DIFFERENT ASPECTS OF DIFFERENTIABILITY
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CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS AND RELATED FUNCTIONAL EQUATIONS

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Abstract. The aim of this survey talk is to show that recently functional equations in a single variable, also called *iterative functional equations*, turned out to be useful as an elementary and handful tool to study peculiar functions, in particular, continuous nowhere differentiable *(cnd)* functions.

Examples. The famous Peano curve (1890) which maps bijectively the interval [0,1] onto the square $[0,1]^2$ may be described parametrically as $x = \phi(t)$, $y = \psi(t)$. The functions ϕ and ψ satisfy 9 functional equations each, of the form

(P) $$\phi((x+k)/9) = a_k\phi(x) + b_k, \qquad k = 0, \ldots, 8$$

where $a_k \in \{-1/3, 1/3\}$, $b_k \in \{0, 1/3, 2/3, 1\}$ (cf. [10]).

It has been proved in 1900 by E. H. Moore [9] that the functions ϕ and ψ are nowhere differentiable (*nd*).

Another curve with the same property as that of Peano is the well-known Sierpiński carpet ([11], cf. also [7], p. 419).

In both the above examples functional equations appear in the definition of the peculiar curve in question.

Most of *cnd* functions, however, are given in the form of the series

(1) $$\phi(x) = \sum_{n=0}^{\infty} a^n h(b^n x), \qquad x \in I \subset \mathbb{R},$$

where I is a closed interval, with an $0 < a < 1$ and a continuous function $h : I \to \mathbb{R}$. Clearly, such functions are continuous on I and among them we may find

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a number of *nd* functions, in particular, the Weierstrass $S_{a,b}$ and $C_{a,b}$ functions (1875, cf. [1])

$$S_{a,b}(x) = \sum_{n=0}^{\infty} a^n \sin(b^{n+1}\pi x); \quad b > 1, \tag{S}$$

$$C_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^{n+1}\pi x); \quad b > 1, \tag{C}$$

and the B. L. van der Waerden function (1930, cf. [2])

$$W(x) = \sum_{n=0}^{\infty} 2^{-n} d(2^n x), \tag{W}$$

where $d(y) = \mathrm{dist}(y, \mathbb{N})$.

All these functions are the unique solutions of "their" functional equations, of the form

$$\phi(x) = a\phi(bx) + h(x), \quad x \in I \tag{2}$$

with suitable h. Indeed, according to the Banach principle, the equation (2) has a unique continuous solution on I whenever h is continuous on I, $0 < a < 1$, which is given by (1). The functions (S), (C) and (W) are *nd* but the equation itself is not powerful enough to provide a proof that they are actually *nd*. Such proofs were given by several authors without using functional equations.

Weierstrass himself was able to prove that the function (C) is *nd* (i.e., it has neither a finite nor an infinite derivative at any point) when b is odd and $ab > 1 + 3\pi/2$. And it was G. H. Hardy who proved in 1916 that neither (S) nor (C) has a finite derivative when $ab \geq 1$ and $b \in \mathbb{R}$ and that there exist cases where they have infinite derivatives (cf. [4]). In the most recent paper [5] M. Hata (1988) used the theory of almost periodic functions to prove *nowhere differentiability* of Weierstrass' functions when $b \in \mathbb{R}$ and $ab > 1 + 1/\cos\psi$, where ψ is the number satisfying $\tan\psi = \pi + \psi$ in $(0, \pi/2)$. (Note that $\psi \cong 5.603\ldots < 5.712\ldots \cong 1 + 3\pi/2$ (Weierstrass).)

The present survey will be concluded by describing a general method invented by R. Girgensohn (1992, cf. [3]) for proving *nowhere differentiability* of a wide class of peculiar functions, including those already discussed here: Peano's, Sierpiński's, Weierstrass' and van der Waerden's.

A generic property of a functional equation. It was the late M. Kuczma who proved in 1988 (cf. [6]) that in most cases the unique continuous solution (1) of equation (2) is *nd*.

Consider an equation which is slightly more general than (2):

$$\phi(x) = a\phi(f(x)) + h(x), \tag{3}$$

and take $I = [0, 1]$, $f \in \mathbf{D} := \{g \in \mathbf{C} : g(I) \subset I\}$, where $a \in (0, 1)$, $h \in \mathbf{C}$ and $\mathbf{C} = \{\phi : I \to \mathbb{R},\ \phi \text{ continuous on } I\}$, $\|\phi\| = \sup_I |\phi(x)|$.

Let $\mathbf{X} = (0,1) \times \mathbf{D} \times \mathbf{C}$ be the space of all triplets $T = (a, f, h)$, representing equation (3), endowed with the product topology. Consider the mapping $T \to F(T) = \phi$, defined by

$$\phi(x) = \sum_{n=0}^{\infty} a^n h[f^n(x)], \qquad x \in I, \tag{4}$$

where f^n denotes the nth iterate of f. Thus $F : \mathbf{X} \to \mathbf{C}$.

M. Kuczma has proved the following

THEOREM 1. *There is a set* $\mathbf{Y} \subseteq \mathbf{X}$, *residual in* $\mathbf{X}$, *such that for every* $T \in \mathbf{Y}$ *the continuous solution* ϕ *of* (3) *given by* (4) *is nd.*

The proof is based on the following result by S. Banach (1931): For every $n \in \mathbb{N}$ the set E_n consisting of those $\phi \in \mathbf{C}$ whose difference $\phi(t + x) - \phi(x)$ is bounded in absolute value by tn for at least one $x \in [0, 1 - 1/n]$ and for every $t \in (0, 1 - x)$, is closed and has no interior points. Thus it is nowhere dense in $\mathbf{C}$.

What is needed to prove is that F is continuous and that the sets $F^{-1}(E_n)$ have empty interiors. Having this, put

$$\mathbf{Y} = \mathbf{X} \setminus \bigcup_{n=1}^{\infty} F^{-1}(E_n).$$

Thus $\mathbf{Y}$ is actually residual in $\mathbf{X}$. Now, if $T \in \mathbf{Y}$, then $F(T) \in E_n$ for any $n \in \mathbb{N}$, whence at least one of the Dini derivatives of ϕ is infinite at every point $x \in (0, 1)$.

M. Kuczma also proved that arbitrarily closely to any equation (3) there is one of this form having a *cnd* solution, i.e.

THEOREM 2. *For every* $(a, f, h) \in \mathbf{X}$ *and* $\varepsilon > 0$ *there exists a function* $\widetilde{h} : I \to \mathbb{R}$ *such that* $(a, f, \widetilde{h}) \in \mathbf{X}$, $\|h - \widetilde{h}\| < \varepsilon$ *and the continuous solution of the corresponding equation* (3) *is nd.*

Van der Waerden's function. The first proof of *nowhere differentiability* making a direct use of functional equations was given in 1989 by W. F. Darsow, M. J. Frank and H.-H. Kairies [2] for the van der Waerden *cnd* function defined by (W). Following [2], we are going to show that (W) satisfies also two other functional equations. We have

$$W(x) = 2\sum_{n=0}^{\infty} 2^{-n-1} d(2^{n+1}x/2) = 2\Big[\sum_{n=-1}^{\infty} 2^{-n-1} d(2^{n+1}x/2) - d(x/2)\Big],$$

whenever $x \in [0, 1]$. Thus, as $d(x/2) = x/2 \in [0, 1/2]$, we get

$$W(x) = 2W(x/2) - x/2, \qquad x \in [0, 1]. \tag{4}$$

On the other hand,

$$W[(x+1)/2] = \sum_{n=0}^{\infty} 2^{-n} d[2^n(x+1)/2] = d[(x+1)/2] + \sum_{n=1}^{\infty} d(2^n x/2 + 2^{n-1})$$
$$= d[(x+1)/2] + W(x/2) - d(x/2).$$

This yields, as $d[(x+1)/2] = 1 - (x+1)/2 \in [1/2, 1]$,

$$W[(x+1)/2] = W(x/2) - x + 1/2, \qquad x \in [0,1]. \tag{5}$$

The equations (4) and (5) rewritten in the form

$$\begin{aligned} \phi(2x) &= 2\phi(x) - 2x, && x \in [0,1/2] \\ \phi(x+1/2) &= \phi(x) - 2x + 1/2, && x \in [0,1/2] \end{aligned} \tag{6}$$

work well in the proof that (cf. [4]) *if* $\phi : [0,1] \to \mathbb{R}$ *satisfies* (6) *then it is nd.*

The proof goes via examining the difference quotient for ϕ. We derive, from (6),

$$\phi(x + 2^{-m}) = \phi(x) - 2x + m2^{-m}, \qquad x \in [0, 2^{-m}].$$

Since $\phi(1) = 0$ implies $\phi(2^{-m}) = m2^{-m}$, and, as $\phi(0) = 0$,

$$2^m[\phi(2^{-m}) - \phi(0)] = m,$$

the function ϕ has no derivative at $x = 0$. Next ϕ is shown to be *nd* for dyadic rationals and finally for other reals from [0,1].

Nowhere differentiability of solutions to a system of functional equations. Quite a new idea of proving *nd* of functions satisfying a system of functional equations has been invented by R. Girgensohn in his 1992 dissertation (cf. [3]).

This system is of the form:

$$\phi[(x+\nu)/b] = a_\nu \phi(x) + h_\nu(x), \qquad \nu = 0, \ldots, b-1, \;\; b \in \mathbb{N} \setminus \{1\}. \tag{7}$$

Here a_ν are some reals and $h_\nu : [0,1] \to \mathbb{R}$ are some given functions.

First of all, it is proved in [3] that a continuous solution to (7) is unique. To formulate a suitable proposition, note that since 0 and 1 are in the domain of ϕ, a necessary condition for the existence of a solution ϕ of (7) in [0,1] is

$$\begin{aligned} (*) \quad & a_{\nu-1} h_{b-1}(1)/(1 - a_{b-1}) + h_{\nu-1}(1) \\ & \qquad = a_\nu h_0(0)/(1 - a_0) + h_\nu(0), \qquad \nu = 1, \ldots, b-1. \end{aligned}$$

PROPOSITION. *If the conditions* $(*)$ *and* $(**)$ $\max\{|a_0|, \ldots, |a_{b-1}|\} < 1$ *are satisfied*, *and* $h_\nu : [0,1] \to \mathbb{R}$ *are continuous*, *then there exists a unique solution of the system* (7) *given by the formula*

$$\phi(x) = \sum_{l=1}^{\infty} \Big(\prod_{k=1}^{l-1} a_{\xi_k} \Big) h_{\xi_l} \Big(\sum_{k=1}^{\infty} b^{-k} \xi_{k+1} \Big), \tag{8}$$

where $x = (0, \xi_1 \xi_2 \ldots)_b$ *is the b-adic representation of* x.

There are several examples of systems (7) satisfied by concrete *cnd* functions. System (6) rewritten as (with $2x$ replaced by x)

$$\phi(x/2) = \phi(x)/2 + x, \qquad x \in [0,1],$$
$$\phi[(x+1)/2] = \phi(x)/2 - x/2 + 1/2, \qquad x \in [1,2],$$

has just the form (7) with $b = 2$, $a_0 = a_1 = 1/2$, and $h_1(x) = x$, $h_2(x) = -x/2 + 1/2$.

One can also check that, e.g., the Weierstrass functions (S) satisfy (7) with $a_\nu = (-1)^{b\nu} a$, $h_\nu(x) = (-1)^\nu \sin \pi x$. The systems (P) for Peano's functions ϕ and ψ also have the form (7). The same applies to Sierpiński's carpet.

Girgensohn uses the Schauder basis of $C[0,1]$ consisting of polygons and the development of any member of the space in this basis by means of a uniformly convergent series. (For Schauder bases cf., e.g., J. T. Marti [7]). He then establishes inequalities for coefficients that ensure *nd* of the function developed and satisfying (7). To check now whether a solution to (7) is *nd* it is enough to calculate its Schauder coefficients. In particular, it has been done in [3] in the case of all *cnd* functions (P), (S), (C), (W) and some others.

A generic property of the system (7). Motivated by the result of M. Kuczma [6], R. Girgensohn also studied generic properties of (7).

Fix a_ν such that $(\min |a_\nu|)b > 1$ and $\max |a_\nu| < 1$ and consider the space $\mathbf{C}^n[0,1]$ of functions with bounded nth derivative and with the norm

$$\|u\|_n = \max\{\|u\|, \|u'\|, \ldots, \|u^{(n)}\|\}.$$

Take a subspace $\mathbf{E} \subseteq \mathbf{C}^n[0,1]$ which contains all affine functions, and the set

$$\mathbf{D} := \{(h_0, \ldots, h_{b-1}) \in \mathbf{E} \text{ such that condition } (*) \text{ is satisfied}\}$$

The set $\mathbf{D}$ with the norm

$$\|(h_0, \ldots, h_{b-1})\| = \max\{\|h_0\|_n, \ldots, \|h_{b-1}\|_n\}$$

is a subspace of $\mathbf{E}$.

Assign to any $(h_0, \ldots, h_{b-1}) \in \mathbf{D}$ the unique continuous solution to (7), $F(h_0, \ldots, h_{b-1})$, given by (8). The following is found in [3]:

PROPOSITION. *The set of those* $(h_0, \ldots, h_{b-1}) \in \mathbf{D}$ *for which* $F(h_0, \ldots, h_{b-1})$ *has at one point a one-sided derivative*, *is of the first Baire category in* $\mathbf{D}$.

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