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A characterization of cubes and spheres

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Introduction

The aim of this paper is to characterize cubes, particularly Euclidean n-cubes, the Hilbert cube and Tychonoff cubes, as well as Euclidean n-spheres, as those topological spaces which have subbases with some prescribed properties. The cubes are characterized by the property of having the so-called BISC-subbases and some additional properties of these subbases allow us to distinguish the cases of Euclidean cubes, the Hilbert cube and Tychonoff cubes. A subbase is called a BISC-subbase if it possesses the properties (B), (I), (S), (C) (see § 1). One of these properties, namely the property (S), the so-called supercompactness, was introduced for the first time by J. de Groot [1], who initiated the study of this subject. In papers [2] and [3] de Groot and Schnare gave also a characterization of cubes in terms of subbases; however, the case of Tychonoff cubes was not included there.

The second part of this paper deals with a characterization of n-spheres, where a characterization of n-cubes in terms of BISC-subbases is applied.

Let us add that topological characterizations of cubes are not known except the subbase characterization and the well-known characterizations of closed interval according to Moore and Lennes [4] (for more information see Kuratowski [4]).

For spheres we know only the characterization of the 1-sphere and the 2-sphere (the last due to Kuratowski [4]).

The situation is different if the space is assumed to be a manifold; then e.g. a characterization of n-cubes due to Doyle [7] is known.

§ 1. BISC-subbases

Let X be a topological space. Let $\mathscr{P}$ be a subbase of the topology on X such that

(B) *if* $U \in \mathscr{P}$, *then* $X \setminus \mathrm{cl}\, U \in \mathscr{P}$;

(I) (*inclusion property*) *if* U, V *are in* $\mathscr{P}$ *and there exists a* G *in* $\mathscr{P}$ *such that*

$$\mathrm{cl}\, U \cap \mathrm{cl} G = \mathrm{cl}\, V \cap \mathrm{cl} G = \emptyset,$$

then $U \subset V$ *or* $V \subset U$;

(S) (*supercompactness property*) *if* $\mathscr{R} \subset \mathscr{P}$ *and* $\bigcup \mathscr{R} = X$, *then there exist* U *and* V *in* $\mathscr{R}$ *such that* $U \cup V = X$;

(C) (*closure separation property*) *if* x *and* y *are in* X *and* $x \neq y$, *then there exists a* U *in* $\mathscr{P}$ *such that* $x \in U$ *and* $y \notin \mathrm{cl}\, U$.

The subbases satisfying the above conditions will be called BISC-*subbases*.

The properties (C) and (S) imply that X is Hausdorff and compact. In virtue of property (B) it suffices to consider only BISC-subbases which consist of regularly open sets and are such that $\emptyset$ and X do not belong to $\mathscr{P}$.

1.1. *If* $\mathscr{R} \subset \mathscr{P}$ *and* $\bigcap \{\mathrm{cl} W\colon W \in \mathscr{R}\} = \emptyset$, *then there exist* U *and* V *in* $\mathscr{R}$ *such that* $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$.

Proof. The above follows immediately by properties (B) and (S) for $\mathscr{P}$.

1.2. *If* $U \in \mathscr{P}$ *and* $x \notin \mathrm{cl}\, U$, *then there exists a* V *in* $\mathscr{P}$ *such that* $x \in V$ *and* $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$.

Proof. If $y \in \mathrm{cl}\, U$ then, by (C), there exists a W in $\mathscr{P}$ such that $x \in W$ and $y \notin \mathrm{cl} W$. Let $\mathscr{R}$ be the family of all such W. Then $\mathrm{cl}\, U \cap \bigcap \{\mathrm{cl} W\colon\ W \in \mathscr{R}\} = \emptyset$. By 1.1, there exist two elements, say V and V', in $\mathscr{R} \cup \{U\}$ such that $\mathrm{cl}\, V \cap \mathrm{cl}\, V' = \emptyset$. Since $x \in \bigcap \{\mathrm{cl} W\colon\ W \in \mathscr{R}\}$, V or V', say V', is U, and the other V belongs to $\mathscr{R}$. Clearly, x belongs to V, the required set in $\mathscr{P}$ for U.

1.3. *If* $U \in \mathscr{P}$ *and* $x \in U$, *then there exists a* V *in* $\mathscr{P}$ *such that* $U \cup V = X$ *and* $x \notin \mathrm{cl}\, V$.

Proof. If $y \notin U$ (such a y exists since $X \notin \mathscr{P}$), then, by (C), there exists a W in $\mathscr{P}$ such that $y \in W$ and $x \notin \mathrm{cl} W$. Let $\mathscr{R}$ be the family of all such W.

Then $U\cup\bigcup\mathscr{R}=X$. By (S), there exist two elements, say V *and* V', in $\mathscr{R}\cup\{U\}$ such that $V\cup V'=X$. Since $x\notin\bigcup\mathscr{R}$ and $U\neq X$, V or V', say V', is U, and the other, V, belongs to $\mathscr{R}$. Clearly $x\notin \mathrm{cl}\,V$ and V is the required set in $\mathscr{P}$ for U and x.

1.4. *If $U\in\mathscr{P}$ and $x\in U$, then there exists a V in $\mathscr{P}$ such that $x\in V\subset \mathrm{cl}\,V\subset U$.*

Proof. Since $U\neq X$, there exists a y in X such that $y\notin U$. By 1.3, there exists a G in $\mathscr{P}$ such that $x\notin \mathrm{cl}G$ and $U\cup G=X$. By 1.2, there exists a V in $\mathscr{P}$ such that $x\in V$ and $\mathrm{cl}\,V\cap\mathrm{cl}G=\emptyset$. Thus, since U, V and G are regulary open, we get $x\in V\subset \mathrm{cl}\,V\subset\mathrm{cl}(X\setminus\mathrm{cl}G)=X\setminus G\subset U$.

1.5. *If $U\in\mathscr{P}$ and $x\notin \mathrm{cl}\,U$, then there exists a V in $\mathscr{P}$ such that $\mathrm{cl}\,U\subset V$ and $x\notin\mathrm{cl}\,V$.*

Proof. Since $x\notin\mathrm{cl}\,U$, by 1.2, there exists a G in $\mathscr{P}$ such that $x\in G$ and $\mathrm{cl}\,U\cap\mathrm{cl}G=\emptyset$. By 1.3, there exists a V in $\mathscr{P}$ such that $x\notin\mathrm{cl}\,V$ and $V\cup G=X$. Thus, $x\notin\mathrm{cl}\,V$ and $\mathrm{cl}\,U\subset X\setminus\mathrm{cl}G\subset X\setminus G\subset V$.

1.6. *If U and V are in $\mathscr{P}$, $U\subset V$ and $U\neq V$, then $\mathrm{cl}\,U\subset V$.*

Proof. Suppose, on the contrary, that (1) $\mathrm{cl}\,U\not\subset V$. Since all members in $\mathscr{P}$ are regularly open, $V\setminus\mathrm{cl}\,U\neq\emptyset$. Let (2) $x\in V\setminus\mathrm{cl}\,U$. By 1.4, there exists a G in $\mathscr{P}$ such that (3) $x\in G$ and (4) $\mathrm{cl}G\subset V$. Let $y\notin\mathrm{cl}\,V$. By 1.5, there exists a W in $\mathscr{P}$ such that (5) $y\notin\mathrm{cl}W$ and (6) $\mathrm{cl}\,V\subset W$. By (B), $X\setminus\mathrm{cl}\,V\in\mathscr{P}$. From the fact that $U\subset V$, (4) and (6) it follows that $\mathrm{cl}G\cap\mathrm{cl}(X\setminus\mathrm{cl}W)=\mathrm{cl}\,U\cap\mathrm{cl}(X\setminus\mathrm{cl}\,W)=\emptyset$. By (I), $G\subset U$ or $U\subset G$. But from (2) and (3) it follows that $G\not\subset U$, and from (1) and (4) it follows that $U\not\subset G$; a contradiction.

1.7. *If U, V are in $\mathscr{P}$* and $\mathrm{cl}\,U\subset V$, *then there exists a G in $\mathscr{P}$ such that $\mathrm{cl}\,U\subset G\subset\mathrm{cl}G\subset V$.*

Proof. Let $x\notin\mathrm{cl}\,U$. By 1.2, there exists a G in $\mathscr{P}$ such that $x\in G$ and $\mathrm{cl}\,U\cap\mathrm{cl}G=\emptyset$. Let $\mathscr{R}$ be the family of all such G. Then $V\cup\bigcup\mathscr{R}=X$. By (S), there exists a G' in $\mathscr{R}$ such that $V\cup G'=X$. In virtue of (B), $X\setminus\mathrm{cl}G'\in\mathscr{P}$. Thus $\mathrm{cl}\,U\subset X\setminus\mathrm{cl}G'\subset\mathrm{cl}(X\setminus\mathrm{cl}G')=X\setminus G'\subset V$.

§ 2. The families $D(U)$ and $C(U)$ and the spaces X_U for a given U

For U belonging to $\mathscr{P}$ let

$$D(U)=\{V\in\mathscr{P}\colon\ V\subset U \text{ or } X\setminus\mathrm{cl}\,U\subset V \text{ or } V\subset X\setminus\mathrm{cl}\,U \text{ or } U\subset V\}.$$

Let us begin from some simple properties of operation D.

2.1. *If $U\in\mathscr{P}$ then $D(U)=D(X\setminus\mathrm{cl}\,U)$.*

2.2. *If $V\in D(U)$ then $X\setminus\mathrm{cl}\,V\in D(U)$.*

2.3. *If U* and *V are in $\mathscr{P}$ and $U\cap V=\emptyset$ then $D(U)=D(V)$.*

Proof. (Only the proof of 2.3 is needed.) Let $G \in D(U)$. If $G \subset U$ then $G \subset (X \setminus \mathrm{cl}\, V)$. Hence $G \in D(V)$. If $U \subset G$, then there exists an H in $\mathscr{P}$ and $x \in U$ such that $x \in H$ and $\mathrm{cl}\, H \subset U$. Hence $\mathrm{cl}\, V \cap \mathrm{cl}\, H = \emptyset = \mathrm{cl}\, H \cap (X \setminus G) = \mathrm{cl}\, H \cap \mathrm{cl}(X \setminus \mathrm{cl}\, G)$. Since V, H and $X \setminus \mathrm{cl}\, G$ belong to $\mathscr{P}$; by (I), $V \subset X \setminus \mathrm{cl}\, G$ or $X \setminus \mathrm{cl}\, G \subset V$. Hence $X \setminus \mathrm{cl}\, G \in D(V)$, and, by 2.2, $G \in D(V)$. If $X \setminus \mathrm{cl}\, U \subset G$, then $X \setminus \mathrm{cl}\, G \subset U$. Hence, $V \subset G$. If $G \subset X \setminus \mathrm{cl}\, U$, then $U \subset X \setminus \mathrm{cl}\, G$. By 1.4, there exists an H in $\mathscr{P}$ such that $\mathrm{cl}\, H \subset U$. Hence $\mathrm{cl}\, V \cap \mathrm{cl}\, H = \emptyset = \mathrm{cl}\, G \cap \mathrm{cl}\, H$. Hence, by (I), $G \in D(V)$. Thus we have $D(U) \subset D(V)$ in all possible cases. By symmetry $D(V) \subset D(U)$.

2.4. *If U, $V \in \mathscr{P}$ and $U \subset V$ then $D(U) = D(V)$.*

Proof. Since $U \subset V$, we have $U \cap (X \setminus \mathrm{cl}\, V) = \emptyset$. By 2.3, $D(U) = D(X \setminus \mathrm{cl}\, V)$ and, by 2.1, $D(X \setminus \mathrm{cl}\, V) = D(V)$; $D(U) = D(V)$.

2.5. *If $U \in \mathscr{P}$ and $U \in D(V)$ then $D(U) = D(V)$.*

2.6. *If U and V are in $\mathscr{P}$ then $D(U) = D(V)$ or $D(U) \cap D(V) = \emptyset$.*

Proof. Let $D(U) \cap D(V) \neq \emptyset$. Then there exists a G in $\mathscr{P}$ such that $G \in D(U) \cap D(V)$. By 2.5, $D(G) = D(U)$ and $D(G) = D(V)$. Hence $D(U) = D(V)$.

In virtue of 2.6 we have a partition of $\mathscr{P}$: U and V are in the same element of the partition iff $D(U) = D(V)$. Then, by the Axiom of Choice, there exists a set $\mathscr{S} \subset \mathscr{P}$ which has exactly one element in each element of the partition.

The cardinality of $\mathscr{S}$ is called the *incomparability number* of $\mathscr{P}$.

If $U \in \mathscr{S}$ then let $C(U) = \{V \in D(U): V \subset U \text{ or } U \subset V\}$. Let $X_U = C(U) \cup \{0\} \cup \{1\}$, where 0 and 1 are elements which do not belong to $C(U)$.

Let us generate in X_U a topology by the sets

$$[0, V) = \{0\} \cup \{G \in C(U): \mathrm{cl}\, G \subset V\},$$

$$(W, 1] = \{G \in C(U): \mathrm{cl}\, W \subset G\} \cup \{1\},$$

where W and V run over $C(U)$.

2.7. *If V and W are in $C(U)$, then $V \subset W$ or $W \subset V$.*

Proof. To prove this, we consider two cases. (The remaining two are trivial.)

1. Let $U \subset V$ and $U \subset W$. By 1.4, there exists an H in $\mathscr{P}$ such that $\mathrm{cl}\, H \subset U$. Hence $\mathrm{cl}\, H \cap \mathrm{cl}(X \setminus \mathrm{cl}\, V) = \mathrm{cl}\, H \cap \mathrm{cl}(X \setminus \mathrm{cl}\, W) = \emptyset$. Hence, by (I), $V \subset W$ or $W \subset V$.

2. Let $V \subset U$ and $W \subset U$. By 1.2, there exists an H in $\mathscr{P}$ such that $\mathrm{cl}\, H \cap \mathrm{cl}\, U = \emptyset$. Then $\mathrm{cl}\, V \cap \mathrm{cl}\, H = \mathrm{cl}\, W \cap \mathrm{cl}\, H = \emptyset$ and, by (I), we get $V \subset W$ or $W \subset V$.

2.8. *If* $U \in C(V)$, $G \in \mathscr{P}$ *and* $U \subset G$ *then* $G \in C(V)$.

Proof. We shall show that $G \subset V$ or $V \subset G$. Since $U \in C(V) \subset D(V)$, we have by 2.5, $D(U) = D(V)$.

Let $V \subset U$. Since $G \in D(V)$, we have (by the definition of $D(V)$) $G \subset V$ or $V \subset G$ or $X \setminus \mathrm{cl}\, V \subset G$ or $G \subset X \setminus \mathrm{cl}\, V$. In the first two cases we get $G \in C(V)$ immediately. The remaining cases are impossible.

Let $G \subset X \setminus \mathrm{cl}\, V$. Then $V \subset X \setminus \mathrm{cl} G$. Hence $X \setminus \mathrm{cl} G \in C(V)$. By 2.7, $U \subset X \setminus \mathrm{cl} G$ or $X \setminus \mathrm{cl} G \subset U$. Since $G \subset U$, we have $G \subset X \setminus \mathrm{cl} G$ or $G \cup (X \setminus \mathrm{cl} G) \subset U$; a contradiction.

Let $X \setminus \mathrm{cl}\, V \subset G$. Then $X \setminus \mathrm{cl} G \subset V$. Hence, $X \setminus \mathrm{cl} G \in C(V)$. By 2.7, $U \subset X \setminus \mathrm{cl} G$ or $X \setminus \mathrm{cl} G \subset U$; a contradiction as before.

2.10. *If* $V, W \in \mathscr{S}$, $V \neq W$, $U \in D(V)$ *and* $H \in D(W)$ *then* $\mathrm{cl}\, U \cap \mathrm{cl} H \neq \emptyset$.

Proof. The proof is obvious by 2.3.

2.11. *If* $U, H \in C(V)$ *for some* V *in* $\mathscr{S}$, *then* $\mathrm{cl}\, U \cap \mathrm{cl} H \neq \emptyset$ *and* $(X \setminus U) \cap (X \setminus H) \neq \emptyset$.

Proof. The proof is obvious by 2.7.

2.12. *The spaces* X_U *are linearly ordered, having the first and the last elements.*

Proof. We define an ordering in X_U as follows:

Let x and y be in X_U and let $x \neq y$. If $x = 0$, then $x < y$. If $y = 1$, then $x < y$. If $0 \neq x \neq 1$ and $0 \neq y \neq 1$, then x and y are in $C(U)$. Then let $x = V$ and $y = W$. We say that $x < y$ iff $\mathrm{cl}\, V \subset W$.

By 2.7 and 1.6, the relation $<$ is a linear ordering of X_U. 0 and 1 are the first and the last elements, respectively. Clearly, $[0, V) = \{x \in U_U : x < V\}$ and $(W, 1] = \{y \in X_U : W < y\}$, so the topology of X_U is the order topology induced by the ordering defined above.

§ 3. A lemma on chains in maximal BISC-subbases of continua

Since the properties which appear in the definition of BISC-subbases depend on finite subfamilies of the BISC-subbase $\mathscr{P}$, by the Kuratowski – Zorn Lemma each BISC-subbase is contained in a maximal one.

Let X be a continuum and let $\mathscr{P}$ be a maximal BISC-subbase for the topology on X. Let G be a fixed member of $\mathscr{P}$. Let $R(G)$ be the family of all U in $\mathscr{P}$ such that $U \cap G = \emptyset$. We shall prove in this section a lemma on chains in $R(G)$.

Before proving the lemma, let us note some preliminary facts.

3.1. *If* L *is a chain contained in* $\mathscr{P}$ *and such that* $\bigcup \{\mathrm{cl}\, U : U \in L\} \cap \mathrm{cl}\, V = \emptyset$ *and* $\mathrm{cl} \bigcup L \cap \mathrm{cl}\, V \neq \emptyset$, *then* $\bigcup L = X \setminus \mathrm{cl}\, V$.

Proof. The inclusion $\bigcup L \subset X \setminus \mathrm{cl}\, V$ is obvious.

To prove that $X \setminus \mathrm{cl}\, V \subset \bigcup L$ let us assume that there exists an x in $X \setminus \mathrm{cl}\, V$ such that $x \notin \bigcup L$. By 1.4, there exists an H in $\mathscr{P}$ such that $x \in H \subset \mathrm{cl} H \subset X \setminus \mathrm{cl}\, V$. Hence $\mathrm{cl} H \cap \mathrm{cl}\, V = \mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$ for each U in L. Hence, by (I) and 1.6, $\mathrm{cl}\, U \subset H$. From this it follows that $\mathrm{cl} \bigcup L \subset \mathrm{cl} H$; a contradiction.

3.2. *If L is a chain contained in $R(G)$, then $\mathscr{P} \cup \{\bigcup L\}$ has property* 1.1.

Proof. Let $\mathrm{cl} \bigcup L \cap \mathrm{cl}\, V_1 \cap \ldots \cap \mathrm{cl}\, V_k = \emptyset$ where $V_i \in \mathscr{P}$ for $i = 1, \ldots, k$. Since $\mathrm{cl}\, U \subset \mathrm{cl} \bigcup L$ for each $U \in L$ and L is a chain, by 1.1 for $\mathscr{P}$ there exists a V_i in $\{V_1, \ldots, V_k\}$ such that $\mathrm{cl}\, U \cap \mathrm{cl}\, V_i = \emptyset$ for each U in L. Clearly, we may assume that $V_l \not\subset V_j$ for $l \neq j$. Hence $\bigcup \{\mathrm{cl}\, U \colon U \in L\} \cap \mathrm{cl}\, V_i = \emptyset$. If $\mathrm{cl} \bigcup L \cap \mathrm{cl}\, V_i \neq \emptyset$, then, by 3.1 $\bigcup L = X \setminus \mathrm{cl}\, V$. Hence $\mathrm{cl}(X \setminus \mathrm{cl}\, V_i) \cap \mathrm{cl}\, V_j = \emptyset$ for some $j \neq i$. Clearly, we may assume that $\mathrm{cl}\, V_1 \cap \ldots \cap \mathrm{cl}\, V_k \neq \emptyset$. Thus $V_j \subset V_i$ for $i \neq j$; a contradiction.

3.3. *If $x \notin \mathrm{cl} \bigcup L$, then there exists a V in $\mathscr{P}$ such that $x \in V$ and*

$$\mathrm{cl} \bigcup L \cap \mathrm{cl}\, V = \emptyset.$$

Proof. By (C), there exist $V_1, \ldots, V_k$ in $\mathscr{P}$ such that $x \in V_1 \cap \ldots \cap V_k$ and $\mathrm{cl} \bigcup L \cap \mathrm{cl}\, V_1 \cap \ldots \cap \mathrm{cl}\, V_k = \emptyset$. By 3.2, there exists a V_i such that $x \in V_i$ and $\mathrm{cl} \bigcup L \cap \mathrm{cl}\, V = \emptyset$.

3.4. *If $x \notin \mathrm{cl} \bigcup L$, then there exists a V in $\mathscr{P}$ such that $\mathrm{cl} \bigcup L \subset V$ and $x \notin \mathrm{cl}\, V$.*

Proof. The proof is obvious by 3.3 and (B) for $\mathscr{P}$.

3.5. *If $V \in \mathscr{P}$ and $\mathrm{cl} \bigcup L \subset V$, then there exists an H in $\mathscr{P}$ such that $\mathrm{cl} \bigcup H \subset H \subset \mathrm{cl} H \subset V$.*

Proof. By 3.3, for each $x \notin \mathrm{cl} \bigcup L$ there exists a W in $\mathscr{P}$ such that $x \in W$ and $\mathrm{cl} \bigcup L \cap \mathrm{cl} W = \emptyset$. Let $\mathscr{R}$ be the family of such sets W in $\mathscr{P}$. Hence $\bigcup \mathscr{R} \cup V = X$. By (S), there exists a W' in $\mathscr{R}$ such that $W' \cup V = X$. By (B) and the definition of family $\mathscr{R}$, we get $X \setminus \mathrm{cl} W' \in \mathscr{P}$ and $\mathrm{cl} W' \cap \mathrm{cl} \bigcup L = \emptyset$. Hence $\mathrm{cl} \bigcup L \subset X \setminus \mathrm{cl} W' \subset \mathrm{cl}(X \setminus \mathrm{cl} W') = X \setminus W' \subset V$.

3.6. *If L is a chain contained in $R(G)$, then $\mathscr{P} \cup \{\bigcup L\}$ has the property* (S).

Proof. Let $X = \bigcup L \cup G_1 \cup \ \ \cup G_k$, and $G_1 \cup \ldots \cup G_k \neq X$. Since $G_i \in \mathscr{P}$, $i = 1, \ldots, k$, and $L \subset \mathscr{P}$, by (S) there exists a G_i and there exists a $U \in L$ such that $U \cup G_i = X$. But $U \subset \bigcup L$, and hence $\bigcup L \cup G_i = X$.

3.7. *If L is a chain contained in $R(G)$, then $\mathscr{P} \cup \{\bigcup L\}$ has property* (I).

Proof. Let H, $H' \in \mathscr{P}$ and $\mathrm{cl} H \cap \mathrm{cl} \bigcup L = \mathrm{cl} H' \cap \mathrm{cl} H = \emptyset$. For each U in L there is $\mathrm{cl}\, U \subset \mathrm{cl} \bigcup L$. Hence $\mathrm{cl} H \cap \mathrm{cl}\, U = \emptyset = \mathrm{cl} H \cap \mathrm{cl} H'$, for each U in L. From (I) it follows that $U \subset H'$ or $H' \subset U$ for $U \in L$.

If for each $U \in L$ there is $U \subset H'$, then $\bigcup L \subset H'$. If there exists an U in L such that $H' \subset U$, then $H' \subset \bigcup L$.

Let $H, H' \in \mathscr{P}$ and $\mathrm{cl}H \cap \mathrm{cl}\bigcup L = \mathrm{cl}H' \cap \mathrm{cl}\bigcup L = \emptyset$. If $U \in L$, then $\mathrm{cl}\,U \subset \mathrm{cl}\bigcup L$. Hence $\mathrm{cl}H \cap \mathrm{cl}\,U = \mathrm{cl}H' \cap \mathrm{cl}\,U = \emptyset$. From (I) it follows that $H \subset H'$ or $H' \subset H$.

3.8. *If L is a chain contained in $R(G)$, then $\bigcup L$ is regularly open.*

Proof. The proof consists in showing that $\mathrm{Int}\,\mathrm{cl}\bigcup L = \bigcup L$; $\bigcup L$ being open.

Let $x \in \mathrm{Int}\,\mathrm{cl}\bigcup L$. Since $\mathscr{P}$ is a subbase in a compact space, hence there exist $G_1, \ldots, G_k$ such that

$$x \in G_1 \cap \ldots \cap G_k \quad \text{and} \quad \mathrm{cl}G_1 \cap \ldots \cap \mathrm{cl}G_k \subset \mathrm{Int}\,\mathrm{cl}\bigcup L.$$

We shall assume that one of G_i, $i = 1, \ldots, k$, is such that $\mathrm{cl}\bigcup L \subset G_i$. By 1.4, there exists an A in $\mathscr{P}$ such that $\mathrm{cl}A \subset G$. Hence we may assume that $X \setminus \mathrm{cl}A$ is one of G_i and we have $\mathrm{cl}\bigcup L \subset X \setminus \mathrm{cl}A$. Since X is connected, we have $\mathrm{cl}\bigcup L \not\subset G_i$ for some $i = 1, \ldots, k$.

Let $G_1, \ldots, G_j$, $1 \leqslant j \leqslant k$, be all those G_i such that $\mathrm{cl}\bigcup L \subset G_i$. By 2.7, those G_i form a chain, say $G_1 \subset \ldots \subset G_j$. By 3.5, there exists an H in $\mathscr{P}$ such that $\mathrm{cl}\bigcup L \subset H \subset \mathrm{cl}H \subset G_1$. From $X \setminus \mathrm{cl}H \in \mathscr{P}$ and $\mathrm{cl}\bigcup L \cap \mathrm{cl}(X \setminus \mathrm{cl}H) = \emptyset$ we get $\mathrm{cl}G_1 \cap \ldots \cap \mathrm{cl}G_j \cap \ldots \cap \mathrm{cl}G_k \cap (X \setminus H) = \emptyset$. By 1.1, there exists a G_l, $l = 1, \ldots, k$, such that $\mathrm{cl}G_l \cap (X \setminus H) = \emptyset$. Since $\mathrm{cl}G_1 \cap (X \setminus H) \neq \emptyset$, we have $\mathrm{cl}\bigcup L \not\subset G_l$.

If $U \in L$, then $\mathrm{cl}\,U \subset \mathrm{cl}\bigcup L$ and $\mathrm{cl}\,U \cap (X \setminus H) = \mathrm{cl}G_l \cap (X \setminus H) = \emptyset$. By (I), for each U in L, there is $U \subset G_l$ or $G_l \subset U$.

If $G_l \subset U$, for some U from L, then $x \in G_l \subset U \subset \bigcup L$. Hence we get the following implication: if $x \in \mathrm{Int}\,\mathrm{cl}\bigcup L$, then $x \in \bigcup L$. Now, suppose that for each $U \in L$ there is a $U \subset G_l$. Hence $\bigcup L \subset G_l$. By 3.4, for $y \notin \mathrm{cl}\bigcup L$, there exists a W in $\mathscr{P}$ such that $y \notin \mathrm{cl}W$ and $\mathrm{cl}\bigcup L \subset W$. Let $\mathscr{R}$ be the family of all such sets W. Hence $\bigcap \mathscr{R} = \mathrm{cl}\bigcup L$. For each set in $\mathscr{R} \cup \{G_l\}$ we have $A \in C(U)$ for some $U \in L$ and $A \in \mathscr{R} \cup \{G_l\}$. By 2.7, we have $H \subset W$ or $W \subset H$ for each H, W in $\mathscr{R} \cup \{G_l\}$.

If $W \subset G_l$ for some W in $\mathscr{R}$, then $\mathrm{cl}\bigcup L \subset G_l$. But this is impossible, because $\mathrm{cl}\bigcup L \not\subset G_l$.

Hence, for each W in $\mathscr{R}$, $G_l \subset W$. Hence, $G_l \subset \bigcap \mathscr{R} = \mathrm{cl}\bigcup L$. But G_l is regularly open; hence $\mathrm{Int}\,\mathrm{cl}\bigcup L \subset \mathrm{Int}\,\mathrm{cl}G_l = G_l$ and $G_l = \mathrm{Int}\,\mathrm{cl}G_l \subset \mathrm{Int}\,\mathrm{cl}\bigcup L$, i.e., $G_l = \mathrm{Int}\,\mathrm{cl}\bigcup L$. By 1.4, there exists a V in $\mathscr{P}$ such that $x \in V \subset \mathrm{cl}\,V \subset G_l$. By 1.4, there exists a T in $\mathscr{P}$ such that $\mathrm{cl}T \subset G$. Since $\mathrm{cl}\bigcup L \cap G = \emptyset$ and $\mathrm{cl}\,V \subset \mathrm{cl}\bigcup L$, for each U in L we have $\mathrm{cl}\,U \cap \mathrm{cl}T = \mathrm{cl}\,V \cap \mathrm{cl}T = \emptyset$. By (I) we have $U \subset V$ or $V \subset U$, for each U in L.

If for each U in L we have $U \subset V$, then $\bigcup L \subset V$. Hence $\mathrm{Int}\,\mathrm{cl}\bigcup L \subset \mathrm{Int}\,\mathrm{cl}\,V = V$ and consequently $G_l \subset V$. But $\mathrm{cl}\,V \subset G_l$ and hence $G_l = \mathrm{cl}G_l$; a contradiction.

Hence it is proved that there exists a U in L such that $V \subset U$. But $x \in V$ and $V \subset U \subset \bigcup L$, and hence $x \in \bigcup L$. We get the required inclusion $\mathrm{Int}\,\mathrm{cl}\bigcup L \subset \bigcup L$, x being an arbitrary point of $\mathrm{Int}\,\mathrm{cl}\bigcup L$.

LEMMA. *If L is a chain contained in $R(G)$, then* $\bigcup L \in R(G)$.

Proof. By 3.4, for each $x \notin \mathrm{cl} \bigcup L$ there exists a V_x such that $x \notin V_x$ and $\mathrm{cl}\, V_x \cap \mathrm{cl} \bigcup L = \emptyset$. Let L' be the family of all such V_x. Since L is a chain, by (I) L' is a chain and $X \setminus \mathrm{cl} \bigcup L = \bigcup L' \in R(U)$ for each $U \in L$. Hence $\mathscr{P} \cup \{\bigcup L, \bigcup L'\}$ has property (B).

From the facts proved above we have $\bigcup L$ and $\bigcup L' \in \mathscr{P}$. Since for each $U \in L$, $U \cap G = \emptyset$, we have $\bigcup L \cap G = \emptyset$. Hence $\bigcup L \in R(G)$.

§ 4. Further properties of the spaces X_U

As before, let X be continuum, $\mathscr{P}$ a maximal BISC-subbase and $\mathscr{S}$ a subfamily of $\mathscr{P}$ defined in § 2. We consider for a given U in $\mathscr{S}$ the subfamilies $C(U)$, $X_U = \{0\} \cup C(U) \cup \{1\}$, with the order topology introduced there.

We shall show that X_U are Hausdorff and compact and that each element from $C(U)$ separates X_U. This will be done by proving some lemmas.

4.1. *If* $V \in C(U)$, *then* $[0, V) \cap (V, 1] = \emptyset$.

Proof. Suppose that there exists a $G \in C(U)$ such that $G \in [0, V) \cap (V, 1]$. By the definition of sets $[0, V)$ and $(V, 1]$ we get $\mathrm{cl}\, V \subset G$ and $\mathrm{cl} G \subset V$. Hence $\mathrm{cl}\, V = V$ is closed-open. But $V \in \mathscr{P}$, and hence $\mathrm{cl}\, V$ is different from X; a contradiction.

4.2. *If* $V \in C(U)$, *then* $X_U \setminus \{V\} = [0, V) \cup (V, 1]$.

Proof. Since $V \notin [0, V) \cup (V, 1]$, we have $[0, V) \cup (V, 1] \subset X_U \setminus \{V\}$. Let $G \in C(U)$ and $G \neq V$. In virtue of 2.7, $\mathrm{cl} G \subset V$ or $\mathrm{cl}\, V \subset G$. If $\mathrm{cl} G \subset V$, then $G \in [0, V)$. If $\mathrm{cl}\, V \subset G$, then $G \in (V, 1]$. Thus $X_U \setminus \{V\} \subset [0, V) \cup (V, 1]$.

4.3. *If* $V \in C(U)$, *then* $\{V\}$ *disconnects* X_U.

4.4. *The spaces* X_U, $U \in \mathscr{S}$, *are Hausdorff*.

Proof. Let x and y be in X_U and let $x \neq y$.

If $x = 0$ and $y = 1$, then $0 \in [0, U)$, $1 \in (U, 1]$ and by 4.2, $[0, U) \cap (U, 1] = \emptyset$.

If $x = 0$ and $y = V \in C(U)$, then there exists a G in $\mathscr{P}$ such that $\mathrm{cl} G \subset V$. Hence $V \in (G, 1]$, $0 \in [0, G)$ and, by 4.1, $[0, G \cap (G, 1] = \emptyset$.

If $x = V \in C(U)$ and $y = 1$, then there exists a G in $\mathscr{P}$ such that $\mathrm{cl}\, V \subset G$. Hence $V \in [0, G)$, $1 \in (G, 1]$ and by 4.1, we get $[0, G) \cap (G, 1] = \emptyset$.

If $x = V$, $y = W$ and V, $W \in C(U)$, then, by 2.7, $\mathrm{cl}\, V \subset W$ or $\mathrm{cl} W \subset V$. Suppose that $\mathrm{cl}\, V \subset W$. By 1.7, there exists a G in $\mathscr{P}$ such that $\mathrm{cl}\, V \subset G \subset \mathrm{cl} G \subset W$. In virtue of 2.8, we have $G \in C(U)$ and $V \in [0, G)$ and $W \in (G, 1]$. By 4.1, we get $[0, G) \cap (G, 1] = \emptyset$.

4.5 *The spaces X_U, $U \in \mathcal{S}$, are compact.*

Proof. Let $U \in \mathcal{S}$ and let $\mathcal{T}$ be an open covering of X_U. By the Alexander Lemma we can assume that the elements are in a subbase consisting of $[0, W)$ and $(V, 1]$ for W and V running over $C(U)$. Decompose $\mathcal{T}$ into two disjoints parts:

$$T_1 = \{W \in T\colon\ W = [0, G) \quad \text{where } G \in C(U)\},$$
$$T_2 = \{WT\colon\ W = (G, 1] \quad \text{where } G \in C(U)\},$$

Let $L_1 = \{G \in C(U)\colon\ [0, G) \in T_1\}$. By 2.7, L_1 is a chain in $\mathcal{P}$. Consider two cases.

1. There exists an H in $\mathcal{P}$ such that for each G in L_1 we have $H \cap G = \emptyset$.

In virtue of the Lemma, we have $\bigcup L_1 \in \mathcal{P}$. By 2.8, $\bigcup L_1 \in C(U)$. Since $\bigcup L_1 \notin \bigcup \{[0, G)\colon G \in L_1\}$ and $\mathcal{T}$ is a covering of X_U, there exists a W' in T_2 such that $\bigcup L_1 \subset W'$. Let $W' = (G', 1]$. From the definition of $(G', 1]$, it follows that $\mathrm{cl}\, G' \subset \bigcup L_1$. But X is compact and L_1 is a chain; hence there exists an V in L_1 such that $\mathrm{cl}\, G' \subset V$.

We shall show that $[0, V) \cup (G', 1] = X_U$. Suppose that there exists a D in $C(U)$ such that $D \notin [0, V)$ and $D \notin (G', 1]$. Since $D \notin [0, V)$, we have $\mathrm{cl}\, D \notin V$. But $D \in C(U)$, and hence, by 2.7, $D \subset V$ or $V \subset D$. Since $D \notin (G', 1]$, we have $\mathrm{cl}\, G' \notin D$. But $D \in C(U)$, and hence, by 2.7, $D \subset G'$ or $G' \subset D$. Since $D \neq G'$ and $D \neq V$, we have $\mathrm{cl}\, D \subset G'$ and $\mathrm{cl}\, V \subset D$ and consequently $D = \mathrm{cl}\, D$ is closed-open. But $\mathrm{cl}\, D$ is different from X as a member of $\mathcal{P}$; a contradiction in virtue of the connectedness of X.

2. For each H in $\mathcal{P}$ there exists a G in L_1 such that $H \cap G \neq \emptyset$. Since $1 \notin \bigcup \{W\colon W \in T_1\}$ and $\mathcal{T}$ is a covering of X_U, we infer that there exists a $W \in T_2 \subset \mathcal{T}$ such that $1 \in W$.

Let $W = (F, 1]$. We shall show that $\mathrm{cl}\, F \subset \bigcup L_1$. Suppose that $\mathrm{cl}\, F \not\subset \bigcup L_1$. Then by 2.7, for each G in L_1 we have $G \subset F$. Hence $\bigcup L_1 \subset F$. Since $X \setminus \mathrm{cl}\, F \in \mathcal{P}$, for G in L_1 we get $(X \setminus \mathrm{cl}\, F) \cap G = \emptyset$; a contradiction.

Thus $\mathrm{cl}\, F \subset \bigcup L_1$ and we infer that there exists a G' in L_1 such that $\mathrm{cl}\, F \subset G'$ (X being a compact space and L_1 being a chain). Hence $[0, G') \cup \cup (F, 1] = X_U$.

Note. In fact, it has been shown that X_U are supercompact.

4.6. *If $U \in \mathcal{P}$, then there exists a $V \in \mathcal{S}$ such that $U \in C(V)$ or $X \setminus \mathrm{cl}\, U \in C(V)$.*

Proof. Let $U \in \mathcal{P}$. Then there exists a V in $\mathcal{S}$ such that $U \in D(V)$. Since $U \in D(V)$, we have $U \subset V$ or $V \subset U$ or $X \setminus \mathrm{cl}\, V \subset U$ or $U \subset X \setminus \mathrm{cl}\, V$. In the first two cases $U \in C(V)$ obviously. In the last two cases we get $X \setminus \mathrm{cl}\, U \in C(V)$, since the members of $\mathcal{P}$ are regularly open.

4.7. *The spaces X_U, $U \in \mathcal{S}$, are linearly ordered Hausdorff compacta with the first and the last elements.*

Proof. This is obvious by 2.2, 2.12, 4.4 and 4.5.

4.8. *If U and V belong to $C(G)$, $G \in \mathscr{S}$ and $x \in V \setminus \mathrm{cl}\, U$, then there exists a $W \in C(G)$ such that $x \in \mathrm{Fr}\, W$.*

Proof. Since U, $V \in C(G)$, we have, by 2.7, $U \subset V$ or $V \subset U$. Since $V \setminus \mathrm{cl}\, U \neq \emptyset$, we have, by 1.6, $\mathrm{cl}\, U \subset V$.

Let $\mathscr{R}$ be a maximal family of elemets of $\mathscr{P}$ such that for each W in $\mathscr{R}$, $x \notin W$ and $U \subset W$. Since $U \in C(G)$ and $U \subset W$ for $W \in \mathscr{R} \subset \mathscr{P}$, we have, by 2.8, $W \in C(G)$ for $W \in \mathscr{R}$. Then, by 2.7, $W \subset V$ or $V \subset W$ for each $W \in \mathscr{R}$. But $x \in V$ and $x \notin W$, and hence, by 1.6, $\mathrm{cl}\, W \subset V$ for $W \in \mathscr{R}$. By 2.7, $\mathscr{R}$ is a chain and $W \cap (X \setminus \mathrm{cl}\, V) = \emptyset$ for $W \in \mathscr{R}$. Then, by the Lemma and 2.8 we have $W = \bigcup \mathscr{R} \in C(G)$, and $x \notin W$. By the maximality of $\mathscr{R}$ and by (B), we get $x \in \mathrm{cl}\, W$.

§ 5. The Main Lemma

Let $X, \mathscr{P}, \mathscr{S} \subset \mathscr{P}$, and X_U for $U \in \mathscr{S}$ be as before.

MAIN LEMMA. *Let Y be the product of X_V, and let $\pi_V\colon Y \to X_V$ be the projection, where $V \in \mathscr{S}$. Then Y is homeomorphic to X.*

Proof. Let $y \in Y$. Decompose $\mathscr{S}$ into three parts, setting

$$\begin{aligned} \mathscr{S}_0(y) &= \{V \in \mathscr{S}\colon \pi_V(y) = 0\}, \\ \mathscr{S}_1(y) &= \{V \in \mathscr{S}\colon \pi_V(y) = 1\}, \\ \mathscr{S}_2(y) &= \{V \in \mathscr{S}\colon \pi_V(y) \in C(V)\}, \end{aligned}$$

Clearly, these parts are disjoint.

If $V \in \mathscr{S}_0(y)$, then let $T_{0V}(y) = \bigcap \{\mathrm{cl}\, U\colon \mathrm{cl}\, U \subset V\}$.

If $V \in \mathscr{S}_1(y)$, then let $T_{1V}(y) = \bigcap \{X \setminus U\colon \mathrm{cl}\, V \subset U\}$.

If $V \in \mathscr{S}_2(y)$, then let $T_{2V}(y) = \mathrm{cl}\, U \cap (X \setminus U) = \mathrm{Fr}\, U$, where $\pi_V(y) = U$.

Let

$$H_y = \bigcap \{T_{0V}(y)\colon V \in \mathscr{S}_0(y)\} \cap \bigcap \{T_{1V}(y)\colon V \in \mathscr{S}_1(y)\} \cap \cap \bigcap \{T_{2V}(y)\colon V \in \mathscr{S}_2(y)\}.$$

By 2.10, 2.11 and 1.1, H_y is non-empty. We shall prove that H_y is a one-point set.

Let us assume, on the contrary, that there exist two different points w and z in H_y. By (C), there exists a G in $\mathscr{P}$ such that $w \in G$ and $z \notin \mathrm{cl}\, G$. By 4.6, there exists a V in $\mathscr{S}$ such that $G \in C(V)$ or $X \setminus \mathrm{cl}\, G \in C(V)$. By symmetry, we can assume that $G \in C(V)$. Then we consider $\pi_V(y)$. There are three possible cases.

Let $\pi_V(y) = 0$. Since $G \in C(V)$, we have $G \subset V$ or $V \subset G$. By 1.4, there exists a W in $\mathscr{P}$ such that $\mathrm{cl}\, W \subset G$ and $\mathrm{cl}\, W \subset V$. Then, by the definition of $C(V)$, $W \in C(V)$. But $\mathrm{cl} W \subset G$ and $z \notin \mathrm{cl} G$, and hence $z \notin \mathrm{cl} W$. But $T_{0V}(y) \subset \mathrm{cl} W$, and hence $z \notin T_{0V}(y)$. But $H_y \subset T_{0V}(y)$, and hence $z \notin H_y$; a contradiction.

Let $\pi_V(y) = 1$. In this case we get a contradiction as before if we pass to complements.

Let $\pi_V(y) = U \in C(V)$. Since $w, z \in H_y \subset \mathrm{Fr}\, U$, we have $G \not\subset U$ and $U \not\subset G$. But $U, G \in C(V)$, and hence, by 2.7, $U \subset G$ or $G \subset U$; a contradiction.

Define $h\colon Y \to X$, letting $h(y)$ be the single point in H_y for $y \in Y$. We shall prove that h is a homeomorphism.

To prove that h is one-to-one, let us assume that there exist two different points w and z in Y such that $h(w) = h(z)$, i.e., that $H_w = H_z$. Since $w, z \in Y$ and $w \neq z$, there exists a V in $\mathscr{S}$ such that $\pi_V(w) \neq \pi_V(z)$.

Let $\pi_V(w) = 0$ and $\pi_V(z) = 1$. By 1.4 and 1.5, there exist G, U in $\mathscr{P}$ such that $\mathrm{cl} G \subset V$ and $\mathrm{cl} V \subset U$. Hence $\mathrm{cl} G \cap (X \setminus U) = \emptyset$. But $H_w \subset \mathrm{cl} G$ and $H_z \subset X \setminus U$. Hence $H_w \cap H_z = \emptyset$ and therefore $h(w) \neq h(z)$.

Let $\pi_V(w) = 0$ and $\pi_V(z) = U \in C(V)$. Since $U \in C(V)$, we have $U \subset V$ or $V \subset U$. Then, by 1.4, there exists a G in $\mathscr{P}$ such that $\mathrm{cl} G \subset V$ and $\mathrm{cl} G \subset U$. Hence, $G \in C(V)$ and $\mathrm{Fr}\, U \cap \mathrm{cl} G = \emptyset$. But $H_w \subset \mathrm{cl} G$ and $H_z \subset \mathrm{Fr}\, U$. Hence $H_w \cap H_z = \emptyset$, i.e., $h(w) \neq h(z)$.

The case where $\pi_V(w) = U \in C(V)$ and $\pi_V(z) = 1$ is dual to the preceding one.

Let $\pi_V(w) = U$ and $\pi_V(z) = G$ and $U, G \in C(V)$. Then, by 1.6 and 2.7, $\mathrm{cl}\, U \subset G$ or $\mathrm{cl} G \subset U$. Hence $\mathrm{Fr}\, U \cap \mathrm{Fr}\, G = \emptyset$. But $H_w \subset \mathrm{Fr}\, U$ and $H_z \subset \mathrm{Fr}\, G$. Hence $H_w \cap H_z = \emptyset$, i.e., $h(w) \neq h(z)$.

To prove that h is onto, let $x \in X$. We define a y in Y, defining its coordinates y_V for each $V \in \mathscr{S}$ as follows. Let $V \in \mathscr{S}$. If for each U in $C(V)$ we have $x \notin U$, then we set $y_V = 1$.

If for each U in $C(V)$ we have $x \in U$, then we set $y_V = 0$.

If there exist U and G in $C(V)$ such that $x \in G \setminus \mathrm{cl}\, U$, then by 4.8, there exists a W in $C(V)$ such that $x \in \mathrm{Fr} W$. Then we set $y_V = W$.

It is obvious that $x \in H_y$ and therefore $h(y) = x$.

To prove the continuity of h let $U \in \mathscr{P}$. There are two cases.

There exists a V in $\mathscr{S}$ such that $U \in C(V)$. We shall prove that $h^{-1}(U) = \pi_V^{-1}([0, U))$. To check this let $y \in h^{-1}(U)$. Hence $h(y) \notin X \setminus U$, i.e., $H_y \cap (X \setminus U) = \emptyset$. Since $X \setminus \mathrm{cl}\, U \in \mathscr{P}$ and each element of the families whose intersection is H_y belongs to $\mathscr{P}$, by (B) there exists a W such that $\mathrm{cl} W \cap (X \setminus U) = \emptyset$, i.e., $\mathrm{cl} W \subset U$. By 2.9, $W \in C(V)$. If $\pi_V(y)$ is different from 0 and 1, then $\pi_V(y) = W$. Hence $\pi_V(y) \in [0, U)$. If $\pi_V(y) = 1$, then $T_{1V}(y) \subset X \setminus U$, i.e., $T_{1V}(y) \cap U = \emptyset$. Hence, $H_y \cap U = \emptyset$, i.e., $h(y) \notin U$; a contradiction. If $\pi_V(y) = 0$, then $y \in \pi_V^{-1}([0, U))$ obviously. Thus the inclusion $h^{-1}(U) \subset \pi_V^{-1}([0, U))$ is proved. Let $y \in \pi_V^{-1}([0, U))$. Then $\pi_V(y) \in [0, U)$. If $\pi_V(y) \neq 0$, then $\pi_V(y) = G$ for some G in $C(V)$. Hence

$\mathrm{cl}G \subset U$. But $H_y \subset \mathrm{cl}G$. Hence $h(y) \in U$. If $\pi_V(y) = 0$, then, by 1.4, there exists a G in $\mathscr{P}$ such that $\mathrm{cl}G \subset U$. But $H_y \subset T_{0V}(y) \subset \mathrm{cl}G$. Hence $h(y) \in U$.

There exists a V in $\mathscr{S}$ such that $X \setminus \mathrm{cl}\, U \in C(V)$. Then we shall prove that $h^{-1}(U) = \pi_V^{-1}\big((X\setminus \mathrm{cl}\, U, 1]\big)$, as in the preceding case, if we pass to the complements.

Thus the continuity of h is proved.

But Y, by 4.4 and 4.5, is compact and Hausdorff, X is Hausdorff, h is continuous, onto and one-to-one, and hence h is a homeomorphism.

§ 6. Application to the characterization of several cubes

An immediate corollary of the Main Lemma is the following

THEOREM 1. *A Hausdorff continuum X is a product of linearly ordered continua iff there exists a* BISC-*subbase for its topology.*

Proof. By the Main Lemma, X is topologically a product of X_V for $V \in \mathscr{S}$. Since X is connected, X_V are connected. By 4.7, X_V are linearly ordered Hausdorff continua.

On the other hand, the linearly ordered continuum C has a subbase consisting of the "half-lines" $C_y^+ = \{x \in C\colon\ x < y\}$ and $C_y^- = \{x \in C\colon\ x > y\}$, where y runs over C. Clearly, this is a BISC-subbase. The product of such C's has also a BISC-subbase, namely that which consists of counter-images under the projections of the "half-lines" on the axes.

In the next characterizations the proof of only one implication is needed.

THEOREM 2. *A metrizable continuum X is topologically a Hilbert cube iff there exists a* BISC-*subbase for its topology and* $\mathrm{card}\mathscr{S} \geqslant \aleph_0$.

Proof. By Theorem 1, X is a topological product of linearly ordered continua X_V, $V \in \mathscr{S}$ and $\mathrm{card}\mathscr{S} \geqslant \aleph_0$.

Since X is metrizable, X_V are metrizable. By Theorem 1 of Kuratowski's book [4], p. 187, X_V are closed segments.

THEOREM 3. *A metrizable continuum X is a Euclidean n-cube iff there exists a* BISC-*subbase for its topology and* $\mathrm{card}\mathscr{S} = n$.

Dyadic spaces have a topological characterization given by Alexandroff and Ponomarev in [6].

THEOREM 4. *X is a Tychonoff cube iff it is a dyadic continuum and if there exists a* BISC-*subbase for its topology.*

Proof. By Theorem 1, X is a product of linearly ordered continua X_V, $V \in \mathscr{S}$. Hence X_V are dyadic. By a theorem of Mardešić and Papić, [5], X_V are topologically closed segments. Thus X is a Tychonoff cube.

Remark 1. Let $\mathscr{P}$ be a BISC-subbase in the continuum X and let $\mathscr{S} \subset \mathscr{P}$ be such as $\mathscr{S}$ for $\mathscr{P}$ in § 2. Then the set

$$B = \bigcup \{\bigcap \{\mathrm{cl}\, U: U \in C(V)\}: V \in \mathscr{S}\}$$

is a boundary of the cube X.

§ 7. Topological characterization of n-spheres

In this section we shall give a topological characterization of n-spheres in terms of subbases.

A family $\mathscr{R}$ of subsets of a topological space X is said to be *n-binary* provided for each subfamily $\mathscr{R}'$ of $\mathscr{R}$ which consists of at most n elements such that $\bigcap \{\mathrm{cl}\, U: U \in \mathscr{R}'\} = \emptyset$ there exist two elements in $\mathscr{R}'$, say U, V, such that $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$.

A family $\mathscr{R}$ of subsets of a topological space X is said to be *binary* if it is n-binary for $n = 1, 2, \ldots$

THEOREM 6. *Let X be a metrizable continuum and let $n \geqslant 2$. Let $\mathscr{P}$ be a subbase of the topology on X such that:*

(B) *if $U \in \mathscr{P}$, then $X \setminus \mathrm{cl}\, U \in \mathscr{P}$;*

(I) *if V, U are in $\mathscr{P}$ and there exists a G in $\mathscr{P}$ such that $\mathrm{cl}\, U \cap \mathrm{cl} G = \mathrm{cl}\, V \cap \cap \mathrm{cl} G = \emptyset$, then $U \subset V$ or $V \subset U$;*

(S'a) *$\mathscr{P}$ is $2n-1$-binary and is not $2n$-binary;*

(S'b) *if $\{V_1, \ldots, V_{2n}\} \subset \mathscr{P}$ are such that $\mathrm{cl}\, V_1 \cap \ldots \cap \mathrm{cl}\, V_{2n} = \emptyset$ and $\mathrm{cl}\, V_i \cap \mathrm{cl}\, V_j \neq \emptyset$ for each $i, j \in \{1, \ldots, 2n\}$, then $\mathrm{cl} G_1 \cap \ldots \cap \mathrm{cl} G_{2n} = \emptyset$ for each $\{G_1, \ldots, G_{2n}\} \subset \mathscr{P}$ such that $V_1 \subset G_1, \ldots, V_{2n} \subset G_{2n}$;*

(S'c) *if $V_1, \ldots, V_k$ are in $\mathscr{P}$ and $k > 2n$ and $\mathrm{cl}\, V_1 \cap \ldots \cap \mathrm{cl}\, V_k = \emptyset$ then there exist V_i, V_j such that $V_i \subset V_j$ or $V_j \subset V_i$, $i \neq j$;*

(C') *if $x \in V \in \mathscr{P}$, then there exists a U in $\mathscr{P}$ such that $x \in U \subset \mathrm{cl}\, U \subset V$.*

Then the space X is homeomorphic to the $n-1$-sphere.

Note. The conditions (C') and (S') for spheres correspond to conditions (C) and (S) for cubes. In virtue of property (B) it suffices only to consider subbases which consist of regularly open sets such that $\emptyset$ and X do not belong to $\mathscr{P}$.

Proof. The proof begins by showing some properties of subbases satisfying the above conditions. Analogously to the properties of BISC-subbases from the beginning of § 1, we have

7.1. *If x and y are in X and $x \neq y$, then there exists a U in $\mathscr{P}$ such that $x \in U$ and $y \notin \mathrm{cl}\, U$.*

7.2. *If U and V are in $\mathscr{P}$, $V \subset U$ and $V \neq U$, then $\mathrm{cl}\, V \subset U$.*

7.3. *If* $\operatorname{cl} U \subset V$, U, $V \in \mathscr{P}$ *then there exists a* G *in* $\mathscr{P}$ *such that* $\operatorname{cl} U \subset G \subset \operatorname{cl} G \subset V$.

Let $V \in \mathscr{P}$. Let $L(V)$ be a maximal family of sets G from $\mathscr{P}$ such that $V \subset G$ or $G \subset V$.

In the sequel the role of $L(V)$'s will be analogous to that of $D(V)$'s for cubes.

7.4. *The family* $L(V)$ *is a chain.*

Proof. Let $U, W \in L(V)$. Then $V \subset U$ or $U \subset V$, and $W \subset V$ or $V \subset W$. If $V \subset U$ and $V \subset W$, then by (C′), there exists an x in V and a G in $\mathscr{P}$ such that $x \in G \subset \operatorname{cl} G \subset V$. By (B) and the assumption that elements of $\mathscr{P}$ are regularly open, $\operatorname{cl}(X \setminus \operatorname{cl} U) = X \setminus U$ and $\operatorname{cl}(X \setminus \operatorname{cl} W) = X \setminus W$ and $X \setminus \operatorname{cl} U, X \setminus \operatorname{cl} W \in \mathscr{P}$. Hence $\operatorname{cl} G \cap (X \setminus U) = \operatorname{cl} G \cap (X \setminus W) = \emptyset$ and, by (I), $X \setminus \operatorname{cl} U \subset X \setminus \operatorname{cl} W$ or $X \setminus \operatorname{cl} W \subset X \setminus \operatorname{cl} U$, i.e., $\operatorname{cl} U \subset \operatorname{cl} W$ or $\operatorname{cl} W \subset \operatorname{cl} U$. Hence $U \subset W$ or $W \subset U$.

If $V \subset U$ and $W \subset V$, then clearly $W \subset U$, and analogously if $U \subset V$ and $V \subset W$, then $U \subset W$.

If $U \subset V$ and $W \subset V$, then, by (B) and (C′), there exists an x in $X \setminus \operatorname{cl} V$ and a G in $\mathscr{P}$ such that $x \in G \subset \operatorname{cl} G \subset X \setminus \operatorname{cl} V$. Hence $\operatorname{cl} G \cap \operatorname{cl} U = \operatorname{cl} G \cap \operatorname{cl} W = \emptyset$. Hence, by (I) $U \subset W$ or $W \subset U$.

7.5. *If* $U \in L(V)$, *then* $L(U) = L(V)$.

Proof. The proof is similar to that of 2.5.

From (S′a) it follows that there exist $U_1, \ldots, U_{2n}$ in $\mathscr{P}$ such that $\operatorname{cl} U_1 \cap \ldots \cap \operatorname{cl} U_{2n} = \emptyset$ and $\operatorname{cl} U_i \cap \operatorname{cl} U_j \neq \emptyset$ for each i, j.

Let us consider the chains $L_i = L(U_i)$.

7.6. *The chains* L_i *are pairwise disjoint and* $\bigcup \{L_i\colon i = 1, \ldots, 2n\} = \mathscr{P}$.

Proof. Let $i \neq j$. Suppose that there exists a V in $\mathscr{P}$ such that $V \in L_i \cap L_j$. Hence, by 7.5, $L(V) = L_i$ and $L(V) = L_j$. Hence $U_i \subset U_j$ or $U_j \subset U_j \subset U_i$ and, by (S′a), there exist two sets in the family $\{U_1, \ldots, U_{2n}\}$ which have an empty intersection of their closures; a contradiction.

Let $V \in \mathscr{P}$. Since $\operatorname{cl} V \cap \operatorname{cl} U_1 \cap \ldots \cap \operatorname{cl} U_{2n} = \emptyset$, by (S′c) there exists a U_i such that $V \subset U_i$ or $U_i \subset V$. Hence $V \in L_i$.

In virtue of (B), $X \setminus \operatorname{cl} U_i \in \mathscr{P}$. Since $X \setminus \operatorname{cl} U_i \notin L_i$, by 7.6 there exists a chain L_j with $j \neq i$ such that $X \setminus \operatorname{cl} U_i \in L_j$. The chain L_j will be called the *opposite chain* for L_i, and will be denoted by L_i'. In the family $\{U_1, \ldots, U_{2n}\}$ let the sets U_{n+i} be such that L_{n+i} is the opposite chain for L_i, i.e., $L_{n+i} = L_i'$.

For any j let $K_j = \bigcap \{\operatorname{cl} V\colon V \in L_j\}$ and $K_j' = \bigcap \{\operatorname{cl} V\colon V \in L_j'\}$.

7.7. *For each* $V \in \mathscr{P}$ *we have* $\operatorname{cl} V \cap K_j = \operatorname{cl}(V \cap K_j)$. *The same holds for* K_j'.

Proof. If $V \cap K_j = \emptyset$, then $V \in L_j'$. Hence $\operatorname{cl} V \cap K_j = \emptyset$, in virtue of 7.2.

Suppose that $V \cap K_j \neq \emptyset$ and let $x \in V \cap K_j$. Hence, by (C′), there exists a U in $\mathscr{P}$ such that $x \in U \subset \operatorname{cl} U \subset V$. Suppose that there exists a $y \in \operatorname{cl} V \cap K_j$ such that $y \notin \operatorname{cl}(V \cap K_j)$. Since X is a Hausdorff compact space, there exist sets $G_1, \ldots, G_l$ in $\mathscr{P}$ such that $y \in G_1 \cap \ldots \cap G_l$ and $x \notin \operatorname{cl} G_1 \cap \ldots \cap \operatorname{cl} G_l$. By (C′) we can assume that $\operatorname{cl} G_1 \cap \ldots \cap \operatorname{cl} G_l \cap V \cap K_j = \emptyset$, the sets G_i being incomparable by inclusion, in virtue of (S′c). Since $\operatorname{cl} U \subset V$, we have $\operatorname{cl} G_1 \cap \ldots \cap \operatorname{cl} G_l \cap \operatorname{cl} U \cap K_j = \emptyset$. Since X is compact and L_j is a chain, there exists an H in L_j such that $\operatorname{cl} G_1 \cap \ldots \cap \operatorname{cl} G_l \cap \operatorname{cl} U \cap \operatorname{cl} H = \emptyset$. None of the sets $G_1, \ldots, G_l, U$ belongs to L_j. Hence, by (S′a), there exists a G_q such that $\operatorname{cl} G_q \cap \operatorname{cl} U = \emptyset$. Since $y \in \operatorname{cl} V$ and $y \in G_q$, we have $V \cap G_q \neq \emptyset$. Let $z \in V \cap G_q$. By (C′), there exists a W in $\mathscr{P}$ such that $z \in W \subset \operatorname{cl} W \subset V$. We have $W \in L(V)$ and $U \in L(V)$. Hence, by 7.4, $W \subset U$ or $U \subset W$. Since $\operatorname{cl} U \cap \operatorname{cl} G_q = \emptyset$, we have $U \subset W$. Since $\operatorname{cl} W \subset V$, we have $\operatorname{cl} G_1 \cap \ldots \cap \operatorname{cl} G_l \cap \operatorname{cl} W \cap \operatorname{cl} H = \emptyset$. Hence, by (S′a), there exists a G_r, $G_r \neq G_q$ such that $\operatorname{cl} G_r \cap \operatorname{cl} W = \emptyset$. Since $U \subset W$, we have $\operatorname{cl} G_r \cap \operatorname{cl} U = \operatorname{cl} U \cap \operatorname{cl} G_q = \emptyset$ and, by (I), $G_q \subset G_r$ or $G_r \subset G_q$; a contradiction.

7.8. *The spaces* K_j *and* K_j' *are continua.*

Proof. Of course, K_j, as a closed subset of a compact space X, is compact.

Suppose, on the contrary, that K_j is a union of two disjoint non-empty and open (in K_j) sets A, B. Since K_j is compact and $\mathscr{P}$ is a subbase, we have $A = (A_1 \cup \ldots \cup A_k) \cap K_j$, $B = (B_1 \cup \ldots \cup B_l) \cap K_j$, where A_r, B_p are finite intersections of elements of $\mathscr{P}$. We can assume that those elements which appear in A_r (respectively B_p) are incomparable by inclusion and do not belong to $L_j \cup L_j'$. This implies that the number of such elements in each A_r (respectively B_p) is not greater that $2n-2$.

Let us consider $A_r = G_1 \cap \ldots \cap G_t$, where $G_i \in \mathscr{P}$. By (C′), we can assume that $\operatorname{cl} G_1 \cap \ldots \cap \operatorname{cl} G_t \cap K_j \subset A$ (this same for B_p). Since $A \cap B = \emptyset$, by (S′a) and the above assumption $A_r \cap B_p = \emptyset$.

The connectedness of X implies that there exists an x which does not belong to $A_1 \cup \ldots \cup A_k \cup B_1 \cup \ldots B_l$. For given A_r and B_p, A_r and B_p being intersections of some elements of $\mathscr{P}$, take those elements to which x does not belong. Let $H_1, \ldots, H_s$ be those elements. Of course $K_j \subset H_1 \cup \ldots \cup H_s$. Since K_j is the intersection of closures of elements of the chain L_j and X is compact, there exists a W in L_j such that $\operatorname{cl} W \cap (X \setminus H_1) \cap \ldots \cap (X \setminus H_s) = \emptyset$; the property (S′c) allows as to reduce the number of sets $\operatorname{cl} W, X \setminus H_1, \ldots, X \setminus H_s$ to $2n$. Hence we can assume that $s \leqslant 2n-1$. Since $H_i \notin L_j \cup L_j'$, $i = 1, \ldots, s$, we have $X \setminus \operatorname{cl} H_i \notin L_j \cup L_j'$. Hence $s \leqslant 2n-2$ and therefore, in virtue of (S′a) and the fact that $x \in (X \setminus H_i)$ for each i, we have $\operatorname{cl} W \cap (X \setminus H_i) = \emptyset$ for some i. Hence $H_i \in L_j$; a contradiction.

7.9. *The subbases* $\mathscr{P}|K_j$ *and* $\mathscr{P}|K_j'$ *are* BISC-*subbases in* K_j *and* K_j', *respectively, and the incomparability number of these subbases is* $n-1$.

Proof. If G is a regular open element of $\mathscr{P}$, then

$$\mathrm{Int}_{K_j}[\mathrm{cl}_{K_j}(G\cap K_j)] = \mathrm{Int}_{K_j}[\mathrm{cl}(G\cap K_j)] = \mathrm{Int}_{K_j}(\mathrm{cl}\,G\cap K_j)$$
$$= \mathrm{Int}\,\mathrm{cl}\,G\cap K_j = G\cap K_j$$

(the equalities follow from 7.7 and the fact that G is regularly open). This means that $G\cap K_j$ is regularly open in K_j.

(C) Follows immediately from (C′).

(I) Let G, H and V be in $\mathscr{P}$ and $\mathrm{cl}(V\cap K_j)\cap\mathrm{cl}(H\cap K_j) = \mathrm{cl}(V\cap K_j)\cap\cap\mathrm{cl}(G\cap K_j) = \emptyset$. By 7.7, $\mathrm{cl}\,V\cap\mathrm{cl}\,H\cap K_j = \mathrm{cl}\,V\cap\mathrm{cl}\,G\cap K_j = \emptyset$. Since X is compact and L_j is a chain, there exists a U in L_j such that $\mathrm{cl}\,V\cap\cap\mathrm{cl}\,H\cap\mathrm{cl}\,U = \mathrm{cl}\,V\cap\mathrm{cl}\,G\cap\mathrm{cl}\,U = \emptyset$. But $n \geqslant 2$ and therefore, in virtue of (S′a), $\mathrm{cl}\,V\cap\mathrm{cl}\,H = \mathrm{cl}\,V\cap\mathrm{cl}\,G = \emptyset$. Hence, $G\subset H$ or $H\subset G$.

(B) Let G be an element of $\mathscr{P}|K_j$. Then $G = G'\cap K_j$, where $G'\in\mathscr{P}$. Then $K_j\setminus\mathrm{cl}_{K_j}G = K_j\setminus\mathrm{cl}_{K_j}(G'\cap K_j) = K_j\setminus\mathrm{cl}(G'\cap K_j) = K_j\setminus\mathrm{cl}\,G'\cap K_j = (X\setminus\mathrm{cl}\,G')\cap K_j$ (the third equality follows from 7.7). Since $X\setminus\mathrm{cl}\,G'\in\mathscr{P}$, the proof of (B) is completed.

(S) Let $G_1, \ldots, G_t$ be in $\mathscr{P}$ and $K_j\subset G_1\cup \quad \cup G_t$. We can assume that G_i are incomparable by inclusion and do not belong to $L_j\cup L_j'$. The above assumption and 7.6 show that $t\leqslant 2n-2$. By the compactness of X it follows that there exists a U in L_j such that $\mathrm{cl}\,U\subset G_1\cup \quad \cup G_t$. Hence, by (S′a) there exist G_i, G_j such that $K_j\subset G_i\cup G_j$.

To prove that the incomparability number of $\mathscr{P}|K_j$ is $n-1$, we shall show that

(∗) *if H, V are in $\mathscr{P}$ and $H, V\notin L_j\cup L_j'$ and $H\cap K_j\subset V\cap K_j$, then $H\subset V$ or $V\subset H$.*

Consider two cases:

(1) $H\cap K_j \neq V\cap K_j$. Hence, by 1.6 for the BISC-subbase $\mathscr{P}|K_j$, $\mathrm{cl}\,H\cap(X\setminus V)\cap K_j = \emptyset$. Since X is compact and L_j is a chain, there exists a U in L_j such that $\mathrm{cl}\,H\cap(X\setminus V)\cap\mathrm{cl}\,U = \emptyset$. Since $n\geqslant 2$, we have by (S′a) and the assumption, $\mathrm{cl}\,H\cap(X\setminus V) = \emptyset$. Hence $\mathrm{cl}\,H\subset V$.

(2) $H\cap K_j = V\cap K_j$. Let $x\in H\cap K_j$. Since $\mathscr{P}|K_j$ is a BISC-subbase, by 1.4 there exists a G in $\mathscr{P}$ such that $x\in G\cap K_j\subset\mathrm{cl}(G\cap K_j)\subset H\cap K_j$. By 7.8, $\mathrm{cl}(G\cap K_j)\neq H\cap K_j$. We have case (1) for G and H and for G and V. Hence $\mathrm{cl}\,G\subset H$ and $\mathrm{cl}\,G\subset V$. Hence $\mathrm{cl}\,G\cap(X\setminus H) = \mathrm{cl}\,G\cap(X\setminus V) = \emptyset$. The property (I) for $\mathscr{P}$ implies that $H\subset V$ or $V\subset H$.

We shall show that if $U\in L_i$ or $U\in L_i'$ and $i\neq j$, then $K_j\not\subset U$ and $U\cap K_j\neq\emptyset$. In fact, let $U\in L_i$. Let us assume that $K_j\subset U$. Since X is compact and L_j is a chain, there exists a V in L_j such that $\mathrm{cl}\,V\subset U$. Hence $U\in L(V)$. But $V\in L_j$; hence, by 7.5, $L(V) = L_j$ and therefore $U\in L_j$; a contradiction.

From this and (*) we infer that in the subbase $\mathscr{P}|K_j$ there are $2n-2$ non-empty chains disjoint, each to other. Let $U \in \mathscr{P} \setminus (L_j \cup L'_j)$. By 7.6, there exists an L_i such that $U \in L_i$. Hence $U \cap K_j \in L_i|K_j$. Consider D (see the definition in § 2) for $\mathscr{P}|K_j$. We have $D(U \cap K_j) = L_i|K_j \cup L'_i|K_j$, for U considered above. Hence the incomparability number for $\mathscr{P}|K_j$ is $n-1$.

Thus, by 7.8, 7.9 and Theorem 3, K_j is a Euclidean $(n-1)$-cube, K_j being metrizable.

7.10. *X is a union of $(n-1)$-cubes K_j and K'_j.*

Proof. Let $x \in X$. Suppose, on the contrary, that $x \notin K_1 \cup \ldots \cup K_n \cup \cup K'_1 \cup \ldots \cup K'_n$. Hence there exist G_i in L_i and G'_i in L'_i such that $x \notin \mathrm{cl}\, G_i \cup \cup \mathrm{cl}\, G'_i$, $i = 1, \ldots, n$. By 7.6 and (B), $X \setminus \mathrm{cl}\, G'_i \in L_i$ and $X \setminus \mathrm{cl}\, G'_i \in L_i$ and $x \in X \setminus \mathrm{cl}\, G_i$, $i = 1, \ldots, n$. Since L_i and L'_i are chains, we have $U_i \subset X \setminus \mathrm{cl}\, G'_i$ or $X \setminus \mathrm{cl}\, G'_i \subset U_i$ and $U_{n+i} \subset X \setminus \mathrm{cl}\, G_i$ or $X \setminus \mathrm{cl}\, G_i \subset U_{n+i}$. Hence, by (S′b), we have $(X \setminus \mathrm{cl}\, G_1) \cap \ldots \cap (X \setminus \mathrm{cl}\, G'_n) = \emptyset$; a contradiction.

Consider the set $L_j \cup L'_j$. Let $\xi \subset L_j \cup L'_j$ be a maximal family such that $\bigcap \{\mathrm{cl}\, V\colon V \in \xi\} \neq \emptyset$. Let $B(K_j)$ denote the set of all such ξ. Let $A \subset X$. Then

$$T(A) = \{\xi \in B(K_j)\colon \bigcap \mathrm{cl}\, \xi \subset A\}, \quad \text{where } \mathrm{cl}\, \xi = \{\mathrm{cl}\, V\colon V \in \xi\}.$$

Let us generate the topology on $B(K_j)$ by the family T_j consisting of sets $T(U)$, where U run over the members of $L_j \cup L'_j$.

7.11. *If G is in $L_j \cup L'_j$, then* $\mathrm{cl}\, T(G) = T(\mathrm{cl}\, G)$.

Proof. Let $\xi \in \mathrm{cl}\, T(G)$ and suppose that $\xi \notin T(\mathrm{cl}\, G)$. This means that $\bigcap \mathrm{cl}\, \xi \not\subset \mathrm{cl}\, G$. Hence $(X \setminus \mathrm{cl}\, G) \cap \bigcap \mathrm{cl}\, \xi \neq \emptyset$. Let $x \in (X \setminus \mathrm{cl}\, G) \cap \bigcap \mathrm{cl}\, \xi$. By (C′), there exists an H in $\mathscr{P}$ such that $x \in H \subset \mathrm{cl}\, H \subset X \setminus \mathrm{cl}\, G$. By 7.4, $H \in L_j \cup L'_j$. Hence $H \in \xi$. Since $\mathrm{cl}\, H \subset X \setminus \mathrm{cl}\, G$, by 7.4 and 7.3 there exists a W in $L_j \cup L'_j$ such that $\mathrm{cl}\, H \subset W \subset \mathrm{cl}\, W \subset X \setminus \mathrm{cl}\, G$. But $H \in \xi$; hence $\bigcap \mathrm{cl}\, \xi \subset \mathrm{cl}\, H \subset W$ and therefore $\xi \in T(W)$. Since $\mathrm{cl}\, W \cap \mathrm{cl}\, G = \emptyset$, we have $T(G) \cap T(W) = \emptyset$; a contradiction.

Let $\xi \in T(\mathrm{cl}\, G)$ and suppose that $\xi \notin \mathrm{cl}\, T(G)$. Hence there exist $G_1, \ldots, G_r$ in $L_j \cup L'_j$ such that $\xi \in T(G_1) \cap \ldots \cap T(G_r)$ and $T(G_1) \cap \ldots \cap T(G_r) \cap T(G) = \emptyset$. Since $\bigcap \mathrm{cl}\, \xi \subset \mathrm{cl}\, G$ and $\bigcap \mathrm{cl}\, \xi \subset G_1 \cap \ldots \cap G_r$, we have $G_1 \cap \quad \cap \cap G_r \cap G \neq \emptyset$. Let $x \in G_1 \cap \quad \cap G_r \cap G$. By (C′), there exist H_i in $\mathscr{P}$, $i = 1, \ldots, r$, and H in $\mathscr{P}$ such that $x \in H \subset \mathrm{cl}\, H \subset G$ and $x \in H_i \subset \mathrm{cl}\, H_i \subset G_i$. Hence $x \in \mathrm{cl}\, H_1 \cap \ldots \cap \mathrm{cl}\, H_r \cap \mathrm{cl}\, H \subset G_1 \cap \ldots \cap G_r \cap G$. Since G_i and G are in $L_j \cup L'_j$, by 7.4 H_i and H are in $L_j \cup L'_j$. Let $\eta \in B(K_j)$ be such that $H_1, \ldots, H_r$, H are in η. Since $\bigcap \mathrm{cl}\, \eta \cap \mathrm{cl}\, H_1 \cap \quad \cap \mathrm{cl}\, H_r \cap \mathrm{cl}\, H \subset G_1 \cap \ldots \cap G_r \cap G$, we have $\eta \in T(G_1) \cap \ldots \cap T(G_r) \cap T(G)$; a contradiction.

7.12. *If $G \in L_j \cup L'_j$ and $\bigcap \mathrm{cl}\, \xi \cap G \neq \emptyset$ for a ξ in $B(K_j)$, then $\xi \in T(G)$.*

Proof. Since $G \in L_j \cup L'_j$ and $\bigcap \mathrm{cl}\, \xi \cap G \neq \emptyset$, we have by the maximality of family ξ, $G \in \xi$. Hence, by (C′) and 7.4, $\xi \in T(G)$.

7.13. *If* $G \in L_j \cup L_j'$, *then* $T(X \setminus \mathrm{cl} G) = B(K_j) \setminus \mathrm{cl} T(G)$.

Proof. $B(K_j) \setminus \mathrm{cl} T(G) = B(K_j) \setminus T(\mathrm{cl} G) = \{\xi \in B(K_j) : \bigcap \mathrm{cl}\,\xi \not\subset \mathrm{cl} G\}$. Hence, by 7.12, $\bigcap \mathrm{cl}\,\xi \cap \mathrm{cl} G = \emptyset$. Hence $\bigcap \mathrm{cl}\,\xi \cap (X \setminus \mathrm{cl} G) \neq \emptyset$. Hence, by 7.12, $\xi \in T(X \setminus \mathrm{cl} G)$.

7.14. *Topology* T_j *on* $B(K_j)$ *generates a Hausdorff topology.*

Proof. Let $\xi, \xi' \in B(K_j)$ and $\xi \neq \xi'$. Hence, by the maximality of the families ξ and ξ', $\bigcap \mathrm{cl}\,\xi \cap \bigcap \mathrm{cl}\,\xi' = \emptyset$. Since X is compact and $\xi, \xi' \subset L_j \cup L_j'$, there exist a G in ξ and a V in ξ' such that $\mathrm{cl} G \cap \mathrm{cl} V = \emptyset$. Hence, by 7.3 and (B), there exists an H in $\mathscr{P}$ such that $\mathrm{cl} V \subset H \subset \mathrm{cl} H \subset X \setminus \mathrm{cl} G$. By 7.4, $H \in L_j \cup L_j'$. Hence, by 7.3 and (B), there exists a W in $\mathscr{P}$ such that $\mathrm{cl} G \subset W \subset \mathrm{cl} W \subset X \setminus \mathrm{cl} H$. By 7.4, $W \in L_j \cup L_j'$. Hence $\mathrm{cl} H \cap \mathrm{cl} W = \emptyset$ and $\mathrm{cl} V \subset H$ and $\mathrm{cl} G \subset W$. By 7.13, $\xi \in T(H)$, $\xi' \in T(W)$ and $T(H) \cap T(W) = \emptyset$.

7.15. *The subbase* T_j *is a* BISC-*subbase in* $B(K_j)$ *with the incomparability number equal to* 1.

Proof. (B) follows immediately from 7.13.

(C) follows immediately from 7.4.

(I) Let G, V and H be in $L_j \cup L_j'$ and let $\mathrm{cl} T(V) \cap \mathrm{cl} T(G) = \mathrm{cl} T(V) \cap \cap \mathrm{cl} T(H) = \emptyset$. We have $\mathrm{cl} V \cap \mathrm{cl} G = \mathrm{cl} V \cap \mathrm{cl} H = \emptyset$. Hence, by (I), $G \subset H$ or $H \subset G$. Hence $T(H) \subset T(G)$ or $T(G) \subset T(H)$.

(S) Let $B(K_j) = \bigcup \{T(G) : G \in R \subset L_j \cup L_j'\}$. Hence, by (B), we have $\bigcup \{G : G \in R\} = X$. Since X is compact and $R \subset L_j \cup L_j'$, there exist G and G' in R such that $G \cup G' = X$. Hence $T(G) \cup T(G') = B(K_j)$.

Since the incomparability number for $L_j \cup L_j'$ is one, the incomparability number of T_j is also one.

Let $I_j = K_1 \cap \ldots \cap K_{j-1} \cap K_{j+1} \cap \ldots \cap K_n$.

7.16 *The space* $B(K_j)$ *is homeomorphic to* I_j.

Proof. Let $\xi \in B(K_j)$ and let $h_j(\xi) = I_j \cap \bigcap \mathrm{cl}\,\xi$. We shall prove that $h_j(\xi)$ is a one-point set. Suppose, on the contrary, that $h_j(\xi)$ is empty. Then, by the compactness of X and 7.4, there are G_i in K_i, $i \neq j$, $i = 1, \ldots, n$, H in L_j and H' in L_j' such that $\mathrm{cl} G_1 \cap \ldots \cap \mathrm{cl} G_n \cap \mathrm{cl} H \cap \mathrm{cl} H' = \emptyset$. But $n+1 < 2n$, for $n \geqslant 2$; hence, by (S'a), two of them, say A, B, are such that $\mathrm{cl} A \cap \mathrm{cl} B = \emptyset$, which is impossible. Now, let us assume that there exist two different points x, y in $h_j(\xi)$. By 7.1, there exists a G in $\mathscr{P}$ such that $x \in G$ and $y \notin \mathrm{cl} G$. By (B), $X \setminus \mathrm{cl} G \in \mathscr{P}$. By 7.6, G or $X \setminus \mathrm{cl} G$ belongs to L_i, $i = 1, \ldots, n$. Hence either x or y does not belong to $h_j(\xi)$; a contradiction.

Thus the map $h_j : B(K_j) \to I_j \subset X$ is defined.

We shall prove that h_j is a homeomorphism.

To prove that h_j is one-to-one, let $\xi, \xi' \in B(K_j)$ and $\xi \neq \xi'$. Since $\xi \neq \xi'$, we have $\bigcap \mathrm{cl}\,\xi \cap \bigcap \mathrm{cl}\,\xi' = \emptyset$ and therefore $h_j(\xi) \neq h_j(\xi')$.

To prove that h_j is onto, let $x \in I_j$. Let $\tilde{\xi} \subset L_j \cup L'_j$ be the family of all those G from $L_j \cup L'_j$ for which $x \in \mathrm{cl}\, G$. Since $x \in \bigcap \mathrm{cl}\, \tilde{\xi}$, there exists a ξ in $B(K_j)$ such that $\tilde{\xi} \subset \xi$. Suppose, on the contrary, that $x \notin \bigcap \mathrm{cl}\, \xi$. Hence there exists an H in ξ such that $x \notin \mathrm{cl}\, H$. By (B) and (C′), there exists a W in $\mathscr{P}$ such that $x \in W$ and $\mathrm{cl}\, H \cap \mathrm{cl}\, W = \emptyset$. By 7.6, $W \in L_j \cup L'_j$. Since $x \in W$, we have $W \in \tilde{\xi} \subset \xi$ (a contrary to $x \notin \bigcap \mathrm{cl}\, \xi$). Hence there exists a ξ in $B(K_j)$ such that $x \in \bigcap \mathrm{cl}\, \xi \cap I_j$. We have $h_j(\xi) = x$ for this ξ.

To prove that h_j is continuous, let $G \in \mathscr{P}$. By 7.6, $G \in L_i \cup L'_i$ for some $i = 1, \ldots, n$. If $i \neq j$, then $G \cap I_j = I_j$ or $G \cap I_j = \emptyset$. Hence $H_j^{-1}(G \cap I_j) = B(K_j)$ or $h_j^{-1}(G \cap I_j) = \emptyset$. Let $G \in L_j \cup L'_j$. We have $h_j^{-1}(G \cap I_j) = T(G)$. In fact, if $\xi \in T(G)$, then $\bigcap \mathrm{cl}\, \xi \subset G$. Hence $\bigcap \mathrm{cl}\, \xi \cap I_j \subset G \cap I_j$ and therefore $h_j(\xi) \in G \cap I_j$, i.e., $\xi \in h_j^{-1}(G \cap I_j)$. If $\xi \in h_j^{-1}(G \cap I_j)$, i.e., $h_j(\xi) \in G \cap I_j$, then $\bigcap \mathrm{cl}\, \xi \cap I_j \subset G \cap I_j$. We have $\bigcap \mathrm{cl}\, \xi \cap G \neq \emptyset$ and, by 7.13, $\xi \in T(G)$.

Recall that by 7.14 and 7.15 $B(K_j)$ is Hausdorff and compact, X is a metrizable continuum and $h_j \colon B(K_j) \to I_j \subset X$ is continuous, onto and one-to-one. Hence h_j is a homeomorphism and therefore $B(K_j)$ is metrizable.

7.17. *The space $B(K_j)$ is connected.*

Proof. Let us assume, on the contrary, that $B(K_j) = A \cup B$, where $A \cap B = \emptyset$, $A \neq \emptyset \neq B$ and A, B are open. Let $i \neq j$, $i = 1, \ldots, n$. Let $K_i^A = \{x \in K_i\colon$ there exists a ξ in A such that $x \in K_i \cap \bigcap \mathrm{cl}\, \xi\}$; similarly we define K_i^B.

Let $x \in K_i^A$. Hence there exists a ξ in A such that $x \in K_i \cap \bigcap \mathrm{cl}\, \xi$. Since A is open in $B(K_j)$, there exist G, H in $L_j \cup L'_j$ such that $\xi \in T(G) \cap \cap T(H) \subset A$. Hence $x \in \bigcap \mathrm{cl}\, \xi \cap K_i \subset G \cap H \cap K_i$. Let $y \in G \cap H \cap K_i$. Hence there exists a η in $B(K_j)$ such that $y \in \bigcap \mathrm{cl}\, \eta \cap K_i$. Hence $\bigcap \mathrm{cl}\, \eta \cap G \cap H \neq \emptyset$. By 7.13, $\eta \in T(G) \cap T(H) \subset A$ and therefore $y \notin K_i^A$. Hence $x \in G \cap \cap K_i \subset K_i^A$. Hence K_i^A is open. Analogously, K_i^B is open. By the maximality of the families ξ in $B(K_j)$ we infer that, if $\xi, \eta \in B(K_j)$ and $\xi \neq \eta$, then $\bigcap \mathrm{cl}\, \xi \cap \bigcap \mathrm{cl}\, \eta$ is empty. Hence $K_i^A \cap K_i^B = \emptyset$. Thus, K_i^A and K_i^B are non-empty, open in K_i and disjoint. Moreover, $K_i = K_i^A \cup K_i^B$. But, by 7.8, K_i is a continuum; a contradiction.

7.18. *The space $B(K_j)$ is a closed segment* $[0, 1]$, *topologically.*

Proof. By 7.15, 7.16 and 7.17, $B(K_j)$ is a metrizable continuum with a BISC-subbase T_j with the incomparability number equal to one. Hence, by Theorem 3, there exists a homeomorphism $H\colon B(K_j) \to [0, 1]$ such that $H(L_j) = 0$ and $H(L'_j) = 1$.

7.19. *Let $x \in K_j$ and let $R_x \subset \mathscr{P} \setminus L_j$ be a maximal family of those V for which $x \in \mathrm{cl}\, V$. Then $A_x = \bigcap \{\mathrm{cl}\, V\colon V \in R_x\} \cap K'_j$ is a one-point set.*

Proof. Suppose, on the contrary, that the intersection is empty. Then, by the compactness of X, 7.4, 7.6 and the fact that $R_x \cap L'_j = \emptyset$, there exist fewer than $2n-2$ sets in R_x having an empty intersection of their closures with K'_j. By 7.4, there exist $2n-1$ sets in $R_x \cup L'_j$ having an empty intersection of their closures. By (S′), there exist two sets in $R_x \cup L'_j$ having an empty intersection of their closures. Since $x \in \bigcap \{\mathrm{cl}\, V\colon V \in R_x\}$, one of them belongs to L'_j. By 7.3 and 7.6, the other belongs to L_j. Hence $R_x \cap L_j \neq \emptyset$; a contradiction.

Suppose that there exist two different points y, z in A_x. Then, by 7.1, there exists a G in $\mathscr{P} \setminus L_j$ such that $z \in G$ and $y \notin \mathrm{cl}\, G$. But $x \in \mathrm{cl}\, G$ or $x \in X \setminus G$ and therefore G or $X \setminus \mathrm{cl}\, G$ belongs to R_x. Hence either z or y does not belong to A_x; a contradiction.

Define $g_j\colon K_j \to K'_j$, letting $g_j(x)$ be the single point in the above A_x.

7.20. *The map g_j is a homeomorphism.*

Proof. To prove that g_j is one-to-one let x, y be in K_j and $x \neq y$. By 7.1, there exists a G in $\mathscr{P}$ such that $x \in G$ and $y \notin \mathrm{cl}\, G$. Hence G belongs to $\mathscr{P} \setminus L_j$ and therefore G belongs to R_x. By (C′), there exists an H in $\mathscr{P}$ such that $x \in H \subset \mathrm{cl}\, H \subset G$. Hence H belongs to R_x. By (2), $X \setminus \mathrm{cl}\, G \in R_y$. Hence, by (1), $\mathrm{cl}\, H \cap \mathrm{cl}(X \setminus \mathrm{cl}\, G) = \emptyset$. But $g_j(x) \in \mathrm{cl}\, H$ and $g_j(y) \in \mathrm{cl}(X \setminus \mathrm{cl}\, G)$. Hence $g_j(x) \neq g_j(y)$.

To prove that g_j is onto let $z \in K'_j$. Let $R'_z \subset \mathscr{P} \setminus L'_j$ be a maximal family of those U for which $z \in \mathrm{cl}\, U$. Hence, by 7.19, $\bigcap \{\mathrm{cl}\, V\colon V \in R'_z\} \cap K_j$ is a one-point set, say x. But $R'_z = R_x$ and therefore $g_j(x) = z$.

To prove that g_j is continuous let G belong to $\mathscr{P}$. We can assume that $G \notin L_j \cup L'_j$. We shall prove the formula $g_j^{-1}(G \cap K'_j) = G \cap K_j$. Let $x \in g_j^{-1}(G \cap K'_j)$. Hence $g_j(x) \in G \cap K'_j$ i.e., $\bigcap \{\mathrm{cl}\, V\colon V \in R_x\} \cap K'_j \subset G \cap K'_j$. Hence $\bigcap \{\mathrm{cl}\, V\colon V \in R_x\} \cap K'_j \cap (X \setminus G) = \emptyset$. But $G \notin L_j \cup L'_j$; hence $X \setminus \mathrm{cl}\, G \notin L_j \cup L'_j$. Hence there exist $2n-1$ sets in $R_x \cup L'_j \cup \{X \setminus \mathrm{cl}\, G\}$, say $H_1, \ldots, H_{2n-1}$, such that the intersection of their closures is empty. Hence, by (S′), there exist two sets in the family $H_1, \ldots, H_{2n-1}$, say H_i, H_j, such that $\mathrm{cl}\, H_i \cap \mathrm{cl}\, H_j = \emptyset$. Since $K'_j \cap \bigcap \{\mathrm{cl}\, V\colon V \in R_x\} \neq \emptyset$, one of them is $X \setminus \mathrm{cl}\, G$. Hence the other belongs to $R_x \cup L'_j$. More precisely, since $X \setminus \mathrm{cl}\, G \notin L_j \cup L'_j$, the other belongs to R_x. Hence we can assume that $H_i = X \setminus \mathrm{cl}\, G$ and $H_j \in R_x$. Hence $x \in \mathrm{cl}\, H_j \subset G$. This implies that $x \in G \cap K_j$ and therefore $g_j^{-1}(G \cap K'_j) \subset G \cap K_j$.

Let $x \in G \cap K_j$. By (C′), there exists an H in $\mathscr{P}$ such that $x \in H \subset \mathrm{cl}\, H \subset G$. Since $G \notin L_j \cup L'_j$, G and H belong to $\mathscr{P} \setminus L_j$. More precisely, G and H belong to R_x. Hence $x \in g_j^{-1}(G \cap K'_j)$ and therefore $G \cap K_j \subset g_j^{-1}(G \cap K'_j)$.

Recall that K_j, K'_j are metrizable continua, and $g_j\colon K_j \to K'_j$ is continuous, onto and one-to-one. Hence g_j is a homeomorphism.

We shall call g_j the *natural homeomorphism*. Recall that

$$S_j = \bigcup \{K_j \cap K_i\colon i \neq j,\ i = 1, \ldots, n\} \cup \bigcup \{K_j \cap K'_i\colon i \neq j,\ i = 1, \ldots, n\}$$

is the boundary of the cube K_j according to Remark 1 in § 6. This means that S_j is an $n-2$-dimensional sphere.

7.21 *Let $x \in S_j$ and let $Q_x \subset \mathscr{P} \setminus L_j$ be a maximal family of those U for which $x \in \mathrm{cl}\, U$. Then $A_x = \bigcap \mathrm{cl}\, \xi \cap \bigcap \mathrm{cl} Q_x$, where $\bigcap \mathrm{cl} Q_x = \bigcap \{\mathrm{cl}\, U: U \in Q_x\}$, is a one-point set for arbitrary ξ in $B(K_j)$.*

Proof. Suppose, on the contrary, that the intersection is empty. Since $x \in S_j$, we have $x \in K_j \cap K_i'$ or $x \in K_j \cap K_i$ for some $i = 1, \ldots, n$ and $i \neq j$. Let us assume that $x \in K_j \cap K_i$. Hence $Q_x \cap L_i' = \emptyset$. By the compactness of X, 7.4, 7.6 and the fact that $Q_x \cap (L_j \cup L_j') = \emptyset$, there exist fewer than $2n-3$ sets in Q_x having an empty intersection of their closures with $\bigcap \mathrm{cl}\, \xi$. By 7.4, there exist $2n-1$ sets in $Q_x \cup L_j \cup L_j'$, say $H_1, \ldots, H_{2n-1}$, such that $\mathrm{cl} H_1 \cap \ldots \cap \mathrm{cl} H_{2n-1} = \emptyset$. By (S'), there exist two sets, say H_p and H_r, such that $\mathrm{cl} H_p \cap \mathrm{cl} H_r = \emptyset$. Since $\bigcap \mathrm{cl} Q_x \neq \emptyset$ and $\bigcap \mathrm{cl}\, \xi \neq \emptyset$, one of them, say H_p, is in Q_x and the other is in ξ. Since $H_r \in \xi \subset L_j \cup L_j'$, we have $H_r \in L_j$ or $H_r \in L_j'$. If $H_r \in L_j$, then, by 7.4 and the fact that $\mathrm{cl} H_r \cap \mathrm{cl} H_p = \emptyset$, we get $H_p \in L_j'$. Hence $Q_x \cap L_j' \neq \emptyset$; a contradiction. If $H_r \in L_j'$, then by 7.4 and the fact that $\mathrm{cl}\, H_r \cap \mathrm{cl} H_p = \emptyset$, we get $H_p \in L_j$. Hence $Q_x \cap L_j \neq \emptyset$; a contradiction.

Suppose that there exist z and y, $z \neq y$, in $\bigcap \mathrm{cl}\, \xi$ such that z and y belong to A_x. Then, by 7.1, there exists a G in $\mathscr{P} \setminus L_j$ such that $z \in G$ and $y \notin \mathrm{cl} G$. But $x \in \mathrm{cl} G$ or $x \in X \setminus \mathrm{cl} G$ and therefore G or $X \setminus \mathrm{cl} G$ belongs to Q_x. Hence either z or y does not belong to A_x; a contradiction.

Define $g_j^\xi: S_j \to \bigcap \mathrm{cl}\, \xi$, letting $g_j^\xi(\xi)$ be the single point in the above A_x.

7.22. *The map $g_j^\xi: S_j \to \bigcap \mathrm{cl}\, \xi$ is a homeomorphism for $\xi \in B(K_j)$ and $L_j \neq \xi \neq L_j'$.*

Proof. To prove that g_j^ξ is one-to-one let $x_1, x_2 \in \mathscr{S}_j$ and $x_1 \neq x_2$. By 7.1, there exists a G in $\mathscr{P}$ such that $x_1 \in G$ and $x_2 \notin \mathrm{cl} G$. Hence G belongs to $\mathscr{P} \setminus L_j$ and therefore $G \in R_{x_1}$. By (C'), there exists an H in $\mathscr{P}$ such that $x_1 \in H \subset \mathrm{cl} H \subset G$. Hence H belongs to R_{x_1}. By (B), $X \setminus \mathrm{cl} G \in R_{x_2}$ and $\mathrm{cl} H \cap \mathrm{cl}(X \setminus \mathrm{cl} G) = \emptyset$. But $g_j^\xi(x_1) \in \mathrm{cl} H$ and $g_j^\xi(x_2) \in \mathrm{cl}(X \setminus \mathrm{cl} G)$. Hence $g_j(x_1) \neq g_j(x_2)$.

To prove that g_j is onto, let $z \in \bigcap \mathrm{cl}\, \xi$. Let $Q_z' \subset \mathscr{P} \setminus (L_j \cup L_j')$ be a maximal family of those U for which $z \in \mathrm{cl}\, U$. Suppose that $\bigcap \mathrm{cl} Q_z' \cap S_j = \emptyset$. Hence $\bigcap \mathrm{cl} Q_z' \cap K_i \cap K_j = \emptyset = \bigcap \mathrm{cl} Q_z' \cap K_i' \cap K_j$ for $i \neq j$ and $i = 1, \ldots, n$. Since $\bigcap \mathrm{cl} Q_z' \cap K_i \cap K_j = \emptyset$ and $Q_z' \cap L_j' = \emptyset$, there exist $2n-1$ sets, say $H_1, \ldots, H_{2n-1}$, in $Q_z' \cup L_i \cup L_j$ such that $\mathrm{cl} H_1 \cap \quad \cap \mathrm{cl} H_{2n-1} = \emptyset$. Hence, by (S'a), there exist two sets, say H_r, H_p, in $\{H_1, \ldots, H_{2n-1}\}$ such that $\mathrm{cl} H_r \cap \mathrm{cl} H_p = \emptyset$. Since $\bigcap \mathrm{cl} Q_z' \cap K_j \neq \emptyset \neq K_i \cap K_j$, one of them, say H_r, is in Q_z' and the other is in L_i. Hence $H_r \in L_i'$. Hence for each $i = 1, \ldots, n$, $i \neq j$, there exists an $H_r \in L_i' \cap Q_z'$. For fixed $i, i \neq j$, we denote this H_r by $H^{i'}$. Accordingly, for each $i = 1, \ldots, n$, $i \neq j$, there exists an H^i in $L_i \cap Q_z'$. Since $L_j \neq \xi \neq L_j'$, there exists a U in $\xi \cap L_j$ and there exists a V in $\xi \cap L_j'$. By (S'b),

$$\operatorname{cl} H^{1'} \cap \ldots \cap \operatorname{cl} H^{j-1'} \cap \operatorname{cl} U \cap \operatorname{cl} H^{j+1'} \cap \ldots \cap \operatorname{cl} H^{n'} \cap \operatorname{cl} H^1 \cap \ldots \cap \operatorname{cl} H^{j-1} \cap \\ \cap \operatorname{cl} V \cap \operatorname{cl} H^{j+1} \cap \ldots \cap \operatorname{cl} H^n = \emptyset.$$

But $\bigcap \operatorname{cl} Q'_z \cap \bigcap \operatorname{cl} \xi \neq \emptyset$; a contradiction. Hence there exists a $x \in \bigcap \operatorname{cl} Q'_z \cap \cap S_j$. Now we prove that $Q_x = Q'_z$. The inclusion $Q'_z \subset Q_x$ is obvious. Let $G \in Q_x$. Suppose that $G \notin Q'_z$. This means that $z \notin \operatorname{cl} G$. By 7.3 and (B), we infer that there exists an H in $\mathscr{P}$ such that $z \in H$ and $\operatorname{cl} H \cap \operatorname{cl} G = \emptyset$. Since $Q'_z \subset Q_x$, we have $H \notin Q'_z$. Since $z \in H$, $x \notin \operatorname{cl} H$ and $H \in Q'_z$, we have $H \in L'_j$. Since $\operatorname{cl} H \cap \operatorname{cl} G = \emptyset$ and $H \in L'_j$, we have $G \in L_j$; a contradiction.

Hence $z \in \bigcap \operatorname{cl} Q_x \cap \bigcap \operatorname{cl} \xi$ and therefore $g_j^\xi(x) = z$.

To prove that g_j is continuous let $G \in \mathscr{P}$. We can assume that $G \notin L_j \cup L'_j$. Then we have the formula

$$(+) \qquad g_j^{\xi-1}(G \cap \bigcap \operatorname{cl} \xi) = G \cap S_j.$$

Let $x \in g_j^{\xi-1}(G \cap \bigcap \operatorname{cl} \xi)$, i.e., $g_j^\xi(x) \in G \cap \bigcap \operatorname{cl} \xi$. Suppose that $x \notin G \cap S_j$. Hence $x \in X \setminus G = \operatorname{cl}(X \setminus \operatorname{cl} G)$. Since $G \notin L_j \cup L'_j$, we have $X \setminus \operatorname{cl} G \notin L_j \cup L'_j$ and therefore $X \setminus \operatorname{cl} G \in Q_x$. Hence, by the definition of the map g_j^ξ, $g_j^\xi(x) \in \operatorname{cl}(X \setminus \operatorname{cl} G) \cap \bigcap \operatorname{cl} \xi = (X \setminus G) \cap \bigcap \operatorname{cl} \xi$. Hence $g_j^\xi(x) \in (X \setminus G) \cap \bigcap \operatorname{cl} \xi \cap \cap G \cap \bigcap \operatorname{cl} \xi = \emptyset$; a contradiction.

Hence $g_j^{\xi-1}(G \cap \bigcap \operatorname{cl} \xi) \subset G \cap S_j$.

Let $x \in G \cap S_j$. By (4), there exists an H in $\mathscr{P}$ such that $x \in H \subset \operatorname{cl} H \subset G$. Since $G \notin L_j \cup L'_j$, we have $H \notin L_j \cup L'_j$ and therefore G and H are in Q_x. Hence

$$g_j^\xi(x) \in \bigcap \operatorname{cl} Q_x \cap \bigcap \operatorname{cl} \xi \subset \operatorname{cl} H \cap \bigcap \operatorname{cl} \xi \subset G \cap \bigcap \operatorname{cl} \xi.$$

Hence $g_j^\xi(x) \in G \cap \bigcap \operatorname{cl} \xi$. Thus the formula (+) is proved. *Eo ipso*, the continuity of the map g_j is proved.

Now, S_j is compact and Hausdorff, and $g_j^\xi \colon S_j \to \bigcap \operatorname{cl} \xi$ is continuous, onto and one-to-one. Therefore g_j is a homeomorphism.

For completeness, we denote by $g_j^{L_j}$ the identity map on K_j and by $g_j^{L'_j}$ the natural homeomorphism defined in 7.20.

For fixed j we shall write only g^ξ for $\xi \in B(K_j)$.

7.23. *If $G \in \mathscr{P} \setminus (L_j \cup L'_j)$, then $g^\xi(y) \in G$ for $y \in G \cap S_j$ and $\xi \in B(K_j)$, $g^{L_j}(y) \in G$ for $y \in G \cap K_j$. Accordingly, $g^\xi(y) \notin G$ for $y \notin G \cap S_j$ and $\xi \in B(K_j)$, $g^{L_j}(y) \notin G$ for $y \notin G \cap K_j$.*

Proof. The proof is obvious by 7.20, 7.22 and the following formulas:

$$G \cap K_j = g^{L_j^{-1}}(G \cap K'_j),$$

$$G \cap S_j = g^{\xi^{-1}}(G \cap \bigcap \operatorname{cl} \xi).$$

Let $I^n \subset R^n$ denote a Euclidean n-cube, i.e., let I^n be a product of n copies of the segment $[0, 1] = I$ and let R^n be a product of n copies of the real line. Let ∂I^n be the boundary of I^n in R^n (thus ∂I^n is the $(n-1)$-sphere, topologically). Denote by I_0^{n-1} and I_1^{n-1} two opposite sides of ∂I^n. We have $I_0^{n-1} = I^{n-1}\times\{0\} \subset I^n$ and $I_1^{n-1} = I^{n-1}\times\{1\} \subset I^n$ and $\partial I^n = I_0^{n-1}\cup I_1^{n-1}\cup\partial I^{n-1}\times I$.

For a fixed j, $1\leqslant j\leqslant n$, let $h\colon K_j\to I^{n-1}$ be a homeomorphism such that $h|S_j$ is a homeomorphism onto ∂I^{n-1} (this follows from Theorem 3).

Define the map $G\colon \partial I^n\to X$. Let $x\in\partial I^n$. Hence $x = (z, t)$, where z is some point in I^{n-1} and $0\leqslant t\leqslant 1$. Then we set

$$G(x) = g^{H^{-1}(t)}[h^{-1}(z)].$$

To prove Theorem 6 it remains to show that the map G is a homeomorphism.

G is one-to-one. Let $x_1\neq x_2$ and $x_1, x_2\in\partial I^n$. Hence $x_1 = (z_1, t_1)$, $x_2 = (z_2, t_2)$. Since $x_1\neq x_2$, we have $z_1\neq z_2$ or $t_1\neq t_2$. Let $t_1\neq t_2$. Hence $H^{-1}(t_1)\neq H^{-1}(t_2)$ and $H^{-1}(t_1)$, $H^{-1}(t_2)\in B(K_j)$. Hence $\bigcap \operatorname{cl} H^{-1}(t_1)\cap \cap\bigcap \operatorname{cl} H^{-1}(t_2) = \emptyset$. But

$$g^{H^{-1}(t_1)}[h^{-1}(z\,)]\in\bigcap \operatorname{cl} H^{-1}(t_1) \quad\text{and}\quad g^{H^{-1}(t_2)}[h^{-1}(z_2)]\in\bigcap \operatorname{cl} H^{-1}(t_2)$$

and therefore $G(x_1)\neq G(x_2)$. Let $z_1\neq z_2$ and suppose that $t_1 = t_2 = t$. Since h is an homeomorphism, we have $h^{-1}(z_1)\neq h^{-1}(z_2)$. Since $g^{H^{-1}(t)}$ is an homeomorphism, we have $G(x_1)\neq G(x_2)$.

G is onto. Let $y\in X$. Let $\zeta\subset L_j\cup L_j'$ be a maximal family of sets U such that $y\in \operatorname{cl} U$. We shall show that $\zeta\in B(K_j)$. In fact, since $y\in\bigcap\operatorname{cl}\zeta$, there exists a ξ in $B(K_j)$ such that $\zeta\subset\xi$. Suppose that $y\notin\bigcap\operatorname{cl}\xi$. Hence there exists a V in $\mathscr{P}\cap\xi$ such that $y\notin\operatorname{cl} V$. By (C′) and 7.3, there exists a W in $\mathscr{P}$ such that $y\in W$ and $\operatorname{cl} W\cap\operatorname{cl} V = \emptyset$. Since $V\in L_j\cup L_j'$ and $\operatorname{cl} W\cap \cap\operatorname{cl} V = \emptyset$, we have $W\in L_j\cup L_j'$. Hence $W\in\zeta\subset\xi$; a contradiction. Hence $\zeta = \xi$.

Consider the point $x = \big(h(g^{\zeta^{-1}}(y)), H(\zeta)\big)$. We shall show that $x\in\partial I^n$.

Let $\zeta = L_j'$. Since $g^{L_j'}$ is the natural homeomorphism and $y\in\bigcap\operatorname{cl}\zeta = \bigcap\operatorname{cl} L_j' = K_j'$, we have $g^{\zeta^{-1}}(y) = g^{L_j'^{-1}}(y)\in K_j$. Since $H(L_j') = 1$, we have $x\in I_1^{n-1}\subset\partial I^n$.

Let $\zeta = L_j$. Since g^{L_j} is the identity on K_j, we have $g^{\zeta^{-1}}(y) = y\in K_j$. Since $H(L_j) = 0$, we have $x\in I_0^{n-1}\subset\partial I^n$.

Let $L_j\neq\zeta\neq L_j'$. Hence $g^{\zeta^{-1}}(y)\in S_j$ and therefore $h(g^{\zeta^{-1}}(y))\in\partial I^{n-1}$ (since $h(S_j) = \partial I^{n-1}$). Hence $x\in\partial I^{n-1}\times I$. This proves that $x\in\partial I^n$. By the definition of the map G we get

$$G(x) = g^{H^{-1}(H(\zeta))}\Big[h^{-1}\Big(h\big(g^{\zeta^{-1}}(y)\big)\Big)\Big] = g^{\zeta}[g^{\zeta^{-1}}(y)] = y.$$

Hence G is onto.

G is continuous. Let $U\in\mathscr{P}$. Consider the following cases.

(a) $U\in L_j\cup L_j'$. Let $\pi\colon I^n\to I^{n-1}$ be the projection given by the formula $\pi(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-1})$ and let $\sigma\colon I^n\to I$ be given by the formula

$\sigma(x_1, \ldots, x_n) = x_n$. We shall show that

$$G^{-1}(U) = \sigma^{-1}\big[H\big(T(U)\big)\big] \cap \partial I^n.$$

Let $x \in G^{-1}(U)$, i.e., $G(x) \in U$. Let $x = (z, t)$. Hence $G(x) = g^{H^{-1}(t)}[h^{-1}(z)] \in U$. But

$$g^{H^{-1}(t)}[h^{-1}(z)] = \bigcap R_{h^{-1}(z)} \cap K'_j$$

when $H^{-1}(t) = L'_j$;

$$g^{H^{-1}(t)}[h^{-1}(z)] = \bigcap Q_{h^{-1}(z)} \bigcap \mathrm{cl}\, H^{-1}(t)$$

when $L_j \neq H^{-1}(t) \neq L'_j$ and

$$g^{H^{-1}(t)}[h^{-1}(z)] = h^{-1}(z)$$

when $H^{-1}(t) = L_j$. This implies that $\bigcap \mathrm{cl}\, H^{-1}(t) \cap U \neq \emptyset$. Hence, by 7.12, $H^{-1}(t) \in T(U)$. Hence $t \in H\big(T(U)\big)$. Hence $x \in \sigma^{-1}\big(H\big(T(U)\big)\big) \cap \partial I^n$.

Let $x \in \sigma^{-1}\big(H\big(T(U)\big)\big) \cap \partial I^n$. Hence $x \in \partial I^n$ and $x = (z, t)$ where $t \in H\big(T(U)\big)$ and $z \in I^{n-1}$. Hence $G(x) = g^{H^{-1}(t)}[h^{-1}(z)]$. Since $t \in H\big(T(U)\big)$, we have $H^{-1}(t) \in T(U)$. Hence $\bigcap \mathrm{cl}\, H^{-1}(t) \subset U$ and therefore

$$g^{H^{-1}(t)}[h^{-1}(z)] \in \bigcap \mathrm{cl}\, H^{-1}(t) \subset U.$$

Hence $G(x) \in U$, i.e., $x \in G^{-1}(U)$.

Since H is a homeomorphism, $T(U)$ is open in $B(K_j)$; $\sigma^{-1}\big(H\big(T(U)\big)\big)$ is open in I^n and therefore $\sigma^{-1}\big(H\big(T(U)\big)\big) \cap \partial I^n$ is open in ∂I^n.

(b) $U \notin L_j \cup L'_j$. We shall show that

$$G^{-1}(U) = \pi^{-1}\big(h(U \cap K_j)\big) \cap \partial I^n.$$

We have

$$G^{-1}(U) = \{x = (z, t) \in \partial I^n \colon g^{H^{-1}(t)}[h^{-1}(z)] \in U\}.$$

Hence, by 7.23,

$$\begin{aligned} G^{-1}(U) &= \{x = (z, t) \in \partial I^n \colon h^{-1}(z) \in U \cap K_j\} \\ &= \{x = (z, t) \in \partial I^n \colon z \in h(U \cap K_j)\} = \pi^{-1}\big(h(U \cap K_j)\big) \cap \partial I^n. \end{aligned}$$

Since h is a homeomorphism, $U \cap K_j$ is open in K_j; hence $h(U \cap K_j)$ is open in I^{n-1} and therefore $\pi^{-1}\big(h(U \cap K_j)\big) \cap \partial I^n$ is open in ∂I^n.

Since $\mathscr{P}$ is a subbase in X, $G^{-1}(U)$ is open in ∂I^n for $U \in \mathscr{P}$; hence G is continuous.

Thus, ∂I^n is compact Hausdorff, and $G\colon \partial I^n \to X$ is continuous, onto and one-to-one. Hence G is a homeomorphism.

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