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TRIANGULATION OF SUBANALYTIC SETS

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1. Introduction

After the considerations of triangulation problem of algebraic set, analytic set, etc. by van der Waerden, Lefschetz, Koopman, Brown and Whitehead around 1930, it was Cairns–Whitehead who first gave a precise proof to the problem in a special case. They proved a unique C^∞ triangulation of a C^∞ manifold (see [15]), that is, for a C^∞ manifold M there exist a simplicial complex K and a homeomorphism $\tau\colon |K| \to M$ such that for each $\sigma \in K$ $\tau|\sigma\colon \sigma \to \tau(\sigma)$ is a C^∞ diffeomorphism. Moreover, such a C^∞ triangulation is unique in the following sense. If there are other K' and $\tau'\colon |K'| \to M$ then $\tau'^{-1} \circ \tau\colon |K| \to |K'|$ is isotopic to a PL homeomorphism. From the uniqueness it automatically follows that $|K|$ is a PL manifold; hence a C^∞ manifold admits a unique PL manifold structure. Furthermore the uniqueness is the key to the proof of existence of C^∞ triangulation because the local existence is trivial.

Łojasiewicz [10] and Giesecke [2] proved a semianalytic (resp. semialgebraic) triangulation of a semianalytic (resp. semialgebraic) set. The reason why it took a long time for the problem to be solved is that the proof needed a detailed study of (semi)-analytic sets. Later Hironaka [5] and Hardt [4] showed that a subanalytic triangulation of a subanalytic set follows in the same way but more easily. Then there was a problem of uniqueness of subanalytic triangulation, which has been recently proved by Yokoi and myself [19] as follows. Let X be a locally compact subanalytic set, Y and Y' be polyhedrons and $\tau\colon Y \to X$, $\tau'\colon Y' \to X$ be subanalytic homeomorphisms. Then $\tau'^{-1} \circ \tau\colon Y \to Y'$ is subanalytically isotopic to a PL homeomorphism.

One of topological spaces known as triangulable sets is a locally compact space with Whitney stratification (see Gorensky [3], Johnson [7], Kato [8] and Verona [20]). The idea of proof of triangulation is clear, because a locally compact Whitney stratified set is locally trivial and moreover locally a join of a sphere and a Whitney stratified set of lower dimension (see [1]). It is easier to show that a set with such properties has subanalytic set structure (Sect. 2). Hence, as a corollary, we obtain a triangulation of a Whitney stratified set, whose proof is the shortest one so far as I know. The former proofs proceed by direct triangulation, and they need a troublesome argument.

When we apply the above triangulation of Whitney stratified set, an inconvenience is that the triangulation is not unique even if we assume that the triangulation is compatible with the stratification and the restriction of the triangulation to each stratum is a C^∞ triangulation in the sense of Cairns–Whitehead. In Section 3 we show how the uniqueness fails, and we sketch the proof of uniqueness of subanalytic triangulation.

Section 4 explains an application of the unique subanalytic triangulation obtained by Matumoto and myself ([12] and [13]). Let G be a compact Lie group acting C^k differentiably on a C^k manifold M with $1 \leqslant k \leqslant \omega$. If $k < \omega$, then (G, M) is C^k equivariant diffeomorphic to an analytic pair uniquely in a sense. Hence we can assume an analytic structure on (G, M). Then the orbit space M/G has naturally a subanalytic set structure and hence a unique triangulation compatible with the orbit type decomposition. Really we obtain a unique triangulation of the orbit space compatible with the orbit type decomposition in a more general condition. This unique triangulation is used to define an equivariant simple homotopy type of a compact C^k G-manifold in the sense of Illman [6].

2. Subanalytic set structure on Whitney stratified sets

A *Whitney stratification* of a set $X \subset \boldsymbol{R}^n$ is a partition of X into C^2 submanifolds X_i of $\boldsymbol{R}^n$ such that $\{X_i\}$ is locally finite at X, $\bar{X}_i \cap X_j \neq \emptyset$ implies $\bar{X}_i \supset X_j$, and $\{X_i\}$ satisfies the Whitney condition (b) (see [1]). If each X_i, moreover, is an analytic manifold (subanalytic (see the below)), we call $\{X_i\}$ *analytic* (resp. *subanalytic*). Let $Y \subset \boldsymbol{R}^n$ be a C^2 manifold. A *tube* at Y is a triple $T = (|T|, \pi, \varrho)$, where $|T|$ is a C^2 tubular neighborhood of Y for some Riemannian metric of $\boldsymbol{R}^n$, $\pi\colon |T| \to X$ is the projection, and ϱ is a nonnegative C^2 function on $|T|$ such that $\varrho^{-1}(0) = X$ and each point $x \in X$ is a nondegenerate critical point of $\varrho|\pi^{-1}(x)$. A *controlled tube system* for a Whitney stratification $\{X_i\}$ consits of one tube $T_i = (|T_i|, \pi_i, \varrho_i)$ at each X_i such that

$$\pi_i \circ \pi_j(x) = \pi_i(x) \quad \text{and} \quad \varrho_i \circ \pi_j(x) = \varrho_i(x)$$

for $x \in |T_i| \cap |T_j| \cap \pi_j^{-1}(|T_i|)$.

A subset X of $\boldsymbol{R}^n$ is called *subanalytic* in $\boldsymbol{R}^n$ if X is a finite union of sets of the form $\operatorname{Im} f_1 - \operatorname{Im} f_2$, where f_1 and f_2 are proper analytic maps from analytic manifolds to R^n. If each point x of X has a neighborhood U in R^n such that $X \cap U$ is subanalytic in R^n then we call X *locally subanalytic*. We remark that when X is closed in R^n and locally subanalytic then X is subanalytic in R^n. A continuous map between subanalytic sets in R^n and in R^m is called *subanalytic* if the graph is subanalytic in $R^n \times R^m$. A *locally subanalytic map* between locally subanalytic sets is relatively defined. Let X be a locally subanalytic C^2 manifold in R^n and let $T = (|T|, \pi, \varrho)$ be a tube at X. If $|T|$, π and ϱ are locally subanalytic, then we call T *locally subanalytic*.

In this section we shall give a subanalytic set structure to a locally compact Whitney stratified set. At first we modify a Whitney stratified set as follows.

LEMMA 2.1. *Let $\{X_i\}$ be a Whitney stratification in R^n. Then there exists a C^2 diffeomorphism τ of R^n such that $\{\tau(X_i)\}$ is an analytic Whitney stratification.*

Proof. Let $\{Y_j\}$ be the collection of strata of $\{X_i\}$ of dimension $< \dim X$. By induction on $\dim X$ we can assume $\{Y_j\}$ is an analytic Whitney stratification. For each X_i not in $\{Y_j\}$, let p_i: $X_i \to R^n$ be the identity map, let U_i be a closed small C^2 tubular neighborhood of X_i in R^n such that $U_i \cap U_{i'} = \emptyset$ for $i \neq i'$ and $\bar{U}_i - U_i = \partial X_i$ $(= \bar{X}_i - X_i)$, let q_i: $U_i \to X_i$ be the C^2 projection, and let α_i be a C^2 function on U_i with support in $\operatorname{Int} U_i$ and with $\alpha_i = 1$ in a neighborhood of X_i. By the approximation theorem of Whitney we have an approximation p'_i of p_i in the Whitney C^2 topology such that $p'_i(X_i)$ is an analytic manifold. Define a C^2 map τ_{i1}: $U_i \to R^n$ by

$$\tau_{i1}(x) = \alpha_i(x)(p'_i \circ q_i(x) + x - q_i(x)) + (1 - \alpha_i(x))x.$$

Then $\tau_{i1} \to$ ident as $p'_i \to p_i$ in the Whitney C^2 topology. Hence we can choose p'_i so that τ_{i1} is a C^2 diffeomorphism of U_i, $\pi_{i1} =$ ident in a neighborhood of $U_i - \operatorname{Int} U_i$, and the jet of τ_{i1}-ident at x converges to 0 as x converges to a point of ∂X_i. By these properties, the extension τ_i of τ_{i1} to $R^n \to R^n$, defined by $\tau_i =$ ident on U_i^c, becomes a C^2 diffeomorphism such that $\{x \in R^n\colon \tau_i(x) \neq x\} \subset \operatorname{Int} U_i$, and $\tau_i(X_i)$ is an analytic manifold. Here we remark that we can choose τ_i arbitrarily close to the identity. Hence $\tau = \prod_i \tau_i$ can be a C^2 diffeomorphism of R^n. Then $\tau(X_i) = \tau_i(X_i)$ are analytic manifolds. It is clear that $\{\tau(X_i)\}$ satisfies the Whitney condition (b) since the condition does not depend on the choice of coordinate system of R^n. Therefore the lemma is proved. □

It is known that a Whitney stratification admits a controlled tube system (see [1]). We need a locally subanalytic controlled tube system.

LEMMA 2. *Let $\{X_i\}$ be an analytic Whitney stratification in R^n. Then there exists a locally subanalytic controlled tube system for $\{X_i\}$.*

Proof. The proof proceeds in the same way as the proof in the C^∞ case in [1]. In [1] the key lemma is (1.6) Theorem in Chapter II, where what we need to pay attention is a partition of unity. There does not exist an analytic partition of unity, but there exists a subanalytic C^r partition of unity for any $0 \leqslant r < \infty$ as follows. Let $\{U_j\}$ be a locally finite open covering of an open set U of $\boldsymbol{R}^n$ such that each U_j is relatively compact in U. Let $0 \leqslant r < \infty$. Then there exists a subanalytic C^r partition of unity $\{\varphi_j\}$ for $\{U_j\}$, i.e. each φ_j is a nonnegative subanalytic C^r function on U with $\operatorname{supp}\varphi_j \subset U_j$ and we have $\sum_j \varphi_j = 1$. Indeed, let $\{\psi_j\}$ be a C^r partition of unity for $\{U_j\}$, let $\{\psi'_j\}$ be analytic approximations of $\{\psi_j\}$ and let χ_j: $\boldsymbol{R} \to \boldsymbol{R}$ be subanalytic C^r functions such that $\chi_j(x) = 0$ for $x \leqslant \varepsilon_j$ and $\chi_j(x) > 0$ for $x > \varepsilon_j$ for some small $\varepsilon_j > 0$. Consider

$$\varphi_j = \chi_j \circ \psi'_j / \sum_k \chi_k \circ \psi'_k$$

in place of ψ_j. Then, if the above approximations are close enough and if ε_j are small enough, $\{\varphi_j\}$ becomes a subanalytic C^r partition of unity. As there is no problem in the rest of proof, we omit the proof. □

If a Whitney stratified set is not locally compact, it is not in general homeomorphic to a subanalytic set. Hence we assume the local compactness from now on.

PROPOSITION 2.3. *Let $\{X_i\}$ be a Whitney stratification of a locally compact set X in $\boldsymbol{R}^n$. Then there exists a homeomorphism τ from X to a subanalytic set in some $\boldsymbol{R}^m$ such that $\tau(X_i)$, for each i, is a subanalytic set in $\boldsymbol{R}^m$.*

Proof. By the local compactness of X, $Y = \bar{X} - X$ is closed in $\boldsymbol{R}^n$. Let ζ be a C^∞ function on $\boldsymbol{R}^n - Y$ such that $\zeta(x) \to \infty$ as x converges to a point of Y. Consider the graphs of $\zeta | X$ and $\{\zeta | X_i\}$ in place of X and $\{X_i\}$. Then we can assume X is closed in $\boldsymbol{R}^n$, and by Lemma 2.1 $\{X_i\}$ is analytic. Moreover we assume each X_i is bounded for the following reason. We need the next fact. Let M be a C^1 manifold closed in $\boldsymbol{R}^n$. Then if we move suitably M a little, M comes to be transversal to all X_i. For closed X_i this is the transversality theorem of Thom. For general X_i we prove this by induction on $\dim X_i$. We assume M is transversal to all $X_{i'}$ with $\dim X_{i'} < \dim X_i$. As X is closed in $\boldsymbol{R}^n$, it follows from the Whitney condition (b) that M is transversal to $X_i \cap U$, where U is a neighborhood of $\bigcup_{\dim X_{i'} < \dim X_i} X_{i'}$. Hence applying the transversality theorem to M, $X_i - U$, we can modify M to be transversal to X_i. Here we used the fact (stability of transversality) that if M is transversal to all $X_{i'}$ with $\dim X_{i'} < \dim X_i$, then M, moved a little, remains to satisfy the same property. Thus M has come to be transversal to all X_i. Once more by the stability of transversality we can choose analytic M; and it is easy to see $\{X_i \cap M, X_i - M\}$ is an analytic Whitney stratification. Let M_j, $j = 1, 2, \ldots$, be the spheres in $\boldsymbol{R}^n$ with center at 0 and of radius j, respectively. Apply the above argument to each M_j and $\{X_i\}$. Then we obtain a refinement of $\{X_i\}$ whose strata are bounded. Hence we can assume, from the beginning, X_i are bounded.

Let $\{T_i = (|T_i|, \pi_i, \varrho_i)\}$ be a locally subanalytic controlled tube system for $\{X_i\}$ (Lemma 2.2). Put $d_i = \dim X_i$ for each i. For small positive numbers ε_i we define inductively

$$U_i = \{x \in X \cap |T_i| - \bigcup_{d_j < d_i} U_j \colon \varrho_i(x) < \varepsilon_i\},$$

$$B_i = \{x \in X \cap |T_i| - \bigcup_{d_j < d_i} U_j \colon \varrho_i(x) = \varepsilon_i\},$$

$$V_i = U_i \cup B_i,$$

$$X'_i = X_i - \bigcup_{d_j < d_i} U_j.$$

Then we have by the properties of controlled tube system [1]

$$X = \bigcup V_i, \qquad B_i = V_i \cap \big(\bigcup_{d_j > d_i} V_j\big),$$

$$\pi_i(B_i) = X'_i \text{ or } \emptyset,$$

$$\pi_i(V_i \cap B_j) = X_i \cap B_j \qquad \text{for } d_i > d_j,$$

$$U_i = \{x \in X \cap |T_i| \colon \varrho_i(x) < \varepsilon_i,\ \pi_i(x) \in X'_i\},$$

$$B_i = \{x \in X \cap |T_i| \colon \varrho_i(x) = \varepsilon_i,\ \pi_i(x) \in X'_i\};$$

and lessenning ε_i if necessary we have homeomorphisms $\tau_i \colon M_i \to V_i$, where M_i is the mapping cylinder $X'_i \cup_{\pi_i} B_i$ of $\pi_i | B_i \colon B_i \to X'_i$, such that

$$\tau_i\big((\pi_i^{-1}(y) \cap B_i) \times (0, 1]\big) = \pi_i^{-1}(y),$$

$$\tau_i(y) = y \qquad \text{for } y \in X'_i,$$

$$\tau_i(x \times 1) = x \qquad \text{for } x \in B_i.$$

For each $0 \leqslant k \leqslant n$ let $\bar{U}_k$ be the union of U_i with $d_i = k$. Define $\bar{B}_k$, $\bar{X}'_k$, $\bar{M}_k$, $\bar{\pi}_k$ $(= \bar{\pi}_k | \bar{B}_k)$: $\bar{B}_k \to \bar{X}'_k$ and $\bar{\tau}_k \colon \bar{M}_k \to \bar{V}_k$ in the same way. Then $\bar{B}_k$, $\bar{X}'_k$, and $\bar{\pi}_k \colon \bar{B}_k \to \bar{X}'_k$ are locally subanalytic since X_i are analytic manifolds and since T_i are locally subanalytic; and furthermore they are subanalytic in $\boldsymbol{R}^n$ because $\bar{B}_k$ and $\bar{X}'_k$ are closed in $\boldsymbol{R}^n$. Put

$$W_k = \bar{V}_n \cup \ldots \cup \bar{V}_k.$$

Remark that $\bar{V}_k$ are not necessarily subanalytic except for $k = n$ and

$$W_0 = X, \qquad \bar{B}_{k-1} = \bar{V}_{k-1} \cap W_k.$$

We want to construct inductively a sequence of subanalytic sets

$S_n \subset \ldots \subset S_0$ and homeomorphisms $\xi_k\colon S_k \to W_k$, $k = n, \ldots, 0$ such that for each $k \geqslant k'$ $\xi_k^{-1}(\bar{B}_{k'} \cap W_k)$ and $\bar{\pi}_{k'} \circ \xi_k\colon \xi_k^{-1}(\bar{B}_{k'} \cap W_k) \to \bar{X}'_{k'}$ are subanalytic, and $\xi_{k'} | S_k = \xi_k$. First put $S_n = W_n (= \bar{V}_n)$, $\xi_n = \text{ident}$, and assume to have constructed $S_{k+1} \subset R^n$ and $\xi_{k+1}\colon S_{k+1} \to W_{k+1}$. Put

$$Z_k = \xi_{k+1}^{-1}(\bar{B}_k)(= \xi_{k+1}^{-1}(\bar{B}_k \cap W_{k+1})) \subset S_{k+1}.$$

Then by assumption Z_k is subanalytic in R^m. Let S_k be the attaching space of S_{k+1} and $\bar{X}'_k$ by $\bar{\pi}_k \circ \xi_{k+1}\colon Z_k \to \bar{X}'_k$ and be given a subanalytic set structure as follows. Assuming $\bar{X}_k$ is contained in $R^m \times 0 \subset R^m \times R$ and S_{k+1} in $R^m \times 1 \subset R^m \times R$, we put

$$S_k = S_{k+1} \cup \bar{X}'_k \cup \{tx + (1-t)\bar{\pi}_k \circ \xi_{k+1}(x)\colon t \in [0, 1], x \in Z_k\}$$

and define $\xi_k\colon S_k \to W_k$ by

$$\xi_k | S_{k+1} = \xi_{k+1}, \qquad \xi_k | \bar{X}'_k = \text{ident}$$

and

$$\xi_k(tx + (1-t)\bar{\pi}_k \circ \xi_{k+1}(x)) = \bar{\tau}_k(t\xi_{k+1}(x) + (1-t)\bar{\pi}_k \circ \xi_{k+1}(x))$$

$$\text{for } t \in [0, 1],\ x \in Z_k.$$

Here we regard naturally $t\xi_{k+1}(x) + (1-t)\bar{\pi}_k \circ \xi_{k+1}(x)$ as an element of $\bar{M}_k$. The S_k is subanalytic in R^{n+1} as Z_k and $\bar{\pi}_k \circ \xi_{k+1}\colon Z_k \to \bar{X}'_k$ are subanalytic, and clearly ξ_k is a homeomorphism. So it is sufficient to see that $\xi_k^{-1}(\bar{B}_{k'} \cap W_k)$ and $\bar{\pi}_{k'} \circ \xi_k\colon \xi_k^{-1}(\bar{B}_{k'} \cap W_k) \to \bar{X}'_{k'}$ are subanalytic for $k' \leqslant k$. By definition of ξ_k $\xi_k^{-1}(\bar{B}_{k'} \cap W_k)$ is the attaching space of $\xi_{k+1}^{-1}(\bar{B}_{k'} \cap W_{k+1})$ and $\bar{B}_{k'} \cap \bar{X}'_k$ by

$$\bar{\pi}_k \circ \xi_{k+1}\colon \xi_{k+1}^{-1}(\bar{B}_{k'} \cap W_{k+1}) \cap Z_k (= \xi_{k+1}^{-1}(\bar{B}_{k'} \cap \bar{B}_k)) \to \bar{B}_{k'} \cap \bar{X}'_k \subset \bar{X}'_k.$$

But by induction hypothesis these sets and the map are subanalytic. Hence $\xi_k^{-1}(\bar{B}_{k'} \cap W_k)$ is subanalytic. The subanalyticness of $\bar{\pi}_{k'} \circ \xi_k\colon \xi_k^{-1}(\bar{B}_{k'} \cap W_k) \to \bar{X}'_k$ follows from this argument and from the fact that a composition of subanalytic maps between compact subanalytic sets is subanalytic. Hence we have constructed the required S_k and ξ_k, which completes the proof.

Corollary 2.4 (triangulation of Whitney stratification). *Let $\{X_i\}$ and X be the same as Proposition 2.3. Then there exist a simplicial complex K and a homeomorphism $\theta\colon |K| \to X$ such that for each X_i, $\xi^{-1}(X_i)$ is a union of open simplexes.*

Proof. Follows from Proposition 2.3 and the triangulation theorem of subanalytic sets. □

3. Uniqueness of triangulation

3.1 (example of distinct triangulations of Whitney stratification due to Milnor [14]). Let M_1 and M_2 denote the C^∞ 3-dimensional lens manifolds of type (7, 1) and (7, 2) and let L_1, L_2 be simplicial complexes of their C^∞ triangulations respectively. Let X_1, X_2 be simplicial complexes obtained from $L_1 \times \sigma$, $L_2 \times \sigma$ by adjoining cones over $L_1 \times \partial\sigma$, $L_2 \times \partial\sigma$ respectively, where σ is a 4-simplex, and let N_1, N_2 be sets in some $\mathbf{R}^n$ defined from M_1, M_2, respectively, in the same way so that $N_1 - a_1$ (vertex) and $N_2 - a_2$ (vertex) are C^∞ smooth and some neighborhoods of a_1 and a_2 in N_1 and N_2, respectively, are cones. Then the natural homeomorphisms τ_1: $|X_1| \to N_1$ and τ_2: $|X_2| \to N_2$ are C^∞ triangulations in the sense of Cairns–Whitehead, and $\{N_1 - a_1, a_1\}$ is a Whitney stratification of N_1. It is known [14] that $|X_1|$ and $|X_2|$ are not PL homeomorphic, and $N_1 - a_1$ and $N_2 - a_2$ are C^∞ diffeomorphic. Let π: $N_1 \to N_2$ denote a homeomorphism such that $\pi | N_1 - a_1$ is a diffeomorphism onto $N_2 - a_2$. Then τ_1: $|X_1| \to N_1$ and $\pi^{-1} \circ \tau_2$: $|X_2| \to N_1$ are distinct triangulations of the Whitney stratification $\{N_1 - a_1, a_1\}$ such that $\tau_1 | |X_1| - \tau_1^{-1}(a_1)$: $|X_1| - \tau_1^{-1}(a_1) \to N_1 - a_1$ and $\tau_2 | |X_2| - \tau_2^{-1}(a_2)$: $|X_2| - \tau_2^{-1}(a_2) \to N_2 - a_2$ are C^∞ triangulations for some subdivision of $|X_1| - \tau_1^{-1}(a_1)$ and $|X_2| - \tau_2^{-1}(a_2)$.

THEOREM 3.2 (uniqueness of subanalytic triangulation ([19])). *Let X be a locally compact subanalytic set in $\mathbf{R}^n$, Y and Y' be polyhedrons in $\mathbf{R}^m$ and τ: $Y \to X$, τ': $Y' \to X$ be subanalytic homeomorphisms. Then there exists an isotopy h_t: $Y \to Y'$, $t \in [0, 1]$, of $\tau'^{-1} \circ \tau$ such that*

(i) *H: $Y \times [0, 1] \to Y'$, defined by $H(x, t) = h_t(x)$, is subanalytic*;

(ii) *h_1 is* PL.

Sketch of proof. It is easy to reduce the problem to the case where Y is a simplex, and by the Alexander trick we only need to prove that Y' is a PL ball. Put $h = \tau'^{-1} \circ \tau$. We want to modify h to a PL homeomorphism. Let Z denote the graph of h, p: $Z \to Y$, p': $Z \to Y'$ the projections, and $\{Z_i\}$ be a subanalytic analytic Whitney stratification of Z such that $p | Z_i$ and $p' | Z'$ are analytic diffeomorphisms onto the images for all i. Put $\{Y_i\} = \{p(Z_i)\}$ and $\{Y_i'\} = \{p'(Z_i)\}$. Then $\{Y_i\}$ and $\{Y_i'\}$ can be subanalytic analytic Whitney stratifications of Y and Y' respectively. For simplicity of notations we assume $\dim Z_i = i$, $i = 0, \ldots, k$.

There is a useful decomposition of Y except a Whitney stratification which was used in proof of Proposition 2.3. Let ε_0 be a small positive number, and put

$$U(Y_0, \varepsilon_0) = \{y \in Y\colon \operatorname{dist}(Y_0, y) < \varepsilon_0\},$$

$$B(Y_0, \varepsilon_0) = \{y \in Y\colon \operatorname{dist}(Y_0, y) = \varepsilon_0\},$$

$$V(Y_0, \varepsilon_0) = U(Y_0, \varepsilon_0) \cup B(Y_0, \varepsilon_0).$$

Choose ε_0 so small that $V(Y_0, \varepsilon_0)$ is homeomorphic to the disjoint union of cones with vertexes Y_0 and with bases $B(Y_0, \varepsilon_0)$. Then, as Y is a polyhedron, $V(Y_0, \varepsilon_0)$ becomes exactly the disjoint union of cones. Next we choose a smaller positive number ε_1 so that $V(Y_1, \varepsilon_1)$ is identical to the mapping cylinder of some projection $B(Y_1, \varepsilon_1) \to Y_1 - U(Y_0, \varepsilon_0)$ where

$$V(Y_1, \varepsilon_1) = \{y \in Y - U(Y_0, \varepsilon_0) \colon \operatorname{dist}(Y_1, y) \leqslant \varepsilon_1\},$$

$$B(Y_1, \varepsilon_1) = \{y \in Y - U(Y_0, \varepsilon_0) \colon \operatorname{dist}(Y_1, y) = \varepsilon_1\}.$$

Repeating this argument we obtain $\varepsilon_0 > \varepsilon_1 > \ldots > \varepsilon_k > 0$, $V(Y_0, \varepsilon_0)$, $V(Y_1, \varepsilon_1), \ldots, V(Y_k, \varepsilon_k)$; and in the same way as proof of Proposition 2.3, each $V(Y_l, \varepsilon_l) \cup \ldots \cup V(Y_k, \varepsilon_k)$ is shown to be an attaching space of $V(Y_{l+1}, \varepsilon_{l+1}) \cup \ldots \cup V(Y_k, \varepsilon_k)$ and $V(Y_l, \varepsilon_l) \cap Y_l$ for a projection $B(Y_l, \varepsilon_l) \to V(Y_l, \varepsilon_l) \cap Y_l$.

Let $V'(Y_0', \varepsilon_0), \ldots, V'(Y_k', \varepsilon_k)$ be the decomposition of Y' obtained in the same way as above for Y' and $\{Y_i'\}$. We define also $B'(Y_l', \varepsilon_l)$. The reasons why we consider $V(Y_l, \varepsilon_l)$, $V'(Y_l', \varepsilon_l)$ in place of $\{Y_i\}$, $\{Y_i'\}$ are that all the family of $V(Y_l, \varepsilon_l)$, $V'(Y_l', \varepsilon_l)$, $B(Y_l, \varepsilon_l)$, $B'(Y_l', \varepsilon_l)$, $V(Y_l, \varepsilon_l) \cap Y_l$ and $V'(Y_l', \varepsilon_l) \cap Y_l'$ are C^∞ triangulable, that is, they are the images of polyhedrons under piecewise C^∞ diffeomorphisms, and that the C^∞ triangulability is a local property by the proof ([15]) of the C^∞ triangulation theorem of Cairns–Whitehead. We can not expect these for $\{Y_i\}$, $\{Y_i'\}$ and C^0 triangulation. For the proof of Theorem we find a C^∞ diffeomorphisms $h_k\colon V(Y_k, \varepsilon_k) \to V'(Y_k', \varepsilon_k)$ and homeomorphisms $h_l\colon \bigcup_{i=l}^{k} V(Y_i, \varepsilon_i) \to \bigcup_{i=l}^{k} V'(Y_i', \varepsilon_i)$ for $0 \leqslant l < k$, inductively, such that each h_l is an extension of h_{l+1} and h_l is a PL homeomorphism for some C^∞ triangulation of $\bigcup_{i=l}^{k} V(Y_i, \varepsilon_i)$ and $\bigcup_{i=l}^{k} V'(Y_i', \varepsilon_i)$. The key to finding of h_k is a version of the following

LEMMA 3.3 ([18]). *Let X be a closed subanalytic subset of $\mathbf{R}^n$. Let f_1 and f_2 be subanalytic functions on X such that for each point $x \in X$, both $f_1(x)$ and $f_2(x)$ have the same sign. Then there exist neighborhoods W_1, W_2 of $f_1^{-1}(0)$ in X and a homeomorphism $\varrho\colon W_1 \to W_2$ such that $f_2 \circ \varrho = f_1$ on W_1.*

It is impossible to require that ϱ is smooth. But under some assumptions we can prove that for any given $\delta > 0$ $\varrho | \{|f_1| > \delta\}\colon \{|f_1| > \delta\} \to \{|f_2| > \delta\}$ is of class C^∞. Put

$$\theta_l(x) = \operatorname{dist}(Y_l, x), \qquad \theta_l'(x) = \operatorname{dist}(Y_l', x)$$

$$f_{l1} = \theta_l \circ p, \qquad f_{l2} = \theta_l' \circ p'$$

for all l. Then f_{l1} and f_{l2} satisfy the conditions on f_1 and f_2 in Lemma 3.3, and we have

$$p^{-1}(V(Y_k, \varepsilon_k)) = Z - \bigcup_{l=0}^{k-1} \{f_{l1} < \varepsilon_l\},$$

$$p'^{-1}(V'(Y_k', \varepsilon_k)) = Z - \bigcup_{l=0}^{k-1} \{f_{l2} < \varepsilon_l\}.$$

Hence, using Lemma 3.3 we obtain a C^∞ diffeomorphism from $p^{-1}(V(Y_k, \varepsilon_k))$ to $p'^{-1}(V'(Y_k', \varepsilon_k))$ and hence h_k: $V(Y_k, \varepsilon_k) \to V'(Y_k', \varepsilon_k)$.

Now we have to extend inductively h_k to h_l: $\bigcup_{i=l}^{k} V(Y_i, \varepsilon_i) \to \bigcup_{i=l}^{k} V'(Y_i', \varepsilon_i)$, $l = k-1, \ldots, 0$. Assume we have defined h_{l+1}. Consider the sets

$$V(Y_l, \varepsilon_l) \cap Y_l = Y_l - \bigcup_{l' \ l} V(Y_{l'}, \varepsilon_{l'})$$

and

$$V'(Y_l', \varepsilon_l) \cap Y_l' = Y_l' - \bigcup_{l'<l} V'(Y_{l'}', \varepsilon_{l'}).$$

We see easily that they are compact analytic manifolds with cornered boundary and C^∞ diffeomorphic. Choose a PL homeomorphism between C^∞ triangulations of the sets so that it is extensible to a C^∞ triangulation of $V(Y_l, \varepsilon_l) \to$ one of $V'(Y_l', \varepsilon_l)$ together with $h_{l+1} | B(Y_l, \varepsilon_l)$. Here we use the Alexander trick and the fact that Y and Y' are polyhedrons. Thus we construct h_l. □

Remark 3.3. In Theorem 3.2, we can choose h_t of form $\tau_t'^{-1} \circ \tau_t$ for some subanalytic isotopies τ_t: $Y \to X$ and τ_t': $Y' \to X$. For the application in Section 4 we need this form.

4. Application to equivariant differential topology ([12], [13])

In this section G denotes a transformation group of a topological space X. We call G *proper* if any x, $y \in X$ have neighborhoods U, V, respectively, such that $\{h \in G\colon hU \cap V \neq \emptyset\}$ is relatively compact in G. This is equivalent to say that $G \times X \ni (g, x) \to (gx, x) \in X \times X$ is proper when G is locally compact and X is Hausdorff (see [9] and [17]). If G and X are contained in some $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively, as subanalytic sets and if the action $G \times X \to X$ is subanalytic, then we call G a *subanalytic transformation group* of a subanalytic set X. Let X/G denote the *orbit space*. For each $x \in X$ the *orbit type* of x or of the orbit Gx means the collection $\{gG_x g^{-1}\colon g \in G\}$, where $G_x = \{g \in G\colon gx = x\}$ is the *isotropy group* at x.

For simplicity of notations we assume X is closed in $\mathbf{R}^m$.

LEMMA 4.1. *Let $G (\subset \mathbf{R}^n)$ be a proper subanalytic transformation group of a subanalytic set X $(\subset \mathbf{R}^m)$ and let $\{X_i\}$ be the decomposition of X by orbit types. Assume X is closed in $\mathbf{R}^m$. Then $\{X_i\}$ is subanalytic and locally finite.*

LEMMA 4.2. *Under the same assumption as above, there exists a subanalytic G-invariant map φ: $X \to \mathbf{R}^{2k+1}$, where $k = \dim X$, such that $\varphi(X)$ is closed and subanalytic in $\mathbf{R}^{2k+1}$ and the induced map $\bar{\varphi}$: $X/G \to \varphi(X)$ is a homeomorphism.*

For the same G and X as above, a *subanalytic triangulation* of X/G means a pair of a simplicial complex K closed in $\mathbf{R}^{m'}$ for some m' and a homeomorphism $\tau\colon |K| \to X/G$ such that $\tau^{-1} \circ q\colon X \to |K|$ is subanalytic, where $q\colon X \to X/G$ is the projection. We call (K, τ) is *compatible* with the orbit type decomposition if for each $\sigma \in K$ all points of $\tau(\operatorname{Int}\sigma)$ have the same orbit type. The following is an immediate consequence of Theorem 3.2, Remark 3.3 and Lemmas 4.1 and 4.2.

COROLLARY 4.3 (unique subanalytic triangulation of orbit space). *Under the same assumption as Lemma* 4.1, *X/G admits a unique subanalytic triangulation compatible with the orbit type decomposition. Here the uniqueness means the following. Assume (K, τ) and (K', τ') are subanalytic triangulations of X/G compatible with the orbit type decomposition. Then we have subanalytic isotopies (K, τ_t) and (K', τ'_t) such that*

(i) $\tau_0 = \tau$ *and* $\tau'_0 = \tau'$;

(ii) (K, τ_t) *and* (K', τ'_t) *are subanalytic triangulations of X/G compatible with the orbit type decomposition for each* $t \in [0, 1]$;

(iii) $\tau_t(\sigma) = \tau(\sigma)$ *and* $\tau'_t(\sigma') = \tau'(\sigma')$ *for each* $\sigma \in K$, $\sigma' \in K'$ *and* $t \in [0, 1]$, *and*

(iv) $(\tau'_1)^{-1} \circ \tau_1\colon |K| \to |K'|$ *is a* PL *map.*

When we consider the equivariant differential topology, we usually treat the (Whitney) C^k topology, $1 \leqslant k \leqslant \infty$. So, to apply the above result, we need the following unique C^ω smoothing. The case of C^∞ smoothing is due to Palais [16], and our proofs are based on it.

LEMMA 4.4. *Let G be a compact Lie group and M a C^r G-manifold, $1 \leqslant r \leqslant \infty$. Then there is a C^ω G-manifold $\tilde{M}$ which is C^r equivariantly diffeomorphic to M.*

LEMMA 4.5. *Let G be a compact Lie group and let M and N be C^∞ G-manifolds. Assume M and N are C^r equivariantly diffeomorphic, $1 \leqslant r \leqslant \infty$. Then they are subanalytically C^k equivariantly diffeomorphic for any $1 \leqslant k < \infty$. Moreover if M is compact M and N are C^ω equivariantly diffeomorphic.*

By Corollary 4.3 and Lemmas 4.4 and 4.5 we can triangulate uniquely the orbit space M/G compatibly with the orbit type decomposition for a compact Lie group G and a C^r G-manifold for $1 \leqslant r \leqslant \infty$. Lifting to M each simplex of the barycentric subdivision of this triangulation of M/G, we obtain a G-CW complex structure on M which is an equivariant case of CW complex (see [11]). The importance is that the G-CW complex structure is unique in a sense, from which we can define equivariant simple homotopy type of M in the sense of Illman [6] at least in the case of compact M.

THEOREM 4.6. *Let G be a compact Lie group and M be a compact C^k G-manifold, $1 \leqslant k \leqslant \infty$. Then we have a well-defined equivariant simple homotopy type of M.*

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