

POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

s. 7133
[250]

DISSERTATIONES MATHEMATICAE (ROZPRAWY MATEMATYCZNE)

KOMITET REDAKCYJNY

BOGDAN BOJARSKI redaktor
ANDRZEJ BIAŁYNIICKI-BIRULA, ZBIGNIEW CIESIELSKI,
JERZY ŁOŚ, ZBIGNIEW SEMADENI

CCL

WITOLD KONDRACKI and JAN ROGULSKI

On the stratification of the orbit space
for the action of automorphisms on connections

Biblioteka Uniwersytecka
w Warszawie



1000676493

WARSZAWA 1986

PAŃSTWOWE WYDAWNICTWO NAUKOWE

5.7133



PRINTED IN POLAND

© Copyright by PWN-Polish Scientific Publishers, Warszawa 1986

ISBN 83-01-06782-9

ISSN 0012-3862

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

BUW-EO-01/41/152

CONTENTS

Introduction .	5
§ 1. Basic notions and notation .	7
1.1. Automorphisms of principal bundles .	7
1.2. Connections and parallel translations	9
1.3. Symmetries and connections .	11
§ 2. The action of the gauge group on connections	14
2.1. The gauge group .	14
2.2. The action of \mathcal{G}^{k+1} on \mathcal{G}^k	17
2.3. Weak and strong invariant metrics on \mathcal{G}^k	20
2.4. The equivariant embedding of \mathcal{G}^k into the space of H^k Riemannian metrics on P	23
§ 3. The Slice Theorem .	30
3.1. The Hodge-Kodaira-like decomposition for $T_e \Phi_A$	30
3.2. The orbits are submanifolds .	36
3.3. The Slice Theorem	38
§ 4. The geometric structure of $\mathcal{A}^k = \mathcal{G}^k / \mathcal{G}^{k+1}$.	43
4.1. Consequences of the Slice Theorem .	44
4.2. The Countability Theorem	47
4.3. Density theorems	49
4.4. The stratification of \mathcal{A}^k	57
References .	61

Introduction

The purpose of this paper is to study the action of the group \mathcal{G} of automorphisms of a principal bundle P on the space \mathcal{C} of connections on P . The physical importance of the orbit space for this action lies in the fact that \mathcal{C}/\mathcal{G} is the proper configuration space for Yang–Mills field theory (see e.g. Gribov [14], Singer [28], Narasimhan and Ramadas [23]).

The study of this action was initiated by Singer [28] in 1978, who announced several interesting results (applicable when the base space of P is S^n and the structure group is a compact semisimple Lie group) concerning the action on the space of irreducible connections. Gribov's ideas and the results announced by Singer have been further investigated by Narasimhan and Ramadas [23] for the case of a trivial $SU(2)$ -bundle P over S^3 and S^4 . The proofs for a more general case, namely, for nontrivial G -bundles P over M (where G is a compact semisimple matrix Lie group and M is a compact manifold without boundary) were presented by Mitter and Viallet [21] in 1981. The authors of the above-mentioned papers considered the following restrictions for the action:

- (i) $\mathcal{G}/(\text{centre}(\mathcal{G}))$ acts on the space $\mathcal{C}_0 \subset \mathcal{C}$ of irreducible connections,
- (ii) the normal subgroup $\mathcal{G}_0 \subset \mathcal{G}$ consisting of automorphisms fixing a given point of P acts on \mathcal{C} .

In both cases such actions are free. It was proved in [21] and [23] for the groups \mathcal{G}^{k+1} and \mathcal{G}_0^{k+1} of automorphisms of Sobolev class H^{k+1} and for the spaces \mathcal{C}^k and \mathcal{C}_0^k of H^k -connections ($k > \frac{1}{2} \dim M + 1$) that $\mathcal{C}_0^k/\mathcal{G}^{k+1}$ and $\mathcal{C}^k/\mathcal{G}_0^{k+1}$ are smooth Hilbert manifolds. Narasimhan and Ramadas proved that $\mathcal{C}_0^k \rightarrow \mathcal{C}_0^k/\mathcal{G}^{k+1}$ and $\mathcal{C}^k \rightarrow \mathcal{C}^k/\mathcal{G}_0^{k+1}$ are smooth principal bundles. Moreover, they proved that $(\mathcal{G}^{k+1}/(\text{centre}))$ -bundle $\mathcal{C}_0^k \rightarrow \mathcal{C}_0^k/\mathcal{G}^{k+1}$ cannot be reduced to \mathcal{G}_0^{k+1} (see [23], Thm. 5.1.). This theorem generalizes the so called "Gribov ambiguity" (consisting in nonexistence of a continuous section $\mathcal{C}_0^k/\mathcal{G}^{k+1} \rightarrow \mathcal{C}_0^k$).

In the present paper we study the action of \mathcal{G}^{k+1} on the whole space \mathcal{C}^k and the "true configuration space" $\mathcal{C}^k/\mathcal{G}^{k+1}$. We deal with a principal G -bundle P over M , where M is a compact manifold without boundary and G is a compact Lie group. The nontrivial principal bundles that we take into account are of physical relevance since they are connected with t'Hooft's

magnetic fluxes (see [15]) and their topological characteristics lead to the instanton numbers (see [2]).

The main difficulty in introducing a differentiable structure onto $\mathcal{C}^k/\mathcal{G}^{k+1}$ lies in the fact that $\mathcal{G}^{k+1}/\text{centre}$ does not act freely on \mathcal{C}^k . In general, \mathcal{G}^{k+1} -action has several orbit types. In § 1 we introduce the notation and a new approach to parallel translations and holonomy groups, especially convenient for our investigations. The evolution bundles, introduced in Section 1.3, are appropriate objects for the description of the isotropy groups of the \mathcal{G}^{k+1} -action on \mathcal{C}^k .

In Section 2.1, applying different techniques from those used by Mitter and Viallet in [21], we prove that \mathcal{G}^{k+1} is a Hilbert–Lie group. We also prove that \mathcal{G}^{k+1} is a smooth submanifold and a topological subgroup of the group $D^{k+1}(P)$ of H^{k+1} -diffeomorphisms of P . We use the techniques of Eells, Palais and their school in order to prove that the action $\mathcal{G}^{k+1} \times \mathcal{C}^k \rightarrow \mathcal{C}^k$ is of class C^∞ . This is not unexpected or unknown, though Mitter and Viallet have given only the proof of its C^1 -smoothness.

\mathcal{G}^{k+1} -invariant, smooth, weak and strong riemannian metrics are constructed in Section 2.3. A smooth, \mathcal{G}^{k+1} -equivariant embedding $F: \mathcal{C}^k \rightarrow \mathcal{M}^k$ into the space of H^k -metrics on P (constructed in Section 2.4) is used to prove that the action is proper. Here we apply the fact that the action of $D^{k+1}(P)$ on \mathcal{M}^k is proper (see Bourguignon [5], II.19, p.8 and Ebin [11]).

In § 3 we prove that every orbit of the action is a closed C^∞ submanifold in \mathcal{C}^k . The existence of a slice at any H^k -connection A is proved. The proofs of existence of a slice at an irreducible connection $A \in \mathcal{C}_0^k \subset \mathcal{C}^k$ can be found in [1] and [23]. However, a generalization of this result to all connections requires some work. We also introduce a notion, which we call the *local slice property*, ensuring the (topological) regularity of the orbit space. This property holds for the $D^{k+1}(P)$ -action on \mathcal{M}^k and it is also satisfied for the \mathcal{G}^{k+1} -action on \mathcal{C}^k .

We refer to [17] for a discussion of slices and slice theorems in general and also for some historical remarks and a generalization of the notion of a slice. In the above-mentioned paper [17] Isenberg and Marsden proved the slice theorem for the action of diffeomorphisms on the space of certain pseudoriemannian metrics. They endowed the orbit space of this action with the stratification structure.

Example (3.3.7) shows that the local slice property (introduced in Section 3.3) is an essential refinement of the classical definition of a slice.

The main results are collected in § 4. We prove that the gauge orbit space is a metrizable, second countable topological space. We endow $\mathcal{C}^k/\mathcal{G}^{k+1}$ with the structure of stratification into C^∞ -Hilbert manifolds. The natural ordering in this stratification (defined in Section 4.4 entirely by the internal topological properties of the orbit space and the strata) is isomorphic to the ordering in the set J of orbit types (of conjugacy classes of

isotropy groups). One of the crucial points in the proof of the final results is Theorem (4.3.2), which states that every connection A can be approximated by connections with an (arbitrarily chosen) holonomy bundle containing the holonomy bundle of A .

§ 1. Basic notions and notation

Let (P, π, M, G) be a smooth G -principal bundle over an n -dimensional C^∞ -manifold M . The map $\pi: P \rightarrow M$ is the projection. We assume that both G and M are compact and connected. We denote the Lie algebra of G by \mathfrak{g} .

1.1. Automorphisms of principal bundles. By an automorphism of P we mean a continuous fibre-preserving map $\varphi: P \rightarrow P$ commuting with the G -action on P , i.e., such that

$$(1.1.1) \quad \pi \circ \varphi = \pi,$$

$$(1.1.2) \quad \forall g \in G \quad \forall p \in P \quad \varphi(p \cdot g) = \varphi(p) \cdot g.$$

Every automorphism appears in a natural way as a section of a certain fibre bundle associated with the principal bundle (P, π, M, G) . The construction of this fibre bundle is as follows. First note that any map $\varphi: P \rightarrow P$ satisfying condition (1.1.1) may be written in the form

$$(1.1.3) \quad \varphi(p) = p \cdot \hat{\varphi}(p),$$

where $\hat{\varphi}: P \rightarrow G$ is uniquely determined by φ . This follows from the fact that the group action on each fibre $P_x = \pi^{-1}(x)$, $x \in M$, of P is free and transitive. If the map φ satisfies (1.1.1) and (1.1.2) then

$$(1.1.4) \quad \hat{\varphi}(p \cdot g) = g^{-1} \hat{\varphi}(p) g.$$

Thus, if φ is an automorphism then the value $\hat{\varphi}(p) \in G$ at fixed point $p \in \pi^{-1}(x)$ determines the values of $\hat{\varphi}$ at each point of $\pi^{-1}(x)$ (by formula (1.1.4)). The above considerations lead to the following construction:

Let

$$\tilde{P} := (P \times G)/G,$$

where the right action of G on $P \times G$ is given by the formula

$$(1.1.5) \quad (p, g_1) \cdot g = (p \cdot g, g^{-1} g_1 g).$$

The action of G on $P \times G$ is free and proper. Hence, $(\tilde{P}, \tilde{\pi}, M)$, where $\tilde{\pi}: \tilde{P} \ni [(p, q)] \mapsto \pi(p) \in M$, is a smooth fibre bundle. This bundle is associated with the principal bundle P (see [6] for the definition), $\tilde{P} = P \times_G G$, where the left action of G on G is defined by

$$G \times G \ni (g, g_1) \mapsto g g_1 g^{-1} \in G.$$

Each fibre $\bar{P}_x = \bar{\pi}^{-1}(x)$ of \bar{P} is a group with the following multiplication:

$$(1.1.6) \quad [(p, g_1)][(p, g_2)] := [(p, g_1 g_2)].$$

Let \mathcal{G}_c denote the group of all (continuous) automorphisms of P and let $\Gamma(\bar{P})$ be the group of all continuous sections of $(\bar{P}, \bar{\pi}, M)$, where the multiplication of sections is pointwise defined.

(1.1.7) PROPOSITION. *There exists a natural isomorphism $\mathcal{G}_c \rightarrow \Gamma(\bar{P})$.*

Proof. Let $\varphi \in \mathcal{G}_c$. We define the section $\tilde{\varphi} \in \Gamma(\bar{P})$ by the formula

$$\tilde{\varphi}(x) := [(p, \hat{\varphi}(p))],$$

where $p \in \pi^{-1}(x)$ is arbitrary and $\hat{\varphi}(p)$ is defined by the given automorphism φ in (1.1.3). The continuity of $\tilde{\varphi}$ can easily be verified with the use of local sections of P and the definition of topology on P . We now show that the map $\mathcal{G}_c \ni \varphi \mapsto \tilde{\varphi} \in \Gamma(\bar{P})$ is a homomorphism. For $\varphi, \psi \in \mathcal{G}_c$ we have

$$\begin{aligned} (\varphi \circ \psi)(p) &= \varphi(\psi(p)) = \psi(p) \cdot \hat{\varphi}(\psi(p)) = p \cdot \hat{\psi}(p) \hat{\varphi}(\psi(p)) \\ &= p \cdot \hat{\psi}(p) \hat{\varphi}(p \cdot \hat{\psi}(p)). \end{aligned}$$

On the other hand,

$$(\varphi \circ \psi)(p) = p \cdot (\varphi \circ \hat{\psi})(p).$$

Combining the above two equalities, we obtain

$$(\varphi \circ \hat{\psi})(p) = \hat{\psi}(p) \hat{\varphi}(p \cdot \hat{\psi}(p))$$

Applying formula (1.1.4) with $g = \hat{\psi}(p)$ to the right-hand side of the above equality, we arrive at the following result:

$$(1.1.8) \quad (\varphi \circ \hat{\psi})(p) = \hat{\varphi}(p) \hat{\psi}(p).$$

From 1.1.8 and the definition of the map $\varphi \mapsto \tilde{\varphi}$ we obtain

$$\begin{aligned} (\varphi \circ \psi)(x) &= [(p, (\varphi \circ \hat{\psi})(p))] = [(p, \hat{\varphi}(p) \hat{\psi}(p))] \\ &= [(p, \hat{\varphi}(p))][[p, \hat{\psi}(p)]] = \tilde{\varphi}(x) \tilde{\psi}(x). \end{aligned}$$

Thus this map is a homomorphism $\mathcal{G}_c \rightarrow \Gamma(\bar{P})$. This homomorphism is in fact an isomorphism since for a given $\tilde{\varphi} \in \Gamma(\bar{P})$ there exists exactly one $\varphi \in \mathcal{G}_c$ corresponding to $\tilde{\varphi}$:

$$\varphi(p) = p \cdot g,$$

where $g \in G$ is such that $\tilde{\varphi}(\pi(p)) = [(p, g)]$. ■

(1.1.9) Remark. If G is abelian or if P is trivial then the bundle \bar{P} is trivial. In general the bundle \bar{P} is not trivial. For instance, if P is a nontrivial $\text{SO}(3)$ – principal bundle over S^4 ($M = S^4$, $G = \text{SO}(3)$) then the corresponding bundle \bar{P} is not trivial. However, the space $\Gamma(\bar{P})$ is always

nonempty, since \bar{P} is the bundle of groups. As in the case of vector bundles, the space $\Gamma(\bar{P})$ is very rich.

We now present another approach to automorphisms. Let us consider the following set of functions:

$$\mathcal{G}_c := \{\hat{\varphi}: P \rightarrow G \mid \hat{\varphi} \text{ is continuous and}$$

$$\forall g \in G \quad \forall p \in P \quad \hat{\varphi}(p \cdot g) = g^{-1} \hat{\varphi}(p) g\}.$$

Of course, \mathcal{G}_c is a subgroup of the group of all continuous G -valued functions on P (the multiplication is pointwise defined). It follows from (1.1.4) and (1.1.8) that the map $\mathcal{G}_c \ni \varphi \mapsto \hat{\varphi} \in \mathcal{G}_c$ defined in (1.1.3) is an isomorphism.

Concluding, we now have three equivalent descriptions of automorphisms:

(i) as fibre preserving homeomorphisms $\hat{\varphi}: P \rightarrow P$ commuting with the G -action,

(ii) as continuous sections $\tilde{\varphi} \in \Gamma(\bar{P})$,

(iii) as continuous functions $\hat{\varphi}: P \rightarrow G$ satisfying condition (1.1.4).

By a morphism of two principal bundles (P_1, π_1, M_1, G_1) and (P_2, π_2, M_2, G_2) with an associated homomorphism $\beta: G_1 \rightarrow G_2$ (or simply: by β -morphism) we mean any (continuous) map $\psi: P_1 \rightarrow P_2$ satisfying the following condition:

$$\forall p_1 \in P_1 \quad \forall g_1 \in G_1 \quad \psi(p_1 \cdot g_1) = \psi(p_1) \cdot \beta(g_1).$$

It is easy to see that every morphism ψ determines uniquely a continuous map $f: M_1 \rightarrow M_2$ such that $\pi_2 \circ \psi = f \circ \pi_1$. A morphism ψ of (P, π, M, G) into itself is an automorphism iff the induced map f is equal to id_M and the associated homomorphism β is equal to id_G .

1.2. Connections and parallel translations. We now recall three equivalent definitions of a connection on a principal bundle P .

(1.2.1) A connection on P is a subbundle $A \subset TP$ satisfying the following conditions:

(i) $\pi_* A_p = T_{\pi(p)} M$ and $A_p \cap T_p P_{\pi(p)} = \{0\}$,

(ii) $A_{p \cdot g} = \bar{g}_* A_p$ (where $\bar{g}: P \ni p \mapsto p \cdot g \in P$),

for any $p \in P$ and $g \in G$.

(1.2.2) Let us consider the set of all n -dimensional subspaces in tangent spaces to P that are transversal to the fibres of P . This set we denote by $G'_n(P)$. As is well known, this set is an open subbundle of the n -th Grassmann bundle over P , and each fibre of this bundle carries the natural structure of $(n \cdot \dim G)$ -dimensional affine space. The right action of G on $G'_n(P)$ is given by the formula

$$G'_n(P) \times G \ni (A_p, g) \mapsto \bar{g}_* A_p \in G'_n(P).$$

Now let $\text{Con } P := G'_n(P)/G$, and $\pi_c: \text{Con } P \ni [A_p] \mapsto \pi(p) \in M$. Since for any fixed $g \in G$ the action $A_p \rightarrow \bar{g}_* A_p$ is an affine mapping from the fibre of $G'_n(P)$ over p onto the fibre over $p \cdot g$, we can see that $\text{Con } P$ is an affine fibre bundle over M . It is obvious that the dimension of fibres in $\text{Con } P$ is also equal to $n \cdot \dim G$. A connection on P is a section \tilde{A} of $\text{Con } P$.

As can easily be seen, any n -dimensional subspace in $T_p P$ that is transversal to the fibre $P_{\pi(p)}$ may be viewed as the kernel of the linear map $\hat{A}_p: T_p P \rightarrow \mathfrak{g}$ such that $\hat{A}_p([p]_* X) = X$, where $[p]: G \ni g \rightarrow p \cdot g \in P$ and $X \in \mathfrak{g} = T_e G$. From this observation we obtain the following definition of a connection on P :

(1.2.3) A connection on P is a \mathfrak{g} -valued 1-form \hat{A} on P such that

$$(i) \quad \forall X \in \mathfrak{g} \quad \forall p \in P \quad \hat{A}([p]_* X) = X,$$

$$(ii) \quad \forall g \in G \quad \forall v \in T_p P \quad \hat{A}_{p \cdot g}(\bar{g}_* v) = \text{Ad } g^{-1} \cdot \hat{A}_p(v).$$

For a vertical tangent vector v the formula in (ii) follows from (i).

The above three definitions of connections are equivalent, and so, in our further considerations, we shall use them alternatively, choosing the most suitable one.

Now we turn our attention to parallel translations and holonomies defined by a connection A of class C^1 . Let $c: [0, 1] \rightarrow M$ be a smooth curve joining $x_0 = c(0) \in M$ and $x = c(1) \in M$. Then the horizontal lifts of c to P by means of A define a map $h_c: P_{x_0} \rightarrow P_x$, called a *parallel translation* along c . The map h_c is a G -morphism of fibres P_{x_0} and P_x of P .

Let us fix a point $x_0 \in M$. Then the set \dot{P} consisting of all G -morphisms $P_{x_0} \rightarrow P$ has the natural structure of a smooth principal bundle over M . The group $\dot{G} := \tilde{P}_{x_0}$ is the structure group of \dot{P} . A smooth structure of \dot{P} can be characterized by the following condition: a local section s of \dot{P} is smooth iff for $p \in P_{x_0}$ the section $x \mapsto s(x)(p) \in P$ is smooth.

The bundle \dot{P} is isomorphic (but not canonically) to P . For a fixed $p \in P_{x_0}$ this isomorphism may be expressed as follows:

$$Q_p: \dot{P} \ni q \mapsto q(p) \in P.$$

The associated isomorphism $\beta: \dot{G} \rightarrow G$ (for Q_p) is then

$$\beta([(p, g)]) = g.$$

The smooth structure on \dot{P} can also be defined by the requirement that Q_p is a diffeomorphism ($Q_{p \cdot g} \circ Q_p^{-1} = \bar{g}$ is a diffeomorphism for any $g \in G$).

All parallel translations $h_c: P_{x_0} \rightarrow P_x$ along all curves c joining x_0 and x form a subset of the fibre \dot{P}_x . The set

$$\dot{H}(A) = \{q \in \dot{P} \mid \text{there exists a curve } c \text{ in } M \text{ such that } q = h_c\}$$

is called a *bundle of parallel translations* of A at $x_0 \in M$. The fibre $\mathring{H}(A)_{x_0}$ of $\mathring{H}(A)$ is clearly a subgroup of $\mathring{G} = \tilde{P}_{x_0}$. The group $\mathring{H}(A)_{x_0}$ is called a *holonomy group* of A at x_0 . We have also $\mathring{H}(A)_x = q\mathring{H}(A)_{x_0}$ for any $q \in \mathring{P}_x \cap \mathring{H}(A)$. Hence $\mathring{H}(A)$ looks like a principal subbundle of \mathring{P} with the holonomy group of A at x_0 as its structure group.

There exists a correspondence between the bundle of parallel translations of A at x_0 and the holonomy bundle of A at $p \in \pi^{-1}(x_0)$. We recall that the holonomy bundle of A at $p \in P$ is defined as a subset $H(A, p) \subset P$ of points that can be joined with p by horizontal curves (see [18] for details).

(1.2.4) **PROPOSITION.** *For any $p \in \pi^{-1}(x_0)$, the set $Q_p(\mathring{H}(A))$ is the holonomy bundle of A at p . In particular, $\mathring{H}(A)$ is a principal subbundle of \mathring{P} of class C^k if and only if $H(A, p)$ is of class C^k . The associated isomorphism $\beta: \mathring{G} \rightarrow G$ maps the holonomy group $\mathring{H}(A)_{x_0}$ at x_0 onto the holonomy group of A at p .*

If M is simply connected and A is a C^1 -connection on P then for any $p \in P$ the holonomy group H_p of A at p is a closed Lie subgroup in G . Moreover, holonomy bundles are then of class C^2 (for the proofs see e.g. [18]). Thus, in this case $\mathring{H}(A)$ is a C^2 -subbundle of \mathring{P} . In general, the holonomy group of A at $p \in P$ is not a closed subgroup of G .

1.3. Symmetries of connections. Now we examine the action of automorphisms on connections. If an automorphism φ of class C^2 and a connection A of class C^1 are given then we can define the subbundle $\varphi \cdot A \subset TP$ in the following way:

$$(1.3.1) \quad (\varphi \cdot A)_{\varphi(p)} := T_p \varphi \cdot A_p.$$

Clearly $\varphi \cdot A$ is also a connection of class C^1 . Expressing the connections A and $\varphi \cdot A$ in terms of \mathfrak{g} -valued 1-forms \hat{A} and $(\varphi \cdot \hat{A})$, respectively (see (1.2.3)), and identifying φ with a G -valued function $\hat{\varphi}$ on P (by (1.1.3)), we obtain from (1.3.1) the following formula:

$$(1.3.2) \quad (\varphi \cdot \hat{A})_p = \text{Ad } \hat{\varphi}(p) \circ \hat{A}_p - R_{\hat{\varphi}(p)^{-1}} \circ T_p \hat{\varphi}.$$

The connection A of class C^1 singles out those C^2 -automorphisms which preserve A , that is, symmetries of A . We denote the set of C^2 symmetries of the connection A by $S(A)$:

$$(1.3.3) \quad S(A) = \{ \varphi \in \mathcal{G}_c \cap C^2(P, P) \mid \varphi \cdot A = A \}.$$

It follows immediately from (1.3.2) that any symmetry $\hat{\varphi}$ of \hat{A} satisfies the following "ordinary" differential equation:

$$(1.3.4) \quad T_p \hat{\varphi} = L_{\hat{\varphi}(p)^{-1}} \circ \hat{A}_p - R_{\hat{\varphi}(p)} \circ \hat{A}_p.$$

It is obvious that any C^1 -solution of the above equation (where \hat{A} is a given C^1 -connection) is also of class C^2 .

The set $S(A) \subset \mathcal{G}_c$ is of course a subgroup, since $(\varphi_1 \circ \varphi_2) \cdot A = \varphi_1 \cdot (\varphi_2 \cdot A)$ which can easily be seen from (1.3.1). Moreover, $S(A)$ is isomorphic to a (closed) subgroup of G . This fact is well known, and therefore we briefly outline only the idea of its proof.

(1.3.5) PROPOSITION. *Let $\varphi \in S(A)$. Then for any $x \in M$ and for any smooth curve $c: [0, 1] \rightarrow M$ joining x_0 and x we have*

$$\tilde{\varphi}(x) = h_c \circ \tilde{\varphi}(x_0) \circ h_c^{-1},$$

where $h_c: P_{x_0} \rightarrow P_x$ is the parallel translation along c with respect to the given C^1 -connection A .

Proof. Let $[0, 1] \ni t \rightarrow p(t) \in P$ be the horizontal lift of the curve c . We see that the curve $[0, 1] \ni t \rightarrow \varphi(p(t))$ is also horizontal for any symmetry φ because $\frac{d}{dt} \varphi(p(t)) = T_{p(t)} \varphi \cdot \frac{dp}{dt} \in A_{\varphi(p(t))}$. The proposition follows from this observation and from the definition of parallel translations. ■

Now we identify $S(A)$ with a subgroup of $\Gamma(\tilde{P})$. Fixing $x_0 \in M$, we define

$$\mathring{S} := \{\tilde{\varphi}(x_0) \in \mathring{G} \mid \tilde{\varphi} \in S(A)\}.$$

Taking $x = x_0$ in (1.3.5), we see that \mathring{S} is contained in the centralizer $C(\mathring{H})$ of the holonomy group $\mathring{H} := \mathring{H}(A)_{x_0}$ in \mathring{G} . Thus, by virtue of Proposition (1.3.5), for any symmetry $\tilde{\varphi} \in \Gamma(\tilde{P})$ of A we obtain

$$\forall x \in M \quad \forall q \in \mathring{H}(A) \cap \mathring{P}_x \quad \forall \mathring{g} \in C(C(\mathring{H})) \quad \tilde{\varphi}(x) = q\mathring{g}\tilde{\varphi}(x_0)\mathring{g}^{-1}q^{-1}$$

(this is clear since the inclusion $\mathring{S} \subset C(\mathring{H})$ implies that $C^2(\mathring{H}) := C(C(\mathring{H})) \subset C(\mathring{S})$). Hence, the section $\tilde{\varphi}$ is determined by its value at a single point x_0 and by the subset

$$(1.3.6) \quad E(A) := \mathring{H}(A)C^2(\mathring{H}) \subset \mathring{P}.$$

For any C^1 -connection A the group $C^2(\mathring{H})$ is a Lie subgroup of G ($C^2(\mathring{H})$ is obviously closed for any subgroup $\mathring{H} \subset \mathring{G}$) and $E(A)$ is a principal subbundle (of class C^2) of \mathring{P} (since P admits local sections of class C^2 taking values in a holonomy bundle—see [18]). We call $E(A)$ an evolution bundle of A .

Now we would like to point out that, in fact, $\mathring{S} = C(\mathring{H})$. If $\mathring{g} \in C(\mathring{H})$ and $\{\chi_j, U_j\}_{j=1}^m$ is a family of local sections of $E(A)$ ($\chi_j: U_j \rightarrow E(A)$ is of class C^2 and $\{U_j\}_{j=1}^m$ is a covering of M), then there exists a global C^2 -section $\tilde{\varphi}_{\mathring{g}}$ of \tilde{P} such that

$$\tilde{\varphi}_{\mathring{g}}(x) = \chi_j(x)\mathring{g}\chi_j(x)^{-1} \quad \text{for } x \in U_j.$$

It can be shown that for any $g \in C(\mathring{H})$ the section $\tilde{\varphi}_g$ is a symmetry of A .

Since $C(\mathring{H})$ is a closed subgroup of \mathring{G} , we can see that \mathring{S} is a compact Lie group.

It appears that we do not need parallel translations of A for the definition of $E(A)$; namely, $E(A)$ is completely determined by $S(A)$. The converse statement also holds: an evolution bundle contains full information about a symmetry group of a connection.

(1.3.7) PROPOSITION. *For any C^1 connection A , the following equalities hold:*

- (i) $E(A) = \{q \in \mathring{P} \mid \forall \tilde{\varphi} \in S(A) \ \tilde{\varphi}(\mathring{\pi}(q)) = q\tilde{\varphi}(x_0)q^{-1}\},$
- (ii) $S(A) = \{\tilde{\varphi} \in \Gamma(\tilde{P}) \mid \forall q \in E(A) \ \tilde{\varphi}(\mathring{\pi}(q)) = q\tilde{\varphi}(x_0)q^{-1}\}.$

Proof. Since $\mathring{S} = C(\mathring{H})$, we infer (by virtue of Proposition (1.3.5)) that the right-hand side of (i) is equal to $\mathring{H}(A) \cdot C^2(\mathring{H}) = E(A)$ (see (1.3.6)). To prove (ii) it is sufficient to show that the values of sections at x_0 form the subgroup $C(\mathring{H}) = \mathring{S} \subset \mathring{G}$. We have $E(A) \cap \mathring{\pi}^{-1}(x_0) = C^2(\mathring{H})$, and therefore the group of values at x_0 of sections belonging to the group on the right-hand side of (ii) is equal to $C^3(\mathring{H})$. Since $C^3(\mathring{H}) = C(\mathring{H})$ for any subgroup $\mathring{H} \subset \mathring{G}$, the proposition follows. ■

(1.3.8) COROLLARY. *Let A_1 and A_2 be C^1 -connections. Then $E(A_1) = E(A_2)$ if and only if $S(A_1) = S(A_2)$.*

It follows by the definition (1.3.6), that $\mathring{H}(A)$ determines $E(A)$. Therefore $\mathring{H}(A)$ also defines $S(A)$. But the converse statement is not true.

(1.3.9) EXAMPLE. Let $M = S^1$, $G = S^1$ and let P be a G -bundle over M (there is no nontrivial S^1 -bundle over S^1). Here $\tilde{P} = M \times G$ (there exists a canonical trivialization). In our case, for any connection A we obtain $E(A) = \mathring{P}$. It is easy to construct a connection with a given subgroup $H \subset S^1$, $H \neq S^1$, as a holonomy group. Moreover, for any A a single fibre of $\mathring{H}(A)$ is finite or countable. Hence, $\mathring{H}(A)$ is not always closed in \mathring{P} . For any A on our P we have $\mathring{S} = S^1$ and $S(A) = \{\tilde{\varphi} \in \mathcal{S}_c \mid \tilde{\varphi} = \text{const}\}$. Therefore, all connections have the same group as a symmetry group.

In the case where the centre $C(G)$ of G is nontrivial, the symmetry group of any connection is also nontrivial. It is easy to see from formula (1.3.2) that any constant function $\hat{\varphi}: P \rightarrow C(G)$ is a symmetry of any connection. On the other hand, it follows from (1.3.2) that if the C^1 -automorphism φ is a symmetry for all smooth connections then $\hat{\varphi}: P \rightarrow G$ is a constant function with values in $C(G)$ (we use here the fact that $\text{Ad } g = \text{id}_\mathfrak{g}$ if and only if $g \in C(G)$). Thus $C(G)$ becomes a subgroup of \mathcal{S} which is responsible for the fact that the action of gauge transformations on connections is not effective.

A connection A is called an irreducible connection if $S(A) = C(G)$. In particular, if the holonomy group $\mathring{H}(A)$ of A is equal to \mathring{G} then A is an irreducible connection.

§ 2. The action of the gauge group on connections

In the previous chapter we have described automorphisms and connections as single objects. Now we begin to study the spaces of automorphisms and connections of a certain Sobolev class. We shall also examine the action of the gauge group (the group of automorphisms) on the space of connections.

2.1. The gauge group. Let $k > n/2$ be an integer. Let us consider the space $H^k(P)$ of all H^k -sections of the bundle P . For the definitions, the basic results and the construction of a C^∞ Hilbert manifold structure in the space of H^k -sections of a fibre bundle we refer the reader to [25]. Under our assumption on k every H^k -section of P is continuous.

(2.1.1) PROPOSITION. *The space $H^k(\tilde{P})$ with the standard Hilbert manifold structure is a Hilbert–Lie group (where the multiplication of sections is pointwise defined).*

Proof. Let $\tilde{P} \times \tilde{P}$ denote the bundle product (i.e., the bundle over M whose fibre at $x \in M$ is $\tilde{P}_x \times \tilde{P}_x$). Consider the map $m: \tilde{P} \times \tilde{P} \rightarrow \tilde{P}$, $m(\varphi_x, \psi_x) = \varphi_x \psi_x^{-1}$ for $\varphi_x, \psi_x \in \tilde{P}_x$, $x \in M$. It is clear that m is a C^∞ map. Let $\tilde{\varphi}, \tilde{\psi} \in H^k(\tilde{P})$. Since $\tilde{\varphi}(x)\tilde{\psi}(x)^{-1} = m(\tilde{\varphi}(x), \tilde{\psi}(x))$ and the section $\tilde{e} \in \Gamma(\tilde{P})$ (defined by $\tilde{e}(x) := [(p, e)]$, $p \in P_x$) is of class C^∞ , we infer in particular that $\tilde{\psi}^{-1} \in H^k(\tilde{P})$ ($\tilde{\psi}^{-1}(x) = m(e(x), \psi(x))$) and, moreover, $\tilde{\varphi}\tilde{\psi} \in H^k(\tilde{P})$ (since $\tilde{\varphi}(x)\tilde{\psi}(x) = m(\tilde{\varphi}(x), \tilde{\psi}^{-1}(x))$). The above properties follow from the fact that the composition of an H^k -map with a smooth map (on the left) is an H^k -map.

Now we consider the following map (between C^∞ Hilbert manifolds):

$$H^k(\tilde{P}) \times H^k(\tilde{P}) \ni (\tilde{\varphi}, \tilde{\psi}) \mapsto \tilde{\varphi}\tilde{\psi}^{-1} \in H^k(\tilde{P}).$$

This map decomposes into the following two maps:

$$H^k(\tilde{P}) \times H^k(\tilde{P}) \ni (\tilde{\varphi}, \tilde{\psi}) \mapsto \tilde{\varphi} \times \tilde{\psi} \in H^k(\tilde{P} \times \tilde{P}),$$

where $(\tilde{\varphi} \times \tilde{\psi})(x) = (\tilde{\varphi}(x), \tilde{\psi}(x))$, and

$$H^k(\tilde{P} \times \tilde{P}) \ni s \mapsto m \circ s \in H^k(\tilde{P}).$$

The smoothness of the first map is evident and the fact that the second map is of class C^∞ follows from the so called α -lemma (see e.g. [12], page 108). This concludes the proof of the smoothness of the map $(\tilde{\varphi}, \tilde{\psi}) \mapsto \tilde{\varphi}\tilde{\psi}^{-1}$. ■

Now let $k > \frac{1}{2} \dim P + 1$ and let $D^k(P)$ be the group of all H^k diffeomorphisms of P . D. Ebin in [11] introduced the structure of a C^∞ Hilbert manifold on $D^k(P)$ and proved that the space $D^k(P)$ is a topological group but is not a Hilbert–Lie group (the left multiplication is not smooth). We now consider the set \mathcal{G}^k of all H^k -automorphisms of P as a subset of $D^k(P)$.

(2.1.2) PROPOSITION. *$\mathcal{G}^k \subset D^k(P)$ is a closed topological subgroup.*

Proof. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{G}^k convergent to $\varphi \in D^k(P)$ with respect to the H^k -topology. Since the convergence with respect to the H^k -topology implies uniform convergence, we obtain for any $p \in P$ and $g \in G$

$$\varphi(p \cdot g) = \lim_{n \rightarrow \infty} \varphi_n(p \cdot g) = \lim_{n \rightarrow \infty} \varphi_n(p) \cdot g = \left[\lim_{n \rightarrow \infty} \varphi_n(p) \right] \cdot g = \varphi(p) \cdot g$$

Here we have also used the continuity of G -action on P . Using the continuity of π , we obtain

$$\pi(\varphi(p)) = \pi\left(\lim_{n \rightarrow \infty} \varphi_n(p)\right) = \lim_{n \rightarrow \infty} \pi(\varphi_n(p)) = \pi(p).$$

Hence, $\varphi \in \mathcal{G}^k$. ■

(2.1.3) PROPOSITION. \mathcal{G}^k is a smooth submanifold in $D^k(P)$.

Proof. First let us note that for any $\varphi \in D^k(P)$ the right multiplication $R_\varphi: D^k(P) \ni \psi \rightarrow \psi \circ \varphi \in D^k(P)$ is a C^∞ map (see [11] or [12]). Therefore, it is sufficient to prove that there exists an open neighbourhood $U \subset D^k(P)$ of id_P such that $U \cap \mathcal{G}^k$ is a smooth submanifold in U .

Let γ be a smooth G -invariant Riemannian metric on P (for the existence of such a metric see [6], Chapter VI, Thm. 2.1). Let $W \subset TP$ be an open neighbourhood of the zero section, such that for any $p \in P$

(i) $W \cap T_p P$ is a ball of radius $r > 0$,

(ii) $\exp_p: W \cap T_p P \rightarrow P$ is a diffeomorphism on its image. The existence of such a radius $r > 0$ follows from the compactness of P . It is clear that W is G -invariant since γ is G -invariant.

Let

$$U := \{\varphi \in D^k(P) \mid \exists X \in H^k(TP) \forall p \in P X(p) \in W \text{ and } \varphi(p) = \exp_p X(p)\}.$$

U is an open neighbourhood of $\text{id}_P \in D^k(P)$ and is also the domain of the local chart:

$$H^k(W) \ni X \mapsto (P \ni p \mapsto \exp_p X(p) \in P) \in U.$$

Let $H_G^k(VTP)$ be the space of G -invariant π -vertical vector fields on P of Sobolev class H^k :

$$H_G^k(VTP) := \{X \in H^k(TP) \mid \pi_* X = 0 \text{ and } \forall g \in G \bar{g}_* X = X\}.$$

It follows from the continuity of $\pi_*: TP \rightarrow TM$ and the continuity of the G -action on TP that $H_G^k(VTP)$ is a closed subspace of $H^k(TP)$ (by virtue of the same argument as in the proof of Proposition (2.1.2))

Now, from the G -invariance of γ it follows that each fibre of P is a totally geodesic submanifold in P . Thus the local chart $H^k(W) \rightarrow U$ defined above gives us a one to one correspondence between π -vertical vector fields and fibre preserving diffeomorphisms. Moreover, the G -invariance of γ

implies that $\mathcal{G}^k \cap U$ is the image of $H_G^k(VTP) \cap H^k(W)$ (by the local chart in question). This concludes the proof. ■

In the previous chapter we presented several approaches to gauge transformations (see Prop. (1.1.7)). We shall now show that the equivalence of $H^k(\tilde{P})$ and \mathcal{G}^k is given by an isomorphism compatible with smooth structures (i.e., by a smooth isomorphism).

(2.1.4) PROPOSITION. *The group \mathcal{G}^k with the smooth structure induced from $D^k(P)$ is a Hilbert–Lie group. The natural isomorphism $H^k(\tilde{P}) \rightarrow \mathcal{G}^k$ is smooth.*

PROOF. We have already proved that $H^k(\tilde{P})$ is a Hilbert–Lie group (2.1.1) and that \mathcal{G}^k is a closed topological subgroup and a submanifold of $D^k(P)$. Therefore it is sufficient to prove that the (algebraic) isomorphism $H^k(\tilde{P}) \rightarrow \mathcal{G}^k$ is also a diffeomorphism.

First, let us introduce the bundle product $\tilde{P} \times P$ and the following (fibre bundle) morphism over id_M :

$$\text{ev}: \tilde{P} \times P \ni ([p, g], p) \mapsto p \cdot g \in P.$$

It is obvious that ev is a C^∞ map. Now the mapping $H^k(\tilde{P}) \ni \tilde{\varphi} \mapsto \varphi \in \mathcal{G}^k$ can be expressed by the following formula:

$$\varphi(p) = \text{ev}(\tilde{\varphi}(\pi(p)), p).$$

For the proof of smoothness we recall some known facts in the theory of manifolds of maps. If F_1, F_2 are fibre bundles over M and the total spaces of F_1, F_2 are compact then the space $H_b^k(F_1, F_2)$ of all H^k bundle maps over id_M is a closed Hilbert submanifold of the Hilbert manifold $H^k(F_1, F_2)$ of all H^k maps $F_1 \rightarrow F_2$. Moreover, if F is another fibre bundle over M (with a compact total space) then $H_b^k(F, F_1 \times F_2)$ is canonically diffeomorphic to $H_b^k(F, F_1) \times H_b^k(F, F_2)$.

We now consider the mapping $H^k(\tilde{P}) \ni \tilde{\varphi} \mapsto \varphi \in \mathcal{G}^k$ as the composition of the following maps:

$$\begin{aligned} H^k(\tilde{P}) \ni \tilde{\varphi} &\mapsto \tilde{\varphi} \circ \pi \in H_b^k(P, \tilde{P}), \\ H_b^k(P, \tilde{P}) \ni \psi &\mapsto (\psi, \text{id}_P) \in H_b^k(P, \tilde{P}) \times H_b^k(P, P), \\ H_b^k(P, \tilde{P}) \times H_b^k(P, P) \ni (\psi, \eta) &\mapsto \psi \times \eta \in H_b^k(P, \tilde{P} \times P), \\ H_b^k(P, \tilde{P} \times P) \ni \zeta &\mapsto \text{ev} \circ \zeta \in H^k(P, P). \end{aligned}$$

The smoothness of the second and the third maps is evident and the smoothness of the fourth map follows from the α -lemma (ev is of class C^∞). The fact that the first map is of class C^∞ can be seen if we use a G -invariant Riemannian metric on P and the metric on M obtained by the standard projection of that on P (or one can prove this immediately by applying a suitably general version of the ω -lemma). The above considerations prove that the map $H^k(\tilde{P}) \ni \tilde{\varphi} \mapsto \varphi = \text{ev} \circ ((\tilde{\varphi} \circ \pi) \times \text{id}_P) \in H^k(P, P)$ is of class C^∞ . It

suffices to show that the map $H^k(\tilde{P}) \ni \tilde{\varphi} \mapsto \varphi \in \mathcal{G}^k$ is smooth since \mathcal{G}^k is a closed, C^∞ submanifold of the open submanifold $D^k(P) \subset H^k(P, P)$.

To end the proof, it is sufficient to verify that the derivative of the map $H^k(\tilde{P}) \ni \tilde{\varphi} \mapsto \varphi \in \mathcal{G}^k$ at $\tilde{\varphi} \in H^k(\tilde{P})$ is bijective. Then, by virtue of the Banach theorem and the inverse function theorem, we infer immediately that this smooth algebraic isomorphism has a smooth inverse. Taking the space $H^k(\tilde{\varphi}^* VT(\tilde{P})) =$ (the space of H^k vertical vector fields on \tilde{P} defined on the image of section $\tilde{\varphi}$) as the tangent space $T_{\tilde{\varphi}} H^k(\tilde{P})$, we find that the inverse linear map

$$T_{\text{id}} \mathcal{G}^k = H_G^k(VTP) \rightarrow H^k(\tilde{\varphi}^* VT(\tilde{P})) = T_{\tilde{\varphi}} H^k(\tilde{P})$$

of the derivative at $\tilde{\varphi}$ is given by the formula

$$X \mapsto \tilde{X}, \quad \tilde{X}(\tilde{\varphi}(\pi(p))) = \text{pr}_*(0_p, [p]^{-1} X(p)),$$

where $\text{pr}: P \times G \rightarrow \tilde{P}$ is the canonical projection $P \times G \rightarrow P \times_G G = \tilde{P}$. Hence, the proposition holds. ■

In order to complete our considerations on gauge transformations we shall say something about the third approach to automorphisms (see (1.1.3) and (1.1.4)). Let G act on G by inner automorphisms, i.e., $G \times G \ni (g, h) \mapsto g^{-1}hg \in G$. Let us consider the set \mathcal{G}^k of all G -equivariant functions $P \rightarrow G$ of Sobolev class H^k :

$$\mathcal{G}^k := \{ \hat{\varphi} \in H^k(P, G) \mid \forall g \in G \forall p \in P \hat{\varphi}(p \cdot g) = g^{-1} \hat{\varphi}(p) g \}.$$

\mathcal{G}^k is a closed topological subgroup of the Hilbert–Lie group $H^k(P, G)$ (with the usual pointwise multiplication of G -valued functions on P). It is also easy to prove that $\mathcal{G}^k \subset H^k(P, G)$ is in fact a Hilbert–Lie subgroup.

We have already proved that the mapping $\mathcal{G}^k \ni \varphi \mapsto \hat{\varphi} \in \mathcal{G}^k$ is an (algebraic) isomorphism (see formula (1.1.8)). One can also prove by using the methods presented above that $\mathcal{G}^k \ni \varphi \mapsto \hat{\varphi} \in \mathcal{G}^k$ is an isomorphism of Hilbert–Lie groups.

Concluding: The Hilbert–Lie groups $H^k(\tilde{P})$, \mathcal{G}^k , and \mathcal{G}^k are canonically isomorphic, and therefore we can identify those spaces.

2.2. The action of \mathcal{G}^{k+1} on \mathcal{G}^k . Let $k > \frac{1}{2} \dim P + 1$. The space $H^k(\mathfrak{g} \otimes T^*P)$ of all \mathfrak{g} -valued, H^k , 1-forms on P (with the standard scalar product defined with the use of a smooth Riemannian metric on P and a scalar product in \mathfrak{g}) is a Hilbert space. It follows from (1.2.3) ((i) and (ii)) that connections of class H^k form a closed affine subspace in that Hilbert space. We denote the space of H^k connections by \mathcal{C}^k . The space \mathcal{C}^k is parallel to the linear subspace of $H^k(\mathfrak{g} \otimes T^*P)$ consisting of all G -equivariant, \mathfrak{g} -valued, H^k , 1-forms on P vanishing on vertical vectors. Namely:

$$\mathcal{C}^k = A + \mathfrak{A}^k \begin{pmatrix} B \\ U \\ W \end{pmatrix}$$

where $A \in \mathcal{C}^k$ is an arbitrary connection and

$$(2.2.1) \quad \mathfrak{A}^k := \{ \alpha \in H^k(\mathfrak{g} \otimes T^*P) \mid (\forall v \in TP \ \forall g \in G \\ \alpha(\bar{g}_* v) = \text{Ad } g^{-1} \alpha(v)) \text{ and } (\forall v \in VTP \ \alpha(v) = 0) \}.$$

It follows from the Sobolev lemma that H^k connections are of class C^1 . Moreover, for $k > \frac{1}{2} \dim P + 1$, H^k connections are of class C^l , where $l = \text{entier}(\frac{1}{2} \dim G) + 1$. This follows from the G -equivariance and the smoothness of the G -action on P . One can also check this property by identifying $\mathcal{C}^k = H^k(\text{Con } P)$ and then applying the Sobolev lemma to that space of H^k -sections of the bundle $\text{Con } P$ over M ($k > \frac{1}{2} \dim M + l$). Hence, the space \mathcal{C}^k is defined for $k > \frac{1}{2} \dim M + 1$ – it contains C^1 connections.

We now examine the action of gauge transformations on \mathcal{C}^k . Let us note that by applying formula (1.3.2) for an H^{k+1} -automorphism φ and an H^k -connection A , we obtain the connection $\varphi \cdot A$ of class H^k . Hence, formula (1.3.2) defines the action

$$\Phi: \mathcal{G}^{k+1} \times \mathcal{C}^k \rightarrow \mathcal{C}^k.$$

(2.2.2) PROPOSITION. *The action Φ of \mathcal{G}^{k+1} on \mathcal{C}^k is affine and is of class C^∞ .*

Proof. It is easy to see from formula (1.3.2) that for any $\varphi \in \mathcal{G}^{k+1}$ the map $\Phi(\varphi, \cdot): \mathcal{C}^k \rightarrow \mathcal{C}^k$ is affine. In order to prove the smoothness of $\Phi: \mathcal{G}^{k+1} \times \mathcal{C}^k \rightarrow \mathcal{C}^k$, we fix a connection $A \in \mathcal{C}^k$ and compose Φ with the affine chart $\mathcal{C}^k \ni A + \alpha \mapsto \alpha \in \mathfrak{A}^k$ defined by A on \mathcal{C}^k . We thus obtain the map $\tilde{\Phi}: \mathcal{G}^{k+1} \times \mathfrak{A}^k \rightarrow \mathfrak{A}^k$, defined by the formula

$$\Phi(\varphi, A + \alpha) =: A + \tilde{\Phi}(\varphi, \alpha).$$

Using (1.3.2), we can express $\tilde{\Phi}$ in the following way:

$$(2.2.3) \quad \tilde{\Phi}(\varphi, \alpha)_p = \text{Ad } \hat{\varphi}(p) \circ \hat{A}_p - \hat{A}_p + R_{\hat{\varphi}(p)^{-1}} T_p \hat{\varphi} + \text{Ad } \hat{\varphi}(p) \circ \alpha_p.$$

Now we have to prove the smoothness of $\tilde{\Phi}$. Since \mathfrak{A}^k is a closed linear subspace of the Hilbert space $H^k(\mathfrak{g} \otimes T^*P)$, it is sufficient to prove that $\tilde{\Phi}$ is a C^∞ function $\mathcal{G}^{k+1} \times \mathfrak{A}^k \rightarrow H^k(\mathfrak{g} \otimes T^*P)$. Formula (2.2.3) allows us to decompose the function $\tilde{\Phi}$ (with values in the linear space $H^k(\mathfrak{g} \otimes T^*P)$) into the following sum of four functions:

$$\tilde{\Phi} = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4,$$

where

$$\begin{aligned} \Phi_1(\varphi, \alpha)_p &= \text{Ad } \hat{\varphi}(p) \circ \hat{A}_p, & \Phi_2(\varphi, \alpha) &= -\hat{A}, \\ \Phi_3(\varphi, \alpha)_p &= R_{\hat{\varphi}(p)^{-1}} T_p \hat{\varphi}, & \Phi_4(\varphi, \alpha)_p &= \text{Ad } \hat{\varphi}(p) \circ \alpha_p. \end{aligned}$$

We examine the smoothness of each function $\Phi_i: \mathcal{G}^{k+1} \times \mathfrak{A}^k \rightarrow H^k(\mathfrak{g} \otimes T^*P)$, $i = 1, 2, 3, 4$, separately. Φ_2 is of class C^∞ since it is constant. The functions

Φ_1 and Φ_3 depend only on $\varphi \in \mathcal{G}^{k+1}$; therefore it is sufficient to prove the smoothness of $\bar{\Phi}_i: \mathcal{G}^{k+1} \rightarrow H^k(\mathfrak{g} \otimes T^*P)$, $i = 1, 3$, defined by the projection $\text{pr}_1: \mathcal{G}^{k+1} \times \mathfrak{A}^k \rightarrow \mathcal{G}^{k+1}$ in the following way: $\bar{\Phi}_i \circ \text{pr}_1 := \Phi_i$, $i = 1, 3$.

Now we define three maps μ , Z and $\tilde{\text{Ad}}$:

$$\mu: G \times G \ni (g_1, g_2) \rightarrow g_1 g_2^{-1} \in G;$$

$$Z: TG \rightarrow TG \text{ is the zero morphism (over } \text{id}_G) \quad Z(v_g) := 0_g;$$

$$\tilde{\text{Ad}}: G \times (\mathfrak{g} \otimes T^*P) \ni (g, y \otimes \beta) \mapsto (\text{Ad } g \cdot y) \otimes \beta \in \mathfrak{g} \otimes T^*P.$$

It is clear that μ , Z and $\tilde{\text{Ad}}$ are of class C^∞ .

Let us consider the composition of the following two maps between Hilbert manifolds:

$$H^{k+1}(P, G) \times H^k(\mathfrak{g} \otimes T^*P) \ni (\hat{\varphi}, \alpha) \rightarrow \hat{\varphi} \times \alpha \in H^k(P, G \times (\mathfrak{g} \otimes T^*P))$$

and

$$H^k(P, G \times (\mathfrak{g} \otimes T^*P)) \ni \zeta \mapsto \tilde{\text{Ad}} \circ \zeta \in H^k(P, \mathfrak{g} \otimes T^*P).$$

These maps are of course smooth, hence their composition is C^∞ . We denote this composition by ψ . The map ψ , when restricted to $\mathcal{G}^{k+1} \times \mathfrak{A}^k$, takes values in $H^k(\mathfrak{g} \otimes T^*P)$ and, moreover, this restriction is then equal to Φ_4 . Composing the map ψ with the map

$$\mathcal{G}^{k+1} \ni \varphi \mapsto (\hat{\varphi}, A) \in H^{k+1}(P, G) \times H^k(\mathfrak{g} \otimes T^*P),$$

we obtain $\bar{\Phi}_1$. Thus, we have proved that Φ_1 and Φ_4 are of class C^∞ .

To complete the proof we have to verify that $\bar{\Phi}_3$ is of class C^∞ . For this purpose we write Φ_3 in the form

$$\bar{\Phi}_3(\varphi) = T\mu \circ (T\hat{\varphi} \times (Z \circ T\hat{\varphi})).$$

The above formula allows us to extend $\bar{\Phi}_3$ onto $H^{k+1}(P, G)$ and to examine its smoothness by a decomposition into simpler maps:

$$H^{k+1}(P, G) \ni \hat{\varphi} \mapsto T\hat{\varphi} \in H_v^k(TP, TG);$$

$$H_v^k(TP, TG) \ni \zeta \mapsto Z \circ \zeta \in H_v^k(TP, TG);$$

$$H_v^k(TP, TG) \times H_v^k(TP, TG) \ni (\zeta, \xi) \mapsto \zeta \times \xi \in H_v^k(TP, T(G \times G));$$

and finally

$$H_v^k(TP, T(G \times G)) \ni \eta \mapsto T\mu \circ \eta \in H_v^k(TP, TG).$$

Here $H_v^k(V_1, V_2)$ denotes the Hilbert manifold of H^k vector bundle morphisms $V_1 \rightarrow V_2$, where V_1, V_2 are vector bundles. It is clear that the above four maps are of class C^∞ and that $\bar{\Phi}_3$ is their composition. This proves that $\bar{\Phi}_3$ is of class C^∞ as a map $H^{k+1}(P, G) \rightarrow H_v^k(TP, TG)$. But the image of (extended) $\bar{\Phi}_3$ is contained in the space $H_v^k(TP, \{e\} \times \mathfrak{g}) = H^k(\mathfrak{g} \otimes T^*P)$, which is a smooth submanifold in $H_v^k(TP, TG)$. Hence, $\bar{\Phi}_3: \mathcal{G}^{k+1} \rightarrow H^k(\mathfrak{g} \otimes T^*P)$ is smooth. ■

2.3. Weak and strong invariant metrics on \mathcal{C}^k . The notions of weak and strong (smooth) Riemannian metrics on Hilbert manifolds are known fairly well – we refer e.g. to [8] and [11] for definitions and important examples.

We construct a weak \mathcal{G}^{k+1} -invariant Riemannian metric on \mathcal{C}^k . For this purpose let us take a G -invariant Riemannian metric γ on P and an $\text{Ad}(G)$ -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} .

Then γ and $\langle \cdot, \cdot \rangle$ define in the standard way a smooth bundle metric (\cdot, \cdot) in the bundle $\mathfrak{g} \otimes T^*P$ and a smooth G -invariant measure μ on P . Now

$$(2.3.1) \quad \mathfrak{A}^k \times \mathfrak{A}^k \ni (\alpha_1, \alpha_2) \rightarrow \int_P (\alpha_1(p), \alpha_2(p)) d\mu(p) =: ((\alpha_1, \alpha_2)) \in \mathbf{R},$$

is a continuous, symmetric and nondegenerate bilinear form. Since \mathcal{C}^k is an affine space modelled on \mathfrak{A}^k (so $T\mathcal{C}^k = \mathcal{C}^k \times \mathfrak{A}^k$), the form $((\cdot, \cdot))$ defines a weak Riemannian metric on \mathcal{C}^k , which is invariant under translations. The smoothness of this metric trivially follows from the construction, namely from its invariance under translations. In order to prove that this weak metric is \mathcal{G}^{k+1} -invariant, we write the formula describing the \mathcal{G}^{k+1} action on $T\mathcal{C}^k$:

$$(2.3.2) \quad T_A \Phi(\varphi, \cdot): T_A \mathcal{C}^k \ni (A, \alpha) \mapsto (\varphi \cdot A, \text{Ad } \hat{\varphi} \cdot \alpha) \in T_{\varphi \cdot A} \mathcal{C}^k.$$

This formula follows immediately from (1.3.2). Now we have

$$\begin{aligned} ((\text{Ad } \hat{\varphi} \cdot \alpha_1, \text{Ad } \hat{\varphi} \cdot \alpha_2)) &= \int_P (\text{Ad } \hat{\varphi}(p) \cdot \alpha_1(p), \text{Ad } \hat{\varphi}(p) \cdot \alpha_2(p)) d\mu(p) \\ &= \int_P (\alpha_1(p), \alpha_2(p)) d\mu(p) = ((\alpha_1, \alpha_2)), \end{aligned}$$

where we use the fact that the metric (\cdot, \cdot) in the bundle $\mathfrak{g} \otimes T^*P$ has been defined with use of the $\text{Ad}(G)$ -invariant scalar product $\langle \cdot, \cdot \rangle$ in \mathfrak{g} . The above equality shows that the weak Riemannian metric introduced here is indeed \mathcal{G}^{k+1} -invariant.

Now we construct a strong metric on \mathcal{C}^k . In our case (here $T\mathcal{C}^k = \mathcal{C}^k \times \mathfrak{A}^k$ is trivial) a strong Riemannian metric may be viewed as a smooth function:

$$\mathcal{C}^k \ni A \rightarrow ((\cdot, \cdot))_A^k \in L(\mathfrak{A}^k, \mathfrak{A}^k; \mathbf{R})$$

with values in the Banach space of continuous bilinear forms on \mathfrak{A}^k . Of course, $((\cdot, \cdot))_A^k$ has to be a symmetric and nondegenerate form defining the original H^k -topology on \mathfrak{A}^k .

The metric $((\cdot, \cdot))_A^k$ constructed below is in fact the scalar product on $H^k(\mathfrak{g} \otimes T^*P) \supset \mathfrak{A}^k$. The definition of $((\cdot, \cdot))_A^k$ is based on the construction of an H^k -scalar product in the space of H^k -sections of a vector bundle defined by using covariant derivatives. The equivalence of the two approaches, the first using covariant derivatives and the second using local trivializations, partial derivatives, partition of unity and so on, is explained in [11].

Let $[\otimes]: (\mathfrak{g} \otimes T^*P) \times (\mathfrak{g} \otimes (\otimes^l T^*P)) \rightarrow (\mathfrak{g} \otimes (\otimes^{l+1} T^*P))$ be the bilinear morphism defined as follows:

$$(y_1 \otimes \alpha_p) [\otimes] (y_2 \otimes \beta_p) := [y_1, y_2] \otimes \alpha_p \otimes \beta_p,$$

where

$$y_1, y_2 \in \mathfrak{g}, \quad \alpha_p \in T_p^*P, \quad \beta_p \in \otimes^l T_p^*P,$$

$l = 1, 2, \dots$, and $[\cdot, \cdot]$ is the Lie bracket in \mathfrak{g} . By ∇ we denote the covariant derivative in the bundles $\mathfrak{g} \otimes (\otimes^l T^*P)$ defined by the metric γ on P :

$$\nabla: H^m(\mathfrak{g} \otimes (\otimes^l T^*P)) \rightarrow H^{m-1}(\mathfrak{g} \otimes (\otimes^{l+1} T^*P)).$$

For the definition of $((\cdot, \cdot))_A^k$ we use another covariant derivative $\overset{A}{\nabla}$ that depends on ∇ and the connection A on P in the following way:

$$(2.3.3) \quad \overset{A}{\nabla} \beta := \nabla \beta + \hat{A} [\otimes] \beta,$$

where β is a section of the bundle $\mathfrak{g} \otimes (\otimes^l T^*P)$. The metric γ and the invariant scalar product $\langle \cdot, \cdot \rangle$ in \mathfrak{g} define metrics in the bundles $\mathfrak{g} \otimes (\otimes^l T^*P)$, $l = 1, 2, \dots$. We denote all of them by the same symbol: (\cdot, \cdot) .

Now we are able to give the formula for $((\cdot, \cdot))_A^k$:

$$(2.3.4) \quad ((\alpha_1, \alpha_2))_A^k := \sum_{l=0}^k \int_P ((\overset{A}{\nabla})^l \alpha_1, (\overset{A}{\nabla})^l \alpha_2) d\mu.$$

(2.3.5) THEOREM. *The strong Riemannian metric*

$$\mathcal{G}^k \ni A \rightarrow ((\cdot, \cdot))_A^k \in L(\mathfrak{A}^k, \mathfrak{A}^k; \mathbb{R})$$

is

- (i) \mathcal{G}^{k+1} -invariant,
- (ii) smooth.

Proof. We first note that properties (i) and (ii) are independent.

(i) We obtain the following formula:

$$(2.3.6) \quad (\nabla \text{Ad } \hat{\varphi} \cdot \beta)_p = \text{Ad } \hat{\varphi}(p) \cdot (\nabla \beta)_p + (R_{\hat{\varphi}(p)-1, T_p \hat{\varphi}}) [\otimes] \text{Ad } \hat{\varphi}(p) \beta_p,$$

where β is a section of $\mathfrak{g} \otimes (\otimes^l T^*P)$, $l = 1, 2, \dots$

Combining formulas (1.3.2), (2.3.3) and (2.3.6), we obtain

$$\overset{\varphi \cdot A}{V} (\text{Ad } \hat{\varphi} \cdot \beta) = \text{Ad } \hat{\varphi} \cdot \overset{A}{V} \beta$$

and further

$$(2.3.7) \quad (\overset{\varphi \cdot A}{V})^l (\text{Ad } \hat{\varphi} \cdot \beta) = \text{Ad } \hat{\varphi} \cdot (\overset{A}{V})^l \beta, \quad l \in \mathbb{N}.$$

Now

$$\begin{aligned} ((\text{Ad } \hat{\varphi} \cdot \alpha_1, \text{Ad } \hat{\varphi} \cdot \alpha_2))_{\varphi \cdot A}^k &= \sum_{l=0}^k \int_P ((\overset{\varphi \cdot A}{V})^l (\text{Ad } \hat{\varphi} \cdot \alpha_1), (\overset{\varphi \cdot A}{V})^l (\text{Ad } \hat{\varphi} \cdot \alpha_2)) d\mu \\ &= \sum_{l=0}^k \int_P (\text{Ad } \hat{\varphi} \cdot (\overset{A}{V})^l \alpha_1, \text{Ad } \hat{\varphi} \cdot (\overset{A}{V})^l \alpha_2) d\mu = \\ &= \sum_{l=0}^k \int_P ((\overset{A}{V})^l \alpha_1, (\overset{A}{V})^l \alpha_2) d\mu = ((\alpha_1, \alpha_2))_A^k. \end{aligned}$$

The first and the fourth equalities hold by virtue of definition (2.3.4), the second is obtained by using (2.3.7) and the third follows from the invariance of the scalar product $\langle \cdot, \cdot \rangle$ in \mathfrak{g} . This calculation and (2.3.2) prove statement (i).

(ii) In order to simplify the notation, we introduce the following abbreviation:

$$H^{k-l} := H^{k-l}(\mathfrak{g} \otimes (\otimes_{l=0}^{l+1} T^* P)), \quad l = 0, 1, 2, \dots, k.$$

The bilinear map

$$H^k \times H^{k-l+1} \ni (\alpha, \beta) \mapsto \alpha [\otimes] \beta \in H^{k-l}$$

is continuous. Hence

$$H^k \ni \alpha \mapsto (\beta \mapsto \alpha [\otimes] \beta) \in L(H^{k-l+1}, H^{k-l})$$

is a continuous linear map (from H^k to the Banach space of all continuous linear operators with the standard norm topology). Since $\overset{A}{V} \in L(H^{k-l+1}, H^{k-l})$ we obtain

$$\mathcal{C}^k \ni A \mapsto \overset{A}{V} \in L(H^{k-l+1}, H^{k-l})$$

is a C^∞ -map (moreover, it is affine).

It is well known that the composition of operators is a continuous multilinear mapping between the spaces of operators. In our case, the map

$$\begin{aligned} L(H^{k-l+1}, H^{k-l}) \times \dots \times L(H^{k-1}, H^{k-2}) \times L(H^k, H^{k-1}) \ni (B_1, \dots, B_l) \\ \mapsto B_1 \circ B_2 \circ \dots \circ B_l \in L(H^k, H^{k-l}) \end{aligned}$$

is a continuous l -linear map, and thus in particular it is of class C^∞ . This fact implies that the map

$$\mathcal{C}^k \ni A \mapsto (\bar{V})^l \in L(H^k, H^{k-l})$$

is of class C^∞ (as the composition of smooth maps).

We now consider the following family of bilinear forms:

$$b_l: H^{k-l} \times H^{k-l} \ni (\beta_1, \beta_2) \mapsto \int_P (\beta_1, \beta_2) d\mu \in \mathbf{R},$$

where $l = 0, 1, \dots, k$. Of course, for any l the form b_l is continuous ($b_l \in L(H^{k-l}, H^{k-l}; \mathbf{R})$). These forms define continuous bilinear forms on the appropriate Banach spaces of operators:

$$\begin{aligned} \tilde{b}_l: L(H^k, H^{k-l}) \times L(H^k, H^{k-l}) \ni (B_1, B_2) \\ \mapsto b_l(B_1(\cdot), B_2(\cdot)) \in L(H^k, H^k; \mathbf{R}). \end{aligned}$$

Since \tilde{b}_l is a continuous bilinear map between Banach spaces, it is of class C^∞ . Composing each \tilde{b}_l with the smooth map $\mathcal{C}^k \ni A \mapsto (\bar{V})^l \in L(H^k, H^{k-l})$ (in both places) we obtain the smooth functions

$$\mathcal{C}^k \ni A \mapsto b_l((\bar{V})^l(\cdot), (\bar{V})^l(\cdot)) \in L(H^k, H^k; \mathbf{R}),$$

where $l = 0, 1, \dots, k$. Since the function $\mathcal{C}^k \ni A \rightarrow ((\cdot, \cdot))_A^k \in L(H^k, H^k; \mathbf{R})$ is the sum of the above $k+1$ smooth functions, it is also of class C^∞ . ■

2.4. The equivariant embedding of \mathcal{C}^k into the space of H^k Riemannian metrics on P . Let \mathcal{M}^k denote the space of all Riemannian metrics on P of Sobolev class H^k . This space is an open and convex cone in the Hilbert space $H^k(T^*P \otimes_s T^*P)$ of all H^k symmetric 2-covariant tensor fields; therefore \mathcal{M}^k carries the natural structure of a C^∞ Hilbert manifold. The group $D^{k+1}(P)$ acts on \mathcal{M}^k (on the left) in the following way:

$$D^{k+1}(P) \times \mathcal{M}^k \ni (\varphi, \gamma) \mapsto \varphi_* \gamma \in \mathcal{M}^k.$$

This action was investigated by D. G. Ebin (see [11]), A. E. Fisher [13] and J. P. Bourguignon [5]. Now we briefly recall some fundamental results contained in those papers. The action of $D^{k+1}(P)$ on \mathcal{M}^k is smooth (C^∞), proper and admits slices at any $\gamma \in \mathcal{M}^k$. There exist also smooth $D^{k+1}(P)$ -invariant weak and strong Riemannian metrics on \mathcal{M}^k . Moreover, the space $\mathcal{M}^k/D^{k+1}(P)$ carries a natural structure of stratification into smooth manifolds.

In this section we shall give the construction of a smooth \mathcal{C}^{k+1} -equivariant embedding $F: \mathcal{C}^k \rightarrow \mathcal{M}^k$. For this construction we need two

objects: a smooth (or H^k) Riemannian metric γ_M on M and an $\text{Ad}(G)$ -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . Now we define the value of the metric $\gamma = F(A)$, $A \in \mathcal{C}^k$, on a pair of vectors $(v_1, v_2) \in T_p P \times T_p P$:

$$(2.4.1) \quad \gamma(v_1, v_2) := \gamma_M(\pi_* v_1, \pi_* v_2) + \langle \hat{A}(v_1), \hat{A}(v_2) \rangle,$$

where $\hat{A} \in H^k(\mathfrak{g} \otimes T^* P)$ is the 1-form of the connection A . It can easily be seen that for any $A \in \mathcal{C}^k$ the resulting bilinear form $F(A)$ is a Riemannian metric of class H^k ; thus $F: \mathcal{C}^k \rightarrow \mathcal{M}^k$. Moreover, $F(A)$ is a G -invariant metric on P :

$$\begin{aligned} F(A)(\bar{g}_* v_1, \bar{g}_* v_2) &= \gamma_M(\pi_* \bar{g}_* v_1, \pi_* \bar{g}_* v_2) + \langle \hat{A}(\bar{g}_* v_1), \hat{A}(\bar{g}_* v_2) \rangle \\ &= \gamma_M((\pi \circ \bar{g})_* v_1, (\pi \circ \bar{g})_* v_2) + \langle \text{Ad } g \hat{A}(v_1), \text{Ad } g \hat{A}(v_2) \rangle \\ &= \gamma_M(\pi_* v_1, \pi_* v_2) + \langle \hat{A}(v_1), \hat{A}(v_2) \rangle = F(A)(v_1, v_2). \end{aligned}$$

Thus, $F(\mathcal{C}^k) \subset \mathcal{M}_G^k$, where \mathcal{M}_G^k denotes a smooth closed submanifold in \mathcal{M}^k consisting of all G -invariant H^k Riemannian metrics on P .

(2.4.2) LEMMA. *The map $F: \mathcal{C}^k \rightarrow \mathcal{M}^k$ is of class C^∞ and is injective.*

Proof. It has already been shown that $F(\mathcal{C}^k) \subset \mathcal{M}^k$. Since \mathcal{M}^k is a submanifold in $H^k(T^* P \otimes T^* P)$, then it is clearly sufficient to examine F as a map $F: \mathcal{C}^k \rightarrow H^k(T^* P \otimes T^* P)$. Now F decomposes into the sum of two functions on \mathcal{C}^k (with values in the Hilbert space $H^k(T^* P \otimes T^* P)$), $F = F_1 + F_2$, where F_1 is the constant function, namely:

$$F_1(A)(v_1, v_2) := \gamma_M(\pi_* v_1, \pi_* v_2).$$

Hence the smoothness of F_1 is trivial. We now verify the smoothness of F_2 , considering F_2 as the composition of simpler maps.

$$F_2(A)(v_1, v_2) = \langle \hat{A}(v_1), \hat{A}(v_2) \rangle.$$

F_2 is the composition of two maps:

$$\mathcal{C}^k \ni A \mapsto \hat{A} \times \hat{A} \in H^k((\mathfrak{g} \otimes T^* P) \times (\mathfrak{g} \otimes T^* P)),$$

where

$$(\hat{A} \times \hat{A})(v_1, v_2) = (\hat{A}(v_1), \hat{A}(v_2)),$$

and

$$\begin{aligned} H^k((\mathfrak{g} \otimes T^* P) \times (\mathfrak{g} \otimes T^* P)) \ni \hat{A}_1 \times \hat{A}_2 \\ \mapsto \langle \cdot, \cdot \rangle \circ (\hat{A}_1 \times \hat{A}_2) \in H^k(T^* P \otimes T^* P). \end{aligned}$$

The first map is evidently of class C^∞ . The second is the left composition of H^k -maps (from M into $(\mathfrak{g} \otimes T^* P) \times (\mathfrak{g} \otimes T^* P)$) with the smooth morphism " $\langle \cdot, \cdot \rangle$ ": $(\mathfrak{g} \otimes T^* P) \times (\mathfrak{g} \otimes T^* P) \rightarrow T^* P \otimes T^* P$,

$$\langle \cdot, \cdot \rangle(\alpha_1, \alpha_2) := \langle \cdot, \cdot \rangle(\alpha_1 \times \alpha_2)$$

for $\alpha_1, \alpha_2 \in \mathfrak{g} \otimes T_p^* P$. Therefore, the second map is also of class C^∞ . Since F_1

and F_2 are smooth, F is also smooth. In order to prove the injectivity of F we take two different connections A_1 and A_2 . Then, there exists a vector $v \in TP$ which is A_1 -horizontal but is not A_2 -horizontal. Then $\hat{A}_1(v) = 0$ and $\hat{A}_2(v) \neq 0$, so $F(A_2)(v, v) - F(A_1)(v, v) = \langle \hat{A}_2(v), \hat{A}_2(v) \rangle > 0$. Thus, we have proved that $F(A_1) \neq F(A_2)$. ■

Let us recall that any G -invariant metric γ on P projects on M and defines a Riemannian metric $\hat{\gamma}$ on M :

$$\hat{\gamma}(\pi_* v_1, \pi_* v_2) := \gamma(v_1, v_2),$$

where $v_1, v_2 \in T_p P$ and v_1, v_2 are γ -orthogonal to the fibre $\pi^{-1}(P)$.

Now let

$$(2.4.3) \quad \mathcal{M}_{G,F}^k := \{ \gamma \in \mathcal{M}_G^k \mid \hat{\gamma} = \gamma_M, \forall p \in P \forall X, Y \in \mathfrak{g} \\ \langle X, Y \rangle = \gamma([p]_* X, [p]_* Y) \}.$$

The metric γ_M and the scalar product $\langle \cdot, \cdot \rangle$ are the same as in (2.4.1). It is clear that $F(\mathcal{C}^k) \subset \mathcal{M}_{G,F}^k$. We shall show that $\mathcal{M}_{G,F}^k$ is precisely the image of F .

Let $\{X_1, X_2, \dots, X_m\}$ be a $\langle \cdot, \cdot \rangle$ -orthonormal basis in \mathfrak{g} . Each element of \mathfrak{g} defines a vector field on P . The vector field on P corresponding to X_i we denote by w_i :

$$w_i(p) := [p]_* X_i, \quad i = 1, 2, \dots, m.$$

The fields $\{w_1, \dots, w_m\}$ are all smooth nonvanishing vertical vector fields. Moreover, they span the bundle VTP of all vertical vectors on P . Let us recall that any G -invariant metric γ defines a connection on P by means of subspaces γ -orthogonal to fibres of P . Using X_i 's and w_i 's, we can express the 1-form of the connection $\bar{F}(\gamma)$ corresponding to the metric $\gamma \in \mathcal{M}_{G,F}^k$ by the explicit formula

$$(2.4.4) \quad \bar{F}(\gamma)(v) := \sum_{i=1}^m \gamma(w_i(p), v) X_i, \quad v \in T_p P.$$

It is easy to verify that \bar{F} maps $\mathcal{M}_{G,F}^k \rightarrow \mathcal{C}^k$ and that

$$F \circ \bar{F} = \text{id}_{\mathcal{M}_{G,F}^k}.$$

This proves that $F(\mathcal{C}^k) \supset \mathcal{M}_{G,F}^k$, and thus

$$\mathcal{M}_{G,F}^k = F(\mathcal{C}^k),$$

and hence \bar{F} is the inverse map of F defined on its image.

(2.4.5) LEMMA. $\mathcal{M}_{G,F}^k$ is a smooth closed submanifold in \mathcal{M}^k .

Proof. We consider the map

$$\tilde{\mathcal{X}}: H^k(T^* P \otimes_s T^* P) \rightarrow H^k(T^* P \otimes_s T^* P)$$

defined by the formula

$$\tilde{\mathcal{X}}(\gamma)(v_1, v_2) := \gamma(v_1 - \sum_{i=1}^m \gamma(w_i(p), v_1)w_i(p), v_2 - \sum_{j=1}^m \gamma(w_j(p), v_2)w_j(p))$$

where $v_1, v_2 \in T_p P$ and the vector fields w_1, \dots, w_m on P are the same as before. We can rewrite the above formula in the following form:

$$\tilde{\mathcal{X}}(\gamma) = \gamma + \sum_{i,j=1}^m (w_j \lrcorner (w_i \lrcorner \gamma))(w_i \lrcorner \gamma) \otimes (w_j \lrcorner \gamma) - 2 \sum_{i=1}^m (w_i \lrcorner \gamma) \otimes (w_i \lrcorner \gamma).$$

Since the multiplication of H^k functions is smooth, it is evident that $\tilde{\mathcal{X}}: H^k(T^*P \otimes_s T^*P) \rightarrow H^k(T^*P \otimes_s T^*P)$ is a C^∞ map.

Let

$$B_2 := \{b \in H^k(T^*P \otimes_s T^*P) \mid b|_{VTP \times VTP} = 0, b \text{ is } G\text{-invariant}\}.$$

Then B_2 is a closed linear subspace in $H^k(T^*P \otimes_s T^*P)$. Now we single out the subset $\mathcal{M}_{G, \langle, \rangle}^k \subset \mathcal{M}^k$ in the following way:

$$\mathcal{M}_{G, \langle, \rangle}^k := (\gamma_1 + B_2) \cap \mathcal{M}^k,$$

where $\gamma_1 \in \mathcal{M}_{G,F}^k$ is fixed and $\gamma_1 + B_2$ denotes the (closed) affine subspace in $H^k(T^*P \otimes_s T^*P)$. It is not difficult to verify that the definition of $\mathcal{M}_{G, \langle, \rangle}^k$ does not depend on the choice of $\gamma_1 \in \mathcal{M}_{G,F}^k$. The space $\mathcal{M}_{G, \langle, \rangle}^k$ consists of all H^k , G -invariant metrics on P such that their restrictions to vertical tangent subspaces look like $\langle \cdot, \cdot \rangle$ (after transport to \mathfrak{g}). Now it becomes clear that $\mathcal{M}_{G, \langle, \rangle}^k$ is closed in \mathcal{M}^k , since B_2 is closed in $H^k(T^*P \otimes_s T^*P)$ and the topology on \mathcal{M}^k is induced from $H^k(T^*P \otimes_s T^*P)$. Moreover, $\mathcal{M}_{G, \langle, \rangle}^k$ is a smooth submanifold in \mathcal{M}^k since \mathcal{M}^k is open in $H^k(T^*P \otimes_s T^*P)$ and $\mathcal{M}_{G, \langle, \rangle}^k$ is the intersection of a closed affine subspace with \mathcal{M}^k .

Now let

$$B_1 := \{b \in B_2 \mid \forall 1 \leq i \leq m \quad w_i \lrcorner b = 0\}.$$

Of course, B_1 is a closed linear subspace in B_2 . B_1 consists of all H^k symmetric 2-covariant G -invariant tensor fields on P that are lifts (by π^*) of H^k 2-covariant symmetric tensor fields on M .

We shall show that the image $\tilde{\mathcal{X}}(\mathcal{M}_{G, \langle, \rangle}^k)$ is contained in B_1 . For $\gamma \in \mathcal{M}_{G, \langle, \rangle}^k$ we have $w_j \lrcorner (w_i \lrcorner \gamma) = \delta_{ij}$ and further

$$\tilde{\mathcal{X}}(\gamma) = \gamma - \sum_{i=1}^m (w_i \lrcorner \gamma) \otimes (w_i \lrcorner \gamma), \quad \gamma \in \mathcal{M}_{G, \langle, \rangle}^k.$$

Thus,

$$w_j \lrcorner \tilde{\mathcal{X}}(\gamma) = 0 \quad \text{for } 1 \leq j \leq m \text{ and for any } \gamma \in \mathcal{M}_{G, \langle, \rangle}^k.$$

To prove the G -invariance of $\tilde{\mathcal{X}}(\gamma)$ (for $\gamma \in \mathcal{M}_{G, \langle, \rangle}^k$), we take $v, v' \in T_p P$ and we decompose these tangent vectors into the vertical and the γ -orthogonal

(to the fibre) parts: $v = v_1 + v_2$, $v' = v'_1 + v'_2$, where $\pi_* v_1 = \pi_* v'_1 = 0$, $\gamma(v_1, v_2) = \gamma(v'_1, v'_2) = 0$. Now

$$\tilde{\mathcal{X}}(\gamma)(v, v') = \tilde{\mathcal{X}}(\gamma)(v_2, v'_2) = \gamma(v_2, v'_2).$$

On the other hand,

$$\tilde{\mathcal{X}}(\gamma)(\bar{g}_* v, \bar{g}_* v') = \tilde{\mathcal{X}}(\gamma)(\bar{g}_* v_2, \bar{g}_* v'_2) = \gamma(\bar{g}_* v_2, \bar{g}_* v'_2) = \gamma(v_2, v'_2).$$

From the above formulas it follows that $\tilde{\mathcal{X}}(\gamma)$ is G -invariant, whence $\tilde{\mathcal{X}}(\gamma) \in B_1$ for $\gamma \in \mathcal{M}_{G, \langle, \rangle}^k$.

Thus, restricting $\tilde{\mathcal{X}}$ to the closed smooth submanifold $\mathcal{M}_{G, \langle, \rangle}^k \subset \mathcal{M}^k$, we obtain the C^∞ function

$$\mathcal{X}: \mathcal{M}_{G, \langle, \rangle}^k \ni \gamma \mapsto \tilde{\mathcal{X}}(\gamma) \in B_1.$$

Since

$$\mathcal{M}_{G, F}^k = \mathcal{X}^{-1}(\pi^* \gamma_M),$$

it is evident that $\mathcal{M}_{G, F}^k$ is a closed subset in $\mathcal{M}_{G, \langle, \rangle}^k$. Thus $\mathcal{M}_{G, F}^k$ is also closed in \mathcal{M}^k . To prove that $\mathcal{M}_{G, F}^k$ is a smooth submanifold, we show that the derivative of \mathcal{X} at any point of $\mathcal{M}_{G, F}^k$ is surjective (onto B_1). It is clear that we can identify

$$T \mathcal{M}_{G, \langle, \rangle}^k = \mathcal{M}_{G, \langle, \rangle}^k \times B_2.$$

We have the following formula for the tangent mapping of \mathcal{X} :

$$T_\gamma \mathcal{X} \cdot b = b - \sum_{i=1}^n [(w_i \lrcorner b) \otimes (w_i \lrcorner \gamma) + (w_i \lrcorner \gamma) \otimes (w_i \lrcorner b)]$$

for $b \in B_2$ and $\gamma \in \mathcal{M}_{G, \langle, \rangle}^k$.

Since $w_i \lrcorner b = 0$ for any $b \in B_1$ and $1 \leq i \leq m$, it is evident that $T_\gamma \mathcal{X} \cdot b = b$ for any $b \in B_1$. Hence $T_\gamma \mathcal{X}$ is a surjective projection onto B_1 . This completes the proof of the lemma. ■

(2.4.6) LEMMA. *The mapping $F: \mathcal{C}^k \rightarrow \mathcal{M}^k$ is equivariant with respect to the actions of \mathcal{G}^{k+1} on \mathcal{C}^k and $\mathcal{G}^{k+1} \subset D^{k+1}(P)$ on \mathcal{M}^k .*

Proof. Let $\varphi \in \mathcal{G}^{k+1}$ and $A \in \mathcal{C}^k$. For $v_1, v_2 \in T_p P$ we have

$$\begin{aligned} (\varphi_* F(A))(v_1, v_2) &= F(A)(\varphi_*^{-1} v_1, \varphi_*^{-1} v_2) \\ &= \gamma_M(\pi_* \varphi_*^{-1} v_1, \pi_* \varphi_*^{-1} v_2) + \langle \hat{A}(\varphi_*^{-1} v_1), \hat{A}(\varphi_*^{-1} v_2) \rangle. \end{aligned}$$

Since $\pi \circ \varphi^{-1} = \pi$ and $\hat{A}(\varphi_*^{-1} v) = (\varphi_* \hat{A})(v)$, we obtain

$$(\varphi_* F(A))(v_1, v_2) = \gamma_M(\pi_* v_1, \pi_* v_2) + \langle (\varphi_* \hat{A})(v_1), (\varphi_* \hat{A})(v_2) \rangle.$$

It follows from the above relation that

$$\varphi_* F(A) = F(\varphi \cdot A). \quad \blacksquare$$

(2.4.7) LEMMA. *The tangent map TF is injective.*

Proof. Let us recall that $T\mathcal{C}^k = \mathcal{C}^k \times \mathfrak{U}^k$ (see (2.2.1)), and \mathcal{C}^k is an affine subspace in $H^k(\mathfrak{g} \otimes T^*P)$. $T\mathcal{M}^k$ is also trivial and $T\mathcal{M}^k = \mathcal{M}^k \times H^k(T^*P \otimes_s T^*P)$. For $\alpha \in \mathfrak{U}^k$, $v_1, v_2 \in T_p P$, we have

$$(T_A F \cdot \alpha)(v_1, v_2) = \langle \alpha(v_1), \hat{A}(v_2) \rangle + \langle \hat{A}(v_1), \alpha(v_2) \rangle.$$

Let $T_A F \cdot \alpha = 0$. This means, in particular, that for any π -vertical vector $v_1 \in T_p P$ and for any $v_2 \in T_p P$

$$\langle \alpha(v_1), \hat{A}(v_2) \rangle + \langle \hat{A}(v_1), \alpha(v_2) \rangle = 0.$$

Since $\alpha(v_1) = 0$ and \hat{A} takes all values in \mathfrak{g} (for the π -vertical vectors), we infer from the above equality that

$$\forall X \in \mathfrak{g} \quad \forall v_2 \in T_p P \quad \langle X, \alpha(v_2) \rangle = 0.$$

This proves that $\alpha = 0$ (for $\langle \cdot, \cdot \rangle$ is non-degenerate). ■

Now we sum up the results obtained in the following theorem:

(2.4.8) **THEOREM.** *Let γ_M be an H^k Riemannian metric on M . Let F be the mapping defined on \mathcal{C}^k with values in $H^k(T^*P \otimes_s T^*P)$ given by formula (2.4.1):*

$$F(A)(v_1, v_2) := \gamma_M(\pi_* v_1, \pi_* v_2) + \langle \hat{A}(v_1), \hat{A}(v_2) \rangle,$$

where $A \in \mathcal{C}^k$, $v_1, v_2 \in T_p P$.

Then $F: \mathcal{C}^k \rightarrow \mathcal{M}^k$ is a C^∞ , \mathcal{G}^{k+1} -equivariant embedding onto the smooth closed submanifold $\mathcal{M}_{G,F}^k \subset \mathcal{M}^k$ defined in (2.4.3). Moreover, F is a C^∞ diffeomorphism on its image.

Proof. The sequence of the Lemmas (2.4.2), (2.4.5), (2.4.6) and (2.4.7) together with the equality $F(\mathcal{C}^k) = \mathcal{M}_{G,F}^k$ (which had been proved before Lemma (2.4.5)) give us the proof of the first statement of the theorem. To prove the rest we have only to show that the inverse mapping $F^{-1} = \bar{F}: \mathcal{M}_{G,F}^k \rightarrow \mathcal{C}^k$ (defined in (2.4.4)) is smooth.

Let us recall that

$$F^{-1}: \mathcal{M}_{G,F}^k \ni \gamma \mapsto \sum_{i=1}^m (w_i \lrcorner \gamma) \otimes X_i \in \mathcal{C}^k,$$

where $\{X_1, \dots, X_m\}$ is a fixed basis in \mathfrak{g} and w_i 's are the smooth vector fields on P (corresponding to X_i 's). Since the maps

$$H^k(T^*P \otimes T^*P) \ni \gamma \mapsto (w_i \lrcorner \gamma) \otimes X_i \in H^k(\mathfrak{g} \otimes T^*P)$$

are linear and continuous (hence of class C^∞), the smoothness of F^{-1} is evident. ■

The above theorem allows us to apply the results obtained by D. G. Ebin, A. E. Fisher and J. P. Bourguignon (mentioned at the beginning of 2.4) to the case of the \mathcal{G}^{k+1} action on \mathcal{C}^k . In particular, if we restrict the weak

and strong $D^{k+1}(P)$ -invariant Riemannian metrics on \mathcal{M}^k (constructed by D. G. Ebin) to the submanifold $\mathcal{M}_{G,F}^k$, then the resulting metrics are \mathcal{G}^{k+1} -invariant. Then, pulling back these metrics by F , we obtain the gauge invariant weak and strong Riemannian metrics on \mathcal{C}^k . It appears that these pull-backs do not depend on the choice of F (on γ_M). Moreover, in this way we get the same structures as in Section 2.3.

Another simple consequence of Theorem (2.4.8) (and the fact, proved by J. P. Bourguignon, that the action of $D^{k+1}(P)$ on \mathcal{M}^k is proper) is the following

(2.4.9) THEOREM. *The action Φ of \mathcal{G}^{k+1} on \mathcal{C}^k is proper.*

Proof. We have to show that the map

$$\tilde{\Phi}: \mathcal{G}^{k+1} \times \mathcal{C}^k \ni (\varphi, A) \mapsto (A, \varphi \cdot A) \in \mathcal{C}^k \times \mathcal{C}^k$$

is proper. We already know that the action Ψ of $D^{k+1}(P)$ on \mathcal{M}^k is proper, i.e., that the map

$$\tilde{\Psi}: D^{k+1}(P) \times \mathcal{M}^k \ni (\varphi, \gamma) \mapsto (\gamma, \varphi_* \gamma) \in \mathcal{M}^k \times \mathcal{M}^k$$

is proper. Now we add to this information the following facts proved in Theorem (2.4.8):

1) the diagram

$$\begin{array}{ccc} D^{k+1}(P) \times \mathcal{M}^k & \xrightarrow{\tilde{\Psi}} & \mathcal{M}^k \times \mathcal{M}^k \\ \text{id} \times F \uparrow & & \uparrow F \times F \\ \mathcal{G}^{k+1} \times \mathcal{C}^k & \xrightarrow{\tilde{\Phi}} & \mathcal{C}^k \times \mathcal{C}^k \end{array}$$

commutes;

2) F is a homeomorphism of \mathcal{C}^k onto its image $F(\mathcal{C}^k)$, which is a closed subset in \mathcal{M}^k .

These two properties imply that for any subset $V \subset \mathcal{G}^{k+1} \times \mathcal{C}^k$ we have

$$\tilde{\Phi}(V) = (F \times F)^{-1}(\tilde{\Psi}((\text{id} \times F)(V))).$$

It is evident from this formula that $\tilde{\Phi}$ is a closed map since $\text{id} \times F$, $\tilde{\Psi}$ are closed maps and $F \times F$ is continuous.

Similarly, for any subset $K \subset \mathcal{C}^k \times \mathcal{C}^k$ we have

$$\tilde{\Phi}^{-1}(K) = (\text{id} \times F)^{-1}(\tilde{\Psi}^{-1}((F \times F)(K))).$$

If K is compact, then $(F \times F)(K)$ is compact. Also $\tilde{\Psi}^{-1}((F \times F)(K))$ is then compact, for $\tilde{\Psi}$ is proper. Since $\text{id} \times F$ is a homeomorphism onto its image and this image is closed in $D^{k+1}(P) \times \mathcal{M}^k$, we infer that $\tilde{\Phi}^{-1}(K)$ is compact for compact K . This proves that Φ is a proper action. ■

From the above theorem we obtain the following corollaries.

(2.4.10) COROLLARY. *The symmetry group of any connection is compact.*

(2.4.11) COROLLARY. *All orbits of the action $\Phi: \mathcal{G}^{k+1} \times \mathcal{C}^k \rightarrow \mathcal{C}^k$ are closed.*

The facts described in (2.4.9), (2.4.10) and (2.4.11) were announced by I. M. Singer [28] and proved by M. S. Narasimhan and T. R. Ramadas [23] (in the case of $M = S^3$, $G = \text{SU}(2)$ and $P = M \times G$) However, the approach presented here is quite different, and the result obtained is more general.

(2.4.12) Remark. The equivariant embedding F converts the affine action Φ into a linear action of \mathcal{G}^{k+1} on $\mathcal{M}^k \subset H^k(T^*P \otimes T^*P)$.

One of the most important consequences of Theorem (2.4.9) is the following statement:

(2.4.13) COROLLARY. *The orbit space $\mathcal{C}^k/\mathcal{G}^{k+1}$ is a Hausdorff space.*

Proof. The orbit space for any continuous proper action of a topological group on a topological space is a Hausdorff space (see [4a] § 4.2, page 49).

§ 3. The Slice Theorem

First we note that the action Φ of \mathcal{G}^{k+1} on \mathcal{C}^k is not free. It is even non-effective, as we already pointed out at the end of Section 1.3. The action of $\mathcal{G}^{k+1}/C(G)$ on \mathcal{C}^k becomes effective, but it turns out that it is still not free. This is the consequence of the fact that connections may have different holonomy (and symmetry) groups. This observation leads to difficulties when one tries to endow the quotient space $\mathcal{C}^k/\mathcal{G}^{k+1}$ with a differentiable structure.

In this chapter we prove that the action Φ admits a slice at any $A \in \mathcal{C}^k$. This is the first step in giving $\mathcal{C}^k/\mathcal{G}^{k+1}$ some kind of a differentiable structure.

3.1. The Hodge-Kodaira-like decomposition for $T_e \Phi_A$. It is evident from the identification $\mathcal{G}^{k+1} \approx \hat{\mathcal{G}}^{k+1} \subset H^{k+1}(P, G)$ given at the end of Section 2.1 that the Lie algebra $\mathcal{L}\mathcal{G}^{k+1}$ of the group \mathcal{G}^{k+1} can be identified with the space of H^{k+1} , \mathfrak{g} -valued, equivariant functions on P , namely:

$$(3.1.1) \quad \mathcal{L}\mathcal{G}^{k+1} = \{ \lambda \in H^{k+1}(P, \mathfrak{g}) \mid \forall g \in G \quad \forall p \in P \\ \lambda(p \cdot g) = \text{Ad } g^{-1} \cdot \lambda(p) \}$$

Translating the above into the terminology of associated bundles, we see that $\mathcal{L}\mathcal{G}^{k+1}$ is the space of all H^{k+1} sections of the vector bundle $P \times_G \mathfrak{g}$ associated with the principal G -bundle P by means of the representation Ad . This bundle is denoted by $\text{Ad } P := P \times_G \mathfrak{g}$. Thus,

$$\mathcal{L}\mathcal{G}^{k+1} = H^{k+1}(\text{Ad } P).$$

Since, for any $g \in G$, $\text{Ad } g: \mathfrak{g} \rightarrow \mathfrak{g}$ is an isomorphism of the Lie algebra, it is clear that any fibre of $\text{Ad } P$ has the structure of a Lie algebra. The Lie multiplication in $H^{k+1}(\text{Ad } P)$ is simply the pointwise Lie multiplication of sections.

We can also use the bundle $\text{Ad } P$ to express the tangent space \mathfrak{A}^k to the space \mathcal{C}^k as a Hilbert space consisting of all H^k -sections of a certain vector bundle (see (2.2.1)):

$$\mathfrak{A}^k = H^k(T^*M \otimes \text{Ad } P).$$

The notions introduced above are useful for the investigation of the derivative $T_e \Phi_A: \mathcal{L}\mathcal{C}^{k+1} \rightarrow \mathfrak{A}^k$, where $\Phi_A: \mathcal{C}^{k+1} \ni \varphi \mapsto \varphi \cdot A \in \mathcal{C}^k$. We shall show further that orbits are smooth submanifolds in \mathcal{C}^k and that the tangent space $T_A(\mathcal{C}^{k+1} \cdot A)$ is equal to $\text{im } T_e \Phi_A$. For this purpose, we have to prove first of all that $\text{im } T_e \Phi_A$ is closed in \mathfrak{A}^k .

Simple calculations give us the following formula for $T_e \Phi_A$:

$$(3.1.2) \quad (T_e \Phi_A \cdot \lambda)_p = \text{ad } \lambda(p) \circ \hat{A}_p - T_p \lambda$$

where λ and \hat{A} are a \mathfrak{g} -valued equivariant function and a connection 1-form on P , respectively. Using (2.3.3), we can write

$$T_e \Phi_A \cdot \lambda = -\overset{\hat{A}}{\nabla} \lambda.$$

The linear operator

$$T_e \Phi_A: H^{k+1}(\text{Ad } P) \rightarrow H^k(T^*M \otimes \text{Ad } P)$$

is a first order differential operator acting between the bundles $\text{Ad } P$ and $T^*M \otimes \text{Ad } P$ over M . $T_e \Phi_A = -\nabla_A$, where ∇_A is the covariant derivative in the associated bundle $\text{Ad } P$ defined by the connection A on P . It is clear that if A is a smooth connection then ∇_A is a differential operator with smooth coefficients; therefore we can treat $T_e \Phi_A$ as the closure of the operator $-\nabla_A: C^\infty(\text{Ad } P) \rightarrow C^\infty(T^*M \otimes \text{Ad } P)$. If A is not smooth then this construction fails. However, we can decompose the operator $T_e \Phi_A$ into two parts: one of them is a first order smooth differential operator and the second is a zero order differential operator with H^k -coefficients. Let A_∞ be a fixed smooth connection on P . Then for any $A \in \mathcal{C}^k$ we have a unique decomposition $A = A_\infty + \alpha$, and from (3.1.2) we obtain

$$(3.1.3) \quad T_e \Phi_A = -\nabla_{A_\infty} + B_\alpha.$$

If we treat α as a section of $T^*M \otimes \text{Ad } P$ then the action of B_α on a section $\tilde{\lambda} \in H^{k+1}(\text{Ad } P)$ is given by the formula

$$(3.1.4) \quad (B_\alpha \cdot \tilde{\lambda})(v_x) = [\tilde{\lambda}(x), \alpha(v_x)]_x,$$

where $v_x \in T_x M$ and $[\ , \]_x$ is the Lie bracket in the fibre $(\text{Ad } P)_x$.

First we prove that for a smooth connection A the operator $T_e \Phi_A$ has a closed image and, moreover, that this image admits a closed, weakly orthogonal, complementary subspace (the weak scalar product on \mathfrak{A}^k was defined by (2.3.1)). The computation of the symbol of this first order

differential operator (acting between vector bundles $\text{Ad } P$ and $T^*M \otimes \text{Ad } P$) gives the following result:

$$\sigma(T_e \Phi_A)(\omega_x): (\text{Ad } P)_x \ni \tilde{\lambda}_x \mapsto -\omega_x \otimes \tilde{\lambda}_x \in T_x^* M \otimes (\text{Ad } P)_x$$

for $\omega_x \in T_x^* M$ and $x \in M$. Thus, it is clear that for $\omega_x \neq 0$ the symbol $\sigma(T_e \Phi_A)(\omega_x)$ is a monomorphism; thus $T_e \Phi_A$ is a differential operator with an injective symbol.

As is well known, a scalar product in a vector space invariant under a representation of the structure group of P defines a smooth bundle metric in the associated vector bundle. In particular, the $\text{Ad}(G)$ -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} defines the metric (\cdot, \cdot) in the bundle $\text{Ad } P$. Taking a smooth measure μ on M and (\cdot, \cdot) one defines the L^2 scalar product $(\cdot, \cdot)_0$ for the sections of $\text{Ad } P$:

$$(\tilde{\lambda}_1, \tilde{\lambda}_2)_0 := \int_M (\tilde{\lambda}_1(x), \tilde{\lambda}_2(x)) d\mu(x).$$

The weak scalar products $(\cdot, \cdot)_0$ on $H^{k+1}(\text{Ad } P)$ and $((\cdot, \cdot))$ on $H^k(T^*M \otimes \text{Ad } P)$ define the formally adjoint operator (to $-\nabla_A$):

$$(-\nabla_A)^*: C^\infty(T^*M \otimes \text{Ad } P) \rightarrow C^\infty(\text{Ad } P).$$

Of course, $(-\nabla_A)^*$ is a first order differential operator, and therefore it has a unique continuous extension to the operator $H^k(T^*M \otimes \text{Ad } P) \rightarrow H^{k-1}(\text{Ad } P)$. This extension of $(-\nabla_A)^*$ we denote by $(T_e \Phi_A)^*$.

(3.1.5) PROPOSITION. *For any smooth connection A the following decomposition holds:*

$$H^k(T^*M \otimes \text{Ad } P) = \text{im}(T_e \Phi_A) \oplus \ker(T_e \Phi_A)^*,$$

where the spaces $\text{im}(T_e \Phi_A)$ and $\ker(T_e \Phi_A)^*$ are both closed in the H^k topology and orthogonal with respect to the weak scalar product $((\cdot, \cdot))$.

Proof. The proof of this proposition follows from the fact that $-\nabla_A$ is a differential operator with an injective symbol and from the general theorems given for such operators in [11] (see Prop. 6.8 and Corollary 6.9. therein). ■

The spaces $\text{im}(T_e \Phi_A)$ and $\ker(T_e \Phi_A)^*$ are also closed in the weak topology given by $((\cdot, \cdot))$. This results from the following lemma:

(3.1.6) LEMMA. *Let \mathcal{X} be a Banach space and let $((\cdot, \cdot))$ be a weak scalar product on \mathcal{X} (i.e., a continuous, symmetric and positive bilinear form). Let us assume that there exist two linear subspaces $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{X}$ such that $\mathcal{X}_1 \cap \mathcal{X}_2 = \{0\}$, $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{X}_1, \mathcal{X}_2$ are $((\cdot, \cdot))$ -orthogonal. Then the subspaces \mathcal{X}_1 and \mathcal{X}_2 are both closed in \mathcal{X} and, moreover, \mathcal{X}_1 and \mathcal{X}_2 are closed in the weak topology given by $((\cdot, \cdot))$.*

Proof. Let us denote $\mathcal{X}_1^\perp := \{y \in \mathcal{X} \mid ((y, y')) = 0 \text{ for any } y' \in \mathcal{X}_1\}$. Since \mathcal{X}_1^\perp is the intersection of kernels of continuous functionals on \mathcal{X} ,

$$\mathcal{X}_1^\perp = \bigcap_{y' \in \mathcal{X}_1} \ker(\mathcal{X} \ni y \mapsto ((y, y')) \in \mathbf{R}),$$

it follows that \mathcal{X}_1^\perp is closed in \mathcal{X} and is also closed in the weak topology. By the assumption, $\mathcal{X}_2 \subset \mathcal{X}_1^\perp$. Now let $y \in \mathcal{X}_1^\perp$. We have $y = y_1 + y_2$, where $y_1 \in \mathcal{X}_1$, $y_2 \in \mathcal{X}_2$ are uniquely determined. Of course, $((y_1, y)) = 0$. On the other hand, $((y_1, y)) = ((y_1, y_1)) + ((y_1, y_2)) = ((y_1, y_1))$. Hence $((y_1, y_1)) = 0$. This means that $y_1 = 0$, and thus $y = y_2 \in \mathcal{X}_2$. Thus, $\mathcal{X}_1^\perp \subset \mathcal{X}_2$, and $\mathcal{X}_1^\perp = \mathcal{X}_2$. Replacing the indices 1 and 2 in the above considerations, we obtain $\mathcal{X}_2^\perp = \mathcal{X}_1$, and the lemma is proved. ■

Now we pass to the general case, where A is of class H^k . Our aim is to prove the decomposition given in (3.1.5) for any $A \in \mathcal{C}^k$. In our consideration we deal with differential operators with coefficients of Sobolev class H^m , $m > n/2$, acting between the vector bundles $\xi = \text{Ad } P$ and $\eta = T^*M \otimes \text{Ad } P$ over M or between ξ and $\eta = \xi$. Such operators are defined as H^m -sections of bundles $\text{Hom}(J^r \xi, \eta)$ $r = 0, 1, 2, \dots$ where $\text{Hom}(J^r \xi, \eta) = (J^r \xi)^* \otimes \eta$ is the vector bundle obtained from the r -jet bundle $J^r \xi$ and η . It is clear that any r th order differential operator D with H^m coefficients naturally defines the linear map $D: C^\infty(\xi) \rightarrow H^m(\eta)$. If $m > n/2 + r$ and the vector bundles ξ, η are equipped with metrics and M with a smooth measure, then there exists a uniquely defined formally adjoint operator D^* which is an r th order differential operator with H^{m-r} coefficients. It is defined in the same way as in the C^∞ case, i.e.,

$$((D\lambda, \alpha)) = (\lambda, D^* \alpha)_0 \quad \text{for any } \alpha \in C^\infty(\eta), \lambda \in C^\infty(\xi).$$

Operators with H^m coefficients have recently been investigated by Y. Choquet-Bruhat and D. Christodoulou [9, 10] and also by M. Cantor [7]. The proofs of decompositions of Hilbert spaces of sections obtained by means of operators with injective symbols may be found in [7] and [27]. We now summarize the main results which we shall need in this and subsequent sections. All proofs of statements (in the setting expressed here) are given in [27].

(3.1.7) **PROPOSITION.** *Let $m > n/2$ and let $D: C^\infty(\xi) \rightarrow H^m(\eta)$ be an r -th order differential operator. Then for any $-m+r \leq l \leq m+r$ there exists a uniquely determined continuous linear operator*

$$D_l: H^l(\xi) \rightarrow H^{l-r}(\eta)$$

which is an extension of D .

Now let $L: C^\infty(\xi) \rightarrow C^\infty(\eta)$ be an elliptic operator of order $r+1$ (not necessarily a differential operator, but $L \in E_{r+1}(\xi, \eta)$ in the terminology of Palais [26]). As is well known, such an operator admits a continuous

extension $L_l: H^l(\xi) \rightarrow H^{l-r-1}(\eta)$ for any $l \in \mathbb{Z}$ and the operator L_l is a Fredholm operator.

(3.1.8) PROPOSITION. *Let $m \geq r \geq 0$ be integers and let $m > n/2$. Let D satisfy the assumptions of Proposition (3.1.7). Then*

$$(L+D)_0: H^0(\xi) \rightarrow H^{-r-1}(\eta)$$

is a Fredholm operator and it is regular in the sense that

$$(L+D)_0^{-1}(H^{l-r-1}(\eta)) \subset H^l(\xi)$$

for any $0 \leq l \leq m+r+1$.

(3.1.9) COROLLARY. $\ker(L+D)_0 \subset H^{m+r+1}(\xi)$, and therefore $\ker(L+D)_0 = \ker(L+D)_{m+r+1}$.

By virtue of Rellich's Lemma and the fact that compact operators form an ideal in the space of continuous operators, we infer that the extensions

$$(L+D)_l: H^l(\xi) \rightarrow H^{l-r-1}(\eta),$$

for $0 \leq l \leq m+r+1$, are Fredholm operators. Here the identity $(L+D)_l = L_l + D_{l-1} \circ \iota$, where $\iota: H^l(\xi) \rightarrow H^{l-1}(\xi)$, is useful. Since $(L+D)_l$ are Fredholm operators, they have closed images $((L+D)_0(H^l(\xi)) \subset H^{l-r-1}(\eta))$ are closed, finite codimensional subspaces).

(3.1.10) THEOREM. *Let $L \in E_{r+1}(\xi, \eta)$ be an elliptic (pseudodifferential) operator and let D be an r th order differential operator with H^m coefficients, where $m > n/2+r$ and $m \geq 2r$. Then, for any $0 \leq l \leq m$,*

$$H^l(\eta) = \text{im}(L+D)_{l+r+1} \oplus \ker(L^*+D^*)_0,$$

where the decomposition is a topological direct sum, and the subspaces are orthogonal with respect to the L^2 scalar product defined by the bundle metric on η .

Now we return to the investigation of the derivative $T_e \Phi_A$ for $A = A_\infty + \alpha$, where A_∞ is a smooth connection and α is of class H^k . We treat the operator B_α (given by (3.1.4)) as a 0-order differential operator with H^k coefficients and $-\nabla_{A_\infty}$ as a first order differential operator with an injective symbol. Thus

$$T_e \Phi_A = (-\nabla_{A_\infty} + B_\alpha)_{k+1}: H^{k+1}(\xi) \rightarrow H^k(\eta),$$

where we denote $\xi := \text{Ad } P$ and $\eta := T^*M \otimes \text{Ad } P$ (compare it with (3.1.3)). Let us consider the second order differential operator

$$\Delta_A := (-\nabla_{A_\infty} + B_\alpha)^* (-\nabla_{A_\infty} + B_\alpha),$$

where the star denotes the formally adjoint operator (with H^{k-1} coefficients).

Of course, we use the weak scalar products $((,))$ and $(,)_0$ to define $(-\nabla_{A_\infty} + B_\alpha)^*$. Now

$$(-\nabla_{A_\infty} + B_\alpha)^*(-\nabla_{A_\infty} + B_\alpha) = \nabla_{A_\infty}^* \nabla_{A_\infty} + (-\nabla_{A_\infty}^* B_\alpha - B_\alpha^* \nabla_{A_\infty} + B_\alpha^* B_\alpha).$$

Since ∇_{A_∞} has the injective symbol, the operator $L := \nabla_{A_\infty}^* \nabla_{A_\infty} : C^\infty(\xi) \rightarrow C^\infty(\xi)$ is elliptic, $L \in E_2(\xi, \xi)$. The operator

$$D := -\nabla_{A_\infty}^* B_\alpha - B_\alpha^* \nabla_{A_\infty} + B_\alpha^* B_\alpha$$

is a first order differential operator with H^{k-1} coefficients (under the assumption $k > n/2 + 1$), $D : C^\infty(\xi) \rightarrow H^{k-1}(\xi)$. Thus, for the operator $\nabla_A = L + D$, Theorem (3.1.10) with $m = k - 1$, $r = 1$ and $l = k - 1$ can be applied.

(3.1.11) COROLLARY. *Let $k > n/2 + 2$. Then*

$$H^{k-1}(\text{Ad } P) = \text{im}(\Delta_A)_{k+1} \oplus \ker(\Delta_A^*)_0,$$

where both spaces on the right-hand side are closed and orthogonal with respect to the weak scalar product $(,)_0$.

As can be easily seen, $\Delta_A^* = \Delta_A$. Therefore, applying (3.1.9) for $\Delta_A = L + D$, $m = k - 1$ and $r = 1$, we obtain $\ker(\Delta_A^*)_0 = \ker(\Delta_A)_0 = \ker(\Delta_A)_{k+1} \subset H^{k+1}(\xi)$. This subspace is finite-dimensional since $(\Delta_A)_0$ is a Fredholm operator (see (3.1.8)).

(3.1.12) PROPOSITION

$$\text{im}(-\nabla_{A_\infty}^* + B_\alpha^*)_k = \text{im}(\Delta_A)_{k+1}.$$

Proof. It is clear that $(\Delta_A)_{k+1} = (-\nabla_{A_\infty}^* + B_\alpha^*)_k \circ (-\nabla_{A_\infty} + B_\alpha)_{k+1}$. Therefore $\text{im}(\Delta_A)_{k+1} \subset \text{im}(-\nabla_{A_\infty}^* + B_\alpha^*)_k$. Since for any $\lambda \in H^k(\eta)$ and $\alpha' \in H^{k+1}(\xi)$

$$((-\nabla_{A_\infty}^* + B_\alpha^*)_k \lambda, \alpha')_0 = ((\lambda, (-\nabla_{A_\infty} + B_\alpha)_{k+1} \alpha')),$$

it is clear that $\text{im}(-\nabla_{A_\infty}^* + B_\alpha^*)_k$ is orthogonal to $\ker(-\nabla_{A_\infty} + B_\alpha)_{k+1}$ with respect to $(,)_0$. Now let $\alpha' \in \ker(\Delta_A)_{k+1}$. We have

$$0 = (\alpha', (\Delta_A)_{k+1} \alpha')_0 = ((-\nabla_{A_\infty} + B_\alpha)_{k+1} \alpha', (-\nabla_{A_\infty} + B_\alpha)_{k+1} \alpha'),$$

whence $(-\nabla_{A_\infty} + B_\alpha)_{k+1} \alpha' = 0$, and thus $\alpha' \in \ker(-\nabla_{A_\infty} + B_\alpha)_{k+1}$. Thus we have proved that $\ker(\Delta_A)_{k+1} \subset \ker(-\nabla_{A_\infty} + B_\alpha)_{k+1}$. The inverse inclusion is trivial. Indeed we have

$$\ker(\Delta_A)_{k+1} = \ker(-\nabla_{A_\infty} + B_\alpha)_{k+1}.$$

As we have pointed out before, $\ker(\Delta_A)_{k+1} = \ker(\Delta_A^*)_0$. Hence, from Corollary (3.1.11) and from the fact that $\text{im}(-\nabla_{A_\infty}^* + B_\alpha^*)_k$ is orthogonal to $\ker(\Delta_A^*)_0$ it

follows that

$$\text{im}(-\nabla_{A_\infty}^* + B_\alpha^*)_k \subset \text{im}(\Delta_A)_{k+1}.$$

This completes the proof. ■

The following theorem extends Proposition (3.1.5) to any $A \in \mathcal{C}^k$:

(3.1.13) THEOREM. *Let $k > n/2 + 2$. Then for $A = A_\infty + \alpha$*

$$H^k(T^*M \otimes \text{Ad } P) = \text{im}(T_e \Phi_A) \oplus \ker(-\nabla_{A_\infty}^* + B_\alpha^*)_k.$$

Both subspaces on the right-hand side are closed and orthogonal with respect to the weak scalar product $((,))$.

Proof. From Proposition (3.1.12) we obtain

$$\begin{aligned} H^k(\eta) &= (-\nabla_{A_\infty}^* + B_\alpha^*)_k^{-1}(H^{k-1}(\zeta)) = (-\nabla_{A_\infty}^* + B_\alpha^*)_k^{-1}(\text{im}(\nabla_{A_\infty}^* + B_\alpha^*)_k) \\ &= (-\nabla_{A_\infty}^* + B_\alpha^*)_k^{-1}(\text{im}(\Delta_A)_{k+1}) \\ &= (-\nabla_{A_\infty}^* + B_\alpha^*)_k^{-1}((-\nabla_{A_\infty}^* + B_\alpha^*)_k(\text{im}(-\nabla_{A_\infty} + B_\alpha)_{k+1})). \end{aligned}$$

Thus $H^k(\eta) = \text{im}(-\nabla_{A_\infty} + B_\alpha)_{k+1} + \ker(-\nabla_{A_\infty}^* + B_\alpha^*)_k$. It is easy to verify that both subspaces are orthogonal with respect to $((,))$, in particular, their intersection is $\{0\}$. Since $(-\nabla_{A_\infty} + B_\alpha)_{k+1} = T_e \Phi_A$, the decomposition is proved. From (3.1.6) it follows that both subspaces are closed. ■

3.2. The orbits are submanifolds. Now our aim is to prove that orbits of the \mathcal{G}^{k+1} -action on \mathcal{C}^k are submanifolds in \mathcal{C}^k .

(3.2.1) THEOREM. *For any $A \in \mathcal{C}^k$ the orbit $\mathcal{G}^{k+1} \cdot A$ is a C^∞ Hilbert submanifold in \mathcal{C}^k .*

The proof of the theorem is based on a few lemmas.

(3.2.2) LEMMA. *For any $A \in \mathcal{C}^k$ the isotropy group $S(A)$ is a Hilbert–Lie subgroup of \mathcal{G}^{k+1} .*

Proof. By (2.4.10) the subgroup $S(A) \subset \mathcal{G}^{k+1}$ is compact in the topology induced from \mathcal{G}^{k+1} ; therefore, in particular, $S(A)$ is a closed and locally compact subgroup in \mathcal{G}^{k+1} . It is known that such subgroups are Hilbert–Lie subgroups (see e.g. [4c] exercise 7b, Chap. III § 8; page 276). ■

(3.2.3) LEMMA. *For any $A \in \mathcal{C}^k$ the coset space $\mathcal{G}^{k+1}/S(A)$ carries the unique structure of a C^∞ Hilbert manifold such that the canonical projection $\mathcal{G}^{k+1} \rightarrow \mathcal{G}^{k+1}/S(A)$ is a submersion.*

Proof. The coset space \mathcal{G}^{k+1}/S , where S is a Hilbert–Lie subgroup (not necessarily finite-dimensional) carries the unique structure of a C^∞ Hilbert manifold (see Bourbaki [4b], § 5, 5.12.4). According to this result and using Lemma (3.2.2), we obtain the lemma. ■

(3.2.4) LEMMA. For any $A \in \mathcal{C}^k$ the canonical map $\iota_A: \mathcal{G}^{k+1}/S(A) \ni \varphi S(A) \mapsto \varphi \cdot A \in \mathcal{C}^k$ is smooth.

Proof. Since the manifold structure on $\mathcal{G}^{k+1}/S(A)$ is the factor structure of the manifold structure on \mathcal{G}^{k+1} (see Bourbaki [4b] § 5, 5.9.5), it is sufficient to prove that the composition of ι_A with the canonical projection $\mathcal{G}^{k+1} \rightarrow \mathcal{G}^{k+1}/S(A)$ is a C^∞ map. But this composition is simply $\mathcal{G}^{k+1} \ni \varphi \mapsto \varphi \cdot A \in \mathcal{C}^k$, and therefore it is smooth as a restriction of Φ to $\mathcal{G}^{k+1} \times \{A\} \subset \mathcal{G}^{k+1} \times \mathcal{C}^k$ (see Proposition (2.2.2)). ■

(3.2.5) LEMMA. For any $A \in \mathcal{C}^k$ the map ι_A is an injective immersion.

Proof. Let $o = eS(A) \in \mathcal{G}^{k+1}/S(A)$. It is sufficient to prove that $\ker T_o \iota_A = \{0\}$ and that the image $\text{im } T_o \iota_A \subset T_A \mathcal{C}^k$ is closed. But for a smooth action such that the isotropy groups are Lie subgroups, the kernel of the tangent map of the canonical map (homogeneous space) \rightarrow (orbit) is trivial. The proof is as follows. Assume that $X \in T_o(\mathcal{G}^{k+1}/S(A))$ satisfies $T_o \iota_A \cdot X = 0$. Since $\pi_{\mathcal{G}}: \mathcal{G}^{k+1} \rightarrow \mathcal{G}^{k+1}/S(A)$ is a submersion, there exists a vector $\lambda \in T_e \mathcal{G}^{k+1}$ such that $T_e \pi_{\mathcal{G}} \cdot \lambda = X$. Let $\varphi_t := \exp t\lambda$ and $\sigma: \mathbf{R} \rightarrow \mathcal{C}^k$ be the curve defined by the formula $\sigma(t) := \iota_A \circ \pi_{\mathcal{G}}(\varphi_t)$. Now

$$\left. \frac{d}{dt} \right|_o \sigma = T_o \iota_A \cdot T_e \pi_{\mathcal{G}} \cdot \left. \frac{d}{dt} \right|_o \varphi_t = T_o \iota_A \cdot T_e \pi_{\mathcal{G}} \cdot \lambda = T_o \iota_A \cdot X = 0.$$

Since

$$\sigma(t) = \varphi_t \cdot A = (\varphi_t \circ \varphi_{t-\tau}) \cdot A = \Phi_{\varphi_\tau}(\sigma(t-\tau)) = (\Phi_{\varphi_\tau} \circ \sigma)(t-\tau),$$

we have

$$\left. \frac{d}{dt} \right|_t \sigma = T_A \Phi_{\varphi_\tau} \left. \frac{d}{dt} \right|_o \sigma = 0, \quad \text{for any } \tau \in \mathbf{R},$$

which proves that, for any $t \in \mathbf{R}$, $\sigma(t) = \sigma(0) = A$. Thus $\varphi_t \in S(A)$ for any $t \in \mathbf{R}$; so $\lambda = \left. \frac{d}{dt} \right|_o \varphi_t \in T_e S(A)$ and further $X = T_e \pi_{\mathcal{G}} \cdot \lambda = 0$. This proves that $\ker T_o \iota_A = \{0\}$.

Since $\pi_{\mathcal{G}}$ is a submersion, we have

$$\text{im } T_o \iota_A = T_o \iota_A (\text{im } T_e \pi_{\mathcal{G}}) = \text{im} (T_o \iota_A \cdot T_e \pi_{\mathcal{G}}),$$

whence

$$\text{im } T_o \iota_A = \text{im } T_e(\iota_A \circ \pi_{\mathcal{G}}).$$

But $\iota_A \circ \pi_{\mathcal{G}} = \Phi_A$ and thus $\text{im } T_o \iota_A = \text{im } T_e \Phi_A$ is closed (and admits a closed complement) by Theorem (3.1.13). The injectivity of ι_A is obvious. ■

(3.2.6) LEMMA. The topology of \mathcal{G}^{k+1} satisfies the Second Axiom of Countability.

Proof. The topology of \mathcal{G}^{k+1} is induced from the space $H^{k+1}(P, P)$ of

all H^{k+1} maps $P \rightarrow P$ (see Proposition (2.1.3)). By the Whitney theorem, the (compact) manifold P can be regarded as a submanifold of \mathbf{R}^N with a suitably chosen N , and so $H^{k+1}(P, P)$ can be regarded as a subset of the Hilbert space $(H^{k+1}(P))^N =$ (the product of N spaces of H^{k+1} real-valued functions on P). Hence $H^{k+1}(P, P)$ and also \mathcal{G}^{k+1} satisfy the Second Countability Axiom since $H^{k+1}(P)$ is a separable Hilbert space. ■

Proof of 3.2.1. For the proof we use the results given in Bourbaki [4b], § 5, 5.12.5 concerning smooth actions of infinite-dimensional Lie groups on infinite-dimensional manifolds.

The properties proved in Lemmas (3.2.2)–(3.2.6) and the fact that orbits are closed (see Corollary (2.4.11)) imply that orbits of Φ are submanifolds in \mathcal{C}^k and, moreover, that for any $A \in \mathcal{C}^k$ the canonical map $\iota_A: \mathcal{G}^{k+1}/S(A) \rightarrow \mathcal{C}^k$ is a diffeomorphism onto its image (which is just the orbit $\mathcal{G}^{k+1} \cdot A$). ■

3.3. The Slice Theorem. Roughly speaking, a tubular neighbourhood of an orbit is an open invariant neighbourhood of that orbit with an equivariant retraction of the neighbourhood onto the orbit. This retraction should be a smooth submersion. Moreover, it is also required that this projection should be locally trivial. Any fibre of the tubular neighbourhood intersects the given orbit at exactly one point and is called a slice at that point. It is also clear that any other orbit contained in the tubular neighbourhood intersects all slices. However, such intersection may contain more than one point. Nevertheless the slice is perhaps the single most important tool in the analysis of local topological and geometric structures on the orbit space.

In this section we shall prove the existence of tubular neighbourhoods for the \mathcal{G}^{k+1} -action on \mathcal{C}^k . We will apply standard constructions based on the existence of an invariant Riemannian metric.

For $A \in \mathcal{C}^k$ let $N_A \subset T\mathcal{C}^k$ denote the set of all tangent vectors orthogonal to the orbit $\mathcal{G}^{k+1} \cdot A$ with respect to the weak Riemannian metric $((,))$.

(3.3.1) LEMMA. N_A is a smooth vector subbundle of $T\mathcal{C}^k$ over $\mathcal{G}^{k+1} \cdot A$.

Proof. Since the orbit $\mathcal{G}^{k+1} \cdot A$ is a smooth submanifold (see (3.2.1)), $T(\mathcal{G}^{k+1} \cdot A)$ and $T\mathcal{C}^k \upharpoonright_{\mathcal{G}^{k+1} \cdot A}$ are smooth subbundles in $T\mathcal{C}^k$. The weak Riemannian metric $((,))$ defines the map $Q: T\mathcal{C}^k \upharpoonright_{\mathcal{G}^{k+1} \cdot A} \rightarrow T(\mathcal{G}^{k+1} \cdot A)$, which is the orthogonal projection of each tangent space onto the tangent space to the orbit (see (3.1.13)). Now we prove that Q is smooth. For this purpose we note that $\mathcal{G}^{k+1} \rightarrow \mathcal{G}^{k+1}/S(A)$ is a locally trivial bundle since for any $A' \in \mathcal{C}^k$ the symmetry group $S(A')$ is the Hilbert–Lie subgroup of \mathcal{G}^{k+1} (see (3.2.2) and Bourbaki [4b] 6.2.3 and 6.2.4a). Let us also recall that the map ι_A may be regarded as a diffeomorphism $\iota_A: \mathcal{G}^{k+1}/S(A) \rightarrow \mathcal{G}^{k+1} \cdot A$. Thus, using ι_A^{-1} and a local section of the bundle $\mathcal{G}^{k+1} \rightarrow \mathcal{G}^{k+1}/S(A)$, we define on a suitably small neighbourhood $V \subset \mathcal{G}^{k+1} \cdot A$ of $A' \in \mathcal{G}^{k+1} \cdot A$ a

smooth mapping $\theta: V \rightarrow \mathcal{G}^{k+1}$ such that

$$\theta(A'') \cdot A' = A'' \quad \text{for any } A'' \in V.$$

Let

$$Q_{A'}: T_{A'} \mathcal{G}^k \rightarrow T_{A'} (\mathcal{G}^{k+1} \cdot A) = \text{im } T_e \Phi_{A'}$$

be the restriction of Q to the fibre $T_{A'} \mathcal{G}^k$, and let

$$\bar{\pi}: T \mathcal{G}^k \upharpoonright_{\mathcal{G}^{k+1} \cdot A} \rightarrow \mathcal{G}^{k+1} \cdot A$$

be the projection. Now we are ready to express the mapping Q on $\bar{\pi}^{-1}(V)$:

$$Q(X) = T_{A'} \Phi_{\theta(\bar{\pi}(X))} \circ Q_{A'} (T \Phi_{(\theta(\bar{\pi}(X)))^{-1}} \cdot X),$$

where $X \in \bar{\pi}^{-1}(V)$ and $\Phi_\varphi: \mathcal{G}^k \ni A \mapsto \varphi \cdot A \in \mathcal{G}^k$ is a diffeomorphism defined by Φ . The formula given above follows from the fact that $((,))$ is \mathcal{G}^{k+1} -invariant.

Now, from the smoothness of Φ , π , θ and from the fact that $Q_{A'}$ is a continuous linear mapping (and thus is also smooth) it follows that Q is smooth on $\bar{\pi}^{-1}(V)$. The above formula shows this fact evidently. Since $A' \in \mathcal{G}^{k+1} \cdot A$ has been chosen arbitrarily, we conclude that Q is of class C^∞ on the whole domain.

N_A is a smooth subbundle in $T \mathcal{G}^k \upharpoonright_{\mathcal{G}^{k+1} \cdot A}$ since it is the kernel of a smooth surjective vector bundle morphism Q . ■

(3.3.2) Remark. The action Φ lifts to the smooth action of \mathcal{G}^{k+1} on $T \mathcal{G}^k$, which we now denote by Φ_* :

$$\Phi_*: \mathcal{G}^{k+1} \times T \mathcal{G}^k \ni (\varphi, X) \mapsto T_{\bar{\pi}(X)} \Phi_\varphi \cdot X \in T \mathcal{G}^k.$$

Since $((,))$ is \mathcal{G}^{k+1} -invariant, the normal bundle N_A to the orbit passing through $A \in \mathcal{G}^k$ is an invariant submanifold in $T \mathcal{G}^k$. In other words, \mathcal{G}^{k+1} acts smoothly on N_A .

In the general case, exponential mappings for weak Riemannian metrics do not exist. However, in our case, the metric $((,))$ defines an exponential mapping. Evidently

$$\exp_A \alpha = A + \alpha.$$

For each $A' \in \mathcal{G}^{k+1} \cdot A$ there exists an open neighbourhood in N_A of the vector $0 \in T_{A'} \mathcal{G}^k$ such that the map \exp , when restricted to this neighbourhood, is a diffeomorphism onto an open (in \mathcal{G}^k) neighbourhood of A' . Thus we can construct an open neighbourhood of the zero section in N_A such that \exp is a local diffeomorphism on it. We shall prove further that it is possible to find an invariant open neighbourhood of the zero section in N_A such that \exp is injective on it.

Using the strong Riemannian metric $((,))^k$ we define the open subsets N_A^ε in N_A in the following way:

$$N_A^\varepsilon := \{X \in N_A \mid ((X, X))^k \leq \varepsilon^2\},$$

where $\varepsilon > 0$. Since the strong metric $((,))^k$ is invariant (see (2.3.5) (i)), the submanifolds N_A^ε are invariant.

(3.3.3) LEMMA. *There exists an $\varepsilon > 0$ such that the map $E = \exp \upharpoonright_{N_A^\varepsilon} : N_A^\varepsilon \rightarrow \mathcal{C}^k$ is an equivariant diffeomorphism onto an open invariant neighbourhood of the orbit $\mathcal{G}^{k+1} \cdot A$.*

Proof. Since $\exp : T\mathcal{C}^k \rightarrow \mathcal{C}^k$ is an equivariant map with respect to the \mathcal{G}^{k+1} -actions Φ_* and Φ , respectively, and N_A^ε is an invariant submanifold in $T\mathcal{C}^k$, it follows that E is equivariant and $E(N_A^\varepsilon)$ is invariant (for any $\varepsilon > 0$).

Let $d : \mathcal{C}^k \times \mathcal{C}^k \rightarrow \mathbf{R}$ be the metric (distance) defined by the Riemannian structure $((,))^k$. Of course, d defines the original topology on \mathcal{C}^k and d is \mathcal{G}^{k+1} -invariant, i.e., $d(\varphi \cdot A_1, \varphi \cdot A_2) = d(A_1, A_2)$ for any $\varphi \in \mathcal{G}^{k+1}$ and $A_1, A_2 \in \mathcal{C}^k$.

We denote by $B(A, \delta)$ the ball with centre $A \in \mathcal{C}^k$ and radius $\delta > 0$ defined by d .

Let $U_1 \ni 0, 0 \in T_A \mathcal{C}^k$, be an open neighbourhood of 0 in N_A such that $\exp \upharpoonright_{U_1}$ is a diffeomorphism onto an open neighbourhood of A in \mathcal{C}^k . Then there exist an open neighbourhood $V_1 \subset \mathcal{G}^{k+1} \cdot A$ of A and $\varepsilon_1 > 0$ such that $\bar{\pi}^{-1}(V_1) \cap N_A^{\varepsilon_1} \subset U_1$ (where $\bar{\pi} : N_A \rightarrow \mathcal{G}^{k+1} \cdot A$ is the projection). Make sure that V_1 , and ε_1 are small enough to give

$$\exp(\bar{\pi}^{-1}(V_1) \cap N_A^{\varepsilon_1}) \cap \mathcal{G}^{k+1} \cdot A = V_1.$$

This is possible since $\mathcal{G}^{k+1} \cdot A$ is a submanifold in \mathcal{C}^k . Of course, $\exp(\bar{\pi}^{-1}(V_1) \cap N_A^{\varepsilon_1})$ is also an open neighbourhood of A , and so there exists a $\delta > 0$ such that $B(A, \delta) \subset \exp(\bar{\pi}^{-1}(V_1) \cap N_A^{\varepsilon_1})$. Let us take $U = (\exp \upharpoonright_{U_1})^{-1}(B(A, \delta/2))$. Since U is an open (in N_A) neighbourhood of $0 \in T_A \mathcal{C}^k$ and $U \subset U_1$, we can find an open neighbourhood $V \subset V_1$ and $\varepsilon > 0$ such that $U \supset (\bar{\pi}^{-1}(V) \cap N_A^\varepsilon)$.

We now proceed to prove that the chosen ε satisfies the lemma. Note that \exp is injective on $\Phi_*(\{\varphi\} \times U)$ for any $\varphi \in \mathcal{G}^{k+1}$. This follows from the fact that $\exp : N_A \rightarrow \mathcal{C}^k$ is equivariant. Hence $\exp(\Phi_*(\mathcal{G}^{k+1} \times U))$ is open in \mathcal{C}^k and \exp is a local diffeomorphism on N_A^ε since $N_A^\varepsilon \subset \Phi_*(\mathcal{G}^{k+1} \times U)$ is an open subset.

Now, to complete the proof it is enough to show that $E = \exp \upharpoonright_{N_A^\varepsilon}$ is injective. Let $X_1, X_2 \in N_A^\varepsilon$ and $\exp X_1 = \exp X_2$.

We have to prove that $X_1 = X_2$. Let $A_i = \bar{\pi}(X_i), i = 1, 2$. Since

$A_1, A_2 \in \mathcal{G}^{k+1} \cdot A$, we can find $\varphi_i \in \mathcal{G}^{k+1}$ such that $\varphi_i \cdot A = A_i$, $i = 1, 2$. Then we obtain:

$$\exp X_i = \varphi_i \cdot \exp(\varphi_i^{-1} \cdot X_i), \quad i = 1, 2.$$

Since d is invariant, we obtain

$$d(E(X_i), A_i) = d(\varphi_i \cdot E(\varphi_i^{-1} \cdot X_i), \varphi_i \cdot A) = d(E(\varphi_i^{-1} \cdot X_i), A) < \delta/2.$$

Then

$$\begin{aligned} d(A_1, A_2) &\leq d(A_1, E(X_1)) + d(E(X_1), A_2) \\ &= d(A_1, E(X_1)) + d(E(X_2), A_2) < \delta, \quad \text{since } E(X_1) = E(X_2). \end{aligned}$$

We know that $\exp \upharpoonright_{\bar{\pi}^{-1}(\varphi_1 \cdot V_1) \cap N_A^{\varepsilon_1}}$ is injective. Also $\exp(\bar{\pi}^{-1}(\varphi_1 \cdot V_1) \cap N_A^{\varepsilon_1}) \supset B(A_1, \delta)$ and therefore

$$A_2 \in \mathcal{G}^{k+1} \cdot A \cap \exp(\bar{\pi}^{-1}(\varphi_1 \cdot V_1) \cap N_A^{\varepsilon_1}) = \varphi_1 \cdot V_1.$$

Thus $X_1, X_2 \in \bar{\pi}^{-1}(\varphi_1 \cdot V_1)$. Since $X_1, X_2 \in N_A^\varepsilon$ and $\varepsilon < \varepsilon_1$, we obtain $X_1, X_2 \in \bar{\pi}^{-1}(\varphi_1 \cdot V_1) \cap N_A^{\varepsilon_1}$, and the equality $\exp(X_1) = \exp(X_2)$ implies that $X_1 = X_2$. ■

(3.3.4) THEOREM. For each $A \in \mathcal{G}^k$ there exists an equivariant map $\mathcal{F}: N_A \rightarrow \mathcal{G}^k$ which is a diffeomorphism onto an open invariant neighbourhood of $\mathcal{G}^{k+1} \cdot A$.

Proof. We choose $\varepsilon > 0$ as in Lemma (3.3.3) and define $F: N_A \rightarrow N_A^\varepsilon$ by the formula

$$F(X) = \varepsilon [((X, X))^k + 1]^{-1/2} X.$$

It is clear that F is a diffeomorphism. Moreover, F is equivariant since $((\ , \))^k$ is invariant. Now we define $\mathcal{F} := E \circ F$, and by Lemma (3.3.3) we obtain the result. ■

The above theorem gives us the construction of a tubular neighbourhood of the orbit passing through A . The invariant open neighbourhood $\mathcal{F}(N_A)$ of $\mathcal{G}^{k+1} \cdot A$ can be equipped with the bundle structure (an equivariant projection $\mathcal{F}(N_A) \rightarrow \mathcal{G}^{k+1} \cdot A$ is given by the composition $\bar{\pi} \circ \mathcal{F}^{-1}$). The slice at $A' \in \mathcal{G}^{k+1} \cdot A$ is the following submanifold $\mathcal{S}_{A'}$:

$$\mathcal{S}_{A'} = \mathcal{F}(T_{A'} \mathcal{G}^k \cap N_A) = \exp(T_{A'} \mathcal{G}^k \cap N_A^\varepsilon) \subset \mathcal{G}^k.$$

From the above theorem we obtain the basic properties of a slice:

(3.3.5) COROLLARY. Let $A' \in \mathcal{G}^{k+1} \cdot A$.

- (i) If $\varphi \in S(A')$ then $\varphi \cdot \mathcal{S}_{A'} = \mathcal{S}_{A'}$.
- (ii) If $\varphi \in \mathcal{G}^{k+1}$, $A_1 \in \mathcal{S}_{A'}$ and $\varphi \cdot A_1 \in \mathcal{S}_{A'}$, then $\varphi \in S(A')$.

Proof. (i)

$$\begin{aligned}\varphi \cdot \mathcal{S}_{A'} &= \varphi \cdot \mathcal{F}(T_{A'} \mathcal{C}^k \cap N_{A'}) = \mathcal{F}(\varphi \cdot (T_{A'} \mathcal{C}^k \cap N_{A'})) \\ &= \mathcal{F}(T_{A'} \mathcal{C}^k \cap N_{A'}) = \mathcal{S}_{A'}.\end{aligned}$$

(ii)

$$\begin{aligned}A' &= \bar{\pi}(\mathcal{F}^{-1}(A_1)) = \bar{\pi}(\mathcal{F}^{-1}(\mathcal{S}_{A'})) = \bar{\pi}(\mathcal{F}^{-1}(\varphi \cdot A_1)) \\ &= \bar{\pi}(\varphi \cdot \mathcal{F}^{-1}(A_1)) = \varphi \cdot \bar{\pi}(\mathcal{F}^{-1}(A_1)) = \varphi \cdot A',\end{aligned}$$

so $\varphi \in S(A')$. Here we have used the fact that both \mathcal{F} and $\bar{\pi}$ are equivariant. ■

Property (3.3.5) (i) says that the symmetry group $S(A')$ acts on the slice $\mathcal{S}_{A'}$. The second property gives information about the intersection of orbits and slices, namely:

$$(\mathcal{G}^{k+1} \cdot A_1) \cap \mathcal{S}_{A'} = S(A') \cdot A_1 \quad \text{for } A_1 \in \mathcal{S}_{A'}.$$

We now formulate the main theorem of this section.

(3.3.6) THEOREM (The Local Slice Theorem). *Let $A \in \mathcal{C}^k$ and let U be an open invariant neighbourhood of $\mathcal{G}^{k+1} \cdot A$. Then there exists a tubular neighbourhood of $\mathcal{G}^{k+1} \cdot A$ such that the complement in U of the closure (in \mathcal{C}^k) of that tubular neighbourhood is not empty, i.e. there exists an equivariant map $\mathcal{F}: N_A \rightarrow U$ such that*

- (i) $\mathcal{F}(N_A)$ is an open set,
- (ii) $\mathcal{F}: N_A \rightarrow \mathcal{F}(N_A)$ is a diffeomorphism,
- (iii) $\overline{\mathcal{F}(N_A)} \subset U$,
- (iv) $U - \overline{\mathcal{F}(N_A)} \neq \emptyset$.

Proof. We construct \mathcal{F} in the same way as in the proof of Theorem (3.3.4). We choose a suitably small $\varepsilon > 0$ such that $\exp(N_A^\varepsilon) \subset U$. Let $\varepsilon_1 > 0$ be as in Lemma (3.3.3).

Then $\exp(N_A^{\varepsilon_1}) \cap U$ is an open invariant neighbourhood of $\mathcal{G}^{k+1} \cdot A$. By virtue of Lemma (3.3.3) $\exp(T_A \mathcal{C}^k \cap N_A^{\varepsilon_1}) \cap U$ is an open neighbourhood of A in $\exp(T_A \mathcal{C}^k \cap N_A^{\varepsilon_1})$ and, moreover, we can choose an $\varepsilon_2 < \varepsilon_1$ such that $\varepsilon_2 > 0$ and $\exp(T_A \mathcal{C}^k \cap N_A^{\varepsilon_2}) \subset U$. Since the strong metric is \mathcal{G}^{k+1} -invariant and \exp is \mathcal{G}^{k+1} -equivariant, we obtain

$$\mathcal{G}^{k+1} \cdot \exp(T_A \mathcal{C}^k \cap N_A^{\varepsilon_2}) = \exp(N_A^{\varepsilon_2}).$$

Hence, $\exp(N_A^{\varepsilon_2}) \subset U$. We now take $\varepsilon = \frac{1}{2}\varepsilon_2$ and (applying the construction given in the proof of Theorem (3.3.4)) we get \mathcal{F} . ■

If a group action admits tubular neighbourhoods satisfying conditions

(i), (ii), (iii) and (iv) of Theorem (3.3.6), we say that the action has the *local slice property*. There exist simple examples of smooth actions admitting slices which do not have the local slice property.

(3.3.7) EXAMPLE. Let us take the natural action of the group \mathbf{R}_+ on \mathbf{R} :

$$\mathbf{R}_+ \times \mathbf{R} \ni (g, x) \mapsto gx \in \mathbf{R}.$$

We have three orbits in \mathbf{R} : $O_+ := (0, \infty)$, $\{0\}$ and $O_- := (-\infty, 0)$. For $x \neq 0$ the slice at x is $\{x\}$. There is only one slice at 0 (obviously \mathbf{R}). Hence, this action does not have the local slice property. The space $\mathbf{R}/\mathbf{R}_+ = \{O_+, O_-, \{0\}\}$ has the topology $\{\{O_+, O_-, \{0\}\}, \emptyset, \{O_+\}, \{O_-\}, \{O_+, O_-\}\}$ which is not T_1 .

(3.3.8) PROPOSITION. *If a smooth action of a Lie group on a manifold has the local slice property then the orbit space is regular.*

Proof. The projection onto the orbit space is an open continuous map in the general case of continuous actions. Hence, it is sufficient to prove that for any orbit o and for any closed invariant subset $D \neq \emptyset$ such that $D \cap o = \emptyset$ there exists an open set U such that $o \subset U$ and $\bar{U} \cap D = \emptyset$. Since the complement of D is an open invariant neighbourhood of o , the local slice property implies that there exists such a U . ■

(3.3.9) COROLLARY. *The orbit space $\mathcal{G}^k/\mathcal{G}^{k+1}$ (with the standard quotient topology) is a regular topological space.*

§ 4. The geometric structure of $\mathcal{R}^k = \mathcal{G}^k/\mathcal{G}^{k+1}$

Let \mathcal{R}^k be the space of \mathcal{G}^{k+1} -orbits in \mathcal{G}^k .

$$\mathcal{R}^k := \{\mathcal{G}^{k+1} \cdot A \mid A \in \mathcal{G}^k\}.$$

We provide \mathcal{R}^k with the usual quotient space topology. In this topology, the canonical projection $\hat{\pi}: \mathcal{G}^k \rightarrow \mathcal{R}^k$ is a continuous and open mapping. The openness of $\hat{\pi}$ is a well-known property of any continuous action of a topological group.

To begin with we show elementary properties of \mathcal{R}^k as a topological space.

(4.0) THEOREM. *The orbit space \mathcal{R}^k is a connected metrizable topological space. Moreover, the topology of \mathcal{R}^k satisfies the Second Countability Axiom.*

Proof. First note that \mathcal{G}^k is homeomorphic to a separable Hilbert space and thus \mathcal{G}^k is connected and satisfies the Second Countability Axiom. Since $\hat{\pi}: \mathcal{G}^k \rightarrow \mathcal{R}^k$ is continuous, \mathcal{R}^k is connected. Furthermore, the map $\hat{\pi}$ projects the basis of topology in \mathcal{G}^k onto the basis in \mathcal{R}^k (since $\hat{\pi}$ is an open map), and so \mathcal{R}^k is second countable. Since \mathcal{R}^k is both regular (see Corollary (3.3.9))

and second countable, it follows that, by virtue of Urysohn's Metrization Theorem, \mathcal{R}^k is metrizable (see [20]).

Remark. By means of standard methods we can endow \mathcal{R}^k with a metric. One can take the strong Riemannian metric $((\cdot, \cdot))^k$ on \mathcal{C}^k and define the \mathcal{G}^{k+1} -invariant (topological) metric on \mathcal{C}^k taking the lower bound (infimum) of integrals along the curves joining corresponding points in \mathcal{C}^k . This metric projects onto \mathcal{R}^k , that is, the distance between two points in \mathcal{R}^k is the lower bound of the distances between points in those orbits (measured in \mathcal{C}^k).

4.1. Consequences of the Slice Theorem

(4.1.1) **PROPOSITION.** *To any smooth invariant submanifold in \mathcal{C}^k and the \mathcal{G}^{k+1} -action on it applies the slice theorem. That is, every orbit admits a tubular neighbourhood in that submanifold.*

Proof. Let us take a smooth invariant submanifold in \mathcal{C}^k and an orbit in it. Restricting the morphism Q (defined in the proof of (3.3.1)) to the tangent bundle of the submanifold, we obtain the smooth surjective morphism of vector bundles (over the orbit). The kernel of this restricted morphism is a smooth vector subbundle of the normal bundle to the orbit (defined previously in $T\mathcal{C}^k$). This subbundle is also \mathcal{G}^{k+1} -invariant. Now, the restriction of \mathcal{F} (\mathcal{F} was defined in the proof of (3.3.4)) to this vector subbundle gives us a tubular neighbourhood of the orbit. ■

Let $S \subset \mathcal{G}^{k+1}$ be a compact Lie subgroup of \mathcal{G}^{k+1} such that there exists a connection A for which $S = S(A)$. Every conjugate subgroup $\varphi S \varphi^{-1}$, $\varphi \in \mathcal{G}^{k+1}$, of S is a symmetry group, namely $\varphi S \varphi^{-1} = S(\varphi \cdot A)$. Obviously, for any subgroup S' from the conjugacy class (S) there exists a connection $A' \in \mathcal{G}^{k+1} \cdot A$ such that $S' = S(A')$. Thus, any orbit singles out a unique conjugacy class of compact subgroups in \mathcal{G}^{k+1} . The set of conjugacy classes of subgroups in \mathcal{G}^{k+1} that correspond to orbits in \mathcal{C}^k is denoted by J . Elements (S) belonging to J are called *orbit types*. In general, there exist many different orbits with a given orbit type (in other words, the canonical map **type**: $\mathcal{R}^k \rightarrow J$ is not injective).

Now we consider the subset $\mathcal{C}_{(S)}^k \subset \mathcal{C}^k$ defined by

$$\mathcal{C}_{(S)}^k := \{A \in \mathcal{C}^k \mid S(A) \in (S)\},$$

where $(S) \in J$. It is clear that $\mathcal{C}_{(S)}^k \subset \mathcal{C}^k$ is an invariant subset and, moreover,

$$\mathcal{C}_{(S)}^k = \hat{\pi}^{-1}(\mathbf{type}^{-1}((S)))$$

By $\mathcal{R}_{(S)}^k$ we denote the subset in \mathcal{R}^k singled out by an orbit type $(S) \in J$:

$$\mathcal{R}_{(S)}^k := \mathbf{type}^{-1}((S)).$$

Of course,

$$\mathcal{R}_{(S)}^k = \hat{\pi}(\mathcal{C}_{(S)}^k).$$

In this section we show that $\mathcal{C}_{(S)}^k$ is a smooth submanifold in \mathcal{C}^k . Then we show that the topological subspace $\mathcal{H}_{(S)}^k \subset \mathcal{H}^k$ can be endowed with the structure of a C^∞ Hilbert manifold such that $\hat{\pi}: \mathcal{C}_{(S)}^k \rightarrow \mathcal{H}_{(S)}^k$ becomes a smooth submersion. First we prove the following technical lemma:

(4.1.2) LEMMA. *Let $A \in \mathcal{C}^k$, $S(A) = S$ and let N_A be the normal bundle to $\mathcal{C}^{k+1} \cdot A$. Let*

$$N_{A(S)} := \{X \in N_A \mid \text{such that the isotropy group at } X \text{ of}$$

the action Φ_ is an element of (S) \}.*

Then $N_{A(S)}$ is a smooth vector subbundle in N_A .

Proof. Let $A' \in \mathcal{C}^{k+1} \cdot A$. Let us observe that for every vector $X \in T_{A'} \mathcal{C}^k \cap N_A$ the isotropy group at X of Φ_* is a subgroup of the group $S' = S(A')$. This follows from the fact that $\bar{\pi}: T\mathcal{C}^k \rightarrow \mathcal{C}^k$ is equivariant with respect to Φ_* and Φ . Thus, $N_{A(S)} \cap T_{A'} \mathcal{C}^k$ consists of all vectors in $N_A \cap T_{A'} \mathcal{C}^k$ whose isotropy groups are equal to S' . Since the action Φ_* of S' on $T_{A'} \mathcal{C}^k \cap N_A$ is linear, the set $N_{A(S)} \cap T_{A'} \mathcal{C}^k$ is a linear subspace. This subspace is closed since

$$N_{A(S)} \cap T_{A'} \mathcal{C}^k = \{X \in N_A \cap T_{A'} \mathcal{C}^k \mid \forall \varphi \in S' \quad \Phi_*(\varphi, X) = X\}.$$

Now we show that $N_{A(S)}$ is a smooth subbundle in $T\mathcal{C}^k \upharpoonright_{\mathcal{C}^{k+1} \cdot A}$. For this purpose let us take an open neighbourhood $V \ni A'$ (in $\mathcal{C}^{k+1} \cdot A$) and a C^∞ -map $\theta: V \rightarrow \mathcal{C}^{k+1}$ such as in the proof of (3.3.1) (i.e., $\theta(A'') \cdot A' = A''$ for any $A'' \in V$). The map

$$V \times T_{A'} \mathcal{C}^k \ni (A'', X) \mapsto \Phi_*(\theta(A''), X) \in \bar{\pi}^{-1}(V)$$

is a diffeomorphism and a trivialization of $\bar{\pi}^{-1}(V)$ because its inverse is given by the formula

$$\bar{\pi}^{-1}(V) \ni X' \mapsto (\bar{\pi}(X'), \Phi_*([\theta(\bar{\pi}(X'))]^{-1}, X')) \in V \times T_{A'} \mathcal{C}^k.$$

The smoothness of this map and its inverse is evident. Of course, for every $\varphi \in \mathcal{C}^{k+1}$ we have

$$N_{A(S)} \cap T_{\varphi \cdot A'} \mathcal{C}^k = \Phi_*([\varphi] \times N_{A(S)} \cap T_{A'} \mathcal{C}^k).$$

Thus $N_{A(S)} \cap \bar{\pi}^{-1}(V)$ is the image of $V \times (N_{A(S)} \cap T_{A'} \mathcal{C}^k)$ by the above trivialization map for $\bar{\pi}^{-1}(V)$. Hence $N_{A(S)} \cap \bar{\pi}^{-1}(V)$ is a smooth subbundle in $\bar{\pi}^{-1}(V)$. Since A' is an arbitrary element in $\mathcal{C}^{k+1} \cdot A$, the lemma is proved. ■

(4.1.3) THEOREM. *For any $(S) \in J$ the set $\mathcal{C}_{(S)}^k$ is a smooth submanifold in \mathcal{C}^k .*

Proof. Let us take $A \in \mathcal{C}_{(S)}^k$, the orbit $\mathcal{C}^{k+1} \cdot A$, the corresponding normal bundle N_A and the equivariant diffeomorphism $\mathcal{F}: N_A \rightarrow \mathcal{F}(N_A) \subset \mathcal{C}^k$. Since $\mathcal{F}(N_A)$ is an open neighbourhood of A in \mathcal{C}^k , it is sufficient to prove that $\mathcal{C}_{(S)}^k \cap \mathcal{F}(N_A)$ is a submanifold in $\mathcal{F}(N_A)$. Since \mathcal{F} is

equivariant, we have

$$\mathcal{C}_{(S)}^k \cap \mathcal{F}(N_A) = \mathcal{F}(N_{A(S)}).$$

Now $\mathcal{F}(N_{A(S)})$ is a smooth submanifold in $\mathcal{F}(N_A)$ because $N_{A(S)}$ is (in particular) a smooth submanifold in N_A (see (4.1.2)) and $\mathcal{F}: N_A \rightarrow \mathcal{F}(N_A)$ is a diffeomorphism. ■

For the construction of a C^∞ -manifold structure on $\mathcal{A}_{(S)}^k$ we use the following lemma:

(4.1.4) LEMMA. *For any $A \in \mathcal{C}^k$ the bundle $N_{A(S)}$ (defined in (4.1.2)) is trivial.*

Proof. We have to construct a smooth isomorphism

$$\chi: N_{A(S)} \rightarrow \mathcal{C}^{k+1} \cdot A \times (N_{A(S)} \cap T_A \mathcal{C}^k).$$

For $X' \in N_{A(S)}$ we define $\chi(X') := (\bar{\pi}(X'), X)$, where $X \in N_{A(S)} \cap T_A \mathcal{C}^k$ is obtained from X' and $\varphi \in \mathcal{C}^{k+1}$ such that $\varphi \cdot A = A' = \bar{\pi}(X')$: $X = \Phi_*(\varphi^{-1}, X')$. Of course, the vector X does not depend on the chosen φ since the symmetry group of A is simultaneously the isotropy group of Φ_* at X . Thus, the map χ is well defined. The smoothness of χ can be seen if we write χ on a suitable neighbourhood of $X' \in N_{A(S)}$ in the form

$$\chi(X'') = (\bar{\pi}(X''), \Phi_*(\varphi^{-1}, \Phi_*([\theta(\bar{\pi}(X''))]^{-1}, X''))),$$

where $\theta: V \rightarrow \mathcal{C}^{k+1}$, $V \ni A' = \bar{\pi}(X')$ are as in the proof of (4.1.2) and $X'' \in \bar{\pi}^{-1}(V) \cap N_{A(S)}$. Here $\varphi \in \mathcal{C}^{k+1}$ is fixed in such a way that $\varphi \cdot A = A'$. It is easy to see that χ is bijective and

$$\chi^{-1}(A'', X) = \Phi_*(\theta(A'') \circ \varphi, X)$$

for $(A'', X) \in V \times (N_{A(S)} \cap T_A \mathcal{C}^k)$. Thus, χ^{-1} is also smooth. ■

(4.1.5) THEOREM. *For any $(S) \in J$ there exists a unique structure of the C^∞ Hilbert manifold on $\mathcal{A}_{(S)}^k$ such that $\hat{\pi}: \mathcal{C}_{(S)}^k \rightarrow \mathcal{A}_{(S)}^k$ is a smooth submersion.⁽¹⁾*

Proof. Let us take an orbit $\mathcal{C}^{k+1} \cdot A \subset \mathcal{C}_{(S)}^k$ and the tubular neighbourhood $\mathcal{F}(N_{A(S)})$ of that orbit in $\mathcal{C}_{(S)}^k$. Let $\hat{\pi}: \mathcal{C}_{(S)}^k \rightarrow \mathcal{C}_{(S)}^k/\mathcal{C}^{k+1}$ be the canonical projection. We define a chart \varkappa on $\hat{\pi}(\mathcal{F}(N_{A(S)}))$ in the following way: first we choose an orbit $\mathcal{C}^{k+1} \cdot A' \in \hat{\pi}(\mathcal{F}(N_{A(S)}))$ and an element $A'' \in \mathcal{C}^{k+1} \cdot A'$; then we define

$$\varkappa(\mathcal{C}^{k+1} \cdot A') := \text{pr}_2(\chi(\mathcal{F}^{-1}(A''))),$$

and so

$$\varkappa: \hat{\pi}(\mathcal{F}(N_{A(S)})) \rightarrow V := T_A \mathcal{C}^k \cap N_{A(S)}.$$

⁽¹⁾ As we show later (Example (4.4.8)), in general the total quotient space \mathcal{A}^k does not admit the structure of a Hilbert manifold. It may be considered as a stratified space.

Since $\varkappa \circ \hat{\pi} = \text{pr}_2 \circ \chi \circ \mathcal{T}^{-1}$ is continuous, \varkappa is also continuous. It is easy to verify that $\varkappa^{-1} = \hat{\pi} \circ \mathcal{T}$, and thus \varkappa^{-1} is also continuous. Hence, \varkappa is a homeomorphism.

Now let \varkappa_1, \varkappa_2 be two charts defined on neighbourhoods of orbits $\mathcal{G}^{k+1} \cdot A_1$ and $\mathcal{G}^{k+1} \cdot A_2$ respectively, and let us assume that $\mathcal{T}_1(N_{A_1(S)}) \cap \mathcal{T}_2(N_{A_2(S)}) \neq \emptyset$. Note that $\varkappa_2 \circ \varkappa_1^{-1}$ maps an open set in V_1 onto an open set in V_2 , where $V_i = T_{A_i} \mathcal{G}^k \cap N_{A_i(S)}$, $i = 1, 2$. Since $\varkappa_2 \circ \varkappa_1^{-1} = \text{pr}_2 \circ \chi_2 \circ \mathcal{T}_2^{-1} \circ \mathcal{T}_1$, this map is of class C^∞ as a composition of smooth maps (here χ_2 is the trivialization of $N_{A_2(S)}$, and $\mathcal{T}_i: N_{A_i(S)} \rightarrow \mathcal{G}_{(S)}^k$, $i = 1, 2$). Thus, the charts we have defined give $\mathcal{H}_{(S)}^k$ the structure of a C^∞ Hilbert manifold. To prove that $\hat{\pi}: \mathcal{G}_{(S)}^k \rightarrow \mathcal{H}_{(S)}^k$ is a smooth submersion it is sufficient to show that for any chart \varkappa the map $\varkappa \circ \hat{\pi}$ is a smooth submersion. But this is clear because $\varkappa \circ \hat{\pi} = \text{pr}_2 \circ (\chi \circ \mathcal{T}^{-1})$ is the composition of the diffeomorphism $\chi \circ \mathcal{T}^{-1}$ and the smooth submersion $\text{pr}_2: \mathcal{G}^{k+1} \cdot A \times V \rightarrow V$. Since $\hat{\pi}: \mathcal{G}_{(S)}^k \rightarrow \mathcal{H}_{(S)}^k$ is a submersion, the uniqueness of the smooth structure on $\mathcal{H}_{(S)}^k$ follows from the general theory of manifolds (see e.g. [4b]). ■

4.2. The Countability Theorem. We prove here that the set J of orbit types is at most countable. We shall also show examples of situations where J is finite and when J is countable.

(4.2.1) THEOREM (The Countability Theorem). *The set J of orbit types of the action Φ is at most countable.*

The proof of this theorem is based on the following lemma:

(4.2.2) LEMMA: *Let $x_0 \in M$ and $A_1, A_2 \in \mathcal{G}^k$. The following conditions are equivalent:*

- (i) *The symmetry groups $S(A_1)$ and $S(A_2)$ are conjugate, in \mathcal{G}^{k+1} .*
- (ii) *There exist an inner automorphism $\beta: \mathring{G} \rightarrow \mathring{G}$ and a β -isomorphism $\iota: \mathring{P} \rightarrow \mathring{P}$ of class H^{k+1} (over id_M) such that $\iota(E(A_1)) = E(A_2)$.*

Proof. We first prove that (i) implies (ii). For this purpose, let us take $\tilde{\psi} \in H^{k+1}(\mathring{P})$ such that $\tilde{\psi} S(A_1) \tilde{\psi}^{-1} = S(A_2)$. By virtue of Proposition (1.3.7) (i), we get

$$\begin{aligned} E(A_2) &= \{q \in \mathring{P} \mid \forall \tilde{\varphi} \in S(A_1) \ (\tilde{\psi} \tilde{\varphi} \tilde{\psi}^{-1})(\hat{\pi}(q)) = q(\tilde{\psi} \tilde{\varphi} \tilde{\psi}^{-1})(x_0) q^{-1}\} \\ &= \{q \in \mathring{P} \mid \forall \tilde{\varphi} \in S(A_1) \ \tilde{\varphi}(\hat{\pi}(q)) = [\tilde{\psi}(\hat{\pi}(q))^{-1} q \tilde{\psi}(x_0)] \tilde{\varphi}(x_0) [\tilde{\psi}(\hat{\pi}(q))^{-1} q \tilde{\psi}(x_0)]^{-1}\} \\ &= \{q \in \mathring{P} \mid \tilde{\psi}(\hat{\pi}(q))^{-1} q \tilde{\psi}(x_0) \in E(A_1)\} = \iota(E(A_1)), \end{aligned}$$

where

$$\iota: \mathring{P} \ni q \mapsto \tilde{\psi}(\hat{\pi}(q)) q \tilde{\psi}(x_0)^{-1} \in \mathring{P}$$

is clearly a β -isomorphism of class H^{k+1} with

$$\beta: \mathring{G} \ni \hat{g} \mapsto \tilde{\psi}(x_0) \hat{g} \tilde{\psi}(x_0)^{-1} \in \mathring{G}.$$

To prove that (ii) implies (i), we first choose an element $\dot{g}_1 \in \dot{G}$ such that for all $\dot{g} \in \dot{G}$ we have $\beta(\dot{g}) = \dot{g}_1 \dot{g} \dot{g}_1^{-1}$. Secondly, we define a section $\tilde{\psi}$ of \tilde{P} by the formula

$$\tilde{\psi}(\dot{\pi}(q)) = \iota(q) \circ \dot{g}_1 \circ q^{-1}, \quad q \in \dot{P}.$$

Taking (smooth) local sections s of \dot{P} , one can verify that $\psi \in \mathcal{G}^{*+1}$ (we have $\tilde{\psi}(x) = \iota(s(x)) \circ \dot{g}_1 \circ (s(x))^{-1}$). Now, applying Proposition (1.3.7) (ii), we prove that the equality $\iota(E(A_1)) = E(A_2)$ implies the equality $\psi S(A_1) \psi^{-1} = S(A_2)$:

$$S(A_2) = \{\tilde{\varphi} \in \Gamma(\tilde{P}) \mid \forall q \in E(A_2)\}$$

$$\begin{aligned} \tilde{\psi}(\dot{\pi}(q))^{-1} \tilde{\varphi}(\dot{\pi}(q)) \tilde{\psi}(\dot{\pi}(q)) &= \tilde{\psi}(\dot{\pi}(q))^{-1} q (\iota(\dot{e}))^{-1} \tilde{\varphi}(x_0) \iota(\dot{e}) q^{-1} \tilde{\psi}(\dot{\pi}(q)) \\ &= \{\tilde{\varphi} \in \Gamma(\tilde{P}) \mid \forall q \in E(A_1)\} \end{aligned}$$

$$\begin{aligned} (\tilde{\psi}^{-1} \tilde{\varphi} \tilde{\psi})(\dot{\pi}(q)) &= \tilde{\psi}(\dot{\pi}(q))^{-1} \iota(q) (\iota(\dot{e}))^{-1} \tilde{\varphi}(x_0) \iota(\dot{e}) (\iota(q))^{-1} \tilde{\psi}(\dot{\pi}(q)) \\ &= \{\tilde{\varphi} \in \Gamma(\tilde{P}) \mid \forall q \in E(A_1)\} \end{aligned}$$

$$(\tilde{\psi}^{-1} \tilde{\varphi} \tilde{\psi})(\dot{\pi}(q)) = q \dot{g}_1^{-1} (\iota(q))^{-1} \iota(q) (\iota(\dot{e}))^{-1} \tilde{\varphi}(x_0) \iota(\dot{e}) (\iota(q))^{-1} \iota(q) \dot{g}_1 q^{-1}.$$

Since $\tilde{\psi}(x_0) = \iota(\dot{g}) \dot{g}_1 \dot{g}^{-1}$ for any $\dot{g} \in \dot{G} = \dot{\pi}^{-1}(x_0)$, we get

$$\tilde{\psi}(x_0) = \iota(\dot{e}) \dot{g}_1 \quad \text{and} \quad (\tilde{\psi}(x_0))^{-1} = \dot{g}_1^{-1} (\iota(\dot{e}))^{-1},$$

and thus

$$\begin{aligned} S(A_2) &= \{\tilde{\varphi} \in \Gamma(\tilde{P}) \mid \forall q \in E(A_1) (\tilde{\psi}^{-1} \tilde{\varphi} \tilde{\psi})(\dot{\pi}(q)) = q \circ (\tilde{\psi}^{-1} \tilde{\varphi} \tilde{\psi})(x_0) \circ q^{-1}\} \\ &= \{\tilde{\varphi} \in \Gamma(\tilde{P}) \mid \tilde{\psi}^{-1} \tilde{\varphi} \tilde{\psi} \in S(A_1)\} \end{aligned}$$

(the last equality follows from Proposition (1.3.7) (ii). Hence

$$S(A_2) = \psi S(A_1) \psi^{-1}. \quad \blacksquare$$

To prove the Countability Theorem, we shall also need the following fact:

(4.2.3) THEOREM. *The set of conjugacy classes of closed subgroups in any (non-finite, of course) compact Lie group is countable.*

The proof of the above theorem is given in [18]. This theorem is a consequence of the Mostov–Palais theorem (see [22] and [24] or G. Bredon [6]).

Proof of 4.2.1. By virtue of Lemma (4.2.2), we obtain a one-to-one correspondence between the set J and the set of classes of β -isomorphic evolution bundles. Now, to prove the theorem, it suffices to show that set of classes of all β -isomorphic subbundles in \dot{P} (where β are inner automorphisms of \dot{G}) is countable.

Let us note that:

(i) If an H_1 -subbundle and an H_2 -subbundle (say, E_1 and E_2 ,

respectively) of \dot{P} are β -isomorphic (where β is an inner automorphism of \dot{G}), then there exists a β -isomorphism $\iota: \dot{P} \rightarrow \dot{P}$ such that $\iota(E_1) = E_2$.

(ii) If $E \subset \dot{P}$ is an H -subbundle ($H \subset \dot{G}$) and β is an inner automorphism of \dot{G} , then there exists a $\beta(H)$ -subbundle $E_1 \subset \dot{P}$ such that E and E_1 are β -isomorphic.

These two simple facts imply that the cardinality of the set of β -isomorphic subbundles does not exceed the sum of the cardinalities of the sets of isomorphic H -bundles over M , where the sum is labelled by the conjugacy classes (H) of closed subgroups of \dot{G} .

The cardinality of the set of classes of isomorphic H -bundles over M is equal to the cardinality of the set of homotopy classes of maps from M into the classifying space of H (see [16]). This set of homotopy classes has the cardinality of the set of arcwise connected components of the space of continuous maps from M into the classifying space. But this space of continuous maps is separable and its arcwise connected components are open (this follows from the compactness of M , the topological properties of classifying spaces and the general theorems for the spaces of continuous maps, given e.g. in [3]). Hence the set of homotopy classes of these maps is countable.

Since for any fixed $H \subset \dot{G}$ the set of H -bundles over M is countable, it follows that by virtue of Theorem (4.2.3) the set of classes of β -isomorphic subbundles of \dot{P} is at most countable. This concludes the proof. ■

If G is abelian (i.e., $G = \mathbb{T}^n$), then J consists exactly of one element, because in this case we have $E(A) = \dot{P}$ for any $A \in \mathcal{C}^k$ (compare this with Example (1.3.9)). If G is not abelian, for instance if $G = \text{SU}(2)$, then a finite number of orbit types may also appear. E.g. in the case of an $\text{SU}(2)$ -bundle P over $M = S^3$ (it is always a trivial bundle) we obtain J containing 3 elements. But if we take $P = S^2 \times \text{SU}(2)$, then J is infinite. In this case, the Hopf bundle $S^3 \rightarrow S^2$ and all other nontrivial S^1 -bundles over $M = S^2$ appear as the subbundles of the (trivial) $\text{SU}(2)$ -bundle \dot{P} . On the other hand, since S^1 is the maximal torus in $\text{SU}(2)$, for any S^1 -subbundle of \dot{P} there exists a connection which has this subbundle as the bundle of parallel translations and also as the evolution bundle ($C^2(S^1) = S^1$ in $\text{SU}(2)$). Hence, for $P = S^2 \times \text{SU}(2)$, the set J is indeed countable.

4.3. Density theorems. Let

$$\mathcal{C}_S^k := \{A \in \mathcal{C}^k \mid S(A) = S\}$$

be the set of connections with a symmetry group $S \subset \mathcal{G}^{k+1}$. We denote by \mathcal{C}_S^k the set of H^k connections preserved by any element of S , namely

$$\mathcal{C}_S^k := \{A \in \mathcal{C}^k \mid S(A) \supset S\}$$

(the set of connections with symmetry equal to or higher than S).

(4.3.1) PROPOSITION. \mathcal{C}_S^k is a closed affine subspace in \mathcal{C}^k . \mathcal{C}_S^k is an open subset in \mathcal{C}_S^k .

Proof. Since Φ is a continuous affine action and

$$\mathcal{C}_S^k = \bigcap_{\varphi \in S} \{A \in \mathcal{C}^k \mid \Phi(\varphi, A) = A\},$$

the first part of the proposition follows. Now, let $A \in \mathcal{C}_S^k$. Then $\mathcal{F}(N_A)$ is an open neighbourhood of $\mathcal{G}^{k+1} \cdot A$ (as in Theorem (3.3.4)); therefore $U = \mathcal{F}(N_A) \cap \mathcal{C}_S^k$ is an open neighbourhood of A in \mathcal{C}_S^k . It is sufficient to show that $U \subset \mathcal{C}_S^k$. For any $A_1 \in U$, we have $S \subset S(A_1)$ because $U \subset \mathcal{C}_S^k$. Since $A_1 \in \mathcal{F}(N_A)$, Corollary (3.3.5) implies that there exists a $\varphi \in \mathcal{G}^{k+1}$ such that $\varphi S(A_1) \varphi^{-1} \subset S(A) = S$. Now the inclusions $\varphi S(A_1) \varphi^{-1} \subset S \subset S(A_1)$ imply that $S(A_1) = S$. This means that $U \subset \mathcal{C}_S^k$, and we conclude the proof.

We shall show later that \mathcal{C}_S^k is indeed a generic subset in \mathcal{C}_S^k , i.e., \mathcal{C}_S^k is open and dense in \mathcal{C}_S^k . The following statement will be a crucial point in the proof of that fact.

(4.3.2) THEOREM. Let $n = \dim M \geq 2$. Let $A \in \mathcal{C}^k$ and let $\dot{H}(A) \subset \dot{P}$ be the bundle of parallel translations. If $\mathcal{H} \subset \dot{P}$ is a connected subbundle such that $\mathcal{H} \supset \dot{H}(A)$, then there exists an $\alpha \in H^k(\text{Ad } P \otimes T^*M)$ such that for every $r \in \mathbb{R} - \{0\}$ we have $\dot{H}(A + r\alpha) = \mathcal{H}$.

Proof. Let us fix a point $p \in \pi^{-1}(x_0)$. Then the image $\mathcal{H}_p = Q_p(\mathcal{H})$ is a connected subbundle in P containing the holonomy bundle $H(A, p)$ (see (1.2.4)). Thus, the connection A reduces to \mathcal{H}_p . This means that the 1-form \hat{A} , when restricted to \mathcal{H}_p , takes values in the Lie algebra \mathfrak{h} of the structure group $H \subset G$ of \mathcal{H}_p . Let $U_0 \subset M$ be an open neighbourhood of x_0 such that there exist a local chart κ and a local section \mathcal{S}_0 of \mathcal{H}_p on U_0 . We choose $\kappa: U_0 \rightarrow \mathbb{R}^n$ such that the local coordinates (y^1, y^2, \dots, y^n) defined by κ vanish at x_0 , $y^1(x_0) = y^2(x_0) = \dots = y^n(x_0) = 0$. Let $m = \dim H$. Now, for a suitably small $\varepsilon > 0$, we define the family of $m+2$ curves $\sigma, \tau_0, \dots, \tau_m$ on the surface $y^3 = \dots = y^n = 0$; σ is given by $y^1 = 0$ and $0 \leq y^2 \leq \varepsilon$; the curve τ_j is defined as follows: $y^2 = j(\varepsilon/m)$, $0 \leq y^1 \leq \varepsilon$, $j = 0, 1, \dots, m$. All these curves form together a kind of comb.

For the construction of α we need a suitable local section of \mathcal{H}_p defined on a neighbourhood of x_0 :

(4.3.3) LEMMA. There exist an open neighbourhood U of x_0 , $U \subset U_0$, and $\varepsilon > 0$ and a local H^{k+1} -section \mathcal{S} of \mathcal{H}_p defined on U such that all curves $\sigma, \tau_0, \dots, \tau_m$ lie in U and the curves $\mathcal{S} \circ \sigma, \mathcal{S} \circ \tau_0, \dots, \mathcal{S} \circ \tau_m$ are horizontal with respect to A .

The proof of Lemma (4.3.3) will be given later.

The section \mathcal{S} defines a local trivialization χ of \mathcal{H}_p and P over U :

$$\chi: U \times G \ni (x, g) \mapsto \mathcal{S}(x) \cdot g \in \pi^{-1}(U).$$

We have $\chi(U \times H) = \mathcal{H}_p \upharpoonright U$. The form $\chi^* \hat{A}$ on $U \times G$ may be written as follows:

$$\chi^* \hat{A} = \omega + \text{pr}_2^* \Theta,$$

where Θ is the canonical left invariant \mathfrak{g} -valued 1-form on G and $\text{pr}_2: U \times G \rightarrow G$ is the projection. The 1-form ω takes values in \mathfrak{h} on the tangent vectors to $U \times H$, because \mathfrak{h} contains the Lie algebra of the holonomy group H_p of A at p . From the properties of \mathcal{S} it follows that for every $g \in G$ and $j = 0, 1, \dots, m$ the curves

$$[0, \varepsilon] \ni t \mapsto (\sigma(t), g) \in U \times G,$$

$$[0, \varepsilon] \ni t \mapsto (\tau_j(t), g) \in U \times G$$

are horizontal with respect to $\chi^* \hat{A}$. This means that

$$\omega \left(\left(\frac{\partial}{\partial y^2} \right)_{(\sigma(t), g)} \right) = 0 \quad \text{and} \quad \omega \left(\left(\frac{\partial}{\partial y^1} \right)_{(\tau_j(t), g)} \right) = 0$$

for each $j = 0, 1, \dots, m$ and for all $t \in [0, \varepsilon]$, $g \in G$.

We now consider the following \mathfrak{g} -valued 1-form on $U \times G$:

$$\tilde{\alpha}_{(x, g)} = f(x) \sum_{l=1}^m (\text{Ad } g^{-1} X_l)(f_l \circ y^2)(x) \text{pr}_1^* dy^1,$$

where $\text{pr}_1: U \times G \rightarrow U$ is the projection; X_1, X_2, \dots, X_m are certain elements of \mathfrak{h} , which we will choose further; f is a smooth function with a compact support contained in U and we require $f \circ \sigma$, $f \circ \tau_0, \dots, f \circ \tau_m$ to be constant functions identically equal to 1; the functions $f_l: \mathbb{R} \rightarrow \mathbb{R}$ are given by the formula

$$f_l(a) := \prod_{\substack{0 \leq j \leq m \\ j \neq l}} \frac{a - j \frac{\varepsilon}{m}}{\frac{\varepsilon}{m} - j \frac{\varepsilon}{m}}.$$

Since $f_l(0) = 0$ and $f_l \left(j \frac{\varepsilon}{m} \right) = \delta_{lj}$ for $l = 1, \dots, m$ and $j = 0, 1, \dots, m$, we obtain

$$\tilde{\alpha} \left(\left(\frac{\partial}{\partial y^1} \right)_{(\tau_l(t), g)} \right) = \text{Ad } g^{-1} X_l$$

and

$$\tilde{\alpha} \left(\left(\frac{\partial}{\partial y^1} \right)_{(\tau_0(t), g)} \right) = 0 \quad \text{for all } t \in [0, \varepsilon], g \in G.$$

Obviously,

$$\tilde{\alpha} \left(\left(\frac{\partial}{\partial y^2} \right)_{(x, g)} \right) = 0 \quad \text{for any } (x, g) \in U \times G.$$

In the second part of the proof we show that for appropriate $X_1, \dots, X_m \in \mathfrak{h}$ and for any $r \neq 0$ the connection $\chi^* \hat{A} + r\tilde{\alpha}$ has \mathfrak{h} as the Lie algebra of the holonomy group at (x_0, e) . For this purpose, let us consider a family of "rectangles" in U , each of them being formed by pieces of curves $\sigma, \tau_l, \sigma'_s$ and τ_0 , where $0 < s < \varepsilon, l = 1, \dots, m$, and the curves σ'_s are given as follows; $y^1 = s, 0 \leq y^2 \leq l(\varepsilon/m), y^3 = \dots = y^n = 0$. We shall now lift these rectangles by means of $\chi^* \hat{A} + r\tilde{\alpha}$. The lift $\hat{\sigma}$ of σ starting from the point (x_0, e) is clearly

$$\left[0, l \frac{\varepsilon}{m}\right] \ni t \mapsto (\sigma(t), e) \in U \times G.$$

The lift $\hat{\tau}_l$ of τ_l starting from the end-point $\left(\sigma\left(l \frac{\varepsilon}{m}\right), e\right)$ of $\hat{\sigma}$ is of the form

$$[0, s] \ni t \mapsto (\tau_l(t), g(t)) \in U \times G,$$

where $g(0) = e$ and

$$(\chi^* \hat{A} + r\tilde{\alpha})(\dot{\tau}_l(t), \dot{g}(t)) = 0 \quad \text{for} \quad 0 \leq t \leq s.$$

By virtue of Lemma (4.3.3), we have

$$\chi^* \hat{A}(\dot{\tau}_l(t), O_g) = 0$$

for any $0 \leq t \leq s$ and for all $g \in G$ ($O_g \in T_g G$ is the origin). Thus

$$0 = (\chi^* \hat{A} + r\tilde{\alpha})(\dot{\tau}_l(t), \dot{g}(t)) = \Theta(\dot{g}(t)) + r \text{Ad } g(t)^{-1} X_l.$$

Therefore, we have

$$\dot{g}(t) = -R_{g(t)} \circ r X_l.$$

The unique solution of this equation, satisfying the condition $g(0) = e$, is given by the formula

$$g(t) = \exp(-rtX_l).$$

Hence the end-point of $\hat{\tau}_l$ is $(\tau_l(s), \exp(-rsX_l))$. Since $\dot{\sigma}'_s(t) = -\frac{\partial}{\partial y^2}$ and $\tilde{\alpha}\left(\frac{\partial}{\partial y^2}\right) = 0$, any lift of σ'_s by means of $\chi^* \hat{A} + r\tilde{\alpha}$ coincides with a lift by means of $\chi^* \hat{A}$. Thus the lift $\hat{\sigma}'_s$ of σ'_s starting from the end-point of $\hat{\tau}_l$ is of the form

$$\left[0, l \frac{\varepsilon}{m}\right] \ni t \mapsto (\sigma'_s(t), \gamma_l(t, s) \exp(-rsX_l)) \in U \times G,$$

where $\gamma_l(0, s) = e$ for $0 < s < \varepsilon$ and $(\sigma'_s(\cdot), \gamma(\cdot, s))$ is the lift of σ'_s (by means of $\chi^* \hat{A}$) starting from $(\tau_l(s), e)$. Finally, the lift $\hat{\tau}_0$ of τ_0 starting from the end

point of $\hat{\sigma}_s^l$ is

$$[0, s] \ni t \mapsto \left(\tau_0(t), \gamma_l \left(l \frac{\varepsilon}{m}, s \right) \exp(-rsX_l) \right) \in U \times G,$$

because $\tilde{\alpha}_{(\tau_0(t), \theta)} = 0$ and $\left(\frac{\partial}{\partial y^1} \right)_{(\tau_0(t), \theta)}$ is horizontal with respect to $\chi^* \hat{A}$ for any $g \in G$. Thus, lifting the single loop (the “rectangle” with indices l and s), we obtain the element $\left(x_0, \gamma_l \left(l \frac{\varepsilon}{m}, s \right) \exp(-rsX_l) \right) \in U \times G$ of the holonomy bundle of $\chi^* \hat{A} + r\tilde{\alpha}$ containing (x_0, e) . Since this occurs for every $s \in (0, \varepsilon)$, we see that the vectors

$$X_l^r := \frac{d}{ds} \Big|_0 \gamma_l \left(l \frac{\varepsilon}{m}, s \right) - rX_l, \quad l = 1, 2, \dots, m$$

are elements of the Lie algebra of the holonomy group. Note that the vectors

$$X_l^r := \frac{d}{ds} \Big|_0 \gamma_l \left(l \frac{\varepsilon}{m}, \varepsilon \right)$$

are elements of \mathfrak{h} , because $\chi^* \hat{A}$ reduces to $U \times H$.

The next step of the proof is purely algebraic:

(4.3.4) LEMMA. *Let $\{X_1^r, \dots, X_m^r\} \subset \mathfrak{h}$ be an arbitrary subset, $m = \dim \mathfrak{h}$. Then there exists a subset $\{X_1, \dots, X_m\} \subset \mathfrak{h}$ such that for every $r \in \mathbf{R} - \{0\}$ the vectors $X_l^r = X_l^r - rX_l$, $l = 1, \dots, m$, span \mathfrak{h} .*

We shall prove this simple lemma later on.

Now, applying Lemma (4.3.4), we see that \mathfrak{h} is contained in the Lie algebra of the holonomy group of $\chi^* \hat{A} + r\tilde{\alpha}$ (for $r \neq 0$). On the other hand, it is obvious that $\chi^* \hat{A} + r\tilde{\alpha}$ reduces to $U \times H$. Hence \mathfrak{h} is the Lie algebra of the holonomy group of $\chi^* A + r\alpha$ at (x_0, e) for all $r \neq 0$. This concludes the second part of the proof.

Finally, in the third part of the proof, we show that the theorem holds for

$$\alpha = \begin{cases} \chi^{-1*} \tilde{\alpha} & \text{on } \pi^{-1}(U), \\ 0 & \text{elsewhere.} \end{cases}$$

It is clear that α is a \mathfrak{g} -valued tensorial 1-form of type Ad on P of Sobolev class H^k (we recall that \mathcal{S} and also χ are of class H^{k+1} , see (4.3.3)). Therefore α corresponds to a section, say α , of $\text{Ad } P \otimes T^*M$. We can see that for each $r \in \mathbf{R}$ the connection $\hat{A} + r\alpha$ reduces to \mathcal{H}_p . Therefore, the holonomy bundle $H(A + r\alpha, p)$ is contained in \mathcal{H}_p . Hence the Lie algebra of the holonomy group of $A + r\alpha$ at p is contained in \mathfrak{h} . But we already know that \mathfrak{h} is the holonomy algebra of $(A + r\alpha) \upharpoonright \pi^{-1}(U)$ at p (for $r \neq 0$); therefore \mathfrak{h} is the holonomy algebra of $A + r\alpha$ at p . This implies that for $r \neq 0$ the holonomy

group of $A+r\alpha$ is a both closed and open subgroup of H . Hence $H(A+r\alpha, p)$ is a closed and open subbundle of \mathcal{H}_p for $r \neq 0$. This means that $\hat{H}(A+r\alpha) = \mathcal{H}$ for all $r \neq 0$, because \mathcal{H} is connected. Thus the theorem is proved. ■

To complete the proof of Theorem (4.3.2), we have to prove the two lemmas formulated earlier.

Proof of (4.3.3). We construct \mathcal{S} by means of the section \mathcal{S}_0 , the local chart κ and the polynomials f_l , $l = 0, 1, \dots, m$, introduced in the proof of (4.3.2). Taking $\tilde{\mathcal{S}} := \mathcal{S}_0 \circ \kappa^{-1}$, we define the functions $h: (-2\varepsilon, 2\varepsilon) \rightarrow G$ and $h_j: (-2\varepsilon, 2\varepsilon) \rightarrow G$, $j = 0, 1, \dots, m$, by the following conditions:

(i) the curves

$$\begin{aligned} (-2\varepsilon, 2\varepsilon) \ni t &\mapsto \tilde{\mathcal{S}}(0, t, 0, \dots, 0) \cdot h(t) \in P \\ (-2\varepsilon, 2\varepsilon) \ni t &\mapsto \tilde{\mathcal{S}}(t, j(\varepsilon/m), 0, \dots, 0) \cdot h_j(t) \in P \end{aligned}$$

are horizontal with respect to A ;

(ii) $h(0) = h_0(0) = \dots = h_m(0) = e$.

We take a local chart κ_G on a neighbourhood of $e \in G$ (such that $\kappa_G(e) = 0$ and $\text{im } \kappa_G = \mathbf{R}^{m'}$, $m' = \dim G$) and we choose an $\varepsilon > 0$ small enough for the G -valued function

$$\tilde{h}(t_1, \dots, t_n) := \left[\kappa_G^{-1} \left(\sum_{j=0}^m f_j(t_2) \kappa_G(h_j(t_1)) \right) \right] h(t_2)$$

to be well defined on the tube

$$\tilde{U} := (-2\varepsilon, 2\varepsilon) \times (2\varepsilon, 2\varepsilon) \times \mathbf{R}^{n-2} \subset \mathbf{R}^n$$

and $U := \kappa^{-1}(\tilde{U}) \subset U_0$. Since $f_i\left(j\frac{\varepsilon}{m}\right) = \delta_{ij}$, it can easily be seen that the local section $\mathcal{S}: U \rightarrow P$, given by the formula

$$\mathcal{S}(x) := \mathcal{S}_0(x) \tilde{h}(\kappa(x)),$$

is horizontal over the curves $\sigma, \tau_0, \dots, \tau_m$. ■

Proof of 4.3.4. The vectors X'_1, \dots, X'_m span a linear subspace in \mathfrak{h} , say \mathfrak{h}_1 . We can choose k vectors ($k = \dim \mathfrak{h}_1 \leq m$) $X'_{i_1}, X'_{i_2}, \dots, X'_{i_k}$ from the given sequence of m vectors, which also span \mathfrak{h}_1 , $1 \leq i_1 < i_2 < \dots < i_k \leq m$. For $l \in I' := \{i_1, \dots, i_k\}$, we take $X_l := 0$. Next we choose $m-k$ vectors $X_l \in \mathfrak{h}$, $l \in I := \{1, \dots, m\} - I'$, such that $\{X_l | l \in I\} \cup \{X'_l | l \in I'\}$ is a basis of \mathfrak{h} . Then we see that

$$X'_l = X'_l \quad \text{for } l \in I',$$

and for any $l \in I$ we have

$$X_l = \frac{-1}{r} X'_l + \frac{1}{r} X'_l = \frac{-1}{r} X'_l + \frac{1}{r} \sum_{i \in I'} a_i X'_i,$$

because $\{X_i: i \in I'\}$ is a basis of \mathfrak{h}_1 . Thus all vectors of the basis $\{X_i | i \in I'\} \cup \{X_i | i \in I\}$ are linear combinations of the vectors X_1^r, \dots, X_m^r , $r \neq 0$. This proves that for any $r \neq 0$ the set $\{X_1^r, \dots, X_m^r\} \subset \mathfrak{h}$ is a basis of \mathfrak{h} . ■

We now obtain the following result as a consequence of Theorem (4.3.2) and Proposition (4.3.1).

(4.3.5) THEOREM. *Let $S \subset \mathcal{G}^{k+1}$ be the symmetry group of a connection $A_1 \in \mathcal{C}^k$. Then \mathcal{C}_S^k is a generic subset in $\tilde{\mathcal{C}}_S^k$, i.e., \mathcal{C}_S^k is open and dense in $\tilde{\mathcal{C}}_S^k$.*

Proof. By virtue of Prop. (4.3.1), \mathcal{C}_S^k is open in $\tilde{\mathcal{C}}_S^k$. If $\mathcal{C}_S^k = \tilde{\mathcal{C}}_S^k$ then the statement is trivial. Otherwise, let us take $A \in \tilde{\mathcal{C}}_S^k - \mathcal{C}_S^k$. For a fixed $x_0 \in M$, the connections A and A_1 define their bundles of parallel translations $\dot{H}(A) \subset \dot{P}$ and $\dot{H}(A_1) \subset \dot{P}$ with the structure groups $\dot{H} = \dot{H}(A)_{x_0} \subset \dot{G}$ and $\dot{H}_1 = \dot{H}(A_1)_{x_0} \subset \dot{G}$, respectively. Recall that $E(A) = \dot{H}(A) \cdot C^2(\dot{H})$ (see (1.3.6)). Let $\mathcal{H} \subset \dot{P}$ be the connected component of $E(A_1)$ containing $\dot{H}(A_1)$ (note that $\dot{H}(A_1)$ is itself connected). Then \mathcal{H} is a principal subbundle of P with the structure group $\dot{G}_1 \subset \dot{G}$. We can see that \dot{G}_1 contains the connected component of $C^2(\dot{H}_1)$, $\dot{G}_1 \supset C^2(\dot{H}_1)_0$. Since $S(A_1) \subset S(A)$, we have $E(A) \subset E(A_1)$ (see Prop. (1.3.7) (i)).

Now $\dot{H}(A) \subset E(A) \subset E(A_1)$, $\dot{H}(A_1) \subset E(A_1)$ and $\dot{H}(A)$, $\dot{H}(A_1)$ are both connected. Since $\dot{H}(A) \cap \dot{H}(A_1) \neq \emptyset$ (because $\dot{H}(A) \cap \dot{H}(A_1) \supset \dot{H} \cap \dot{H}_1 \neq \emptyset$) and $\dot{H}(A) \cup \dot{H}(A_1) \subset E(A_1)$, it follows that $\dot{H}(A) \cup \dot{H}(A_1)$ is connected and hence it is contained in \mathcal{H} . Therefore $\dot{H}(A) \subset \mathcal{H}$. By virtue of Theorem (4.3.2), there exists a section $\alpha \in H^k(\text{Ad } P \otimes T^*M)$ such that for any real number $r \neq 0$

$$\dot{H}(A + r\alpha) = \mathcal{H}.$$

Now, for $r \neq 0$,

$$E(A + r\alpha) = \mathcal{H} \cdot C^2(\dot{G}_1) \supset \mathcal{H} \cdot C^2(\dot{H}_1) = \dot{H}(A) \cdot \dot{G}_1 \cdot C^2(\dot{H}_1) = E(A_1)$$

since $\dot{G}_1 \supset \dot{H}$ and $\dot{G}_1 \subset C^2(\dot{H}_1)$.

On the other hand,

$$E(A + r\alpha) \subset \mathcal{H} \cdot C^2(C^2(\dot{H}_1)) = \mathcal{H} \cdot C^2(\dot{H}_1) = E(A_1)$$

since $\dot{G}_1 \subset C^2(\dot{H}_1)$, and hence $C^2(\dot{G}_1) \subset C^4(\dot{H}_1) = C^2(\dot{H}_1)$. Thus we obtain the equality

$$E(A + r\alpha) = \begin{cases} E(A) & \text{for } r = 0, \\ E(A_1) & \text{for } r \neq 0. \end{cases}$$

Hence, by virtue of Corollary (1.3.8), we obtain

$$S(A + r\alpha) = \begin{cases} S(A) & \text{for } r = 0, \\ S(A_1) = S & \text{for } r \neq 0. \end{cases}$$

Since the curve $\sigma: \mathbf{R} \ni r \mapsto A + r\alpha \in \overline{\mathcal{C}_S^k}$ is continuous and for $r \neq 0$ we have $\sigma(r) \in \mathcal{C}_S^k$, it is clear that $A = \sigma(0) \in \overline{\mathcal{C}_S^k}$. ■

The connection between \mathcal{C}_S^k and $\mathcal{C}_{(S)}^k$ is clear:

$$\mathcal{C}_{(S)}^k = \mathcal{G}^{k+1} \cdot \mathcal{C}_S^k.$$

As a simple consequence of the continuity of the \mathcal{G}^{k+1} action, we obtain by virtue of Thm. (4.3.5) the following result:

(4.3.6) COROLLARY. $\mathcal{C}_{(S)}^k$ is dense in $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$.

Let us observe that, for any continuous action of a topological group \mathbf{G} on a topological space X , if $Y \subset Z \subset X$ and Y is dense in Z then $\mathbf{G} \cdot Y$ is dense in $\mathbf{G} \cdot Z$. On the other hand, if Y is open in Z then $\mathbf{G} \cdot Y$ is not necessarily open in $\mathbf{G} \cdot Z$. A simple example is the following: $\mathbf{G} = \mathbf{Z}_2 = \{-1, 1\}$, $X = S^1$, the action is $\mathbf{G} \times X \ni (a, z) \mapsto az \in X$,

$$Z := \{1\} \cup \{z \in S^1 \mid \text{Im } z > 0\} \quad \text{and} \quad Y := \{z \in S^1 \mid \text{Im } z \geq 0, \text{Re } z > 0\}.$$

In our case, however we can prove the following statement:

(4.3.7) PROPOSITION $\mathcal{C}_{(S)}^k$ is open in $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$.

Proof. ad absurdum. Assume that $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k} - \mathcal{C}_{(S)}^k$ is not closed in $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. Then there exists a sequence $\{A_n\}_{n \in \mathbf{N}}$ in $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k} - \mathcal{C}_{(S)}^k$ such that $\lim_{n \rightarrow \infty} A_n = A \in \mathcal{C}_{(S)}^k$. Let us take an invariant tubular neighbourhood of $\mathcal{G}^{k+1} \cdot A$ open in \mathcal{C}^k (see Thm. (3.3.4)) and the intersection of this neighbourhood of A with $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. From (3.3.5) (ii) we deduce that this intersection is contained in $\mathcal{C}_{(S)}^k$, and thus $A_n \in \mathcal{C}_{(S)}^k$ for all sufficiently large n . This contradicts our assumption. ■

Another consequence of the slice theorem is the following fact:

(4.3.8) PROPOSITION. $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$ is a closed subset of \mathcal{C}^k .

Proof. Let $A' \in \overline{\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}}$. Then the slice $\mathcal{S}_{A'}$ at A' contains a point $A_1 \in \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. By virtue of (3.3.5) (ii), we obtain $S(A') \supset S(A_1)$. Since $A_1 \in \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$, there exists a $\varphi \in \mathcal{G}^{k+1}$ such that $S(A_1) \supset \varphi^{-1} S\varphi$. This implies that $S(A') \supset \varphi^{-1} S\varphi$. Therefore $S(\varphi \cdot A') \supset S$, which means that $\varphi \cdot A' \in \overline{\mathcal{C}_S^k}$. Hence $A' \in \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. ■

Thus, (4.3.6)–(4.3.8) give us the result:

(4.3.9) THEOREM. $\mathcal{C}_{(S)}^k$ is a generic subset of $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. Moreover,

$$\overline{\mathcal{C}_{(S)}^k} = \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k} \text{ (the closure in } \mathcal{C}^k \text{)}.$$

We now show that $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$ is the sum of subsets of the form $\mathcal{C}_{(S')}^k$ for some $(S') \in J$.

(4.3.10) PROPOSITION. Let $A' \in \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$ and $S' = S(A')$. Then $\mathcal{C}_{(S')}^k \subset \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$.

Proof. It is clear that there exists a $\varphi \in \mathcal{G}^{k+1}$ such that $\varphi \cdot A' \in \overline{\mathcal{C}_S^k}$ and thus $S_1 := S(\varphi \cdot A) \supset S$. Since $S_1 \in (S')$, each orbit lying in $\mathcal{C}_{(S')}^k$ contains a connection with the symmetry group equal to S_1 , that is, every orbit from $\mathcal{C}_{(S')}^k$ intersects $\overline{\mathcal{C}_S^k}$. This means that $\mathcal{C}_{(S')}^k \subset \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. ■

We recall a simple property of continuous actions:

(4.3.11) **PROPOSITION.** *Let a topological group \mathbf{G} act continuously on a topological space X . Then for any subset $V \subset X/\mathbf{G}$ we have*

$$\bar{V} = \hat{\pi}(\overline{\hat{\pi}^{-1}(V)}),$$

where $\hat{\pi}: X \rightarrow X/\mathbf{G}$ is the projection.

Proof. Since the action is continuous, $\overline{\hat{\pi}^{-1}(V)}$ is an invariant subset in X . Hence $\hat{\pi}(X - \overline{\hat{\pi}^{-1}(V)}) = X/\mathbf{G} - \hat{\pi}(\overline{\hat{\pi}^{-1}(V)})$. This set is open since $\hat{\pi}$ is open. Clearly, $\hat{\pi}(\overline{\hat{\pi}^{-1}(V)}) \supset \hat{\pi}(\hat{\pi}^{-1}(V)) = V$ and hence $\hat{\pi}(\overline{\hat{\pi}^{-1}(V)}) \supset \bar{V}$. On the other hand, $\hat{\pi}^{-1}(V) \subset \hat{\pi}^{-1}(\bar{V})$ and, by the continuity of $\hat{\pi}$, the set $\hat{\pi}^{-1}(\bar{V})$ is closed. Therefore, $\hat{\pi}^{-1}(\bar{V}) \supset \overline{\hat{\pi}^{-1}(V)}$. Hence $\bar{V} \supset \hat{\pi}(\overline{\hat{\pi}^{-1}(V)})$. ■

(4.3.12) **PROPOSITION.** *For any $(S) \in J$, we have*

$$\overline{\mathcal{P}_{(S)}^k} = \hat{\pi}(\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}).$$

Proof. This is a simple consequence of Proposition (4.3.11) and Theorem (4.3.9).

4.4. The stratification of \mathcal{P}^k . A decomposition of a topological space into parts homeomorphic to well-known geometrical figures is one of the most useful tools in topology. Our aim here is to construct a decomposition of \mathcal{P}^k into smooth manifolds. This decomposition satisfies some regular frontier properties, which enable us to call this partition a regular smooth stratification.

Stratifications first arose in connection with describing singularities of differentiable maps (Thom [29], Whitney [30]). The definition of a stratification for infinite dimensional spaces can be found in Fisher's paper [13]. Since this notion is perhaps not sufficiently familiar, we shall formulate briefly its definition.

(4.4.1) Let \mathbf{D} be a countable (or finite) family of nonempty subsets of a topological space X ($\mathbf{D} \subset 2^X$). We consider the following conditions:

- (i) $\bigcup \mathbf{D} = X$.
- (ii) $\forall (\Omega, \Omega') \in \mathbf{D} \times \mathbf{D} \quad \Omega \cap \Omega' \neq \emptyset \Rightarrow \Omega = \Omega'$.
- (iii) $\forall (\Omega, \Omega') \in \mathbf{D} \times \mathbf{D} \quad \bar{\Omega} \cap \Omega' \neq \emptyset \Rightarrow \bar{\Omega} \supset \Omega'$.
- (iv) $\forall (\Omega, \Omega') \in \mathbf{D} \times \mathbf{D} \quad \bar{\Omega} \cap \Omega' \neq \emptyset \Rightarrow \bar{\Omega}' \cap (\Omega \cup \bar{\Omega}') = \Omega'$.

If \mathbf{D} satisfies conditions (i) and (ii) then it is called a *partition*. A partition is a *manifold decomposition* if each $\Omega \in \mathbf{D}$ is a manifold. A manifold

decomposition is a *stratification* if it satisfies condition (iii). Then an element of \mathbf{D} is called a *stratum*.

(4.4.2) DEFINITION. A regular stratification is a stratification satisfying condition (iv) of (4.4.1).

(4.4.3) Remark. Any stratification \mathbf{D} of X distinguishes a partial ordering (quasi-ordering in the terminology of Kuratowski [20]) in \mathbf{D} :

$$\Omega < \Omega' \quad \text{iff} \quad \bar{\Omega} \cap \Omega' \neq \emptyset.$$

It is easy to see that $<$ is a reflexive and transitive relation. A partial ordering distinguished by a regular stratification is an order, i.e., it satisfies the asymmetry condition

$$(\Omega < \Omega' \text{ and } \Omega' < \Omega) \Rightarrow \Omega = \Omega'.$$

For any stratification \mathbf{D} of X and for every stratum $\Omega \in \mathbf{D}$, we have

$$\bar{\Omega} = \bigcup_{\Omega' < \Omega} \Omega'.$$

The definition of a regular stratification of X may also be formulated as follows:

(4.4.4) DEFINITION. A countable (or finite) set \mathbf{D} of nonempty subsets of a topological space X is called a *regular stratification* iff

(i) each $\Omega \in \mathbf{D}$ is endowed with a manifold structure compatible with the topology induced from X ,

(ii) $X = \bigcup \mathbf{D}$,

(iii) $\forall (\Omega, \Omega') \in \mathbf{D} \times \mathbf{D} \quad \bar{\Omega} \cap \Omega' \neq \emptyset \Rightarrow \bar{\Omega} \supset \Omega'$,

(iv) $\forall (\Omega, \Omega') \in \mathbf{D} \times \mathbf{D} \quad (\Omega \neq \Omega' \wedge \bar{\Omega} \cap \Omega' \neq \emptyset) \Rightarrow \bar{\Omega}' \cap \Omega = \emptyset$.

Let us point out that condition (iv) of Definition (4.4.4) implies conditions (ii) and (iv) of (4.4.1).

(4.4.5) THEOREM (The Stratification Theorem). *The family $\mathbf{D} = \{\mathcal{R}_{(S)}^k \mid (S) \in J\}$ is a regular stratification of \mathcal{R}^k .*

Proof. Since \mathbf{D} is bijective with J , it follows that \mathbf{D} is countable or finite (see Theorem (4.2.1)). We have also proved that each $\mathcal{R}_{(S)}^k$ admits a structure of a C^∞ Hilbert manifold compatible with the induced topology (see Theorem (4.1.5)). It is easily seen that

$$\bigcup \mathbf{D} = \bigcup_{(S) \in J} \mathcal{R}_{(S)}^k = \mathcal{R}^k.$$

Now it remains to verify the frontier properties (iii) and (iv) of Definition (4.4.4). If $\overline{\mathcal{R}_{(S)}^k} \cap \mathcal{R}_{(S')}^k \neq \emptyset$ then we can choose a connection A' such that $\mathcal{G}^{k+1} \cdot A' \in \mathcal{R}_{(S)}^k \cap \mathcal{R}_{(S')}^k$ and $S(A') = S'$. Since

$$\hat{\pi}^{-1}(\overline{\mathcal{R}_{(S)}^k}) = \hat{\pi}^{-1}(\hat{\pi}(\mathcal{G}^{k+1} \cdot \mathcal{G}_S^k)) = \mathcal{G}^{k+1} \cdot \mathcal{G}_S^k$$

(by Prop. (4.3.12)), we see that $A' \in \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. Therefore, by virtue of Proposition (4.3.10), we obtain $\mathcal{C}_{(S')}^k \subset \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. This implies that

$$\mathcal{R}_{(S')}^k = \hat{\pi}(\mathcal{C}_{(S')}^k) \subset \hat{\pi}(\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}) = \overline{\mathcal{R}_{(S)}^k}$$

and proves that condition (iii) of Definition (4.4.4) is satisfied.

We prove condition (iv) of (4.4.4) as follows. If $\mathcal{R}_{(S)}^k \neq \mathcal{R}_{(S')}^k$ and $\overline{\mathcal{R}_{(S)}^k} \cap \overline{\mathcal{R}_{(S')}^k} \neq \emptyset$ then $\mathcal{C}_{(S)}^k \cap \mathcal{C}_{(S')}^k = \emptyset$ and $\mathcal{C}_{(S')}^k \subset \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$ (as above). Since $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$ is closed (Prop. (4.3.8)), we see that $\overline{\mathcal{C}_{(S')}^k} \subset \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$. But $\mathcal{C}_{(S)}^k$ is open in $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k}$ (Prop. (4.3.7)) and $\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k} - \mathcal{C}_{(S)}^k \supset \mathcal{C}_{(S')}^k$. Hence $\overline{\mathcal{C}_{(S)}^k} \cap \overline{\mathcal{C}_{(S')}^k} = \emptyset$. This means that

$$\hat{\pi}^{-1}(\overline{\mathcal{R}_{(S)}^k}) \cap \hat{\pi}^{-1}(\overline{\mathcal{R}_{(S')}^k}) = \emptyset.$$

Applying Proposition (4.3.12) and Theorem (4.3.9), we obtain

$$\hat{\pi}^{-1}(\overline{\mathcal{R}_{(S)}^k}) = \hat{\pi}^{-1}(\hat{\pi}(\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k})) = \mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k} = \overline{\mathcal{C}_{(S)}^k} = \overline{\hat{\pi}^{-1}(\mathcal{R}_{(S)}^k)}.$$

Therefore

$$\emptyset = \hat{\pi}^{-1}(\overline{\mathcal{R}_{(S)}^k}) \cap \hat{\pi}^{-1}(\overline{\mathcal{R}_{(S')}^k}) \doteq \hat{\pi}^{-1}(\overline{\mathcal{R}_{(S)}^k}) \cap \hat{\pi}^{-1}(\overline{\mathcal{R}_{(S')}^k}) = \hat{\pi}^{-1}(\overline{\mathcal{R}_{(S)}^k} \cap \overline{\mathcal{R}_{(S')}^k})$$

and we conclude that $\overline{\mathcal{R}_{(S)}^k} \cap \overline{\mathcal{R}_{(S')}^k} = \emptyset$. ■

Let us point out that we have one to one correspondence between \mathbf{D} and J (defined by the map **type**). We can endow J with a natural order \prec' :

$$(S) \prec' (S') \Leftrightarrow [\exists \varphi \in \mathcal{G}^{k+1} \quad \varphi S \varphi^{-1} \subset S'].$$

Since $\mathbf{D} = \{\mathcal{R}_{(S)}^k \mid (S) \in J\}$ is a regular stratification, it has also a natural order \prec (see Remark (4.3.3)). We have the following agreement between \prec and \prec' :

(4.4.6) PROPOSITION. *The bijection $J \ni (S) \mapsto \mathcal{R}_{(S)}^k \in \mathbf{D}$ is an isomorphism of ordered sets (J, \prec') and (\mathbf{D}, \prec) .*

Proof. Let us first note that

$$\mathcal{G}^{k+1} \cdot \overline{\mathcal{C}_S^k} = \bigcup_{(S) \prec' (S')} \mathcal{C}_{(S')}^k = \bigcup_{(S) \prec' (S')} \hat{\pi}^{-1}(\mathcal{R}_{(S')}^k)$$

is the sum of disjoint sets. If $\mathcal{R}_{(S)}^k \prec \mathcal{R}_{(S_1)}^k$, then, applying Propositions (4.3.12) and (4.3.10) in the same way as in the proof of Theorem (4.4.5), we obtain

$$\hat{\pi}^{-1}(\mathcal{R}_{(S_1)}^k) \subset \bigcup_{(S) \prec' (S')} \hat{\pi}^{-1}(\mathcal{R}_{(S')}^k).$$

Therefore

$$\mathcal{R}_{(S_1)}^k \subset \bigcup_{(S) \prec' (S')} \mathcal{R}_{(S')}^k.$$

Since \mathbf{D} is a partition, we see that $(S) \prec' (S_1)$.

On the other hand, if $(S) \prec' (S_1)$ then, by virtue of Theorem (4.3.9), we

have

$$\overline{\mathcal{C}_{(S)}^k} = \bigcup_{(S') < (S)} \hat{\pi}^{-1}(\mathcal{R}_{(S')}) \supset \hat{\pi}^{-1}(\mathcal{R}_{(S)}).$$

Hence $\overline{\mathcal{R}_{(S)}^k} = \hat{\pi}(\overline{\mathcal{C}_{(S)}^k}) \supset \mathcal{R}_{(S)}^k$ (the equality follows from Proposition (4.3.12)), and therefore $\mathcal{R}_{(S)}^k < \mathcal{R}_{(S_1)}^k$. ■

The notion of a regular stratification, introduced in the present paper (Def. (4.4.2) and Def. (4.4.4)), is studied in detail in [19b]. We show there some examples of stratifications which are not regular. But we also prove that every stratification of a Hausdorff space is a regular stratification. Therefore, the regularity of the stratification $\{\mathcal{R}_{(S)}^k \mid (S) \in J\}$ (proved here directly) is a straightforward consequence of the theorem mentioned above and Theorem (4.0).

(4.4.8) EXAMPLE. Consider the action Φ of gauge transformations on connections in the case: $M = S^1$, $G = \text{SU}(2)$ and $P = M \times G$. Let $k > 2$. One can see that the orbit space $\mathcal{C}^k/\mathcal{G}^{k+1}$ in this case ($M = S^1$) may be identified with the orbit space $G/\text{Aut } G$ of the action of the group of inner automorphisms of G on G . For $G = \text{SU}(2)$ we see that $G/\text{Aut } G$ is homeomorphic to a closed interval $[-1, 1]$. Hence $\mathcal{C}^k/\mathcal{G}^{k+1}$ does not admit a manifold structure.

Acknowledgments. Our warmest thanks are due to K. Gawędzki for illuminating discussions and several important remarks. We are also grateful to W. Szczyrba (who was the first to read the whole manuscript) for his kind interest and his comments upon it. Lastly we thank A. Białynicki-Birula, A. E. Fisher, S. Klimek, K. Maurin, M. S. Narasimhan, T. R. Ramadas, R. Rączka, S. Rolewicz, R. Rubinstein, A. Strasburger, J. Śniatycki, H. Toruńczyk and A. Wawrzyńczyk for their attention and their helpful observations.

References

- [1] M. F. Atiyah, N. J. Hitchin, I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Soc. (London) A³⁶² (1978), 425–461.
- [2] M. F. Atiyah, J. D. S. Jones, *Topological aspects of Yang–Mills theory*, Commun. Math. Phys. 61 (1978), 97–118.
- [3] K. Borsuk, *Theory of retracts*, PWN, Warszawa 1967.
- [4] N. Bourbaki, *Éléments de mathématique*, Paris: Hermann, [a] Fasc. III, *Topologie générale. Groupes topologiques* (1960), [b] Fasc. XXXIII, *Variétés différentielles et analytiques. Fascicule de résultats* (Paragraphes 1 a 7) (1971), [c] Fasc. XXXVII, *Groupes et algèbres de Lie* (1972).
- [5] J. P. Bourguignon, *Une stratification de l'espace des structures riemanniennes*, Comp. Math. 30 (1975), 1–41.
- [6] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York 1972.
- [7] M. Cantor, *Elliptic operators and the decomposition of tensor fields*, Bull. Amer. Math. Soc. 5 (1981), 235–262.
- [8] P. R. Chernoff, J. E. Marsden, *Properties of infinite dimensional Hamiltonian systems*. In: *Lecture notes in mathematics*. Vol. 425 Springer, Berlin–Heidelberg–New York 1974.
- [9] Y. Choquet–Bruhat, D. Christodoulou, *Elliptic systems in Hilbert spaces on manifolds which are euclidean at infinity*, Acta Math. 146 (1981), 129–150.
- [10] Y. Choquet–Bruhat, D. Christodoulou: *Systèmes elliptiques sur une variété euclidienne à l'infini*, C. R. Acad. Sci. Paris 290 Ser. A. (1980), 781–785.
- [11] D. G. Ebin, *The manifold of Riemannian metrics*, Proc. Symp. Pure Math. Amer. Math. Soc. XV (1970), 11–40.
- [12] D. G. Ebin, J. E. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. 92 (1970), 102–163.
- [13] A. E. Fisher, *The theory of superspace*. In: *Relativity* (M. Carmeli; S. Fickler and L. Witten eds), Plenum Press, New York 1970.
- [14] V. N. Gribov, *Instability of non-abelian gauge theories and impossibility of choice of Coulomb gauge*, SLAC Translation 176 (1977). See also: *Quantization of non-abelian gauge theories*, Nuclear Phys. B 139 (1978), 1–19.
- [15] G. 't Hooft, *Aspects of quark confinement*, Phys. Scripta 24 (1981), 841–846.
- [16] D. Husemoller, *Fibre bundles*, Mc Graw–Hill Book Co. New York 1966.
- [17] J. Isenberg, J. Marsden, *A slice theorem for the space of solutions of Einstein equations*, Phys. Rep. 89 (1982), 179–222.
- [18] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York 1963.
- [19] W. Kondracki, J. S. Rogulski, [a] *On conjugacy classes of closed subgroup*, [b] *On the notion of stratification*, Preprint 281 PAN, Warszawa 1983.
- [20] K. Kuratowski, *Introduction to set theory and topology* PWN, Warszawa 1972.
- [21] P. K. Mitter, C. M. Viallet, *On the bundle of connections and the gauge orbit manifold in Yang–Mills theory*, Commun. Math. Phys. 79 (1981), 457–472.

- [22] G. D. Mostov, *Equivariant embeddings in euclidean space*, Ann. of Math. 65 (1957), 432–446.
- [23] M. S. Narasimhan, T. R. Ramadas, *Geometry of SU(2) gauge fields*, Commun. Math. Phys. 67 (1979), 121–136.
- [24] R. S. Palais, *Embedding of compact differentiable transformation groups in orthogonal representations*, J. Math. Mech. 6 (1957), 673–678.
- [25] R. S. Palais, *Foundations of global non linear analysis*, Benjamin Company Inc, New York 1968.
- [26] –, *Seminar on the Atiyah–Singer index theorem*, Princeton Univ. Press (1965).
- [27] J. S. Rogulski, *Operators with H^k coefficients and generalized Hodge–de Rham decompositions*, Demonstratio Mathematica vol. XVIII, No 1. (1985), 77–89.
- [28] I. M. Singer, *Some remarks on the Gribov ambiguity*, Commun. Math. Phys. 60 (1978), 7–12.
- [29] R. Thom, *Les singularités des applications différentiables*, Annales de l'Institut Fourier VI (1956), 43–87.
- [30] H. Whitney, *Elementary structures of real algebraic varieties*, Ann. of Math. 66 (1957), 546–556.

INSTYTUT MATEMATYCZNY PAN
Śniadeckich 8, 00-950 Warszawa

and

INSTYTUT MATEMATYKI POLITECHNIKI WARSZAWSKIEJ
Plac Jedności Robotniczej 1, 00-661 Warszawa
