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ON THE COVERAGE PROBABILITY OF CONFIDENCE SETS BASED ON A PRIOR DISTRIBUTION

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1. Introduction

For definiteness consider the situation where we have a sample $X_1, \ldots, X_n$ of independent identically distributed random variables whose distribution depends on an unknown real parameter θ. Suppose we choose a prior density π_0 for θ with respect to Lebesgue measure and determine $\phi_\alpha(X)$ so that the posterior probability that $\theta < \phi_\alpha(X)$ is α, a fixed number in the interval (0, 1). Welch and Peers (1963) indicated that under appropriate regularity conditions, if π_0 is the square root of Fisher's information, then for any θ the probability that $\theta < \phi_\alpha(X)$ differs from α by a term of the order of $1/n$. Some further results are contained in Welch (1965) and Peers (1965). My aim in this paper is to extend and clarify this work.

The method of proof attempted here is quite different from that of Welch and Peers and the other relevant papers that will be referred to below. The basic idea is to compare the posterior probability that $\theta < \phi_\alpha(X)$, under largely arbitrary prior density π with the posterior probability under π_0. This is done using only an expansion of the logarithm of the likelihood function in a Taylor series, and consequently it does not require a special structure such as independently identically distributed observations. The unconditional probability under π is then obtained with the aid of an integration by parts. This approach postpones the difficulty caused by end effects and possible irregularities in the sampling distribution. However these do cause difficulty in a final step of unsmoothing.

The work described above is carried out in Sections 2 and 3. In Section 4 I use the same method to obtain an additional term in the expansion without even an attempt at rigor. In Section 5 I look at an analogous question arising in the study of confidence sets for a single function of the parameters in the multiparameter case. This section is also not at all rigorous and it is somewhat unsatisfactory even as heuristic work but it may be a useful starting point for a more serious approach. Two examples are discussed in Sections 6 and 7. In Section 8, following a suggestion by Efron, I indicate an argument that the basic result also holds conditionally given an approximately ancillary statistic. It was the work by Hinkley (1980) on this question that first called my attention to this whole set of problems. He also pointed out a serious blunder in my original version of Section 5. Some closing remarks are also included in Section 8.

There are no real proofs in this paper. However, in Sections 2 and 3 I have attempted to indicate, non-rigorously but with some care, the conditions needed for the validity of the results obtained there. It would be highly desirable to obtain mathematically correct versions of these results but I am not at all sure that I shall do so.

This work has some resemblance to the development by Le Cam of his notion of contiguity of probability measures. A good exposition of this theory is that of Roussas (1972).

I am indebted to Professor Zieliński and to the directors of the Banach Center for their invitation to present this work at the Center during the fall semester of 1981.

2. The basic result in the one-parameter case

Let $(\mathscr{X}, \mathscr{B}, \mu)$ be a σ-finite measure space, $\mathscr{T}$ an open subinterval of the real line and p a function on $\mathscr{T}$ to the set of all probability density functions with respect to the measure μ. For $\theta \in \mathscr{T}$ I shall denote by P_θ the probability measure that is the indefinite integral of p_θ with respect to μ. My aim is to discuss a method of obtaining an approximate upper α confidence point $\phi_\alpha(X)$ for θ, that is a random point such that, for all $\theta \in \mathscr{T}$

$$P_\theta\{\theta < \phi_\alpha(X)\} \approx \alpha. \tag{1}$$

As I have already indicated in the introduction I have not obtained rigorous proofs of any of the readily interpretable results. However the arguments leading to (40) with the associated formulas for remainders in (33)–(39) and earlier definitions are intended to be rigorous.

Both the definition of the approximate confidence points and the attempt to verify (1) will use prior distributions for θ. For any probability density function π with respect to Lebesgue measure in $\mathscr{T}$, let P_π denote the probability distribution of a pair (Θ, X) of random variables where Θ is distributed in $\mathscr{T}$ according to π and the conditional p.d.f. (wrt μ) of X given Θ is p_Θ. Conditional probability given X under this distribution will be denoted by P_π^X. We choose a particular p.d.f. π_0 with respect to Lebesgue measure in $\mathscr{T}$ and determine ϕ_α to satisfy

$$P_{\pi_0}^X\{\Theta < \phi_\alpha(X)\} = \alpha. \tag{2}$$

Unfortunately it is impracticable to study the approximate identity (1) directly. Instead we shall ask when it is true that for a largely arbitrary prior density π, we have, to a good approximation,

$$P_\pi\{\Theta < \phi_\alpha(X)\} \approx \alpha. \tag{3}$$

The question of the extent to which this implies (1) will be postponed to the next section. Roughly speaking, we shall see that under suitable regularity conditions,

$$P_\pi\{\Theta < \phi_\alpha(X)\} \approx \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \int_{\mathscr{T}} \frac{1}{\pi_0(\theta)} \frac{d}{d\theta}[\pi_0(\theta) I^{-1/2}(\theta)]\, \pi(\theta)\, d\theta \tag{4}$$

where $I(\theta)$ is Fisher's information, given by

$$I(\theta) = E_\theta\left[\frac{\partial \log p_\theta(X)}{\partial\theta}\right]^2. \tag{5}$$

The second term on the r.h.s. of (4) vanishes if we choose π_0 proportional to $I^{1/2}$.

The result will be obtained by comparing $P_\pi^X\{\Theta < \phi_\alpha(X)\}$ with $\alpha = P_{\pi_0}^X\{\Theta < \phi_\alpha(X)\}$ and then taking unconditional expectation under P_π. In this way a version of (4) will be obtained in the form of an identity. The remainder is somewhat complicated but it should not be difficult to make a rough judgment of its order of magnitude.

It is assumed that π and π_0 are twice continuously differentiable and that π_0 does not vanish. With

$$\mathscr{T} = (a, b) \tag{6}$$

where we may have $a = -\infty$ or $b = \infty$ or both, it is also assumed that

$$\lim_{\theta\to a} \pi(\theta) I^{-1/2}(\theta) = \lim_{\theta\to b} \pi(\theta) I^{-1/2}(\theta). \tag{7}$$

This will be required for an integration by parts and, as will be discussed in Section 3, is related to the fact that (1) cannot be valid for θ that are, in an appropriate sense, close to an endpoint of $\mathscr{T}$. It is also assumed that, for all $x \in \mathscr{X}$, the function $\theta \mapsto p_\theta(x)$ is twice continuously differentiable. I shall use the abbreviations

$$M_\theta = \log p_\theta(X) + \log \pi_0(\theta) \tag{8}$$

and

$$\varrho(\theta) = \frac{\pi(\theta)}{\pi_0(\theta)}. \tag{9}$$

We want to compare

$$P_\pi^X\{\Theta < \phi_\alpha(X)\} = \frac{\mathcal{N}^*}{\mathcal{D}^*} \tag{10}$$

where

$$\mathcal{N}^* = \int_a^{\phi_\alpha(X)} e^{M_\theta} \varrho(\theta)\, d\theta \tag{11}$$

and

$$\mathcal{D}^* = \int_a^b e^{M_\theta} \varrho(\theta)\, d\theta, \tag{12}$$

with

$$\alpha = P_{\pi_0}^X\{\Theta < \phi_\alpha(X)\} = \frac{\mathcal{N}}{\mathcal{D}} \tag{13}$$

where

$$\mathcal{N} = \int_a^{\phi_\alpha(X)} e^{M_\theta}\, d\theta \tag{14}$$

and

$$\mathcal{D} = \int_a^b e^{M_\theta}\, d\theta. \tag{15}$$

Let us look at the details of the approximation of $\mathcal{N}^*$ since it is somewhat more complicated than the other numerator and the denominators. We have

$$\begin{aligned}
\mathcal{N}^* &= \int_a^{\phi_\alpha(X)} e^{M_\theta} \varrho(\theta)\, d\theta \\
&= \int_a^{\phi_\alpha(X)} e^{M_\theta}\big[\varrho(\hat\Theta) + (\theta - \hat\Theta)\, \dot\varrho(\hat\Theta) + \big(\varrho(\theta) - \varrho(\hat\Theta) - (\theta - \hat\Theta)\, \dot\varrho(\hat\Theta)\big)\big]\, d\theta \\
&= \varrho(\hat\Theta)\mathcal{N} + \dot\varrho(\hat\Theta)\Big[\int_a^{\phi_\alpha(X)} \exp\big(M_{\hat\Theta} + \tfrac{1}{2}(\theta - \hat\Theta)^2 \ddot M_{\hat\Theta}\big)(\theta - \hat\Theta)\, d\theta + \\
&\qquad + \int_a^{\phi_\alpha(X)} \big(\exp(M_\theta) - \exp\big(M_{\hat\Theta} + \tfrac{1}{2}(\theta - \hat\Theta)^2 \ddot M_{\hat\Theta}\big)\big)(\theta - \hat\Theta)\, d\theta\Big] + \\
&\qquad\qquad + \int_a^{\phi_\alpha(X)} e^{M_\theta}\big[\varrho(\theta) - \varrho(\hat\Theta) - (\theta - \hat\Theta)\, \dot\varrho(\hat\Theta)\big]\, d\theta,
\end{aligned} \tag{16}$$

where $\hat{\Theta}$ is chosen to maximize the function $\theta \mapsto M_\theta$. Similarly

$$\begin{aligned}\mathscr{S}^* &= \int_a^b e^{M_\theta} \varrho(\theta)\, d\theta \\ &= \varrho(\hat{\Theta})\, \mathscr{S} + \dot{\varrho}(\hat{\Theta}) \Big[\int_a^b \exp\big(M_{\hat{\Theta}} + \tfrac{1}{2}(\theta - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)(\theta - \hat{\Theta})\, d\theta + \\ &\quad + \int_a^b \big(\exp(M_\theta) - \exp\big(M_{\hat{\Theta}} + \tfrac{1}{2}(\theta - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)\big)(\theta - \hat{\Theta})\, d\theta \Big] + \\ &\quad + \int_a^b e^{M_\theta} [\varrho(\theta) - \varrho(\hat{\Theta}) - (\theta - \hat{\Theta})\, \dot{\varrho}(\hat{\Theta})]\, d\theta. \end{aligned} \tag{17}$$

Continuing to pursue our aim of comparing (10) with (13) we shall need, in addition to (16) and (17), an approximation to $\mathscr{S}$ defined by (15). We have

$$\begin{aligned}\mathscr{S} &= \int_a^b e^{M_\theta}\, d\theta \\ &= \int_a^b \exp\big(M_{\hat{\Theta}} + \tfrac{1}{2}(\theta - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)\, d\theta + \int_a^b \big(\exp(M_\theta) - \exp\big(M_{\hat{\Theta}} + \tfrac{1}{2}(\theta - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)\big)\, d\theta. \end{aligned} \tag{18}$$

It will be useful to evaluate some of the terms in the above expressions explicitly. For the first term in $\mathscr{S}$ we have

$$\int_a^b \exp\big(M_{\hat{\Theta}} + \tfrac{1}{2}(\theta - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)\, d\theta = \sqrt{\frac{2\pi}{-\ddot{M}_{\hat{\Theta}}}}\, e^{M_{\hat{\Theta}}} \cdot \big[\Phi\big((b - \hat{\Theta})\sqrt{-\ddot{M}_{\hat{\Theta}}}\big) - \Phi\big((a - \hat{\Theta})\sqrt{-\ddot{M}_{\hat{\Theta}}}\big)\big]. \tag{19}$$

For the second factor in the second term in the final expression for $\mathscr{N}^*$ in (16) we have

$$\int_a^{\phi_\alpha(X)} \exp\big(M_{\hat{\Theta}} + \tfrac{1}{2}(\theta - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)(\theta - \hat{\Theta})\, d\theta = \frac{1}{-\ddot{M}_{\hat{\Theta}}}\, e^{M_{\hat{\Theta}}} \big(\exp\big(\tfrac{1}{2}(\phi_\alpha(X) - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big) - \exp\big(\tfrac{1}{2}(a - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)\big). \tag{20}$$

Similarly, for the corresponding term in the expression (17) for $\mathscr{S}^*$ we have

$$\int_a^b \exp\big(M_{\hat{\Theta}} + \tfrac{1}{2}(\theta - \hat{\Theta})^2 M_{\hat{\Theta}}\big)(\theta - \hat{\Theta})\, d\theta = \frac{1}{-\ddot{M}_{\hat{\Theta}}}\, e^{M_{\hat{\Theta}}} \big(\exp\big(\tfrac{1}{2}(b - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big) + \exp\big(\tfrac{1}{2}(a - \hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\big)\big). \tag{21}$$

I shall write R_1, R_2, and R_3 for the remainders in (16), (17), and (18), that is

$$R_1 = \dot{\varrho}(\hat{\Theta}) \int_a^{\phi_\alpha(X)} (\theta-\hat{\Theta})\left[\exp(M_\theta)-\exp\left(M_{\hat{\Theta}}+\tfrac{1}{2}(\theta-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\right)\right] d\theta +$$

$$+ \int_a^{\phi_\alpha(X)} [\varrho(\theta)-\varrho(\hat{\Theta})-(\theta-\hat{\Theta})\,\dot{\varrho}(\hat{\Theta})]\, e^{M_\theta}\, d\theta, \tag{22}$$

$$R_2 = \dot{\varrho}(\hat{\Theta})\int_a^b (\theta-\hat{\Theta}) e^{M_\theta} d\theta + \int_a^b [\varrho(\theta)-\varrho(\hat{\Theta})-(\theta-\hat{\Theta})\,\dot{\varrho}(\hat{\Theta})]\, e^{M_\theta}\, d\theta, \tag{23}$$

and

$$R_3 = \int_a^b \left(\exp(M_\theta)-\exp\left(M_{\hat{\Theta}}+\tfrac{1}{2}(\theta-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\right)\right) d\theta. \tag{24}$$

Putting things together from (10)–(24) we find that

$$P_\pi^X\{\Theta < \phi_\alpha(X)\} - P_{\pi_0}^X\{\Theta < \phi_\alpha(X)\} = \frac{\mathcal{N}^*}{\mathcal{D}^*} - \frac{\mathcal{N}}{\mathcal{D}}$$

$$= \frac{-\dfrac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})(-\ddot{M}_{\hat{\Theta}})}\exp\left(M_{\hat{\Theta}}+\tfrac{1}{2}(\phi_\alpha(X)-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\right)+R_1^*+R_4^*-\mathcal{N}R_2^*}{\mathcal{D}(1+R_2^*)}, \tag{25}$$

where

$$R_1^* = \frac{R_1}{\varrho(\hat{\Theta})}, \tag{26}$$

$$R_2^* = \frac{R_2}{\varrho(\hat{\Theta})\,\mathcal{D}}, \tag{27}$$

and

$$R_4^* = \frac{\dot{\varrho}(\hat{\Theta})\sqrt{2\pi}}{\varrho(\hat{\Theta})(-\ddot{M}_{\hat{\Theta}})}\exp\left(M_{\hat{\Theta}}+\tfrac{1}{2}(a-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\right). \tag{28}$$

Using (18), (19), and (24) we also have for $\mathcal{D}(1+R_2^*)$ the expression

$$\mathcal{D}(1+R_2^*) = \sqrt{\frac{2\pi}{-\ddot{M}_{\hat{\Theta}}}}\, e^{M_{\hat{\Theta}}}(1+R_5^*) \tag{29}$$

where

$$R_5^* = -\Phi\big((a-\hat{\Theta})\sqrt{-\ddot{M}_{\hat{\Theta}}}\big)-\big[1-\Phi\big((b-\hat{\Theta})\sqrt{-\ddot{M}_{\hat{\Theta}}}\big)\big]+\sqrt{\frac{-\ddot{M}_{\hat{\Theta}}}{2\pi}}\,e^{-M_{\hat{\Theta}}}\mathscr{D}R_2^*. \tag{30}$$

Finally it will be convenient to define

$$R_6^* = \frac{1}{1+R_5^*}-1 = -\frac{R_5^*}{1+R_5^*} \tag{31}$$

so that we can rewrite (25) in the form

$$P_\pi^X\{\Theta < \phi_\alpha(X)\} = \alpha-\frac{1}{\sqrt{-\ddot{M}_{\hat{\Theta}}}}\frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})}\frac{\exp\big(\frac{1}{2}(\phi_\alpha(X)-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\big)}{\sqrt{2\pi}}+R \tag{32}$$

where

$$R = R_1^{**}+R_4^{**}+R_7^{**}+R_6^{**} \tag{33}$$

with

$$R_1^{**} = \frac{R_1^*}{\mathscr{D}(1+R_2^*)}, \tag{34}$$

$$R_4^{**} = \frac{R_4^*}{\mathscr{D}(1+R_2^*)}, \tag{35}$$

$$R_7^{**} = -\frac{R_2^*}{\mathscr{D}(1+R_2^*)}, \tag{36}$$

and

$$R_6^{**} = -R_6^*\frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})}\frac{\exp\big(\frac{1}{2}(\phi_\alpha(X)-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\big)}{\sqrt{2\pi(-\ddot{M}_{\hat{\Theta}})}}. \tag{37}$$

It follows from (32) that

$$\begin{aligned}P_\pi\{\Theta < \phi_\alpha(X)\} &= E_\pi P_\pi^X\{\Theta < \phi_\alpha(X)\}\\ &= \alpha-\int E_\theta\frac{1}{\sqrt{-\ddot{M}_{\hat{\Theta}}}}\frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})}\frac{\exp\big(\frac{1}{2}(\phi_\alpha(X)-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\big)}{\sqrt{2\pi}}\pi(\theta)\,d\theta+E_\pi R.\end{aligned} \tag{38}$$

Let

$$\begin{aligned}R_8(\theta) = E_\theta&\frac{1}{\sqrt{-\ddot{M}_{\hat{\Theta}}}}\frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})}\frac{\exp\big(\frac{1}{2}(\phi_\alpha(X)-\hat{\Theta})^2\ddot{M}_{\hat{\Theta}}\big)}{\sqrt{2\pi}}-\\ &-\frac{1}{\sqrt{2\pi}}\exp\big(-\tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\big)\frac{1}{\sqrt{I(\theta)}}\frac{\dot{\varrho}(\theta)}{\varrho(\theta)}.\end{aligned} \tag{39}$$

Then (38) yields

$$P_\pi\{\Theta < \phi_\alpha(X)\} = \alpha - \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \int_a^b \frac{1}{\sqrt{I(\theta)}}\frac{\dot{\varrho}(\theta)}{\varrho(\theta)}\pi(\theta)\,d\theta - $$
$$- \int_a^b R_8(\theta)\,d\theta + E_\pi R$$
$$= \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \int_a^b \frac{1}{\pi_0(\theta)}\frac{d}{d\theta}[\pi_0(\theta) I^{-1/2}(\theta)]\,\pi(\theta)\,d\theta - $$
$$- \int_a^b R_8(\theta)\,\pi(\theta)\,d\theta + E_\pi R. \tag{40}$$

The final equality uses an integration by parts.

3. The case of independently identically distributed observations

Although it is clear that the results of Section 2 and their implied consequences do not even require that the observations be independent it may be useful to make these results explicit in the case of independently identically distributed observations. I shall first bound the remainders without any attempt at rigor and then look at some other aspects of the problem.

Thus we consider the case where $X_1, \ldots, X_n$ are independently identically distributed random variables, each with probability density function p_θ^* with respect to a σ-finite measure μ. Here θ is an unknown real parameter in the interval (a, b) where, as before, we may have $a = -\infty$ or $b = \infty$ or both. Fisher's information for a single observation will be denoted by I^*, that is

$$I^*(\theta) = E_\theta\left(\frac{\partial \log p^*(X_1)}{\partial\theta}\right)^2 = -E_\theta \frac{\partial^2 \log p_\theta^*(X_1)}{\partial\theta^2}. \tag{1}$$

Differentiability of the likelihood function and existence of moments of the resulting derivatives will be assumed freely in the course of the argument. Later I shall try to summarize the assumptions actually used. I shall also assume that $a = -\infty$ and $b = +\infty$, which can always be achieved by a change of parameter.

Some simplification is achieved by the assumption that $a = -\infty$ and $b =$

$+\infty$. In equation (2.19) the expression in brackets becomes 1 and consequently the first two terms in the expression (2.30) for R_5^* vanish. Also R_4^* vanishes. These simplifications and others that will be pointed out later do not affect the order of magnitude of the remainder in (2.40).

Now let us look at the remainders of Section 2 systematically in their final form. By (2.27) and (2.23) we have

$$R_2^* = \frac{R_2}{\varrho(\hat{\Theta})\mathscr{I}}$$

$$= \frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})} \frac{\int_{-\infty}^{\infty} (\theta-\hat{\Theta}) e^{M_\theta} d\theta}{\int_{-\infty}^{\infty} e^{M_\theta} d\theta} + \frac{\int_{-\infty}^{\infty} [\varrho(\theta)-\varrho(\hat{\Theta})-(\theta-\hat{\Theta})\dot{\varrho}(\hat{\Theta})] e^{M_\theta} d\theta}{\varrho(\hat{\Theta}) \int_{-\infty}^{\infty} e^{M_\theta} d\theta,}$$

$$= O\left(\frac{1}{n}\right) \tag{2}$$

In evaluating the first term in the next to the last expression I have used traditional large sample theory to the effect that

$$\ddot{M}_{\hat{\Theta}} = -nI^*(\Theta) + O(n^{1/2}) \tag{3}$$

and

$$\int_{-\infty}^{\infty} (\theta-\hat{\Theta}) e^{M_\theta} d\theta = \int_{-\infty}^{\infty} (\theta-\hat{\Theta}) \{\exp(M_{\hat{\Theta}}+\tfrac{1}{2}(\theta-\hat{\Theta})^2 \ddot{M}_{\hat{\Theta}})$$

$$+[\exp(M_\theta)-\exp(M_{\hat{\Theta}}+\tfrac{1}{2}(\theta-\hat{\Theta})^2 \ddot{M}_{\hat{\Theta}})]\} d\theta = \left(\int_{-\infty}^{\infty} e^{M_\theta} d\theta\right) \cdot O(1/n). \tag{4}$$

In (4) the integral resulting from the first term in braces and that resulting from the term in brackets are roughly of the order of

$$\int_{-\infty}^{\infty} [\theta-\hat{\Theta}]^4 \ddot{M}_{\hat{\Theta}} e^{M_\theta} d\theta = \left(\int_{-\infty}^{\infty} e^{M_\theta} d\theta\right) O(1/n). \tag{5}$$

The second term in the next to the last expression in (2) is evaluated similarly by observing that it is of the order of

$$\frac{\int_{-\infty}^{\infty} (\theta-\hat{\Theta})^2 \ddot{\varrho}(\hat{\Theta}) e^{M_\theta} d\theta}{\varrho(\hat{\Theta}) \int_{-\infty}^{\infty} e^{M_\theta} d\theta} \pm O(1/n). \tag{6}$$

Next from (2.34), (2.26), and (2.22) we obtain

$$R_1^{**} = \frac{R_1}{\varrho(\hat{\Theta})\,\mathscr{D}(1+R_2^*)}$$

$$= \frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})} \frac{\int_{-\infty}^{\phi_\alpha} (\theta-\hat{\Theta})\left[\exp(M_\theta) - \exp\left(M_{\hat{\Theta}} + \frac{1}{2}(\theta-\hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\right)\right] d\theta}{(1+R_2^*) \int_{-\infty}^{\infty} e^{M_\theta}\, d\theta} +$$

$$+ \frac{\int_{-\infty}^{\infty} [\varrho(\theta) - \varrho(\hat{\Theta}) - (\theta-\hat{\Theta})\,\dot{\varrho}(\hat{\Theta})]\, e^{M_\theta}\, d\theta}{\varrho(\hat{\Theta})(1+R_2^*) \int_{-\infty}^{\infty} e^{M_\theta}\, d\theta} = O(1/n). \tag{7}$$

The argument is essentially the same as in the case of R_2^*, except that we also use the fact that R_2^* is negligible.

Trivially

$$R_7^{**} = -\frac{\mathscr{N} R_2^*}{\mathscr{D}(1+R_2^*)} = -\frac{\alpha R_2^*}{1+R_2^*} = O(1/n) \tag{8}$$

by (2.36), (2.13), and (2). We must also look at

$$R_6^{**} = \frac{R_5^*}{1+R_5^*} \frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})} \frac{\exp\left(\frac{1}{2}(\phi_\alpha(X)-\hat{\Theta})^2 \ddot{M}_{\hat{\Theta}}\right)}{\sqrt{2\pi(-\ddot{M}_{\hat{\Theta}})}} = O(1/n). \tag{9}$$

The first equality uses (2.37) and (2.31). The evaluation of the order of magnitude uses (2.30) which, with the first two terms vanishing as indicated earlier, yields

$$R_5^* = \sqrt{\frac{-\ddot{M}_{\hat{\Theta}}}{2\pi}}\, e^{-M_{\hat{\Theta}}}\, \mathscr{D} R_2^* = \sqrt{\frac{-\ddot{M}_{\hat{\Theta}}}{2\pi}}\, e^{-M_{\hat{\Theta}}} \cdot \frac{R_2}{\varrho(\hat{\Theta})} = O\left(\frac{1}{\sqrt{n}}\right), \tag{10}$$

as in the verification of (2). This completes the sketch of the proof that the remainder R in (2.32) is of the order of n^{-1}. This suggests that its expectation $E_\pi R$ occurring in (2.40) is also of the order of n^{-1}.

In order to complete the evaluation of the remainder in (2.40) in the i.i.d. case we must look at $\int_{\infty}^{\infty} R_8(\theta)\pi(\theta)\, d\theta$ where R_8 is defined by (2.39). From the asymptotic normality of the posterior distribution of Θ with error of the order of $n^{-1/2}$ we have

$$(\phi_\alpha(X)-\hat{\Theta})^2 \ddot{M}_{\hat{\Theta}} = -[\Phi^{-1}(\alpha)]^2 + O\left(\frac{1}{\sqrt{n}}\right). \tag{11}$$

Then, using (3) and the fact that

$$\Theta - \hat{\Theta} = O(n^{-1/2}), \tag{12}$$

we see that

$$\int_{-\infty}^{\infty} R_8(\theta)\,\pi(\theta)\,d\theta = O(n^{-1}). \tag{13}$$

Thus it follows that, in the i.i.d. case,

$$\int_{-\infty}^{\infty} P_\theta\{\theta < \phi_\alpha(X)\}\,\pi(\theta)\,d\theta = P_\pi\{\Theta < \phi_\alpha(X)\}$$

$$= \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\pi_0(\theta)} \frac{d}{d\theta}[\pi_0(\theta) I^{-1/2}(\theta)]\,\pi(\theta)\,d\theta + O\left(\frac{1}{n}\right). \tag{14}$$

It remains to look at the question of whether this implies that, for fixed θ,

$$P_\theta\{\theta < \phi_\alpha(X)\} = \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \cdot \frac{1}{\pi_0(\theta)} \frac{d}{d\theta}[\pi_0(\theta) I^{-1/2}(\theta)] + O\left(\frac{1}{n}\right). \tag{15}$$

My answer to this will perhaps be even vaguer than the treatment up to this point. First we must try to be a bit more precise about the dependence of the remainder in (14) on π. Looking back over the detailed computations leading to (14) I am inclined to believe that, under appropriate regularity conditions on the function $\theta \mapsto p_\theta^*$ I could prove that, assuming $\log \pi_0$ twice uniformly continuously differentiable, we have, for all π satisfying the same condition as π_0,

$$\int_{-\infty}^{\infty} P_\theta\{\theta < \phi_\alpha(X)\}\,\pi(\theta)\,d\theta$$

$$= \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\pi_0(\theta)} \frac{d}{d\theta}[\pi_0(\theta) I^{-1/2}(\theta)]\,\pi(\theta)\,d\theta +$$

$$+ O\left(\frac{1}{n}\right) \int_{-\infty}^{\infty} \left(\left|\frac{d}{d\theta}\log\frac{\pi(\theta)}{\pi_0(\theta)}\right| + \left|\frac{d^2}{d\theta^2}\log\frac{\pi(\theta)}{\pi_0(\theta)}\right|\right)\pi(\theta)\,d\theta, \tag{16}$$

with the implied constant in the factor $O(1/n)$ not depending on π. Without further conditions, this does not imply (15). For example, in the binomial case, (16) can be verified directly subject to appropriate end conditions on π and π_0 but instead of the remainder $O(n^{-1})$ we have an error of the exact order of $n^{-1/2}$ since the left hand side has jumps of this order but the two leading terms on the right hand side are smooth.

4. The next term in the approximation

Let us compute the next term in the approximation of Section 2 without any attempt at rigor. The notation and assumptions are as in that section except that π, π_0 and $\theta \mapsto p_\theta(X)$ are assumed to be thrice continuously differentiable. The result is

$$P_\pi\{\theta < \phi_\alpha(X)\} - \alpha$$
$$\approx \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \int \left\{\frac{d}{d\theta}\frac{\pi_0(\theta)}{I^{1/2}(\theta)}\left(1+\frac{1}{6}M_3^*(\theta)\Phi^{-1}(\alpha)\right) - \right.$$
$$\left. -\frac{\Phi^{-1}(\alpha)}{2}\frac{d}{d\theta}\frac{\pi_0(\theta)}{I(\theta)}\right\}\frac{\pi(\theta)}{\pi_0(\theta)}d\theta \tag{1}$$

where

$$M_3^* = \frac{E_\theta\left(\frac{\partial^3 M_\theta}{\partial\theta^3}\right)\Big|_{\theta=\hat\theta}}{I^{3/2}(\theta)} \tag{2}$$

with

$$M_\theta = \log p_\theta(X) + \log \pi_0(\theta). \tag{3}$$

It is left to the reader to verify that the error is formally $O(n^{-3/2})$ in the case where $X = (X_1, \ldots, X_n)$ with X_i independent identically distributed. I shall need the integrals

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} t e^{-\frac{1}{2}t^2} dt = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}, \tag{4}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} t^2 e^{-\frac{1}{2}t^2} dt = \Phi(u) - \frac{1}{\sqrt{2\pi}} u e^{-\frac{1}{2}u^2}, \tag{5}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} t^3 e^{-\frac{1}{2}t^2} dt = -\frac{1}{\sqrt{2\pi}} (2+u^2) e^{-\frac{1}{2}u^2} \tag{6}$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} t^4 e^{-\frac{1}{2}t^2} dt = 3\Phi(u) - \frac{1}{\sqrt{2\pi}}(u^3+3u)e^{-\frac{1}{2}u^2}. \tag{7}$$

As in Section 2, we suppose $\phi_\alpha(X)$ chosen to satisfy

$$\alpha = \frac{\mathcal{N}}{\mathcal{D}}, \tag{8}$$

where

$$\begin{aligned}
\mathcal{N} &= \int_a^{\phi_\alpha(X)} e^{M_\theta} d\theta \\
&\approx \int_{-\infty}^{\phi_\alpha(X)} \exp\big((M_{\hat\theta} + \tfrac{1}{2}(\theta-\hat\Theta)^2 M_2 + \tfrac{1}{6}(\theta-\hat\Theta)^3 M_3 + \ldots\big) d\theta \\
&\approx e^{M_{\hat\theta}} \int_{-\infty}^{\phi_\alpha(X)} \exp\big(\tfrac{1}{2}(\theta-\hat\Theta)^2 M_2\big)\big[1 + \tfrac{1}{6}(\theta-\hat\Theta)^3 M_3\big] d\theta \\
&= e^{M_{\hat\theta}} \sqrt{2\pi(-M_2)^{-1}} \times \\
&\quad \times \left[\Phi(\Psi_\alpha(X)) - \frac{1}{6}\frac{M_3}{(-M_2)^{3/2}}\frac{1}{\sqrt{2\pi}}(2+\Psi_\alpha^2(X))\exp(-\tfrac{1}{2}\Psi_\alpha^2(X))\right]
\end{aligned} \tag{9}$$

with

$$\Psi_\alpha(X) = \sqrt{-M_2}\,(\phi_\alpha(X) - \hat\Theta), \tag{10}$$

and

$$\mathcal{D} = \int_a^b e^{M_\theta} d\theta \approx \int_{-\infty}^{\infty} \exp\big(M_{\hat\theta} + \tfrac{1}{2}(\theta-\hat\Theta)^2 M_2\big) d\theta = e^{M_{\hat\theta}} \sqrt{\frac{2\pi}{-M_2}}. \tag{11}$$

Thus, to this approximation

$$\alpha \approx \Phi(\Psi_\alpha(X)) - \frac{1}{6}\frac{M_3}{(-M_2)^{3/2}}\frac{1}{\sqrt{2\pi}}(2+\Psi_\alpha^2(X))\exp(-\tfrac{1}{2}\Psi_\alpha^2(X)). \tag{12}$$

We can solve this approximately for $\Psi_\alpha(X)$, obtaining

$$\Psi_\alpha(X) \approx \Phi^{-1}(\alpha) + \frac{1}{6}\frac{M_3}{(-M_2)^{3/2}}(2+\Phi^{-1}(\alpha))^2. \tag{13}$$

As in Section 2, let us write $\mathscr{N}^*$ and $\mathscr{D}^*$ for the analogue of (9) and (11) with an extra factor of

$$\varrho(\theta) = \frac{\pi(\theta)}{\pi_0(\theta)} \tag{14}$$

under the integral sign. Then

$$P_\pi^X\{\Theta < \phi_\alpha(X)\} = \frac{\mathscr{N}^*}{\mathscr{D}^*}, \tag{15}$$

and

$$\begin{aligned}
\mathscr{N}^* &= \int_a^{\phi_\alpha(X)} e^{M_\theta}\varrho(\theta)\,d\theta \\
&\approx \int_a^{\phi_\alpha(X)} e^{M_\theta}[\varrho(\hat\Theta)+(\theta-\hat\Theta)\,\dot\varrho(\hat\Theta)+\tfrac{1}{2}(\theta-\hat\Theta)^2\,\ddot\varrho(\hat\Theta)]\,d\theta \\
&\approx \varrho(\hat\Theta)\mathscr{N} + \int_{-\infty}^{\phi_\alpha(X)} \exp\big(M_{\hat\Theta}+\tfrac{1}{2}(\theta-\hat\Theta)^2 M_2\big)\times \\
&\quad\times[(\theta-\hat\Theta)\,\dot\varrho(\hat\Theta)+\tfrac{1}{2}(\theta-\hat\Theta)^2\,\ddot\varrho(\hat\Theta)+\tfrac{1}{6}(\theta-\hat\Theta)^4\,\dot\varrho(\hat\Theta)\,M]\,d\theta \\
&= \varrho(\hat\Theta)\mathscr{N}+(-M_2)^{-\frac{1}{2}}e^{M_{\hat\Theta}}\Big\{-\frac{\dot\varrho(\hat\Theta)}{(-M_2)^{1/2}}\exp\big(-\tfrac{1}{2}\Psi_\alpha^2(X)\big)+ \\
&\quad+\frac{1}{2}\frac{\ddot\varrho(\hat\Theta)}{-M_2}[\sqrt{2\pi}\,\alpha-\Psi_\alpha(X)\exp\big(-\tfrac{1}{2}\Psi_\alpha^2(X)\big)]+ \\
&\quad+\frac{1}{6}\frac{\dot\varrho(\hat\Theta)\,M_3}{(-M_2)^2}[3\sqrt{2\pi}\,\alpha-\big(\Psi_\alpha^3(X)+3\Psi_\alpha(X)\big)\exp\big(-\tfrac{1}{2}\Psi_\alpha^2(X)\big)]\Big\},
\end{aligned} \tag{16}$$

where Ψ_α, defined by (10), is given approximately by (14), and similarly

$$\mathscr{D}^* \approx \varrho(\hat\Theta)\,\mathscr{D}+(-M_2)^{-\frac{1}{2}}\left[\frac{1}{2}\frac{\ddot\varrho(\hat\Theta)}{-M_2}\sqrt{2\pi}+\frac{1}{6}\frac{\dot\varrho(\hat\Theta)\,M_3}{(-M_2)^2}3\sqrt{2\pi}\right]. \tag{17}$$

It will be convenient to define remainders R_1 and R_2 by

$$\mathscr{N}^* = \varrho(\hat\Theta)\mathscr{N}+R_1 \tag{18}$$

and

$$\mathscr{D}^* = \varrho(\hat\Theta)\,\mathscr{D}+R_2. \tag{19}$$

From the above we obtain, to the desired approximation,

$$P_\pi^X\{\hat{\Theta} < \phi_\alpha(X)\} = \frac{\mathcal{N}^*}{\mathcal{D}^*} = \frac{\varrho(\hat{\Theta})\alpha\mathcal{D}+R_1}{\varrho(\hat{\Theta})\mathcal{D}+R_2} \approx \alpha + \frac{R_1-\alpha R_2}{\varrho(\hat{\Theta})\mathcal{D}}$$

$$\approx \alpha - \frac{\exp(-\frac{1}{2}\Psi_\alpha^2(X))}{\sqrt{2\pi}\,\varrho(\hat{\Theta})}\times$$

$$\times\left\{\frac{\dot{\varrho}(\hat{\Theta})}{(-M_2)^{1/2}} + \frac{1}{2}\frac{\ddot{\varrho}(\hat{\Theta})}{-M_2}\Psi_\alpha(X) + \frac{1}{6}\frac{\dot{\varrho}(\hat{\Theta})M_3}{(-M_2)^2}(\Psi_\alpha^3(X)+3\Psi_\alpha(X))\right\}$$

$$\approx \alpha - \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}\,\varrho(\hat{\Theta})}\times$$

$$\times\left\{\frac{\dot{\varrho}(\hat{\Theta})}{(-M_2)^{1/2}}\left(1+\frac{1}{6}\frac{M_3}{(-M_2)^{3/2}}\Phi^{-1}(\alpha)\right)+\frac{1}{2}\frac{\ddot{\varrho}(\hat{\Theta})}{-M_2}\Phi^{-1}(\alpha)\right\}. \tag{20}$$

The first equality is (15) and the second uses (18), (19), and (8). The first approximate equality uses the fact that R_2 is small compared to $\varrho(\hat{\Theta})\mathcal{D}$, the second uses (16)–(19), and the final approximate equality uses (13).

Taking unconditional expectation we obtain

$$P_\pi\{\Theta < \phi_\alpha(X)\} = E_\pi P_\pi^X\{\Theta < \phi_\alpha(X)\}$$

$$\approx \alpha - \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} E_\pi \frac{1}{\varrho(\hat{\Theta})}\times$$

$$\times\left\{\frac{\dot{\varrho}(\hat{\Theta})}{(-M_2)^{1/2}}\left[1+\frac{1}{6}\frac{M_3}{(-M_2)^{3/2}}\Phi^{-1}(\alpha)\right]+\frac{1}{2}\frac{\ddot{\varrho}(\hat{\Theta})}{-M_2}\Phi^{-1}(\alpha)\right\} \tag{21}$$

Let us first look at the leading term in the expectation.

$$E_\pi\left(\frac{1}{(-M_2)^{1/2}}\frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})}\right)$$

$$= E_\pi \frac{1}{[I(\Theta)-(M_2+I(\Theta))]^{1/2}}\left[\frac{\dot{\varrho}(\Theta)}{\varrho(\Theta)}+\left(\frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})}-\frac{\dot{\varrho}(\Theta)}{\varrho(\Theta)}\right)\right]$$

$$\approx E_\pi \frac{1}{I^{1/2}(\Theta)}\left\{\frac{\dot{\varrho}(\Theta)}{\varrho(\Theta)}\left(1+\frac{1}{2}E_\pi^\Theta\left(\frac{M_2}{I(\Theta)}+1\right)\right)+E_\pi^\Theta\left(\frac{\dot{\varrho}(\hat{\Theta})}{\varrho(\hat{\Theta})}-\frac{\dot{\varrho}(\Theta)}{\varrho(\Theta)}\right)\right\}$$

$$\approx E_\pi \frac{1}{I(\Theta)}\frac{\dot{\varrho}(\Theta)}{\varrho(\Theta)} = \int_a^b \frac{\pi_0(\Theta)}{I^{1/2}(\Theta)}\,d\frac{\pi(\theta)}{\pi_0(\theta)} = -\int_a^b \frac{d}{d\theta}\left(\frac{\pi_0(\theta)}{I^{1/2}(\theta)}\right)\frac{\pi(\theta)}{\pi_0(\theta)}\,d\theta. \tag{22}$$

The remaining, smaller, terms on the right hand side of (21) are treated in a similar way but without the need to introduce conditional expectations. This completes the argument for (1).

From (1) it is plausible that, to the same approximation, under appropriate regularity conditions,

$$P_\theta\{\theta < \phi_\alpha(X)\} \approx \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \frac{1}{\pi_0(\theta)} \times$$

$$\times \frac{d}{d\theta}\cdot\left[\frac{\pi_0(\theta)}{I^{1/2}(\theta)}\left(1+\frac{1}{6}M_3^*(\theta)\Phi^{-1}(\alpha)\right) - \frac{\Phi^{-1}(\alpha)}{2}\frac{d}{d\theta}\frac{\pi_0(\theta)}{I(\theta)}\right]. \tag{23}$$

If, as in earlier sections, we choose

$$\pi_0(\theta) = \sqrt{I(\theta)} \tag{24}$$

it becomes

$$P_\theta\{\theta < \phi_\alpha(X)\}$$

$$\approx \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}}\cdot\frac{\Phi^{-1}(\alpha)}{2}\cdot\frac{1}{\sqrt{I(\theta)}}\left[\frac{1}{3}\frac{d}{d\theta}M_3^*(\theta) - \frac{d^2}{d\theta^2}I^{-\frac{1}{2}}(\theta)\right]. \tag{25}$$

5. A multiparameter case

Again without any attempt at rigor, I shall look at a multiparameter analogue of the result of Section 2 in the special case where the confidence sets are, to a first approximation, half spaces. The notation is essentially that of Section 2 with appropriate modifications. In particular the parameter point θ is assumed to lie in a specified convex subset $\mathscr{T}$ of R^p (rather than R) and $I(\theta)$ is Fisher's information matrix, defined by

$$I_{ij}(\theta) = E_\theta \frac{\partial \log p_\theta(X)}{\partial\theta_i}\frac{\partial \log p_\theta(X)}{\partial\theta_j}. \tag{1}$$

I shall choose a random set $S_\alpha(X)$ to satisfy

$$P_{\pi_0}^X\{\Theta \in S_\alpha(X)\} = \alpha, \tag{2}$$

and I shall also suppose that, as a crude approximation

$$S_\alpha(X) \approx \{\theta:\ \eta'(\hat\Theta)I(\hat\Theta)(\theta-\hat\Theta) < \Phi^{-1}(\alpha)\}, \tag{3}$$

where $\hat\Theta$ is chosen to maximize $\theta \to \pi_0(\theta)p_\theta(X)$ and where η is a function on $\mathscr{T}$ to R^p such that

$$\eta'(\theta)I(\theta)\eta(\theta) = 1 \qquad \text{for all } \theta. \tag{4}$$

The analogue of (2.4) is

$$P_\pi\{\Theta \in S_\alpha(X)\} \approx \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \int_{\mathscr{T}} \frac{1}{\pi_0(\theta)}(\nabla'(\pi_0\eta))(\theta)\pi(\theta)\,d\theta. \tag{5}$$

Here of course ∇ denotes the vector differential operator

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial \theta_1} \\ \vdots \\ \frac{\partial}{\partial \theta_p} \end{bmatrix} \tag{6}$$

and consequently

$$(\nabla'(\pi_0 \eta))(\theta) = \pi_0(\theta) \sum_{i=1}^{p} \frac{\partial \eta_i(\theta)}{\partial \theta_i} + \sum_{i=1}^{p} \eta_i(\theta) \frac{\partial \pi_0(\theta)}{\partial \theta_i}. \tag{7}$$

As in Section 2, (5) is interesting primarily because it suggests that in order to achieve confidence coefficient α to within the accuracy indicated it suffices to choose π_0 so that

$$\nabla'(\pi_0 \eta) = 0. \tag{8}$$

We proceed as in Section 4 except that we are dealing with the multiparameter case and do not require as much accuracy. As I have already indicated in (2), we choose the $S_\alpha(X)$ so that

$$\alpha = P^X_{\pi_0}\{\Theta \in S_\alpha(X)\} = \frac{\mathcal{N}}{\mathcal{D}}, \tag{9}$$

where

$$\mathcal{N} = \int_{S_\alpha(X)} e^{M_\theta} d\theta \tag{10}$$

and

$$\mathcal{D} = \int_{\mathcal{T}} e^{M_\theta} d\theta, \tag{11}$$

with

$$M_\theta = \log[\pi_0(\theta) p_\theta(X)]. \tag{12}$$

Our aim is to approximate $P_\pi\{\Theta \in S_\alpha(X)\}$ where π is a largely arbitrary probability density (for the random variable Θ) with respect to Lebesgue measure in $\mathcal{T}$.

In order to do this, let

$$\varrho(\theta) = \frac{\pi(\theta)}{\pi_0(\theta)}, \tag{13}$$

and consider the posterior probability (under π)

$$P^X_\pi\{\Theta \in S_\alpha(X)\} = \frac{\mathcal{N}^*}{\mathcal{D}^*}, \tag{14}$$

where

$$\mathcal{N}^* = \int_{S_\alpha(X)} e^{M_\theta} \varrho(\theta)\, d\theta \tag{15}$$

and

$$\mathcal{D}^* = \int_{\mathcal{T}} e^{M_\theta} \varrho(\theta)\, d\theta. \tag{16}$$

We approximate $\mathcal{N}^*$ by

$$\mathcal{N}^* = \int_{S_\alpha(X)} e^{M_\theta}\big[\varrho(\hat{\Theta}) + (\varrho(\theta) - \varrho(\hat{\Theta}))\big]\, d\theta$$

$$\approx \varrho(\hat{\Theta})\mathcal{N} + \int_{S_\alpha^*(X)} \exp\big(M_{\hat{\Theta}} + (\theta - \hat{\Theta})' \ddot{M}_{\hat{\Theta}}(\theta - \hat{\Theta})\big)\big(\nabla\varrho(\hat{\Theta})\big)'(\theta - \hat{\Theta})\, d\theta \tag{17}$$

where

$$S_\alpha^*(X) = \{\theta\colon\ \eta'(\hat{\Theta}) I(\hat{\Theta})(\theta - \hat{\Theta}) < \Phi^{-1}(\alpha)\}, \tag{18}$$

which was assumed to approximate $S_\alpha(X)$ in (3). But the integral in (17) can readily be evaluated by imagining a random vector Θ^* normally distributed with mean $\hat{\Theta}$ (treated as a constant) and covariance matrix $-\ddot{M}_{\hat{\Theta}}^{-1}$ (which, for the limited precision needed here, may be identified with $I^{-1}(\hat{\Theta})$. We introduce two real random variables

$$Y = \eta'(\hat{\Theta}) I(\hat{\Theta})(\Theta^* - \hat{\Theta}) \tag{19}$$

and

$$Z = \big(\nabla\varrho(\hat{\Theta})\big)'(\Theta^* - \hat{\Theta}). \tag{20}$$

These are jointly normally distributed with mean 0 and covariance matrix

$$E\binom{Y}{Z}(Y Z) = \begin{pmatrix} 1 & \eta'(\hat{\Theta})\, \nabla\varrho(\hat{\Theta}) \\ \eta'(\hat{\Theta})\, \nabla\varrho(\hat{\Theta}) & \big(\nabla\varrho(\hat{\Theta})\big)' I^{-1}(\hat{\Theta})\big(\nabla\varrho(\hat{\Theta})\big) \end{pmatrix}. \tag{21}$$

Then

$$\frac{\sqrt{\det I(\hat{\Theta})}}{(2\pi)^{p/2}} \int_{S_\alpha^*(X)} \exp\big(-\tfrac{1}{2}(\theta - \hat{\Theta})' I(\hat{\Theta})(\theta - \hat{\Theta})\big)\big(\nabla\varrho(\hat{\Theta})\big)'(\theta - \hat{\Theta})\, d\theta$$

$$= EZ\mathcal{I}\{Y < \Phi^{-1}(\alpha)\} = E(E^Y Z)\mathcal{I}\{Y < \Phi^{-1}(\alpha)\}$$

$$= E\eta'(\hat{\Theta})\, \nabla\varrho(\hat{\Theta})\, Y\mathcal{I}\{Y < \Phi^{-1}(\alpha)\}$$

$$= \eta'(\hat{\Theta})\, \nabla\varrho(\hat{\Theta}) \cdot \left(-\frac{1}{\sqrt{2\pi}} \exp\big(-\tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\big)\right). \tag{22}$$

Thus (17) yields

$$\mathcal{N}^* \approx \varrho(\hat{\Theta})\mathcal{N} - \frac{(2\pi)^{(p-1)/2}}{I^{1/2}(\hat{\Theta})}\eta'(\hat{\Theta})\,\nabla\varrho(\hat{\Theta})\exp\left(M_{\hat{\Theta}} - \tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right). \qquad (23)$$

Similarly, to the same approximation, we have for $\mathcal{D}^*$ defined by (16)

$$\mathcal{D}^* \approx \varrho(\hat{\Theta})\,\mathcal{D}. \qquad (24)$$

Thus (14) and (9) yield

$$\begin{aligned}
P_\pi^X\{\Theta \in S_\alpha(X)\} &= \frac{\mathcal{N}^*}{\mathcal{D}^*} \\
&\approx \frac{\varrho(\hat{\Theta})\mathcal{N} - \dfrac{(2\pi)^{(p-1)/2}}{I^{1/2}(\hat{\Theta})}\eta'(\hat{\Theta})\,\nabla\varrho(\hat{\Theta})\exp\left(M_{\hat{\Theta}} - \tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right)}{\varrho(\hat{\Theta})\,\mathcal{D}} \\
&= \alpha - \frac{1}{\mathcal{D}}\frac{(2\pi)^{(p-1)/2}}{I^{1/2}(\hat{\Theta})}\eta'(\hat{\Theta})\nabla\log\varrho(\hat{\Theta})\exp\left(M_{\hat{\Theta}} - \tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right). \qquad (25)
\end{aligned}$$

Finally we approximate $\mathcal{D}$, defined by (11), by

$$\mathcal{D} = \int_{\mathcal{T}} e^{M_\theta}\,d\theta \approx \int \exp\left(M_{\hat{\theta}} + \tfrac{1}{2}(\theta - \hat{\theta})'\,\ddot{M}_{\hat{\theta}}(\theta - \hat{\theta})\right)d\theta = e^{M_{\hat{\theta}}}\cdot\frac{(2\pi)^{p/2}}{\sqrt{\ddot{M}_{\hat{\theta}}}}. \qquad (26)$$

Then (25) becomes

$$P_\pi^X\{\Theta \in S_\alpha(X)\} \approx \alpha - \frac{1}{\sqrt{2\pi}}\eta'(\hat{\Theta})\,\nabla\log\varrho(\hat{\Theta})\exp\left(-\tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right). \qquad (27)$$

Taking unconditional expectation under π and integrating by parts, assuming $\eta\pi$ vanishes on the boundary of $\mathcal{T}$, we obtain

$$\begin{aligned}
P_\pi\{\Theta \in S_\alpha(X)\} &= E_\pi P_\pi^X\{\Theta \in S_\alpha(X)\} \\
&\approx \alpha - \frac{\exp\left(-\tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right)}{\sqrt{2\pi}}E_\pi\,\eta'(\hat{\Theta})\,\nabla\log\varrho(\hat{\Theta}) \\
&\approx \alpha - \frac{\exp\left(-\tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right)}{\sqrt{2\pi}}E_\pi\,\eta'(\Theta)\,\nabla\log\varrho(\Theta) \\
&= \alpha - \frac{\exp\left(-\tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right)}{\sqrt{2\pi}}\int_{\mathcal{T}}\eta'(\theta)\left[\frac{\nabla\pi(\theta)}{\pi(\theta)} - \frac{\nabla\pi_0(\theta)}{\pi_0(\theta)}\right]\pi(\theta)\,d\theta \\
&= \alpha + \frac{\exp\left(-\tfrac{1}{2}[\Phi^{-1}(\alpha)]^2\right)}{\sqrt{2\pi}}\int_{\mathcal{T}}\frac{1}{\pi_0(\theta)}(\nabla'(\pi_0\eta))(\theta)\,\pi(\theta)\,d\theta, \qquad (28)
\end{aligned}$$

which is (5).

As in earlier sections this suggests that for θ not too close to the boundary of $\mathscr{T}$, subject to appropriate smoothness conditions on the distribution of X, we have

$$P_\theta\{\theta \in S_\alpha(X)\} \approx \alpha + \frac{\exp(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2)}{\sqrt{2\pi}} \frac{1}{\pi_0(\theta)}(V'(\pi_0\eta))(\theta). \tag{29}$$

6. The first-order autoregressive process

A good illustration of the results of Sections 2–4 would be provided by a careful detailed treatment of the problem of estimating the autoregression coefficient of a first-order autoregressive Gaussian process. The present section is merely a sketch of the more obvious features of the problem. I start with the usually unrealistic case where both the mean and the residual variance are known. This simple case gives a roughly correct picture of more realistic models as long as we are not interested in an autoregression coefficient close to ± 1. Then I consider the case where the residual variance is known but the mean is unknown, using Pitman's method to eliminate the unknown mean. The computations are heavy and fairly difficult to interpret. I have not attempted to discuss the case where both the mean and the residual variance are unknown.

The stationary sequence of real random variables $\{X_t\}$, for integral t, is assumed to be distributed in such a way that the

$$U_t = (X_t - \xi) - \alpha(X_{t-1} - \xi) \tag{1}$$

are conditionally normally distributed with mean 0 and variance 1 given the past, $(X_{t-1}, X_{t-2}, \ldots)$. Here α is an unknown real number with

$$-1 < \alpha < 1, \tag{2}$$

and for the present I assume ξ known. From the fact that the variance of X_t is $(1-\alpha^2)^{-1}$ we see that the joint density $p_{\alpha,\xi}$ of $X_1 \ldots X_n$ with respect to Lebesgue measure in R^n is given by

$$\begin{aligned} p_{\alpha,\xi}(x) &= \frac{\sqrt{1-\alpha^2}}{(2\pi)^{n/2}} \exp\Big(-\tfrac{1}{2}\Big[(1-\alpha^2)(x_1-\xi)^2 + \sum_{t=2}^{n} ((x_t-\xi)\alpha(x_{t-1}-\xi))^2\Big]\Big) \\ &= \frac{\sqrt{1-\alpha^2}}{(2\pi)^{n/2}} \exp\Big(-\tfrac{1}{2}\Big[(x_1-\xi)^2 + (x_n-\xi)^2 + (1+\alpha^2)\sum_{t=2}^{n-1}(x_t-\xi) - \\ &\qquad\qquad - 2\alpha \sum_{t=2}^{n} (x_t-\xi)(x_{t-1}-\xi)\Big]\Big). \end{aligned} \tag{3}$$

Without essential loss of generality I take $\xi = 0$.

Now let us compute Fisher's information $I(\alpha)$. Since

$$\frac{\partial^2}{\partial\alpha^2}\log p_\alpha(x) = -\frac{1+\alpha^2}{(1-\alpha^2)^2} - \sum_{t=2}^{n-1} x_t^2, \tag{4}$$

we have

$$I(\alpha) = -E\frac{\partial^2}{\partial\alpha^2}\log p_\alpha(X) = \frac{1+\alpha^2}{(1-\alpha^2)^2} + \frac{n-2}{1-\alpha^2}. \tag{5}$$

Thus the choice of π_0 to which we are led by the considerations of Section 2 is

$$\begin{aligned}\pi_0(\alpha) &= \sqrt{\frac{1+\alpha^2}{(1-\alpha^2)^2} + \frac{n-2}{1-\alpha^2}} \\ &= \frac{\sqrt{n-2}}{\sqrt{1-\alpha^2}}\sqrt{1+\frac{1+\alpha^2}{(n-2)(1-\alpha^2)}} \\ &= \frac{\sqrt{1+\alpha^2}}{1-\alpha^2}\sqrt{1+\frac{(n-2)(1-\alpha^2)}{1+\alpha^2}}.\end{aligned} \tag{6}$$

The second line of (6) indicates that, for large n, π_0 can be approximated by

$$\pi_0(\alpha) \approx \frac{C}{\sqrt{1-\alpha^2}}, \tag{7}$$

except in the neighborhood of $\alpha = \pm 1$. The third line suggests, less firmly, that in the neighborhood of $\alpha = \pm 1$, π_0 can be approximated very roughly by

$$\pi_0(\alpha) \approx \frac{C}{1-\alpha^2}. \tag{8}$$

We shall see later that, when ξ is unknown, the behaviour of the information is drastically different from the above in the neighborhood of $\alpha = +1$, but rather similar in the neighborhood of $\alpha = -1$. Presumably this reflects the fact that if α is close to -1 we can estimate the unknown mean quite accurately but if α is close to $+1$ we cannot. In the latter case most of the information about α (when ξ is known to be 0) comes from the magnitude of the average of the X_t or even from any one of the X_t. The function I given by (5) is symmetric about 0, reflecting the fact that if we define

$$Y_t = (-1)^t X_t, \tag{9}$$

the autocorrelation coefficient of the $\{Y_t\}$ is $-\alpha$. Of course this symmetry is lost when the known $\xi = 0$ is replaced by unknown ξ.

Now let us look briefly at the difference (in the case of large n) between the choice of π_0 given by the approximation (7) and the often thoughtless choice

$$\pi_0 = C, \tag{10}$$

a constant. From (3) with $\xi = 0$ we see that the choice (7) leads to a posterior distribution of α (assumed not to be close to ± 1) that is normal with mean

$$\hat{\alpha} = \frac{\sum_2^n X_t X_{t-1}}{\sum_2^{n-1} X_t^2} \tag{11}$$

and variance

$$\sigma_{\hat{\alpha}}^2 \approx \Big(\sum_2^{n-1} X_t^2\Big)^{-1} \approx \frac{1-\alpha^2}{n}. \tag{12}$$

But, in the neighborhood of $\hat{\alpha}$,

$$\sqrt{1-\alpha^2} = \exp\left(\frac{1}{2}\log(1-\alpha^2)\right) \approx \exp\left(\frac{1}{2}\log(1-\hat{\alpha}^2) - \frac{\hat{\alpha}}{1-\hat{\alpha}^2}(\alpha-\hat{\alpha})\right), \tag{13}$$

so that, still with $\xi = 0$, the posterior density of α is roughly proportional to

$$\exp\left(-\frac{1}{2}\left[\alpha^2 \sum_2^{n-1} X_t^2 - 2\alpha\left(\sum_2^n X_t X_{t-1} - \frac{\hat{\alpha}}{1-\hat{\alpha}^2}\right)\right]\right). \tag{14}$$

Thus to a first approximation, the effect of the choice of (10) is to move the center of the posterior distribution from (11) to

$$\hat{\alpha}^* = \hat{\alpha} - \frac{\hat{\alpha}}{(1-\hat{\alpha}^2)\sum_2^{n-1} X_t^2} \approx \left(1-\frac{1}{n}\right)\hat{\alpha}. \tag{15}$$

The ratio of the magnitude of the shift of the center to the standard deviation of the posterior distribution is roughly $[n(1-\alpha^2)]^{-1/2}$.

For the case of unknown ξ, I shall merely sketch the computation of the information and comment on it very briefly. By a well-known argument of Pitman (1939), the joint density of the differences

$$Z_t = X_{t+1} - X_t, \tag{16}$$

for $t \in \{1, \ldots, n-1\}$, with respect to Lebesgue measure in R^{n-1}, expressed in terms of the original variables, is given by

$$q_\alpha(x) = \int_{-\infty}^{\infty} p_{\alpha,\xi}\, d\xi = \frac{1}{(2\pi)^{(n-1)/2}} \sqrt{\frac{1+\alpha}{n-(n-2)\alpha}} \times$$

$$\times \exp\left(-\frac{1}{2}\left\{x_1^2 + x_n^2 + (1+\alpha^2)\sum_2^{n-1} x_t^2 - 2\alpha \sum_2^n x_t x_{t-1} - \right.\right.$$

$$\left.\left. - \frac{(1-\alpha)[x_1 + x_n + (1-\alpha)\sum_2^{n-1} x_t]^2}{n-(n-2)\alpha}\right\}\right). \tag{17}$$

Thus

$$\frac{\partial^2 \log q_\alpha(x)}{\partial \alpha^2} = -\frac{1}{2(1+\alpha)^2} + \frac{(n-2)^2}{[n-(n-2)\alpha]^2} - \sum_2^{n-1} x_t^2 +$$

$$+ \frac{(1-\alpha)(\sum_2^{n-1} x_t)^2}{n-(n-2)\alpha} + \frac{4(\sum_2^{n-1} x_t)[x_1 + x_n + (1-\alpha)\sum_2^{n-1} x_t]}{[n-(n-2)\alpha]^2} -$$

$$- \frac{2(n-2)}{[n-(n-2)\alpha]^3}[x_1 + x_n + (1-\alpha)\sum_2^{n-1} x_t]^2. \tag{18}$$

After quite a bit of computation we find that the information for this problem is given by

$$I(\alpha) = -E_\alpha \frac{\partial^2 \log q_\alpha(X)}{\partial \alpha^2}$$

$$= \frac{n-2}{1-\alpha^2} - \frac{2(n-2)}{[n-(n-2)\alpha](1-\alpha)} + \frac{2\alpha(1-\alpha^{n-2})}{[n-(n-2)\alpha](1-\alpha^2)(1-\alpha)} + \frac{1}{2(1+\alpha)^2}. \tag{19}$$

Let us look at the order of magnitude of (19) when n is large. The leading term for α bounded away from ± 1 is the third term $\frac{n-2}{1-\alpha^2}$, in agreement with the result (5) for the case of known ξ. The last term in (19) is important only in the neighborhood of -1, where the behaviour of I is also essentially the same as in the case of known ξ. However, in the neighborhood of 1 the present situation is quite different from the case of known ξ.

In order to study the behaviour of I in the neighborhood of 1 it is convenient to put

$$\alpha = 1 - \frac{\beta}{n-2}, \tag{20}$$

and, in evaluating the third term on the r.h.s. of (19) to use

$$1-\alpha^{n-2} = (1-e^{-\beta})+\left[e^{-\beta}-\left(1-\frac{\beta}{n-2}\right)^{n-2}\right]. \tag{21}$$

After some computation we find that

$$\begin{aligned} I\left(1-\frac{\beta}{n-2}\right) &= \frac{2(n-2)^2\left[\left(1-\frac{\beta}{n-2}\right)(1-\beta)+\frac{\beta^2}{2}-\left(1-\frac{\beta}{n-2}\right)e^{-\beta}\right]}{(2+\beta)\left(2-\frac{\beta}{n-2}\right)\beta^2} \\ &+\frac{2(n-2)^2\left(1-\frac{\beta}{n-2}\right)\left[e^{-\beta}-\left(1-\frac{\beta}{n-2}\right)^{n-2}\right]}{(2+\beta)\left(2-\frac{\beta}{n-2}\right)\beta^2}+\frac{1}{2\left(2-\frac{\beta}{n-2}\right)^2} \\ &= T_1+T_2+T_3, \end{aligned} \tag{22}$$

say. For $\alpha \geqslant 0$, that is $\beta \leqslant n-2$, all terms are positive and, subject to an unimportant qualification, the second and third terms are negligible.

For small β,

$$T_1 \sim \frac{(n-2)^2\beta}{12} \tag{23}$$

while for large β,

$$T_1 \sim \frac{(n-2)^2}{\beta\left(2-\frac{\beta}{n-2}\right)} = \frac{n-2}{1-\alpha^2}. \tag{24}$$

It may be useful to observe that

$$T_1 \sim \frac{(n-2)^2(1-\beta+\frac{1}{2}\beta^2-e^{-\beta})}{(2+\beta)\beta^2}, \tag{25}$$

uniformly for bounded β. For the second term in (22) we have

$$T_2 = O\left(\frac{n-2}{2+\beta}e^{-\beta}\right), \tag{26}$$

uniformly for $\beta \leqslant n-2$. Finally, for the third term we have

$$T_3 = O(1), \tag{27}$$

again uniformly for $\beta \leqslant n-2$. Thus T_2 and T_3 are negligible compared to T_1, for $\alpha \geqslant 0$, that is $\beta \leqslant n-2$, with the unimportant exception that for $\beta = O(n^{-1})$, $T_1 = O(n)$ while T_2 is of the exact order of n and T_3 of the exact order of 1.

7. The squared distance of the mean of a multivariate normal distribution from the origin

In this section, my aim is merely to indicate that, at least in this special case, the approximate result of Section 5 is not hopelessly inaccurate. For this purpose I shall try to compare two approaches to the problem described in the title of this section:

(i) the multiparameter approach of Section 5.

(ii) the one-parameter approach of Sections 2 and 3 applied to the sample point $|X|^2$.

In Stein (1959) two other approaches are contrasted:

(iii) confidence sets having the correct probability of coverage, at least asymptotically in the dimension p.

(iv) sets obtained from the posterior distribution corresponding to constant prior "density" for ξ.

Since the case of n observations is isomorphic to that of one dimension I shall, for notational simplicity, consider only the latter case at first. The notational changes required for comparison with the asymptotic theory (in n) will be indicated at the end of this section. In the present paper I shall only sketch the comparison of the two methods described above. I believe that the second approach gives essentially correct results (in the sense of (iii)) except for the end effect at $|\xi| = 0$. I hope to complete a more careful treatment of this question and the material of Section 6 before too long.

Let X be a p-dimensional normal random vector with unknown mean ξ and the identity as covariance matrix. We want to obtain confidence sets for $|\xi|^2$, or equivalently, confidence sets for ξ of the form

$$S_\alpha(X) = \{\xi: |\xi|^2 \leqslant \phi_\alpha(X)\}. \tag{1}$$

Because of the invariance of the problem under orthogonal transformations, it is reasonable to choose ϕ_α of the form

$$\phi_\alpha(X) = \phi_\alpha^*(|X|^2). \tag{2}$$

Let us first look at the approach labelled (i) above, that of Section 5. Fisher's information matrix is the identity matrix and the vector field η for which the confidence sets described by (1) and (2) can be approximated crudely by (5.3), that is

$$S_\alpha(X) \approx \{\xi: \eta'(\xi)(\xi - \Xi) \leqslant \Phi^{-1}(\alpha)\}, \tag{3}$$

is given by

$$\eta(\xi) = \frac{\xi}{|\xi|}. \tag{4}$$

Thus $\eta(\xi)$ is a unit vector in the direction of ξ. Of course, in (3), $\hat{\Xi}$ is chosen to maximize

$$\xi \to \pi_0^*(|\xi|^2)\exp(-\tfrac{1}{2}|X-\xi|^2) \tag{5}$$

where the prior density π_0 of Section 5 is given by

$$\pi_0(\xi) = \pi_0^*(|\xi|^2). \tag{6}$$

With

$$\beta(|\xi|^2) = \frac{\pi_0^*(|\xi|^2)}{|\xi|}, \tag{7}$$

the condition (5.8) becomes

$$0 = \nabla'\left[\pi_0^*(|\xi|^2)\frac{\xi}{|\xi|}\right] = \nabla'(\beta(|\xi|^2)\xi) = p\beta(|\xi|^2) + 2|\xi|^2\beta'(|\xi|^2). \tag{8}$$

The solution of this is

$$\beta(|\xi|^2) = |\xi|^{-p}, \tag{9}$$

that is

$$\pi_0(\xi) = \pi_0^*(|\xi|^2) = |\xi|^{-(p-1)}. \tag{10}$$

Thus the suggested confidence sets are

$$S_\alpha(X) = \{\xi\colon |\xi|^2 \leqslant \phi_\alpha^*(|X|^2)\} \tag{11}$$

where $\phi_\alpha^*(|X|^2)$ is chosen to satisfy

$$\begin{aligned}
\alpha &= P_{\pi_0}^X\{|\xi|^2 \leqslant \phi_\alpha^*(|X|^2)\} \\
&= \frac{\int_{|\xi|^2 \leqslant \phi_\alpha(|X|)^2)} |\xi|^{-(p-1)} e^{-|X-\xi|^2/2}\, d\xi}{\int_{R^p} |\xi|^{-p-1} e^{-|X-\xi|^2/2}\, d\xi} \\
&= \frac{\int_{-\infty}^{\phi_\alpha^*(|X|^2)} \theta^{-(p-1)/2} e^{-|X|^2/2} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}|X|^2)^k}{k!} \frac{\theta^{p/2-k-1}}{\Gamma(\frac{1}{2}p+k)} e^{-\theta}\, d\theta}{\int_{-\infty}^{\infty} \theta^{-(p-1)/2} e^{-|X|^2/2} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}|X|^2)^k}{k!} \frac{\theta^{p/2-k-1}}{\Gamma(\frac{1}{2}p+k)} e^{-\theta}\, d\theta}
\end{aligned}$$

$$= \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(\frac{1}{2}|X|^2)^k}{k!\,\Gamma(\frac{1}{2}p+k)} \int_{-\infty}^{\phi_\alpha^*(|X|^2)} \theta^{k-\frac{1}{2}} e^{-\theta}\, d\theta}{\displaystyle\sum_{k=0}^{\infty} \frac{(\frac{1}{2}|X|^2)^k}{k!\,\Gamma(\frac{1}{2}p+k)} \cdot \Gamma(k+\tfrac{1}{2})}. \tag{12}$$

The approach labelled (ii) at the beginning of this section, the reduction to a one-dimensional problem by use of orthogonal invariance, is presumably better. We consider $|X|^2$ as the observed sample point and note that its distribution depends on ξ only through

$$\theta = |\xi|^2. \tag{13}$$

Although it does not seem easy to compute the information in this problem, we can approximate it by the reciprocal of the variance of $|X|^2 - p$ which is an unbiased estimate of $\theta = |\xi|^2$. This approximate value is

$$I^*(\theta) = (2p+4\theta)^{-1}. \tag{14}$$

The density of

$$Y = |X|^2 \tag{15}$$

is given by

$$p_\theta(y) = e^{-\theta/2} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}\theta)^k}{k!} \frac{y^{p/2+k-1}}{\Gamma(\frac{1}{2}p+k)} e^{-y/2} \tag{16}$$

for $y > 0$. Thus, in the spirit of Section 2, the approximate upper confidence point is given by $\phi_\alpha^{**}(|X|^2)$, determined by the condition that, with

$$\pi_1(\theta) = (2p+4\theta)^{-1/2}, \tag{17}$$

we have

$$\alpha = \frac{\displaystyle\int_0^{\phi_\alpha^{**}(|X|^2)} p_\theta(|X|^2)\, \pi_1(\theta)\, d\theta}{\displaystyle\int_0^{\infty} p_\theta(|X|^2)\, \pi_1(\theta)\, d\theta}. \tag{18}$$

This equation for the determination of ϕ^{**} differs from the equation (12) for the determination of ϕ, which can be rewritten as

$$\alpha = \frac{\displaystyle\int_0^{\phi(|X|^2)} p_\theta(|X|^2)\, d\theta}{\displaystyle\int_0^{\infty} p_\theta(|X|^2)\, d\theta} \tag{19}$$

only by the presence of the factor $\pi_1(\theta)$.

It remains only to verify that, when these results are expressed in terms of large samples, the first and second approaches differ only by the order of n^{-1}. Thus let $Y_1, \ldots, Y_n$ be independently normally distributed p-dimensional random vectors with common unknown mean η and covariance matrix equal to the identity matrix. Then

$$X = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \tag{20}$$

will be a normally distributed p-dimensional random vector with unknown mean

$$\xi = \sqrt{n}\,\eta \tag{21}$$

and covariance matrix equal to the identity matrix. Thus θ, defined by (13) is

$$\theta = |\xi|^2 = n|\eta|^2, \tag{22}$$

and consequently

$$\pi_1(\theta) = (2p + 4n|\eta|^2)^{-1/2}. \tag{23}$$

Locally this differs from a constant by a term of relative order n^{-1}.

8. Closing remarks

In this section I shall try to indicate some of the directions in which I think it may be useful to carry out further work on this subject. Before doing so I shall sketch an argument which indicates that the results of this paper, with the possible exception of Section 4, also hold conditionally given an approximately ancillary statistic. As I mentioned earlier, the applicability of the present method to this question was suggested by Brad Efron when I lectured on this question at Stanford University.

For definiteness I shall consider the case of Section 3 where we have one unknown real parameter and a large number n of independent identically distributed observations. The result sketched here was obtained by Hinkley (1980) by a different method. See also Cox (1980). Of course the fact that I consider this question should not be interpreted as advocacy of the principle of conditionality.

With the notation of Section 3, let

$$T^{(n)} = \tau(X_1, \ldots, X_n) \tag{1}$$

be a real-valued approximately ancillary statistic in the sense that it has a probability density function $p_\theta^{(n)}$ that is nearly independent of θ. A more nearly precise meaning will be given to this statement in the course of the discussion. The conditional density $(x, t) \mapsto \bar{p}^{(n)}(x|t, \theta)$ of $X = (X_1 \ldots X_n)$ given

T, with respect to an arbitrary measure, has the form

$$\bar{p}^{(n)}(x|t,\theta) = v(x)\frac{\prod_{1}^{n} p_\theta^*(x_i)}{p_\theta^{(n)}(t)} \tag{2}$$

where the function v does not depend on θ. We want to know whether the conditional probability (for each θ) of coverage given T of the confidence sets determined from π_0, defined by

$$\pi_0(\theta) = \sqrt{I(\theta)} = \sqrt{nI^*(\theta)}, \tag{3}$$

still differs from α by $O\left(\frac{1}{n}\right)$ under appropriate regularity conditions.

The regularity conditions sketched roughly in Section 2 and, for the case of independent, identically distributed random variables, in Section 3 depended on the derivatives of the likelihood function and on expectations of these derivatives. Let us go through the final remainder terms in Section 2 and study the effect of the transition from the unconditional to the conditional distribution. From (2.27) and (2.23) we see that, for this transition to have no substantial effect on R_2^* it will suffice that for every pair $\theta, \theta' \in \mathscr{T}$, with high probability

$$\left|\log \frac{p_{\theta'}^{(n)}(T^{(n)})}{p_\theta^{(n)}(T^{(n)})}\right| < 1 \tag{4}$$

(or any other constant), and of course this could be weakened. The same is true of R_1^{**} as discussed, for example in (3.7), provided the $O\left(\frac{1}{n}\right)$ is achieved even with $\theta - \hat{\Theta}$ replaced by its absolute value. The same is also true of R_7^{**} as discussed in (3.8).

For R_5^*, and thus also for R_6^*, as discussed in (3.9) and (3.10), the orders of magnitude will remain the same provided that for all θ, with high probability,

$$\left|\frac{\partial^2 \log p_\theta^{(n)}(T^{(n)})}{\partial\theta^2}\right| < 1. \tag{5}$$

Finally we must look at the more difficult $R_8(\theta)$ defined in (2.39) and evaluated around (3.13). For (3.13) to remain valid, after the transition from unconditional to conditional distribution, it should suffice that, again with high probability,

$$\frac{\partial^2 \log p_\theta^{(n)}(T^{(n)})}{\partial\theta^2} = O\left(\frac{1}{\sqrt{n}}\right). \tag{6}$$

In summary, conditions (3) and (5) should suffice for the conditional probability of coverage to remain $\alpha+O\left(\frac{1}{n}\right)$ when the same is true of the unconditional probability of coverage.

The most urgent need for extension of this work occurs in the multiparameter case because it is here that crude asymptotic theory usually fails most spectacularly. It would be very useful to work out a number of examples, even in the same rough way as in Section 7. The work of Section 5 ought also to be developed to at least the (still low) level of rigor of the treatment of the one-parameter case in Sections 2–4. It would also be desirable to study confidence sets of arbitrary shape, especially the roughly elliptic case centered near the maximum likelihood estimate. Here the error in the probability of coverage is $O\left(\frac{1}{n}\right)$ but this is still often unsatisfactory. Thus this situation should be studied with at least the precision of Section 4.

Much remains to be done even in the one-parameter case. First, a number of special cases should be worked out carefully and the results should be compared with serious Monte Carlo studies or direct computation of the probability of coverage. This work has not yet been carried out even in the case of the autoregression problem studied in Section 6. In addition the program of Section 3 should be carried out rigorously.

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