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On density theorems for outer measures

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INTRODUCTION

In recent years, great advances were made in Surface Area Theory and its generalizations to Euclidean spaces of arbitrary dimension. However, the ever-increasing technical complexity of the researches which yielded these advances seems to indicate that a systematic effort to organize, simplify, and improve the technical background involved is in order if progress in this field is to be maintained and accelerated. Also, one may feel that the extreme technical complexity of the researches just referred to is perhaps due to the fact that some simple fundamental device has eluded so far the attention of the workers in this field. The purpose of this monograph is to contribute to further progress in Surface Area Theory by a systematic presentation of density theorems for outer measures. While the literature on measure theory and its applications contains an enormous store of information on density theorems, we were handicapped considerably by two facts. First, most of the results available in the literature had to be modified and generalized to some extent to meet our needs. Second, most of the relevant results occur as auxiliary lemmas in papers dealing with applications of many types, and thus they are difficult to locate. Furthermore, we found that various results related to density theorems admit of substantial generalizations which increase their scope and utility considerably. Motivated by these considerations, we felt that a systematic presentation of density theorems for outer measures, arranged in proper logical sequence and formulated in appropriate generality, should be a worthwhile undertaking. The purpose of this paper is to give such a systematic presentation. Since several basic open problems in surface area theory can be reduced to questions concerning the validity of certain density theorems, we hope in particular that the present systematic study will be of material assistance to workers in this field. We shall give presently an outline of the contents of this paper.

Various generalizations of the classical Vitali covering theorem are of course fundamental in dealing with density theorems. A general study of such covering theorems has been made by A. P. Morse [1] (numbers in square brackets refer to the bibliography at the end of this paper), and in our paper [2] we proved a very general covering theorem, invol-

ving two abstract binary relations, which includes as special cases the general covering theorems of A. P. Morse. A brief survey of these matters is given in Section 1.

Many results relating to Carathéodory outer measures Γ are proved in the literature under the assumption that for every set E there exists a Borel set B such that $E \subset B$ and $\Gamma(E) = \Gamma(B)$. Some such results have been generalized by replacing Borel sets by analytic sets. In our own work we found that in a variety of important contexts one can replace Borel sets by the so-called *absolutely measurable sets*. While this term apparently does not occur actually in print, we are indebted to A. P. Morse for the information that the term (as well as the concept) was communicated to him by J. D. Tamarkin over 20 years ago, and that A. P. Morse himself made extensive use of absolutely measurable sets in his lectures. In Section 2 we collect a number of theorems involving absolutely measurable sets which we came across in our own work.

Density theorems are usually of the *almost everywhere* type in the sense that they involve an exceptional set of measure zero with respect to an appropriate measure. In order to characterize such exceptional sets, we found it convenient to introduce certain auxiliary measures of the Hansdorff type. In Section 3 we collect the necessary information on such measures. We also introduce there the so-called "$5r$-condition" in a more complicated form than it had in previous literature. The reason for this is to have a condition which is satisfied not only by n-dimensional Lebesgue measure in Euclidean n-space but also by various measures occurring in integral geometry. In particular, we wished to cover the case of Haar measure in the space of oriented lines in Euclidean 3-space which seems to be of importance in certain questions relating to surface area.

In Sections 4, 5, 6 we take up the actual study of density theorems. All the main results are proved for general separable metric spaces, making no use of compactness assumptions, and many theorems previously proved by various authors for Euclidean spaces are obtained as very special applications. In particular, Section 5 and Section 6 contain such generalizations of previous results of ourselves [5] and of H. Federer [6].

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SECTION 1

COVERING THEOREMS

1.1. Throughout the present Section 1, X denotes a separable metric space with distance-function d. We put, for $x \in X$, $0 < r < \infty$,

$$c(x, r) = \{y \mid d(x, y) \leqslant r\}, \quad c^0(x, r) = \{y \mid d(x, y) < r\}.$$

A subset E of X will be termed a *closed sphere* if there exists a point $x \in X$ and a finite positive real number r such that

$$E = c(x, r). \tag{1}$$

In a general metric space X, a closed sphere may admit of several representations of the form (1); in other words, the *center* x and the *radius* r are generally not unique, as simple examples show. Similar comments apply to *open spheres*, that is, sets E which admit of representations of the form

$$E = c^0(x, r), \quad x \in X, \quad 0 < r < \infty.$$

However, one verifies readily that a closed sphere is a non-empty closed set, an open sphere is a non-empty open set, and

$$\operatorname{diam} c(x, r) \leqslant 2r, \quad \operatorname{diam} c^0(x, r) \leqslant 2r.$$

Since X is separable, it is easy to see that every family $\mathcal{F}$ of pair-wise disjoint closed (or open) spheres in X is countable. Furthermore, if E is a subset of X and $\mathcal{G}$ is a family of open sets O covering E, then there exists a sequence O_i, $i = 1, 2, \ldots$, in $\mathcal{G}$ such that $E \subset \bigcup O_i$.

Our purpose in the present Section 1 is to state certain covering theorems in terms of closed spheres. Due to the ambiguity of *center* and *radius* referred to above, the statements below are formulated with appropriate care.

1.2. Let E be a subset of X, and let $\mathcal{F}$ be a family of closed spheres in X. Then $\mathcal{F}$ is said to cover E if

$$E \subset \bigcup_{C \in \mathcal{F}} C.$$

Furthermore, $\mathcal{F}$ is said to cover E in the Vitali sense if for every point $x \in E$ and every $\varepsilon > 0$ there exists in $\mathcal{F}$ a closed sphere C (which depends upon both x and ε) such that $x \in C$ and $\operatorname{diam} C < \varepsilon$.

The following two covering theorems are corollaries of very general theorems proved by A. P. Morse (see [1]):

THEOREM 1. *Let $\mathcal{F}$ be a non-empty family of closed spheres in X which covers a subset E of X. Assume further that the diameters of the closed spheres $C \in \mathcal{F}$ are less than some finite positive constant K. Then $\mathcal{F}$ contains a sequence C_i, $i = 1, 2, \ldots$, of pairwise disjoint closed spheres such that if for each i an arbitrary representation $C_i = c(x_i, r_i)$ is selected, one has the inclusion $E \subset \bigcup_i c(x_i, 5r_i)$.*

THEOREM 2. *Let $\mathcal{F}$ be a non-empty family of closed spheres in X which covers a subset E of X in the Vitali sense. Then $\mathcal{F}$ contains a sequence C_i, $i = 1, 2, \ldots$, of pairwise disjoint closed spheres such that if for each i an arbitrary representation $C_i = c(x_i, r_i)$ is selected, one has the inclusion*

$$E \subset \Big[\bigcup_{i=1}^{N} c(x_i, r_i)\Big] \cup \Big[\bigcup_{i=N+1}^{\infty} c(x_i, 5r_i)\Big], \tag{2}$$

for every positive integer N.

1.3. As noted above, the preceding covering theorems are corollaries of very general theorems of A. P. Morse. On the other hand, the general theorems of A. P. Morse are corollaries of a covering theorem in terms of abstract binary relations which we shall formulate presently (for further details, see [2]).

Let S be a non-empty set. Let there be given over S two binary relations σ, δ. We shall write $s' \sigma s''$ to state that the elements s', s'' of S satisfy the binary relation σ, with a similar convention for δ. For each element s of S we define the following sets:

$$N_\sigma(s) = \{s' \mid s' \in S, s' \sigma s\}, \qquad N_\delta(s) = \{s' \mid s' \in S, s' \delta s\},$$

$$N(s) = N_\sigma(s) \cap N_\delta(s).$$

For each subset E of S we define

$$N_\sigma(E) = \bigcup_{s \in E} N_\sigma(s), \qquad N_\delta(E) = \bigcup_{s \in E} N_\delta(s), \qquad N(E) = \bigcup_{s \in E} N(s),$$

with the understanding that for the empty set $\emptyset$ we have $N_\sigma(\emptyset) = N_\delta(\emptyset) = N(\emptyset) = \emptyset$.

Regarding the binary relations σ, δ we make the following assumptions:

(i) σ and δ are reflexive. That is, $s \sigma s$ and $s \delta s$ for every $s \in S$.

(ii) σ is symmetric. That is, $s'\sigma s''$ if and only if $s''\sigma s'$.

(iii) If E is any non-empty subset of S, then there exists at least one element s such that $s \in E \subset N_\delta(s)$. Such an element s will be termed a *dominant element* of E.

A subset E of S is termed *scattered* if E contains no pair of distinct elements s', s'' such that $s'\sigma s''$. Thus the empty set is scattered, and so is every set containing a single element.

THE (σ, δ)-COVERING THEOREM. *Under the circumstances just described, there exists a scattered subset E of S such that $S = N(E)$.*

1.4. The proofs of all the covering theorems referred to above depend upon Zorn's lemma or some equivalent statement. Accordingly, it is a matter of some interest that the (σ, δ)-covering theorem implies Zorn's lemma (see [2]).

SECTION 2

ABSOLUTELY MEASURABLE SETS

2.1. Throughout the present Section 2, X will denote a metric space with distance-function d. If E_1, E_2 are non-empty subsets of X, then their distance $d(E_1, E_2)$ is defined as the greatest lower bound of the distance $d(x_1, x_2)$ for $x_1 \in E_1$, $x_2 \in E_2$.

2.2. A real-valued set-function $\Gamma(E)$, defined for all the subsets E of X, is termed an *outer measure* in X if the following conditions are satisfied:

(i) Always $0 \leqslant \Gamma(E) \leqslant \infty$.

(ii) $\Gamma(\emptyset) = 0$, where $\emptyset$ denotes the empty set.

(iii) If $E_1 \subset E_2$, then $\Gamma(E_1) \leqslant \Gamma(E_2)$.

(iv) If E_n, $n = 1, 2, \ldots$, is any sequence of subsets of X, then $\Gamma(\bigcup E_n) \leqslant \sum \Gamma(E_n)$.

An outer measure Γ is termed a *Carathéodory outer measure* if it satisfies the following additional condition:

(v) If E_1, E_2 are non-empty subsets of X such that $d(E_1, E_2) > 0$, then $\Gamma(E_1) + \Gamma(E_2) = \Gamma(E_1 \cup E_2)$.

2.3. A set E of X is termed *measurable* with respect to a Carathéodory outer measure Γ (or briefly Γ-measurable) if $\Gamma(P) + \Gamma(Q) = \Gamma(P \cup Q)$ for every pair of sets P, Q such that $P \subset E$, $Q \subset \mathcal{C}E$ (where $\mathcal{C}E$ denotes the complement $X - E$ of E). The family of Γ-measurable subsets of X will be denoted by $\mathfrak{M}(\Gamma)$.

A subset E of X will be termed *absolutely measurable* if E is measurable with respect to every Carathéodory outer measure Γ in X. The family of the absolutely measurable sets in X will be denoted by $\mathcal{U}$. The main objective in the present Section 2 is to point out that in various important contexts the family $\mathcal{U}$ plays a role analogous to that of the family of Borel sets.

2.4. Given a Carathéodory outer measure Γ in X, we shall say that a set $E \subset X$ is *almost closed with respect to* Γ, or *Γ-almost closed,* if

for every $\varepsilon > 0$ there exists a closed set C such that $C \subset E$ and $\Gamma(E-C) < \varepsilon$. The family of the almost closed sets with respect to Γ will be denoted by $\mathfrak{C}(\Gamma)$. To state some well-known facts about the family $\mathfrak{C}(\Gamma)$, we need some definitions.

DEFINITION. A Carathéodory outer measure Γ in X is termed *bounded* if $\Gamma(X) < \infty$.

DEFINITION. A Carathéodory outer measure Γ in X is termed *Borel-regular* if for every set $E \subset X$ there exists a Borel set B such that $E \subset B$ and $\Gamma(E) = \Gamma(B)$.

The following three theorems were proved in [3] for the case when X is a Euclidean plane; however, the proof given there remains obviously valid for a general metric space X.

THEOREM 1. *If the Carathéodory outer measure Γ in X is bounded, then $\mathfrak{C}(\Gamma)$ contains all Borel sets.*

THEOREM 2. *If Γ is a Carathéodory outer measure in X, then $\mathfrak{C}(\Gamma)$ contains all Borel sets B such that $\Gamma(B) < \infty$.*

THEOREM 3. *If Γ is a Borel-regular Carathéodory outer measure in X, then $\mathfrak{C}(\Gamma)$ contains all Γ-measurable sets E such that $\Gamma(E) < \infty$.*

We shall show below, among other things, that in the preceding three theorems one can replace Borel sets by absolutely measurable sets.

2.5. A family $\mathcal{F}$ of subsets of X will be termed *Borel-closed* if the following conditions are satisfied:

(i) $\mathcal{F}$ contains the empty est $\emptyset$.

(ii) If $E \in \mathcal{F}$, then $\mathcal{C}E \in \mathcal{F}$.

(iii) If $E_n \in \mathcal{F}$, $n = 1, 2, \ldots$, then $\bigcup E_n \in \mathcal{F}$.

Given any non-empty family $\mathcal{F}_0$ of subsets of X, there exists a unique smallest Borel-closed family $\mathcal{F}$ of subsets of X which contains $\mathcal{F}_0$; namely, the intersection of all the Borel-closed families which contain $\mathcal{F}_0$. This unique family will be termed the Borel class over $\mathcal{F}_0$.

If Γ is any Carathéodory outer measure in X, then (see [4]) the family $\mathfrak{M}(\Gamma)$ of Γ-measurable sets is Borel-closed, contains all Borel sets and all analytic sets, and is closed under the operation A. Hence it is obvious that the family $\mathcal{U}$ of absolutely measurable sets in X is Borel-closed, contains all Borel sets and all analytic sets, and is closed under the operation A.

2.6. A Carathéodory outer measure Γ in X will be termed *boundedly finite* if $\Gamma(E) < \infty$ whenever E is a bounded set in X, and *locally finite* if for every point $x \in X$ there exists an open set O_x such that $x \in O_x$ and $\Gamma(O_x) < \infty$. If X is separable and Γ is locally finite, then clearly there

exists a sequence of bounded open sets O_i, $i = 1, 2, \ldots$, such that $X = \bigcup O_i$ and $\Gamma(O_i) < \infty$. If Γ is boundedly finite then for $x_0 \in X$ and $O_n = c^0(x_0, n)$, $n = 1, 2, \ldots$, we have $X = \bigcup O_n$ and $\Gamma(O_n) < \infty$, $n = 1, 2, \ldots$ In view of these two situations we introduce the following definition. A Carathéodory outer measure Γ in X will be termed *O-sigma finite* if there is a countable sequence of open sets O_n, $n = 1, 2, \ldots$, such that $X = \bigcup O_n$ and $\Gamma(O_n) < \infty$ for $n = 1, 2, \ldots$

2.7. A Carathéodory outer measure Γ in X will be termed *regular* with respect to a family $\mathcal{F}$ of subsets of X (briefly, $\mathcal{F}$-regular) if for every set $E \subset X$ there exists a set $F \in \mathcal{F}$ such that $E \subset F$ and $\Gamma(E) = \Gamma(F)$.

2.8. LEMMA. *Let Γ be a Carathéodory outer measure in X which is regular with respect to a Borel-closed family $\mathcal{F}$ of subsets of X. Let E be a Γ-measurable subset of X such that $\Gamma(E) < \infty$. Then there exists a set $E^* \in \mathcal{F}$ such that $E^* \subset E$ and $\Gamma(E^*) = \Gamma(E)$.*

Proof. Since Γ is $\mathcal{F}$-regular, there exists a set E' such that $E \subset E' \in \mathcal{F}$ and $\Gamma(E') = \Gamma(E) < \infty$. Since E is Γ-measurable, we have $\Gamma(E) + \Gamma(E' - E) = \Gamma(E') = \Gamma(E)$, and since $\Gamma(E) < \infty$ it follows that $\Gamma(E' - E) = 0$. Thus $(E' - E)$ is Γ-measurable, and hence $E' = (E' - E) \cup E$ is also Γ-measurable. Now since $\Gamma(E' - E) = 0$ and Γ is $\mathcal{F}$-regular, there exists a set E'' such that

$$E' - E \subset E'' \in \mathcal{F}, \qquad \Gamma(E'') = \Gamma(E' - E) = 0.$$

Note that E'' is Γ-measurable since $\Gamma(E'') = 0$. Now define $E^* = E' - E''$. Then $E^* \in \Gamma$ since $\mathcal{F}$ is Borel-closed, and clearly $E \subset E^*$. As $E' \subset E^* \subset E''$ and $\Gamma(E'') = 0$, it follows that $\Gamma(E') \leqslant \Gamma(E^*)$. Also, $E^* \subset E$ implies that $\Gamma(E^*) \leqslant \Gamma(E)$. Since $\Gamma(E) = \Gamma(E')$, we conclude that

$$\Gamma(E) = \Gamma(E') \leqslant \Gamma(E^*) \leqslant \Gamma(E),$$

and hence $\Gamma(E^*) = \Gamma(E)$. Thus the lemma is proved, since $E \supset E^* \in \mathcal{F}$.

2.9. The concept of almost closed sets (see 2.4) suggests the consideration of *almost open* sets defined as follows: a set $E \subset X$ is termed almost open relative to a Carathéodory outer measure Γ in X if for every $\varepsilon > 0$ there exists an open set O such that $E \subset O$ and $\Gamma(O - E) < \varepsilon$. The family of almost open sets relative to Γ will be denoted by $\mathfrak{O}(\Gamma)$. The following properties of the families $\mathfrak{C}(\Gamma)$ and $\mathfrak{O}(\Gamma)$ can be easily verified by the reader:

(i) $E \in \mathfrak{C}(\Gamma)$ if and only if $\mathcal{C}E \in \mathfrak{O}(\Gamma)$.

(ii) $\mathfrak{C}(\Gamma) \subset \mathfrak{M}(\Gamma)$ and $\mathfrak{O}(\Gamma) \subset \mathfrak{M}(\Gamma)$.

(iii) If $E_n \in \mathfrak{O}(\Gamma)$ for $n = 1, 2, \ldots$, then $\bigcup E_n$ is in $\mathfrak{O}(\Gamma)$.

(iv) If $E_1 \in \mathfrak{O}(\Gamma)$, $E_2 \subset E_1$ and $\Gamma(E_1 - E_2) = 0$, then $E_2 \in \mathfrak{O}(\Gamma)$.

(v) If $E_1 \in \mathfrak{C}(\Gamma)$, $E_2 \supset E_1$ and $\Gamma(E_2 - E_1) = 0$, then $E_2 \in \mathfrak{C}(\Gamma)$.

(vi) If for each set $E \subset X$ we define $\Gamma_0(E)$ to be the greatest lower bound of $\Gamma(O)$ for all open sets O such that $E \subset O$, then $\Gamma_0(E)$ is a Carathéodory outer measure in X. Furthermore, if $E \in \mathfrak{M}(\Gamma_0)$ and $\Gamma_0(E) < \infty$ then $E \in \mathfrak{O}(\Gamma)$.

2.10. THEOREM. *If Γ is a Carathéodory outer measure in X that is O-sigma finite, then $\mathcal{U} \subset \mathfrak{C}(\Gamma) \cap \mathfrak{O}(\Gamma)$.*

Proof. We first show that

$$\mathcal{U} \subset \mathfrak{O}(\Gamma). \tag{1}$$

Consider a set $E \in \mathcal{U}$. Since Γ is O-sigma finite there is a seqeunce of open sets $O_1, O_2, \ldots$, such that

$$X = \bigcup O_n, \quad \Gamma(O_n) < \infty \quad \text{for} \quad n = 1, 2, \ldots$$

Set $E_n = E \cap O_n$, $n = 1, 2, \ldots$ Since $E_n \in \mathcal{U}$, for the set function Γ_0 defined in (vi) of 2.9, $E_n \in \mathfrak{M}(\Gamma_0)$ and $\Gamma_0(E_n) \leqslant \Gamma(O_n) < \infty$. Thus, by (vi) of 2.9, $E_n \in \mathfrak{O}(\Gamma)$. Since $E = \bigcup E_n$, by (iii) of 2.9, $E \in \mathfrak{O}(\Gamma)$ and (1) holds.

Consider a set $E \in \mathcal{U}$. Then $\mathcal{C}E \in \mathcal{U}$ and by (1), $\mathcal{C}E \in \mathfrak{O}(\Gamma)$. By (i) of 2.9 we thus have $E \in \mathfrak{C}(\Gamma)$. Therefore

$$\mathcal{U} \subset \mathfrak{C}(\Gamma). \tag{2}$$

(1) and (2) imply that $\mathcal{U} \subset \mathfrak{C}(\Gamma) \cap \mathfrak{O}(\Gamma)$.

2.11. THEOREM. *If Γ is a $\mathcal{U}$-regular Carathéodory outer measure in X that is O-sigma finite, then $\mathfrak{C}(\Gamma) = \mathfrak{O}(\Gamma) = \mathfrak{M}(\Gamma)$.*

Proof. We first show that

$$\mathfrak{M}(\Gamma) \subset \mathfrak{O}(\Gamma). \tag{3}$$

Since Γ is O-sigma finite, a set $E \in \mathfrak{M}(\Gamma)$ is the union of a finite or countable number of sets in $\mathfrak{M}(\Gamma)$ each of which is of finite Γ-measure. Hence, for $E \in \mathfrak{M}(\Gamma)$ there is a set $U \in \mathcal{U}$ such that $E \subset U$ and $\Gamma(U - E) = 0$. By the theorem in 2.10, $U \in \mathfrak{O}(\Gamma)$ and by (iv) of 2.9, $E \in \mathfrak{O}(\Gamma)$. Thus (3) holds.

We next show that

$$\mathfrak{M}(\Gamma) \subset \mathfrak{C}(\Gamma). \tag{4}$$

Consider a set $E \in \mathfrak{M}(\Gamma)$. Then $\mathcal{C}E \in \mathfrak{M}(\Gamma)$ and by (3), $\mathcal{C}E \in \mathfrak{O}(\Gamma)$. By (i) of 2.9, we then have that $E \in \mathfrak{C}(\Gamma)$. Thus (4) holds.

(3), (4) and (ii) of 2.9 imply that $\mathfrak{C}(\Gamma) = \mathfrak{M}(\Gamma) = \mathfrak{O}(\Gamma)$.

2.12. THEOREM. *If Γ is a Carathéodory outer measure in X then $E \in \mathcal{U}$ with $\Gamma(E) < \infty$ implies that $E \in \mathfrak{C}(\Gamma)$.*

Proof. For $E_0 \in \mathcal{U}$ with $\Gamma(E_0) < \infty$ we define $\Gamma^*(E) = \Gamma(E \cap E_0)$ for $E \subset X$. Then Γ^* is a Carathéodory outer measure in X and $\Gamma^*(X) = \Gamma(E_0) < \infty$. By the theorem in 2.10 we have $E_0 \in \mathfrak{C}(\Gamma^*)$. Thus for $\varepsilon > 0$ given there is a closed set $C \subset E_0$ such that $\Gamma(E_0 - C) = \Gamma^*(E_0 - C) < \varepsilon$ and $E_0 \in \mathfrak{C}(\Gamma)$.

2.13. THEOREM. *If Γ is a $\mathcal{U}$-regular Carathéodory outer measure in X then $E \in \mathfrak{M}(\Gamma)$ with $\Gamma(E) < \infty$ implies that $E \in \mathfrak{C}(\Gamma)$.*

Proof. Consider a set $E \in \mathfrak{M}(\Gamma)$ with $\Gamma(E) < \infty$. By the lemma in 2.8 there is a set $U \in \mathcal{U}$ such that $U \subset E$ and $\Gamma(U) = \Gamma(E)$. By the theorem in 2.12, $U \in \mathfrak{C}(\Gamma)$ and, since $\Gamma(E - U) = 0$, by (v) of 2.9 we have $E \in \mathfrak{C}(\Gamma)$.

2.14. Since the family $\mathcal{U}$ contains all Borel sets, it is clear that a Borel regular Carathéodory outer measure is also $\mathcal{U}$-regular. The following theorem yields a special situation where the converse is true.

THEOREM. *If the Carathéodory outer measure Γ in X is $\mathcal{U}$-regular and 0-sigma finite, then it is also Borel regular.*

Proof. Let E be a set in X. If $\Gamma(E) = \infty$ then $E \subset X$ and $\Gamma(X) = \Gamma(E)$. If $\Gamma(E) < \infty$ then there is a set $U \in \mathcal{U}$ such that $E \subset U$ and $\Gamma(U) = \Gamma(E)$. By the theorem in 2.10, $U \in \mathfrak{O}(\Gamma)$. Hence, for each positive integer n there is an open set O_n such that $U \subset O_n$ and $\Gamma(O_n - U) < 1/n$. Set $B = \bigcap O_n$. Then B is a Borel set, $E \subset U \subset B$ and

$$\Gamma(E) = \Gamma(U) = \Gamma(O_n) - \Gamma(O_n - U) > \Gamma(B) - 1/n$$

for $n = 1, 2, \ldots$ Thus $\Gamma(E) \geqslant \Gamma(B)$ and, since $E \subset B$, we have that $\Gamma(E) = \Gamma(B)$. Therefore, Γ is Borel regular.

2.15. It may be of interest to note that the family $\mathcal{U}$ can be characterized in terms of bounded Carathéodory outer measures. Indeed, let $\mathcal{U}^*$ be the family of those subsets of X which are measurable with respect to every Caratheodory outer measure Γ in X such that $\Gamma(X) = 1$.

THEOREM. $\mathcal{U}^* = \mathcal{U}$.

Proof. Obviously $\mathcal{U}^* \supset \mathcal{U}$. To establish the complementary inclusion, consider any set $E^* \subset X$ such that $E^* \notin \mathcal{U}$. Then there exists a Carathéodory outer measure Γ^* in X such that $E^* \notin \mathfrak{M}(\Gamma^*)$. Hence there exist sets P^*, Q^*, such that

$$(5) \qquad P^* \subset E^*, \quad Q^* \subset \mathcal{C}E^*, \quad \Gamma^*(P^* \cup Q^*) < \Gamma^*(P^*) + \Gamma^*(Q^*).$$

Clearly (5) implies that for $1/a = \Gamma^*(P^* \cup Q^*)$ we have $0 < a < \infty$, and hence we can introduce an auxiliary set-function Γ by the formula

$$(6) \qquad \Gamma(E) = a\Gamma^*[E \cap (P^* \cup Q^*)].$$

It is easy to check that Γ is a Carathéodory outer measure. Clearly

$$\Gamma(X) = 1, \quad \Gamma(P^*) = a\Gamma^*(P^*), \quad \Gamma(Q^*) = a\Gamma^*(Q^*). \tag{7}$$

From (5), (6) and (7) it follows that

$$\Gamma(P^* \cup Q^*) = a\Gamma^*(P^* \cup Q^*) < a\Gamma^*(P^*) + a\Gamma^*(Q^*) = \Gamma(P^*) + \Gamma(Q^*),$$

and hence $E^* \notin \mathfrak{M}(\Gamma)$. Since $\Gamma(X) = 1$, it is thus shown that $E^* \notin \mathcal{U}$ implies that $E^* \notin \mathcal{U}^*$. Hence $\mathcal{U} \supset \mathcal{U}^*$, and the theorem is proved.

SECTION 3

GENERALIZED SPHERICAL HAUSDORFF MEASURES

3.1. We consider a fixed separable metric space X. Let $\varphi = \varphi(x, r)$ be a real-valued function defined for $x \in X$ and $0 < r < \infty$, such that always $0 \leqslant \varphi(x, r) < \infty$. In terms of φ we define in X a Carathéodory outer measure S as follows. Consider a set $E \subset X$. Assign $\varepsilon > 0$. Let $c(x_i, r_i)$, $i = 1, 2, \ldots$, be a finite or infinite sequence of closed spheres such that

$$E \subset \bigcup_i c(x_i, r_i), \quad 0 < r_i < \varepsilon. \tag{1}$$

Note that such sequences $c(x_i, r_i)$ do exist because X is separable. We define

$$S_\varepsilon^\varphi(E) = \text{gr. l. b.} \sum \varphi(x_i, r_i),$$

where the greatest lower bound is taken with respect to all the sequences $c(x_i, r_i)$ satisfying (1). Clearly $S_\varepsilon^\varphi(E)$ increases (in the wide sense) if ε decreases, and hence for $\varepsilon \to 0$ the number $S_\varepsilon^\varphi(E)$ converges to a (finite or infinite) limit. Accordingly, we can define

$$S^\varphi(E) = \lim_{\varepsilon \to 0} S_\varepsilon^\varphi(E). \tag{2}$$

We shall term $S^\varphi(E)$ a generalized spherical Hausdorff measure. We agree to put $S^\varphi(\emptyset) = 0$. It is easily verified that S^φ is a Borel-regular Carathéodory outer measure (see 2.4).

3.2. The function $\varphi = \varphi(x, r)$ was completely general so far, except for the condition $0 \leqslant \varphi(x, r) < \infty$. However, in all the applications considered in the sequel, φ will satisfy the following

5r-CONDITION. *For every bounded set $E \subset X$ there exist two finite positive constants $k(E)$, $K(E)$ such that*

$$\varphi(x, 5r) < K(E)\varphi(x, r) \quad \textit{for} \quad x \in E, \quad 0 < r < k(E). \tag{3}$$

Clearly, (3) *holds then also if $K(E)$ is replaced by any larger constant and $k(E)$ is replaced by any smaller constant. Thus we can assume that*

$$0 < k(E) < 1 < K(E) < \infty. \tag{4}$$

If $\varphi(x, r)$ satisfies the $5r$-condition, then we have in particular for the case when E reduces to a single point x the inequality

$$\varphi(x, 5r) < K(x)\varphi(x, r) \quad \text{for} \quad 0 < r < k(x).$$

As an immediate consequence, it follows that

$$0 < \varphi(x, r) < \infty \quad \text{for} \quad 0 < r < k(x)/5. \tag{5}$$

3.3. Let σ be a Carathéodory outer measure in X. Then σ will be said to satisfy the $5r$-condition if for every bounded set $E \subset X$ there exist two positive finite constants $k(E), K(E)$ such that

$$\sigma[c(x, 5r)] < K(E)\sigma[c(x, r)] \quad \text{for} \quad x \in E, \quad 0 < r < k(E). \tag{6}$$

On setting

$$\sigma(x, r) = \sigma[c(x, r)], \tag{7}$$

clearly the condition is equivalent to the $5r$-condition for the function $\sigma(x, r)$ as defined in 3.2. Hence, as noted there, we have

$$0 < \sigma[c(x, r)] < \infty \quad \text{for} \quad 0 < r < k(x)/5. \tag{8}$$

Hence also

$$\sigma[c^0(x, r)] < \infty \quad \text{for} \quad 0 < r < k(x)/5.$$

Thus the $5r$-condition implies that σ is locally finite and hence, since X is assumed to be separable, that σ is O-sigma finite.

The function $\sigma(x, r) = \sigma[c(x, r)]$ gives rise to the Carathéodory outer measure S^σ (see 3.1), and there arises the question as to how σ and S^σ compare with each other. The following theorem yields a partial answer:

THEOREM. *Let σ be a $\mathcal{U}$-regular Carathéodory outer measure in X which satisfies the $5r$-condition. Then $S^\sigma = \sigma$.*

Proof. We first show that

$$\sigma \leqslant S^\sigma. \tag{9}$$

Indeed, consider any set $E \neq \emptyset$. Assign $\varepsilon > 0$, and let $c(x_i, r_i)$, $(i = 1, 2, \ldots,$ be a sequence of closed spheres such that

$$E \subset \bigcup_i c(x_i, r_i), \quad 0 < r_i < \varepsilon.$$

We then have

$$\sigma(E) \leqslant \sum_i \sigma[c(x_i, r_i)] = \sum_i \sigma(x_i, r_i),$$

and hence $\sigma(E) \leqslant S^\sigma_\varepsilon(E) \leqslant S^\sigma(E)$. Thus (9) is verified. Next we show that (see 2.3)

$$S^\sigma(U) \leqslant \sigma(U) \quad \text{for} \quad U \in \mathcal{U}. \tag{10}$$

Since X is separable and σ is locally finite, it follows that X is the union of a finite or countable sequence of Borel sets that are pairwise disjoint, bounded and of finite σ-measure and hence any set $U \in \mathcal{U}$ is the union of a finite or countable sequence of sets in $\mathcal{U}$ that are pairwise disjoint, bounded and of finite σ-measure. Thus it is sufficient to prove (10) in the case where U is bounded and $\sigma(U) < \infty$. Hence the constants $k(U)$, $K(U)$ are available. By the theorem in 2.10, $U \in \mathfrak{O}(\sigma)$. Accordingly, if $\eta > 0$ is assigned, then there is an open set O such that

$$U \subset O, \quad \sigma(O) < \sigma(U) + \eta < \infty. \tag{11}$$

Keep $\eta > 0$ fixed and assign $\varepsilon > 0$. Then the closed spheres $c(x, r)$ such that

$$0 < r < \varepsilon/5, \quad 0 < r < k(U), \quad x \in U, \quad c(x, r) \subset O \tag{12}$$

cover U in the Vitali sense. Hence (see 1.2, theorem 2) there exists a disjoint sequence $c(x_i, r_i)$, $i = 1, 2, \ldots$, of closed spheres such that

$$U \subset \Big[\bigcup_{i=1}^{N} c(x_i, r_i)\Big] \cup \Big[\bigcup_{i=N+1}^{\infty} c(x_i, 5r_i)\Big]$$

for every positive integer N. In view of (12), the preceding inclusion yields a covering of U by closed spheres of radius less than ε. Hence

$$S_\varepsilon^\sigma(U) \leqslant \sum_{i=1}^{N} \sigma[c(x_i, r_i)] + K(U) \sum_{i=N+1}^{\infty} \sigma[c(x_i, r_i)]. \tag{13}$$

Now since the closed spheres $c(x_i, r_i)$ are pairwise disjoint and are contained in O, we have (see (11))

$$\sum_{i=1}^{\infty} \sigma[c(x_i, r_i)] \leqslant \sigma(O) < \sigma(U) + \eta < \infty.$$

Thus the infinite series on the left is convergent, and hence

$$\sum_{i=N+1}^{\infty} \sigma[c(x_i, r_i)] \to 0 \quad \text{for} \quad N \to \infty.$$

Thus (13) yields, for $N \to \infty$, the inequality

$$S_\varepsilon^\sigma(U) \leqslant \sigma(U) + \eta.$$

On making first $\varepsilon \to 0$ and then $\eta \to 0$, (10) follows. Since S^σ and σ are both $\mathcal{U}$-regular, it follows that $S^\sigma(E) \leqslant \sigma(E)$ for every set $E \subset X$ and the theorem is proved in view of (9).

3.4. Lebesgue n-dimensional outer measure L_n in Euclidean n-space R^n is an important special instance of a Carathéodory outer measure satisfying the $5r$-condition. Indeed, if $a(n)$ denotes the L_n-measure of the solid unit sphere in R^n, then

$$L_n[c(x, r)] = a(n)r^n,$$

and hence

$$L_n[c(x, 5r)] = 5^n L_n[c(x, r)].$$

Thus L_n satisfies the $5r$-condition with the constant $2\cdot 5^n$, for example. Also, it is well known that L_n is Borel-regular. Accordingly, the theorem in 3.4 appears as a generalization of the well-known fact that

$$L_n = S^{a(n)r^n}.$$

Further important instances occur in integral geometry, as we shall point out later on.

3.5. Returning to the general separable metric space X, let k be a finite positive real number. The k-dimensional Hausdorff measure H^k in X is then defined as follows. Consider a set $E \subset X$. Assign $\varepsilon > 0$. Let e_i, $i = 1, 2, \ldots$, be a sequence of subsets of X such that

$$E \subset \bigcup e_i, \quad \operatorname{diam} e_i < \varepsilon. \tag{14}$$

Clearly such sequences e_i exist since X is separable. We define

$$H_\varepsilon^k(E) = \text{gr. l. b.} \sum_i (\operatorname{diam} e_i)^k,$$

where the greatest lower bound is taken with respect to all the sequences e_i satisfying (14). Clearly $H_\varepsilon^k(E)$ increases (in the wide sense) if ε decreases. Hence we can define

$$H^k(E) = \lim_{\varepsilon\to 0} H_\varepsilon^k(E). \tag{15}$$

It is well known, and easily proved, that H^k is a Borel-regular Carathéodory outer measure. H^k is termed the k-dimensional Hausdorff measure in X. Actually, if X coincides with Euclidean n-space R^n, an appropriate normalizing factor is used to achieve agreement with standard "area-formulas", as follows. Assume that $X = R^n$, and let now k be a positive integer. Denote by $a(k)$ the L_k-measure of the solid unit sphere in R^k (see 3.5). Then the normalized k-dimensional Hausdorff measure in R^n, to be denoted by H_n^k, is defined by

$$H_n^k = \frac{a(k)}{2^k} H^k,$$

where H^k is given by (15).

3.6. Returning to the general separable metric space X, for $0 < k < \infty$ we shall consider along with H^k the Carathéodory outer measure

$$S^k = S^{\varphi = r^k},$$

that is, the set-function corresponding to the choice $\varphi(x, r) = r^k$. We leave to the reader the verification of the relations

$$S^k \leqslant H^k \leqslant 2^k S^k. \tag{16}$$

From (16) there follows directly the following

LEMMA. *$S^k(E) = 0$ if and only if $H^k(E) = 0$, and $S^k(E) = \infty$ if and only if $H^k(E) = \infty$.*

3.7. In the separable metric space X consider a function $\varphi = \varphi(x, r)$ which satisfies the $5r$-condition and is also subadditive in the following sense: if $c(x_i, r_i)$, $i = 1, 2, \ldots$, is any finite or infinite sequence of pairwise disjoint closed spheres and $c(x, r)$ is any closed sphere such that $c(x_i, r_i) \subset c(x, r)$, for $i = 1, 2, \ldots$, then

$$\sum_i \varphi(x_i, r_i) \leqslant \varphi(x, r).$$

We want to verify that under these conditions S^φ is locally finite. Indeed, consider any point $x \in X$. Since φ satisfies the $5r$-condition, by 3.2 there follows the existence of a finite positive number r_x such that $\varphi(x, r_x) < \infty$. Now the finite positive constants $k = k[c(x, r_x)]$, $K = K[c(x, r_x)]$ occuring in the definition of the $5r$-condition are available. Accordingly, the local finiteness of S^φ will be shown if we verify that

$$S^\varphi[c^0(x, r_x)] \leqslant K\varphi(x, r_x). \tag{17}$$

Assign now $\varepsilon > 0$. Then the family of closed spheres $c(y, r)$ such that

$$c(y, r) \subset c^0(x, r_x), \quad 0 < r < \varepsilon/5, \quad 0 < r < k, \tag{18}$$

cover $c^0(x, r_x)$. By theorem 1 in 1.2, there exists a pairwise disjoint sequence $c(y_i, r_i)$, $i = 1, 2, \ldots$, of such closed spheres that

$$c^0(x, r_x) \subset \bigcup_i c(x_i, 5r_i). \tag{19}$$

Since (19) yields a covering of $c^0(x, r_x)$ in terms of closed spheres of radius less than ε, in view of (18), the $5r$-condition and the subadditivity of φ we have

$$S_\varepsilon^\varphi[c^0(x, r_x)] \leqslant \sum_i \varphi(x_i, 5r_i) \leqslant K \sum_i \varphi(x_i, r_i) \leqslant K\varphi(x, r_x). \tag{20}$$

Since $\varepsilon > 0$ is arbitrary, (20) implies (17).

Thus it is proved that S^φ is a Borel regular Carathéodory outer measure in X (see 3.1) that is locally finite (and hence O-sigma finite). As a direct application of the theorem in 2.11 we have therefore the following

LEMMA. *If X is a separable metric space and $\varphi(x, r)$ satisfies the $5r$-condition and is subadditive, then $\mathfrak{O}(S^\varphi) = \mathfrak{C}(S^\varphi) = \mathfrak{M}(S^\varphi)$.*

SECTION 4

DENSITY THEOREMS FOR SUBADDITIVE SET FUNCTIONS

4.1. Throughout the present Section 4 the following notation will be used: X will denote a separable metric space; $\varphi = \varphi(x, r)$ will denote a non-negative real-valued function of the point $x \in X$ and of the finite positive real number r which satisfies the $5r$-condition (see 3.2).

4.2. Let $\omega(E)$ be a non-negative, real-valued function of the set $E \subset X$. We shall say that ω is a *subadditive* set function in X if for any set $E \subset X$ and any sequence of pairwise disjoint closed spheres $c(x_i, r_i)$, $i = 1, 2, \ldots$, we have

$$\sum_i \omega[c(x_i, r_i) \cap E] \leqslant \omega(E).$$

We shall say that a set E is *ω-almost closed* if for $\varepsilon > 0$ given there is a closed set $C \subset E$ such that $\omega(E - C) < \varepsilon$. We shall say that ω is *O-sigma finite* if there is a sequence of open sets O_i, $i = 1, 2, \ldots$, such that $X = \bigcup O_i$ and $\omega(O_i) < \infty$ for $i = 1, 2, \ldots$

4.3. It follows from [4], p. 44, that if Γ is a Carathéodory outer measure in X, then Γ is a subadditive set function in X in the sense of 4.2. As a second example we consider the following situation. Let $\mathcal{F}$ be a family of subsets of X such that: (i) $\mathcal{F}$ contains all open sets and (ii) $\mathcal{F}$ is closed under countable unions and finite intersections. Let $\omega_0(E)$ be a non-negative, real-valued function of the sets $E \in \mathcal{F}$ which is subadditive on $\mathcal{F}$ in the following sense: if E_i, $i = 1, 2, \ldots$, is any finite or infinite sequence of pairwise disjoint sets in $\mathcal{F}$ and $E_i \subset E \in \mathcal{F}$, $i = 1, 2, \ldots$, then

$$\sum_i \omega_0(E_i) \leqslant \omega_0(E).$$

For such a set function we define, for each set $E \subset X$,

$$\omega(E) = \text{gr. l. b.}\, \omega_0(F) \quad \text{for} \quad E \subset F \in \mathcal{F}. \tag{1}$$

LEMMA. *The set function ω defined in* (1) *is a subadditive set function in X in the sense of* 4.2 *and* $\omega(E) = \omega_0(E)$ *for* $E \in \mathcal{F}$.

Proof. Since $E_1 \subset E_2$, $E_1, E_2 \in \mathcal{F}$ implies that $\omega_0(E_1) \leqslant \omega_0(E_2)$, it follows that $\omega(E) = \omega_0(E)$ for $E \in \mathcal{F}$. For $E \subset X$ let $c(x_i, r_i)$, $i = 1, 2, \ldots$, be a sequence of pairwise disjoint closed spheres. Let $F \in \mathcal{F}$ be such that $E \subset F$. For a positive integer N there is a sequence of pairwise disjoint open sets $O_1, \ldots, O_N$ such that $c(x_i, r_i) \subset O_i$ for $i = 1, \ldots, N$. Then $c(x_i, r_i) \cap E \subset (O_i \cap F) \in \mathcal{F}$ for $i = 1, \ldots, N$ and hence

$$\sum_{i=1}^{N} \omega[c(x_i, r_i) \cap E] \leqslant \sum_{i=1}^{N} \omega_0(O_i \cap F) \leqslant \omega_0(F).$$

Since this holds for every positive integer N we have

$$\sum_{i=1}^{\infty} \omega[c(x_i, r_i) \cap E] \leqslant \omega_0(F).$$

Since this holds for every $F \in \mathcal{F}$ such that $E \subset F$, we have

$$\sum_{i=1} \omega[c(x_i, r_i) \cap E] \leqslant \omega(E).$$

Thus ω is subadditive in the sense of 4.2.

4.4. THEOREM. *Let ω be a subadditive set function in X and for $E \subset X$ and $0 < t < \infty$ set* (see 4.1)

$$G_t = \{x \mid \limsup_{r \to 0} \omega[c(x, r) \cap E]/\varphi(x, r) > t\}.$$

Then

$$S^{\varphi}(G_t) \leqslant \omega(E)/t. \tag{2}$$

Proof. Since (2) is obvious if $\omega(E) = \infty$ we can assume that $\omega(E) < \infty$.

Step 1. E_0 is a bounded subset of G_t. Then the finite positive constants $k(E_0)$, $K(E_0)$ occurring in the $5r$-condition are available (see 3.2). Assign $\varepsilon > 0$. The family of closed spheres $c(x, r)$ such that

$$x \in E_0, \quad 0 < r < \varepsilon/5, \quad 0 < r < k(E_0), \quad \omega[c(x, r) \cap E] > t\varphi(x, r), \tag{3}$$

cover E_0 in the sense of Vitali. Hence (see 1.2, theorem 2) there exists a pairwise disjoint sequence $c(x_i, r_i)$, $i = 1, 2, \ldots$, of such closed spheres that, for any positive integer N,

$$E_0 \subset [\bigcup_{i=1}^{N} c(x_i, r_i)] \cup [\bigcup_{i=N+1}^{\infty} c(x_i, 5r_i)].$$

In view of (3) the preceding inclusion yields a covering of E_0 by closed spheres of radius less than ε. Hence (see 3.1)

$$(4)\quad S_\varepsilon^\varphi(E_0) \leqslant \sum_{i=1}^{\infty} \varphi(x_i, r_i) + \sum_{i=N+1}^{\infty} \varphi(x_i, 5r_i)$$

$$\leqslant \sum_{i=1}^{\infty} \varphi(x_i, r_i) + K(E_0) \sum_{i=N+1}^{\infty} \varphi(x_i, r_i)$$

$$\leqslant \frac{1}{t} \sum_{i=1}^{\infty} \omega[c(x_i, r_i) \cap E] + \frac{K(E_0)}{t} \sum_{i=N+1}^{\infty} \omega[c(x_i, r_i) \cap E].$$

Since the closed spheres $c(x_i, r_i)$ are pairwise disjoint and ω is subadditive we have

$$(5)\qquad \sum_{i=1}^{\infty} \omega[c(x_i, r_i) \cap E] \leqslant \omega(E) < \infty.$$

Thus the infinite series on the left of (5) is convergent. Hence

$$(6)\qquad \sum_{i=N+1}^{\infty} \omega[c(x_i, r_i) \cap E] \to 0 \quad \text{for} \quad N \to \infty.$$

From (4), (5) and (6) we infer now, for $N \to \infty$, that $S_\varepsilon^\varphi(E_0) \leqslant \omega(E)/t$. For $\varepsilon \to 0$ it follows that

$$S^\varphi(E_0) \leqslant \omega(E)/t.$$

Step 2. Fix a point $x_0 \in X$ and put

$$E_n = G_t \cap c(x_0, n), \qquad n = 1, 2, \ldots$$

Then $E_1 \subset E_2 \subset \ldots$, $\bigcup E_n = G_t$. It follows (see [4], p. 46) that

$$(7)\qquad S^\varphi(G_t) = \lim_{n\to\infty} S^\varphi(E_n).$$

Since $S^\varphi(E_n) \leqslant \omega(E)/t$ by step 1, (2) follows from (7).

4.5. THEOREM. *Let ω be a subadditive set function in X. If $E \subset X$ is ω-almost closed then*

$$\lim_{r\to 0} \frac{\omega[c(x, r) \cap E]}{\varphi(x, r)} = 0 \quad \textit{for} \quad S^\varphi\text{-a. e. } x \in \mathcal{C}E.$$

Explicitly the lim *is equal to zero S^φ-almost everywhere on $\mathcal{C}E$ (that is, the set of those points $x \in \mathcal{C}E$ where the* lim sup *is positive has S^φ-measure zero).*

Proof. For $0 < t < \infty$ put

$$H_t = \left\{x \mid x \in \mathcal{C}E,\ \limsup_{r\to 0} \frac{\omega[c(x,r) \cap E]}{\varphi(x,r)} > t\right\}.$$

Clearly it is sufficient to show that

$$S^\varphi(H_t) = 0 \tag{8}$$

for $0 < t < \infty$. Now fix t and assign $\eta > 0$. Since E is ω-almost closed, there exists a closed set $C \subset E$ such that $\omega(E-C) < \eta$. Put

$$G_t = \left\{x \mid \limsup_{r\to 0} \frac{\omega[c(x,r) \cap (E-C)]}{\varphi(x,r)} > t\right\}.$$

By the theorem in 4.4 we have

$$S^\varphi(G_t) \leqslant \omega(E-C)/t < \eta/t \tag{9}$$

since $\omega(E-C) < \eta$. Now consider a point $x \in \mathcal{C}E$. Then $x \notin C$ and since C is closed, it follows that

$$c(x,r) \cap E = c(x,r) \cap (E-C)$$

for r sufficiently small and consequently

$$\limsup_{r\to 0} \frac{\omega[c(x,r) \cap E]}{\varphi(x,r)} = \limsup_{r\to 0} \frac{\omega[c(x,r) \cap (E-C)]}{\varphi(x,r)}$$

for $x \in \mathcal{C}E$. Thus clearly $H_t \subset G_t$ and hence

$$S^\varphi(H_t) < \eta/t$$

by (9). Since $\eta > 0$ was arbitrary, (8) follows and the theorem is proved.

4.6. THEOREM. *Let Γ be a Carathéodory outer measure in X (assumed to be separable) and let E be a subset of X. Then*

$$\lim_{r\to 0} \frac{\Gamma[c(x,r) \cap E]}{\varphi(x,r)} = 0 \quad \textit{for} \quad S^\varphi\text{-a. e. } x \in \mathcal{C}E$$

under each of the following conditions:

(a) *E is Γ-almost closed.*

(b) *Γ is $\mathcal{U}$-regular, E is Γ-measurable and $\Gamma(E) < \infty$. Furthermore, if Γ is assumed to be locally finite (and hence 0-sigma finite) the assumption $\Gamma(E) < \infty$ can be dropped.*

(c) *$\varphi(x,r) = \sigma[c(x,r)]$ where σ is a $\mathcal{U}$-regular Carathéodory outer measure in X that satisfies a 5r-condition, E is σ-measurable and $\Gamma(E) < \infty$.*

Furthermore, if Γ is locally finite then the assumption that $\Gamma(E) < \infty$ can be dropped.

Proof. (a) follows from the theorem in 4.5. (b) follows from (a) since, by the theorems in 2.11 and 2.13, E is Γ-almost closed. Concerning (c), since E is σ-measurable, by the theorem in 2.11, E is σ-almost open. Hence there is a Borel set $B \supset E$ such that $\sigma(B-E) = 0$. Set

$$\Gamma^*(A) = \Gamma(A \cap E) \quad \text{for} \quad A \subset X.$$

Then Γ^* is a Carathéodory outer measure in X, $\Gamma^*(X) = \Gamma(E)$ and Γ^* is O-sigma finite if either $\Gamma(E) < \infty$ or Γ is O-sigma finite. By the theorem in 2.10 B is Γ^*-almost closed. By (a)

$$\lim_{r\to 0} \frac{\Gamma[c(x,r) \cap E]}{\varphi(x,r)} = \lim_{r\to 0} \frac{\Gamma[c(x,r) \cap B]}{\varphi(x,r)} = 0$$

for $S^\varphi = \sigma$-a. e. $x \in \mathcal{C}B$. Thus (c) holds since $\sigma(B-E) = 0$.

4.7. THEOREM. *Let σ be a $\mathfrak{U}$-regular Carathéodory outer measure which satisfies the 5r-condition and let E be a σ-measurable set. Then*

$$\lim_{r\to 0} \frac{\sigma[c(x,r) \cap E]}{\sigma[c(x,r)]} = \begin{cases} 0 & \textit{for} \quad \sigma\text{-a. e. } x \in \mathcal{C}E, \\ 1 & \textit{for} \quad \sigma\text{-a. e. } x \in E. \end{cases} \tag{10}$$

Proof. Since σ is locally finite (see 3.3) the first line of (10) follows from (b) of the theorem in 4.5 by setting $\varphi(x,r) = \sigma[c(x,r)]$. Applying this result to the set $\mathcal{C}E$ we obtain

$$\lim_{r\to 0} \frac{\sigma[c(x,r) \cap \mathcal{C}E]}{\sigma[c(x,r)]} = 0 \quad \text{for} \quad \sigma\text{-a. e. } x \in E. \tag{11}$$

We now have (note that σ is locally finite) that for r sufficiently small

$$\frac{\sigma[c(x,r) \cap E]}{\sigma[c(x,r)]} = 1 - \frac{\sigma[c(x,r) \cap \mathcal{C}E]}{\sigma[c(x,r)]}$$

and the second line on the right of (10) follows in view of (11). This completes the proof of the theorem.

We noted in 3.4 that the Lebesgue n-dimensional outer measure in Euclidean n-space R^n satisfies the condition placed upon σ above. Noting that $L_n c(x,r) = a(n) r^n$, the preceding theorem yields the following classical statement:

COROLLARY. *If E is an L_n-measurable set in Euclidean n-space R^n, then*

$$\lim_{r\to 0} \frac{L_n[c(x,r) \cap E]}{\sigma(n) r^n} = \begin{cases} 0 & \textit{for} \quad L_n\text{-a. e. } x \in \mathcal{C}E, \\ 1 & \textit{for} \quad L_n\text{-a. e. } x \in E. \end{cases}$$

4.8. We now give a more general result than that given in (c) of the theorem in 4.6.

THEOREM. *Let Γ be a Carathéodory outer measure in X, let σ be a $\mathcal{U}$-regular Carathéodory outer measure in X that satisfies a $5r$-condition and let E be a σ-measurable set. Then*

$$\lim_{r\to 0}\frac{\Gamma[c(x,r)\cap E]}{\sigma[c(x,r)]} = 0 \text{ or } \infty \quad \text{for} \quad \sigma\text{-a. e. } x\in \mathcal{C}E.$$

More precisely, on setting

$$H = \{x \mid \Gamma[c(x,r)\cap E] = \infty \text{ for } 0 < r < \infty\}$$

we have

$$\lim_{r\to 0}\frac{\Gamma[c(x,r)\cap E]}{\sigma[c(x,r)]} = 0 \quad \text{for} \quad \sigma\text{-a. e. } x\in \mathcal{C}H\cap \mathcal{C}E, \tag{12}$$

$$\lim_{r\to 0}\frac{\Gamma[c(x,r)\cap E]}{\sigma[c(x,r)]} = \infty \quad \text{for} \quad x\in H\cap \mathcal{C}E. \tag{13}$$

Proof. (13) follows immediately from the definition of H. It is also evident that $\mathcal{C}H$ is open. Thus (12) follows by an elementary application of (c) in the theorem in 4.6.

4.9. THEOREM. *Let ω be a subadditive set function in X that is O-sigma finite. If every open set on which ω is finite is ω-almost closed, then for every open set O we have*

$$\lim_{r\to 0}\frac{\omega[c(x,r)\cap O]}{\varphi(x,r)} = 0 \quad \text{for} \quad S^{\varphi}\text{-a. e. } x\in \mathcal{C}O. \tag{14}$$

Proof. Consider an open set O. Since ω is O-sigma finite there are open sets O_n^*, $n = 1, 2, \ldots$, such that $X = \bigcup O_n^*$ and $\omega(O_n^*) < \infty$ for $n = 1, 2, \ldots$ Set $O_n = O\cap O_n^*$ for $n = 1, 2, \ldots$ Then $O = \bigcup O_n$ and each O_n is an open set such that $\omega(O_n) < \infty$. Hence O_n is ω-almost closed by assumption. By the theorem in 4.5 applied to O_n there follows for each n the existence of a set e_n such that

$$S^{\varphi}(e_n) = 0 \quad \text{and} \quad \lim_{r\to 0}\frac{\omega[c(x,r)\cap O_n]}{\varphi(x,r)} = 0 \quad \text{for} \quad x\in \mathcal{C}O_n - e_n. \tag{15}$$

Put $e = \bigcup e_n$. Then $S^{\varphi}(e) = 0$. Consider now a point $x\in \mathcal{C}O_n - e$. Then, since $X = \bigcup O_n^*$, we have $x\in O_n^*$ for some n. Let n be so chosen. Then $c(x,r)\subset O_n^*$ for small r and hence

$$c(x,r)\cap O = c(x,r)\cap O_n^*\cap O = c(x,r)\cap O_n$$

for small r. Thus

$$\omega[c(x,r)\cap O] = \omega[c(x,r)\cap O_n] \quad \text{for small } r.$$

Also, since $x\in \mathcal{C}O-e$, we have $x\in \mathcal{C}O_n-e_n$. Hence, by (15),

$$\lim_{r\to 0}\frac{\omega[c(x,r)\cap O]}{\varphi(x,r)} = \lim_{r\to 0}\frac{\omega[c(x,r)\cap O_n]}{\varphi(x,r)} = 0 \quad \text{for} \quad x\in \mathcal{C}O-e,$$

and (14) is proved since $S^\varphi(e) = 0$.

4.10. An important application of the preceding result arises as follows. Let $\varphi = \varphi(x,r)$ satisfy the $5r$-condition and in addition is subadditive in the sense of 3.7. Let us now define a function $\omega^\varphi(O)$ of the open sets O in X as follows. Given an open set $O\subset X$, put

$$\omega^\varphi(O) = \text{l. u. b.} \sum \varphi(x_i, r_i),$$

where the least upper bound is taken with respect to all sequences $c(x_i, r_i)$, $i = 1, 2, \ldots$, of pairwise disjoint closed spheres contained in O. Clearly, always $0 \leqslant \omega^\varphi(O) \leqslant \infty$. It follows readily from the definition of ω^φ that if O_n, $n = 1, 2, \ldots$, is a sequence of pairwise disjoint open sets contained in an open set O, then

$$\omega^\varphi(O_n) \leqslant \omega^\varphi(O).$$

By 4.3 (for the case where $\mathcal{F}$ is the family of open sets of X) ω^φ can be extended to all the subsets of X such that this extended ω^φ is a subadditive set function in X in the sense of 4.2.

Now let $x\in X$. Since φ satisfies a $5r$-condition, there is a positive real number r_x such that $\varphi(x, r_x) < \infty$. Since φ is subadditive, it follows that $\omega^\varphi[c^0(x, r_x)] \leqslant \varphi(x, r_x)$ and hence ω^φ is locally finite. Since X is separable, we have that ω^φ is O-sigma finite.

Consider now an open set O such that $\omega^\varphi(O) < \infty$. We proceed now to show that O is ω^φ-almost closed. Assign $\varepsilon > 0$. By the definition of ω^φ there exists in O a sequence of pairwise disjoint closed spheres, $c(y_j, s_j)$, $j = 1, 2, \ldots$, such that

$$\sum \varphi(y_j, s_j) > \omega^\varphi(O) - \varepsilon.$$

Since $\omega^\varphi(O) < \infty$, we can choose a positive integer N such that

$$\omega^\varphi(O) \geqslant \sum_{j=1}^{N} \varphi(y_j, s_j) > \omega^\varphi(O) - \varepsilon. \tag{16}$$

Then $F = c(y_1, s_1) \cup \ldots \cup c(y_N, s_N)$ is a closed subset of O and we shall now show that

$$\omega^\varphi(O-F) < \varepsilon. \tag{17}$$

Indeed, let $c(x_i, r_i)$, $i = 1, 2, \ldots$, be any sequence of pairwise disjoint closed spheres in the open set $O-F$. Then the closed spheres $c(y_1, s_1)$, $\ldots, c(y_N, s_N)$, $c(x_1, r_1), \ldots, c(x_i, r_i), \ldots$, are pairwise disjoint and are contained in 0. Hence

$$\sum_{j=1}^{N} \varphi(y_j, s_j) + \sum_i \varphi(x_i, r_i) \leqslant \omega^\varphi(O) < \infty.$$

In view of (16) it follows that

$$\sum \varphi(x_i, r_i) < \varepsilon,$$

and (17) follows.

We have thus shown that ω^φ is a subadditive set function in X that is O-sigma finite and is such that every open set O with $\omega^\varphi(O) < \infty$ is ω^φ-almost closed. Accordingly, the theorem in 4.9 yields the following result:

THEOREM. *If O is any open set then*

$$\lim_{r\to 0} \frac{\omega^\varphi[c(x, r) \cap O]}{\varphi(x, r)} = 0 \quad \textit{for} \quad S^\varphi\text{-}a.\ e.\ x \in \mathcal{C}O.$$

4.11. Returning to the theorem in 4.4 we have the following refinement in the case where $\omega = \Gamma$ is a $\mathcal{U}$-regular Carathéodory outer measure in X, E is a Γ-measurable set such that $\Gamma(E) < \infty$ and F is a Γ-measurable set such that $F \subset G_t$.

THEOREM. *Under the above conditions*

$$S^\varphi(F) \leqslant \Gamma(E \cap F)/t. \tag{18}$$

Proof. Since $F \subset G_t$ we have

$$\limsup_{z\to 0} \frac{\Gamma[c(x, r) \cap E]}{\varphi(x, r)} > t \quad \text{for} \quad x \in F.$$

Since $E = (E-F) \cup (E \cap F)$, it follows that

$$\limsup_{r\to 0} \frac{\Gamma[c(x, r) \cap (E-F)]}{\varphi(x, r)} + \limsup_{r\to 0} \frac{\Gamma[c(x, r) \cap (E \cap F)]}{\varphi(x, r)} > t \tag{19}$$

for $x \in F$. Now by (b) of the theorem in 4.6 applied to $E-F$, the first limsup in (19) is equal to zero for S^φ-a. e. $x \in \mathcal{C}(E-F) = F \cup \mathcal{C}E$, hence a fortiori for S^φ-a. e. $x \in F$. Thus (19) yields

$$\limsup_{r\to 0} \frac{\Gamma[c(x, r) \cap (E \cap F)]}{\varphi(x, r)} > t \quad \text{for} \quad S^\varphi\text{-a. e. } x \in F,$$

and (18) follows from the theorem in 4.4.

SECTION 5

A DENSITY THEOREM FOR OUTER MEASURES

5.1. We shall consider now an outer measure ψ (see 2.2) in the separable metric space X. Our objective in the present Section 5 is to prove the following

THEOREM. *Let ψ be an outer measure in the separable metric space X, and let $\varphi = \varphi(x, r)$ be a function which satisfies the $5r$-condition and is subadditive* (see 3.7). *Then for every S^φ-measurable set E we have*

$$\limsup_{r\to 0} \frac{\psi[c(x, r) \cap E]}{\varphi(x, r)} = 0 \text{ or } \infty \quad \text{for} \quad S^\varphi\text{-a. e. } x \in \mathcal{C}E.$$

For clarity, the proof is subdivided into a number of steps. Let us note that the preceding is a generalization of a result in [5]. In view of the theorem in 4.8 one may wonder if in the above theorem the lim sup may be replaced by lim. This, however, is not the case (see [5]).

5.2. Let us denote by $\mathcal{F}(\psi)$ the family of those S^φ-measurable sets E for which

$$\limsup_{r\to 0} \frac{\psi[c(y, r) \cap \mathcal{C}E]}{\varphi(x, r)} = 0 \quad \text{for} \quad S^\varphi\text{-a. e. } x \in E.$$

For $0 < M < \infty$, $0 < \delta < \infty$ we put

$$H(M, \delta) = \left\{x \,\middle|\, \frac{\psi[c(y, r)]}{\varphi(y, r)} < M \quad \text{if} \quad 0 < r < \delta \quad \text{and} \quad x \in c^0(y, r)\right\}.$$

Finally we put

$$G(\psi) = \left\{x \,\middle|\, \limsup_{r\to 0} \frac{\psi[c(x, r)]}{\varphi(x, r)} < \infty\right\}.$$

5.3. LEMMA. *The set $H(M, \delta)$ is closed.*

Indeed, take a point $x_0 \in \mathcal{C}H(M, \delta)$. Then there exists an open sphere $c^0(y, r)$ such that

$$(1) \qquad 0 < r < \delta, \quad \frac{\psi[c(y, r)]}{\varphi(y, r)} \geqslant M, \quad x_0 \in c^0(y, r).$$

Take now any point $x \in c^0(y, r)$. Then the first two conditions in (1) imply that $x \in \mathcal{C}H(M, \delta)$. Thus for every point $x_0 \in \mathcal{C}H(M, \delta)$ there exists an open sphere $c^0(y, r)$ such that

$$x_0 \in c^0(y, r) \subset \mathcal{C}H(M, \delta).$$

Thus clearly $\mathcal{C}H(M, \delta)$ is open, and hence $H(M, \delta)$ is closed,

5.4. Lemma. $G(\psi) = \bigcup_{n=1}^{\infty} H(n, 1/n)$.

Proof. Consider a point $x_0 \in G(\psi)$. Then clearly there exists a positive integer m such that

$$\frac{\psi[c(x_0, \varrho)]}{\varphi(x_0, \varrho)} < m \quad \text{for} \quad 0 < \varrho < \frac{1}{m}. \tag{2}$$

Next consider the closed sphere $c(x_0, 1)$. This is a bounded set, and hence the corresponding finite positive constants

$$k = k[c(x_0, 1)], \quad K = K[c(x_0, 1)]$$

are available (see 3.2). Then we have

$$\varphi(y, 5r) < K\varphi(y, r) \quad \text{if} \quad 0 < r < k, \quad y \in c(x_0, 1). \tag{3}$$

We choose now a positive integer n such that

$$n > Km + 2m + 1/k + 4. \tag{4}$$

Consider now any closed sphere $c(y, r)$ such that

$$0 < r < 1/n, \quad x_0 \in c^0(y, r). \tag{5}$$

Clearly we have then the inclusions

$$c(y, r) \subset c(x_0, 2r) \subset c(y, 5r). \tag{6}$$

In view of (5) and (4) it follows that $y \in c(x_0, 1)$ and $0 < r < k$. Hence, by (3),

$$\varphi(y, 5r) < K\varphi(y, r). \tag{7}$$

Next we have, by (6),

$$\frac{\psi[c(y, r)]}{\varphi(y, r)} \leqslant \frac{\psi[c(x_0, 2r)]}{\varphi(x_0, 2r)} \cdot \frac{\varphi(x_0, 2r)}{\varphi(y, r)}. \tag{8}$$

By (4) and (5) we have $0 < 2r < 1/m$. Hence, by (2),

$$\frac{\psi[c(x_0, 2r)]}{\varphi(x_0, 2r)} < m. \tag{9}$$

Also, since φ is subadditive, (6) and (7) imply that

$$\varphi(x_0, 2r) \leqslant \varphi(y, 5r) < K\varphi(y, r).$$

In view of (4), (8) and (9) the preceding inequality yields

$$\frac{\psi[c(y, r)]}{\varphi(y, r)} \leqslant Km < n.$$

Since this holds whenever (5) holds, it is shown (see 5.2) that $x_0 \in H(n, 1/n)$. As x_0 was an arbitrary point of $G(\psi)$, it follows that

$$G(\psi) \subset \bigcup_{n=1}^{\infty} H(n, 1/n).$$

To derive the complementary inclusion, consider a point x_0 such that $x_0 \in H(n, 1/n)$ for some n. Then we have, in particular,

$$\frac{\psi[c(x_0, r)]}{\varphi(x_0, r)} < n \quad \text{for} \quad 0 < r < \frac{1}{n},$$

and hence obviously

$$\limsup_{r \to 0} \frac{\psi[c(x_0, r)]}{\varphi(x_0, r)} \leqslant n < \infty.$$

Thus $x_0 \in G(\psi)$ and hence

$$\bigcup_{n=1}^{\infty} H(n, 1/n) \subset G(\psi).$$

5.5. Lemma. *If $S^{\varphi}(E) = 0$, then $E \in \mathcal{F}(\psi)$.*

This is an obvious consequence of the definition of $\mathcal{F}(\psi)$ (see 5.2).

5.6. Lemma. *If $E_n \in \mathcal{F}(\psi)$, $n = 1, 2, \ldots$, then $\bigcup E_n \in \mathcal{F}(\psi)$.*

Proof. Put $E = \bigcup E_n$. Since each E_n is S^{φ}-measurable, E is also S^{φ}-measurable. Furthermore, for each n there exists a set $e_n \subset E_n$ such that $S^{\varphi}(e_n) = 0$ and

$$\limsup_{r \to 0} \frac{\psi[c(x, r) \cap \complement E_n]}{\varphi(x, r)} = 0 \quad \text{for} \quad x \in E_n - e_n. \tag{10}$$

Put $e = \bigcup e_n$. Then

$$e \subset E, \quad S^{\varphi}(e) = 0. \tag{11}$$

Consider now any point $x \in E - e$. Then $x \in E_n - e_n$ for some n. Then, since $\mathcal{C}E \subset \mathcal{C}E_n$, we have by (10)

$$\limsup_{r \to 0} \frac{\psi[c(x, r) \cap \mathcal{C}E]}{\varphi(x, r)} \leqslant \limsup_{r \to 0} \frac{\psi[c(x, r) \cap \mathcal{C}E_n]}{\varphi(x, r)} = 0.$$

Thus the first limsup is equal to zero for $x \in E - e$, and in view of (11) it follows that $E \in \mathcal{F}(\psi)$.

5.7. Lemma. *If F is a non-empty closed subset of X, $0 < M < \infty$, $0 < \delta < \infty$, and*

$$\frac{\psi[c(y, \varrho)]}{\varphi(y, \varrho)} < M \quad \textit{for} \quad 0 < \varrho < \delta, \quad c^0(y, \varrho) \cap F \neq \emptyset, \tag{12}$$

then $F \in \mathcal{F}(\psi)$.

Proof. Case 1. F is bounded. We introduce the auxiliary set

$$S = \{x \mid d(x, F) < 1\}. \tag{13}$$

Then S is also a bounded set, and hence (see 3.2) the finite positive constants $k(S)$, $K(S)$ are available. We have then

$$\varphi(y, 5r) < K(S)\varphi(y, r) \quad \text{if} \quad 0 < r < k(S), \quad y \in S. \tag{14}$$

Consider any point x and any number r such that

$$x \in F, \quad 0 < r < k(S), \quad 0 < r < \tfrac{1}{2}, \quad 0 < r < \delta/5. \tag{15}$$

We put $\mathcal{C}F = O$. For each point $y \in O \cap c(x, r)$ we put

$$r_y = \tfrac{1}{2} d(y, F). \tag{16}$$

Note that $0 < r_y < \infty$ since F is closed and non-empty. From (13), (15) and (16) there follow readily the inclusions and inequalities

$$c(y, r_y) \subset O \cap c^0(x, 2r) \subset S, \quad 0 < r_y < k(S), \tag{17}$$

$$c^0(y, 5r_y) \cap F \neq \emptyset, \quad 0 < 5r_y < 5r < \delta. \tag{18}$$

Clearly $O \cap c(x, r) \subset \bigcup c(y, r_y)$ for $y \in O \cap c(x, r)$. Since the radii r_y are bounded by $k(S) < \infty$ (see (17)), by 1.2, theorem 1, we conclude that from the family of the closed spheres $c(y, r_y)$, $y \in O \cap c(x, r)$, we can extract a disjoint sequence $c(y_i, r_{y_i})$, $i = 1, 2, \ldots$, such that

$$O \cap c(x, r) \subset \bigcup_i c(y_i, 5r_{y_i}).$$

We have then

$$\psi[O \cap c(x, r)] \leqslant \sum_i \psi[c(y_i, 5r_{y_i})]. \tag{19}$$

We have then (see (17), (18))

$$0 < 5r_{y_i} < \delta, \quad c^0(y_i, 5r_{y_i}) \cap F \neq \emptyset, \quad 0 < r_{y_i} < k(S), \quad y_i \in S.$$

Hence, by (12) and (14),

$$\psi[c(y_i, 5r_{y_i})] < M\varphi(y_i, 5r_{y_i}) < MK(S)\varphi(y_i, r_{y_i}). \tag{20}$$

Now the closed spheres $c(y_i, r_{y_i})$, $i = 1, 2, \ldots$, are pairwise disjoint and are contained in the open set $O \cap c^0(x, 2r)$ (see (17)). Hence (see 4.10) we have the inequality

$$\sum_i \varphi(y_i, r_{y_i}) \leqslant \omega^\varphi[O \cap c^0(x, 2r)].$$

In view of (19) and (20) it follows that

$$\frac{\psi[O \cap c(x, r)]}{\varphi(x, r)} < MK(S) \frac{\omega^\varphi[O \cap c^0(x, 2r)]}{\varphi(x, 2r)} \cdot \frac{\varphi(x, 2r)}{\varphi(x, r)}.$$

Now since $x \in F \subset S$ and $0 < r < k(S)$, we have (see (14))

$$\varphi(x, 2r) \leqslant \varphi(x, 5r) < K(S)\varphi(x, r)$$

where we used the fact that φ is subadditive. Hence

$$\frac{\psi[O \cap c(x, r)]}{\varphi(x, r)} < M[K(S)]^2 \frac{\omega^\varphi[O \cap c^0(x, 2r)]}{\varphi(x, 2r)}. \tag{21}$$

By the theorem in 4.10 it follows that

$$\limsup_{r \to 0} \frac{\omega^\varphi[O \cap c^0(x, 2r)]}{\varphi(x, 2r)} = 0 \quad \text{for} \quad S^\varphi\text{-a. e. } x \in \mathcal{C}O.$$

Since $\mathcal{C}O = F$, by (21) it follows that

$$\limsup_{r \to 0} \frac{\psi[c(x, r) \cap \mathcal{C}F]}{\varphi(x, r)} = 0 \quad \text{for} \quad S^\varphi\text{-a. e. } x \in F.$$

Hence (see 5.2) $F \in \mathcal{F}(\psi)$.

Case 2. We now drop the assumption that F is bounded. We pick a point $x_0 \in F$ and put

$$F_n = F \cap c(x_0, n), \quad n = 1, 2, \ldots$$

Then F_n is a non-empty, bounded closed subset of F, and by (12) clearly

$$\frac{\psi[c(y,\varrho)]}{\varphi(y,\varrho)} < M \quad \text{for} \quad 0 < \varrho < \delta, \quad c^0(y,\varrho) \cap F_n \neq \emptyset.$$

Hence, by Case 1, we have $F_n \in \mathcal{F}(\psi)$. Since $\bigcup F_n = F$, by 5.6 we conclude that $F \in \mathcal{F}(\psi)$, and the lemma is proved.

5.8. Lemma. *Let F be a closed subset of $H(M,\delta)$. Then $F \in \mathcal{F}(\psi)$.*

Proof. Since the assertion is obvious if $F = \emptyset$, we can assume that $F \neq \emptyset$. Consider any closed sphere $c(y,\varrho)$ such that

$$0 < \varrho < \delta \quad \text{and} \quad c^0(y,\varrho) \cap F \neq \emptyset. \tag{22}$$

Then there exists a point $x \in c^0(y,\varrho) \cap F$. Then we have also $x \in H(M,\delta)$, since $F \subset H(M,\delta)$. Thus we have the relations

$$0 < \varrho < \delta, \quad x \in c^0(y,\varrho), \quad x \in H(M,\delta).$$

By the definition of $H(M,\delta)$ (see 5.2) it follows that

$$\frac{\psi[c(y,\varrho)]}{\varphi(y,\varrho)} < M. \tag{23}$$

Thus (23) holds whenever (22) holds, showing that F satisfies the assumptions of the lemma in 5.7. Hence $F \in \mathcal{F}(\delta)$ by that lemma.

5.9. Lemma. *Let F be a closed subset of the set $G(\psi)$* (see 5.2). *Then $F \in \mathcal{F}(\psi)$.*

Proof. Since $F \subset G(\psi)$, we have (see 5.4)

$$F = F \cap G(\psi) = F \cap [\bigcup_n H(n,1/n)] = \bigcup_n [F \cap H(n,1/n)]. \tag{24}$$

Now since $H(n,1/n)$ is closed by 5.3 the set $F \cap H(n,1/n)$ is a closed subset $H(n,1/n)$, and hence

$$F \cap H(n,1/n) \in \mathcal{F}(\psi)$$

by 5.8. Hence, in view of (24) it follows by 5.6 that $F \in \mathcal{F}(\psi)$.

5.10. Lemma. *Let E be an S^φ-measurable subset of the set $G(\psi)$* (see 5.2). *Then $E \in \mathcal{F}(\psi)$.*

Proof. By the lemma in 3.7 the set E is S^φ-almost closed. Hence for each positive integer n there exists a closed set $F_n \subset E$ such that $S^\varphi(E - F_n) < 1/n$. On setting

$$e = E - \bigcup F_n,$$

we have $S^\varphi(e) \leqslant S^\varphi(E-F_n) < 1/n$, $n = 1, 2, \ldots$ Thus $S^\varphi(e) = 0$ and $E = e \cup F_1 \cup F_2 \cup \ldots$ Also, $e \in \mathcal{F}(\psi)$ by 5.5, and $F_n \in \mathcal{F}(\psi)$ by 5.9 (since $F_n \subset E \subset G(\psi)$). Hence $E \in \mathcal{F}(\psi)$ by 5.6.

5.11. Lemma. *Let E be an S^φ-measurable set. Then*

$$\limsup_{r\to 0} \frac{\psi[c(x,r) \cap E]}{\varphi(x,r)} = 0 \quad \textit{for} \quad S^\varphi\text{-a. e. } x \in G(\psi) \cap \complement E. \tag{25}$$

Proof. $G(\psi)$ is a Borel set (see 5.2, 5.3, 5.4). Hence the set

$$E' = G(\psi) \cap \complement E$$

is S^φ-measurable. By 5.10, applied to E', it follows that

$$\limsup_{r\to 0} \frac{\psi[c(x,r) \cap \complement E']}{\varphi(x,r)} = 0 \quad \text{for} \quad S^\varphi\text{-a. e. } x \in E'. \tag{26}$$

Now $\complement E' = E \cup \complement G(\psi) \supset E$. Thus (26) yields (25).

5.12. We are now ready to prove the theorem stated in 5.1. Let E be an S^φ-measurable set. For subsets S of X, let us define

$$\psi^*(S) = \psi(S \cap E). \tag{27}$$

It follows readily that ψ^* is again an outer measure. On setting (see 5.2)

$$G(\psi^*) = \left\{x \,\middle|\, \limsup_{r\to 0} \frac{\psi^*[c(x,r)]}{\varphi(x,r)} < \infty\right\}, \tag{28}$$

we have

$$\limsup_{r\to 0} \frac{\psi^*[c(x,r)]}{\varphi(x,r)} = \infty \quad \text{for} \quad x \in \complement G(\psi^*),$$

hence a fortiori

$$\limsup_{r\to 0} \frac{\psi^*[c(x,r)]}{\varphi(x,r)} = \infty \quad \text{for} \quad x \in \complement G(\psi) \cap \complement E. \tag{29}$$

Also, by 5.11 (applied to ψ^*), we have

$$\limsup_{r\to 0} \frac{\psi^*[c(x,r) \cap E]}{\varphi(x,r)} = 0 \quad \text{for} \quad S^\varphi\text{-a. e. } x \in G(\psi^*) \cap \complement E. \tag{30}$$

In view of (27), we have

$$\psi^*[c(x,r)] = \psi^*[c(x,r) \cap E] = \psi[c(x,r) \cap E],$$

and thus (29) and (30) show that

$$\limsup_{r\to 0}\frac{\psi[c(x,r)\cap E]}{\varphi(x,r)} = 0 \text{ or } \infty \quad \text{for} \quad S^{\varphi}\text{-a. e. } x\in \mathcal{C}E.$$

Thus the theorem in 5.1 is proved.

5.13. Let now σ be a $\mathcal{U}$-regular Carathéodory outer measure which satisfies the $5r$-condition. Consider any sequence $c(x_i, r_i)$, $i = 1, 2, \ldots$, of pairwise disjoint closed spheres contained in a closed sphere $c(x, r)$. Then, on setting $\sigma(x, r) = \sigma[c(x, r)]$, we have

$$\sum_i \sigma(x_i, r_i) = \sum_i \sigma[c(x_i, r_i)] \leqslant \sigma[c(x, r)] = \sigma(x, r).$$

Thus $\sigma(x, r)$ is subadditive. Also, since σ satisfies the $5r$-condition clearly $\sigma(x, r) = \sigma[c(x, r)]$ also satisfies the $5r$-condition. Accordingly, we can apply the theorem in 5.1 with $\varphi(x, r) = \sigma(x, r)$.. Also, by 3.3 we have $S^{\sigma} = \sigma$. Accordingly, the theorem in 5.1 yields the following

COROLLARY 1. *Let ψ be an outer measure in the separable metric space X, and let σ be a $\mathcal{U}$-regular Carathéodory outer measure in X which satisfies the $5r$-condition. Then for every σ-measurable set E we have,*

$$\limsup_{r\to 0}\frac{\psi[c(x,r)\cap E]}{\sigma[c(x,r)]} = 0 \text{ or } \infty \quad \text{for} \quad \sigma\text{-a.e. } x\in \mathcal{C}E.$$

We observed in 3.4 that n-dimensional Lebesgue exterior measure L_n in Euclidean n-space R^n satisfies the conditions placed upon σ in the preceding corollary 1. Recalling that $L_n[c(x, r)] = a(n)r^n$, we have threfore the following

COROLLARY 2. *Let ψ be an outer measure in R^n. Then for every L_n-measurable set E we have*

$$\limsup_{r\to 0}\frac{\psi[c(x,r)\cap E]}{a(n)r^n} = 0 \text{ or } \infty \quad \text{for} \quad L_n\text{-a. e. } x\in \mathcal{C}E.$$

Remark. This result has been established in [5].

SECTION 6

MISCELLANEOUS DENSITY THEOREMS

6.1. The collection of results discussed in the present section 6 includes various minor generalizations and refinements of lemmas and theorems scattered in the literature. Also, we use this opportunity to dispose of certain gaps and discrepancies occurring in previous treatments. Throughout the present Section 6, X will denote a separable metric space.

6.2. LEMMA. *Let $0 < k < \infty$, and let $f(x, r)$ be a non-negative real-valued function, defined for $x \in X$ and $0 < r < \infty$, which is monotone in the following sense: if $c(x', r') \subset c(x'', r'')$, then $f(x', r') \leqslant f(x'', r'')$. For $0 \leqslant b < \infty$, $0 < a < \infty$ put*

$$G(a, b, f) = \{x \mid f(x, r) \leqslant br^k \text{ for } 0 < r < a\}.$$

Then $G(a, b, f)$ is a closed set.

Proof. Let $x_j \in G(a, b, f)$, $j = 1, 2, \ldots$, and $x_j \to x_0$. We have to show that $x_0 \in G(a, b, f)$. Consider any real number r such that $0 < r < a$. This number r being fixed, select R to satisfy $r < R < a$. For j sufficiently large we have then $c(x_0, r) \subset c(x_j, R)$ and hence $f(x_0, r) \leqslant f(x_j, R)$. Consequently, since $x_j \in G(a, b, f)$ and $0 < R < a$,

$$f(x_0, r) \leqslant f(x_j, R) \leqslant bR^k = br^k(R/r)^k \quad \text{for} \quad j \text{ large.}$$

Letting $R \to r$ we obtain $f(x_0, r) \leqslant br^k$ if $0 < r < a$, showing that $x_0 \in G(a, b, f)$.

6.3. LEMMA. *Given k and $f(x, r)$ as in 6.2, put*

$$G(t, f) = \left\{x \mid \limsup_{r \to 0} \frac{f(x, r)}{r^k} > t\right\},$$

where $0 < t < \infty$. Then $G(t, f)$ is a Borel set.

Proof. Using the set $G(a, b, f)$ of 6.2, one verifies readily the identity

$$G(t, f) = \bigcup_{m=1} \bigcap_{n=1} \mathcal{C}G(1/n, t+1/m, f).$$

Since $G(1/n, t+1/m, f)$ is closed by the lemma in 6.2, this identity shows that $G(t, f)$ is a Borel set.

6.4. To simplify notation, in the special case when $\varphi(x, r) = r^k$, $0 < k < \infty$, we write S^k for the corresponding S^φ (see 3.1). Then S^k is Borel regular and hence also $\mathcal{U}$-regular. Now let Γ be a $\mathcal{U}$-regular Carathéodory outer measure in X and let E be a Γ-measurable set such that $\Gamma(E) < \infty$. The set

$$G_t = \left\{x \,|\, \limsup_{r\to 0} \frac{\Gamma[c(x, r)\cap E]}{r^k} > t\right\}, \quad 0 < t < \infty,$$

is now a Borel set by the lemma in 6.3 since on setting $f(x, r) = \Gamma[c(x, r)\cap E]$ and using the notation of 6.3 we have $G_t = G(t, f)$. Accordingly, we can apply the theorem in 4.11 with $F = G_t$ obtaining the inequality

$$S^k(G_t) \leqslant \Gamma(E\cap G_t)/t \leqslant \Gamma(E) < \infty. \tag{1}$$

From (1) it follows that $S^k(G_t) \leqslant \Gamma(G_t)$ and if we take $\Gamma = S^k$,

$$S^k(G_t) < \infty, \quad S^k(G_t) \leqslant S^k(G_t)/t.$$

It thus follows that

$$S^k(G_t) = 0 \quad \text{if} \quad \Gamma = S^k \quad \text{and} \quad 1 < t < \infty. \tag{2}$$

From this result we shall derive presently the following remarkable theorem which has been proved by Federer ([6], 3.5) for the special case $X = R^n$:

THEOREM. *In the separable metric space X let E be a set such that $S^k(E) < \infty$, where $0 < k < \infty$. Then*

$$\limsup_{r\to 0} \frac{S^k[c(x, r)\cap E]}{r^k} \leqslant 1 \quad \textit{for} \quad S^k\text{-a. e. } x\in X.$$

Proof. Clearly it is sufficient to show that on setting

$$G_t = \left\{x \,|\, \limsup_{r\to 0} \frac{S^k[c(x, r)\cap E]}{r^k} > t\right\}, \tag{3}$$

one has $S^k(G_t) = 0$ for $1 < t < \infty$. Now since S^k is Borel-regular and $S^k(E) < \infty$, there exists a Borel set E' such that $E \subset E'$ and $S^k(E') = S^k(E) < \infty$. On setting

$$G_t' = \left\{x \,|\, \limsup_{r\to 0} \frac{S^k[c(x, r)\cap E']}{r^k} > t\right\},$$

we have clearly

$$G_t \subset G'_t. \tag{4}$$

Also, since E' is a Borel set, E' is S^k-measurable. As $S^k(E') < \infty$, application of (2) to E' yields that $S^k(G'_t) = 0$ if $1 < t < \infty$. By (4) it follows that $S^k(G_t) = 0$ if $1 < t < \infty$, and the theorem is proved.

6.5. Let us recall the following general fact about Carathéodory outer measures Γ in X (see Saks [4], p. 44). Let S_j, $j = 1, 2, \ldots$, be a sequence of pairwise disjoint Γ-measurable sets. Put $S = \bigcup S_j$, and let Q be an arbitrary set. Then

$$\Gamma(Q \cap S) = \sum_j \Gamma(Q \cap S_j).$$

6.6. We shall now prove the following theorem which has been established by Federer ([6], 3.6), for the special case $X = R^n$:

THEOREM. *Let ψ be an outer measure in the separable metric space X. Let λ and k be finite positive real numbers, and let E be a subset of X such that*

$$\limsup_{r \to 0} \frac{\psi[c(x, r) \cap E]}{r^k} < \lambda \quad \textit{for} \quad x \in E. \tag{5}$$

Then (see 3.5)

$$\psi(E) \leqslant \lambda H^k(E). \tag{6}$$

We divide the proof into several steps.

Step 1. We introduce the auxiliary sets

$$S = \left\{x \,|\, \limsup_{r \to 0} \frac{\psi[c(x, r) \cap E]}{r^k} < \lambda\right\}, \tag{7}$$

$$S_j = \left\{x \,|\, \frac{\psi[c(x, r) \cap E]}{r^k} \leqslant \lambda - \frac{1}{j} \quad \text{for} \quad 0 < r < \frac{1}{j}\right\}, \tag{8}$$

where j is a positive integer. One verifies readily the relations

$$S_1 \subset \ldots \subset S_j \subset \ldots, \tag{9}$$

$$S = \bigcup S_j. \tag{10}$$

Furthermore, each set S_j is closed. Indeed, on setting $f(x, r) = \psi[c(x, r) \cap E]$, the set S_j coincides with the set $G(1/j, \lambda - 1/j, f)$ of 6.2, and hence S_j is closed by the lemma in 6.2. From (9) and (10) we have

$$S = S_1 \cup (S_2 - S_1) \cup \ldots \cup (S_{j+1} - S_j) \cup \ldots, \tag{11}$$

a representation of S as a countable union of pairwise disjoint Borel sets. In view of (5), clearly $E \subset S$, and hence $E = E \cap S$. Thus (11) yields

$$(12) \quad E = E \cap S = [E \cap S_1] \cup [E \cap (S_2 - S_1)] \cup \ldots \cup [E \cap (S_{j+1} - S_j)] \cup \ldots$$

By 6.5, applied with $\Gamma = H^k$, we conclude that

$$(13) \quad H^k(E) = H^k[E \cap S_1] + H^k[E \cap (S_2 - S_1)] + \ldots + \\ + H^k[E \cap (S_{j+1} - S_j)] + \ldots$$

Since ψ is an outer measure, (12) yields

$$(14) \quad \psi(E) \leqslant \psi[E \cap S_1] + \psi[E \cap (S_2 - S_1)] + \ldots + \\ + \psi[E \cap (S_{j+1} - S_j) + \ldots$$

Step 2.

LEMMA. *Let μ be a finite positive number, and let A be a set such that*

$$(15) \quad \frac{\psi[c(x, r) \cap A]}{r^k} < \lambda \quad \textit{for} \quad x \in A, \quad 0 < r < \mu.$$

Then $\psi(A) \leqslant \lambda H^k(A)$.

To show this, assign ε such that $0 < \varepsilon < \mu$, and let the sequence of sets e_i, $i = 1, 2, \ldots$, be such that

$$(16) \quad A \subset \bigcup e_i, \quad e_i \cap A \neq \emptyset, \quad \operatorname{diam} e_i < \varepsilon.$$

For each i, pick a point $x_i \in e_i \cap A$. Then $e_i \subset c(x_i, \operatorname{diam} e_i)$. Thus

$$(17) \quad A \subset \bigcup c(x_i, \operatorname{diam} e_i), \quad \operatorname{diam} e_i < \varepsilon < \mu, \quad x_i \in A.$$

From (17) clearly

$$A \subset \bigcup [c(x_i, \operatorname{diam} e_i) \cap A].$$

In view of (17) and (15) it follows that

$$\psi(A) \leqslant \sum \psi[c(x_i, \operatorname{diam} e_i) \quad A] < \lambda \sum (\operatorname{diam} e_i)^k.$$

Since this holds for every sequence e_i satisfying (16), we conclude (see 3.5) that

$$\psi(A) \leqslant \lambda H_\varepsilon^k(A) \leqslant \lambda H^k(A),$$

and the lemma is proved.

Step 3. We can now prove the theorem itself as follows. We can apply, in view of (8), the preceding lemma to the sets

$$E \cap S_1, \; E \cap (S_2 - S_1), \; \ldots, \; E \cap (S_{j+1} - S_j), \; \ldots,$$

since

$$\frac{\psi[c(x,r)\cap E\cap S_1]}{r^k} \leqslant \frac{\psi[c(x,r)\cap E]}{r^k} < \lambda \quad \text{for} \quad x\in E\cap S_1, \quad 0<r<1,$$

$$\frac{\psi[c(x,r)\cap E\cap(S_{j+1}-S_j)]}{r^k} \leqslant \frac{\psi[c(x,r)\cap E]}{r^k} < \lambda$$

$$\text{for} \quad x\in E\cap(S_{j+1}-S_j),\ 0<r<\frac{1}{j+1}.$$

Thus the preceding lemma yields

$$\psi[E\cap S_1] \leqslant \lambda H^k[E\cap S_1],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\psi[E\cap(S_{j+1}-S_j)] \leqslant \lambda H^k[E\cap(S_{j+1}-S_j)],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Addition yields, in view of (14) and (13), the inequality (6).

6.7. For $X = R^n$ and k a positive integer, we obtain the following statement. Let E be a subset of R^n such that

$$\limsup_{r\to 0} \frac{\psi[c(x,r)\cap E]}{a(k)r^k} < \lambda \quad \text{for} \quad x\in E. \tag{18}$$

Then

$$\psi(E) \leqslant \lambda 2^k H_n^k(E). \tag{19}$$

Indeed, (18) yields

$$\limsup_{r\to 0} \frac{\psi[c(x,r)\cap E]}{r^k} < a(k)\lambda \quad \text{for} \quad x\in E.$$

Hence the preceding theorem (with λ replaced by $a(k)\lambda$) yields

$$\psi(E) \leqslant a(k)\lambda H^k(E),$$

and (19) follows, since $H_n^k(E) = \dfrac{a(k)}{2^k} H^k(E)$.

Similarly, the theorem in 6.4 yields, for $X = R^n$ and k a positive integer, the following statement. Let E be a subset of R^n such that $H_n^k(E) < \infty$. Then

$$\limsup_{r\to 0} \frac{H_n^k[c(x,r)\cap E]}{a(k)r^k} \leqslant 1 \quad \text{for} \quad H_n^k\text{-a. e. } x\in R^n. \tag{20}$$

Indeed, since $H_n^k = \frac{a(k)}{2^k} H^k$, we have $H^k(E) < \infty$, and hence also (see 3.6) $S^k(E) < \infty$. Hence the theorem in 6.4 yields

$$\limsup_{r\to 0} \frac{S^k[c(x,r)\cap E]}{r^k} \leqslant 1 \quad \text{for} \quad S^k\text{-a. e. } x\in R^n. \tag{21}$$

Now (see 3.5, 3.6), we have

$$\frac{H_n^k[c(x,r)\cap E]}{a(k)r^k} = \frac{H^k[c(x,r)\cap E]}{2^k r^k} \leqslant \frac{S^k[c(x,r)\cap E]}{r^k}$$

and thus (20) follows from (21), noting that S^k, H^k, and H_n^k have the same nullsets (see 3.5, 3.6).

6.8. THEOREM. *Let $0 < k < \infty$, and let E be a subset of the separable metric space X. Then* (see 3.5)

$$\limsup_{r\to 0} \frac{H^k[c(x,r)\cap E]}{r^k} \geqslant 1 \quad \textit{for} \quad H^k\text{-a. e. } x\in E. \tag{22}$$

Proof. Since X is separable, we have a countable subset C of X whose closure coincides with X. Let $\mathcal{F}$ be the family of those closed spheres $c(y,\varrho)$ which satisfy the following conditions:

$$y\in C, \quad \varrho \text{ is rational}, \quad H^k[c(y,\varrho)\cap E] < \infty.$$

If this family $\mathcal{F}$ is empty, then clearly

$$H^k[c(x,r)\cap E] = \infty \quad \text{for} \quad x\in E, \quad 0 < r < \infty,$$

and thus (22) is obvious, since the limsup is infinite for every $x\in E$. So we can assume that $\mathcal{F}$ is non-emtpy. Then $\mathcal{F}$ is clearly a countable family, and we can arrange the closed spheres in $\mathcal{F}$ into a sequence $c(y_i,\varrho_i)$, $i = 1, 2, \ldots$ If

$$x\in E - \bigcup_i c^0(y_i,\varrho_i),$$

then clearly $H^k[c(x,r)\cap E] = \infty$ for $0 < r < \infty$, and thus the limsup in (22) is infinite at the point x. Hence it is sufficient to show that

$$\limsup_{r\to 0} \frac{H^k[c(x,r)\cap E]}{r^k} \geqslant 1 \quad \text{for} \quad H^k\text{-a. e. } x\in E\cap\Big[\bigcup_i c^0(y_i,\varrho_i)\Big]. \tag{23}$$

Let us put

$$E_i = E\cap c^0(y_i,\varrho_i). \tag{24}$$

Since $H^k[c(x,r)\cap E_i] \leqslant H^k[c(x,r)\cap E]$, in view of (23) it is sufficient to show that

$$\limsup_{r\to 0} \frac{H^k[c(x,r)\cap E_i]}{r^k} \geqslant 1 \quad \text{for} \quad H^k\text{-a. e. } x\in E_i. \tag{25}$$

Let us note that $H^k(E_i) \leqslant H^k[E\cap c(y_i, \varrho_i)] < \infty$, by the definition of the family $\mathcal{F}$ (see above). For each positive integer $n > 1$, let us put

$$E_i^n = \left\{x \,|\, x\in E_i,\ \limsup_{r\to 0} \frac{H^k[c(x,r)\cap E_i]}{r^k} < 1-\frac{1}{n}\right\}. \tag{26}$$

Clearly (25) follows if we show that

$$H^k(E_i^n) = 0. \tag{27}$$

Now since $E_i^n \subset E_i$, in view of (26) we have a fortiori

$$\limsup_{r\to 0} \frac{H^k[c(x,r)\cap E_i^n]}{r^k} < 1-\frac{1}{n} \quad \text{for} \quad x\in E_i^n. \tag{28}$$

Also, since $H^k(E_i) < \infty$, we have

$$H^k(E_i) < \infty. \tag{29}$$

From (28) we conclude, by means of 6.6, that

$$H^k(E_i^n) \leqslant \left(1-\frac{1}{n}\right) H^k(E_i^n),$$

and (27) follows in view of (29). Thus the theorem is proved.

6.9. Let us apply the preceding theorem in the special case when X coincides with Euclidean n-space R^n and k is a positive integer. Recalling (see 3.5) that

$$H_n^k = \frac{a(k)}{2^k} H^k,$$

the preceding theorem yields the following statement, which has been proved by Federer (see [6], 3.7):

THEOREM. *Let k be a positive integer, and let E be a subset of R^n. Then*

$$\limsup_{r\to 0} \frac{H_n^k[c(x,r)\cap E]}{a(k)r^k} \geqslant \frac{1}{2^k} \quad \textit{for} \quad H_n^k\text{-a. e. } x\in E.$$

6.10. Throughout the present section 6.10, ψ will denote an outer measure in the separable metric space X, and σ will denote a $\mathcal{U}$-regular Carathéodory outer measure satisfying the $5r$-condition (see 2.3, 3.3).

THEOREM. *Let E be a subset of X such that*

$$\liminf_{r\to 0}\frac{\psi[c(x,r)\cap E]}{\sigma[c(x,r)]}\leqslant t \quad \textit{for} \quad x\in E, \tag{30}$$

where t is a finite positive constant. Then

$$\psi(E)\leqslant t\sigma(E). \tag{31}$$

We divide the proof into several steps.

LEMMA 1. *Let E be a subset of X such that $\sigma(E)=0$ and*

$$\liminf_{r\to 0}\frac{\psi[c(x,r)]}{\sigma[c(x,r)]}\leqslant t \quad \textit{for} \quad x\in E, \tag{32}$$

where t is a finite positive constant. Then $\psi(E)=0$.

Proof. Case 1. E is bounded. Then the finite positive constants $k(E)$, $K(E)$ are available (see 3.3). Assign now $\eta>0$. By 2.11 and 3.3 there exists an open set O such that

$$E\subset O, \quad \sigma(O)<\eta. \tag{33}$$

Then in view of (32) the family of closed spheres $c(x,r)$ such that

$$x\in E, \quad c(x,5r)\subset O, \quad 0<r<k(E), \quad \frac{\psi[c(x,5r)]}{\sigma[c(x,5r)]}<t+\eta, \tag{34}$$

covers E. Hence (see 1.2, theorem 1) this family contains a sequence $c(x_i,r_i)$, $i=1,2,\ldots$, of pairwise disjoint closed spheres such that

$$E\subset\bigcup_i c(x_i,5r_i).$$

In view of (33) and (34) if follows that

$$\begin{aligned}\psi(E)&\leqslant\sum_i\psi[c(x_i,5r_i)]<(t+\eta)\sum_i\sigma[c(x_i,5r_i)]\\&<(t+\eta)K(E)\sum_i\sigma[c(x_i,r_i)]\leqslant(t+\eta)K(E)\sigma(O)\\&<(t+\eta)K(E)\eta.\end{aligned}$$

Since $\eta>0$ was arbitrary, it follows that $\psi(E)=0$.

Case 2. We now drop the assumption that E is bounded. We pick a point $x_0\in X$ and put

$$E_n=E\cap c(x_0,n), \quad n=1,2,\ldots$$

Then E_n is a bounded subset of E, and hence clearly

$$\sigma(E_n) = 0, \quad \liminf_{r\to 0} \frac{\psi[c(x,r)]}{\sigma[c(x,r)]} \leqslant t \quad \text{for} \quad x \in E_n.$$

By Case 1 it follows that $\psi(E_n) = 0$. Since $E = \bigcup E_n$, we conclude that $\psi(E) = 0$, and the lemma is proved.

LEMMA 2. *Let E be a subset of X such that*

$$\liminf_{r\to 0} \frac{\psi[c(x,r)]}{\sigma[c(x,r)]} \leqslant t \quad \textit{for} \quad x \in E, \tag{35}$$

where $0 < t < \infty$. Then

$$\psi(E) \leqslant t\sigma(E). \tag{36}$$

Proof. Since (36) is obvious if $\sigma(E) = \infty$, we can assume that $\sigma(E) < \infty$. We make the proof in two steps.

Case 1. E is bounded. Then the finite positive constants $k(E)$, $K(E)$ are available (see 3.3). Now assign $\eta > 0$. Now there is a set $U \in \mathcal{U}$ such that $E \subset U$ and $\sigma(U) = \sigma(E)$. By the theorem in 2.10 there is an open set $O \supset U$ such that $\sigma(O-U) < \eta$. Thus

$$E \subset O, \quad \sigma(O) = \sigma(U) + \sigma(O-U) < \sigma(E) + \eta < \infty. \tag{37}$$

Then, in view of (35), the family of closed spheres $c(x,r)$ such that

$$x \in E, \quad c(x,r) \subset O, \quad 0 < r < k(E), \quad \frac{\psi[c(x,r)]}{\sigma[c(x,r)]} < t+\eta, \tag{38}$$

covers E in the Vitali sense. Hence (see 1.2, theorem 2) this family contains a sequence $c(x_i, r_i)$, $i = 1, 2, \ldots$, of pairwise disjoint closed spheres such that

$$E - \bigcup_i c(x_i, r_i) \subset \bigcup_{i=N+1}^{\infty} c(x_i, 5r_i)$$

for every positive integer N. There follows the inequality

$$\sigma[E - \bigcup_i c(x_i, r_i)] \leqslant \sum_{i=N+1}^{\infty} \sigma[c(x_i, 5r_i)] < K(E) \sum_{i=N+1}^{\infty} \sigma[c(x_i, r_i)]. \tag{39}$$

Now since the closed spheres $c(x_i, r_i)$, $i = 1, 2, \ldots$, are pairwise disjoint σ-measurable subsets of O (see (38)), we have

$$\sum_i \sigma[c(x_i, r_i)] \leqslant \sigma(O) < \sigma(E) + \eta < \infty$$

by (37). Thus the infinite series on the left is convergent, and hence

$$\sum_{i=N+1}^{\infty} \sigma[c(x_i, r_i)] \to 0 \quad \text{for} \quad N \to \infty. \tag{40}$$

From (39) and (40) we infer that

$$\sigma[E - \bigcup_i c(x_i, r_i)] = 0. \tag{41}$$

On setting

$$E' = E - \bigcup_i c(x_i, r_i),$$

we have

$$E \subset E' \cup [\bigcup_i c(x_i, r_i)], \tag{42}$$

and in view of (35) and (41) also

$$\sigma(E') = 0, \quad \liminf_{r\to 0} \frac{\psi[c(x, r)]}{\sigma[c(x, r)]} \leqslant t \quad \text{for} \quad x \in E'. \tag{43}$$

By lemma 1, applied to the set E', (43) implies that $\psi(E') = 0$. In view of (37), (38), and (42) it follows that

$$\begin{aligned}\psi(E) &\leqslant \psi(E') + \sum_i \psi[c(x_i, r_i)] = 0 + \sum_i \psi[c(x_i, r_i)] \\ &< (t+\eta) \sum \sigma[c(x_i, r_i)] \leqslant (t+\eta)\sigma(O) < (t+\eta)[\sigma(E)+\eta],\end{aligned}$$

and (36) follows since $\eta > 0$ was arbitrary.

Case 2. We now drop the assumption that E is bounded. Since X is separable and σ is locally finite, there exists a sequence O_n, of bounded open sets such that

$$X = \bigcup_i O_n, \quad \sigma(O_n) < \infty.$$

Let us put

$$B_1 = O_1, \quad B_n = O_n - \bigcup_{m=1}^{n-1} O_m \quad \text{for} \quad n > 1.$$

Then B_n, $n = 1, 2, \ldots$, is a sequence of pairwise disjoint bounded Borel sets such that $X = \bigcup_n B_n$. By 6.5 we have therefore

$$\sigma(E) = \sum_n \sigma(E \cap B_n). \tag{44}$$

Now since $E \cap B_n \subset E$, by (35) we have

$$\liminf_{r\to 0} \frac{\psi[c(x, r)]}{\sigma[c(x, r)]} \leqslant t \quad \text{for} \quad x \in E \cap B_n.$$

Also, $E \cap B_n$ is bounded since B_n is bounded. Hence, by Case 1, applied to the set $E \cap B_n$, we have

$$\psi(E \cap B_n) \leqslant t\sigma(E \cap B_n), \qquad n = 1, 2, \ldots$$

Since $E = \bigcup_n (E \cap B_n)$, in view of (44) it follows that

$$\psi(E) \leqslant \sum_n \psi(E \cap B_n) \leqslant t \sum_n \sigma(E \cap B_n) = t\sigma(E),$$

and the lemma is proved.

We can now prove the theorem itself as follows. For $S \subset X$ we put

$$\psi^*(S) = \psi(S \cap E).$$

Then ψ^* is again an outer measure, and by (30) we have

$$\liminf_{r \to 0} \frac{\psi^*[c(x, r)]}{\sigma[c(x, r)]} \leqslant t \quad \text{for} \quad x \in E.$$

By lemma 2, applied to ψ^*, there follows the inequality $\psi^*(E) \leqslant t\sigma(E)$, and the theorem is proved since $\psi^*(E) = \psi(E)$.

Remark. If X and σ coincide with n-dimensional Euclidean space R^n and n-dimensional Lebesgue outer measure L_n respectively, then we obtain from the preceding theorem the following result. Let ψ be an outer measure in R^n and let E be a subset of R^n such that

$$\liminf_{r \to 0} \frac{\psi[c(x, r) \cap E]}{a(n) r^n} \leqslant t \quad \text{for} \quad x \in E,$$

where t is a finite positive constant. Then $\psi(E) \leqslant tL_n(E)$.

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