(t_0,y)\,d\mu - \eta,
\end{aligned}$$

where η is a nonnegative number which can be sufficiently small while the set K is suitable chosen. As before from $\int_{\phi_{t_0,t}(A_{2+})} f(t,y)\,d\mu \leqslant 0$ and $\int_{\phi_{t,t_0}(K)} f(t_0,y)\,d\mu \geqslant 0$, we obtain $\int_{\phi_{t_0,t}(A_{2+})} f(t,y)\,d\mu = 0$.

For the closed and bounded set $K \in A_{2+}$, we also get

$$\int_{A_{2+}} f(t_0, y)\, d\mu = \int_K f(t_0, y)\, d\mu + \int_{A_{2+}\setminus K} f(t_0, y)\, d\mu$$

$$= \int_{\phi_{t_0,t}(K)} f(t, y)\, d\mu + \eta,$$

where η is a nonnegative number which can be sufficiently small while the set K is suitable chosen, $\int_{A_{2+}} f(t_0, y)\, d\mu \geqslant 0$ and $\int_{\phi_{t_0,t}(K)} f(t, y)\, d\mu \leqslant 0$. Hence $\int_{A_{2+}} f(t_0, y)\, d\mu = 0$ and consequently also the set A_{2+} satisfies the condition (3).

Arguing as for the sets A_{1+} and A_{2+} we deduce that the sets A_{1-} and A_{2-} satisfy condition (3), too. Hence the condition (3) is valid for every bounded set $A \in L_n$, $A \subset G$.

If the condition (4) of Theorem 1 holds and the set $A \in L_n$, $A \subset G$ is unbounded then the fact that A satisfies the condition (3) is a conclusion drawn from the following facts

(a) The space $\mathbb{R}^n$ can be represented as a countable sum of pairwise disjoint sets from L_n.

(b) Every bounded set $A \subset G$, $A \in L_n$, $A \subset G$ satisfies the condition (3).

(c) The Lebesque integral is additive with respect to a set of integration.

This complets the proof of Theorem 2. ∎

Let μ_1 be an absolutely continuous (with respect to the Lebesque measure) measure given by the formula

$$\mu_1(A) = \int_A g(y)\, d\mu \ for\ A \in L_n, \tag{14}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuous function with nonnegative values and having continuous partial derivatives on $\mathbb{R}^n$.

The following corollary is an immediate consequence of the definition of the local integral invariant of system (1), Theorem 2 and the condition $\frac{\partial g}{\partial t}(y) = 0$ for each $y \in \mathbb{R}^n$ and $t \in [a, b]$.

Corollary 4. *The measure defined by the formula* (14) *is an invariant measure of the local dynamical system generated by* (1) *if and only if* $\sum_{i=1}^{n} \frac{\partial (gF_i)}{\partial y_i} = 0$.

References

[1] F. Calgero and A. Degasperis, Spectral Transform and Solitons: Tools to Slove and Investigate Nonlinear Evotion Equations (New York, North-Holland, 1982).

[2] D.G. Crowdy, *General solutions to the 2D Liouville equation*, Int. J. Engng Sci. **35** (1997) 141–149. doi:10.1016/S0020-7225(96)00080-8.

[3] R. Kubo, *Stochastic Liouville equations*, J. Math. Phys. **4** (1963) 174–183. doi:10.1063/1.1703941.

[4] Y. Matsumo, *Exact solution for the nonlinear Klein-Gordon and Liouville equations in four – dimensional Euklidean space*, J. Math. Phys. **28** (1987) 2317–2322. doi:10.1063/1.527764.

[5] P.K. Sahoo and T. Riedel, Mean Value Theorems and Functional Equations (World Scientific Publishing, Singapore, 1998).

[6] V.E. Tarasov, *Stationary solutions of Liouville equations for non-Hamiltonian systems*, Ann. Phys. **316** (2005) 393–413. doi:10.1016/j.aop.2004.11.001.

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