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1. Introduction

Problems of the geometrical structure of sets of solutions of partial differential equations arise in many branches of mathematical physics. In recent years methods of global analysis have enabled us to solve several difficult classical problems—cf. papers of Chernoff, Ebin, Marsden and others [13]. These papers treat a partial differential equation as a Hamiltonian (dynamical) system in an infinite-dimensional space, i.e., as an ordinary differential equation in an appropriate function space. The main tool in such considerations is the (weak) riemannian metric in the corresponding function space. In our paper we investigate another structure which plays an important role in mathematical physics. It turns out that under some reasonable assumptions the set of all solutions of a given differential equation has a natural pre-symplectic structure. This fact has been recently proved by J. Kijowski and the present author in [26]. In this paper we investigate the general scheme given in [26] for the most beautiful and difficult classical field theory—the Einstein theory of gravitation. The problem of the existence of a symplectic structure in the set of solutions of Einstein equations is connected with the canonical formulation of General Relativity (cf. [26] for a general discussion of this problem). Therefore the theory presented here enables us to elucidate many problems considered earlier in the classical works of Dirac, Arnowitt–Deser–Misner, Wheeler, De Witt and others [12], [3], [41], [11]. Our results give a new insight into the nature of many known problems, considering them from a geometrical point of view. The main goal of the paper is a construction of the phase space for General Relativity. It is known (cf. [1], [35]) that the Hamilton formulation of mechanics is a consequence of the existence of a symplectic structure in the cotangent bundle of an n-dimensional manifold V. In the phase space $\mathscr{P}_{2n} = T^*V$ of a mechanical system with n degrees of freedom we have the symplectic form Ω. If (q^i) are local coordinates in V and (p_j, q^i) are local coordinates in T^*V, then

$$\Omega = \sum_{i=1}^{n} dp_i \wedge dq^i. \tag{1.1}$$

The symplectic form Ω defines a Lie algebra structure in the set of all smooth functions on $\mathscr{P}_{2n}$. A skew symmetric form $\{\cdot\,,\cdot\}$ in $\mathscr{F} = C^\infty(\mathscr{P}_{2n})$

determining this structure is called the *Poisson bracket.* In local coordinates we have:

$$\{f_1, f_2\} = \sum_{i=1}^{n} \left[\frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q^i} - \frac{\partial f_2}{\partial p_i} \frac{\partial f_1}{\partial q^i} \right]; \quad f_1, f_2 \in \mathscr{F}. \tag{1.2}$$

In [26] J. Kijowski and the present author have found a generalization of formulas (1.1) and (1.2) for systems with an infinite number of degrees of freedom. The theory presented in [26] takes as a starting point the geometric formulation of the field equations developed in [24]. We assume that there exists a finite-dimensional manifold $\mathscr{P}$ and a closed $(n+1)$-differential form ω on $\mathscr{P}$ such that submanifolds $C \subset \mathscr{P}$ which satisfy the condition that for every vector field X on C

$$(X \lrcorner\, \omega)|C = 0 \tag{1.3}$$

are in a one-to-one correspondence with the set of solutions of given field equations (partial differential equations). Therefore the first task is to find the couple $(\mathscr{P}, \omega)$ for gravidynamics. This is done in Section 5, where the effective formulas for the 5-form $\overset{5}{\omega}$ and its primitive $\overset{4}{\theta}$ are given.

In the set $\mathscr{H}$ of all submanifolds of $\mathscr{P}$ which satisfy condition (1.3) we define the structure of an inductive differential manifold (cf. [26]). By virtue of the results obtained in [26] the 5-form $\overset{5}{\omega}$ on $\mathscr{P}$ defines a closed differential 2-form Ω on $\mathscr{H}$.

If we take an ADMW coordinate system connected with a spacelike surface σ in space-time M (cf. [3], [41]), then we obtain a "diagonal" expression for Ω in terms of infinitesimal translations δg_{ij}, $\delta \pi^{ij}$, where g_{ij} is the metric tensor on σ and π^{ij} are expressed by its second fundamental form K_{ij} by the formula:

$$\pi^{ij} = -\sqrt{\bar{g}}(K_{pq} - g_{pq} K_{ab} \bar{g}^{ab}) \bar{g}^{ip} \bar{g}^{jq}. \tag{1.4}$$

The diagonal expression for Ω corresponds to the Darboux theorem for finite-dimensional symplectic manifolds. This enables us to find the degeneracy distribution of Ω. In Section 9 we prove that this distribution is completely determined by an action of the diffeomorphism group of space-time in the space of states $\mathscr{H}$. If we divide the space $\mathscr{H}$ by this action, we obtain the superphase space $\tilde{\mathscr{H}}$ for General Relativity. In this way we obtain the space $\tilde{\mathscr{H}}$ endowed with the non-degenerate 2-form $\tilde{\Omega}$ which was proposed earlier by A. Fischer and J. Marsden [17] as a superphase space for Einstein dynamics.

In Section 11 we discuss a Lie algebra structure defined by the form Ω. The degeneracy of Ω imposes restrictions on the set of functionals to be considered. A smooth functional $F\colon \mathscr{H} \to \boldsymbol{R}$ is called a *physical quantity*

if there exists a smooth vector field $\hat{Y}$ in $\mathscr{H}$ such that

$$dF(\hat{X}) = -\Omega(\hat{Y}, \hat{X}), \quad \hat{X} \in T_C(\mathscr{H}). \tag{1.5}$$

If W is the degeneracy (gauge) distribution of Ω, then (1.5) implies that F is constant in directions belonging to W. We say that physical quantities are independent of gauge. The distribution W is involutive and therefore $T(\tilde{\mathscr{H}}) = T(\mathscr{H})/W$.

The multisymplectic approach to gravidynamics gives an elegant formulation of the Hilbert–Palatini action principle. We discuss this problem in Section 12. Sections 13 and 14 are devoted to the theory of the Hamilton–Jacobi equation in lagrangian field theories and in General Relativity. The results of Section 14 generalize those obtained in [33] and [41].

The theory of multiphase spaces which is the background of our approach is based on the geometric theory of the calculus of variations (cf. [10], [25], [22], [37]). Therefore we give in Section 3 a brief survey of this theory. As an example of a Lagrangian theory we consider electrodynamics. It turns out that from our point of view electrodynamics and gravidynamics are similar theories.

All the proofs are presented in Section 15. Of course, it is impossible to give full calculations: we present only the final results.

It would be interesting to give a detailed comparison of our theory with the results contained in [3], [11], [12], [41] and especially with the recent results of Fadeev, Popov and Kuchar [14], [15], [29] and Fischer–Marsden [17]. We hope to do that elsewhere.

The principal results of this paper were obtained in Trieste (spring–summer–autumn 1974), where the present author was a visiting scientist at the University of Trieste and at the International Centre for Theoretical Physics. The work was supported financially by the Consiglio Nazionale delle Ricerche–Roma. The author would like to express his thanks to these institutions for their hospitality and the excellent conditions for research.

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2. Notation and preliminary remarks

In the present paper we consider problems of finite and infinite-dimensional differential geometry and therefore the reader should be familiar with the basic concepts of these theories, e.g [27], [31], [36].

We consider only objects of the class C^∞ but a generalization for C^k classes can also be given.

Let Q_1, Q_2 be differential manifolds and let f be a smooth mapping from Q_1 to Q_2. For every $q_1 \in Q_1$ we have a linear mapping between the tangent spaces

$$f_*\colon T_{q_1}(Q_1) \to T_{f(q_1)}(Q_2) \tag{2.1}$$

and a linear mapping

$$f^*\colon T^*_{f(q_1)}(Q_2) \to T^*_{q_1}(Q_1). \tag{2.2}$$

The mapping (2.2) generates a linear mapping between spaces of differential forms on Q_2 and Q_1, respectively:

$$f^*\colon C^\infty\big(T^*(Q_2)\big) \to C^\infty\big(T^*(Q_1)\big). \tag{2.2'}$$

Let $\pi\colon W \to B$ be a bundle with local coordinates (x^j, u^k), where (x^j) are local coordinates in B. In the space $T(W)$ we have a singled-out class of π-vertical vectors, i.e., vectors belonging to the kernel of π_*. We have also the dual notion of π-horizontal covectors in W, i.e., the subspace of T^*W consisting of covectors which anihilate elements of π-ver TW.

For a manifold M with local coordinates (x^i) we denote a tangent vector (vector field) by X (or $\boldsymbol{X}$). In local coordinates we have

$$X = \sum_{i=1}^{n} \delta x^i \frac{\partial}{\partial x^i}. \tag{2.3}$$

Let N be a submanifold of M and let $i\colon N \to M$ be the canonical injection. A vector field X tangent to M at points $i(x) \in M$, $x \in N$, is called a *vector field on* N. If, additionally, there exists a vector field X' in N such that $i_* X' = X$, then we call X a vector field *in* N.

We have similar definitions for covectors and differential forms.

In local coordinates of a special type we have:

$$N = \{x \in M\colon x^i = 0,\ n < i \leqslant m\}_{\substack{n=\dim N\\ m=\dim M,}} \tag{2.4}$$

and a vector field in N has the form

$$X = \sum_{i=1}^{n} \delta x^i \frac{\partial}{\partial x^i} \tag{2.5}$$

and a differential form in N does not contain differentials dx^i, $n < i \leqslant m$.

For a differential form α on M we denote by $\alpha|N$ its pull-back to N

$$\alpha|N = i^*\alpha. \tag{2.6}$$

The symbols $\wedge$, $\lrcorner$ denote operations of exterior and interior products, for instance

$$(2.7)\qquad \langle X_1\wedge \ldots \wedge X_p|\ X_0 \lrcorner \overset{p+1}{a}\rangle = \langle X_0\wedge X_1\wedge \ldots \wedge X_p|\ \overset{p+1}{a}\rangle,$$
$$X_i\in T_x(M),\ \overset{p+1}{a}\in \overset{p+1}{\bigwedge} T_x^*(M).$$

These definitions are the same as in [36]. In the above notation we have the formula for Lie derivative:

$$(2.8)\qquad \mathscr{L}_X \overset{p+1}{a} = d(X \lrcorner \overset{p+1}{a}) + X \lrcorner d\overset{p+1}{a}.$$

In Section 6 we consider a four-dimensional manifold M with local coordinates $(x^\mu)_{\mu=0}^3$ and a 3-dimensional submanifold $\sigma\subset M$. A metric $\boldsymbol{g} = (g_{\mu\nu})$ on M generates by its pull-back a metric on σ. Local coordinates in which $\sigma = \{x\in M\colon x^0 = 0\}$ generate local coordinates $(x^k)_{k=1}^3$ in σ. We denote by $\bar{g}_{ij}$ the components of the metric tensor on σ. It is obvious that $\bar{g}_{ij} = g_{ij}$. By $\bar{g}^{ij}$ we denote the matrix inverse to g_{ij} and by $\bar{\Gamma}_{ij}^k$, $\bar{R}_{ijk}^p$, $\bar{R}_{ij}$, $\bar{R}$ the coefficients of the Riemannian connection, the Riemann tensor, the Ricci tensor and the scalar curvature corresponding to the metric $\bar{g}_{ij}$ on σ. In papers [3], [41] these objects are denoted by ${}^{(3)}g_{ij}$, ${}^{(3)}g^{ij}$, ${}^{(3)}\Gamma_{ij}^k$, ${}^{(3)}R_{ijk}^p$, ..., respectively; we prefer our notation for technical reasons.

Generally we prefer the classical coordinate notation instead of the modern coordinate free approach. It is more useful for calculation. Nevertheless all quantities in question have a clear geometric sense. The notation in the present paper is generally the same as in papers [24], [26], [37] but we denote the multisymplectic $(n+1)$-form by $\overset{n+1}{\omega}$ ($\overset{n+1}{\gamma}$ in the previous papers) its primitive by $\overset{n}{\theta}$ ($\overset{n}{\omega}$ previously) and the presymplectic 2-form by Ω (instead of Γ in [26]). We also use the Einstein summation convention (for Greek indices 0, 1, 2, 3 and for Latin indices 1, 2, 3).

3. A geometrical approach to the calculus of variations

Connections between variational principles and differential equations for physical systems were already known in the 19th century. The works of Lagrange, Legendre, Poisson, Hamilton and others gave the mathematical structure of mechanics. Those classical results allow us to construct a Lie algebra structure in the set of all smooth functions on the phase space. This structure is the starting point for a quantization of mechanical systems [21], [28], [30], [35]. However, direct generaliz-

ations of those results do not allow us to achieve similar results for field theories. One of the known difficulties is the absence of a singled-out coordinate in the space-time which could play the role of time t in mechanics. Therefore the first task before a generalization of such classical notions as the Legendre transformation and the Poisson bracket is to express mechanics in an appropriate form. Such a formulation was found in the fifties in the language of modern differential geometry. The main concept in that formulation is the phase space of a system, i.e., the cotangent bundle $P = T^*V$ of an n-dimensional manifold V (V is the configuration space of the given system). It is known, [1], [24], [35], that P has a natural symplectic structure, i.e., there exists in P a non-degenerate, closed differential 2-form Ω. If $(q^i)_{i=1}^n$ are local coordinates in V and (q^i, p_j) are local coordinates in P, then

$$\Omega = \sum_{i=1}^{n} dp_i \wedge dq^i. \tag{3.1}$$

It is easy to see that

$$\Omega = d\theta, \quad \text{where} \quad \theta = \sum_{i=1}^{n} p_i dq^i, \tag{3.2}$$

is the canonical 1-form on P. The non-degeneracy of Ω provides an isomorphism between the tangent and the cotangent bundle of P given by the formula:

$$T_p(P) \ni X \to X^\flat \in T_p^*(P), \quad \text{where} \tag{3.3}$$
$$X^\flat(Y) = \langle Y, X^\flat \rangle = -\Omega(X, Y); \; Y \in T_p(P).$$

We denote by $\#$ the inverse operator:

$$T_p^*(P) \ni \alpha \to \alpha^\# \in T_p(P). \tag{3.4}$$

The vector space $C^\infty(P)$ of all smooth functions on P has a Lie algebra structure. The Lie bracket is defined by

$$C^\infty(P) \times C^\infty(P) \ni (f_1, f_2) \to \{f_1, f_2\} = \Omega\big((df_1)^\#, (df_2)^\#\big) \in C^\infty(P). \tag{3.5}$$

In local coordinates we have the classical formula

$$\{f_1, f_2\} = \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q^i} - \frac{\partial f_2}{\partial p_i} \frac{\partial f_1}{\partial q^i} \right). \tag{3.6}$$

For our purposes it is more convenient to consider the homogeneous formulation of mechanics including time among dynamical variables. Let $\mathscr{P}$ be a submanifold of codimension one, of the manifold $T^*(V \times \boldsymbol{R})$

which is given by a constraint equation $H = H(p_i, q^j, t)$, where $-H$ is the coordinate conjugate to a coordinate t in $\boldsymbol{R}$. The canonical 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i - dH \wedge dt \tag{3.7}$$

in $T^*(V \times \boldsymbol{R})$ generates a closed, differentiable 2-form in $\mathscr{P}$. We denote it also by ω. We have

$$\omega = d\theta_{\text{hom}} = d\Big(\sum_{i=1}^{n} p_i dq^i - H(p_i, q^j, t)dt\Big). \tag{3.8}$$

The form ω in $\mathscr{P}$ is degenerate and we define the singular distribution of ω, i.e., such a subspace $Z_p \subset T_p(V \times \boldsymbol{R})$ that,

$$\text{for every } Y \in T_p(V \times \boldsymbol{R}) \text{ and } X \in Z_p,\ \omega(X, Y) = 0. \tag{3.9}$$

DEFINITION. A 1-dimensional submanifold $C \subset \mathscr{P}$ is called *ω-singular* if, for every $p \in C$, $T_p(C) \subset Z_p$.

It is easy to see that an ω-singular submanifold must be transversal with respect to the projection $\varrho\colon V \times \boldsymbol{R} \to \boldsymbol{R}$. Therefore we can consider only those submanifolds which are sections of ϱ. The above definition of the ω-singularity can be written in an equivalent form: A submanifold $C \subset \mathscr{P}$ is called ω-singular if for every vector field X on C (tangent to $\mathscr{P}$)

$$(X \lrcorner\, \omega)|C = 0. \tag{3.10}$$

In local coordinates C can be described (locally) by a parametrization

$$C = \{(p_i(t), q^j(t), t) \in \mathscr{P}\colon\ t \in \boldsymbol{R}\}.$$

Equation (3.10) reads:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{3.11}$$

The Hamilton equations (3.11) are equivalent to the Euler–Lagrange equations

$$\dot{q}^i = \frac{dq^i}{dt}, \quad \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \tag{3.12}$$

where $(q^i, \dot{q}^j)$ are local coordinates in $T(V)$ and L is a function on $T(V) \times \boldsymbol{R}$ such that the matrix $\left[\dfrac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right]$ is invertible.

The functions L and H are connected by the Legendre transformation:

$$(3.13)\qquad \begin{aligned} p_i &= \frac{\partial L(q^i, \dot{q}^j, t)}{\partial \dot{q}^i}, \\ H(p_i, q^j, t) &= p_i \dot{q}^i(q^k, p_k, t) - L\big(q^i, \dot{q}^j(q^k, p_k, t), t\big). \end{aligned}$$

Equations (3.12) determine extremals of the variational problem

$$\delta S(t_0, t_1) = \delta \int_{t_0}^{t_1} L\big(q^i(t), \dot{q}^j(t), t\big)\, dt = 0$$

with a fixed boundary

$$(3.14)\qquad \delta q^i(t_0) = \delta q^i(t_1) = 0.$$

Instead of considering the variational problem with a fixed boundary (3.14), we consider the variational problem

$$\delta S(t_0, t_1) = \delta \int_{t_0}^{t_1} \big(p_i dq^i - H\big(p_j(t), q^i(t), t\big)\, dt\big) = 0$$

with (q^i, p_j) as independent variables, and with boundary conditions

$$(3.15)\qquad \delta q^i(t_0) = \delta q^i(t_1) = 0, \qquad \delta p_j(t_0) \text{ and } \delta p_j(t_1) \text{ are arbitrary}.$$

The variational principle (3.15) gives the Hamilton equations (3.11) for extremals. We can write it also in the form

$$\delta S(t_0, t_1) = \delta \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \dot{q}^i} dq^i - \left(\frac{\partial L}{\partial \dot{q}^j} \dot{q}^j - L\right) dt\right) = 0$$

with (q^i, p_j) as independent variables and the boundary conditions

$$(3.16)\qquad \delta q^i(t_0) = \delta q^i(t_1) = 0.$$

Let

$$(3.17)\qquad \psi = \frac{\partial L}{\partial \dot{q}^i} dq^i - \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L\right) dt$$

be a differential 1-form on $T(V) \times \boldsymbol{R}$. It is obvious that 1-dimensional, $d\psi$-singular submanifolds of $T(V) \times \boldsymbol{R}$ are in a one-to-one correspondence with the solutions of system (3.12). The formulation (3.16) of the variational problem turns out to be the best for a generalization in several dimensions. Instead of the manifold $T(V) \times \boldsymbol{R}$ we have in that case an appropriate Grassmannian bundle with a differential n-form ψ.

A geometric formulation of the calculus of variations for multiple integrals was made by Caratheodory, Cartan, Weyl, De Donder, Lepage. The modern version of those results was presented by Dedecker [10]

in 1953. However, the most interesting case is where the manifold W under consideration (which in mechanics is $V \times \boldsymbol{R}$) has a bundle structure over an n-dimensional basis M. This assumption allows us to simplify Dedecker's theory. The further results of the present section are based on the present author's paper [37]. They permit the understanding of the axiomatic approach to multiphase spaces, which we present in Section 4. An equivalent approach to the problems considered below can also be found in paper [22], which contains the same results in the language of differential forms with values in vector bundles.

Let W be an r-dimensional, smooth bundle over an n-dimensional, smooth orientable manifold M $(n < r)$. Let $K^n(W)$ be the bundle of all simple, non-zero n-vectors tangent to W. Let ω_M be an n-form giving an orientation of M. By ν_M we denote the dual field of n-vectors tangent to M, i.e., $\langle \nu_M | \omega_M \rangle = 1$.

DEFINITION. A *lagrangian function* $\mathscr{L}$ is a positive homogeneous function on $K^n(W)$, i.e., $\mathscr{L}(\lambda v) = \lambda \mathscr{L}(v)$; $v \in K^n(W)$, $\lambda \in \boldsymbol{R}_+$. We assume that $\mathscr{L}$ is at least of the class C^2. Let B be a compact submanifold (with a boundary) of M. For every smooth imbedding $f\colon B \to W$ (i.e., if f is a diffeomorphism B onto $f(B)$) we define the action integral

$$S_f = \int_B \mathscr{L}(f_* \nu_M)\, \omega_M \tag{3.18}$$

where f_* is the tangent map to f.

It follows from the positive homogeneity of $\mathscr{L}$ that S_f is independent of the choice of the form ω_M in a given orientation class of M. It can also be proved that S_f depends only on a submanifold $f(B)$ and does not depend on a particular imbedding f.

Let $\pi\text{-tr}\, K^n(W)$ be the subbundle of $K^n(W)$ consisting of all π-transversal, simple, non-zero n-vectors tangent to W (π is the projection $W \to M$, $v \in K^n(W)$ is transversal if $\pi_* v \neq 0$). Let $w \in W$ and $v \in \pi\text{-tr}\, K^n_w(W)$. The submanifold $\pi\text{-tr}\, K^n_w(W)$, i.e., the fibre of $\pi\text{-tr}\, K^n(W)$ over the point $w \in W$, is $\big(n(r-n)+1\big)$-dimensional. Its tangent space at a point v is isomorphic to the subspace $A(v) \subset \bigwedge^n T_w(W)$ spanned by the following n-vectors:

$$v_1 \wedge \ldots \wedge v_n; \quad v_1 \wedge \ldots \wedge \underbrace{v_k}_{m} \wedge \ldots \wedge v_n, \qquad 1 \leqslant m \leqslant n,\ n+1 \leqslant k \leqslant r, \tag{3.19}$$

where $v = v_1 \wedge \ldots \wedge v_n$ and $(v_k)_{k=n+1}^r$ are arbitrary linear independent π-vertical vectors tangent to W at w.

This definition of the subspace $A(v)$ is independent of the representation of v in the form $v = v_1 \wedge \ldots \wedge v_n$ and does not depend on the choice

of π-vertical vectors $(v_k)_{k=n+1}^r$. We have the decomposition

$$\bigwedge^n T_w(W) = A(v) \oplus A_2 \tag{3.20}$$

where A_2 is the subspace of all at least 2-vertical n-vectors and is spanned by all n-vectors of the form

$$v_1 \wedge \ldots \wedge \underbrace{v_{k_1}}_{j_1} \wedge \ldots \wedge \underbrace{v_{k_s}}_{j_s} \wedge \ldots \wedge v_n; \tag{3.21}$$

$$1 \leqslant j_1 < j_2 < \ldots < j_s \leqslant n, \; n+1 \leqslant k_1 < k_2 < \ldots < k_s \leqslant r; \; s \geqslant 2.$$

The subspace A_2 does not depend on the n-vector v or on the choice of π-vertical vectors $(v_k)_{k=n+1}^r$.

For every $v \in K^n(W)$ we have a uniquely determined element $\bar{v} \in G^n(W)$ where τ: $G^n(W) \to W$ is the Grassmannian bundle of all oriented n-planes tangent to W. Let π_1: $\pi\text{-tr}K^n(W) \to W$ and π_2: $\pi\text{-tr}K^n(W) \to \pi\text{-tr}G^n(W)$,

$$\pi_2(v) = \bar{v}. \tag{3.22}$$

For every section f: $M \to W$ we define the lift of f to $\pi\text{-tr}G^n(W)$

$$\bar{f} = (\pi_2 \circ f_*)\nu_M. \tag{3.23}$$

Of course, $\bar{f}$ does not depend on the choice of ν_M. We have

$$f = \tau \circ \bar{f}. \tag{3.24}$$

A section ξ: $M \to \pi\text{-tr}G^n(W)$ is called *integrable* if

$$\overline{\tau \circ \xi} = \xi. \tag{3.25}$$

The cotangent space of the manifold $\pi\text{-tr}K_w^n(W)$ at a point v is isomorphic to the subspace of one-vertical n-covectors 1-ver$\bigwedge^n T_w^*(W)$ which is generated by:

$$v^{*1} \wedge \ldots \wedge v^{*n}; \quad v^{*1} \wedge \ldots \wedge \underbrace{v^{*k}}_{j} \wedge \ldots \wedge v^{*n}, \quad 1 \leqslant j \leqslant n, \; n+1 \leqslant k \leqslant r, \tag{3.26}$$

where the covectors $(v^{*p})_{p=1}^r$ form the dual basis to the basis $(v_p)_{p=1}^r$ of $T_w(W)$. This definition of the subspace 1-ver$\bigwedge^n T_w^*(W)$ does not depend on v or on the choice of a basis $(v_k)_{k=n+1}^r$ in the fibre $W_{\pi(w)}$. It is easy to see that the covectors $(v^{*p})_{p=1}^n$ are π-horizontal.

We consider the map

$$\pi\text{-tr}K_w^n(W) \ni v \to \mathscr{L}(v) \in \boldsymbol{R}. \tag{3.27}$$

Its derivative is called the *fibre derivative* of $\mathscr{L}$ and we denote it by $\mathscr{L}'_{\text{ver}}$. For every $v \in \pi\text{-tr}K_w^n(W)$, $\mathscr{L}'_{\text{ver}}(v) \in$ 1-ver$\bigwedge^n T_w^*(W)$. The following formula

holds:

$$\mathscr{L}'_{\mathrm{ver}}(v) = \mathscr{L}(v)v^{*1}\wedge \ldots \wedge v^{*n} + \sum_{\substack{1\leqslant m\leqslant n\\ n+1\leqslant k\leqslant r}} B^m_k v^{*1}\wedge \ldots \wedge \underbrace{v^{*k}}_{m}\wedge \ldots \wedge v^{*n} \tag{3.28}$$

where

$$\begin{aligned} B^m_k &= \frac{d}{dt}\mathscr{L}\left(v_1\wedge \ldots \wedge (v_m+tv_k)\wedge \ldots \wedge v_n\right)\Big|_{t=0}\\ &= \langle v_1\wedge \ldots \wedge \underbrace{v_k}_{m}\wedge \ldots \wedge v_n|\mathscr{L}'_{\mathrm{ver}}(v)\rangle. \end{aligned}$$

The positive homogeneity of $\mathscr{L}$ implies the following property:

$$\mathscr{L}'_{\mathrm{ver}}(\lambda v) = \mathscr{L}'_{\mathrm{ver}}(v), \qquad \lambda\in \boldsymbol{R}_+. \tag{3.29}$$

By virtue of (3.29) we can define the map

$$\pi\text{-}\mathrm{tr}G^n_w(W)\ni \bar{v}\to\overline{\mathscr{L}'}_{\mathrm{ver}}(\bar{v}) = \mathscr{L}'_{\mathrm{ver}}(v)\in 1\text{-}\mathrm{ver}\bigwedge^n T^*_w(W). \tag{3.30}$$

We define a τ-horizontal n-form on π-$\mathrm{tr}G^n(W)$ by

$$\psi(\cdot) = \tau^*\circ\overline{\mathscr{L}'}_{\mathrm{ver}}(\cdot). \tag{3.31}$$

The form ψ plays an essential role in our approach. It has the following property: for every section $f\colon M\to W$,

$$(\bar{f})^*\psi = \mathscr{L}(f_*\nu_M)\,\omega_M, \tag{3.32}$$

where $\bar{f}$ is the lift of f to the Grassmannian bundle defined by (3.23).

In local coordinates (x^j, u^k) in W and local coordinates (x^j, u^k, γ^k_j) in π-$\mathrm{tr}G^n(W)$ we have

$$v_s = a^j_s\frac{\partial}{\partial x^j} + \beta^k_s\frac{\partial}{\partial u^k}, \qquad 1\leqslant s\leqslant n, \qquad \det[a^j_s]\neq 0, \tag{3.33}$$

$$v = v_1\wedge \ldots \wedge v_n = \det[a^j_s]\left(\frac{\partial}{\partial x^1}+\gamma^k_1\frac{\partial}{\partial u^k}\right)\wedge \ldots \wedge\left(\frac{\partial}{\partial x^n}+\gamma^k_n\frac{\partial}{\partial u^k}\right), \tag{3.34}$$

where

$$\gamma^k_j = \overset{-1}{a}{}^m_j\beta^k_m \tag{3.35}$$

are local coordinates in a fibre of the bundle $\tau\colon \pi\text{-}\mathrm{tr}G^n(W)\to W$. A lagrangian function

$$\mathscr{L}(v) = \mathscr{L}(x^j, u^k, \gamma^k_j, \det[a]), \tag{3.36}$$

where the right side of (3.36) is positive homogeneous with respect to the last variable. We have also

$$v_p = \eta^k_p\frac{\partial}{\partial u^k}, \qquad n+1\leqslant p\leqslant r, \qquad \det[\eta^k_p]\neq 0, \tag{3.37}$$

$$v^{*j} = \overset{-1}{a}{}^j_s dx^s,\ 1\leqslant j\leqslant n; \qquad v^{*p} = \overset{-1}{\eta}{}^p_k du^k - \overset{-1}{\eta}{}^p_k\gamma^k_s dx^s,\ n+1\leqslant p\leqslant r. \tag{3.38}$$

We define a local lagrangian density:

$$\overline{\mathscr{L}}(x^j, u^k, \gamma_j^k) = \mathscr{L}(x^j, u^k, \gamma_j^k, 1). \tag{3.39}$$

In these coordinates we have

$$\begin{aligned}\psi(x^j, u^k, \gamma_j^k) = {} & \frac{\partial \overline{\mathscr{L}}(x^j, u^k, \gamma_j^k)}{\partial \gamma_s^p} dx^1 \wedge \ldots \wedge \underbrace{du^p}_{s} \wedge \ldots \wedge dx^n - \\ & - \left(\frac{\partial \overline{\mathscr{L}}(x^j, u^k, \gamma_j^k)}{\partial \gamma_s^p} \gamma_s^p - \overline{\mathscr{L}}(x^j, u^k, \gamma_j^k) \right) dx^1 \wedge \ldots \wedge dx^n.\end{aligned} \tag{3.40}$$

EXAMPLE. *Non-linear electrodynamics.* The manifold M is a 4-dimensional pseudo-Riemannian manifold with a metric tensor $(g_{\mu\nu})$ and the Riemannian connection $\Gamma^\lambda_{\mu\nu}$. Let $W = T^*M$ with local coordinates (x^μ, A_ν). Let $v \in \pi\text{-tr}K^4(W)$,

$$v = v_0 \wedge \ldots \wedge v_3, \quad \text{where} \quad v = a^\nu_\mu \frac{\partial}{\partial x^\nu} + b_{\mu\nu} \frac{\partial}{\partial A_\nu}, \quad \det[a^\nu_\mu] \neq 0. \tag{3.41}$$

We have local coordinates $(x^\mu, A_\nu, \gamma_{\mu\nu}, \lambda)$, where $\lambda = \det[a^\nu_\mu]$, in the bundle $\pi\text{-tr}K^4(W)$ and local coordinates $(x^\mu, A_\nu, \gamma_{\mu\nu})$ in the bundle $\pi\text{-tr}G^4(W)$,

$$\gamma_{\mu\nu} = \overset{-1}{a}{}^\lambda_\mu b_{\lambda\nu}. \tag{3.42}$$

Let

$$\begin{gathered} f_{\mu\nu} = \gamma_{\mu\nu} - \gamma_{\nu\mu}; \quad \check{f}_{\mu\nu} = \frac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\tau\varrho} f^{\tau\varrho}; \\ (f|f) = f_{\mu\nu} f^{\mu\nu}, \quad (f|\check{f}) = f_{\mu\nu} \check{f}^{\mu\nu}. \end{gathered} \tag{3.43}$$

In the above notation we define a lagrangian function

$$\mathscr{L}(v) = \det[a^\nu_\mu] \sqrt{-g} \left(\mathscr{L}_0\big((f|f), (f|\check{f})\big) + \mathscr{L}_{\mathrm{I}}(x^\mu, A_\nu) \right), \tag{3.44}$$

where $\mathscr{L}_0$ is an arbitrary function of two real variables. For a section $F\colon M \to W$, $F(x^\mu) = \big(x^\mu, A_\nu(x^\mu)\big)$ we have

$$\mathscr{L}(F_* \nu_M)\omega_M = \sqrt{-g} \left(\mathscr{L}_0\big((f|f), (f|\check{f})\big) + \mathscr{L}_{\mathrm{I}}(x^\mu, A_\nu) \right), \tag{3.45}$$

where

$$\omega_M = \sqrt{-g}\, dx^0 \wedge \ldots \wedge dx^3; \quad f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Formula (3.45) is the well-known expression for the lagrangian density in electrodynamics (cf. [5]). In the most important case of linear (Maxwell) electrodynamics we have

$$\mathscr{L}_0 = -\frac{1}{16\pi} f_{\mu\nu} f^{\mu\nu}; \quad \mathscr{L}_{\mathrm{I}} = j^\nu(x^\lambda) A_\nu, \quad \text{where} \quad \nabla_\nu j^\nu(x^\lambda) = 0. \tag{3.46}$$

By the general formula (3.40) we obtain

$$(3.47)\qquad \psi(x, A, \gamma_{\mu\nu}) = \frac{\partial \mathscr{L}_0}{\partial f_{\mu\nu}}\sqrt{-g}\,dx^0\wedge \ldots \wedge \underbrace{dA_\nu}_{\mu}\wedge \ldots \wedge dx^3 + \\ + \left(-\frac{1}{2}\frac{\partial \mathscr{L}_0}{\partial f_{\mu\nu}}f_{\mu\nu}+\mathscr{L}_0+\mathscr{L}_{\mathrm{I}}\right)\sqrt{-g}\,dx^0\wedge \ldots \wedge dx^3.$$

In the Maxwell case:

$$(3.48)\qquad \psi(x^\lambda, A_\nu, \gamma_{\mu\nu}) = -\frac{1}{4\pi}f^{\mu\nu}\sqrt{-g}\;dx^0\wedge \ldots \wedge \underbrace{dA_\nu}_{\mu}\wedge \ldots \wedge dx^3 + \\ + \left(\frac{1}{16\pi}f_{\mu\nu}f^{\mu\nu}+j^\nu(x^\lambda)A_\nu\right)\sqrt{-g}\;dx^0\wedge \ldots \wedge dx^3.$$

We now return to the general theory of the variational problem with a fixed boundary. Let $D \subset M$ be a compact n-dimensional submanifold with a boundary and let B be an $(n-1)$-dimensional submanifold of W such that π is a diffeomorphism of B onto ∂D. We consider sections of W such that $f(\partial D) \subset B$. The set of such sections we denote $C^\infty(D, W; \partial D, B)$. The classical problem with a fixed boundary is to find an extremal of the action functional

$$(3.49)\qquad C^\infty(D, W; \partial D, B) \ni f \to S_D(f) = \int_D \mathscr{L}(f_* \nu_M)\,\omega_M \in \boldsymbol{R}.$$

By virtue of (3.32) the action functional (3.49) can be expressed by the formula

$$(3.50)\qquad C^\infty(D, W; \partial D, B) \ni f \to S_D(f) = \int_{\bar{f}(D)} \psi \in \boldsymbol{R}.$$

The main idea of Dedecker's theory consists in considering (3.50) instead of (3.49) and in an extension of the set of the sections which are under consideration. Let $C^\infty\big(D, \pi\text{-tr}G^n(W); \partial D, \tau^{-1}(B)\big)$ be the set of all sections ξ of the bundle $\pi\text{-tr}G^n(W) \to D$ such that $\tau \circ \xi \in C^\infty(D, W; \partial D, B)$. Of course, these sections are in general non-integrable. The action functional is

$$(3.51)\qquad C^\infty\big(D, \pi\text{-tr}G^n(W); \partial D, \tau^{-1}(B)\big) \ni \xi \to \bar{S}_D(\xi) = \int_{\xi(D)} \psi \in \boldsymbol{R}.$$

PROPOSITION 3.1 (cf. [10], [22], [25], [37]).

(i) *A section* $\xi \in C^\infty\big(D, \pi\text{-tr}G^n(W); \partial D, \tau^{-1}(B)\big)$ *is an extremal of the functional* (3.51) *if and only if for every vector field* X *tangent to* $\pi\text{-tr}G^n(W)$ *and defined on* $\xi(D)$ *we have*

$$(3.52)\qquad (X \lrcorner\, d\psi)|_{\xi(D)} = 0.$$

BU W

(ii) *If ξ is an extremal of* (3.51), *then $\tau \circ \xi$ is an extremal of the functional* (3.49).

(iii) *If the matrix* $\left[\dfrac{\partial^2 \bar{\mathscr{L}}(x^j, u^k, \gamma_j^k)}{\partial \gamma_q^p \partial \gamma_s^m}\right]$ *is non-singular, then the equation* (3.52) *implies the integrability of ξ.*

It follows from Proposition 3.1 that in the case of a non-singular lagrangian, equation (3.52) gives the kinematical equations $\gamma_j^k = \dfrac{\partial u^k}{\partial x^j}$ (the condition of integrability) and dynamical equations. For instance, in electrodynamics we have

$$\gamma_{\mu\nu} = \partial_\mu A_\nu, \qquad \nabla_\mu \frac{\partial \mathscr{L}}{\partial f_{\mu\nu}} = \frac{\partial \mathscr{L}_{\mathrm{I}}}{\partial A_\nu}. \tag{3.53}$$

The geometric form (3.52) of the equations for extremals generalizes equation (3.10). A closer correspondence between (3.52) and the Hamilton formulation of mechanics is seen when we pass from the bundle $\pi\text{-tr}G^n(W)$ to a submanifold of $\overset{n}{\wedge} T^*(W)$. If condition (iii) of Proposition 3.1 is fulfilled, then the map

$$\pi\text{-tr}G^n(W) \ni \bar{v} \to L(\bar{v}) = \bar{\mathscr{L}}'_{\mathrm{ver}}(\bar{v}) \in \overset{n}{\wedge} T^*(W) \tag{3.54}$$

gives (locally) an isomorphism of $\pi\text{-tr}G^n(W)$ with a submanifold of $\overset{n}{\wedge} T^*(W)$. We call the map (3.54) the *Legendre transformation* and its image an *n-phase space* $\mathscr{P}$. Let θ be the canonical n-form on $\overset{n}{\wedge} T^*(W)$, i.e., for $z \in \overset{n}{\wedge} T^*(W)$, let

$$\theta_z = \sigma^* z \tag{3.55}$$

where $\sigma\colon \overset{n}{\wedge} T^*(W) \to W$ is the projection. We denote the pull-back of θ into $\mathscr{P}$ also by θ. It can be proved that $L^*\theta = \psi$ and the equations of extremals are: for every vector field X on C

$$(X \lrcorner\, d\theta)|C = 0 \tag{3.56}$$

where C is an n-dimensional submanifold in $\mathscr{P}$.

As an example, let us take non-linear electrodynamics. The 4-phase space $\mathscr{P}$ consist of all 4-covectors in T^*M of the special form

$$\mathscr{P} = \Big\{\nu \in \overset{4}{\wedge} T^*(T^*M)\colon\ \nu(x^\mu, A_\nu) = -\frac{1}{4\pi} h^{\mu\nu}\sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{dA_\nu}_{\mu} \wedge \ldots \wedge dx^3 + H(x^\lambda, A_\nu, h^{\mu\nu})\sqrt{-g}\, dx^0 \wedge \ldots \wedge dx^3\Big\}, \tag{3.57}$$

where

$$h^{\mu\nu} = -4\pi \frac{\partial \mathscr{L}_0}{\partial f_{\mu\nu}};$$

$$H(x^\lambda, A_\mu, h^{\mu\nu}) = \frac{1}{8\pi} h^{\mu\nu} f_{\mu\nu}(h^{\lambda\varrho}) + \mathscr{L}_0\big(f_{\mu\nu}(h^{\lambda\varrho})\big) + \mathscr{L}_1(x^\lambda, A_\nu).$$

$$\theta(x^\lambda, A_\nu, h^{\mu\nu}) = -\frac{1}{4\pi} h^{\mu\nu}\sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{dA_\nu}_{\mu} \wedge \ldots \wedge dx^3 + \tag{3.58}$$
$$+ H(x^\lambda, A_\mu, h^{\mu\nu})\sqrt{-g}\, dx^0 \wedge \ldots \wedge dx^3.$$

The geometric theory of the calculus of variations and especially equation (3.56) have inspired the formulation of an axiomatic theory of multiphase spaces. This formulation is due to Kijowski [24]. We outline it in the next section.

Remark. The image of the Legendre transformation L consists of 1-vertical n-covectors on W (an n-covector $\nu \in \bigwedge^n T^*_w(W)$ is 1-vertical if for every $X_1, X_2 \in \operatorname{ver} T_w(W)$ we have $(X_1 \wedge X_2) \lrcorner\, \nu = 0$). Usually $\operatorname{im} L$ is a submanifold of codimension one in this space. This is not a case of electrodynamics, where the antisymmetric properties of $h^{\mu\nu}$ imply $\operatorname{codim} \mathscr{P} > 1$.

4. Multisymplectic manifolds and a multiphase structure of a classical field theory

The geometric form (3.52), (3.56) of the equations for extremals in the calculus of variations enables us to formulate an axiomatic approach to a wide class of quasi-linear differential equations of the second order. Let $\pi\colon W \to M$ be a bundle and let $J_k(W)$ be the bundle of k-jets of local sections of W. For $f \in C^\infty(W)$, $j_k(f)$ denotes the k-jet of f. Let $V \to M$ be a vector bundle and let F be a fibre preserving map from $J_2(W)$ to V. We call the system (W, V, M, F) a *differential equation of the second order*. A section $f \in C^\infty(W)$ is its solution if

$$F \circ j_2(f) = 0 \quad \text{(the zero section of } V). \tag{4.1}$$

It is interesting to note that for a comprehensive class of differential equations of the type (4.1) the set of solutions is in a one-to-one correspondence with a singled-out subset of submanifolds in $G^n(W)$. We shall now explain this correspondence.

Definition ([24], [38]). An *n-symplectic manifold* is an m-dimensional differentiable manifold $\mathscr{P}$ $(n < m)$ with a closed differential $(n+1)$-form ω in $\mathscr{P}$ (i.e., $d\omega = 0$).

A submanifold $C \in \mathscr{P}$ is called *ω-singular* if for every vector field X tangent to $\mathscr{P}$ and defined on C we have

$$(X \lrcorner\, \omega)|C = 0. \tag{4.2}$$

In the following we shall consider maximal ω-singular submanifolds of $\mathscr{P}$, i.e., ω-singular submanifolds which are not contained in any other, larger ω-singular submanifold of $\mathscr{P}$.

In lagrangian theories with a Lagrange function $\mathscr{L}$ fulfilling the non-degeneracy condition (condition (iii) of Proposition 3.1) $\mathscr{P}$ is a submanifold of $\bigwedge^n T^*(W)$ given as the image of the Legendre transformation (3.54). The form $\omega = d\theta$, where θ is the canonical n-form in $\mathscr{P}$ defined by (3.55). ω is obviously closed.

DEFINITION. We say that a multisymplectic manifold $(\mathscr{P}, \omega)$ determines a *multiphase structure* for the differential equation (W, V, M, F) of the solutions of this equation are in a one-to-one correspondence with the ω-singular, maximal submanifolds of $\mathscr{P}$.

The set $\mathscr{H}$ of all ω-singular, maximal submanifolds of $\mathscr{P}$ is called the *space of states* of a given field theory and its elements are called *states*. We assume also a regularity condition: all ω-singular, maximal submanifolds in $\mathscr{P}$ are of the same dimension $\geqslant n$. The notion of a multisymplectic manifold has been introduced by W. Tulczyjew and J. Kijowski [38], [24]. The results of Kijowski [24] show that this approach is an appropriate tool for understanding the structure of classical field theories. By a classical field theory we mean a quasi-linear differential equation of the second order. It turns out that almost all theories important from the physical point of view have multiphase structures. In Section 5 we show that gravidynamics, i.e., the General Theory of Relativity has a multiphase structure. This fact proves that the multisymplectic approach is more general than the lagrangian approach. From the multiphase point of view gravidynamics does not differ from electrodynamics, although there exists no Lagrangian for the General Relativity depending on first derivatives of a metric tensor. In [20], [24], [26] it has been shown that many difficulties of the canonical formalism for classical field theories can be solved if we take as the background a multiphase structure of a given field theory instead of a Lagrangian or a Hamilton function. In particular, the multisymplectic approach enables us to endow the space $\mathscr{H}$ with a presymplectic structure generated by the multisymplectic form ω in $\mathscr{P}$. This construction in a very general case has been performed in [26]. In the subsequent sections we investigate the presymplectic structure in the space $\mathscr{H}$ for gravidynamics.

Having a multisymplectic manifold $(\mathscr{P}, \omega)$ which gives a multiphase structure of a given field theory, we can ask the following questions:

1. Does there exist a closed differential $(n+1)$-form $\omega' \neq \omega$ in $\mathscr{P}$ such that the couple $(\mathscr{P}, \omega')$ gives an equivalent multiphase structure of the given field theory?

2. Does there exist a multisymplectic manifold $(\mathscr{P}, \omega_1)$ which determines an equivalent multiphase structure of the field theory?

3. In the set of all multisymplectic manifolds generating the same multiphase structure we define the structure of a partially ordered set. We have $(\mathscr{P}_2, \omega_2) \succ (\mathscr{P}_1, \omega_1)$ if there exists a diffeomorphism φ transforming $\mathscr{P}_2$ onto a submanifold of $\mathscr{P}_1$ such that $\omega_2 = \varphi^* \omega_1$.

The following problem arises: Has this partially ordered set maximal elements? We do not know full solutions of the above problems. We summarize here only some results known to us.

1. It seems that the answer to the problem is negative, i.e., only $\omega' = a\omega$, where a is a real constant, gives an equivalent multiphase structure.

2. Let $(\mathscr{P}_1, \omega_1)$ be a multisymplectic manifold. We define the primitive gauge distribution of ω_1:

$$D_p^0 = \{X_p \in T_p(\mathscr{P}) \colon X_p \lrcorner\, \omega_1 = 0\}. \tag{4.3}$$

It has been shown in [24] that D^0 is involutive and therefore we can construct the quotient space $\mathscr{P} = \mathscr{P}_1/D^0$. It is easy to see that the form ω_1 can be projected on $\mathscr{P}$ and we obtain a multisymplectic manifold $(\mathscr{P}, \omega)$. Roughly speaking, the manifold $\mathscr{P}_1$ contains more dimensions than $\mathscr{P}$ but the differential form ω_1 does not contain differentials in those additional dimensions. The couples $(\mathscr{P}_1, \omega_1)$ and $(\mathscr{P}, \omega)$ give the same multiphase structure.

3. The reduction of the primitive gauge for a given multisymplectic manifold gives a majorizing element in the partially ordered set of multisymplectic manifolds which has been defined above. But we do not know how to construct a maximal element.

In the present paper we consider hyperbolic field theories. This means that solutions of the field equations are determined by the initial values of fields and their time-derivatives. In the general case initial values do not determine a solution uniquely. Only in theories without gauge the correspondence between initial values and solutions in one-to-one. A geometric axiomatics of hyperbolicity is due to J. Kijowski [24]. In the multisymplectic manifold $\mathscr{P}$ we distinguish a subset of $(n-1)$-dimensional submanifolds $\mathscr{C}$. The elements of the set $\mathscr{C}$ are called *admissible initial surfaces*. In [24] the following system of axioms has been postulated:

1. For every $p \in \mathscr{P}$ and $C \in \mathscr{H}$ which contains p there exists a $c \in \mathscr{C}$ such that $p \in c \subset C$.

2. If an $(n-1)$-dimensional submanifold $c \subset \mathscr{P}$ has the property that for any $p \in c$ there exists a $c_p \in \mathscr{C}$ such that $c \cap c_p$ is a neighbourhood of p in c, then $c \in \mathscr{C}$.

3. If $C \in \mathscr{H}$ and c_1, c_2 are admissible initial surfaces contained in C, and $K_i \subset c_i$ are compact sets, then there exists a $c \subset C$ such that $K_1 \subset (c_1-(c-c_1))$, $K_2 \subset (c \cap c_2)$ and $(c_1-(c-c_1))$ is a set relatively compact in c_1.

4. For every $c_1, c_2 \in \mathscr{C}$, $c_1, c_2 \subset C$ there exists a homotopy in C joining c_1 and c_2.

5. If $c \in \mathscr{C}$, $c \subset C$ and φ is a diffeomorphism of C, then $\varphi(c) \in \mathscr{C}$.

Remark. In the general case, for $c \in \mathscr{C}$ there exist many states containing c.

In our further applications we shall consider a multiphase space $\mathscr{P}$ which is a bundle over a 4-dimensional pseudo-riemannian manifold M (π: $\mathscr{P} \to M$). The projection π and the set $\mathscr{C}$ generate in M a set of 3-dimensional submanifolds, such that if $c \in \mathscr{C}$, then $\sigma = \pi(c) \subset M$. An admissible initial surface c determines the initial values of fields and their normal derivatives in the submanifold σ.

5. A multiphase structure of General Relativity

For a 4-dimensional smooth manifold M and a metric tensor $g_{\mu\nu}$ on M we define the Riemann tensor

$$R^{\alpha}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\tau\beta}\Gamma^{\tau}_{\mu\nu} - \Gamma^{\alpha}_{\tau\nu}\Gamma^{\tau}_{\mu\beta}, \tag{5.1}$$

the Ricci tensor $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ and the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$, where $g^{\mu\nu}$ is a 2-contravariant tensor on M such that $g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}_{\lambda}$ and $\Gamma^{\alpha}_{\mu\nu}$ are components of the Riemannian connection

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{5.2}$$

Definition. A metric $g_{\mu\nu}$ on M is called an *Einstein metric* if the signature of $g_{\mu\nu}$ is $(-1, +3)$, and

$$R_{\mu\nu} = \lambda g_{\mu\nu}, \quad \text{where } \lambda \text{ is a real constant.} \tag{5.3}$$

In this section we construct a multisymplectic manifold $(\mathscr{P}, \overset{5}{\omega})$ such that $\overset{5}{\omega}$-singular sections of the bundle π: $\mathscr{P} \to M$ are in a one-to-one correspondence with the set of Einstein metrics on M. In the further considerations we fix a real number λ.

Let π_1: $S^2_* T^*M \to M$ be the bundle of symmetric, 2-covariant nondegenerate tensors (metrics) on M with a negative determinant $g = \det g_{\mu\nu}$.

Let π_1-tr$G^4(S^2_* T^* M)$ be the Grassmannian bundle of π_1-transversal planes tangent to $S^2_* T^* M$. If (x^μ) are local coordiantes in M, and $(x^\mu, g_{\mu\nu})$ are local coordinates in $S^2_* T^* M$, then local coordinates in π_1-tr$G^4(S^2_* T^* M)$ are $(x^\mu, g_{\mu\nu}, \gamma_{\lambda\mu\nu})$, where $\gamma_{\lambda\mu\nu} = \gamma_{\lambda\nu\mu}$. We have the following transformation properties for these coordinates:

$$
\begin{aligned}
g_{\mu'\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}, \\
\gamma_{\lambda'\mu'\nu'} &= \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \gamma_{\lambda\mu\nu} + \frac{\partial}{\partial x^{\lambda'}} \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \right) g_{\mu\nu}.
\end{aligned}
\tag{5.4}
$$

For the purposes of General Relativity it is more convenient to consider the bundle of Christoffel symbols (the bundle of the Riemannian connection) which is isomorphic (as a bundle over $S^2_* T^* M$) to the bundle π_1-tr$G^4(S^2_* T^* M)$. Local coordinates in Ch are $(x^\mu, g_{\mu\nu}, \Gamma^\lambda_{\mu\nu})$ with the coordinate transformation law

$$
\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Gamma^\lambda_{\mu\nu} + \frac{\partial^2 x^\sigma}{\partial x^{\mu'} \partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^\sigma}. \tag{5.5}
$$

An isomorphism between Ch and π_1-tr$G^4(S^2_* T^* M)$ is given by the formulas:

$$
\begin{aligned}
\Gamma^\lambda_{\mu\nu} &= \tfrac{1}{2} g^{\lambda\sigma} (\gamma_{\mu\nu\sigma} + \gamma_{\nu\mu\sigma} - \gamma_{\sigma\mu\nu}), \\
\gamma_{\lambda\mu\nu} &= g_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} + g_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}.
\end{aligned}
\tag{5.6}
$$

We would like to emphasize that, in the bundle Ch, $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ are independent variables. In future we shall consider all sections of the bundle Ch$\to M$. For some of them equation (5.2) holds. Those sections are of a special interest, because they correspond to the integrable sections of the bundle π_1-tr$G^4(S^2_* T^* M) \to M$. We define $\mathscr{P} =$ Ch and

$$
\begin{aligned}
\overset{4}{\theta} = {} & g^{\alpha\beta} \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{d\Gamma^\tau_{\alpha\beta}}_{\tau} \wedge \ldots \wedge dx^3 - \\
& - g^{\alpha\tau} \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{d\Gamma^\beta_{\alpha\beta}}_{\tau} \wedge \ldots \wedge dx^3 - \\
& - \left(g^{\nu\varrho} (\Gamma^\mu_{\tau\nu} \Gamma^\tau_{\mu\varrho} - \Gamma^\mu_{\tau\mu} \Gamma^\tau_{\nu\varrho}) + 2\lambda \right) \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,
\end{aligned}
\tag{5.7}
$$

where λ is a fixed real constant.

PROPOSITION 5.1. *Formula* (5.7) *defines a* 4-*form in* $\mathscr{P}$.

Proof. We have to check that (5.7) is covariant with respect to the coordinate transformations (5.4), (5.5).

DEFINITION. $\overset{5}{\omega} = d\overset{4}{\theta}$;

$$
\begin{aligned}
(5.8)\qquad \overset{5}{\omega} = {} & \big(-(g^{\beta\nu}\Gamma^{\mu}_{\beta\lambda}+g^{\beta\mu}\Gamma^{\nu}_{\beta\lambda})+\tfrac{1}{2}(g^{\beta\varrho}\Gamma^{\mu}_{\beta\varrho}\delta^{\nu}_{\lambda}+g^{\beta\varrho}\Gamma^{\nu}_{\beta\varrho}\delta^{\mu}_{\lambda})+g^{\mu\nu}\Gamma^{\beta}_{\lambda\beta}\big)\times \\
& \times\sqrt{-g}\,d\Gamma^{\lambda}_{\mu\nu}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3+ \\
& +\big(\tfrac{1}{2}(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu})\delta^{\varrho}_{\lambda}-\tfrac{1}{4}(g^{\alpha\mu}g^{\beta\varrho}\delta^{\nu}_{\lambda}+g^{\alpha\nu}g^{\beta\varrho}\delta^{\mu}_{\lambda}+g^{\alpha\varrho}g^{\beta\mu}\delta^{\nu}_{\lambda}+g^{\alpha\varrho}g^{\beta\nu}\delta^{\mu}_{\lambda})\big)\times \\
& \times\sqrt{-g}\,d\Gamma^{\lambda}_{\mu\nu}\wedge dx^0\wedge\ldots\wedge\underbrace{dg_{\alpha\beta}}_{\varrho}\wedge\ldots\wedge dx^3+ \\
& +\big(-\tfrac{1}{2}g^{\alpha\beta}g^{\mu\nu}\delta^{\varrho}_{\lambda}+\tfrac{1}{4}g^{\alpha\beta}(g^{\mu\varrho}\delta^{\nu}_{\lambda}+g^{\nu\varrho}\delta^{\mu}_{\lambda})\big)\times \\
& \times\sqrt{-g}\,d\Gamma^{\lambda}_{\mu\nu}\wedge dx^0\wedge\ldots\wedge\underbrace{dg_{\alpha\beta}}_{\varrho}\wedge\ldots\wedge dx^3+ \\
& +\big((\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho}-\Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho})g^{\nu\alpha}g^{\varrho\beta}-\tfrac{1}{2}(\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho}-\Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho})g^{\varrho\nu}g^{\alpha\beta}-\lambda g^{\alpha\beta}\big)\times \\
& \times\sqrt{-g}\,dg_{\alpha\beta}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3.
\end{aligned}
$$

PROPOSITION 5.2. *The form $\overset{5}{\omega}$ is locally an exterior derivative of the form $\overset{4}{\psi}$, where*

(5.9)

$$
\begin{aligned}
\overset{4}{\psi} = {} & \big(g^{\alpha\tau}g^{\beta\zeta}\Gamma^{\varrho}_{\tau\zeta}-\tfrac{1}{2}(g^{\alpha\tau}g^{\beta\varrho}\Gamma^{\lambda}_{\tau\lambda}+g^{\alpha\varrho}g^{\beta\tau}\Gamma^{\lambda}_{\tau\lambda})-\tfrac{1}{2}g^{\alpha\beta}g^{\tau\xi}\Gamma^{\varrho}_{\tau\xi}+\tfrac{1}{2}g^{\alpha\beta}g^{\tau\varrho}\Gamma^{\xi}_{\tau\xi}\big)\times \\
& \times\sqrt{-g}\,dx^0\wedge\ldots\wedge\underbrace{dg_{\alpha\beta}}_{\varrho}\wedge\ldots\wedge dx^3- \\
& -\big(g^{\nu\varrho}(\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho}-\Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho})+2\lambda\big)\sqrt{-g}\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3.
\end{aligned}
$$

Expression (5.9) is not covariant with respect to the coordinate transformations (5.4), and (5.5) and does not determine any geometric object. The above formula is valid only in a one-coordinate chart. Formally (5.9) can be obtained by a general procedure from a Lagrange function $\mathscr{L}$. We know (cf. [2]) that the formula

$$
(5.10)\qquad \overline{\mathscr{L}}(g_{\mu\nu},\Gamma^{\lambda}_{\mu\nu}) = \big((\Gamma^{\mu}_{\tau\nu}\Gamma^{\tau}_{\mu\varrho}-\Gamma^{\mu}_{\tau\mu}\Gamma^{\tau}_{\nu\varrho})g^{\nu\varrho}+2\lambda\big)\sqrt{-g}
$$

gives locally a non-covariant lagrangian density for the Einstein equations. Locally we have (cf. the formula (3.40))

$$
\begin{aligned}
(5.11)\qquad \overset{4}{\psi} = {} & \sum_{\mu\leqslant\nu}\frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu,\lambda}}\,dx^0\wedge\ldots\wedge\underbrace{dg_{\mu\nu}}_{\lambda}\wedge\ldots\wedge dx^3- \\
& -\Big(\sum_{\mu\leqslant\nu}\frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu,\lambda}}g_{\mu\nu,\lambda}-\overline{\mathscr{L}}\Big)dx^0\wedge dx^1\wedge dx^2\wedge dx^3.
\end{aligned}
$$

Let $\varphi\colon M\to\mathscr{P}$ be a global section of the bundle $\pi\colon\mathscr{P}\to M$ such that $C=\varphi(M)$ is a 4-dimensional imbedded submanifold of $\mathscr{P}$.

PROPOSITION 5.3. *A submanifold* $C = \varphi(M)$ *is* $\overset{5}{\omega}$*-singular if and only if the Einstein equations* (5.2) *and* (5.3) *hold.*

PROPOSITION 5.4. *A* $\overset{5}{\omega}$*-singular submanifold* $C \subset \mathscr{P}$ *which is a graph of a section of* π *is a maximal* $\overset{5}{\omega}$*-singular submanifold.*

The proofs of the above propositions consist in purely algebraic calculations. See Section 15.

By virtue of Propositions 5.3 and 5.4 the couple $(\mathscr{P}, \overset{5}{\omega})$ gives a multiphase structure for the Einstein equations.

Proposition 5.2 and formula (5.11) give a connection between the multisymplectic formulations and the classical lagrangian formulation of General Relativity. However, this connection is not exactly the same as in lagrangian theories; the form $\overset{4}{\psi}$ is not defined globally.

The couple $(\mathscr{P}, \overset{5}{\omega})$ is the starting point of our further considerations. The $\overset{5}{\omega}$-singular sections of $\mathscr{P}$ form a prephase space $\mathscr{H}$ for General Relativity. In the subsequent sections we show that the group of diffeomorphisms of the manifold M acts in the space $\mathscr{H}$. The orbits of this action form the phase space for gravidynamics. It has a natural symplectic structure.

Remark 1. For every section $\varphi\colon M \to \mathscr{P}$ fulfilling (5.2)

$$\varphi^* \overset{4}{\theta} = (R-2\lambda)\sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{5.12}$$

Therefore

$$\int_C \overset{4}{\theta} = \int_M (R-2\lambda)\sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{5.13}$$

Formula (5.13) shows that the differential form $\overset{4}{\theta}$ is an appropriate geometrical object for the formulation of the Palatini variational principle in General Relativity. We discuss that problem in Section 12.

Remark 2. Having a multiphase structure for gravidynamics without matter, one can easily find a multisymplectic structure of the interacting gravitational and electromagnetic field. This problem will be considered elsewhere.

6. The Cauchy problem and ADMW coordinates in General Relativity

The Einstein equations (5.2), (5.3) form a system of second order differential equations for components of a metric tensor on the manifold M. It has been shown by A. Lichnerowicz and Y. Bruhat–Choquet that

these equation have a hyperbolic character, and therefore we can consider the initial problem for them. The choice appropriate coordinates in the manifold $\mathscr{P}$ is very important for this discussion. Arnowitt–Deser–Misner and Wheeler [3], [41] have shown that a $3+1$ decomposition of geometrical objects connected with a given space-like surface σ in M provides an elegant and relatively simple description of the Cauchy problem. Recently a profound discussion of the ADMW coordinates has been supplied by A. Fischer and J. Marsden [17].

We present here briefly the ADMW construction.

Let σ be a 3-dimensional surface in M and let $i\colon \sigma \to M$ be the canonical injection. Let $\boldsymbol{g} = (g_{\mu\nu})$ be a metric tensor in M with a signature $(+3, -1)$ such that the metric tensor $\bar{\boldsymbol{g}} = i^*\boldsymbol{g}$ is positively defined. We call σ a *space-like surface* with respect to $\boldsymbol{g}$. Let $\boldsymbol{\tau}$ be a transversal vector field on σ and $\boldsymbol{n} = (n^\mu)$ the unit vector field normal to σ, $(\boldsymbol{n}|\boldsymbol{n}) = g_{\mu\nu}n^\mu n^\nu = -1$. We assume also a consistency of orientation condition

$$(\boldsymbol{n}|\boldsymbol{\tau}) = g_{\mu\nu}n^\mu\tau^\nu < 0. \tag{6.1}$$

We define a vector field

$$\boldsymbol{X} = \boldsymbol{\tau} + \boldsymbol{n}(\boldsymbol{n}|\boldsymbol{\tau}). \tag{6.2}$$

By virtue of $(\boldsymbol{X}|\boldsymbol{n}) = 0$, the vector field $\boldsymbol{X}$ is tangent to σ. Let $\boldsymbol{N}$ be a vector field in σ such that $i_*\boldsymbol{N} = \boldsymbol{X}$, and let

$$N = -(\boldsymbol{n}|\boldsymbol{\tau}) \tag{6.3}$$

be a function on σ.

The vector field $\boldsymbol{\tau}$ defines locally a coordinate system (x^0, x^1, x^2, x^3) in a neighbourhood of σ in M in which $\sigma = \{x \in M\colon\ x^0 = 0\}$ and $\boldsymbol{\tau} = \dfrac{\partial}{\partial x^0}$. If σ is compact, $\boldsymbol{\tau}$ defines (globally) in a neighbourhood of σ such a coordinate system that σ is a zero-section of a line bundle with local coordinates (x^0, x^1, x^2, x^3) having the following transformations properties:

$$x^{0'} = x^0, \quad x^{k'} = x^{k'}(x^k). \tag{6.4}$$

In this case we have a neighbourhood of σ in M which is diffeomorphic with the product $\boldsymbol{R} \times \sigma$. In the coordinate system (6.4)

$$\boldsymbol{\tau} = (1, 0, 0, 0), \quad \boldsymbol{n} = (1, -\bar{g}^{ks}g_{0s})\sqrt{-g^{00}}, \tag{6.5}$$

where $\bar{g}_{ij} = g_{ij}$, $\bar{g}^{ij}$ is the matrix inverse to $\bar{g}_{ij}$ and $g^{00} < 0$. The last inequality follows from the conditions

$$g = \det g_{\mu\nu} < 0, \quad \bar{g} = \det g_{ij} > 0.$$

Let $N_k = g_{0k}$ be a covector field in σ; then

$$N = (-g^{00})^{-1/2}, \quad N^k = \bar{g}^{ks}g_{0s} = \bar{g}^{ks}N_s. \tag{6.6}$$

We have also

$$
(6.7)\qquad
\begin{gathered}
g^{0k} = \frac{N^k}{N^2}, \quad g_{00} = -N^2 + N^k N_k, \quad g^{pq} = \bar{g}^{pq} - \frac{N^p N^q}{N^2}, \\
\sqrt{-g} = N\sqrt{\bar{g}}, \quad \boldsymbol{n} = \left(\frac{1}{N}, -\frac{N^k}{N}\right).
\end{gathered}
$$

According to Wheeler we call N a *lapse function* and $\boldsymbol{N}$ a *shift vector*, cf. [41]. We have seen that a coordinate system (6.4) (in a neighbourhood of σ) determines a scalar function $N > 0$ on σ and a vector field $\boldsymbol{N}$ in σ. Conversely, if we have a positive scalar function N on σ and a vector field $\boldsymbol{N}$ in σ, then $\boldsymbol{\tau} = \boldsymbol{N} + N\boldsymbol{n}$ is a transversal vector field on σ which defines, at least when σ is compact, a coordinate system of type (6.4) in a neighbourhood of σ. However, generally in such a coordinate system submanifolds $x^0 = \text{const}$ are not space-like. If σ is compact, we can find such an $\varepsilon > 0$ that the submanifolds $\sigma_t = \{x \in M\colon x^0 = t\}$, $-\varepsilon < t < \varepsilon$ are space-like. This fails in a non-compact case because small deformations of σ caused by the field $\boldsymbol{\tau}$ do not preserve that property at spatially distant points.

A metric tensor $g_{\mu\nu}$ in M defines a metric tensor $\bar{g}_{ij}$ in σ (the components of $\bar{g}_{ij}$ are equal to the spatial components of $g_{\mu\nu}$) and the second fundamental form of σ by the formula (cf. [27]):

$$
(6.8)\qquad K_{ij} = -g_{i\mu}\nabla_j n^\mu.
$$

It can be proved (see (6.11)) that K_{ij} is a symmetric 2-covariant tensor in σ. Let

$$
\Gamma^k_{ij} = \tfrac{1}{2}\bar{g}^{ka}(\partial_j g_{ia} + \partial_i g_{ja} - \partial_a g_{ij})
$$

be the Riemannian connection corresponding to the metric g_{ij} in σ. We have the following relations:

$$
(6.9)\qquad
\begin{aligned}
\Gamma^k_{rs} &= \bar{\Gamma}^k_{rs} - N^k \Gamma^0_{rs}, \\
\Gamma^k_{r0} &= N^2 \bar{g}^{kp}\Gamma^0_{rp} + \bar{\nabla}_r N^k - N^k N^p \Gamma^0_{rp} - \frac{N^k}{N}\partial_r N, \\
\Gamma^\lambda_{p\lambda} &= \frac{\partial_p N}{N} + \Gamma^s_{ps}, \\
\Gamma^0_{p0} &= N^s \Gamma^0_{ps} + \frac{\partial_p N}{N}, \\
\Gamma^k_{00} &= (\bar{g}^{kp} - N^k N^p/N^2)\partial_0 N_p - \tfrac{1}{2}(\bar{g}^{kp} - N^k N^p/N^2)\partial_p(-N^2 + N^s N_s) + \\
&\qquad + \frac{1}{2}\frac{N^k}{N^2}\partial_0(-N^2 + N^s N_s), \\
\Gamma^0_{00} &= \frac{N^p}{N^2}\partial_0 N_p - \frac{1}{2}\frac{N^p}{N^2}\partial_p(-N^2 + N^s N_s) - \frac{1}{2N^2}\partial_0(-N^2 + N^s N_s),
\end{aligned}
$$

and

$$\partial_0 g_{ij} = \overline{\nabla}_i N_j + \overline{\nabla}_j N_i + 2N^2 \Gamma^0_{ij}, \tag{6.10}$$
$$\partial_k g_{00} = 2(-N\partial_k N + N^p \overline{\nabla}_k N_p),$$

where $\overline{\nabla}_r$ is the covariant derivative in σ with respect to the connection Γ^k_{ij}.

Using (6.9), we obtain

$$K_{ij} = -N\Gamma^0_{ij}. \tag{6.11}$$

It follows from (6.9) and (6.11) that Γ^0_{ij}, Γ^k_{r0}, Γ^0_{p0} are the appropriate tensor fields in σ. We define a 2-contravariant tensor density in σ:

$$\pi^{ij} = -\sqrt{\bar{g}}\,(K_{pq} - g_{pq}K_{rs}\bar{g}^{rs})\bar{g}^{pi}\bar{g}^{qj}. \tag{6.12}$$

We call this quantity the ADMW *tensor density* on σ.

By (6.11) and (6.12) we have

$$\pi^{ij} = \sqrt{-g}\,(\Gamma^0_{pq} - g_{pq}\Gamma^0_{rs}\bar{g}^{rs})\bar{g}^{pi}\bar{g}^{qj}, \tag{6.13}$$

$$\Gamma^0_{pq} = \frac{1}{\sqrt{-g}}(\pi_{pq} - \tfrac{1}{2}g_{pq}\operatorname{tr}\pi), \tag{6.14}$$

where $\operatorname{tr}\pi = g_{ij}\pi^{ij}$, $\pi_{pq} = g_{ip}g_{jq}\pi^{ij}$.

Having the above formulas, we can discuss the Einstein equations in the ADMW coordinate system.

PROPOSITION 6.1. *If equations* (5.2) *are fulfilled, then the following conditions are equivalent:*

$$R_{\mu\nu} = \lambda g_{\mu\nu}, \tag{6.15}$$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 0, \tag{6.16}$$

$$R_{ks} = \lambda g_{ks}, \tag{6.17a}$$

$$G^0_\mu = 0. \tag{6.17b}$$

The proof of this proposition is purely algebraic and can be found in [2].

PROPOSITION 6.2. *System* (6.17) *is equivalent to the system*

$$R_{ks} = \lambda g_{ks}, \tag{6.18a}$$

$$G^0_\mu = 0 \quad \text{on} \quad \sigma. \tag{6.18b}$$

For the proof see Section 15.

In local coordinates of type (6.4) G^0_μ does not depend on $g_{0\nu,0}$, $g_{0\nu,0\lambda}$, $g_{ij,00}$. It is easy to see that G^0_μ on σ depends only on the values g_{ij}, π^{ij}, N_k, N on σ and their spatial derivatives. Therefore system (6.18) consists of 6 dynamical equations (6.18a) and 4 constraints on the initial data (6.18b).

In the ADMW coordinate system we have

$$\partial_0 \pi^{ij} = -N\sqrt{\bar{g}}\,(\bar{R}^{ij} - \bar{g}^{ij}\bar{R}) - \frac{2N}{\sqrt{\bar{g}}}\,(\pi^i_q \pi^{qj} - \tfrac{1}{2}\mathrm{tr}\,\pi\pi^{ij}) + \tag{6.19a}$$
$$+\sqrt{\bar{g}}\,(\bar{\nabla}^i \bar{\nabla}^j N - \bar{g}^{ij}\bar{\nabla}^s \bar{\nabla}_s N) + \bar{\nabla}_u (N^u \pi^{ij}) -$$
$$-\bar{\nabla}_s N^i \pi^{sj} - \bar{\nabla}_s N^j \pi^{si} - 2\lambda N\sqrt{\bar{g}}\,\bar{g}^{ij},$$

$$\bar{\nabla}_i \pi^{ij} = 0 \quad \text{on} \quad \sigma, \tag{6.19b$'$}$$

$$\bar{R} - 2\lambda - \frac{1}{\bar{g}}\left(\pi_{pq}\pi^{pq} - \frac{1}{2}(\mathrm{tr}\,\pi)^2\right) = 0 \quad \text{on} \quad \sigma. \tag{6.19b$''$}$$

Having the initial data on σ, i.e., a positive defined metric g_{ij} and a 2-contravariant tensor density π^{ij} which satisfy the constraint equations (6.19b), we have to choose N_k and N in a neighbourhood of σ in M. Then equations (6.19a) have a unique solution in a neighbourhood of σ which satisfies those Cauchy data. We have to determine N and N_k also beyond σ because equations (6.19a) do not contain the time derivatives of these quantities. This fact is connected with the invariance of the Einstein equations with respect to the action of the diffeomorphisms of the manifold M (cf. Sections 9 and 10). A detailed discussion of the existence, maximal prolongation and regularity of solutions can be found in [7], [9], [17], [18], [23], [32].

Remark. The constraint equations (6.19b) do not contain N and N_k and therefore we have no restrictions on the choice of these quantities beyond σ.

It has been postulated in [3], [41] that the components of the ADMW density are the conjugate variables to the components of the metric tensor g_{ij} on σ. In Section 8 we show that such a choice of variables follows from a diagonalization of the (pre)symplectic form Ω. It turns out that in terms of (π^{ij}, g_{ij}) the form Ω has a diagonal expression which corresponds to the diagonal formula $\overset{2}{\omega} = \sum_{i=1}^{n} dp_i \wedge dq^i$ for the canonical 2-form in mechanics.

The set Cd of all 3-dimensional surfaces in M with given positive defined metric tensors g_{ij} and 2-contravariant tensor densities π^{ij} on them which fulfil the constraint equations (6.19b) generates the set of 3-dimensional submanifolds of $\mathscr{P}$. This set contains all admissible initial surfaces of $(\mathscr{P}, \overset{5}{\omega})$.

7. A symplectic structure in the set of solutions of field equations

The multiphase approach to classical field theories enables us to define a natural presymplectic structure in the space $\mathscr{H}$ of all states of a given multisymplectic manifold $(\mathscr{P}, \omega)$. The construction has recently

been presented in the paper of J. Kijowski and the present author [26]. We outline there the main ideas of this construction.

Let $(\mathscr{P}, {}^{n+1}\omega)$ give a multiphase structure of a given field theory (of a given differential equation). We assume that the theory under consideration is of a hyperbolic character in the sense of Section 4. We denote by $\mathscr{C}$ the set of all admissible initial surfaces on $\mathscr{P}$. For the sake of simplicity we consider only a situation where $\mathscr{P}$ is a bundle over an n-dimensional manifold M and states are sections of $\pi\colon \mathscr{P}\to M$ which are diffeomorphisms of M onto their images. A more general situation has been discussed in [26]. We intend to define at every point $C\in\mathscr{H}$ the tangent space to $\mathscr{H}$.

Let $C = f(M)$, where f is a section of π and, for every vector field X on C,

$$(X \lrcorner\, {}^{n+1}\omega)|C = 0. \tag{7.1}$$

Let G_t be a one-parameter family of diffeomorphisms of $\mathscr{P}$ which transforms the state C onto submanifold $G_t(C)$ such that for every $\tilde{X}$

$$\frac{d}{dt}\{(\tilde{X} \lrcorner\, {}^{n+1}\omega)|G_t(C)\}_{t=0} = 0. \tag{7.2}$$

In general $G_t(C)$ is not a state, but we can say that the curve $t\to G_t(C)$ is tangent to the state C. We assume also that G_t is a fibre-preserving transformation in $\mathscr{P}$ and therefore it generates a π-vertical vector field Y on C:

$$Y = \frac{d}{dt}G_t(C)|_{t=0}. \tag{7.3}$$

PROPOSITION 7.1 ([26]). *For any extension $\tilde{Y}$ of Y onto a neighbourhood of C in $\mathscr{P}$ and an arbitrary vector field $\tilde{X}$ defined in this neighbourhood we have*

$$\pounds_{\tilde{Y}}(\tilde{X} \lrcorner\, {}^{n+1}\omega)|C = 0. \tag{7.4}$$

Remark 1. If (7.4) holds for an extension $\tilde{Y}$ of Y, it holds also for another extension, $\tilde{Y}_1$.

Remark 2. It is sufficient to consider π-vertical vector fields $\tilde{X}$.

PROPOSITION 7.2 ([26]). *Condition* (7.4) *is equivalent to the following: For every π-vertical vector field $\tilde{X}$ in a neighbourhood U of C in $\mathscr{P}$ and vector fields $\tilde{Z}_1, \ldots, \tilde{Z}_n$ in U which are tangent to C on C*

$$\Big(\tilde{Y}\big((\tilde{X} \lrcorner\, {}^{n+1}\omega)(\tilde{Z}_1, \ldots, \tilde{Z}_n)\big) + \\ + \sum_{k=1}^{n}(-1)^k(\tilde{X} \lrcorner\, {}^{n+1}\omega)([\tilde{Y}, \tilde{Z}_k], \tilde{Z}_1, \ldots, \tilde{Z}_n)\Big)|C = 0. \tag{7.5}$$

Writing down (7.5) in local coordinates in $\mathscr{P}$, we see that equation (7.5) involves only the values of $\tilde{Y}$ on C and its derivatives in directions tangent to C. Therefore (7.5) represents a system of linear differential equations on the manifold C. We call equation (7.5) the *linearized equation of motion*.

We define two subspaces of vector fields on C:

(a) the subspace $T_C(\mathscr{H})$ consisting of all π-vertical vector fields on C generated by families of diffeomorphisms of $\mathscr{P}$ satisfying (7.3);

(b) the subspace $\tilde{T}_C(\mathscr{H})$ consisting of all π-vertical vector fields on C satisfying the linearized equation of motion (7.5).

By virtue of Propositions 7.1 and 7.2 $T_C(\mathscr{H}) \subset \tilde{T}_C(\mathscr{H})$. We call $T_C(\mathscr{H})$ the *tangent space to* $\mathscr{H}$ at C and $\tilde{T}_C(\mathscr{H})$ the *extended tangent space to* $\mathscr{H}$ at C. The problem arises when $T_C(\mathscr{H}) = \tilde{T}_C(\mathscr{H})$. This difficult mathematical problem of integrating vector fields over a non-compact manifold cannot be solved generally: we have to investigate it in a given concrete case. In our further considerations subspaces $T_C(\mathscr{H})_{\mathrm{sb}} \subset T_C(\mathscr{H})$ and $\tilde{T}_C(\mathscr{H})_{\mathrm{sb}} \subset \tilde{T}_C(\mathscr{H})$ consisting of vector fields with "spatially bounded supports" are of special interest. We say that a vector field Y on C has a *spatially bounded support* if the intersection of its support with every admissible initial surface $c \subset C$ is compact. For hyperbolic theories in the sense of Section 4, the linearized equations of motion (7.5) form a hyperbolic system of linear differential equations. Therefore if the field Y on C has a compact support on an admissible initial surface $c \subset C$, it has also a compact support on any other admissible initial surface $c_1 \subset C$ (a finite speed of propagation for hyperbolic equations). It is known from the paper of Y. Choquet-Bruhat [7] that the Einstein equations have this property.

We define a bilinear skew-symmetric form Ω on $\tilde{T}_C(\mathscr{H})_{\mathrm{sb}}$:

$$(7.6)\qquad \tilde{T}_C(\mathscr{H})_{\mathrm{sb}} \times \tilde{T}_C(\mathscr{H})_{\mathrm{sb}} \ni (\hat{Y}_1, \hat{Y}_2) \to \Omega(\hat{Y}_1, \hat{Y}_2) = \int_{c \subset C} (Y_1 \wedge Y_2) \lrcorner \overset{n+1}{\omega} = \int_{c \subset C} Y_2 \lrcorner Y_1 \lrcorner \overset{n+1}{\omega},$$

where Y_1, Y_2 are π-vertical vector fields on C representing vectors $\hat{Y}_1$, $\hat{Y}_2$, and $c \subset C$ is an admissible initial surface contained in C.

PROPOSITION 7.3. *The value of Ω given by* (7.6) *does not depend on the choice of an admissible initial surface $c \subset C$.*

This proposition follows from the formulas $d\overset{n+1}{\omega} = 0$ and (7.5). Those equations imply the following:

$$(7.7)\qquad d\big((Y_1 \wedge Y_2) \lrcorner \overset{n+1}{\omega}\big)|C\big) = 0.$$

By virtue of (7.7) and the axioms of the family $\mathscr{C}$ we obtain Proposition 7.3 as a consequence of the Stokes theorem.

The 2-form Ω defined by (7.6) on the set $\tilde{T}_C(\mathscr{H})_{sb}$ can be extended to a larger subspace of $\tilde{T}_C(\mathscr{H})$. We have to impose some conditions which provide the convergence of the integral in formula (7.6). Therefore we can consider only those vectors $\hat{Y}$ which are represented by vector fields Y on C vanishing rapidly at "spatial infinity".

In [26] the space $\mathscr{H}$ has been endowed with a pseudodifferential structure which enables us to define in $\mathscr{H}$ the basic notions of differential geometry: vector fields in $\mathscr{H}$, their commutators, differential forms and their exterior derivatives. In that structure formula (7.6) defines a differential 2-form in $\mathscr{H}$. It has been proved in [26] that Ω is closed, i.e., $d\Omega = 0$. The 2-form Ω generates a Lie algebra structure in some subset of all smooth functionals on $\mathscr{H}$. We discuss this problem in Section 11.

In mechanics the space $\mathscr{H}$ consists of 1-dimensional trajectories in the phase space $T^*V \times \boldsymbol{R}$ satisfying the equation of motion $(X \lrcorner\, d\theta_{\text{hom}})|C = 0$. The set $\mathscr{H}$ is $2n$-dimensional, because it is parametrized by the initial values $p_i(0)$, $q^i(0)$. The form Ω is equal to the canonical 2-form in T^*V, i.e., $\Omega = \sum_{i=1}^{n} dp_i \wedge dq^i$. We see that in mechanics the theory developed in [26] coincides with the symplectic approach presented in [1], [35]. In this case the form Ω is non-degenerate but in general it is not true. We define the gauge subspace of $\tilde{T}_C(\mathscr{H})_{sb}$:

$$W_C = \{\hat{Y} \in \tilde{T}_C(\mathscr{H})_{sb}\colon\ \Omega(\hat{X}, \hat{Y}) = 0,\ \hat{X} \in \tilde{T}_C(\mathscr{H})_{sb}\}. \tag{7.8}$$

The corresponding distribution W is called the *gauge distribution* of Ω. It has been proved in [26] that W is involutive, i.e., if $\hat{X}_1$, $\hat{X}_2$ are vector fields with values in W, then $[\hat{X}_1, \hat{X}_2]$ is a vector field with values in W (cf. Section 11).

A degeneracy of the 2-form Ω is caused by:

(i) The existence of constraints in the space of Cauchy data for the equation of motion.

(ii) The fact that an admissible initial surface $c \in \mathscr{C}$ does not determine a state uniquely.

As an example we consider non-linear electrodynamics in a curved space. Let M be a 4-dimensional pseudoriemannian manifold with a metric tensor $g_{\mu\nu}$ having the signature $(-1, +3)$.

The bundle $\pi\colon \mathscr{P} \to M$ is a submanifold of $\bigwedge^4 T^*(T^*M)$ consisting of all 4-covectors on T^*M having the form:

$$v = -\frac{1}{4\pi} h^{\mu\nu} \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{dA_\nu}_{\mu} \wedge \ldots \wedge dx^3 + \\ + H(x^\mu, A_\nu, h^{\mu\nu}) \sqrt{-g}\, dx^0 \wedge \ldots \wedge dx^3 \tag{7.9}$$

where $h^{\mu\nu} = -h^{\nu\mu}$ and the 6×6 matrix $\left[\frac{\partial^2 H}{\partial h^{\mu\nu}\partial h^{\lambda\varrho}}\right]_{\substack{\mu<\nu\\ \lambda<\varrho}}$ is invertible (cf. (3.57)).

The canonical 4-form in $\mathscr{P}$ is:

$$(7.10)\quad \overset{4}{\theta}(x^\lambda, A_\nu, h^{\mu\nu}) = -\frac{1}{4\pi} h^{\mu\nu}\sqrt{-g}\,dx^0\wedge \ldots \wedge \underbrace{dA_\nu}_{\mu}\wedge \ldots \wedge dx^3 + \\ + H(x^\lambda, A_\nu, h^{\mu\nu})\sqrt{-g}\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3.$$

The equations of motion are (cf. (3.53)):

$$(7.11a)\qquad f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 4\pi\frac{\partial H}{\partial h^{\mu\nu}},$$

$$(7.11b)\qquad \nabla_\nu h^{\mu\nu} = 4\pi\frac{\partial H}{\partial A_\mu} = 4\pi j^\mu(x^\lambda), \quad \text{where} \quad \nabla_\mu j^\mu(x^\lambda) = 0.$$

Let $\sigma\subset M$ be a space-like surface such that the values A_ν and $h^{\mu\nu}$ on σ determine an admissible initial surface $c\subset\mathscr{P}$.

LEMMA 7.1. *The system of differential equations*

$$\nabla_\nu h^{\mu\nu} = 4\pi j^\mu(x^\lambda), \quad \textit{where} \quad \nabla_\mu j^\mu(x^\lambda) = 0$$

is equivalent to the system

$$(7.12a)\qquad \nabla_\nu h^{k\nu} = 4\pi j^k(x^\lambda) \quad \textit{on} \quad M,$$

$$(7.12b)\qquad \nabla_k h^{0k} = 4\pi j^0(x^\lambda) \quad \textit{on} \quad \sigma.$$

For the proof see Section 15.

A space-like surface $\sigma\subset M$ gives a $(3+1)$ decomposition of the antisymmetric tensor $h^{\mu\nu}$. This tensor determines a vector field

$$(7.13)\qquad D^i = h^{0i} \quad \text{in} \quad \sigma$$

and a pseudo-covector field

$$(7.14)\qquad H_i = \tfrac{1}{2}\sqrt{|\bar g|}\,\varepsilon_{ijk}h^{jk} \quad \text{in} \quad \sigma.$$

For orientable σ we have the appropriate densities:

$$(7.15a)\qquad \mathscr{D}^i = \sqrt{-g}\,D^i = \sqrt{\bar g}\,N D^i,$$

$$(7.15b)\qquad \mathscr{H}^i = \sqrt{-g}\,H^i = \sqrt{\bar g}\,N H^i, \quad \text{where} \quad H^i = \bar g^{ij}H_j.$$

Equations (7.12) read:

$$(7.16a)\qquad \partial_0\mathscr{D}^i = -4\pi j^i\sqrt{\bar g}\,N + (\operatorname{rot}\mathscr{H})^i \quad \text{on} \quad M,$$

$$(7.16b)\qquad \bar\nabla_k\mathscr{D}^k = 4\pi j^0\sqrt{\bar g}\,N \quad \text{on} \quad \sigma,$$

where

$$(\mathrm{rot}\,\mathscr{H})^i = \varepsilon^{iks}\,\overline{\nabla}_k(NH_s). \tag{7.17}$$

The kinematical equations (7.11a) have in the (3+1) approach the following form:

$$\partial_0 A_k - \partial_k A_0 = 4\pi\frac{\partial H(x^\lambda, A_\nu, h^{0i}, h^{pq})}{\partial h^{0k}} = 4\pi\frac{\partial H(x^\lambda, A_\nu, \mathscr{D}^i, \mathscr{H}^s)}{\partial h^{0k}}, \tag{7.18a}$$

$$\partial_r A_k - \partial_k A_r = 4\pi\frac{\partial H(x^\lambda, A_\nu, h^{0i}, h^{pq})}{\partial h^{rk}} = 4\pi\frac{\partial H(x^\lambda, A_\nu, \mathscr{D}^i, \mathscr{H}^s)}{\partial h^{rk}}. \tag{7.18b}$$

If the 3×3 matrix $\left[\dfrac{\partial^2 H}{\partial h^{rk}\partial h^{pq}}\right]_{\substack{r<k\\ p<q}}$ is invertible, equation (7.18b) determines

$$\mathscr{H}^s = \mathscr{H}^s(x^\lambda, A_\nu, \mathscr{D}^i, \partial_r A_k - \partial_k A_r). \tag{7.19}$$

Using (7.19), we obtain the system (7.16a), (7.18a) of differential equations for $A_k, A_0, \mathscr{D}^i$. This system does not contain the derivative $\partial_0 A_0$. Therefore we cannot determine A_0 from its initial value. We have to assign a function A_0 in a neighbourhood of an initial surface $\sigma \subset M$. Then the covector field A_k and the vector field $\mathscr{D}^i$ on σ fulfilling the constraint equation (7.16b) give the initial data for the system (7.16a), (7.18a). We see that the initial data on σ do not determine a state uniquely. Now we pass to the tangent space of $\mathscr{H}$. A π-vertical vector field on $C \in \mathscr{H}$ is given by

$$X = \sum_{\mu<\nu} \delta h^{\mu\nu}\frac{\partial}{\partial h^{\mu\nu}} + \delta A_\nu \frac{\partial}{\partial A_\nu}. \tag{7.20}$$

The linearized equations of motion in the case of vanishing currents are:

$$\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = 4\pi\frac{\partial^2 H}{\partial h^{\mu\nu}\partial h^{\lambda\varrho}}\,\delta h^{\lambda\varrho}, \tag{7.21a}$$

$$\partial_0 \delta\mathscr{D}^i = (\mathrm{rot}\,\delta\,\mathscr{H})^i \quad \text{on} \quad M, \tag{7.21b}$$

$$\overline{\nabla}_k \delta\mathscr{D}^k = 0 \quad \text{on} \quad \sigma, \tag{7.21c}$$

where

$$\begin{aligned} \delta\mathscr{D} &= \delta(N\sqrt{\bar g}\,h^{0i}) = N\sqrt{\bar g}\,\delta h^{0i},\\ \delta\mathscr{H}^i &= \delta(N\sqrt{\bar g}\,H^i) = \tfrac{1}{2}N\bar g\,\bar g^{ij}\varepsilon_{jab}\,\delta h^{ab}. \end{aligned} \tag{7.22}$$

Remark. It follows from the π-verticality of X that δA_k is a covector field in σ, and $\delta\mathscr{D}^i$ and $\delta\mathscr{H}^i$ are vector densities in σ. By arguments similar to those applied to the system (7.16a), (7.18a), we can assign δA_k on σ in an arbitrary way and $\delta\mathscr{D}^i$ in such a way that (7.21c) holds.

If

$$(7.23)\qquad X_i = \underset{i}{\delta} A_\mu \frac{\partial}{\partial A_\mu} + \underset{i}{\delta}\mathscr{D}^k \frac{\partial}{\partial \mathscr{D}^k} + \underset{i}{\delta}\mathscr{H}^s \frac{\partial}{\partial \mathscr{H}^s}, \qquad i = 1, 2,$$

represent vectors $\hat{X}_i \in \tilde{T}_c(\mathscr{H})_{\mathrm{sh}}$, then:

$$(7.24)\qquad \Omega(\hat{X}_1, \hat{X}_2) = -\frac{1}{4\pi}\int_\sigma (\underset{1}{\delta}\mathscr{D}^k \underset{2}{\delta} A_k - \underset{2}{\delta}\mathscr{D}^k \underset{1}{\delta} A_k)\, dx^1 \wedge dx^2 \wedge dx^3.$$

Using the Hodge decomposition for covector fields, we can easily see (at least when σ is compact) that all vectors belonging to the gauge distribution of the form Ω are of the type

$$(7.25)\qquad Y = \delta A_0 \frac{\partial}{\partial A_0} + \delta A_k \frac{\partial}{\partial A_k}$$

where $\delta A_0|_\sigma$ is an arbitrary function on σ and $\delta A_k|_\sigma$ is the gradient of a function $\psi \in C^\infty(\sigma)$. We decompose the initial values δA_k (i.e., a covector field in σ) into a gradient part and a free divergence part

$$\delta A_k = \partial_k \psi + \delta' A_k, \qquad \text{where} \qquad \bar{\nabla}^k \delta' A_k = 0.$$

In this way a vector $\hat{X}$ representing a class of the equivalence relation $\tilde{T}_C(\mathscr{H})/W_C$ is determined by six quantities δA_k, $\delta\mathscr{D}^i$ in σ which fulfil 2 constraint equations:

$$(7.26)\qquad \bar{\nabla}^k \delta A_k = 0, \qquad \bar{\nabla}_k \delta\mathscr{D}^k = 0.$$

Thus, we have 4 degrees of freedom in the phase space, and 2 degrees of freedom in the configuration space.

Let G be a group of tranformations in $\mathscr{P}$ defined by the formula

$$(7.27)\qquad A_\mu \to A_\mu + \partial_\mu \chi, \qquad h^{\mu\nu} \to h^{\mu\nu}$$

where χ is a smooth function on M.

The Lie algebra of G consists of all vector fields in $\mathscr{P}$ of the type

$$(7.28)\qquad X = \delta A_\mu \frac{\partial}{\partial A_\mu}, \qquad \text{where} \qquad \delta A_\mu = \partial_\mu \delta\chi.$$

The action

$$(7.29)\qquad G \times \mathscr{H} \ni (\chi, C) = \big(\chi, \{(x^\lambda, A_\nu(x^\lambda), h^{\mu\nu}(x^\lambda))\}\big) \to R_\chi(C)$$
$$= \{(x^\lambda, A_\nu(x^\lambda) + \partial_\nu \chi, h^{\mu\nu}(x^\lambda))\} \in \mathscr{H}$$

generates the action of the Lie algebra $\mathfrak{G}$:

$$(7.30)\qquad \mathfrak{G} \ni v \to dR_0(C)\, v \in \tilde{T}_C(\mathscr{H}).$$

It is easy to see that

$$\operatorname{im} dR_0(C) = W_C. \tag{7.31}$$

In Sections 8 and 9 we discuss the same problem for gravidynamics. Instead of the gauge transformation grup $A_\mu \to A_\mu + \partial_\mu \chi$ we have an action of the group of all diffeomorphisms of the manifold M. The corresponding action of the Lie algebra of the diffeomorphism group determines the gauge distribution of the 2-form Ω. This result is analogous to (7.31) where the action (7.30) determines the gauge distribution for the 2-form Ω in electrodynamics (without currents).

8. A symplectic structure in the set of Einstein metrics

Let $(\mathscr{P}, \overset{5}{\omega})$ be the 4-symplectic manifold defined in Section 5. The set $\mathscr{H}$ of all $\overset{5}{\omega}$-singular section of the bundle $\pi\colon \mathscr{P} \to M$ is in a one-to-one correspondence with the set of Einstein metrics in M. The Einstein equations (5.2), (5.3) generate linearized equations for the components of a π-vertical vector field Y on a given $\overset{5}{\omega}$-singular submanifold $C \subset \mathscr{P}$. In local coordinates

$$Y = \sum_{\mu \leqslant \nu} \delta g_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}} + \sum_{\mu \leqslant \nu} \delta \Gamma^\lambda_{\mu\nu} \frac{\partial}{\partial \Gamma^\lambda_{\mu\nu}}. \tag{8.1}$$

The field Y on C fulfils equation (7.5) if and only if:

$$\begin{aligned} \delta \Gamma^\lambda_{\mu\nu} &= \delta\left(\tfrac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\right) \\ &= \tfrac{1}{2} g^{\lambda\sigma}(\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu}) = 0, \end{aligned} \tag{8.2}$$

$$\delta(R_{\mu\nu} - \lambda g_{\mu\nu}) = \sum_{\alpha \leqslant \beta} \frac{\partial R_{\mu\nu}}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} + \sum_{\alpha \leqslant \beta} \frac{\partial R_{\mu\nu}}{\partial \Gamma^\tau_{\alpha\beta}} \delta \Gamma^\tau_{\alpha\beta} - \lambda \delta g_{\mu\nu} = 0. \tag{8.3}$$

Remark. If Y is a π-vertical vector field on C, then $\delta g_{\mu\nu}$ and $\delta \Gamma^\lambda_{\mu\nu}$ form components of appropriate tensor fields in C (or equivalently in M). It is not true for non-vertical vector fields Y (see Section 14).

Vector fields Y on C satisfying (8.2) and (8.3) represent vectors $\hat{Y} \in \tilde{T}_C(\mathscr{H})$. If $\sigma \subset M$ is a space-like surface (with respect to the metric $g_{\mu\nu}$ determined by C), we have a 3-dimensional submanifold $c = f(\sigma)$, where f is a section of $\mathscr{P}$ such that $C = f(M)$. The submanifold $c \subset \mathscr{P}$ is an admissible initial surface contained in C. We define a bilinear skew-symmetric map on $\tilde{T}_C(\mathscr{H})_{\mathrm{sb}}$

$$\tilde{T}_C(\mathscr{H})_{\mathrm{sb}} \times \tilde{T}_C(\mathscr{H})_{\mathrm{sb}} \ni (\hat{X}_1, \hat{X}_2) = \Omega(\hat{X}_1, \hat{X}_2) = \int_c X_2 \lrcorner X_1 \lrcorner \overset{5}{\omega}. \tag{8.4}$$

The above formula does not depend on the choice of an admissible initial surface $c \subset C$ (cf. Proposition 7.3), or equivalently it does not depend on the choice of a space-like surface $\sigma \subset M$. X_1, X_2 are π-vertical vector fields on C whose components satisfy (8.2), (8.3). In local coordinates $(x^\lambda, g_{\mu\nu}, \Gamma^\lambda_{\mu\nu})$, (8.4) is a very complicated expression (cf. (5.8)) and is inconvenient for a detailed analysis. We would like to find a diagonal form for Ω which corresponds to formula (7.24) for electrodynamics. It turns out that the $(3+1)$ decomposition of geometrical objects connected with a given space-like surface $\sigma \subset M$ is an appropriate tool for solutions of that problem. In a fibre of the bundle $\pi\colon \mathscr{P} \to M$ we have the ADMW coordinates $(N, N_k, g_{ij}, \pi^{ij}, M_\mu, M_{\mu k}, \Gamma^k_{ij})$ instead of the Christoffel coordinates $(g_{\mu\nu}, \Gamma^\lambda_{\mu\nu})$. The correspondence between these systems is given by the formulas (6.6), (6.7), (6.13), (6.9), (6.14), and

$$M_\mu = (\partial_\mu N)(g_{\alpha\beta}, \Gamma^\tau_{\alpha\beta}); \qquad M_{\mu k} = (\partial_\mu N_k)(g_{\alpha\beta}, \Gamma^\tau_{\alpha\beta}). \tag{8.5}$$

A π-vertical vector field Y on C representing a vector $\hat{Y} \in \tilde{T}_C(\mathscr{H})$ has the form

$$Y = \delta N \frac{\partial}{\partial N} + \delta N_k \frac{\partial}{\partial N_k} + \sum_{i \leq j} \delta g_{ij} \frac{\partial}{\partial g_{ij}} + \sum_{i \leq j} \delta \pi^{ij} \frac{\partial}{\partial \pi^{ij}} + \\ + \delta M_\mu \frac{\partial}{\partial M_\mu} + \delta M_{\mu k} \frac{\partial}{\partial M_{\mu k}} + \sum_{i \leq j} \delta \Gamma^k_{ij} \frac{\partial}{\partial \Gamma^k_{ij}}, \tag{8.6}$$

where the components $(\delta N, \delta N_k, \delta g_{ij}, \delta \pi^{ij}, \delta M_\mu, \delta M_{\mu k}, \delta \Gamma^k_{ij})$ fulfil the linearized equations of motion:

$$\delta M_\mu = \partial_\mu \delta N, \qquad \delta M_{\mu k} = \partial_\mu \delta N_k, \tag{8.7}$$

$$\delta \Gamma^j_{ks} = \tfrac{1}{2} \bar{g}^{ja} (\bar{\nabla}_k \delta g_{sa} + \bar{\nabla}_s \delta g_{ka} - \bar{\nabla}_a \delta g_{ks}), \tag{8.8}$$

$$\partial_0 \delta g_{ij} = \delta \left(\bar{\nabla}_i N_j + \bar{\nabla}_j N_i + \frac{2N}{\sqrt{\bar{g}}} (g_{ip} g_{jq} \pi^{pq} - \tfrac{1}{2} g_{ij} \operatorname{tr} \pi) \right), \tag{8.9}$$

$$\partial_0 \delta \pi^{ij} = \delta \left(-N \sqrt{\bar{g}} (\bar{R}^{ij} - \bar{g}^{ij} \bar{R}) - \frac{2N}{\sqrt{\bar{g}}} (\pi^i_q \pi^{qj} - \tfrac{1}{2} \pi^{ij} \operatorname{tr} \pi) \right) + \\ + \delta \left(\sqrt{\bar{g}} (\bar{\nabla}^i \bar{\nabla}^j N - \bar{g}^{ij} \bar{\nabla}^s \bar{\nabla}_s N) + \bar{\nabla}_u (N^u \pi^{ij}) \right) + \\ + \delta (- \bar{\nabla}_s N^i \pi^{sj} - \bar{\nabla}_s N^j \pi^{si} - 2\lambda N \sqrt{\bar{g}}\, \bar{g}^{ij}), \tag{8.10a}$$

$$\delta (\bar{\nabla}_j \pi^{ij}) = \bar{\nabla}_j \delta \pi^{ij} + \delta \Gamma^i_{ks} \pi^{ks} = 0 \quad \text{on} \quad \sigma, \tag{8.10b'}$$

$$\delta \left(\bar{R} - 2\lambda - \frac{1}{\bar{g}} \left(\pi_{pq} \pi^{pq} - \tfrac{1}{2} (\operatorname{tr} \pi)^2 \right) \right) = 0 \quad \text{on} \quad \sigma. \tag{8.10b''}$$

Equations (8.7), (8.8) and (8.9) are equivalent to the linearized kinematical equations (8.2). They are obtained from (8.5), (6.9), (6.10) and (6.14). Equations (8.10) correspond to equations (8.3) and are obtained by the linearization of system (6.19).

Remark. The right sides of the equations (8.9) and (8.10) have to be expressed in terms of $(\delta N, \delta N_k, \delta M_\mu, \delta M_{\mu k}, \delta g_{ij}, \delta\pi^{ij}, \delta\Gamma^k_{ij})$. We compute these expressions in Section 9.

It follows from the verticality of the vector field Y on C that $(\delta N, \delta N_k, \delta M_\mu, \delta M_{\mu k}, \delta g_{ij}, \delta\pi^{ij}, \delta\Gamma^k_{ij})$ have tensor transformation properties and therefore δN is a function on σ, δN_k is a covector field in σ, δg_{ij} is a 2-covariant tensor field in σ, and $\delta\pi^{ij}$ is a 2-contravariant tensor density in σ.

Having a state $C \in \mathscr{H}$, i.e., a solution of the Einstein equations, and a space-like surface $\sigma \subset M$, we can assign a 2-covariant tensor field δg_{ij} in σ, and a 2-contravariant tensor density $\delta\pi^{ij}$ in σ, in such a way that the couple $(\delta\pi^{ij}, \delta g_{ij})$ fulfils the constraint equations (8.10b), a one-parameter family of functions δN on σ and a one-parameter family of covector fields δN_k in σ. These initial values determine a unique solution for equations (8.7)–(8.10), where x^0 is considered as a parameter. Of course, we have only a solution for small values of the parameter x^0, i.e., a solution in a neighbourhood of the admissible initial surface $c = f(\sigma)$ in C. The problem arises whether this solution can be extended over the whole of C. We know no satisfactory answer to that question but we see that an element $\hat{X} \in \tilde{T}_C(\mathscr{H})$ gives 12 quantities $(\delta\pi^{ij}, \delta g_{ij})$ in σ (or equivalently in $c \subset \mathscr{P}$) fulfilling the constraint equations (8.10b) and 4 quantities δN, δN_k in a neighbourhood of c in C. The correspondence

$$\tilde{T}_C(\mathscr{H}) \ni X \to \big((\delta\pi^{ij}, \delta g_{ij}) \text{ in } c, (\delta N, \delta N_k) \text{ in } C\big)$$

is injective and therefore instead of the elements of $\tilde{T}_C(\mathscr{H})$ we shall often consider the appropriate systems $(\delta\pi^{ij}, \delta g_{ij})$ in c, $(\delta N, \delta N_k)$ in C.

The following subspace of $\tilde{T}_C(\mathscr{H})$ plays an important role in our considerations:

Definition. $\overset{0}{\tilde{T}}_C(\mathscr{H}, c)$ is the linear subspace of $\tilde{T}_C(\mathscr{H})$ consisting of such $\hat{Y} \in \tilde{T}_C(\mathscr{H})$ that $\hat{Y}$ is represented by a vector field Y on C of the form (8.6) such that:

$$(8.11)\quad \begin{cases} \text{(i)}\ \delta N, \delta N_k, \delta M_0, \delta M_{0k} \text{ are arbitrary on } c, \\ \text{(ii)}\ \delta M_k = \partial_k \delta N,\ \delta M_{sk} = \partial_s \delta N_k \text{ on } c, \\ \text{(iii)}\ \delta\pi^{ij} = 0,\ \delta g_{ij} = 0 \text{ on } c. \end{cases}$$

It is easy to see that conditions (8.11) are consistent with (8.7) and (8.10b). The definition of the subspace $\overset{0}{\tilde{T}}_C(\mathscr{H}, c)$ essentially depends on an admissible initial surface $c \subset C$.

PROPOSITION 8.1. *For every* $\hat{X} \in \tilde{T}_C(\mathscr{H})_{\mathrm{sb}}$ *and* $\hat{Y} \in \overset{0}{\tilde{T}}_C(\mathscr{H}, c)$,

$$\Omega(\hat{X}, \hat{Y}) = 0.$$

The proof is given in Section 15.

Proposition 8.1 shows that the subspace $\overset{0}{\tilde{T}}_C(\mathscr{H}, c)$ belongs to the gauge subspace W_C of Ω defined by (7.8). It is connected with the fact that the Einstein equations do not determine N and N_k by their initial values on σ (and the same is true for the linearized Einstein equations). This in turn is related to an invariance of the Einstein equations with respect to the action of the diffeomorphism group of the space-time M.

THEOREM 8.1. *Let* $\hat{X}_1, \hat{X}_2 \in \tilde{T}_C(\mathscr{H})$ *be represented by vector fields* X_1, X_2 *of the form* (8.6), *one of them having a compact support on* c. *Then*:

$$\Omega(\hat{X}_1, \hat{X}_2) = \int_c (\underset{1}{\delta}\pi^{ij} \underset{2}{\delta} g_{ij} - \underset{2}{\delta}\pi^{ij} \underset{1}{\delta} g_{ij}) dx^1 \wedge dx^2 \wedge dx^3. \tag{8.12}$$

The proof is given in Section 15.

Theorem 8.1 shows that the quantities π^{ij} are in some sense conjugate to the spatial components of a metric tensor $g_{\mu\nu}$. However, we must remember that π^{ij} and g_{ij} are not independent but fulfil the constraint equations (6.19b). In the next section we show that these equations determine the gauge distribution of the form Ω. Proposition 8.1 is the first step in that direction.

The form Ω is a geometric object independent of the choice of coordinates. Therefore the diagonal expression (8.12) distinguishes the ADMW coordinates. In particular, it shows why we should take the ADMW density as momenta for General Relativity (cf. [3], [41]).

9. The gauge distribution and the action of the diffeomorphism group

Considering the general approach in Section 7, we have emphasized that the 2-form Ω is in general degenerate. In the present section we prove that the gauge distribution of Ω in gravidynamics is determined by the action of the diffeomorphism group of M in the space of states $\mathscr{H}$. In the language of classical physics this action is called "an active change of coordinates in the space-time M". The Einstein equations $R_{\mu\nu} = \lambda g_{\mu\nu}$ are invariant with respect to such a transformation (cf. [2], [33]). Therefore a physical meaning can be assigned to not Einstein metrics in M but to their equivalence classes with respect to the natural action of $\mathrm{Diff}(M)$, cf. [16], [33], [41].

To provide the convergence of integrals in further considerations we take in account only elements $\hat{X} \in \tilde{T}_C(\mathscr{H})_{sb}$. As we have mentioned before, a finite speed of propagation for the linearized Einstein equations provides that if a vector field X on C representing a vector $\hat{X} \in \tilde{T}_C(\mathscr{H})$ has a compact support on an admissible initial surface $c \subset C$ (or equivalently on $\sigma \subset M$), then it has also a compact support on another admissible initial surface $c_1 \subset C$ (or on $\sigma_1 \subset M$), cf. [7], [23]. Let

$$W_C = \{\hat{Y} \in \tilde{T}_C(\mathscr{H}) \colon\ \Omega(\hat{Y}, \hat{X}) = 0,\ \hat{X} \in \tilde{T}_C(\mathscr{H})_{sb}\}. \tag{9.1}$$

We know by Proposition 8.1 that the subspace $\overset{0}{\tilde{T}}_C(\mathscr{H}, c)$ defined by (8.11) is contained in W_C. We are looking for its algebraic complement in $\tilde{T}_C(\mathscr{H})$.

DEFINITION. $\overset{1}{\tilde{T}}_C(\mathscr{H}, c)$ is a subspace of $\tilde{T}_C(\mathscr{H})$ consisting of all vectors $\hat{Y}$ which are represented by vector fields Y on C such that

$$\delta N = 0, \quad \delta N_k = 0 \quad \text{on} \quad C. \tag{9.2}$$

We know that the constraint equations (8.10b) do not contain the quantities δN, δN_k. This fact implies

PROPOSITION 9.1. *For every admissible initial surface $c \subset C$ we have decompositions into direct sums*:

$$\begin{aligned} \tilde{T}_C(\mathscr{H}) &= \overset{1}{\tilde{T}}_C(\mathscr{H}, c) \oplus \overset{0}{\tilde{T}}_C(\mathscr{H}, c), \\ \tilde{T}_C(\mathscr{H})_{sb} &= \overset{1}{\tilde{T}}_C(\mathscr{H}, c)_{sb} \oplus \overset{0}{\tilde{T}}_C(\mathscr{H}, c)_{sb}. \end{aligned} \tag{9.3}$$

Remark. The subspace $\overset{1}{\tilde{T}}_C(\mathscr{H}, c)$ depends on an ADMW system connected with a space-like submanifold $\sigma = \pi(c) \subset M$. If we take a family of admissible initial surfaces $c_t \subset C$ and such an ADMW system that projections $\pi(c_t)$ are space-like submanifolds in M described by the equation

$$\sigma_t = \pi(c_t) = \{x \in M \colon\ x^0 = t\},$$

then all spaces $\overset{1}{\tilde{T}}_C(\mathscr{H}, c_t)$ coincide.

For a given space-like surface $\sigma \subset M$ (or equivalently for a given admissible initial surface $c \subset C$) we define a vector space

$$C_\sigma = C^\infty\big(\operatorname{den} S^2T(\sigma) \oplus S^2T^*(\sigma)\big)$$

consisting of couples $\mathfrak{X} = (\delta\pi^{ij}, \delta g_{ij})$, where $\delta\pi^{ij}$ is a symmetric 2-contravariant tensor density in σ and δg_{ij} is a symmetric 2-covariant tensor field in σ. The space C_σ plays the role of Cauchy data for the linearized Einstein equations. A vector $\hat{X} \in \overset{1}{\tilde{T}}_C(\mathscr{H}, c)$ determines a couple $\mathfrak{X} =$

$= (\delta\pi^{ij}, \delta g_{ij})$ which fulfils the constraint equations (8.10b) and an element $\mathfrak{X} = (\delta\pi^{ij}, \delta g_{ij}) \in C_\sigma$ satisfying (8.10b) generates a solution of the system (8.10a), (8.9).

We define a scalar product in C_σ:

$$(9.4)\qquad C_\sigma \times C_\sigma \ni (\mathfrak{X}_1, \mathfrak{X}_2) = \big((\underset{1}{\delta\pi^{ij}}, \underset{1}{\delta g_{ij}}), (\underset{2}{\delta\pi^{ij}}, \underset{2}{\delta g_{ij}})\big) \to g(C, \sigma)(\mathfrak{X}_1, \mathfrak{X}_2)$$

$$= \int_\sigma \left(\frac{1}{\sqrt{\bar g}}\, \underset{1}{\delta\pi^{ij}} g_{ip} g_{jq} \underset{2}{\delta\pi^{pq}} + \sqrt{\bar g}\, \underset{1}{\delta g_{ij}} \bar g^{ip} \bar g^{jq} \underset{2}{\delta g_{pq}}\right) dx^1 \wedge dx^2 \wedge dx^3.$$

Remark. We assume that at least one of the elements $\mathfrak{X}_1, \mathfrak{X}_2$ has a compact support.

A connection between the scalar product (9.4) and the form Ω (cf. (8.12)) is given by the operator J: $C_\sigma \to C_\sigma$

$$(9.5)\qquad J(\delta\pi^{ij}, \delta g_{ij}) = \left(\sqrt{\bar g}\, \delta g_{pq} \bar g^{pi} \bar g^{qj}, -\frac{1}{\sqrt{\bar g}}\, \delta\pi^{pq} g_{pi} g_{qj}\right).$$

It is easy to see that

$$(9.6)\qquad J^2 = -\mathrm{id}.$$

Formula (9.4) defines a degenerate scalar product in $\tilde T_C(\mathscr{H})_{\mathrm{sb}}$:

$$(9.7)\qquad \tilde T_C(\mathscr{H})_{\mathrm{sb}} \times \tilde T_C(\mathscr{H})_{\mathrm{sb}} \ni (\hat X_1, \hat X_2) \to \tilde g(C, \sigma)(\hat X_1, \hat X_2) = g(C, \sigma)(\mathfrak{X}_1, \mathfrak{X}_2).$$

Remark. The definition of the scalar product $g(C, \sigma)$ essentially depends on the choice of $\sigma \subset M$.

Comparing (9.4) with (8.12) and using (9.5), we have:

$$(9.8)\qquad -g(C, \sigma)(J\mathfrak{X}_1, \mathfrak{X}_2) = \Omega(\hat X_1, \hat X_2) = g(C, \sigma)(\mathfrak{X}_1, J\mathfrak{X}_2).$$

Let $C^\infty\big(T(\sigma)\oplus \boldsymbol{R}\big)$ be a vector space consisting of couples $U = (u^j, \chi)$, where u^j is a vector field in σ and χ is a real function on σ. That space has a natural scalar product

$$(9.9)\qquad g_1(C, \sigma)(U_1, U_2) = \int_\sigma \big(\underset{1}{u_j} \underset{2}{u^j} \sqrt{\bar g} + \chi_1 \chi_2 \sqrt{\bar g}\big)\, dx^1 \wedge dx^2 \wedge dx^3$$

where $u_j = g_{js} u^s$.

Remark. We have to assume that one of the sections U_1, U_2 has a compact support.

For a given $C \in \mathscr{H}$, i.e., if (π^{ij}, g_{ij}) satisfy the constraint equations (6.16b), we define a differential operator A generated by the linearized constraint equations (8.10b):

$$C_\sigma \ni \mathfrak{X} \to A\mathfrak{X} \in C^\infty\big(T(\sigma)\oplus \boldsymbol{R}\big), \quad \text{if} \quad \mathfrak{X} = (\delta\pi^{ij}, \delta g_{ij});$$

then

$$(9.10)\quad A\mathfrak{X} = \left(\frac{1}{\sqrt{\bar g}}(\bar\nabla_j\delta\pi^{ij}+\delta\bar\Gamma^i_{ks}\pi^{ks}),\ \delta\bar R+\frac{1}{\bar g}(\bar R-2\lambda)\delta\bar g-\tfrac12\delta\left(\pi_{ks}\pi^{ks}-\tfrac12(\operatorname{tr}\pi)^2\right)\right)$$

where

$$(9.11)\quad \begin{aligned}\bar\nabla_j\delta\pi^{ij} &= \partial_j\delta\pi^{ij}+\bar\Gamma^i_{js}\delta\pi^{js},\\ \delta\bar\Gamma^k_{ij} &= \tfrac12\bar g^{ka}(\bar\nabla_j\delta g_{ia}+\bar\nabla_i\delta g_{ja}-\bar\nabla_a\delta g_{ij}),\\ \delta\bar R &= -\bar R^{pq}\delta g_{pq}+\bar\nabla^j\bar\nabla^k\delta g_{jk}-\bar g^{ij}\bar\nabla_k\bar\nabla^k\delta g_{ij},\\ \delta\bar g &= \bar g\bar g^{jk}\delta g_{jk}.\end{aligned}$$

The kernel of the operator A consists of those $\mathfrak{X}=(\delta\pi^{ij},\ \delta g_{ij})$ which give Cauchy data for equations (8.9), (8.10a).

By means of the scalar products (9.4) and (9.9) we define the adjoint operator A^*:

$$A^*:\ C^\infty\big(T(\sigma)\oplus\boldsymbol{R}\big)\to C_\sigma,$$

$$(9.12)\quad g(C,\sigma)(A^*U,\mathfrak{X}) = g_1(C,\sigma)(U,A\mathfrak{X});\qquad U\in C^\infty\big(T(\sigma)\oplus\boldsymbol{R}\big),\ \mathfrak{X}\in C_\sigma.$$

Integrating equality (9.12) by parts, we obtain

$$A^*(u^j,\chi) = (\delta\pi^{ij},\ \delta g_{ij}),$$

where

$$(9.13)\quad \begin{aligned}\delta\pi^{ij} &= -\tfrac12(\bar\nabla^i u^j+\bar\nabla^j u^i)\sqrt{\bar g}-2(\pi^{ij}-\tfrac12\operatorname{tr}\pi\bar g^{ij})\chi,\\ \delta g_{ij} &= -\frac{1}{2\sqrt{\bar g}}\left(\pi_{ai}\bar\nabla^a u_j+\pi_{aj}\bar\nabla^a u_i-\bar\nabla_a(\pi_{ij}u^a)\right)-\\ &\quad-\frac{2}{\bar g}\left(\pi_{is}\bar g^{sk}\pi_{kj}-\frac12\operatorname{tr}\pi\pi_{ij}\right)\chi+\bar\nabla_i\bar\nabla_j\chi-g_{ij}\bar\nabla^k\bar\nabla_k\chi-\bar R_{ij}\chi+\\ &\quad+g_{ij}(\bar R-2\lambda)\chi.\end{aligned}$$

It is obvious that the definition of the operators A, A^* depends on the choice of state $C\in\mathscr{H}$ and on the choice of a space-like surface $\sigma\subset M$.

PROPOSITION 9.2.

$$\operatorname{im}JA^*\subset\ker A.$$

The proof is presented in Section 15.

The above proposition is very important for our further considerations. It enables us to construct vector fields on $C\in\mathscr{H}$ representing vectors of $\overset{1}{\tilde T}_C(\mathscr{H},c)$. In fact, every $\mathfrak{X}\in\operatorname{im}JA^*$ generates such a field.

DEFINITION. $\overset{0}{W}_C(c) \subset \overset{1}{\tilde{T}}_C(\mathscr{H}, c)$ consists of vectors $\hat{Y} \in \overset{1}{\tilde{T}}_C(\mathscr{H}, c)$ which are represented by vector fields on C generated by $\operatorname{im} JA^*$.

PROPOSITION 9.3.

$$\overset{0}{W}_C(c) \subset W_C.$$

Proof. This proposition follows from formula (9.8) and the orthogonality of $\ker A$ and $\operatorname{im} A^*$.

DEFINITION.

$$\overset{1}{W}_C(c) = \overset{0}{W}_C(c) \oplus \overset{0}{\tilde{T}}_C(\mathscr{H}, c) \quad \text{(a direct sum)}. \tag{9.14}$$

It turns out that the subspace $\overset{1}{W}_C(c) \subset \tilde{T}_C(\mathscr{H})$ does not depend on the choice of an admissible initial surface $c \subset C$ (or equivalently on the choice of a space-like surface $\sigma \subset M$). We shall obtain this result as a consequence of Proposition 9.7. A direct discussion of the definition for $\overset{1}{W}_C(c)$ is given in Section 15.

PROPOSITION 9.4.

$$\overset{1}{W}_C(c) \subset W_C.$$

The above fact is a consequence of Propositions 8.1 and 9.3.

We cannot assert that $\overset{1}{W}_C(c) = W_C$ because we do not know whether $\ker A \oplus \operatorname{im} A^*$ is equal to the whole space C_σ. It is true when σ is compact.

PROPOSITION 9.5. *Let $C \in \mathscr{H}$ and σ be a compact space-like surface (without boundary) in M. We have the decomposition into an orthogonal sum:*

$$C^\infty\big(\operatorname{den} S^2 T(\sigma) \oplus S^2 T^*(\sigma)\big) = \ker A \oplus \operatorname{im} A^*. \tag{9.15}$$

The proof of the proposition is based on the theory of differential operators with injective symbols ([4], [34]) and is given in Section 15. It seems that decomposition (9.15) can also be proved in the non-compact case but we have to impose appropriate boundary conditions.

COROLLARY 1. *If decomposition* (9.15) *holds, then*

$$\ker A = (\ker A \cap \ker AJ) \oplus \operatorname{im} JA^* \quad \textit{(an orthogonal sum)}. \tag{9.16}$$

DEFINITION. $F_C(c)$ is a subspace of $\overset{1}{\tilde{T}}_C(\mathscr{H}, c)$ generated by elements belonging to $\ker A \cap \ker AJ$.

COROLLARY 2. *If* (9.15) *holds, then*

$$\begin{gathered} \overset{1}{W}_C(c) = W_C, \\ \tilde{T}_C(\mathscr{H}) = F_C(c) \oplus W_C \quad \textit{(a direct sum)}. \end{gathered} \tag{9.17}$$

Corollary 2 shows that in the case of a compact σ the subspace $\overset{1}{W}_C(c)$ is independent of the choice of $c \subset C$ because it is equal to the subspace W_C, whose definition does not contain c. We see also that the subspace F_C is dependent on the choice of $c \subset C$. We discuss this problem in Section 15, where the proof of the above corollaries are given.

For every admissible initial surface $c \subset C$ formula (9.17) gives a complementary subspace to the gauge distribution W_C. The subspace $F_C(c)$ determines degrees of freedom for the gravitational field (cf. Section 10). However, we cannot find a canonical decomposition of $\tilde{T}_C(\mathscr{H})$ into W_C and a complementary part.

We shall now explain a connection between the gauge distribution W and the action of the diffeomorphism group $\mathrm{Diff}(M)$ in the space $\mathscr{H}$. The group of diffeomorphisms of M acts on the right in the set of all pseudo-riemannian metrics in M having the signature $(-1, +3)$

$$\mathrm{Diff}(M) \times C^\infty(S^2T^*M) \ni (\varphi, \boldsymbol{g}) \to R_\varphi \boldsymbol{g} = \varphi^* \boldsymbol{g} \in C^\infty(S^2T^*M). \tag{9.18}$$

Action (9.18) generates a left action in the bundle $\pi\colon \mathscr{P} \to M$. In local coordinates $x^\mu \to \varphi^{\lambda'}(x^\mu) = y^{\lambda'}$,

$$\begin{aligned} g_{\mu\nu} &\to g_{\alpha'\beta'} = \frac{\partial(\varphi^{-1})^\mu}{\partial y^{\alpha'}} \frac{\partial(\varphi^{-1})^\nu}{\partial y^{\beta'}} g_{\mu\nu}, \\ \Gamma^\lambda_{\mu\nu} &\to \Gamma^{\tau'}_{\alpha'\beta'} = \frac{\partial(\varphi^{-1})^\mu}{\partial y^{\alpha'}} \frac{\partial(\varphi^{-1})^\nu}{\partial y^{\beta'}} \frac{\partial \varphi^{\tau'}}{\partial x^\lambda} \Gamma^\lambda_{\mu\nu} + \frac{\partial^2(\varphi^{-1})^\lambda}{\partial y^{\alpha'} \partial y^{\beta'}} \frac{\partial \varphi^{\tau'}}{\partial x^\lambda}. \end{aligned} \tag{9.19}$$

PROPOSITION 9.6. *Formulas* (9.19) *define a fibre preserving left action in* $\mathscr{P}$:

$$\mathrm{Diff}(M) \times \mathscr{P} \ni (\varphi, p) \to L_\varphi p \in \mathscr{P}. \tag{9.20}$$

The forms $\overset{4}{\theta}$ *and* $\overset{5}{\omega}$ *are invariant with respect to this action.*

This proposition follows from the transformation properties of $g_{\mu\nu}$, $\Gamma^\lambda_{\mu\nu}$, cf. (5.4), (5.5). The invariance of the form $\overset{4}{\theta}$ under the action (9.20) is a consequence of Proposition 5.1. The invariance of $\overset{5}{\omega}$ follows by the properties of exterior derivative.

The left action in the bundle $\mathscr{P}$ defines a right action in the space of sections of $\mathscr{P}$. By virtue of Proposition 9.6 we have the right action in the space $\mathscr{H}$:

$$\mathrm{Diff}(M) \times \mathscr{H} \ni (\varphi, C) \to \hat{R}_\varphi(C) = \varphi^* C \in \mathscr{H}. \tag{9.21}$$

(In local coordinates, if $R_{\mu\nu}(\boldsymbol{g}) = \lambda g_{\mu\nu}$, then $R_{\mu\nu}(\varphi^*\boldsymbol{g}) = \lambda(\varphi^*\boldsymbol{g})_{\mu\nu}$.)

It is known (cf. [13]) that the Lie algebra of the group $\mathrm{Diff}(M)$ can be identified with the Lie algebra of smooth vector fields in M (with the commutator as a Lie bracket). The action (9.21) generates the action of the Lie algebra of $\mathrm{Diff}(M)$:

$$C^\infty(TM) \ni v \to d\hat{R}_{\mathrm{id}}(C)v \in \tilde{T}_C(\mathscr{H}) \tag{9.22}$$

where $d\hat{R}_{\mathrm{id}}(C)$ is the derivative of (9.21) with respect to the first variable at the point $(\mathrm{id}, C) \in \mathrm{Diff}(M) \times \mathscr{H}$.

We have the following result, which connects the gauge distribution of Ω with the action (9.21):

PROPOSITION 9.7. *For every admissible initial surface $c \subset C$*

$$\mathrm{im}\, d\hat{R}_{\mathrm{id}}(C) = \overset{1}{W}_C(c).$$

For the proof see Section 15.

This proposition shows that the subspaces $\overset{1}{W}_C(c)$ are isomorphic for any $c \subset C$ (cf. also the discussion in Section 15).

In the case where admissible initial surfaces in $\mathscr{P}$ are compact we can combine Propositions 9.5 (Corollary 2) and 9.7 into

THEOREM 9.1. *For a manifold M with compact spatial sections (i.e., if admissible initial surfaces contained in $C \in \mathscr{H}$ are compact manifolds without boundary) we have:*

$$\mathrm{im}\, d\hat{R}_{\mathrm{id}}(C) = W_C.$$

This result shows that the degeneracy of Ω is fully determined by the action of $\mathrm{Diff}(M)$ in $\mathscr{H}$. Therefore it is natural to pass to the quotient space $\mathscr{H}$/action of $\mathrm{Diff}(M)$. We discuss such a construction in Section 10.

The results obtained in the present section enable us to describe isotropic submanifolds of Ω.

DEFINITION. A linear subspace $H \subset \tilde{T}_C(\mathscr{H})$ is called *isotropic* if for $\hat{X}_1, \hat{X}_2 \in H$, $\Omega(\hat{X}_1, \hat{X}_2) = 0$.

The following two subspaces of $\tilde{T}_C(\mathscr{H})$ are isotropic: the subspace ${}_\pi\tilde{T}_C(\mathscr{H}, c) \subset \tilde{T}_C(\mathscr{H})$ consisting of vectors $\hat{Y}$ which are represented by vector fields Y on C such that

$$\begin{aligned} \delta N = 0, \quad \delta N_k = 0 \quad &\text{in} \quad C, \\ \delta g_{ij} = 0 \quad &\text{in} \quad c; \end{aligned} \tag{9.23}$$

The subspace ${}_g\tilde{T}_C(\mathscr{H}, c) \subset \tilde{T}_C(\mathscr{H})$ consisting of vectors $\hat{Y}$ which are represented by vector fields Y on C such that

$$\begin{aligned} \delta N = 0, \quad \delta N_k = 0 \quad &\text{in} \quad C, \\ \delta \pi^{ij} = 0 \quad &\text{in} \quad c. \end{aligned} \tag{9.24}$$

In the first case the couple $(\delta\pi^{ij}, 0)$ must satisfy the constraint equations

(8.10b), which have the form

$$\bar{\nabla}_j \delta\pi^{ij} = 0 \quad \text{in} \quad c,$$

(9.25a)

$$(\pi_{ks}\delta\pi^{ks} - \tfrac{1}{2} g_{ks} \operatorname{tr}\pi\, \delta\pi^{ks}) = 0 \quad \text{in} \quad c.$$

In the second case the couple $(0, \delta g_{ij})$ must satisfy the equations

$$\delta\bar{\Gamma}^i_{ks}\pi^{ks} = 0 \quad \text{in} \quad c,$$

(9.25b)

$$\delta\bar{R} - \frac{1}{\bar{g}}(\bar{R} - 2\lambda)\,\delta\bar{g} - \frac{2}{\bar{g}}(\pi^i_k \pi^{kj}\,\delta g_{ij} - \tfrac{1}{2}\operatorname{tr}\pi\pi^{ks}\,\delta g_{ks}) = 0 \quad \text{in} \quad c.$$

We have

$${}_\pi\tilde{T}_C(\mathscr{H}, c) \oplus {}_g\tilde{T}_C(\mathscr{H}, c) \subset \overset{1}{\tilde{T}}_C(\mathscr{H}, c).$$

PROPOSITION 9.8. *If admissible initial surfaces in $\mathscr{P}$ are compact, then for every admissible initial surface $c \subset C$ the subspaces*

$$H_1(c) = \big({}_\pi\tilde{T}_C(\mathscr{H}, c) \cap F_C(c)\big) \oplus \overset{0}{\tilde{T}}_C(\mathscr{H}, c) \oplus \overset{0}{W}_C(c),$$

$$H_2(c) = \big({}_g\tilde{T}_C(\mathscr{H}, c) \cap F_C(c)\big) \oplus \overset{0}{\tilde{T}}_C(\mathscr{H}, c) \oplus \overset{0}{W}_C(c)$$

are maximal isotropic subspaces of Ω.

The proof is given in Section 15.

According to A. Weinstein [40] we call a maximal isotropic subspace $H \subset \tilde{T}_C(\mathscr{H})$ a *lagrangian subspace* if it has a complement H' which is also a maximal isotropic subspace. However, the degeneracy of Ω gives rise to the non-existence of lagrangian subspaces. In the next section we pass to the quotient space $\tilde{\mathscr{H}} = \mathscr{H}/\mathrm{Diff}(M)$. The 2-form Ω generates a non-degenerate 2-form $\tilde{\Omega}$ in $\tilde{\mathscr{H}}$. In the space $\tilde{\mathscr{H}}$ we have two families of lagrangian submanifolds: those generated by $\big({}_\pi\tilde{T}_C(\mathscr{H}, c) \cap F_C(c)\big)$ and those generated by $\big({}_g\tilde{T}_C(\mathscr{H}, c) \cap F_C(c)\big)$.

10. Degrees of freedom and a superphase space for General Relativity

The space $\mathscr{H}$ introduced in previous sections is too large for a description of dynamics in General Relativity. A physical meaning can be assigned not to different metrics in M but to their equivalency classes with respect to the action (9.18). The corresponding action (9.21) in the space $\mathscr{H}$ splits it into classes of equivalent states. In terms of tangent spaces $\tilde{T}_C(\mathscr{H})$, $C \in \mathscr{H}$, we have equivalency classes generated by the action (9.22). In the case of compact admissible initial surfaces in $\mathscr{P}$ we can give a complete discussion. It follows by Corollary 2 of Proposition 9.5 and by Theorem 9.1

that the space $F_C(c)$ determines the degrees of freedom for the gravitational field. By virtue of the definition of $F_C(c)$ we have 8 constraint equations for 12 components

$$(\delta\pi^{ij}, \delta g_{ij}) \in C^\infty\big(\operatorname{den} S^2T(\sigma) \oplus S^2T^*(\sigma)\big).$$

Four of them are the constraint equations (8.10b). The remaining four are obtained from (8.10b) by the transformation

$$\begin{aligned} \delta\pi^{ij} \to \delta'\pi^{ij} &= \sqrt{\bar{g}}\,\bar{g}^{ip}\bar{g}^{jq}\,\delta g_{pq}, \\ \delta g_{ij} \to \delta' g_{ij} &= -\frac{1}{\sqrt{\bar{g}}}\, g_{ip} g_{jp}\,\delta\pi^{pq}. \end{aligned} \tag{10.1}$$

In this sense we say that there are 4 independent degrees of freedom (in the phase space) for the gravitational field (2 degrees of freedom in the configuration space).

We now discuss briefly the global problem. Let $\sigma \subset M$ be a 3-dimensional compact submanifold and let $C^\infty_{(E,\sigma)}(S^2T^*M)$ be a set of Einstein metrics in M for which σ is space-like. We assume also that σ determines correctly the Cauchy problem, i.e., for every $\boldsymbol{g} \in C^\infty_{(E,\sigma)}(S^2T^*M)$ the natural lift of σ to $\mathscr{P}$ is an admissible initial surface. If $\mathcal{O}$ is a sufficiently small neighbourhood of the identity in $\operatorname{Diff}(M)$ (in a suitable topology), then we have the action:

$$\mathcal{O} \times C^\infty_{(E,\sigma)}(S^2T^*M) \ni (\varphi, \boldsymbol{g}) \to R_\varphi \boldsymbol{g} = \varphi^*\boldsymbol{g} \in C^\infty_{(E,\sigma)}(S^2T^*M). \tag{10.2}$$

One can divide the space $C^\infty_{(E,\sigma)}(S^2T^*M)$ by the action (10.2) to obtain the superphase space for General Relativity. If we express $g_{\mu\nu}$ in terms of g_{ij}, π^{ij} in σ and N, N_k in M, we do not know whether the action (10.2) allows us to change N, N_k in an arbitrary way. This fact is possible, because it is satisfied in the tangent space (Proposition 9.7 and Theorem 9.1). This problem is difficult and therefore it is better to define the superphase space axiomatically. Let Cd a subset of $C_\sigma = C^\infty\big(\operatorname{den} S^2T(\sigma) \oplus S^2T^*(\sigma)\big)$ consisting of couples (π^{ij}, g_{ij}) satisfying the constraint equations (6.19b). Cd is the set of Cauchy data for the Einstein equations. We define the action of $\mathcal{O} \subset \operatorname{Diff}(M)$ in the set Cd:

$$\mathcal{O} \times \mathrm{Cd} \ni (\varphi, \boldsymbol{\pi}, \bar{\boldsymbol{g}}) \to R_\varphi(\boldsymbol{\pi}, \bar{\boldsymbol{g}}) = \big((\varphi^{-1})_*\boldsymbol{\pi}, \varphi^*\bar{\boldsymbol{g}}\big) \in \mathrm{Cd}. \tag{10.3}$$

The couple $\big((\varphi^{-1})_*\boldsymbol{\pi}, \varphi^*\bar{\boldsymbol{g}}\big)$ is defined in the following way: let $\boldsymbol{g}$ be the metric in M determined by $(\boldsymbol{\pi}, \bar{\boldsymbol{g}})$ (we take $N = 1$, $N_k = 0$); the pull-back $\varphi^*\boldsymbol{g}$ generates in σ the couple consisting of tensor density $(\varphi^{-1})_*\boldsymbol{\pi}$ and tensor field $(\varphi)^*\bar{\boldsymbol{g}}$. It is obvious that this couple satisfies also the constraint equations (6.19b).

The superphase space $\tilde{\mathscr{H}}$ is the quotient space–Cd divided by the action (10.3).

Remark. We can define other actions of the type (10.3) taking different lapse functions and shift vectors in M. If the transformation (10.2) acts transitively in variables N, N_k, then all those actions give the same space $\tilde{\mathscr{H}}$.

Recently a similar object has been proposed by A. Fischer and J. Marsden [17] as a possible choice of a superphase space for Einstein dynamics. The structure of such an object is very complicated (cf. [16], [17]) and requires detailed investigations. We do not consider this problem here.

11. A pseudo-differential structure in the space $\mathscr{H}$ A Lie algebra of functionals on $\mathscr{H}$

It has been shown in [26] that for a given field theory with a multiphase structure $(\mathscr{P}, \omega)$ ($\mathscr{P}$ is a bundle over M) the set of states $\mathscr{H}$ can be endowed with a kind of pseudodifferential structure. We need such a structure for defining such notions as smooth vector fields in $\mathscr{H}$, their commutators and the calculus of the exterior forms. In the structure described in [26] Ω becomes a 2-differential form in $\mathscr{H}$. The main theorem of the above-mentioned paper states that $d\Omega = 0$. We are going to describe briefly the results of [26]. Moreover, we give an alternative approach to those problems, which is simpler and probably more useful for nonlinear theories.

Let P_k be an open k-dimensional cube in $\boldsymbol{R}^k$ $(k = 0, 1, 2, \ldots)$

$$P_k = \{\boldsymbol{t} = (t^1, \ldots, t^k) \in \boldsymbol{R}^k\colon\ |t^i| < 1,\ 1 \leqslant i \leqslant k\}.$$

For every $C \in \mathscr{H}$ we consider k-parameter families of diffeomorphisms of $\mathscr{P}$; $\big(\psi(\boldsymbol{t}; \cdot)\big)_{\boldsymbol{t} \in P_k}$ $(k = 0, 1, 2, \ldots)$ which transform the submanifold $C \subset \mathscr{P}$ onto a submanifold $C_{\boldsymbol{t}} \subset \mathscr{P}$ which is a ω-singular section of $\mathscr{P}$. In general, such a family does not transform all states onto states; we only know that it transforms the state C onto the state $C_{\boldsymbol{t}}$.

We assume also the following non-degeneracy condition:

$$(11.1)\quad \begin{cases} \text{(i) } \psi(0; \cdot) = \mathrm{id}_{\mathscr{P}}, \\ \text{(ii) (a) if } k = 1, \text{ then for every } t \in P_1 \text{ there exists a } p \in C \text{ such} \\ \qquad \text{that the vertical vector } \dfrac{\partial}{\partial t}\psi(t; p) \text{ is not equal to zero,} \\ \qquad \text{(b) if } k > 1, \text{ and } \big(t^1(\tau), \ldots, t^k(\tau)\big) \text{ is a curve in } P_k \text{ with} \\ \qquad\quad \text{a non-vanishing tangent vector, then the mapping} \\ \qquad\qquad\qquad \varphi(\tau; \cdot) = \psi\big(t^1(\tau), \ldots, t^k(\tau); \cdot\big) \\ \qquad\quad \text{satisfies condition (a).} \end{cases}$$

Every family $\big((\psi(t;\cdot))\big)_{t\in P_k}$ of type (11.1) generates a mapping $P_k \ni t \to \varkappa(t) \in \mathscr{H}$

$$\varkappa(t) = \{\psi(t;p) \in \mathscr{P}\colon\ p \in C\}. \tag{11.2}$$

Taking all families of type (11.1), we obtain a set of mappings $\varkappa\colon P_k \to \mathscr{H}$, $k = 0, 1, 2, \ldots$ We denote it by $\mathscr{A}$.

Condition (i) of (11.1) provides the injectivity of these mappings and therefore the image $\varkappa(P_k)$ is called a *k-dimensional surface* in $\mathscr{H}$.

Subsequent geometric applications of the theory require the following axioms:

$$\text{(11.3)}\quad \begin{cases} A.\ \text{For every } C \in \mathscr{H} \text{ there exists at least one mapping } (\varkappa, P_k) \in \mathscr{A} \\ \quad\ \text{such that } C \subset \varkappa(P_k); \\ B.\ \text{For every two mappings } (\varkappa_1, P_{k_1}),\ (\varkappa_2, P_{k_2}) \in \mathscr{A}, \text{ having} \\ \quad\ \text{the property } \varkappa_1(P_{k_1}) \cap \varkappa_2(P_{k_2}) \neq \emptyset \text{ there exists a mapping} \\ \quad\ (\varkappa, P_k) \in \mathscr{A} \text{ such that} \\ \qquad\qquad \varkappa_1(P_{k_1}) \subset \varkappa(P_k), \quad \varkappa_2(P_{k_2}) \subset \varkappa(P_k). \end{cases}$$

The set of injective mappings $\varkappa \in \mathscr{A}$ generates in $\mathscr{H}$ a topology of an inductive limit. It has recently been proved that the space is a Hausdorff space in this topology (J. Czyż — to appear in Bull. Acad. Polon. Sci.). The couple $(\mathscr{H}, \mathscr{A})$ is called an *inductive differential manifold* [26].

The vector space $T_C(\mathscr{H})$ tangent to $\mathscr{H}$ at C is generated by curves

$$P_1 \ni t \to \varkappa(t) \in \mathscr{H}, \quad \text{where} \quad (\varkappa, P_1) \in \mathscr{A}.$$

It has been proved in [26] that a vector $\hat{Y} \in T_C(\mathscr{H})$ is represented by a vertical vector field Y on C fulfilling the linearized equations of motion (7.5). In a natural way the definition of a smooth vector field in $\mathscr{H}$ can be given. A map $\mathscr{H} \ni C \to \hat{X}(C) \in T_C(\mathscr{H})$ is a smooth vector field in $\mathscr{H}$ if for every $C \in \mathscr{H}$ there exists such a mapping $(\varkappa, P_1)$ that $\hat{X}(C_t)$ is represented by the vector field

$$X\big(\psi(t;p)\big) = \frac{\partial}{\partial t}\psi(t;p), \quad p \in C_0 = C, \quad \text{on} \quad C_t. \tag{11.4}$$

In a similar way mappings $(\varkappa, P_k)$ generate k-vector fields in a k-dimensional surface $\varkappa(P_C) \subset \mathscr{H}$. The axiom B enables us to define the commutator of two and more vector fields. Having this definition, we can introduce the notion of the exterior derivative for differential forms in $\mathscr{H}$ (cf. [26]).

The construction described depends essentially on the axioms (11.3). It seems that in non-linear theories it is difficult to check their validity. Therefore we propose a simpler approach. Instead of mappings $(\varkappa, P_k)$ generated by k-parameter families of diffeomorphisms of $\mathscr{P}$ we shall consider k-parameter families of states:

$$P_k \ni t \to \xi(t) \in \mathscr{H},$$

generated by k-parameter families of sections of $\mathscr{P}$: φ: $P_k \times M \to \mathscr{P}$

$$\xi(t) = \{\varphi(t; x) \in \mathscr{P};\ x \in M\}. \tag{11.5}$$

Every family $(\xi(t))_{t \in P_k}$ of the type (11.5) defines k-vector fields on $C_t = \xi(t)$ satisfying (7.5). The set of such vector fields on $C = \xi(0)$ is called the *tangent space* at $C \in \mathscr{H}$ and denoted by $T_C(\mathscr{H})$. Instead of (11.3) we postulate the following axiom:

For every two families of states $(\xi_1(t))_{t \in P_{k_1}}$, $(\xi_2(t))_{t \in P_{k_2}}$ of the type (11.5) there exists a family $(\xi(t))_{t \in P_k}$ of the same type such that

$$\xi_1(P_{k_1}) \subset \xi(P_k), \qquad \xi_2(P_{k_2}) \subset \xi(P_k). \tag{11.6}$$

The image $\xi(P_k)$ (for a family $(\xi(t))_{t \in P_k}$) determines a subset $\varphi(P_k, M) \subset \mathscr{P}$. In general, (P_k, M) is not an $(n+k)$-dimensional submanifold of $\mathscr{P}$ because at some points it can have a dimension $n \leqslant m < n+k$ and at some points it can be singular (i.e. the tangent space is not well determined). In spite of these difficulties we can define in $\varphi(P_k; M)$ vector fields generated by partial derivatives $\frac{\partial}{\partial t^s}\varphi$, $1 \leqslant s \leqslant k$. Having two vector fields $\hat{X}_1, \hat{X}_2$ in $\mathscr{H}$ and the corresponding generating mappings $(\xi_1, P_{1'})$, $(\xi_2, P_{1''})$, we have vector fields $\tilde{X}_1, \tilde{X}_2$ in the subset $\varphi_1(P_{1'}; M) \cap \varphi_2(P_{1''}; M) \subset \mathscr{P}$. Their commutator defines the commutator of $\hat{X}_1$, $\hat{X}_2$:

$$[\hat{X}_1, \hat{X}_2] = [\tilde{X}_1, \tilde{X}_2]^{\wedge}. \tag{11.7}$$

For the 2-form Ω defined by (7.6) we compute its exterior derivative. According with the well-known formula (cf. [36]) we have for vector fields $\hat{X}_1, \hat{X}_2, \hat{X}_3$ in $\mathscr{H}$:

$$\begin{aligned} 3d\Omega(\hat{X}_1, \hat{X}_2, \hat{X}_3) &= \hat{X}_1\Omega(\hat{X}_2, \hat{X}_3) - \hat{X}_2\Omega(\hat{X}_1, \hat{X}_3) + \hat{X}_3\Omega(\hat{X}_1, \hat{X}_2) - \\ &\quad - \Omega([\hat{X}_1, \hat{X}_2], \hat{X}_3) + \Omega([\hat{X}_1, \hat{X}_3], \hat{X}_2) - \Omega([\hat{X}_2, \hat{X}_3], \hat{X}_1) \end{aligned} \tag{11.8}$$

where $\hat{X}_{i_1}\Omega(\hat{X}_{i_2}, \hat{X}_{i_3})$ is the Gateaux derivative of the function $\Omega(\hat{X}_{i_2}, \hat{X}_{i_3})$ in the direction of the field $\hat{X}_{i_1}$.

Geometric considerations together with the definition of the Lie derivative give:

LEMMA 11.1. *If $\tilde{X}_{i_1}, \tilde{X}_{i_2}, \tilde{X}_{i_3}$ are vector fields in the subset*

$$\varphi_1(P_{k_1}; M) \cap \varphi_2(P_{k_2}; M) \cap \varphi_3(P_{k_3}; M)$$

of $\mathscr{P}$ generated by fields $\hat{X}_{i_1}, \hat{X}_{i_2}, \hat{X}_{i_3}$, then

$$\big(\hat{X}_{i_1}\Omega(\hat{X}_{i_2}, \hat{X}_{i_3})\big)(C) = \int_{c \subset C} \pounds_{\tilde{X}_{i_1}}\big((\tilde{X}_{i_2} \wedge \tilde{X}_{i_3}) \lrcorner\, \omega\big) \tag{11.9}$$

where c is an admissible initial surface contained in C.

For a detailed proof of the lemma see [26].

LEMMA 11.2. *If $\tilde{X}_i$, $i = 1, 2, 3$, are vector fields in a finite-dimensional manifold $\mathscr{K}$ and ω is a closed differential $(n+1)$-form in $\mathscr{K}$ $(n > 1)$, then*

$$(11.10)\qquad \pounds_{\tilde{X}_1}\big((\tilde{X}_2\wedge\tilde{X}_3)\ \lrcorner\ \omega\big)-\pounds_{\tilde{X}_2}\big((\tilde{X}_1\wedge\tilde{X}_3)\ \lrcorner\ \omega\big)+\pounds_{\tilde{X}_3}\big((\tilde{X}_1\wedge\tilde{X}_2)\ \lrcorner\ \omega\big)-$$
$$-([\tilde{X}_1, \tilde{X}_2]\wedge\tilde{X}_3)\ \lrcorner\ \omega+([\tilde{X}_1, \tilde{X}_3]\wedge\tilde{X}_2)\ \lrcorner\ \omega-([\tilde{X}_2, \tilde{X}_3]\wedge\tilde{X}_1)\ \lrcorner\ \omega$$
$$= d\big((\tilde{X}_1\wedge\tilde{X}_2\wedge\tilde{X}_3)\ \lrcorner\ \omega\big).$$

For the proof see [26].

THEOREM 11.1. *The form Ω is closed, i.e., $d\Omega = 0$.*

Proof. By (7.6), (11.8), (11.7), (11.9) and (11.10) we have

$$(11.11)\qquad 3d\Omega(\hat{X}_1, \hat{X}_2, \hat{X}_3)(C) = \int_c d\big((\tilde{X}_1\wedge\tilde{X}_2\wedge\tilde{X}_3)\ \lrcorner\ \omega\big).$$

Therefore the theorem follows from the Stockes theorem and the boundary conditions for fields $\tilde{X}_i$.

We know that in lagrangian theories a multisymplectic form ω is exact, i.e., $\omega = d\theta$. In this case we can give an effective formula for a primitive 1-form Θ of Ω.

PROPOSITION 11.1. *If $\omega = d\theta$ and $c \subset C$ is an admissible initial surface, then*

$$(11.12)\qquad \Theta_C(\hat{X}) = 2\int_c X\ \lrcorner\ \theta,$$

where a vector field X on C represents a vector $\hat{X} \in T_C(\mathscr{H})$, defines a 1-form Θ in $\mathscr{H}$ such that $d\Theta = \Omega$.

The proof is based on the following results:

$$(11.13)\qquad 2d\Theta(\hat{X}_1, \hat{X}_2) = \hat{X}_1\Theta(\hat{X}_2)-\hat{X}_2\Theta(\hat{X}_1)-\Theta([\hat{X}_1, \hat{X}_2]).$$

LEMMA 11.3. *If $\tilde{X}_i$, $i = 1, 2$, are vector fields in the subset $\varphi_1(P_{1'}; M)\cap\varphi_2(P_{1''}; M) \subset \mathscr{P}$ generated by fields $\hat{X}_1$, $\hat{X}_2$ in $\mathscr{H}$, then*

$$(11.14)\qquad \big(\hat{X}_{i_1}\Theta(\hat{X}_{i_2})\big)(C) = \int_{c\subset C} \pounds_{\tilde{X}_{i_1}}(\tilde{X}_{i_2}\ \lrcorner\ \theta).$$

The proof is analogous to the proof of Lemma 11.1.

LEMMA 11.4. *If $\tilde{X}_i$, $i = 1, 2$, are vector fields in a finite-dimensional manifold $\mathscr{K}$ and θ is an n-form in $\mathscr{K}$, then*

$$(11.15)\qquad \pounds_{\tilde{X}_1}(\tilde{X}_2\ \lrcorner\ \theta)-\pounds_{\tilde{X}_2}(\tilde{X}_1\ \lrcorner\ \theta)-[\tilde{X}_1, \tilde{X}_2]\ \lrcorner\ \theta$$
$$= (\tilde{X}_1\wedge\tilde{X}_2)\ \lrcorner\ d\theta-d\big((\tilde{X}_1\wedge\tilde{X}_2)\ \lrcorner\ \theta\big).$$

For the proof see Section 15.

By virtue of (11.13), (11.14) and (11.15) we have

$$(11.16)\qquad 2d\Theta(\hat{X}_1, \hat{X}_2)(C) = \int_{c\subset C} 2(\tilde{X}_1\wedge\tilde{X}_2) \lrcorner\, d\theta - 2\int_{c\subset C} d\big((\tilde{X}_1\wedge\tilde{X}_2) \lrcorner\, \theta\big).$$

Integrating (11.16) and using the boundary conditions for fields $\tilde{X}_i$, we obtain $d\Theta = \Omega$.

Remark. We know (cf. Section 7) that the definition (7.6) of the 2-form Ω does not depend on the choice of an admissible initial surface $c \subset C$. It is not true for formula (11.12), which defines different 1-forms Θ in $\mathscr{H}$ for different $c \subset C$. Of course, all these 1-forms have the same exterior derivative, equal to Ω.

Example. *Non-linear electrodynamics* (cf. Section 7). Let $(\mathscr{P}, \overset{5}{\omega})$ be a 4-symplectic manifold for electrodynamics described in Section 7. Let $\overset{4}{\theta}$ be a primitive form of $\overset{5}{\omega}$ defined by (7.10). For a vector $\hat{X}\in T_C(\mathscr{H})$ and a vector field X, representing it, of type (7.20) on C we have

$$(11.17)\qquad \Theta_C(\hat{X}) = -\frac{1}{2\pi}\int_\sigma \delta A_k h^{0k}\sqrt{-g}\,dx^1\wedge dx^2\wedge dx^3.$$

In the $(3+1)$ formalism the vector field X assumes the form (7.23) and we have, instead of (11.17),

$$(11.17')\qquad \Theta_C(\hat{X}) = -\frac{1}{2\pi}\int_\sigma \delta A_k \mathscr{D}^k dx^1\wedge dx^2\wedge dx^3.$$

We define also another 4-form in $\mathscr{P}$:

$$(11.18)\qquad {}'\overset{4}{\theta} = \frac{1}{4\pi} A_\nu dx^0\wedge\ldots\wedge d\big(h^{\mu\nu}\sqrt{-g(x^\lambda)}\big)\wedge\ldots\wedge dx^3 + \\ + H(x^\lambda, A_\nu, h^{\mu\nu})\sqrt{-g}\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3,$$

having the property $d'\overset{4}{\theta} = d\overset{4}{\theta} = \overset{5}{\omega}$. (It is easy to check that (11.18) defines a 4-form in $\mathscr{P}$.) The 4-form ${}'\overset{4}{\theta}$ defines another primitive 1-form of Ω. For $\hat{X}\in T_C(\mathscr{H})$ represented by a vector field on C of type (7.23) we have

$$(11.19)\qquad {}'\Theta_C(X) = \frac{1}{2\pi}\int_\sigma \delta\mathscr{D}^k A_k dx^1\wedge dx^2\wedge dx^3.$$

We now pass to General Relativity. The 4-form $\overset{4}{\theta}$ defined by (5.7) determines a primitive form Θ of the presymplectic form Ω:

$$(11.20)\qquad \Theta_C(\hat{X}) = 2\int_{c\subset C} X \lrcorner\, \overset{4}{\theta},$$

where X is a vector field of type (8.1) on C.

PROPOSITION 11.2. *If we take an* ADMW *coordinate system determined by a given space-like surface* $\sigma \subset M$ *and a vector field* X *on* C *of type* (8.6) *which represents a vector* $\hat{X} \in T_C(\mathscr{H})$, *then*

$$\Theta_C(\hat{X}) = -\int_\sigma \delta\pi^{ij} g_{ij}\, dx^1 \wedge dx^2 \wedge dx^3. \tag{11.20'}$$

We can also define another primitive 1-form for Ω, which in the ADMW coordinate system is defined in the following way:

$$'\Theta_C(\hat{X}) = \int_\sigma \pi^{ij}\, \delta g_{ij}\, dx^1 \wedge dx^2 \wedge dx^3, \tag{11.21}$$

where a vector $\hat{X} \in T_C(\mathscr{H})$ is represented by a vector field of type (8.6). However, we do not know how to express (11.21) in an arbitrary coordinate system analogous to (11.20).

The presymplectic form Ω in $\mathscr{H}$ defines a Lie algebra structure in some subset of smooth functionals in $\mathscr{H}$. This structure is a generalization of the Poisson algebra of functions in the phase space. Contrary to the 1-dimensional case (mechanics), we do not take into account all smooth functionals in $\mathscr{H}$ but only a subset. We know (cf. Section 3) that a symplectic 2-form Ω in mechanics gives an isomorphism between the tangent and the cotangent space of the manifold $\mathscr{P}_{2n} \cong \mathscr{H}$. In the general situation, for a field theory $(\mathscr{P}, \omega)$, the form Ω is degenerate and therefore the mapping

$$T_C(\mathscr{H}) \ni \hat{X} \to \hat{X}^\flat \in T_C^*(\mathscr{H}), \tag{11.12}$$

where

$$\langle \hat{Y}, \hat{X}^\flat \rangle = \hat{X}^\flat(\hat{Y}) = \Omega(\hat{X}, \hat{Y}), \qquad \hat{Y} \in T_C(\mathscr{H}),$$

is not injective. The inverse mapping $\#$ is determined up to vectors belonging to the gauge subspace W_C.

In the set of smooth functionals in $\mathscr{H}$ we distinguish a subset whose elements are called physical quantities (observables).

DEFINITION. A smooth functional $F\colon \mathscr{H} \to \boldsymbol{R}$ is called a *physical quantity* if there exists a smooth vector field $\hat{X}$ in $\mathscr{H}$ such that

$$dF = -\hat{X}^\flat, \tag{11.23}$$

or equivalently, if for every vector field $\hat{Y}$ in $\mathscr{H}$

$$dF(\hat{Y}) = -\Omega(\hat{X}, \hat{Y}). \tag{11.24}$$

Formula (11.24) implies that, for $\hat{Y} \in W_C$, $dF(\hat{Y}) = 0$, i.e., the functional F is independent of the gauge. We cannot assert (in the general case) that every smooth functional in $\mathscr{H}$ independent of the gauge is a physical quantity, i.e., satisfies (11.23) for a vector field $\hat{X}$ in $\mathscr{H}$. In electro-

dynamics and gravidynamics we can define a wide class of functionals called regular and every smooth regular functional which does not depend on the gauge is a physical quantity (cf. Proposition 11.5).

In the set $\mathscr{F}$ of physical quantities in $\mathscr{H}$ we introduce a bilinear skew-symmetric form

$$(11.25)\qquad \mathscr{F}\times\mathscr{F}\ni(F_1,F_2)\to\{F_1,F_2\}\in\mathscr{F},\quad \{F_1,F_2\}=\Omega(\hat{X}_1,\hat{X}_2)$$

where $dF_1=-\hat{X}_1^\flat$, $dF_2=-\hat{X}_2^\flat$.

The functional $\{F_1,F_2\}$ is called the *Poisson bracket* of F_1,F_2.

The property $d\Omega=0$ implies

PROPOSITION 11.3 (cf. [26]). (i) *If* $dF_1=-\hat{X}_1^\flat$, $dF_2=-\hat{X}_2^\flat$; $\hat{X}_1,\hat{X}_2\in C^\infty(T(\mathscr{H}))$, *then*

$$(11.26)\qquad d(\{F_1,F_2\})=-[\hat{X}_1,\hat{X}_2]^\flat.$$

(ii) *The Poisson bracket of two physical quantities is also a physical quantity.*

(iii) *The Jacobi identity holds*:

$$(11.27)\qquad \big\{\{F_1,F_2\},F_3\big\}+\big\{\{F_2,F_3\},F_1\big\}+\big\{\{F_3,F_1\},F_2\big\}=0.$$

PROPOSITION 11.4 (cf. [26]). *The gauge distribution W of Ω is involutive, i.e., if $\hat{X}_1$, $\hat{X}_2$ are vector fields in $\mathscr{H}$ with values in W, then $[\hat{X}_1,\hat{X}_2]$ is also a vector field with values in W.*

The statement of the above proposition is nothing else than the Frobenius integrability condition [31]. Although $\mathscr{H}$ is not a Banach manifold, we can try to construct a space $\tilde{\mathscr{H}}$ such that $T(\tilde{\mathscr{H}})=T(\mathscr{H})/W$ and a non-degenerate 2-form $\tilde{\Omega}$ in $\tilde{\mathscr{H}}$. We have discussed this problem for General Relativity in the previous section.

We refer the reader to [26], where examples of physical quantities and Poisson brackets in electrodynamics and scalar field theory are given. Independence of gauge implies that for instance in electrodynamics the potentials A_μ do not generate physical quantities but $\boldsymbol{B}=\operatorname{rot}\boldsymbol{A}$ and $D^k=h^{0k}$ do. It can also be shown that in the case of a homologically non-trivial 3-dimensional surface $\sigma\subset M$ the harmonic part of the covector field A_k in σ determines a physical quantity.

We shall now give the definition of a regular functional in gravidynamics.

DEFINITION. A functional $F\colon\mathscr{H}\to\boldsymbol{R}$ is called *regular* if for every $\sigma\subset M$ there exists an element $\mathfrak{X}_1\in C_\sigma$ such that for every $\hat{X}\in T_C(\mathscr{H})$ and the element $\mathfrak{X}\in C_\sigma$ corresponding to $\hat{X}$ (cf. Definitions in Section 9) we have

$$(11.28)\qquad dF_C(\hat{X})=g(C,\sigma)(\mathfrak{X}_1,\mathfrak{X}).$$

PROPOSITION 11.5. *A regular functional F independent of gauge is a physical quantity.*

For the proof see Section 15.

In particular, the local observables discussed in [24] are regular functionals (for instance in electrodynamics).

Remark. A similar result can be obtained in electrodynamics.

It is easier to check that a functional is regular because it requires the Hilbert space technique.

An exact discussion of physical quantities in General Relativity is beyond the aims and possibilities of the present paper. The problem is connected with the theory of invariants in differential geometry. We hope to deal with it elsewhere.

In Section 7 we introduced two spaces, $T_C(\mathscr{H})$ and $\tilde{T}_C(\mathscr{H})$. The space $T_C(\mathscr{H})$ consists of all vectors generated by curves in $\mathscr{H}$ and $\tilde{T}_C(\mathscr{H})$ consists of vector fields on C fulfilling (7.5) (in General Relativity (8.2), (8.3) or (8.9), (8.10)). We know that $T_C(\mathscr{H}) \subset \tilde{T}_C(\mathscr{H})$.

According to [6], [8], [19] we call a state $C \in \mathscr{H}$ *linearization stable* if $T_C(\mathscr{H}) = \tilde{T}_C(\mathscr{H})$, i.e., every solution of the linearized Einstein equations generates a curve in the set of Einstein metrics in M. It turns out that under a natural assumption a state C is linearization stable [19]. In particular the Minkowski space, i.e., $\boldsymbol{R}^4$ with the trivial Lorentz metric is linearization stable [8]. However, counterexamples can also be given [6], [19] but these cases are rather exceptional.

12. A variational principle for General Relativity

In spite of the non-existence of a lagrangian formulation of gravidynamics in the sense of Section 3, i.e., the non-existence of an invariant lagrangian function depending on first derivatives of components of a metric tensor $g_{\mu\nu}$, it is possible to obtain dynamical equations from a variational principle. In the present section we discuss a geometric approach to the Hilbert–Palatini action principle. In our formulation the Palatini method corresponds to Dedecker's theory for lagrangian systems.

We consider the bundle $\pi\colon \mathscr{P} \to M$ defined in Section 5. The transformation properties (5.4), (5.5) show that this bundle is a fibre product of the tensor bundle $\pi_1\colon S^2T^*M \to M$ and the affine bundle of coefficients of an affine connection $\pi_2\colon \Gamma \to M$. (In the bundle Γ we have coordinates $(x^\lambda, \Gamma^\tau_{\mu\nu})$.) Therefore a section f of $\mathscr{P}$ generates sections f_1 of π_1 and a section f_2 of π_2. We write $f_1 = \mathrm{pr}_1 f$, $f_2 = \mathrm{pr}_2 f$.

Let D be a relatively compact domain in M. We consider a set of sections of $\mathscr{P}$ over D with fixed values of the projection pr_2 on the

boundary of D. The action functional for gravidynamics is

$$C^\infty(D, \mathscr{P}) \ni f \to S_D(f) = \int_D f^* \overset{4}{\theta} = \int_{f(D)} \overset{4}{\theta} \tag{12.1}$$

where $\overset{4}{\theta}$ is the 4-form in $\mathscr{P}$ defined by (5.7). We know by (5.12), (5.13) that formula (12.1) defines the Hilbert–Palatini action functional (cf. [33]).

We say that a section f of $\mathscr{P}$ is an *extremal section* of the functional S with fixed boundary values of coefficients of an affine connection if for every one-parameter family $(f_\varepsilon)_{\varepsilon \in]-\delta,\delta[}$ of sections of $\mathscr{P}$ having the properties:

$$\text{(i) } f_0 = f, \qquad \text{(ii) } \mathrm{pr}_2 f_\varepsilon|_{\partial D} = \mathrm{pr}_2 f|_{\partial D} \tag{12.2}$$

the condition

$$\frac{d}{d\varepsilon} \int_D f_\varepsilon^* \overset{4}{\theta}|_{\varepsilon=0} = 0$$

holds.

PROPOSITION 12.1. *A section f of $\mathscr{P}$ is an extremal of the functional S (with fixed boundary values of* pr_2*) if and only if it is $\overset{5}{\omega}$-singular.*

For the proof see Section 15.

We can consider also a weaker variational principle, taking into account families of type (12.2) consisting of integrable sections (i.e., sections fulfilling (5.2)). It turns out that such a problem has the same extremals. We prove this fact in Section 15. The equivalence of both variational principles is in fact the equivalence of the Hilbert and Palatini methods of variation of the scalar curvature R ([33], [41]).

The Palatini variational principle has a clear geometric interpretation in the $3+1$ formulation. We take a domain $D \subset M$ whose boundary consists of two 3-dimensional surfaces $\sigma_1, \sigma_2 \subset M$. We assume that there exists an ADMW coordinate system in which $\sigma_i = \{x \in M; x^0 = t_i\}$, $i = 1, 2$, and consider only such metrics $g_{\mu\nu}$ in M (integrable sections of $\mathscr{P}$) for which σ_i are space-like. If σ_i are compact submanifolds of M without boundary or if the metrics in question are asymptotically flat, then we obtain:

PROPOSITION 12.2. *Let σ_i, $i = 1, 2$, be two 3-dimensional surfaces in M with given* ADMW *densities $\underset{1}{\pi}^{ij}$, $\underset{2}{\pi}^{ij}$ in σ_1, σ_2, respectively. Then an extremal f of functional* (12.1) *with respect to all integrable sections of $\mathscr{P}$ which determine the given* ADMW *densities in σ_1, σ_2 is a $\overset{5}{\omega}$-singular section of $\mathscr{P}$.*

For the proof see Section 15.

We know (cf. Section 8) that the components of the ADMW density π^{ij} in a given surface $\sigma \in M$ can be treated as momenta in General Relativity. By virtue of Proposition 12.2 we call the variational principle (12.1), (12.2) a *problem with fixed momenta* ([33], [41]).

13. The Hamilton-Jacobi equation in lagrangian field theories

This section is devoted to a discussion of the Hamilton–Jacobi equation in field theories with a Lagrange function. The idea consists in considering the action functional S as a functional in the space of initial and final values of fields. The simplest, well-known example is mechanics (cf. [1], [35]). In this case the Hamilton–Jacobi equation is simply a first order partial differential equation. We recall its construction.

In the 1-phase space $\mathscr{P} \subset T^*(V \times \boldsymbol{R})$ we have the canonical 1-form θ and the canonical 2-form ω (cf. (3.7), (3.8)). Every state, i.e., a ω-singular section f of $\mathscr{P} \to \boldsymbol{R}$, generates a section $\varrho \circ f$ of the bundle $\pi\colon V \times \boldsymbol{R} \to \boldsymbol{R}$ ($\varrho\colon \mathscr{P} \to V$ is the natural projection).

Let $(\overset{0}{q}{}^i, t_0)$, $(\overset{1}{q}{}^i, t_1)$ be two fixed points in $V \times \boldsymbol{R}$.

We consider a family (f_ε) of states

$$(f_\varepsilon) = \{t \to (p_j(t, \varepsilon), q^i(t, \varepsilon), t) \in \mathscr{P}\}, \qquad \varepsilon \in]1-\delta, 1+\delta[\tag{13.1}$$

such that

$$q^i(t_0, \varepsilon) = q^i(t_0, 1) = \overset{0}{q}{}^i, \qquad q^i(t_\varepsilon, \varepsilon) = \overset{\varepsilon}{q}{}^i,$$

where $\varepsilon \to (\overset{\varepsilon}{q}{}^i, t_\varepsilon)$ is an arbitrary curve in the neighbourhood of $(\overset{1}{q}{}^i, t_1)$ in $V \times \boldsymbol{R}$.

In other words, we consider solutions of the Hamilton equations (3.11) which have fixed initial values of coordinates at an instant t_0 and the final values of coordinates as well as the final instant of time are variable.

A family (f_ε) of type (13.1) generates a vertical vector field Y' on the submanifold $f_1(\boldsymbol{R}) = C$:

$$Y' = \left(\delta q^i - \frac{\partial q^i}{\partial t}\,\delta t\right)\frac{\partial}{\partial q^i} + \left(\delta p_j - \frac{\partial p_j}{\partial t}\,\delta t\right)\frac{\partial}{\partial p_j} \tag{13.2}$$

where

$$\begin{aligned} \delta q^i &= \left.\frac{dq^i(t_\varepsilon, \varepsilon)}{d\varepsilon}\right|_{\varepsilon=1} = \left.\frac{d\overset{\varepsilon}{q}{}^i}{d\varepsilon}\right|_{\varepsilon=1}; \\ \delta p_j &= \left.\frac{dp_j(t_\varepsilon, \varepsilon)}{d\varepsilon}\right|_{\varepsilon=1}; \qquad \delta t = \left.\frac{dt_\varepsilon}{d\varepsilon}\right|_{\varepsilon=1}. \end{aligned} \tag{13.3}$$

We have also a non-vertical vector field Y on C

$$Y = \delta p_j \frac{\partial}{\partial p_j} + \delta q^i \frac{\partial}{\partial q^i} + \delta t \frac{\partial}{\partial t}. \tag{13.4}$$

The vector field Y is the generator of a transformation G_ε which carries the solution with final values $(\overset{1}{q}{}^i, t_1)$ into a solution with final values $(\overset{\varepsilon}{q}{}^i, t_\varepsilon)$. The fields Y' and Y both fulfil the linearized equations of motion (7.5) and the difference $Y - Y'$ is a vector field tangent to C. Therefore Y and Y' determine the same vector $\hat{Y} \in T_C(\mathscr{H})$ (cf. [26]) tangent to the curve $\varepsilon \to f_\varepsilon$. The action functional

$$f_\varepsilon \to S(f_\varepsilon) = \int_{f_\varepsilon(R)} \theta \in \boldsymbol{R} \tag{13.5}$$

can be considered as a function depending on the final values $(\overset{\varepsilon}{q}{}^i, t_\varepsilon)$

$$S(f_\varepsilon) = S(\overset{\varepsilon}{q}{}^i, t_\varepsilon). \tag{13.6}$$

We know that

$$\left.\frac{dS(f_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=1} = \int_C \pounds_Y \theta = \int_C Y \lrcorner\, d\theta + \int_{\partial C} Y \lrcorner\, \theta. \tag{13.7}$$

By virtue of equation (3.10)

$$\left.\frac{dS(f_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=1} = \int_{\partial C} Y \lrcorner\, \theta. \tag{13.8}$$

∂C consists of two points $(\overset{0}{p}_j, \overset{0}{q}{}^i, t_0)$, $(\overset{1}{p}_j, \overset{1}{q}{}^i, t_1)$ but $Y \lrcorner\, \theta = 0$ at the first. Therefore

$$\left.\frac{dS(f_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=1} = \overset{1}{p}_j \left.\frac{d\overset{\varepsilon}{q}{}^j}{d\varepsilon}\right|_{\varepsilon=1} - H(\overset{1}{p}_j, \overset{1}{q}{}^i, t_1) \left.\frac{dt_\varepsilon}{d\varepsilon}\right|_{\varepsilon=1}. \tag{13.9}$$

By (13.6) and (13.9) we have

$$\frac{\partial S}{\partial q^i} = p_i; \qquad \frac{\partial S}{\partial t} = -H(p_j, q^i, t) \tag{13.10}$$

or

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q^j}, q^i, t\right) = 0. \tag{13.11}$$

Equation (13.11) is known as the Hamilton–Jacobi equation. In field theories a generalization of (13.11) is a functional equation for the action functional (3.50).

We consider a multiphase manifold $(\mathscr{P}, \omega)$, where $\mathscr{P} \subset \overset{4}{\wedge} T^* W$ and $\pi\colon W \to M$ is a bundle over the space-time M. We assume that $\mathscr{P}$ consists of all 4-covectors having the form

$$\tilde{p}^\mu_a dx^0 \wedge \ldots \wedge \underbrace{du^a}_{\mu} \wedge \ldots \wedge dx^3 - \tilde{H}(\tilde{p}^\mu_a, u^a, x^\lambda)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{13.12}$$

((x^λ, u^a) are local coordinates in W).

The canonical 4-form in $\mathscr{P}$ defined by (3.55) is equal to

$$\overset{4}{\theta}(\tilde{p}^{\mu}_{a}, u^{a}, x^{\lambda}) = \tilde{p}^{\mu}_{a} dx^{0} \wedge \ldots \wedge \underbrace{du^{a}}_{\mu} \wedge \ldots \wedge dx^{3} - \tilde{H}(\tilde{p}^{\mu}_{a}, u^{a}, x^{\lambda}) dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3}. \tag{13.13}$$

Let $\varrho\colon \mathscr{P} \to W$ be the projection.

DEFINITION. A 3-dimensional submanifold $\varphi \subset W$ which is a section of π over a 3-dimensional submanifold $\sigma \subset M$ is called an *admissible initial field* if there exists an admissible initial surface $c \in \mathscr{P}$ such that $\varrho(c) = \varphi$.

We denote the set of admissible initial fields Φ. In important examples we have no constraints for the initial values of field strengths u^{a} and therefore in such a case

$$\Phi = \bigcup_{\sigma \subset M} C^{\infty}(\sigma, W).$$

We fix an admissible initial field φ_0 and vary another φ. For every couple (φ_0, φ) we have a one-state $C \in \mathscr{H}$ such that the projection $\varrho(C)$ contains both φ_0 and φ.

We define the action integral

$$\Phi \ni \varphi \to S(\varphi) = \int_{C(\varphi_0, \varphi)} \overset{4}{\theta} \in \boldsymbol{R} \tag{13.14}$$

where $C(\varphi_0, \varphi)$ is a subset of C confined by lifts of φ_0, φ to $\mathscr{P}$. For a one-parameter family $(\varphi_\varepsilon)_{\varepsilon \in]1-\delta, 1+\delta[}$ we have

$$\frac{d}{d\varepsilon} S(\varphi_\varepsilon)\Big|_{\varepsilon=1} = \frac{d}{d\varepsilon} \int_{C_\varepsilon(\varphi_0, \varphi_\varepsilon)} \overset{4}{\theta}\Big|_{\varepsilon=1} = \int_{C(\varphi_0, \varphi)} \pounds_Y \overset{4}{\theta} \tag{13.15}$$

where

$$Y = \delta\tilde{p}^{\mu}_{a} \frac{\partial}{\partial \tilde{p}^{\mu}_{a}} + \delta u^{a} \frac{\partial}{\partial u^{a}} + \delta x^{\nu} \frac{\partial}{\partial x^{\nu}} \tag{13.16}$$

is the vector field generated by the family of states C_ε. The field Y on C satisfies equation (7.5).

The formula $\pounds_Y \overset{4}{\theta} = Y \lrcorner\, d\overset{4}{\theta} + d(Y \lrcorner\, \overset{4}{\theta})$ and the equation of motion (3.56) give

$$\frac{d}{d\varepsilon} S(\varphi_\varepsilon)\Big|_{\varepsilon=1} = \int_{\partial C(\varphi_0, \varphi)} Y \lrcorner\, \overset{4}{\theta}. \tag{13.17}$$

The vector field Y on C generates a vector field $\tilde{\tilde{Y}} = \varrho_* Y$ on $\varrho(C)$. In local coordinates, if Y is given by (13.16) then

$$\tilde{\tilde{Y}} = \delta u^{a} \frac{\partial}{\partial u^{a}} + \delta x^{\nu} \frac{\partial}{\partial x^{\nu}}. \tag{13.18}$$

The ϱ-horizontality of the form $\overset{4}{\theta}$ implies

PROPOSITION 13.1. *There exists a 3-form* $a_{\tilde{\tilde{Y}}}$ *in* $\varrho(C)$ *such that*

$$\int_{\partial C(\varphi_0,\varphi)} Y \lrcorner \overset{4}{\theta} = \int_{\varrho(\partial C(\varphi_0,\varphi))} a_{\tilde{\tilde{Y}}}. \tag{13.19}$$

In local coordinates:

$$\begin{aligned} a_{\tilde{\tilde{Y}}} = & \sum_{\mu=0}^{3} (-1)^{\mu} \tilde{p}^{\mu}_{a} \delta u^{a} dx^{0} \wedge \ldots \underset{\hat{\mu}}{\ldots} \wedge dx^{3} - \\ & - \sum_{\mu=0}^{3} (-1)^{\mu} \tilde{H}(\tilde{p}^{\mu}_{a}, u^{a}, x^{\lambda}) \delta x^{\mu} dx^{0} \wedge \ldots \underset{\hat{\mu}}{\ldots} \wedge dx^{3} + \\ & + \sum_{\mu=0}^{3} (-1)^{\mu+1} \left(\tilde{p}^{\mu}_{a} \delta x^{\nu} \frac{\partial u^{a}}{\partial x^{\nu}} - \tilde{p}^{\nu}_{a} \delta x^{\mu} \frac{\partial u^{a}}{\partial x^{\nu}} \right) dx^{0} \wedge \ldots \underset{\hat{\mu}}{\ldots} \wedge dx^{3} \end{aligned} \tag{13.20}$$

where $\tilde{p}^{\mu}_{a} = \tilde{p}^{\mu}_{a}(x^{\lambda})$, $u^{a} = u^{a}(x^{\lambda})$.

The proof of the proposition consists in elementary calculations.

COROLLARY. *If the vector field* $\tilde{\tilde{Y}}$ *is tangent to* φ, *then* $a_{\tilde{\tilde{Y}}}|\varphi = 0$.

Proof. A vector field tangent to φ has the form

$$\tilde{\tilde{Y}} = \frac{\partial u^{a}}{\partial x^{\nu}} \delta x^{\nu} \frac{\partial}{\partial u^{a}} + \delta x^{\nu} \frac{\partial}{\partial x^{\nu}}, \tag{13.21}$$

where $\delta x^{\nu} \dfrac{\partial}{\partial x^{\nu}}$ is a vector field tangent to $\sigma = \pi(\varphi)$. In a coordinate system in which $\sigma = \{x \in M;\ x^{0} = 0\}$ $\delta x^{0} = 0$ and δx^{k} are arbitrary. Then if we put (13.21) into (13.20) we obtain the result.

A vector field $\tilde{\tilde{Y}}$ (tangent to W) on φ represents a vector tangent to the space Φ. If we have a Riemannian structure in the space-time M, we can choose a representative Y_n in the space $T_{\varphi}(\Phi) = C^{\infty}(\varphi, TW)/C^{\infty}(\varphi, T\varphi)$ in such a way that its projection $\pi_{*}Y_n$ is normal to $\sigma = \pi(\varphi)$. By virtue of Proposition 13.1 and the Corollary we have

$$\frac{d}{d\varepsilon} S(\varphi_{\varepsilon}) \bigg|_{\varepsilon=1} = \int_{\varphi_1} a_{\tilde{\tilde{Y}}_n}. \tag{13.22}$$

In a coordinate system where $\sigma = \pi(\varphi_1) = \{x \in M\colon x^{0} = 0\}$ the normal unit vector has components $n^{\mu} = \left(\dfrac{1}{N}, -\dfrac{N^{k}}{N}\right)$ (cf. (6.7)). If we write $\tilde{p}^{\mu}_{a} = p^{\mu}_{a}\sqrt{-g}$, $\tilde{H} = H\sqrt{-g}$, $\delta x^{\nu} = n^{\nu}\delta\tau$, then

$$\begin{aligned} a_{\tilde{\tilde{Y}}_n}|\varphi = & \, p^{0}_{a} \delta u^{a} N \sqrt{g}\, dx^{1} \wedge dx^{2} \wedge dx^{3} - \\ & - \left(H(p^{\mu}_{a}, u^{a}, x^{\lambda}) - p^{0}_{a} N^{k} \frac{\partial u^{a}}{\partial x^{k}} - p^{k}_{a} \frac{\partial u^{a}}{\partial x^{k}} \right) \delta\tau \sqrt{g}\, dx^{1} \wedge dx^{2} \wedge dx^{3}. \end{aligned} \tag{13.23}$$

A geometric discussion of the above formula in particular cases gives the Hamilton–Jacobi equation for a given theory.

EXAMPLE. *Electrodynamics.* Let $\overset{4}{\theta}$ be the 4-form defined by (7.10) in the manifold $\mathscr{P}$ (cf. Section 7). If

$$\tilde{\tilde{Y}}_n = \delta A_\nu \frac{\partial}{\partial A_\nu} + \delta\tau n^\nu \frac{\partial}{\partial x^\nu}, \tag{13.24}$$

then formula (13.23) reads:

$$\begin{aligned} a_{\tilde{\tilde{Y}}_n} = &-\frac{1}{4\pi} h^{0k}(\delta A_k - n^\nu \partial_k A_\nu \delta\tau) N\sqrt{\bar{g}}\, dx^1 \wedge dx^2 \wedge dx^3 + \\ &+ \left(H(h^{\mu\nu}, A_\nu, x^\lambda) - \frac{1}{8\pi} h^{\alpha\beta} f_{\alpha\beta} + \frac{1}{4\pi} h^{0k} N n^\nu f_{\nu k}\right) \delta\tau \sqrt{\bar{g}}\, dx^1 \wedge dx^2 \wedge dx^3. \end{aligned} \tag{13.25}$$

Using the equation of motion (7.12), we have

$$\begin{aligned} h^{0k} n^\nu \partial_k A_\nu N = &\overline{\nabla}_k (h^{0k} A_0) - \overline{\nabla}_k (h^{0k} N^s A_s) - 4\pi j^0 N A_\mu n^\mu + \\ &+ h^{0k}(A_\mu n^\mu \partial_k N + \partial_k N^s A_s). \end{aligned} \tag{13.26}$$

If we use (13.25), (13.26) and neglect the divergences, we have

$$\begin{aligned} \int_\sigma a_{\tilde{\tilde{Y}}_n} = &-\int_\sigma \frac{1}{4\pi} h^{0k} \hat{\delta} A_k N \sqrt{\bar{g}}\, dx^1 \wedge dx^2 \wedge dx^3 + \\ &+ \int_\sigma \left(H(h^{\mu\nu}, A_\nu, x^\lambda) - j^0 N (A_\mu n^\mu) - \frac{1}{8\pi} h^{\alpha\beta} f_{\alpha\beta} + \frac{1}{4\pi} h^{0k} N n^\nu f_{\nu k}\right) \times \\ &\times \delta\tau \sqrt{\bar{g}}\, dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \tag{13.27}$$

where

$$\hat{\delta} A_k = \delta A_k - A_\mu n^\mu \frac{\partial_k N}{N} \delta\tau - \frac{\partial_k N^s}{N} A_s \delta\tau. \tag{13.28}$$

The vector fields $\tilde{\tilde{Y}}_n$ is not vertical and therefore δA_k do not represent a differential 1-form on the manifold $\sigma \subset M$. However, it turns out that the quantities $\hat{\delta} A_k$ defined by (13.28) determine a covector field in σ. It is defined by the following geometric construction: a state C determines a 3-dimensional submanifold φ (an initial value of the field — an admissible initial field) which is a section of the bundle $W = T^*M$ over a 3-dimensional space-like surface $\sigma \subset M$. In other words we have a covector field (tangent to M) on σ. The injection i: $\sigma \to M$ generates a covector field $i^*\varphi$ in σ. Let G_s be a family of transformations of W generated by a field $\tilde{\tilde{Y}}$ of type (13.24). We have a family $\varphi_s = G_s \varphi$ of

submanifolds in W and the corresponding family of 1-forms $i^*\varphi_\varepsilon$ in the manifolds $\sigma_\varepsilon = \pi(\varphi_\varepsilon)$. In such a situation the formula

$$\zeta = \frac{d}{d\varepsilon}(G_\varepsilon^* i^*\varphi_\varepsilon - i^*\varphi)|_{\varepsilon=0} \tag{13.29}$$

defines a 1-form (a covector field) in σ.

In local coordinates in M for which $\sigma = \{x \in M\colon x^0 = 0\}$

$$\varphi = \{((0, x^k), A_\mu(x^k)) \in W\}, \qquad i^*\varphi = \{(x^k, A_s(x^k)) \in T^*\sigma\}. \tag{13.30}$$

PROPOSITION 13.2. *In such local coordinates* $\zeta = \hat{\delta}A_k dx^k$, *where* $\hat{\delta}A_k$ *are defined by* (13.28).

The proof is given in Section 15.

The above geometric construction shows that $\hat{\delta}A_k$ have tensor transformation properties.

By virtue of (13.22) and (13.27) we write

$$\frac{\delta S}{\hat{\delta}A_k} = -\frac{1}{4\pi} h^{0k} N \sqrt{g} = -\frac{1}{4\pi}\mathscr{D}^k, \tag{13.31a}$$

$$\frac{\delta S}{\delta\tau} = \left(H(h^{\mu\nu}, A_\nu, x^\lambda) - j^0 N(A_\mu n^\mu) - \frac{1}{8\pi} h^{\alpha\beta} f_{\alpha\beta} + h^{0k} N n^\nu f_{\nu k}\right)\sqrt{g}. \tag{13.31b}$$

Formulas (13.31) generalize the results presented in [33], pp. 496–498. The quantity on the right side of (13.31b) is the density of energy for an electromagnetic field. We see that a variation of the action functional S with respect to potentials gives the density of electromagnetic induction and a variation with respect to "time" gives energy.

14. The Hamilton–Jacobi equation in General Relativity

The system of functional equations (13.31) is called the *Hamilton–Jacobi equation* in electrodynamics. It gives a dependence of the action S on a final (initial) surface $\sigma \subset M$ and on the value of a covector field A_k in σ. In the present section we discuss a similar problem in gravidynamics. The action functional (12.1) depends on initial and final surfaces $\sigma_0, \sigma \subset M$ and on ADMW densities $\underset{0}{\pi}^{ij}, \pi^{ij}$ in σ_0, σ. It turns out that a dependence on σ is trivial (cf. the discussion of that fact in [33], Chapter 21) and the functional derivative with respect to the ADMW density π^{ij} gives the conjugate quantities g_{ij}.

We begin with some geometric notions. In a neighbourhood of σ in M we take a coordinate system in which $\sigma = \{x \in M\colon x^0 = 0\}$. Such a neighbourhood has a structure of a bundle determined by the vector field

$$\xi = \frac{\partial}{\partial x^0} \tag{14.1}$$

(in Section 6 we used the symbol τ instead of ξ).

A metric $g_{\mu\nu}$ in M determines a vector normal to σ, $\boldsymbol{n}$.

A lapse function and a shift vector are defined by

$$N = -(\boldsymbol{n}|\xi), \tag{14.2}$$

$$\boldsymbol{N} = \xi + (\boldsymbol{n}|\xi)\boldsymbol{n} \tag{14.3}$$

(cf. Section 6).

They determine in a neighbourhood of c in $\mathscr{P}$ an ADMW coordinate system (c is the lift of σ to $\mathscr{P}$). A vector field

$$Y = \sum_{\mu\leqslant\nu} \delta\Gamma^{\lambda}_{\mu\nu}\frac{\partial}{\partial\Gamma^{\lambda}_{\mu\nu}} + \sum_{\mu\leqslant\nu}\delta g_{\mu\nu}\frac{\partial}{\partial g_{\mu\nu}} + \delta x^{\nu}\frac{\partial}{\partial x^{\nu}} \tag{14.4}$$

generates a family G_ε of transformations in $\mathscr{P}$. Let $c_\varepsilon = G_\varepsilon c$, $\sigma_\varepsilon = \pi(G_\varepsilon c)$ ($\pi\colon \mathscr{P}\to M$). We consider only those fields Y which transform a state C (determined by a metric $g_{\mu\nu}$ in M) onto sections of $\mathscr{P}$ which are integrable (i.e., fulfilling (5.2)). This implies for $\delta x^{\nu} = n^{\nu}\delta\tau$.

LEMMA 14.1.

$$\delta\partial_k g_{\mu\nu} = \partial_k\delta g_{\mu\nu} + \partial_p g_{\mu\nu}\partial_k\left(\frac{N^p}{N}\right)\delta\tau - \partial_0 g_{\mu\nu}\partial_k\left(\frac{1}{N}\right)\delta\tau \quad \text{on} \quad \sigma. \tag{14.5}$$

In the ADMW *system*

$$Y = \delta N\frac{\partial}{\partial N} + \delta N_k\frac{\partial}{\partial N_k} + \delta M_\mu\frac{\partial}{\partial M_\mu} + \delta M_{\mu_k}\frac{\partial}{\partial M_{\mu_k}} + \tag{14.4'}$$
$$+ \sum_{i\leqslant j}\delta g_{ij}\frac{\partial}{\partial g_{ij}} + \sum_{i\leqslant j}\delta\pi^{ij}\frac{\partial}{\partial\pi^{ij}}\sum_{i\leqslant j}\delta\Gamma^k_{ij}\frac{\partial}{\partial\Gamma^k_{ij}} + \delta x^{\nu}\frac{\partial}{\partial x^{\nu}}.$$

We intend to compute the normal vector to σ_ε, a lapse function, a shift vector, the metric in σ_ε, its second fundamental form and the ADMW density. Of course, we can find only the solution up to linear terms. The diffeomorphism $\pi\circ G_\varepsilon|_\sigma$ enables us to parametrize the submanifold σ_ε by coordinates x^1, x^2, x^3 in σ. In subsequent considerations we always use such a coordinate system.

PROPOSITION 14.1. *If Y is of type* (14.4′) *and* $\delta x^\nu = n^\nu \delta\tau$, $\delta\tau$ *is a constant, then*

$$(14.6)\qquad \begin{cases} \underset{\sigma_\varepsilon}{n^0} \cong \dfrac{1}{N} - \dfrac{1}{N^2}\varepsilon\,\delta N, \\[2ex] \underset{\sigma_\varepsilon}{n^p} \cong -\dfrac{N^p}{N} - \varepsilon\dfrac{\delta N^p}{N} - \dfrac{\varepsilon}{N}\bar{g}^{pk}\partial_k N\,\delta\tau + \dfrac{\varepsilon}{N^2}N^p\,\delta N; \end{cases}$$

$$(14.7)\qquad \begin{cases} \underset{\sigma_\varepsilon}{N} \cong N + \varepsilon\hat{\delta}N, \\[2ex] \hat{\delta}N = \delta N + \dfrac{N^k\partial_k N}{N}\,\delta\tau; \end{cases}$$

$$(14.8)\qquad \begin{cases} \underset{\sigma_\varepsilon}{N^p} \cong N^p + \varepsilon\hat{\delta}N^p, \\[2ex] \hat{\delta}N^p = \delta N^p + \bar{g}^{pk}\partial_k N\,\delta\tau + N^k\partial_k N^p\,\delta\tau; \end{cases}$$

$$(14.9)\qquad \begin{cases} \underset{\sigma_\varepsilon}{g_{ij}} \cong g_{ij} + \varepsilon\hat{\delta}g_{ij}, \\[2ex] \hat{\delta}g_{ij} = \delta g_{ij} - \dfrac{1}{N}(g_{kj}\partial_i N^k + g_{ki}\partial_j N^k)\,\delta\tau; \end{cases}$$

$$(14.10)\qquad \begin{cases} \underset{\sigma_\varepsilon}{\bar{g}^{ij}} \cong \bar{g}^{ij} + \varepsilon\hat{\delta}\bar{g}^{ij}, \\[2ex] \hat{\delta}\bar{g}^{ij} = -\bar{g}^{ia}\,\delta g_{ab}\,\bar{g}^{bj} + \dfrac{1}{N}(\bar{g}^{ia}\partial_a N^j + \bar{g}^{ja}\partial_a N^i)\,\delta\tau; \end{cases}$$

$$(14.11)\qquad \begin{cases} \underset{\sigma_\varepsilon}{\pi^{ij}} \cong \pi^{ij} + \varepsilon\hat{\delta}\pi^{ij}, \\[2ex] \hat{\delta}\pi^{ij} = \delta\pi^{ij} + \dfrac{1}{N}(\pi^{ja}\partial_a N^i + \pi^{ia}\partial_a N^j)\,\delta\tau - \dfrac{1}{N}\pi^{ij}\partial_a N^a\,\delta\tau - \\[2ex] \qquad\qquad - \sqrt{\bar{g}}\,\dfrac{1}{N}(\bar{\nabla}^i\bar{\nabla}^j N - \bar{g}^{ij}\bar{\nabla}^s\bar{\nabla}_s N)\,\delta\tau. \end{cases}$$

The proofs of (14.6)–(14.10) are rather easy, the proof of (14.11) is very long, we have to use Lemma 14.1. See Section 15.

Remark. The vector field Y is not vertical and therefore quantities δN^p, δg_{ij}, $\delta\bar{g}^{ij}$, $\delta\pi^{ij}$ do not define appropriate tensor densities or fields in σ. The geometric construction of $\hat{\delta}N^p$, $\hat{\delta}g_{ij}$, $\hat{\delta}\bar{g}^{ij}$, $\hat{\delta}\pi^{ij}$ shows that they determine in σ tensor quantities. This can be proved directly by checking their transformation properties.

LEMMA 14.2. *For a vector field Y of type* (14.4) *and a state $C \in \mathscr{H}$ we have*

$$(14.12)\qquad (Y \lrcorner \overset{4}{\theta})|C = \sum_{\nu=0}^{3} (-1)^{\nu} g^{\alpha\beta}\, \delta\Gamma^{\nu}_{\alpha\beta} \sqrt{-g}\, dx^0 \wedge \ldots \underset{\hat{\nu}}{\ldots} \wedge dx^3 -$$

$$- \sum_{\nu=0}^{3} (-1)^{\nu} g^{\alpha\nu}\, \delta\Gamma^{\beta}_{\alpha\beta} \sqrt{-g}\, dx^0 \wedge \ldots \underset{\hat{\nu}}{\ldots} \wedge dx^3 +$$

$$+ \sum_{\nu=0}^{3} (-1)^{\nu} 2\lambda\, \delta x^{\nu} \sqrt{-g}\, dx^0 \wedge \ldots \underset{\hat{\nu}}{\ldots} \wedge dx^3 +$$

$$+ \sum_{\nu=0}^{3} (-1)^{\nu+1} (g^{\alpha\beta} \partial_{\tau} \Gamma^{\nu}_{\alpha\beta} - g^{\alpha\nu} \partial_{\tau} \Gamma^{\beta}_{\alpha\beta})\, \delta x^{\tau} \sqrt{-g}\, dx^0 \wedge \ldots \underset{\hat{\nu}}{\ldots} \wedge dx^3 .$$

LEMMA 14.3. *In the* ADMW *coordinate system, for an admissible initial surface $c \subset C$ (c is the lift of σ), we have*

$$(14.13)\qquad (Y \lrcorner \overset{4}{\theta})|c = \Big\{ \sqrt{\bar{g}}\, \bar{\nabla}_p \Big(\frac{\hat{\delta} N^p}{N}\Big) - \sqrt{\bar{g}}\, \bar{\nabla}_p \Big(\frac{1}{N} \bar{\nabla}_p N\Big) \delta\tau - g_{ij}\, \hat{\delta}\pi^{ij} +$$

$$+ 2\Big(\bar{R} - 2\lambda - \frac{1}{\bar{g}} \big(\pi^{ij}\pi_{ij} - \tfrac{1}{2}(\operatorname{tr}\pi)^2\big)\Big) \sqrt{\bar{g}}\, \delta\tau - \sqrt{\bar{g}}\, \bar{\nabla}_p \Big(\frac{\partial_0 N^p}{N^2}\Big) \delta\tau \Big\}\, dx^1 \wedge dx^2 \wedge dx^3$$

where $\hat{\delta} N^p$, $\hat{\delta}\pi^{ij}$ are defined by (14.8), (14.11).

The proof of Lemma 14.2 consists in elementary calculations using only the equations $R_{\mu\nu} = \lambda g_{\mu\nu}$. Formula (14.13) is a consequence of (14.12) and equations (6.19a).

We consider the action integral (12.1)

$$(14.14)\qquad S(\sigma_\varepsilon, \underset{\varepsilon}{\pi}^{ij}) = \int_{C(\sigma_\varepsilon, \underset{\varepsilon}{\pi}^{ij})} \overset{4}{\theta}$$

where $C(\sigma_\varepsilon, \underset{\varepsilon}{\pi}^{ij})$ is a subset of C confined between admissible initial surfaces c_0 and c_ε.

In the same way as in Section 13 we obtain

$$(14.15)\qquad \frac{dS(\sigma_\varepsilon, \underset{\varepsilon}{\pi}^{ij})}{d\varepsilon}\Big|_{\varepsilon=1} = \int_{\partial C} Y \lrcorner \overset{4}{\theta} .$$

In the case where the admissible initial surface $c \subset C$ is a compact manifold without boundary or the solution C is assymptotically flat (i.e., $N \equiv 1$, $N_k \equiv 0$ in the "spatial infinity") integrating (14.13) we obtain

$$(14.16)\qquad \begin{aligned} \frac{\delta S}{\hat{\delta}\pi^{ij}} &= -g_{ij}, \\ \frac{\delta S}{\delta\tau} &= 2\Big(\bar{R} - 2\lambda - \frac{1}{\bar{g}} \big(\pi^{ij}\pi_{ij} - \tfrac{1}{2}(\operatorname{tr}\pi)^2\big)\Big) \sqrt{\bar{g}} . \end{aligned}$$

By virtue of (6.19b) we obtain

PROPOSITION 14.2. *For the functional S*

$$\frac{\delta S}{\delta \pi^{ij}} = -g_{ij}, \qquad \frac{\delta S}{\delta \tau} = 0. \tag{14.17}$$

We call system (14.17) the *Hamilton–Jacobi equation* (*relation*) in the momentum representation. It is interesting to note that the quantity $\frac{\delta S}{\delta \tau}$ which corresponds to the density of hamiltonian in electrodynamics is in gravitation equal to zero.

It is possible to discuss the action functional S in a coordinate representation but we do not tackle that problem here.

15. Proofs

Proof of Proposition 5.3. Let $x^\lambda \to (x^\lambda, g_{\mu\nu}(x^\lambda), \Gamma^\tau_{\mu\nu}(x^\lambda))$ be a $\overset{5}{\omega}$-singular section of $\mathscr{P}$. The condition

$$(X_1 \lrcorner \overset{5}{\omega})|C = 0, \quad \text{where} \quad X_1 = \sum_{\mu \leqslant \nu} \delta\Gamma^\lambda_{\mu\nu} \frac{\partial}{\partial \Gamma^\lambda_{\mu\nu}},$$

gives the equations

$$\begin{aligned} [\tfrac{1}{2}(g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu})\,\delta^\varrho_\lambda - \tfrac{1}{4}(g^{\alpha\mu} g^{\beta\varrho}\,\delta^\nu_\lambda + g^{\alpha\nu} g^{\beta\varrho}\,\delta^\mu_\lambda + g^{\alpha\varrho} g^{\beta\mu}\,\delta^\nu_\lambda + g^{\alpha\varrho} g^{\beta\nu}\,\delta^\mu_\lambda) - \\ - \tfrac{1}{2} g^{\alpha\beta} g^{\mu\nu}\,\delta^\varrho_\lambda + \tfrac{1}{4} g^{\alpha\beta}(g^{\mu\varrho}\,\delta^\nu_\lambda + g^{\nu\varrho}\,\delta^\mu_\lambda)]\,\partial_\varrho g_{\alpha\beta} \\ = g^{\beta\nu}\Gamma^\mu_{\beta\lambda} + g^{\beta\mu}\Gamma^\nu_{\beta\lambda} - \tfrac{1}{2}(g^{\beta\varrho}\Gamma^\mu_{\beta\varrho}\,\delta^\nu_\lambda + g^{\beta\varrho}\Gamma^\nu_{\beta\varrho}\,\delta^\mu_\lambda) - g^{\mu\nu}\Gamma^\beta_{\beta\lambda}. \end{aligned} \tag{15.1}$$

Contracting equations (15.1) with respect to μ, ν and with respect to μ, λ, we obtain a set of equations which gives

$$\begin{aligned} g^{\alpha\beta}\partial_\lambda g_{\alpha\beta} &= 2\Gamma^\mu_{\mu\lambda}, \\ g^{\beta\varrho}\partial_\varrho g_{\lambda\beta} &= \Gamma^\mu_{\mu\lambda} + g_{\lambda\mu}\Gamma^\mu_{\beta\varrho} g^{\beta\varrho}. \end{aligned} \tag{15.2}$$

From (15.1) we have

$$\begin{aligned} \partial_\lambda g_{\mu\nu} = g_{\mu\tau}\Gamma^\tau_{\nu\lambda} + g_{\nu\tau}\Gamma^\tau_{\mu\lambda} - \tfrac{1}{2}(g_{\mu\tau}\Gamma^\tau_{\alpha\beta} g^{\alpha\beta} g_{\nu\lambda} + g_{\nu\tau}\Gamma^\tau_{\alpha\beta} g^{\alpha\beta} g_{\mu\lambda}) - g_{\mu\nu}\Gamma^\beta_{\beta\lambda} + \\ + \tfrac{1}{2}(\partial_\alpha g_{\mu\beta} g^{\alpha\beta} g_{\lambda\nu} + \partial_\alpha g_{\nu\beta} g^{\alpha\beta} g_{\lambda\mu}) + \\ + \tfrac{1}{2} g^{\alpha\beta}\partial_\lambda g_{\alpha\beta} g_{\mu\nu} - \tfrac{1}{4}(g^{\alpha\beta}\partial_\mu g_{\alpha\beta} g_{\nu\lambda} + g^{\alpha\beta}\partial_\nu g_{\alpha\beta} g_{\mu\lambda}). \end{aligned} \tag{15.3}$$

Equations (15.2) and (15.3) give

$$\partial_\lambda g_{\mu\nu} = g_{\mu\tau}\Gamma^\tau_{\nu\lambda} + g_{\nu\tau}\Gamma^\tau_{\mu\lambda}. \tag{15.4}$$

We know that (15.4) is equivalent to (5.2).

The condition

$$(X_2 \lrcorner \overset{5}{\omega})|C = 0, \quad \text{where} \quad X_2 = \sum_{\mu\leqslant\nu} \delta g_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}},$$

gives rise to equations (5.3).

Proof of Proposition 5.4. Let $p \in \mathscr{P}$ and C be an $\overset{5}{\omega}$-singular submanifold passing through p which is the graph of a section of $\mathscr{P}$. If the tangent space $T_p(C)$ is not a maximal $\overset{5}{\omega}$-singular subspace of $T_p(\mathscr{P})$, then there exists a vector $Y \in T_p(\mathscr{P})$ but $Y \notin T_p(C)$ such that for every vector $X \in T_p(\mathscr{P})$ and vectors $Z_i \in T_p(C)$, $i = 1, 2, 3$, we have

$$(X \lrcorner \overset{5}{\omega})(Y, Z_1, Z_2, Z_3) = 0. \tag{15.5}$$

By virtue of (4.2) we assume that vectors X, Y are π-vertical. Let

$$\begin{aligned} X &= \sum_{\mu\leqslant\nu} \underset{1}{\delta}\Gamma^{\lambda}_{\mu\nu} \frac{\partial}{\partial \Gamma^{\lambda}_{\mu\nu}} + \sum_{\mu\leqslant\nu} \underset{1}{\delta} g_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}}, \\ Y &= \sum_{\mu\leqslant\nu} \underset{2}{\delta}\Gamma^{\lambda}_{\mu\nu} \frac{\partial}{\partial \Gamma^{\lambda}_{\mu\nu}} + \sum_{\mu\leqslant\nu} \underset{2}{\delta} g_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}}. \end{aligned} \tag{15.6}$$

In local coordinates equation (15.5), which is equivalent to

$$\big((X \wedge Y) \lrcorner \overset{5}{\omega}\big)(Z_1, Z_2, Z_3) = 0, \tag{15.7}$$

gives

$$\underset{1}{\delta}\Gamma^{\varrho}_{\mu\nu} (\underset{2}{\delta} g)^{\mu\nu} - \underset{2}{\delta}\Gamma^{\varrho}_{\mu\nu} (\underset{1}{\delta} g)^{\mu\nu} - \big(\underset{1}{\delta}\Gamma^{\lambda}_{\mu\lambda} (\underset{2}{\delta} g)^{\mu\varrho} - \underset{2}{\delta}\Gamma^{\lambda}_{\mu\lambda} (\underset{1}{\delta} g)^{\mu\varrho}\big) = 0 \tag{15.8}$$

for every $\underset{1}{\delta}\Gamma^{\varrho}_{\mu\nu}$, $(\underset{1}{\delta} g)^{\mu\nu} = g^{\mu\alpha} \underset{1}{\delta} g_{\alpha\beta} g^{\beta\nu}$.

Elementary considerations show that (15.8) has only a trivial solution for $\underset{2}{\delta}\Gamma^{\lambda}_{\mu\nu}$, $(\underset{2}{\delta} g)^{\mu\nu}$. The proposition is proved.

Proof of Proposition 6.2. It is known (cf. [2]) that the Einstein tensor $G_{\mu\nu}$ defined by (6.16) from any metric $g_{\mu\nu}$ in M satisfies the contracted Bianchi identities:

$$\nabla_\nu G^\nu_\mu = 0. \tag{15.9}$$

We write (15.9) in the form

$$-\partial_0 G^0_\mu = \partial_k G^k_\mu + \Gamma^\nu_{\nu\lambda} G^\lambda_\mu - \Gamma^\lambda_{\mu\nu} G^\nu_\lambda. \tag{15.10}$$

If equation (6.18a) holds, then we have

$$G^k_j = \frac{1}{g^{00}} g^{k0} G^0_j - \delta^k_j G^0_0 - \frac{1}{g^{00}} \delta^k_j g^{0s} G^0_s. \tag{15.11}$$

By (15.10) and (15.11) we obtain a system of linear differential equations

$$\partial_0 G^0_\mu = A^\nu_\mu(g_{\alpha\beta}, \Gamma^\tau_{\alpha\beta}) G^0_\nu + B^{k\nu}_\mu(g_{\alpha\beta}, \Gamma^\tau_{\alpha\beta}) \partial_k G^0_\nu. \tag{15.12}$$

We consider (15.12) as a linear differential equations in the functional spaces of quantities G^0_μ (if we take the ADMW coordinate system, we have a scalar function G^0_0 on σ depending on a parameter x^0 and a covector field G^0_k in σ depending on x^0). Equations (15.12) together with the initial data (6.18b) imply $G^0_\mu = 0$ in M.

Proof of Lemma 7.1. For an antisymmetric tensor field $a^{\mu\nu}$ in M we have

$$\nabla_\mu \nabla_\nu a^{\mu\nu} = 0. \tag{15.13}$$

Therefore we have

$$\nabla_0(\nabla_\nu h^{0\nu}) + \nabla_k(\nabla_\nu h^{k\nu}) = 0. \tag{15.14}$$

Equations (15.14), (7.12a) together with the continouity equation $\nabla_\nu j^\nu = 0$ give

$$\partial_0(\nabla_k h^{0k} - 4\pi j^0) + \Gamma^0_{00}(\nabla_k h^{0k} - 4\pi j^0) = 0. \tag{15.15}$$

Equation (15.15) is a linear differential equation in the space of function on σ. It has a unique, trivial solution satisfying the initial data (7.12b). The lemma is proved.

LEMMA 15.1. *The equations*

$$\begin{aligned} \xi &= \left((\Gamma^0_{0r} - \Gamma^p_{pr})\frac{N^r}{N^3} - \Gamma^0_{jr}\left(\bar{g}^{jr} + \frac{N^j N^r}{N^2}\right)\frac{1}{N} + \Gamma^p_{p0}\frac{1}{N^3}\right)\sqrt{-g}, \\ \xi^k &= \left((\Gamma^0_{0r} - \Gamma^p_{pr})\left(-\frac{1}{N^2}\bar{g}^{kr}\right) + 2\Gamma^0_{jr}\bar{g}^{kj}\frac{N^r}{N^2}\right)\sqrt{-g} \end{aligned} \tag{15.16}$$

give a one-to-one correspondence between the variables ξ, ξ^k and $(\Gamma^0_{0r} - \Gamma^p_{pr})$, Γ^p_{p0}. In such coordinates we have:

$$\xi = \frac{\sqrt{\bar{g}}}{N^2}\partial_k N^k; \qquad \xi^k = -\frac{1}{N^2}\sqrt{\bar{g}}\ \bar{g}^{kr}(\partial_r N - \Gamma^j_{jr} N); \tag{15.17}$$

and the multisymplectic form $\overset{5}{\omega}$ is equal to

$$\begin{aligned} \overset{5}{\omega} = {} & d\xi \wedge dN \wedge dx^1 \wedge dx^2 \wedge dx^3 + d\xi^k \wedge dN_k \wedge dx^1 \wedge dx^2 \wedge dx^3 + \\ & + \left(-(\xi^s dN^k + N^k d\xi^s) - \tfrac{1}{2}(N d\xi + \xi dN)\bar{g}^{ks} - \tfrac{1}{2}\xi N\, d\bar{g}^{ks} + d\pi^{ks}\right) \wedge \\ & \wedge dg_{ks} \wedge dx^1 \wedge dx^2 \wedge dx^3 + \\ & + (\text{terms containing the differential } dx^0). \end{aligned} \tag{15.18}$$

Expression (15.18) for the 5-form $\overset{5}{\omega}$ is obtained from the formula

$$(15.19)\qquad \begin{aligned}\psi = {} & \xi\, dN \wedge dx^1 \wedge dx^2 \wedge dx^3 + \xi^k dN_k \wedge dx^1 \wedge dx^2 \wedge dx^3 + \\ & + \left(-\tfrac{1}{2}(\xi^s N^k + \xi^k N^s) - \tfrac{1}{2}\xi N \bar{g}^{ks} + \pi^{ks}\right) dg_{ks} \wedge dx^1 \wedge dx^2 \wedge dx^3 + \\ & + (\text{terms containing } dx^0),\end{aligned}$$

where ψ is the form defined locally by (5.9). The non-covariant formula (15.19) gives a covariant expression (15.18). We shall use (15.18) in the subsequent proofs.

Proof of Proposition 8.1. We divide the proof into two parts. First we prove

LEMMA 15.2. *For every two vectors* $\hat{Y}_1, \hat{Y}_2 \in \overset{0}{\hat{T}}_C(\mathscr{H}, c)$,

$$(15.20)\qquad \Omega(\hat{Y}_1, \hat{Y}_2) = 0.$$

Proof. Let Y_1, Y_2 be vector fields of type (8.6) on C representing vectors $\hat{Y}_1, \hat{Y}_2$ and fulfilling conditions (8.11). Using (15.18) and (8.7), (8.8), (15.17), we have

$$(15.21)\qquad \Omega(\hat{Y}_1, \hat{Y}_2) = \int_c (Y_1 \wedge Y_2) \lrcorner \overset{5}{\omega} = \int_c (d\mu_1 - d\mu_2),$$

where

$$(15.22)\qquad \begin{aligned}\mu_1 &= \sum_{r=1}^{3} (-1)^{r+1} \left(\frac{1}{N^2} \sqrt{\bar{g}}\, \bar{g}^{kr} \underset{1}{\delta} N \underset{2}{\delta} N_k\right) dx^1 \wedge \ldots \underset{\hat{r}}{} \ldots \wedge dx^3, \\ \mu_2 &= \sum_{r=1}^{3} (-1)^{r+1} \left(\frac{1}{N^2} \sqrt{\bar{g}}\, \bar{g}^{kr} \underset{2}{\delta} N \underset{1}{\delta} N_k\right) dx^1 \wedge \ldots \underset{\hat{r}}{} \ldots \wedge dx^3\end{aligned}$$

are 2-forms in the manifold $c \subset C$ (or in the manifold $\sigma \subset M$). The boundary conditions for the fields Y_1, Y_2 give (15.20).

LEMMA 15.3. *If* $\hat{Y}_1 \in \overset{0}{\hat{T}}_C(\mathscr{H}, c)$, $\hat{Y}_2 \in \overset{1}{\hat{T}}_C(\mathscr{H}, c)$, *then*

$$(15.23)\qquad \Omega(\hat{Y}_1, \hat{Y}_2) = 0.$$

Let Y_1, Y_2 be vector fields on C representing $\hat{Y}_1, \hat{Y}_2$. Conditions (8.11) and (9.2) together with formulas (15.18), (8.7), (8.8), (15.17) give

$$(15.24)\qquad \Omega(\hat{Y}_1, \hat{Y}_2) = \int_c (Y_1 \wedge Y_2) \lrcorner \overset{5}{\omega} = \int_c (d\eta_1 - d\eta_2),$$

where

$$
(15.25)\qquad
\begin{aligned}
\eta_1 &= \sum_{r=1}^{3}(-1)^{r+1}\left(\frac{1}{N^2}\sqrt{\bar g}\,\bar g^{ku}N_u\bar g^{rs}\underset{1}{\delta N}\underset{2}{\delta g_{ks}}\right)dx^1\wedge\ldots\underset{\hat r}{\ldots}\wedge dx^3,\\
\eta_2 &= \sum_{r=1}^{3}(-1)^{r+1}\left(\frac{1}{2N}\sqrt{\bar g}\,\bar g^{kr}\bar g^{ij}\underset{2}{\delta g_{ij}}\underset{1}{\delta N_k}\right)dx^1\wedge\ldots\underset{\hat r}{\ldots}\wedge dx^3
\end{aligned}
$$

are 2-forms in the manifold $c\subset C$ (in $\sigma\subset M$).

Therefore (15.23) follows from the boundary conditions for Y_1, Y_2.

By virtue of Proposition 9.1 we can decompose a vector $\hat X\in\tilde T_C(\mathscr{H})$ into a sum $\hat X=\hat X_1+\hat X_2$, where $\hat X_1\in\overset{0}{\tilde T}_C(\mathscr{H},c)$, $\hat X_2\in\overset{1}{\tilde T}_C(\mathscr{H},c)$. Thus, Proposition 8.1 is a direct consequence of Lemmas 15.2, 15.3.

Proof of Theorem 8.1. By virtue of Proposition 8.1 and decomposition (9.3) it is sufficient to prove formula (8.12) for $\hat X_1,\hat X_2\in\overset{1}{\tilde T}_C(\mathscr{H},c)$. Direct calculations, formulas (8.7), (8.8), (15.17) and (15.18) give

$$
(15.26)\qquad \Omega(\hat X_1,\hat X_2)=\int_c(\underset{1}{\delta\pi^{ij}}\underset{2}{\delta g_{ij}}-\underset{2}{\delta\pi^{ij}}\underset{1}{\delta g_{ij}})dx^1\wedge dx^2\wedge dx^3+\int_c(d\nu_1-d\nu_2)
$$

where

$$
(15.27)\qquad
\begin{aligned}
\nu_1 &= \sum_{r=1}^{3}(-1)^{r+1}\left(\frac{N^k}{2N}\sqrt{\bar g}\,\bar g^{sr}\bar g^{ij}\underset{1}{\delta g_{ij}}\underset{2}{\delta g_{sk}}\right)dx^1\wedge\ldots\underset{\hat r}{\ldots}\wedge dx^3,\\
\nu_2 &= \sum_{r=1}^{3}(-1)^{r+1}\left(\frac{N^k}{2N}\sqrt{\bar g}\,\bar g^{sr}\bar g^{ij}\underset{2}{\delta g_{ij}}\underset{1}{\delta g_{sk}}\right)dx^1\wedge\ldots\underset{\hat r}{\ldots}\wedge dx^3
\end{aligned}
$$

are 2-forms in c (in $\sigma\subset M$). The boundary conditions for fields X_i and the Stockes theorem give the result.

Proof of Proposition 9.2. Let $(\delta\pi^{ij},\delta g_{ij})=JA^*(u^j,\chi)$. Then we have

$$
(15.28)\qquad
\begin{aligned}
\delta\pi^{ij} &= -\tfrac12\big(\pi^{ia}\overline{\nabla}_a u^j+\pi^{ja}\overline{\nabla}_a u^i-\overline{\nabla}_a(\pi^{ij}u^a)\big)-\\
&\quad-\frac{2}{\sqrt{\bar g}}(\pi^{is}g_{sk}\pi^{kj}-\tfrac12\operatorname{tr}\pi\pi^{ij})\chi+\sqrt{\bar g}\,(\overline{\nabla}^i\overline{\nabla}^j\chi-\bar g^{ij}\overline{\nabla}^k\overline{\nabla}_k\chi)+\\
&\quad+\sqrt{\bar g}\,\big(-\bar R^{ij}\chi+\bar g^{ij}(\bar R-2\lambda)\chi\big),\\
\delta g_{ij} &= \frac12(\overline{\nabla}_j u_i+\overline{\nabla}_i u_j)+\frac{2}{\sqrt{\bar g}}(\pi_{ij}-\tfrac12\operatorname{tr}\pi g_{ij})\chi.
\end{aligned}
$$

Using the constraint equations (6.19b) and the formulas

$$\overline{\nabla}_j\overline{\nabla}^j\overline{\nabla}^i\chi-\overline{\nabla}^i\overline{\nabla}_j\overline{\nabla}^j\chi = \bar{R}^{si}\overline{\nabla}_s\chi, \qquad \overline{\nabla}_j\bar{R}^{ij} = \tfrac{1}{2}\overline{\nabla}^i\bar{R},$$

$$(15.29)\qquad (\overline{\nabla}_i\overline{\nabla}_j-\overline{\nabla}_j\overline{\nabla}_i)u^k = \bar{R}^k_{sij}u^s, \qquad (\overline{\nabla}_i\overline{\nabla}_j-\overline{\nabla}_j\overline{\nabla}_i)u_k = -\bar{R}^s_{kij}u_s,$$

$$(\overline{\nabla}_i\overline{\nabla}_j-\overline{\nabla}_j\overline{\nabla}_i)v^{ks} = \bar{R}^k_{pij}v^{ps}+\bar{R}^s_{qij}v^{kq},$$

we obtain by a direct computation $AJA^*(u^j, \chi) = 0$.

Proof of Proposition 9.5. We define two differential operators:

A_1: $C^\infty\big(\operatorname{den} S^2T(\sigma)\oplus S^2T^*(\sigma)\big)\to C^\infty\big(T(\sigma)\big)$,

A_2: $C^\infty\big(\operatorname{den} S^2T(\sigma)\oplus S^2T^*(\sigma)\big)\to C^\infty(\sigma)$,

$$(15.30)\qquad A_1(\delta\pi^{ij}, \delta g_{ij}) = \frac{1}{\sqrt{\bar{g}}}(\overline{\nabla}_j\delta\pi^{ij}+\delta\bar{\Gamma}^i_{ks}\pi^{ks}),$$

$$(15.31)\qquad A_2(\delta\pi^{ij},\delta g_{ij}) = \delta\bar{R}+\frac{1}{\bar{g}}(\bar{R}-2\lambda)\delta\bar{g}-\frac{1}{\bar{g}}\delta\big(\pi^{ks}\pi_{ks}-\tfrac{1}{2}(\operatorname{tr}\pi)^2\big)$$

(cf. (9.11)).

Formulas (9.11) show that A_1 is a first order linear differential operator and A_2 is a second order linear differential operator. We have

$$(15.32)\qquad A = A_1\oplus A_2.$$

The scalar products (9.4) and (9.9) define the corresponding adjoint operators:

A_1^*: $C^\infty\big(T(\sigma)\big)\to C^\infty\big(\operatorname{den} S^2T(\sigma)\oplus S^2T^*(\sigma)\big)$,

A_2^*: $C^\infty(\sigma)\to C^\infty\big(\operatorname{den} S^2T(\sigma)\oplus S^2T^*(\sigma)\big)$,

$$(15.33)\qquad A_1^*(u^j) = \Big(-\tfrac{1}{2}(\overline{\nabla}^iu^j+\overline{\nabla}^ju^i)\sqrt{\bar{g}}\,,\ -\frac{1}{2\sqrt{\bar{g}}}(\pi_{ia}\overline{\nabla}^au_j+\pi_{ja}\overline{\nabla}^au_i)-$$
$$-\overline{\nabla}_a(\pi_{ij}u^a)\Big).$$

$$(15.34)\qquad A_2^*(\chi) = \Big(-2(\pi^{ij}-\tfrac{1}{2}\operatorname{tr}\pi\bar{g}^{ij})\chi\,,\ -\frac{2}{\bar{g}}(\pi_{is}\bar{g}^{sk}\pi_{kj}-\tfrac{1}{2}\operatorname{tr}\pi\pi_{ij})+$$
$$+\overline{\nabla}_i\overline{\nabla}_j\chi-g_{ij}\overline{\nabla}_k\overline{\nabla}^k\chi-\bar{R}_{ij}\chi+g_{ij}(\bar{R}-2\lambda)\chi\Big).$$

LEMMA 15.4. *The operators A_1^*, A_2^* have injective symbols.*

We refer the reader to [34] for the definition of the symbol.

The proof of the lemma is elementary.

LEMMA 15.5. *If σ is a compact manifold without a boundary, then we have the decompositions in orthogonal sums:*

$$(15.35a)\qquad C^\infty\big(\operatorname{den} S^2T(\sigma)\oplus S^2T^*(\sigma)\big) = \ker A_1\oplus\operatorname{im}A_1^*,$$

$$(15.35b)\qquad C^\infty\big(\operatorname{den} S^2T(\sigma)\oplus S^2T^*(\sigma)\big) = \ker A_2\oplus\operatorname{im}A_2^*.$$

Decompositions (15.35) are proved in [4] for any differential operator with an injective symbol on a compact manifold. We are going to prove (9.15) using the theory of elliptic differential operators. The detailed analysis of the Berger–Ebin proof [4] of decompositions of the type (15.35) shows that we need only the operator AA^* be elliptic. Unfortunately, AA^* fails be an elliptic operator in the usual sense because it consists of four differential operators

$$L_{11}\colon C^\infty\big(T(\sigma)\big) \to C^\infty\big(T(\sigma)\big) \quad \text{of order } 2, \tag{15.36}$$

$$L_{12}\colon C^\infty(\sigma) \to C^\infty\big(T(\sigma)\big) \quad \text{of order } 3, \tag{15.37}$$

$$L_{21}\colon C^\infty\big(T(\sigma)\big) \to C^\infty(\sigma) \quad \text{of order } 3, \tag{15.38}$$

$$L_{22}\colon C^\infty(\sigma) \to C^\infty(\sigma) \quad \text{of order } 4, \tag{15.39}$$

$$AA^* = (L_{11}, L_{12}) \oplus (L_{21}, L_{22}). \tag{15.40}$$

This system turns out to be elliptic in the generalized sense of Douglis–Nirenberg (cf. Appendix for the complete proof). The ellipticity of AA^* gives raise to the decomposition (9.15) by arguments presented in [4].

Proof of Corollary 1. Proposition 9.2 implies

$$(\ker A \cap \ker AJ) \oplus \operatorname{im} JA^* \subset \ker A.$$

Let $\mathfrak{X} \in \ker A$. We decompose $-J\mathfrak{X} = \mathfrak{Y}_1 + \mathfrak{Y}_2$ where $\mathfrak{Y}_1 \in \ker A$, $\mathfrak{Y}_2 \in \operatorname{im} A$. Thus, $\mathfrak{X} = J\mathfrak{Y}_1 + J\mathfrak{Y}_2$. But $J\mathfrak{Y}_2 \in \operatorname{im} JA^* \subset \ker A$. Therefore, $J\mathfrak{Y}_1 \in \ker A$. Moreover, $J(J\mathfrak{Y}_1) = -\mathfrak{Y}_1 \in \ker A$.

Proof of Corollary 2. The first statement of corollary follows from (9.15), (9.8) and Proposition 9.2. The second part is the consequence of (9.3), (9.16).

Proof of Proposition 9.7.

LEMMA 15.6. *In local coordinates*

$$C = \big\{\big(x^\lambda, g_{\mu\nu}(x^\lambda), \Gamma^\tau_{\mu\nu}(x^\lambda)\big) \in \mathscr{P}\colon (x^\lambda) \in M\big\} \quad \textit{and} \quad v = v^\mu \frac{\partial}{\partial x^\mu}.$$

Then $d\hat{R}_{\mathrm{id}}(C)v = \hat{X}$, *where* $\hat{X} \in \tilde{T}_C(\mathscr{H})$ *is represented by the vector field* X *on* C:

$$X = \sum_{\mu \leqslant \nu} (\nabla_\mu v_\nu + \nabla_\nu v_\mu) \frac{\partial}{\partial g_{\mu\nu}} + \sum_{\mu \leqslant \nu} (\nabla_\mu \nabla_\nu v^\lambda + R^\lambda_{\nu\sigma\mu} v^\sigma) \frac{\partial}{\partial \Gamma^\lambda_{\mu\nu}}. \tag{15.41}$$

Proof. If $t \to \varphi_t$ is a one-parameter family of diffeomorphisms of M such that $\varphi_0 = \mathrm{id}$ and $\frac{d}{dt}\varphi_t\Big|_{t=0} = v$, then the infinitesimal change of $g_{\mu\nu}$ is given by the Killing formula (cf. [42]):

$$\Delta_t g_{\mu\nu} = t\delta g_{\mu\nu} = t\pounds_v g_{\mu\nu} = t(\nabla_\mu v_\nu + \nabla_\nu v_\mu). \tag{15.42}$$

The infinitesimal change of $\Gamma^{\lambda}_{\mu\nu}$ is equal to (cf. (8.2))

$$\Delta_t \Gamma^{\lambda}_{\mu\nu} = t\delta\Gamma^{\lambda}_{\mu\nu} = t\tfrac{1}{2} g^{\lambda\alpha}(\nabla_\mu \delta g_{\nu\alpha} + \nabla_\nu \delta g_{\mu\alpha} - \nabla_\alpha \delta g_{\mu\nu}). \tag{15.43}$$

Substituting (15.42) into (15.43), we obtain

$$\delta\Gamma^{\lambda}_{\mu\nu} = \nabla_\mu \nabla_\nu v^\lambda + R^{\lambda}_{\nu\sigma\mu} v^\sigma.$$

We intend to express (15.41) in the ADMW coordinates. We pick a space-like surface $\sigma \subset M$ and the admissible initial surface $c \subset \mathscr{P}$ corresponding to it. In a neighbourhood of σ in M we have the vector field $n^\mu = \left(\frac{1}{N}, -\frac{N^k}{N}\right)$ which is normal to σ. The vector field v can be decomposed into two components:

$$v^\mu = \beta n^\mu + (0, a^k), \tag{15.44}$$

where β is a scalar function on σ depending on the parameter x^0 and a^k is a vector field in σ depending on x^0. We have

$$\begin{aligned} \beta &= N v^0, \quad & a^k &= v^k + v^0 N^k, \\ v^0 &= \frac{\beta}{N}, \quad & v^k &= a^k - \frac{\beta}{N} N^k. \end{aligned} \tag{15.45}$$

In ADMW coordinates (15.42) takes the form

$$\delta g_{ij} = (\bar{\nabla}_i a_j + \bar{\nabla}_j a_i) + \frac{2}{\sqrt{\bar{g}}}(\pi_{ij} - \tfrac{1}{2} g_{ij} \operatorname{tr}\pi)\beta, \tag{15.46a}$$

where $a_j = g_{js} a^s = g_{js} v^s + g_{j0} v^0$.

According to (6.13) we have

$$\delta\pi^{ij} = \delta\left(\sqrt{-g}(\bar{g}^{ip}\bar{g}^{jq}\Gamma^0_{pq} - \bar{g}^{ij}\bar{g}^{pq}\Gamma^0_{pq})\right). \tag{15.47}$$

Using (15.42), (15.43) and the equations of motion (6.19a), we get

$$\begin{aligned} \delta\pi^{ij} = {} & -\left(\pi^{ib}\bar{\nabla}_b a^j + \pi^{jb}\bar{\nabla}_b a^i - \bar{\nabla}_b(\pi^{ij} a^b)\right) + \\ & + \sqrt{\bar{g}}\left(\bar{\nabla}^i\bar{\nabla}^j\beta - \bar{g}^{ij}\bar{\nabla}^r\bar{\nabla}_r\beta - \bar{R}^{ij}\beta + \bar{g}^{ij}(\bar{R} - 2\lambda)\beta\right) - \\ & - \frac{2}{\sqrt{\bar{g}}}\left(\pi^{im} g_{mn} \pi^{nj} - \frac{1}{2}\operatorname{tr}\pi\pi^{ij}\right)\beta. \end{aligned} \tag{15.46b}$$

We have also:

$$\begin{aligned} \delta N_k = \delta g_{0k} = {} & g_{ks}\partial_0 a^s + a^p \bar{\nabla}_p N_k + N_s \bar{\nabla}_k a^s + \bar{\nabla}_k N \cdot \beta - N\bar{\nabla}_k\beta + \\ & + \frac{2}{\sqrt{\bar{g}}} N^s(\pi_{ks} - \tfrac{1}{2} g_{ks}\operatorname{tr}\pi)\beta, \end{aligned} \tag{15.48a}$$

$$\delta N = \partial_0\beta - N^k\bar{\nabla}_k\beta + \bar{\nabla}_k N \cdot a^k. \tag{15.48b}$$

Comparing (15.28) with (15.46), we see that $\operatorname{im} d\hat{R}_{\mathrm{id}}(C) \subset \overset{1}{W}_C(c)$. Let $\hat{X} \in \overset{1}{W}_C(c)$ be determined by four quantities, $\delta N_k = \delta N_k(x^0)$, $\delta N = \delta N(x^0)$, depending on a parameter x^0 and 12 quantities $(\delta\pi^{ij}, \delta g_{ij})$ which are generated by a vector field u^j in σ and a function χ on σ (cf. (15.28)). System (15.48) of linear differential equations has (for the given δN_k, δN) a unique solution $a^k = a^k(x^0)$, $\beta = \beta(x^0)$ satisfying the initial data $a^k(0) = u^k$, $\beta(0) = \chi$. Thus, $\hat{X} \in \operatorname{im} d\hat{R}(C)$.

Proposition 9.7 shows that the subspace $\overset{1}{W}_C(c)$ does not depend on a choice of an admissible initial surface $c \subset C$. Now we shall give a direct proof of that statement.

LEMMA 15.7. *Let $c_1, c_2 \subset C$ be admissible initial surfaces in $\mathscr{P}$ and σ_1, σ_2 the corresponding space-like surfaces in M. Let an* ADMW *coordinate system in $\mathscr{P}$ be given and let $\sigma_i = \{x \in M\colon x^0 = t_i\}$, $i = 1, 2$. In such an* ADMW *coordinate system $\overset{0}{W}_C(c_1) = \overset{0}{W}_C(c_2)$.*

Proof. Suppose $\hat{Y} \in \overset{0}{W}_C(c_1)$ and $\hat{Y}$ is determined by the quantities u^j, χ in σ (cf. (15.28)). Let $a^k = a^k(x^0)$, $\beta = \beta(x^0)$ be the solution of the system of linear differential equations

$$g_{ks}\partial_0 a^s = -a^p \overline{\nabla}_p N_k - N_s \overline{\nabla}_k a^s - \overline{\nabla}_k N \cdot \beta + N \overline{\nabla}_k \beta - \frac{2}{\sqrt{\bar{g}}} N^s (\pi_{ks} - \tfrac{1}{2} g_{ks} \operatorname{tr} \pi)\beta,$$

(15.49)

$$\partial_0 \beta = N^k \overline{\nabla}_k \beta - \overline{\nabla}_k N \cdot a^k$$

satisfying the initial conditions $a^k(t_1) = u^k$, $\beta(t_1) = \chi$. It is easy to show by a direct calculation that the quantities $\big(\delta\pi^{ij}(x^0), \delta g_{ij}(x^0)\big)$, obtained from $a^k(x^0)$, $\beta(x^0)$ by means of formulas (15.28), satisfy the linearized equations of motion (8.9) and (8.10a) (for $\delta N_k \equiv 0$, $\delta N \equiv 0$). Therefore the values $\delta\pi^{ij}(t_2)$, $\delta g_{ij}(t_2)$ are initial data on σ_2 for a vector field Y on C representing $\hat{Y}$. Thus, we have a one-to-one correspondence between elements of $\overset{0}{W}_C(c_1)$ and $\overset{0}{W}_C(c_2)$.

The definition of the subspace $\overset{0}{W}_C(c)$ depends on an ADMW coordinate system connected with the surface $\sigma = \pi(c) \subset M$, i.e., it depends on the choice of a lapse function N and a shift vector N^k. Therefore we cannot assert that $\overset{0}{W}_C(c)$ are the same for all $c \subset C$ (if $\delta N = 0$, $\delta N_k = 0$ in one ADMW system, they can be different from zero in another). However, the subspace $\overset{1}{W}_C(c) = \overset{0}{W}_C(c) \oplus \overset{0}{\tilde{T}}_C(\mathscr{H}, c)$ does not depend on the choice of an ADMW coordinate system. Therefore, by arguments similar to those used in the proof of Lemma 15.7, we infer that all subspaces $\overset{1}{W}_C(c)$ are the same.

Proof of Proposition 9.8. It is easy to see that $H_1(c)$ is an isotropic subspace (Proposition 9.4 and formula (8.12)). We have to prove that $H_1(c)$ is a maximal isotropic subspace. If this is not true, then there exists a vector $\hat{Y}_1 \in \overset{1}{\tilde{T}}_C(\mathscr{H}, c)$ represented by a couple $(\underset{1}{\delta\pi^{ij}}, \underset{1}{\delta g_{ij}})$ such that, for every vector $\hat{Y} \in_\pi T_C(\mathscr{H}, c)$,

$$\Omega(\hat{Y}, \hat{Y}_1) = 0. \tag{15.50}$$

If $\hat{Y}$ is represented by a couple $(\delta\pi^{ij}, 0)$, then (15.50) implies

$$\int_c \delta\pi^{ij} \underset{1}{\delta g_{ij}} dx^1 \wedge dx^2 \wedge dx^3 = 0. \tag{15.50'}$$

The constraint equations (9.25a) and arguments given in the proof of Proposition 9.5 imply that

$$\underset{1}{\delta g_{ij}} = -\tfrac{1}{2}(\bar{\nabla}_i u_j + \bar{\nabla}_j u_i) - \frac{2}{\sqrt{\bar{g}}}(\pi_{ij} - \tfrac{1}{2}\operatorname{tr}\pi g_{ij})\chi, \tag{15.51}$$

where u_j is a covector field in σ and χ is a function on σ. We have no special conditions for quantities $\underset{1}{\delta\pi^{ij}}$, we have to assume only that $(\underset{1}{\delta\pi^{ij}}, \underset{1}{\delta g_{ij}})$ satisfy the constraint equations (8.10b). Let $(\underset{2}{\delta\pi^{ij}}, \underset{2}{\delta g_{ij}})$ be the couple generated from (u_j, χ) by means of formulas (15.28). We know that such a couple determines a vector $\hat{Y}_2 \in \overset{0}{W}_C(c)$. By (15.51) we have also $\underset{2}{\delta g_{ij}} = -\underset{1}{\delta g_{ij}}$. Therefore we have

$$(\underset{1}{\delta\pi^{ij}}, \underset{1}{\delta g_{ij}}) = (\underset{1}{\delta\pi^{ij}} + \underset{2}{\delta\pi^{ij}}, 0) - (\underset{2}{\delta\pi^{ij}}, \underset{2}{\delta g_{ij}}). \tag{15.52}$$

It is easy to check that the quantities $\underset{3}{\delta\pi^{ij}} = \underset{1}{\delta\pi^{ij}} + \underset{2}{\delta\pi^{ij}}$ satisfy (9.25a) and therefore the couple $(\underset{1}{\delta\pi_{ij}}, \underset{1}{\delta g_{ij}})$ represents a vector belonging to the subspace ${}_\pi T_C(\mathscr{H}, c) + \overset{0}{W}_C(c)$, i.e., $\hat{Y}_1 \in H_1(c)$.

In a similar way the maximality of the subspace $H_2(c)$ can be proved.

Proof of Lemma 11.4. We know (cf. [36]) that the Lie derivative is a homomorphism of the tensor (exterior) algebra on $\mathscr{K}$. We also have the formulas

$$\pounds_{\tilde{X}} \tilde{Y} = [\tilde{X}, \tilde{Y}], \quad \tilde{X}, \tilde{Y} \text{ are vector fields in } \mathscr{K}, \tag{15.53}$$

$$\pounds_X \overset{p}{a} = d(X \lrcorner \overset{p}{a}) + X \lrcorner d\overset{p}{a}, \quad \overset{p}{a} \text{ is a } p\text{-form in } \mathscr{K}. \tag{15.54}$$

Let

$$\mu = \pounds_{\tilde{X}_1}(\tilde{X}_2 \lrcorner \theta) - \pounds_{\tilde{X}_2}(\tilde{X}_1 \lrcorner \theta) - [\tilde{X}_1, \tilde{X}_2] \lrcorner \theta. \tag{15.55}$$

The homomorphism property gives

$$(15.56)\qquad \mu = \pounds_{\tilde{X}_1}\tilde{X}_2 \lrcorner\, \theta + \tilde{X}_2 \lrcorner\, \pounds_{\tilde{X}_1}\theta - \pounds_{\tilde{X}_2}\tilde{X}_1 \lrcorner\, \theta - \tilde{X}_1 \lrcorner\, \pounds_{\tilde{X}_2}\theta - [\tilde{X}_1, \tilde{X}_2] \lrcorner\, \theta.$$

Using (15.53) we have

$$(15.57)\qquad \mu = 2(\tilde{X}_2 \lrcorner\, \tilde{X}_1 \lrcorner\, d\theta) + \tilde{X}_2 \lrcorner\, d(\tilde{X}_1 \lrcorner\, \theta) - \tilde{X}_1 \lrcorner\, d(\tilde{X}_2 \lrcorner\, \theta) - [\tilde{X}_2, \tilde{X}_1] \lrcorner\, \theta.$$

On the other hand, applying (15.54) to (15.55), we obtain

$$(15.58)\qquad \mu = \tilde{X}_1 \lrcorner\, d(\tilde{X}_2 \lrcorner\, \theta) + d(\tilde{X}_1 \lrcorner\, \tilde{X}_2 \lrcorner\, \theta) - \tilde{X}_2 \lrcorner\, d(\tilde{X}_1 \lrcorner\, \theta) - d(\tilde{X}_2 \lrcorner\, \tilde{X}_1 \lrcorner\, \theta) - [\tilde{X}_1, \tilde{X}_2] \lrcorner\, \theta.$$

By (15.57) and (15.58) we have

$$(15.59)\qquad 2\mu = 2(\tilde{X}_2 \lrcorner\, \tilde{X}_1 \lrcorner\, d\theta) - 2d(\tilde{X}_2 \lrcorner\, \tilde{X}_1 \lrcorner\, \theta).$$

Lemma is proved.

Proof of Proposition 11.2. Let θ_0 be that part of formula (5.7) which does not contain the differential dx^0. We have

$$(15.60)\qquad \theta_0 = \sqrt{-g}(g^{k0} d\Gamma^0_{0k} + g^{ks} d\Gamma^0_{ks} - g^{00} d\Gamma^p_{p0} - g^{k0} d\Gamma^p_{pk}) \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

Of course, expression (15.60) is not covariant with respect to an arbitrary coordinate transformation in $\mathscr{P}$. In the ADMW coordinate-system (15.60) reads:

$$(15.60')\qquad \theta_0 = \Big[-g_{ij} d\pi^{ij} + \sqrt{\bar{g}} \frac{1}{N} \bar{g}^{ku} \big(d(\partial_u N_k) - \bar{\Gamma}^s_{ku} dN_s - N_s d\bar{\Gamma}^s_{ku}\big) + \sqrt{\bar{g}} \left(\partial_k \left(\frac{1}{N}\right) \bar{g}^{ku} dN_u - \partial_k \left(\frac{1}{N}\right) \bar{g}^{qu} \bar{g}^{kr} N_u dg_{qr} \right) - \sqrt{\bar{g}} \left(\frac{1}{N} \bar{\nabla}_u N_k \bar{g}^{kr} \bar{g}^{qu} dg_{rq} + \frac{N^k}{N} d\bar{\Gamma}^p_{pk} \right) \Big] \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

In the ADMW system, (15.60′) is a differential 4-form. If X is a vector field on C of type (8.6), then

$$(15.61)\qquad (X \lrcorner\, \theta_0)|C = -\delta\pi^{ij} g_{ij} dx^1 \wedge dx^2 \wedge dx^3 + \bar{\nabla}_k \left(\frac{1}{N} \delta N_u \bar{g}^{uk} \right) \sqrt{\bar{g}}\, dx^1 \wedge dx^2 \wedge dx^3 - \bar{\nabla}_u \left(N_k \bar{g}^{ka} \delta g_{ab} \bar{g}^{bu} \frac{1}{N} \right) \sqrt{\bar{g}}\, dx^1 \wedge dx^2 \wedge dx^3.$$

Thus, Proposition 11.2 follows from (15.61) and boundary conditions for the field X.

Proof of Proposition 11.5. We can assume that in formula (11.28) $\mathfrak{X}_1$ belongs to the subspace $\ker A \cap \ker AJ$. Thus $J\mathfrak{X}_1$ also belongs to this subspace. By virtue of (9.8) and (11.28) the proposition is proved.

Proof of Proposition 12.1. We have

$$\frac{dS_D(f_\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_{f(D)} \pounds_X \overset{4}{\theta}, \tag{15.62}$$

where $X = \dfrac{df_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}$ is a π-vertical vector field on $C = f(D)$. We consider all sections over D which have a fixed value of pr_2 on ∂D and therefore we have no restrictions for components $\delta g_{\mu\nu}$, $\delta \Gamma^\lambda_{\mu\nu}$ of X at interior points of D. If f is an extremal, then by virtue of (15.62), (15.54) and the boundary conditions we have

$$\int_{f(D)} X \lrcorner \overset{5}{\omega} = 0 \quad \text{for every } \pi\text{-vertical field } X \text{ on } f(D). \tag{15.63}$$

Therefore $(X \lrcorner \overset{5}{\omega})|f(D) = 0$ for every vector field X on $f(D)$.

Remark. We can consider the variational principle (12.1)–(12.2) for the class of integrable sections of $\mathscr{P}$, i.e., sections satisfying (5.2). In that case instead of (15.63) we obtain the following condition: for every integrable vector field Y on $f(D)$ (i.e., for a vector field of the type (8.1) satisfying (8.2)) we have

$$\int_{f(D)} Y \lrcorner \overset{5}{\omega} = 0. \tag{15.64}$$

But for every π-vertical vector field X on $f(D)$ there exists a integrable vector field Y such that $Y - X$ is ϱ-vertical (ϱ: $\mathscr{P} \to S^2T^*M$). By virtue of the proof of Proposition 5.3

$$\big((Y - X) \lrcorner \overset{5}{\omega}\big)|f(D) = 0 \tag{15.65}$$

for an integrable section f. Formulas (15.64) and (15.65) imply (15.63). We conclude that an extremal of the problem is an $\overset{5}{\omega}$-singular section of $\mathscr{P}$.

Proof of Proposition 12.2. By virtue of formulas (15.62), (15.54) and (15.61) we have

$$\frac{dS(f_\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_{f(D)} (Y \lrcorner \overset{5}{\omega}) - \int_{\partial f(D)} g_{ij}\, \delta\pi^{ij}\, dx^1 \wedge dx^2 \wedge dx^3, \tag{15.66}$$

where Y is an field of type (8.6) on $f(D)$ which is integrable (i.e. satisfying (8.7), (8.8)) and $\partial f(D)$ consists of two 3-dimensional surfaces in $\mathscr{P}$ whose projections onto M are space-like surfaces σ_1, σ_2. The boundary conditions (12.2) (ii) give $\delta\pi^{ij} = 0$ on $\partial f(D)$. Thus, formula (15.66) implies (15.64) and the above arguments show that f is an $\overset{5}{\omega}$-singular section of $\mathscr{P}$.

Remark. We have omitted the divergences in formula (15.61) because of the boundary conditions for the field Y in "spatial infinity". These boundary conditions are consequences of restrictions which have to be imposed for the class of sections under consideration.

Proof of Proposition 13.2. In local coordinates of type (13.30)

$$\sigma_\varepsilon = \left\{\tilde{x}^0 \cong \frac{1}{N(x^k)}\,\varepsilon\delta\tau,\ \tilde{x}^k \cong x^k - \frac{N^k(x^s)}{N(x^s)}\,\varepsilon\delta\tau\right\}, \tag{15.67}$$

$$\varphi_\varepsilon \cong \big(A_\mu(0, x^s) + \varepsilon\delta A_\mu(0, x^s)\big)\,d\tilde{x}^\mu. \tag{15.68}$$

Formulas (15.67) and (15.68) give

$$\begin{aligned} G_\varepsilon^* i^* \varphi_\varepsilon \cong{}& \big(A_0(x^u) + \varepsilon\delta A_0(x^u)\big)\left(-\frac{1}{N^2}\,\partial_k N\varepsilon\,\delta\tau\,dx^k\right) + \\ &+ \big(A_s(x^u) + \varepsilon\delta A_s(x^u)\big)\left(dx^s - \frac{1}{N}\,\partial_k N^s\,\varepsilon\delta\tau\,dx^k + \frac{1}{N^2}\,\partial_k N\,N^s\varepsilon\delta\tau\,dx^k\right). \end{aligned} \tag{15.69}$$

Expressions (13.29) and (15.69) imply

$$\zeta = \left(\delta A_k - (A_\mu n^\mu)\,\frac{1}{N}\,\partial_k N\delta\tau - A_s\,\frac{1}{N}\,\partial_k N^s\,\delta\tau\right)dx^k = \hat{\delta} A_k\,dx^k.$$

Proof of Lemma 14.1. Let σ_ε be described by (15.67). The quantities $g_{\alpha\beta} + \varepsilon\delta g_{\alpha\beta}$ are components of the metric tensor g at points of σ_ε. We have

$$\partial_\lambda g_{\mu\nu}(\tilde{x}^\lambda)\,\frac{\partial\tilde{x}^\lambda}{\partial x^k} = \frac{\partial}{\partial x^k}\,(g_{\mu\nu} + \varepsilon\delta g_{\mu\nu}). \tag{15.70}$$

(15.70) and (15.67) give

$$\begin{aligned} (\partial_0 g_{\mu\nu} + \varepsilon\delta\partial_0 g_{\mu\nu})\,\varepsilon\partial_k\left(\frac{1}{N}\right)\delta\tau + (\partial_p g_{\mu\nu} + \varepsilon\delta\partial_p g_{\mu\nu})\left(\delta_k^p - \varepsilon\partial_k\left(\frac{N^p}{N}\right)\delta\tau\right) \\ = \partial_k g_{\mu\nu} + \varepsilon\partial_k\delta g_{\mu\nu}. \end{aligned} \tag{15.71}$$

It is easy to see that (15.71) gives (14.5).

Proof of Proposition 14.1. For a surface σ_ε defined by (15.67) we denote by $\underset{\sigma_\varepsilon}{n^\mu}$ the unit normal vector. We have the conditions

$$g_{\mu\nu}(\tilde{x}^\lambda)\,\underset{\sigma_\varepsilon}{n^\mu}(\tilde{x}^\lambda)\,\frac{\partial\tilde{x}^\nu}{\partial x^k} = 0, \tag{15.72}$$

$$g_{\mu\nu}(x^\lambda)\,\underset{\sigma_\varepsilon}{n^\mu}(\tilde{x}^\lambda)\,\underset{\sigma_\varepsilon}{n^\nu}(\tilde{x}^\lambda) = -1. \tag{15.73}$$

Condition (15.72) gives

$$\big(g_{\mu\nu}(0, x^k) + \varepsilon\delta g_{\mu\nu}\big)\,\underset{\sigma_\varepsilon}{n^\mu}(\tilde{x}^\lambda)\,\frac{\partial\tilde{x}^\nu}{\partial x^k} \cong 0. \tag{15.74}$$

Using (15.67), we obtain the second of formulas (14.6). The normalization condition (15.73) takes the form

$$(g_{\mu\nu}+\varepsilon\delta g_{\mu\nu})n^{\mu}(\tilde{x}^{\lambda})n^{\nu}(\tilde{x}^{\lambda})\cong 1. \tag{15.75}$$

Using (15.67), we obtain the first of the formulas (14.6).

Formulas (14.7) and (14.8) follows from (14.2), (14.3) and (14.6).

We have also

$$\underset{\sigma_\varepsilon}{g_{ij}} = g_{\mu\nu}(\tilde{x}^{\lambda})\frac{\partial\tilde{x}^{\mu}}{\partial x^{i}}\frac{\partial\tilde{x}^{\nu}}{\partial x^{j}}\cong(g_{\mu\nu}+\varepsilon\delta g_{\mu\nu})\frac{\partial\tilde{x}^{\mu}}{\partial x^{i}}\frac{\partial\tilde{x}^{\nu}}{\partial x^{j}}. \tag{15.76}$$

By virtue of (15.67) we have (14.8). Formula (14.9) is a consequence of (14.8).

We have to find a formula which generalizes formula (6.8) because σ_ε is not a surface of constant x^0. The geometric definition of the second fundamental for K of σ_ε (cf. [27], [33]) gives

$$K_{ij} = -\frac{\partial\tilde{x}^{\beta}}{\partial x^{i}}g_{\beta\alpha}(\tilde{x}^{\lambda})\left(\partial_j \underset{\sigma_\varepsilon}{n^{\alpha}}(\tilde{x}^{\lambda})+\Gamma^{\alpha}_{\nu\mu}(\tilde{x}^{\lambda})\underset{\sigma_\varepsilon}{n^{\nu}}(\tilde{x}^{\lambda})\frac{\partial\tilde{x}^{\mu}}{\partial x^{j}}\right). \tag{15.77}$$

By virtue of (15.67), (14.6), (14.5), (8.2) we have

$$K_{ij} = -N\Gamma^0_{ij}-\varepsilon\delta(N\Gamma^0_{ij})+\varepsilon(\Gamma^0_{iQ}\partial_j N^Q+\Gamma^0_{jQ}\partial_i N^Q)\,\delta\tau+ \\ +\varepsilon\frac{1}{N}\bar{\nabla}_i\bar{\nabla}_j N\,\delta\tau. \tag{15.78}$$

Formulas (15.78), (6.13), (14.9) and (14.10) give (14.11).

Appendix

Proof of the ellipticity of the operator AA^*

Let $V_i\to B$, $i=1,2$, be vector bundles and L_{ij} be linear differential operators from V_i to V_j. We say that the system $L=(L_{ij})$ is *elliptic in the sense of Douglis–Nirenberg* if there exist non-negative integers s_1, s_2, t_1, t_2 such that the degree of the operator L_{ij} does not exceed s_i-t_j and for every $p\in B$ and for every non-zero covector $\xi\in T^*_p(B)$ the matrix

$$\sigma(L)(\xi) = \begin{bmatrix}\sigma_{s_1-t_1}(L_{11})(\xi) & \sigma_{s_1-t_2}(L_{12})(\xi)\\ \sigma_{s_2-t_1}(L_{21})(\xi) & \sigma_{s_2-t_2}(L_{22})(\xi)\end{bmatrix}$$

provides an isomorphism between $V_{p1}\oplus V_{p2}$ and $V_{p1}\oplus V_{p2}$. Here $\sigma_{s_i-t_j}(L_{ij})(\xi)$ stands for the symbol of the order s_i-t_j of the operator L_{ij}.(1)

(1) Compare: S. Agmon, A. Douglis and L. Nirenberg, Comm. Pure Appl. Math. 17 (1964), pp. 35–92; L. Hörmander, *Linear partial differential operators*, Berlin 1963; R. Palais, *Seminar on the Atiyah–Singer index theorem*, Princenton 1965.

In our case $V_1 = T(\sigma)$, $V_2 = \boldsymbol{R} \times \sigma$. Using (9.10), (9.11), and (9.13), we have

$L_{11}(u^j) = v^i$,

$$v^i = -\tfrac{1}{2}\bar{g}^{ia}\bar{\nabla}_j\bar{\nabla}_a u^j - \tfrac{1}{2}\bar{g}^{ja}\bar{\nabla}_j\bar{\nabla}_a u^i -$$

$$-\frac{1}{2\bar{g}}\pi^b_s\pi^{sk}\bar{\nabla}_k\bar{\nabla}_b u^i - \frac{1}{2\bar{g}}\pi^{ib}\pi^k_s\bar{\nabla}_k\bar{\nabla}_b u^s +$$

$$+\frac{1}{2\bar{g}}\pi^{is}\pi^k_s\bar{\nabla}_k\bar{\nabla}_b u^b + \frac{1}{4\bar{g}}\bar{g}^{ai}\pi^b_k\pi^k_s\bar{\nabla}_a\bar{\nabla}_b u^s + \frac{1}{4\bar{g}}\bar{g}^{ai}\pi^b_s\pi^s_k\bar{\nabla}_a\bar{\nabla}_b u^k -$$

$$-\frac{1}{4\bar{g}}\bar{g}^{ia}\pi_{ks}\pi^{ks}\bar{\nabla}_a\bar{\nabla}_b u^b + \text{(terms of the lower order)};$$

$L_{12}(\chi) = v^i$,

$$v^i = \frac{1}{\sqrt{\bar{g}}}\bar{g}^{ia}\pi^{ks}\bar{\nabla}_k\bar{\nabla}_s\bar{\nabla}_a\chi - \frac{1}{\sqrt{\bar{g}}}\bar{g}^{ab}\pi^{ik}\bar{\nabla}_k\bar{\nabla}_a\bar{\nabla}_b\chi - \frac{1}{2\sqrt{\bar{g}}}\bar{g}^{ia}\pi^{ks}\bar{\nabla}_a\bar{\nabla}_k\bar{\nabla}_s\chi +$$

$$+\frac{1}{2\sqrt{\bar{g}}}\operatorname{tr}\pi\bar{g}^{ai}\bar{g}^{bc}\bar{\nabla}_a\bar{\nabla}_b\bar{\nabla}_c\chi + \text{(terms of the lower order)};$$

$L_{21}(u^i) = \psi$,

$$\psi = -\frac{1}{2\sqrt{\bar{g}}}\pi^{ap}\bar{\nabla}_p\bar{\nabla}_q\bar{\nabla}_a u^q - \frac{1}{2\sqrt{\bar{g}}}\pi^{aq}\bar{\nabla}_p\bar{\nabla}_q\bar{\nabla}_a u^p +$$

$$+\frac{1}{2\sqrt{\bar{g}}}\pi^{pq}\bar{\nabla}_p\bar{\nabla}_q\bar{\nabla}_a u^a + \frac{1}{\sqrt{\bar{g}}}\pi^a_b\bar{\nabla}^k\bar{\nabla}_k\bar{\nabla}_a u^b -$$

$$-\frac{1}{2\sqrt{\bar{g}}}\operatorname{tr}\pi\bar{\nabla}_k\bar{\nabla}^k\bar{\nabla}_a u^a + \text{(terms of the lower order)};$$

$L_{22}(\chi) = \psi$,

$$\psi = \bar{\nabla}^p\bar{\nabla}^q\bar{\nabla}_p\bar{\nabla}_q\chi + \bar{\nabla}^q\bar{\nabla}_q\bar{\nabla}^p\bar{\nabla}_p\chi + \text{(terms of the lower order)}.$$

We put $s_1 = 3$, $t_1 = 1$, $s_2 = 4$, $t_2 = 0$. For $\xi = (\xi_i)$

$$\sigma(L)(\xi) = \begin{bmatrix} P_{11}{}^1{}_1(\xi) & P_{11}{}^1{}_2(\xi) & P_{11}{}^1{}_3(\xi) & P_{12}{}^1(\xi) \\ P_{11}{}^2{}_1(\xi) & P_{11}{}^2{}_2(\xi) & P_{11}{}^2{}_3(\xi) & P_{12}{}^2(\xi) \\ P_{11}{}^3{}_1(\xi) & P_{11}{}^3{}_2(\xi) & P_{11}{}^3{}_3(\xi) & P_{12}{}^3(\xi) \\ P_{21\,1}(\xi) & P_{21\,2}(\xi) & P_{21\,3}(\xi) & P_{22}\ (\xi) \end{bmatrix},$$

where

$$P_{11}{}^{i}{}_{j}(\xi) = -\frac{1}{2}\xi^i\xi_j - \frac{1}{2}\delta^i_j\|\xi\|^2 - \frac{1}{2\bar{g}}\pi^b_s\pi^{sk}\xi_b\xi_k\delta^i_j - \frac{1}{2\bar{g}}\pi^{ib}\pi^k_j\xi_k\xi_b +$$

$$+\frac{1}{2\bar{g}}\pi^{is}\pi^k_s\xi_k\xi_j + \frac{1}{2\bar{g}}\xi^i\pi^b_k\pi^k_j\xi_b - \frac{1}{4\bar{g}}\pi^{ks}\pi_{ks}\xi^i\xi_j,$$

$$P_{12}{}^{i}(\xi) = \frac{1}{2\sqrt{\bar{g}}}\pi^{ks}\xi_k\xi_s\xi^i - \frac{1}{\sqrt{\bar{g}}}\pi^{ik}\xi_k\|\xi\|^2 + \frac{1}{2\sqrt{\bar{g}}}\operatorname{tr}\pi\xi^i\|\xi\|^2,$$

$$P_{21\,j}(\xi) = -\frac{1}{2\sqrt{\bar{g}}}\pi^{ks}\xi_k\xi_s\xi_j + \frac{1}{\sqrt{\bar{g}}}\pi^a_j\xi_a\|\xi\|^2 - \frac{1}{2\sqrt{\bar{g}}}\operatorname{tr}\pi\xi_j\|\xi\|^2,$$

$$P_{22}(\xi) = 2\|\xi\|^4.$$

We have to prove that the system of linear equations given by the matrix $\sigma(L)(\xi)$ has only the trivial solution. One can easily eliminate the last equation. The remaining system of 3 linear equations has only the zero solution. An elementary but non-trivial proof of this fact consists of choosing such a coordinate system in which g_{ij} is the unit matrix. Then we diagonalize the symmetric matrix π^{ij} by an orthogonal transformation and consider the matrix equation in the space $\boldsymbol{R}^3$. It turns out that this equation has only the trivial solution.

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