POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCXXIX

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Some applications of the topological degree theory to multi-valued boundary value problems

WARSZAWA 1984

PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-05103-5 ISSN 0012-3862

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CONTENTS

Introduction

The fundamental boundary value problem in the theory of ordinary differential equations with multi-valued right-hand side is the Cauchy initial value problem, which is formulated as follows:

Let U be an open subset of $R^1 \times R^n$ (or let $U = [0, a] \times R^n$), $(t_0, x_0) \in U$ and let $F\colon U \to R^n$ be a multi-valued mapping. The question is what conditions about F are sufficient for the existence of an open interval $J \subset R^1$ and an absolutely continuous function $x\colon R^1 \to R^n$ satisfying the following:

(i) $t_0 \in J$ and $x(t_0) = x_0$;

(ii) for each $t \in J$, $(t, x(t)) \in U$;

(iii) $x'(t) \in F(t, x(t))$ almost everywhere on J.

Forty five years ago A. Marchaud [72], [73] and S. K. Zaremba [102], [103] showed that if F is a multi-valued mapping with convex compact values and it is continuous (with respect to the Hausdorff metric), then the above question has a positive answer.

In 1961, T. Ważewski [97] gave the solution to this problem for convex-valued upper semi-continuous mappings.

The above problem for convex-valued mappings, which satisfies the Carathéodory type conditions was first studied by A. F. Filippov [30] and T. Ważewski [99], and developped by A. Pliś [84], C. Castaing [16], [17], A. Lasota and Z. Opial [67], A. Lasota [62], J. M. Lasry and R. Robert [69] and others.

In 1970, H. Hermes asked about this problem for mappings which are not necessarily convex-valued. The first to give a positive answer to Hermes's question was A. F. Filippov. Note that Hermes's question has been studied by several authors, for example see: [4], [11], [52], [70], [81], [90], [93].

One of the most remarkable methods in the theory of differential equations (with a single-valued right side) consists in the applications of the topological degree theory or some consequences of this theory, for example topological fixed point theorems, the Browder invariance of domain theorem or the Borsuk antipodes theorem.

It is known that the topological degree theory is well developed for multi-valued mappings (comp. [40], [41], [22], [71], [14], [10]).

Therefore, it is quite a natural question how to apply this theory to differential equations with a multi-valued right side. The first results in this direction belong to A. Lasota and Z. Opial [67], [68]. Later, some results of this type were given by A. Lasota [62], J. P. Aubin and A. Cellina [5], J. M. Lasry and R. Robert [69], U. G. Borisovic, B. D. Gelman, A. D. Myskis and V. V. Obuchovskii [10].

Note that in all of the above papers use was made only of some consequences of the topological degree theory for first order boundary value multi-valued problems.

In this paper we present a systematic study of multi-valued boundary valued problems (not necessarily of first order) by using the Leray–Schauder degree theory.

In this order we introduce the notion of admissible multi-valued boundary value problems (comp. [91]-[92]) and we define the Leray–Schauder degree theory for such problems. Several applications of this method are presented.

The paper is arranged as follows. Chapter I contains some preliminaries from functional analysis. Chapter II is devoted to multi-valued mappings. In Chapter III we introduce a class of admissible boundary value problems and we develop the Leray–Schauder degree theory for such problems.

Finally, Chapters IV, V and VI contains applications of the results given in Chapter III. In particular, we obtain the following existence theorems for multi-valued boundary value problems:

1. The Cauchy, the Nicoletti and the Floquet boundary value problem for first order differential equations (not necessarily with a convex-valued right side).

2. Problems with nonlinear boundary conditions for first order differential equations (not necessarily with a convex-valued right side).

3. The Picard boundary value problem for second order differential equations (not necessarily with a convex-valued right side).

4. The Darboux problem for hyperbolic partial differential equations with a convex-valued right side.

5. Problems with nonlinear boundary conditions for hyperbolic partial differential equations with a convex-valued right side.

6. The general boundary value problem for elliptic partial differential equations with a convex-valued right side.

Acknowledgement. The author should like to express his thanks to Professor Lech Górniewicz for inspiring his interest in the above problems and continued help during the preparation of this paper.

I. Preliminaries

In this chapter we give a few basic definitions and facts concerning functional and mathematical analysis, and measure theory which will be used in our later considerations. By space we will always mean a real Banach space.

1. Strong convergence and weak convergence in Banach spaces. Let $(E, \|\cdot\|)$ be a Banach space with the norm $\|\cdot\|$ and let E^* denote the conjugate space of all linear continuous functionals from E into the set R of real numbers.

A sequence $\{x_n\} \subset E$ is said to be *convergent*, or *strongly convergent, to a point* $x \in E$ (written $\{x_n\} \to x$) if $\lim_n \|x_n - x\| = 0$. Then the point x is called a *strong limit of the sequence* $\{x_n\}$.

A sequence $\{x_n\} \subset E$ is said to be *weakly convergent to a point* $x \in E$ (written $\{x_n\} \underset{w}{\to} x$) if, for every functional $f \in E^*$, $\lim_n f(x_n) = f(x)$. Then the point x is called a *weak limit of the sequence* $\{x_n\}$.

It is clear that every strongly convergent sequence in E is also weakly convergent.

If the points $x_1, x_2, \ldots, x_n$ are in the space E and the non-negative coefficients $c_1, c_2, \ldots, c_n$ satisfy $c_1 + \ldots + c_n = 1$, then the linear combination $c_1 x_1 + \ldots + c_n x_n$ is called a *convex combination of the points* $x_1, x_2, \ldots, x_n$. Recall the following connection between the above two types of convergence.

(1.1) (Mazur Theorem, [100]). *Let a sequence $\{x_n\}$ be weakly convergent in the space E to a point $x \in E$. Then for every n there is a convex combination $y_n = c_1 x_1 + \ldots + c_n x_n$ of the points $x_1, \ldots, x_n$ such that $\{y_n\} \to x$.*

2. Compact and weakly compact sets in Banach spaces. A subset X of the space E is called *relatively compact* if its closure $\bar{X}$ in E is compact.

A subset X of E is called *relatively weakly compact* if every sequence $\{x_n\}$ in X contains a subsequence which converges weakly to a point x in E.

A subset X of E is called *weakly compact* if every sequence $\{x_n\}$ in X contains a subsequence which converges weakly to a point x in X.

Let $T\colon E \to E_1$ be a linear mapping from E into a Banach space E_1. For any $X \subset E$, $T(X)$ denotes the set $\{y \in E_1\colon y = T(x), x \in X\}$ and is called the image of X under T. In particular, $\operatorname{Im} T$ denotes the image $T(E)$ of E under T and is called the image of T, and $\operatorname{Ker} T$ denotes the set $\{x \in E\colon T(x) = 0\}$ and is called the kernel of T. Note that $\operatorname{Im} T$ and $\operatorname{Ker} T$ are linear subspaces of E_1 and E respectively.

The following fact clearly results from the respective definitions.

(2.1) Proposition. *Let $T\colon E \to E_1$ be a linear continuous mapping and let X and Y be two relatively weakly compact subsets of E. Then:*

(2.1.1) *the set* $X+Y=\{x+y\colon x\in X \text{ and } y\in Y\}$ *is a weakly relatively compact subset of* E,

(2.1.2) *the image* $T(X)$ *of* X *under* T *is a relatively weakly compact subset of* E_1.

Let E^{**} be the conjugate of the Banach space E^*. The mapping $\varkappa\colon E \to E^{**}$, given by the following condition: for every $x\in E$, $\varkappa(x)(f)=f(x)$ for every $f\in E^*$ is called the *natural embedding of* E *into* E^{**}.

The space E is called *reflexive* if $\operatorname{Im}\varkappa = E^{**}$.

For an arbitrary subset X of E we will denote by $\operatorname{conv}(X)$ the intersection of all closed convex sets containing X. It is easy to see that $\operatorname{conv}(X)$ is a minimal closed convex set containing X.

In what follows we will make use of the following two facts from functional analysis:

(2.2) (Mazur Theorem, [100]). *If* X *is compact subset of the Banach space* E, *then the set* $\operatorname{conv}(X)$ *is compact.*

(2.3) (Banach–Bourbaki Theorem, [3]). *The Banach space* E *is reflexive if and only if its closed unit ball* $B_1=\{x\in E\colon \|x\|\leqslant 1\}$ *is weakly compact.*

3. Weakly compact sets in the space of integrable functions. Let R^n be an n-dimensional euclidean space with the norm given by putting $|x|=(x_1^2+\ldots+x_n^2)^{1/2}$, where $x\in R^n$, $x=(x_1,\ldots,x_n)$. Let U be an open bounded domain in R^n and let $\mathscr{L}^p(U;R^k)$, $1\leqslant p<\infty$, be the Banach space of all Lebesgue measurable functions (equivalence classes) $w\colon U\to R^k$ for which $\int_U |w(x)|^p\,dx < \infty$, with the norm

$$\|w\|_p = \Big(\int_U |w(x)|^p\,dx\Big)^{1/p}.$$

Recall that the space $\mathscr{L}^p(U;R^1)$ is reflexive iff $1<p<\infty$.

From the above and Theorem (2.3) we obtain the following two facts.

(3.1) Proposition. *Let a real number* p *satisfy* $1<p<\infty$. *Then the space* $\mathscr{L}^p(U;R^k)$ *is reflexive.*

(3.2) Proposition. *Let a real number* p *satisfy* $1<p<\infty$. *Then every bounded subset of* $\mathscr{L}^p(U;R^k)$ *is weakly relatively compact.*

We note the following fact.

(3.3) Proposition [28]. *If a sequence* $\{w_n\}$ *in* $\mathscr{L}^p(U;R^k)$ *is strongly convergent to a function* $w\in\mathscr{L}^p(U;R^k)$, *then there exists a subsequence* $\{w_{n(j)}\}$ *which is convergent almost everywhere to* w, *i.e.* $\lim_j w_{n(j)}(x)=w(x)$ *a.e. on* U.

The following theorem gives a characterization of weak compactness in $\mathscr{L}^1(U;R^1)$.

(3.4) (Dunford–Pettis Theorem, [29]). *A set* $X\subset\mathscr{L}^1(U;R^1)$ *is relatively* *weakly compact if and only if the following three conditions are satisfied*:

(i) *the set* X *is bounded in* $\mathscr{L}^1(U; R^1)$;

(ii) *for every number* $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that if* $w \in X$ *and* A *is a measurable subset of* U *with the Lebesgue measure* $\mu(A) < \delta$, *then*

$$\int_A |w(x)|\,dx < \varepsilon;$$

(iii) *for every number* $\varepsilon > 0$ *there exists a compact subset* K *of* U *such that if* $w \in X$, *then*

$$\int_{U \setminus K} |w(x)|\,dx < \varepsilon.$$

In what follows we will make use of the absolute continuity of the integral with respect to measure.

(3.5) THEOREM [27]. *Let* $w\colon U \to R^k$ *be an integrable function. Then for every number* $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that if* A *is a measurable subset of* U *with the Lebesgue measure* $\mu(A) < \delta$ *then* $\int_A |w(x)|\,dx < \varepsilon$.

A subset X of the space $\mathscr{L}^1(U; R^k)$ is called *integrably bounded* if there exists a function $m \in \mathscr{L}^1(U; R)$ such that for every $w \in X$, $|w(x)| \leqslant m(x)$ for each $x \in U$.

(3.6) PROPOSITION. *Any integrably bounded subset of* $\mathscr{L}^1(U; R^k)$ *is weakly relatively compact.*

Proof. Let us notice that every function $w \in \mathscr{L}^1(U; R^k)$ has the form $w = (w_1, \ldots, w_k)$, where $w_i \in \mathscr{L}^1(U; R^1)$, $(i = 1, \ldots, k)$. Let $\pi_i\colon \mathscr{L}^1(U; R^k) \to \mathscr{L}^1(U; R^k)$, $(i = 1, \ldots, k)$, be a linear continuous mapping given by the following condition:

$\pi_i(w) = (0, \ldots, w_i, 0, \ldots, 0)$ is the element whose i-th component is w_i, and all other components are zero.

Moreover, let T_i: $\operatorname{Im} \pi_i \to \mathscr{L}^1(U; R^1)$ be an isomorphism such that $T_i \circ \pi_i(w) = w_i$ $(i = 1, \ldots, k)$ and let Y be an integrably bounded subset of $\mathscr{L}^1(U; R^k)$.

We can see that the image $T_i \circ \pi_i(Y)$ of Y under $T_i \circ \pi_i$ is a bounded subset of $\mathscr{L}^1(U; R^1)$ and, moreover, in virtue of (3.5), satisfies conditions (ii)–(iii) of (3.4). Therefore, by the Dunford–Pettis Theorem, the set $T_i \circ \pi_i(Y)$, $i = 1, 2, \ldots, k$, is relatively weakly compact in $\mathscr{L}^1(U; R^1)$. Since $Y \subset \pi_1(Y) + \ldots + \pi_k(Y)$, it follows from (2.1) that Y is relatively weakly compact in $\mathscr{L}^1(U; R^k)$.

The proof is complete.

4. Compact sets in the space of continuous functions. Let U be a bounded open domain in R^n and let $(C(\bar{U}; R^k), |\cdot|_0)$ be the Banach space of all continuous mappings from $\bar{U}$ into R^k with the sup norm, $|\cdot|_0$.

The following theorem gives a characterization of compactness in $C(\bar{U}; R^k)$.

(4.1) (ARZELA–ASCOLI THEOREM, [28]). *A set X in $C(\bar{U}; R^k)$ is relatively compact if and only if the following two conditions are satisfied:*

(i) *the set X is bounded in $C(\bar{U}; R^k)$;*

(ii) *for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that*

$$\sup_{u \in X} \sup \{|u(x)-u(\bar{x})|: x, \bar{x} \in \bar{U} \text{ and } |x-\bar{x}| < \delta\} < \varepsilon.$$

From (3.5) and (4.1) we obtain the following two facts.

(4.2) PROPOSITION. *Let $[0, a]$ be a compact real interval, let $T\colon \mathscr{L}^1((0, a); R^k) \to C([0, a]; R^k)$ be a linear continuous mapping given by*

$$T(w)(t) = \int_0^t w(\tau)\,d\tau \quad \text{for each } t \in [0, a]$$

and let Y be an integrably bounded subset of $\mathscr{L}^1((0, a); R^k)$. Then the image $T(Y)$ of Y under T is a relatively compact subset of $C([0, a]; R^k)$.

(4.3) PROPOSITION. *Let $\Delta = [0, a] \times [0, a]$, let $T\colon \mathscr{L}^1(\Delta; R^k) \to C(\Delta; R^k)$ be a linear continuous mapping given by*

$$T(w)(s, t) = \int_0^s \int_0^t w(\tau, \eta)\,d\tau\,d\eta \quad \text{for each } s, t \in [0, a]$$

and let Y be an integrably bounded subset of $\mathscr{L}^1(\Delta; R^k)$. Then the image $T(Y)$ of Y under T is a relatively compact subset of $C(\Delta; R^k)$.

5. Basic integral and differential inequalities. A function $f\colon [0, a] \to R^1$ is called *absolutely continuous* provided there exists an integrable function $g\colon [0, a] \to R^1$ such that

$$f(t) = f(0) + \int_0^t g(\tau)\,d\tau \quad \text{for every } t \in [0, a].$$

Note that the absolutely continuous function $f\colon [0, a] \to R^1$ is differentiable almost everywhere (written a.e.) on $[0, a]$ and $\frac{df}{dt} = g$.

Let $x\colon [0, a] \to R^k$ be a function of the form $x(t) = (x_1(t), x_2(t), \ldots, x_k(t))$ for each $t \in [0, a]$, where $x_i\colon [0, a] \to R^1$ ($i = 1, 2, \ldots, k$). The function $x\colon [0, a] \to R^k$ is called *absolutely continuous* provided that for every $i = 1, 2, \ldots, k$, $x_i\colon [0, a] \to R^1$ is absolutely continuous.

In what follows for any absolutely continuous function $x\colon [0, a]$

$\to R^k$, $x = (x_1, x_2, \ldots, x_k)$, and for an integrable function $y\colon [0, a] \to R^k$, $y = (y_1, y_2, \ldots, y_k)$, we will use the following notation:

$$x'(t) = \left(\frac{dx_1}{dt}(t), \ldots, \frac{dx_k}{dt}(t)\right) \quad \text{for almost every } t \in [0, a]$$

and

$$\int_0^t y(\tau)\,d\tau = \left(\int_0^t y_1(\tau)\,d\tau, \ldots, \int_0^t y_k(\tau)\,d\tau\right) \quad \text{for each } t \in [0, a].$$

Finally, by R_+ we will denote the set of all non-negative real numbers.

(5.1) (Gronwall inequality, [47]). *Let $p\colon [0, a] \to R_+$ be a continuous function and let $q\colon [0, a] \to R_+$ be an integrable function. If there exists a non-negative real number M such that*

$$p(t) \leqslant M + \int_0^t q(\tau)\cdot p(\tau)\,d\tau \quad \textit{for every } t \in [0, a],$$

then

$$p(t) \leqslant M \cdot \exp\left[\int_0^t q(\tau)\,d\tau\right] \quad \textit{for every } t \in [0, a].$$

From (5.1) we obtain the following

(5.2) Proposition. *Let $q\colon [0, a] \to R_+$ be an integrable function and let $x\colon [0, a] \to R^k$ be an absolutely continuous function such that*

$$|x'(t)| \leqslant q(t)\cdot|x(t)| \quad \textit{a.e. on } [0, a],$$

$$x(t_0) = 0 \quad \textit{for some } t_0 \in [0, a].$$

Then $x(t) = 0$ for every $t \in [0, a]$.

(5.3) (Lasota, Olech; [64]). *Let $q\colon [0, a] \to R_+$ be an integrable function such that $\int_0^a q(t)\,dt < \pi/2$, let $t_1, t_2, \ldots, t_k$ be real numbers such that $0 \leqslant t_1 < t_2 < \ldots < t_k \leqslant a$ and let $x\colon [0, a] \to R^k$, $x = (x_1, x_2, \ldots, x_k)$ be an absolutely continuous function satisfying the system*

$$|x'(t)| \leqslant q(t)\cdot|x(t)| \quad \textit{a.e. on } [0, a],$$

$$x_i(t_i) = 0 \quad \textit{for } i = 1, 2, \ldots, k.$$

Then $x(t) = 0$ for every $t \in [0, a]$.

(5.4) (Kasprzyk, Myjak; [54]). *Let λ be a positive real number, let $q\colon [0, a] \to R_+$ be an integrable function such that*

$$\int_0^a q(t)\,dt < \sqrt{\pi^2 + \log^2 \lambda}$$

and let $x\colon [0, a] \to R^k$ *be an absolutely continuous function such that*

$$|x'(t)| \leqslant q(t)\cdot|x(t)| \quad \text{a.e. on } [0, a],$$

$$x(0)+\lambda\cdot x(a) = 0.$$

Then $x(t) = 0$ *for every* $t\in[0, a]$.

The following fact is well known; see for instance, [89].

(5.5) *Let* $q\colon \Delta \to R_+$ *be an integrable function and let* $u\colon \Delta \to R^k$ *be a continuous function such that the derivative* u_{xy} *is an integrable function and*

$$|u_{xy}(x, y)| \leqslant q(x, y)\cdot|u(x, y)| \quad \text{a.e. on } \Delta,$$

$$u(x, 0) = 0 \text{ and } u(0, y) = 0 \quad \text{for all } x, y\in[0, a].$$

Then $u(x, y) = 0$ *for every* $(x, y)\in\Delta$.

II. Multi-valued mappings

In this chapter we present the well-known basic properties of multi-valued completely continuous mappings and the basic properties of the Leray–Schauder degree for convex-valued vector fields. Moreover, we specify some classes of multi-valued mappings from $[0, a]\times R^n$ into R^k, for which there exist convex-valued weakly compact selectors.

We also give the definition and basic properties of L–compact (L–completely continuous) multi-valued mappings which play a significant role in the application of the Leray–Schauder degree to boundary value problems.

In what follows E_1 and E_2 will denote two Banach spaces and $X \subset E_1$ will denote a closed non-empty subset of E_1.

1. Upper semi-continuous, compact and weakly compact mappings. Let $\varphi\colon X \to E_2$ be a multi-valued mapping. For any $A \subset X$, the set $\varphi(A) = \bigcup_{x\in A} \varphi(x)$ is called the *image of A under* φ. In particular, Im φ denotes the image $\varphi(X)$ of X under φ and is called the *image of* φ. For any $B \subset E_2$ the set $\varphi^{-1}(B) = \{x\in X\colon \varphi(x) \subset B\}$ is called a *counter image of B under* φ.

In what follows the symbols $\varphi, \psi, \chi, \Phi, \Psi, F, H$ will be reserved for multi-valued mappings; singe-valued mappings will be denoted by $f, g, h, l, m, p, q, L, T$.

A multi-valued mapping $\varphi\colon X \to E_2$ is called *upper semi-continuous* (u.s.c.) provided the following two conditions are satisfied:

(i) $\varphi(x)$ is compact for each $x\in X$,

(ii) for each open set $U \subset E_2$ the counter-image $\varphi^{-1}(U)$ of U under φ is an open subset in X.

It is clear now (see for instance [71]) that a convex-valued mapping $\varphi\colon X \to E_2$ is u.s.c. iff for each point $x \in X$ and for each open convex subset U of E_2 containing $\varphi(x)$, there exists an open ball $B_x \subset E_1$, with the centre in x, such that $\varphi(B_x \cap X) \subset U$.

In what follows we will always consider multi-valued mappings with compact values.

The following facts are well known (see, for instance, [37, 71]

(1.1) *Let $\varphi\colon X \to E_2$ be a u.s.c. multi-valued mapping. Then:*

(i) *the graph*

$$\Gamma_\varphi = \{(x, y) \in X \times E_2\colon\ y \in \varphi(x)\}$$

is a closed subset of $X \times E_2$,

(ii) *if A is a compact subset of X, then the image $\varphi(A)$ of A under φ is compact.*

(1.2) *Let $\varphi\colon X \to E_2$ be a multi-valued mapping such that for every bounded subset $B \subset X$, $\overline{\varphi(B)}$ is a compact set. Then φ is a u.s.c. mapping iff the graph Γ_φ is closed subset of $X \times E_2$.*

A u.s.c. multi-valued mapping $\varphi\colon X \to E_2$ is called *compact* provided the image $\operatorname{Im} \varphi$ of φ is a relatively compact subset of E_2.

A u.s.c. multi-valued mapping $\varphi\colon X \to E_2$ is called *completely continuous* provided for every bounded subset $B \subset X$ the image $\varphi(B)$ of B under φ is a relatively compact subset of E_2.

Let E be a Banach space and let $\varphi\colon X \to E_2$ and $\psi\colon E_2 \to E$ be two multi-valued mappings. Then the composition $\psi \circ \varphi\colon X \to E$ of φ and ψ is defined by $\psi \circ \varphi(x) = \psi(\varphi(x))$ for all $x \in X$.

We will need the following properties of the completely continuous mappings.

(1.3) *Let $\varphi\colon X \to E_2$ and $\psi\colon E_2 \to E$ be two u.s.c. mappings. If φ or ψ is a completely continuous mapping, then the composition $\psi \circ \varphi$ of φ and ψ is completely continuous.*

(1.4) *Let $\varphi\colon X \to E_2$ and $\psi\colon X \to E_2$ be two completely continuous mappings. Then the mapping $(\varphi + \psi)\colon X \to E_2$ given by*

$$(\varphi+\psi)(x) = \{y+z\colon\ y \in \varphi(x) \text{ and } z \in \psi(x)\}$$

is completely continuous.

(1.5) *Let $\varphi\colon X \to E_2$ be a completely continuous multi-valued mapping and let $m\colon X \to R^1$ be a continuous single-valued mapping. Then the mapping $m \cdot \varphi\colon X \to E_2$ given by*

$$(m \cdot \varphi)(x) = \{m(x) \cdot y\colon\ y \in \varphi(x)\}$$

is completely continuous.

The above facts (1.3)–(1.5) for compact multi-valued mappings are well known (see, for instance, [37, 71, 91]), whereas for completely continuous mappings they are an immediate consequence of the definition of completely continuous mappings.

A multi-valued mapping $\varphi\colon X \to E_2$ is called *weakly upper semi-continuous* (w-u.s.c.) provided for all sequences $\{x_n\} \subset X$ and $\{y_n\} \subset E_2$ the conditions $\{x_n\} \to x$, $\{y_n\} \underset{w}{\to} y$ and $y_n \in \varphi(x_n)$, for every n, imply $y \in \varphi(x)$.

(1.6) DEFINITION. Let $\varphi\colon X \to E_2$ be a w-u.s.c. multi-valued mapping.

(i) The mapping φ is called *weakly compact* provided $\varphi(X)$ is a relatively weakly compact subset of E_2.

(ii) The mapping φ is called *weakly completely continuous* provided that, for every bounded subset B of X, $\varphi(B)$ is a relatively weakly compact subset of E_2.

The following fact is a consequence of the above definition.

(1.7.) PROPOSITION. *Let $\varphi\colon X \to E_2$ be a weakly completely continuous multi-valued mapping and let $T\colon E_2 \to E$ be a continuous linear mapping from E_2 into a space E. Then the graph $\Gamma_{T\circ\varphi}$ of the mapping $T\circ\varphi\colon X \to E$ is a closed subset of $X \times E$.*

Now, by (1.2) we obtain

(1.8) PROPOSITION. *Let $\varphi\colon X \to E_2$ be a weakly compact multi-valued mapping and let $T\colon E_2 \to E$ be a linear continuous mapping from E_2 into a space E such that $\overline{\operatorname{Im} T\circ\varphi}$ is a compact subset of E. Then the composition $T\circ\varphi\colon X \to E$ of φ and T is compact.*

2. L-compact mappings. Let dom L be a linear subspace of E_1 and let L: dom $L \to E_2$ be a linear (not necessarily continuous) mapping such that $\operatorname{Im} L = E_2$ and $\operatorname{Ker} L$ is a finite dimensional space.

We will need the following algebraic facts.

(2.1) PROPOSITION [34]. *Let the mapping L: dom $L \to E_2$ be as above. Then*:

(2.1.1) *There exists a continuous linear projection $P\colon E_1 \to E_1$ such that* $\operatorname{Im} P = \operatorname{Ker} L$.

(2.1.2) *If $P_1, P_2\colon E_1 \to E_1$ are two linear projections onto* Ker L, *then for each $s \in [0, 1]$ the mapping $s\cdot P_1 + (1-s)\cdot P_2$ is also a linear projection from E_1 onto* Ker L.

(2.1.3) *With a linear projection $P\colon E_1 \to E_1$ onto* Ker L *there corresponds a right inverse $T_P\colon E_2 \to E_1$ given as follows*

$$T_P(y) = x \quad \textit{iff} \quad P(x) = 0 \textit{ and } L(x) = y.$$

(2.1.4) *Let $P_1, P_2\colon E_1 \to E_1$ be two linear projections onto* Ker L *and let $P(s) = s\cdot P_1 + (1-s)\cdot P_2$ for $s \in [0, 1]$.*

Then

(i) $T_{P_1} \circ L(x) = x - P_1(x)$ *for each* $x \in \operatorname{dom} L$,

(ii) $T_{P_2}(y) = T_{P_1}(y) - P_2 \circ T_{P_1}(y)$ *for each* $y \in E_2$,

(iii) $T_{P(s)} = s \cdot T_{P_1} + (1-s) \cdot T_{P_2}$.

(2.2) Definition ([74], [91]). *Let the mapping* $L\colon \operatorname{dom} L \to E_2$ *be as above, let* $T\colon E_2 \to E_1$ *be a right inverse to* L *and let* X *be a closed subset of* E_1.

(i) A convex-valued mapping $\varphi\colon X \to E_2$ is called *L-compact* (*L-completely continuous*) provided the composition $T \circ \varphi\colon X \to E_1$ of φ and T is a compact (a completely continuous) mapping.

(ii) A convex-valued mapping $\varphi\colon X \to E_2$ is called *L-bounded* provided the image $\operatorname{Im} T \circ \varphi$ of $T \circ \varphi$ is a bounded subset of E_1.

It follows from (2.1.4) and (1.3)–(1.4) that the above definition does not depend upon the choice of the right inverse T.

From the above definition we obtain

(2.3) Proposition. *If* $f, g\colon X \to R^1$ *are two single-valued mappings and* $\varphi, \psi\colon X \to E_2$ *are two L-compact (L-completely continuous) mappings, then the mapping* $(f \cdot \varphi + g \cdot \psi)\colon X \to E_2$ *is also L-compact (L-completely continuous).*

The following fact immediately follows from (2.1.4 (i)).

(2.4) Proposition. *Let* $\varphi\colon X \to E_2$ *be an L-completely continuous mapping, let* $P\colon E_1 \to E_1$ *be a linear continuous projection onto* $\operatorname{Ker} L$ *and let* $T\colon E_2 \to E_1$ *be a right inverse to* L *such that* $\operatorname{Im} P \circ T = 0$. *Then* $L(x) \in \varphi(x)$ *iff the point* $x \in \operatorname{dom} L$ *is a fixed point of the convex-valued completely continuous mapping*

$$(P + T \circ \varphi)\colon X \to E_1.$$

(2.5) Theorem. *Let* X *be a closed subset of* E_1 *and let* $\varphi\colon X \to E_2$ *be an L-compact mapping. Then there exists an L-compact mapping* $\tilde{\varphi}\colon E_1 \to E_2$ *such that*

$$\tilde{\varphi}(x) = \varphi(x) \quad \text{for every } x \in X \text{ and } \operatorname{Im} \tilde{\varphi} \subset \operatorname{conv} \varphi(X).$$

Proof. Let us put, for every $y \in E_1 \setminus X$,

$$\varrho(y, X) = \inf\{\|y - x\|\colon x \in X\}.$$

Let $r\colon E_1 \setminus X \to R_+$ be a real function satisfying

$$0 < r(y) < \tfrac{1}{2}\varrho(y, X) \quad \text{for every } y \in E_1 \setminus X$$

and let $B_r(y)$ be an open ball with centre y and radius $r(y)$.

Since the Banach space E_1 is paracompact, there exists a locally finite refinement $\{Q_\alpha\}_{\alpha \in A}$ of $E_1 \setminus X$ such that:

for every $\alpha \in A$ there exists a point $y \in E_1 \setminus X$ such that $Q_\alpha \subset B_r(y)$.

Let us define the functions q_α: $E_1 \setminus X \to R_+$ and p_α: $E_1 \setminus X \to [0, 1]$, for $\alpha \in A$, by the following formulas:

$$q_\alpha(x) = \begin{cases} 0 & \text{if } x \notin Q_\alpha, \\ \varrho(x, \partial Q_\alpha) & \text{if } x \in Q_\alpha \ (\partial Q_\alpha \text{ denotes the boundary of } Q_\alpha \text{ in } E_1) \end{cases}$$

and

$$p_\alpha(x) = q_\alpha(x) \cdot [\sum_{\beta \in A} q_\beta(x)]^{-1}.$$

Since $\{Q_\alpha\}_{\alpha \in A}$ is locally finite, each p_α is correctly defined. Now, for every $\alpha \in A$ let us choose a couple of the points $y_\alpha \in Q_\alpha$ and $x_\alpha \in X$ such that $\|y_\alpha - x_\alpha\| < 2 \cdot \varrho(y_\alpha, X)$. Define $\tilde{\varphi}$: $E_1 \to E_2$ by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in X, \\ \sum_{\alpha \in A} p_\alpha(x) \cdot \varphi(x_\alpha) & \text{if } x \in E_1 \setminus X. \end{cases}$$

We see that $\operatorname{Im} \tilde{\varphi} \subset \operatorname{conv} \varphi(X)$ and therefore, for a right inverse T: $E_2 \to E_1$ to L, $\overline{T \circ \tilde{\varphi}(E_1)}$ is a compact subset of E_1. We will show that $T \circ \tilde{\varphi}$: $E_1 \to E_1$ is a u.s.c. mapping.

1° Let $u \in E_1 \setminus X$ and let $\{Q_{\alpha_1}, \ldots, Q_{\alpha_n}\}$ be a finite family of all sets from $\{Q_\alpha\}_{\alpha \in A}$, containing the point u. Then for every $x \in X$ we have

$$(*) \qquad \|x - x_{\alpha_i}\| \leqslant 9 \cdot \|u - x\| \quad \text{for} \quad i = 1, 2, \ldots, n.$$

In fact,

$$\|x - x_{\alpha_i}\| \leqslant \|x - y_{\alpha_i}\| + \|y_{\alpha_i} - x_{\alpha_i}\| \leqslant 3 \cdot \|x - y_{\alpha_i}\|$$
$$\leqslant 3 \cdot [\|x - u\| + \|u - y_{\alpha_i}\|] \leqslant 9 \cdot \|x - u\|,$$

because for $i = 1, 2, \ldots, n$ there exists a $y_i \in E_1 \setminus X$ such that $Q_{\alpha_i} \subset B_r(y_i)$ and therefore

$$\|u - y_{\alpha_i}\| \leqslant 2r(y_i) \leqslant 2[\varrho(y_i, X) - r(y_i)] \leqslant 2[\|y_i - x\| - \|y_i - u\|] \leqslant 2\|x - u\|.$$

2° Let $x \in X$ and let $U \subset E_1$ be an open convex set containing $T \circ \varphi(x)$. Since $T \circ \varphi$ is a convex-valued u.s.c. mapping on X, there exists a ball $B_\tau(x)$ such that $T \circ \varphi[B_\tau(x) \cap X] \subset U$. Hence, by $(*)$, we get

$$T \circ \tilde{\varphi}(B_\eta(x)) \subset U \quad \text{for} \quad \eta = \tfrac{1}{9}\tau.$$

3° On the other hand, if $x \in E_1 \setminus X$, then there exists an open neighbourhood $U_x \subset E_1 \setminus X$ of x, such that only a finite number of sets Q_α, $\alpha \in A$, convers U_x. By (2.3) the mapping ψ: $U_x \to E_2$, given by $\psi(y) = \tilde{\varphi}(y)$ for $y \in U_x$, is L-compact. Therefore, for the point x and a convex set $U \subset E_1$ containing $T \circ \tilde{\varphi}(x)$, there exists a ball $B_\eta(x) \subset E_1 \setminus X$ such that $T \circ \tilde{\varphi}(B_\eta(x)) \subset U$.

The proof is completed.

3. Carathéodory conditions for convex-valued mappings. By $\mathscr{B}(R^n)$ we will denote the family of all non-empty, compact subsets of the n-dimensional Euclidean space R^n. For two sets $A, B \in \mathscr{B}(R^n)$ we will denote by $d(A, B)$ the Hausdorff distance between A and B. Recall that $(\mathscr{B}(R^n), d)$ is a metric space. In particular, we put $|A| = d(A, \{0\})$.

Let $A \in \mathscr{B}(R^n)$, let $F\colon A \times R^l \to R^k$ be a multi-valued mapping and let (x, u) be a point in $A \times R^l$. By $F(x, \cdot)$ we will denote a multi-valued mapping $F_x\colon R^l \to R^k$ given by $F_x(v) = F(x, v)$ for each $v \in R^l$ and by $F(\cdot, u)$ we will denote a multi-valued mapping $F_u\colon A \to R^k$ given by $F_u(y) = F(y, u)$ for each $y \in A$.

(3.1) DEFINITION. Let $A \subset \mathscr{B}(R^n)$. A multi-valued mapping $H\colon A \to R^n$ is called *measurable* provided that, for every open set $U \subset R^k$, the set $\{x \in R^n \cap A\colon H(x) \cap U \neq \emptyset\}$ is Lebesgue measurable.

(3.2) DEFINITION. Let $A \subset \mathscr{B}(R^n)$. We say that a convex-valued mapping $F\colon A \times R^l \to R^k$ *satisfies the Carathéodory conditions* if:

(c_1) for each $u \in R^l$ the mapping $F(\cdot, u)$ is measurable;

(c_2) for each $x \in A$ the mapping $F(x, \cdot)$ is u.s.c.

The following two facts are well known (see, for instance, [18], [84]).

(3.3) PROPOSITION. *Let $A \subset \mathscr{B}(R^n)$ and let a multi-valued mapping $H\colon A \to R^k$ be measurable. Then there exists a measurable single-valued mapping $h\colon A \to R^k$ such that $h(x) \in H(x)$ a.e. on A.*

(3.4) PROPOSITION. *Let $A \subset \mathscr{B}(R^n)$, let $u\colon A \to R^l$ be a single-valued continuous mapping and let $F\colon A \times R^l \to R^k$ be a convex-valued mapping satisfying the Carathéodory conditions* (comp. (3.2)). *Then the convex-valued mapping $H\colon A \to R^k$ given by*

$$H(x) = F(x, u(x)) \quad \textit{for each } x \in A$$

is measurable.

The following fact is fundamental for applications.

(3.5) PROPOSITION [62, 92]. *Let $U \subset R^n$ be a bounded open domain and let a convex-valued mapping $F\colon \bar{U} \times R^l \to R^k$ satisfy the following*:

(i) *F satisfies the Carathéodory conditions* (comp. (3.2)),

(ii) *for every bounded domain $B \subset R^l$ there exists a function $m_B \in \mathscr{L}^p(B; R^1)$ such that*

$$|F(x, u)| \leqslant m_B(x) \quad \textit{for all } x \in \bar{U} \textit{ and } u \in B.$$

Then the mapping $\varphi_F\colon C(\bar{U}; R^l) \to \mathscr{L}^p(\bar{U}; R^k)$ given by

(3.5.1) $\quad \varphi_F(u) = \{w \in \mathscr{L}^p(\bar{U}; R^k)\colon w(x) \in F(x, u(x)) \text{ a.e. on } \bar{U}\}$

is weakly completely continuous.

(3.6) PROPOSITION. *Let* $U \subset R^n$ *be a bounded open domain, let* $F\colon \bar{U} \times R^i \to R^k$ *be a convex-valued mapping satisfying conditions* (3.5) (i)–(ii) *and let* $\varphi_F\colon C(\bar{U}; R^i) \to \mathscr{L}^p(\bar{U}; R^k)$ *be a convex-valued mapping given in* (3.5.1) *for F. Assume moreover that* $T_1\colon E_1 \to C(\bar{U}; R^i)$ *and* $T_2\colon \mathscr{L}^p(\bar{U}; R^k) \to E_2$ *are two linear continuous mappings. Then, if for each bounded subset* $Y \subset C(\bar{U}; R^i)$ *the set* $T_2 \circ \varphi_F(Y)$ *is a relatively compact subset of* E_2, $T_2 \circ \varphi_F \circ T_1\colon E_1 \to E_2$ *is a completely continuous convex-valued mapping from the space* E_1 *into the space* E_2.

Proof. It follows from (1.7) that the graph $\Gamma_{(T_2 \circ \varphi_F \circ T_1)}$ is a closed subset of $E_1 \times E_2$. On the other hand, by our assumptions, for every bounded set $X \subset E_1$, the set $T_2 \circ \varphi_F \circ T_1(X)$ is relatively compact. Therefore, from (1.2) we obtain (3.6).

Now, in virtue of (I.4.2) and (I.4.3) respectively, the following two facts are an immediate consequence of (3.6).

(3.7) PROPOSITION. *Let* $F\colon [0, a] \times R^i \to R^k$ *be a convex-valued mapping satisfying the following:*

(i) *F satisfies the Carathéodory conditions* (comp. (3.2));

(ii) *there exist two integrable functions* $p, q\colon [0, a] \to R_+$ *such that*

$$|F(t, x)| \leqslant p(t) + q(t) \cdot |x| \quad \textit{for all } t \in [0, a] \textit{ and } x \in R^i.$$

Assume, moreover, that the convex-valued mapping $\varphi_F\colon C([0, a]; R^i) \to \mathscr{L}^1((0, a); R^k)$ *is given in* (3.5.1) *for F and the linear continuous mapping* $T\colon \mathscr{L}^1((0, a); R^k) \to C([0, a]; R^k)$ *is given by*

$$T(w)(t) = \int_0^t w(s)\,ds \quad \textit{for} \quad t \in [0, a].$$

Then the composition $T \circ \varphi_F\colon C([0, a]; R^i) \to C([0, a]; R^k)$ *of* φ_F *and T is completely continuous.*

(3.8) PROPOSITION. *Let* $\Delta = [0, a] \times [0, a]$ *and let* $F\colon \Delta \times R^i \to R^k$ *be a convex-valued mapping satisfying the following:*

(i) *F satisfies the Carathéodory conditions* (comp. (3.2));

(ii) *there exist two integrable functions* $p, q\colon \Delta \to R_+$ *such that*

$$|F(x, y, u)| \leqslant p(x, y) + q(x, y) \cdot |u| \quad \textit{for all } (x, y) \in \Delta \textit{ and } u \in R^i.$$

Assume, moreover, that the convex-valued mapping $\varphi_F\colon C(\Delta; R^i) \to \mathscr{L}^1(\Delta; R^k)$ *is given in* (3.5.1) *for F and the linear continuous mapping* $T\colon \mathscr{L}^1(\Delta; R^k) \to C(\Delta; R^k)$ *is given by*

$$T(w)(x, y) = \int_0^x \int_0^y w(s, t)\,ds\,dt \quad \textit{for} \quad (x, y) \in \Delta.$$

Then the composition $T \circ \varphi_F\colon C(\Delta; R^i) \to C(\Delta; R^k)$ *of* φ_F *and T is completely continuous.*

4. Convex-valued, weakly compact selectors. We start with the following definition.

(4.1) DEFINITION. Let $U \subset R^l$ be a bounded open domain. We say that a multi-valued mapping $F\colon \bar{U} \times R^k \to R^n$ *admits a convex-valued, weakly compact selector* provided that for every compact set $X \subset C(\bar{U}; R^k)$ there exists a convex-valued and weakly compact mapping $\varphi\colon X \to \mathcal{L}^1(\bar{U}; R^n)$ such that for each function $u \in X$ the following inclusion holds

(4.1.1) $\{y(t)\colon y \in \varphi(x)\} \subset F(t, x(t))$ for almost every $t \in \bar{U}$.

A multi-valued mapping $F\colon [0, a] \times R^k \to R^n$ is called *integrably bounded* if there exists an integrable function $m\colon [0, a] \to R_+$ such that

$$|F(t, x)| \leqslant m(t) \quad \text{for all } (t, x) \in [0, a] \times R^k.$$

The following fact immediately follows from the construction given by Antosiewicz and Cellina (see [4], Th. 2, p. 391 and (i)–(ii), p. 392).

(4.2) PROPOSITION. *Suppose some multi-valued integrably bounded mapping $F\colon [0, a] \times R^k \to R^n$ satisfies the following conditions:*

(i) *for every $t \in [0, a]$, $F(t, \cdot)$ is a continuous mapping from R^k into $(\mathcal{B}(R^n), d)$;*

(ii) *for every $x \in R^k$, $F(\cdot, x)$ is a measurable mapping from $[0, a]$ into R^n.*

Then the mapping F admits a single-valued, weakly compact selector.

The following fact is an immediate consequence of (3.5).

(4.3) PROPOSITION. *Suppose some convex-valued integrably bounded mapping $F\colon [0, a] \times R^k \to R^n$ satisfies the following Carathéodory conditions:*

(i) *for every $t \in [0, a]$, $F(t, \cdot)$ is a u.s.c. mapping from R^k into R^n;*

(ii) *for every $x \in R^k$, $F(\cdot, x)$ is a measurable mapping from $[0, a]$ into R^n.*

Then the mapping F admits a convex-valued, weakly compact selector.

Let us put, for $A \in \mathcal{B}(R^k)$, $x \in R^k$ *and a real number* $\varepsilon > 0$,

$$\varrho(x, A) = \inf\{|x-y|\colon y \in A\} \quad \text{and} \quad O_\varepsilon(A) = \{u \in R^k\colon \varrho(u, A) \leqslant \varepsilon\}.$$

A multi-valued mapping $F\colon [0, a] \times R^k \to R^n$ is called *lower semicontinuous* if for every point $(t_0, x_0) \in [0, a] \times R^k$ and for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$F(t_0, x_0) \subset O_\varepsilon(F(t, x)) \quad \text{provided} \quad |(t_0, x_0) - (t, x)| < \delta.$$

The following fact immediately follows from the construction given by Bressan [10] and Łojasiewicz (Jr) [70].

(4.4) PROPOSITION. *If a multi-valued integrably bounded mapping $F\colon [0, a] \times R^k \to R^n$ is lower semicontinuous, then the mapping F admits a single-valued, weakly compact selector.*

5. Compact convex-valued vector fields. Let U be an open neighbourhood of zero in a Banach space E and let ∂U denote the boundary of U in E. In what follows we will denote by I the identity mapping on $\bar{U}$.

A multi-valued mapping $\Phi\colon \bar{U} \to E$ is called a *convex-valued vector field on the set* $\bar{U}$ provided the following two conditions are satisfied:

(i) there exists a compact convex-valued mapping $\varphi\colon \bar{U} \to E$ such that

$$\Phi = I - \varphi;$$

(ii) $\Phi(\partial U) \subset E \setminus \{0\}$.

Two convex-valued vector fields $\Phi_1, \Phi_2\colon \bar{U} \to E$ are called *homotopic* (written $\Phi_1 \sim \Phi_2$), provided there exists a compact convex-valued mapping $\Psi\colon [0, 1] \times \bar{U} \to E$ such that the following two conditions are satisfied:

(iii) for every $t \in [0, 1]$ and $x \in \partial U$, $[x - \Psi(t, x)] \subset E \setminus \{0\}$;

(iv) $\Phi_1(x) = x - \Psi(0, x)$ and $\Phi_2(x) = x - \Psi(1, x)$ for $x \in \bar{U}$.

If $\Phi\colon \bar{U} \to E$ is a convex-valued vector field, there is an integer defined, called the *Leray–Schauder degree of Φ on U* and written $\deg(\Phi, U, 0)$. For the definition of the Leray–Schauder degree for compact convex-valued vector fields and for full statements of the topological results, see [10, 22, 40, 41, 69, 71].

For our purposes we will need the following properties of the topological degree $\deg(\Phi, U, 0)$.

(5.1) *Let $\Phi = I - \varphi$ be a convex-valued vector field on $\bar{U}$ such that* $\deg(I - \varphi, U, 0) \neq 0$. *Then, φ has a fixed point.*

(5.2) *If Φ_1 and Φ_2 are two convex-valued vector fields on $\bar{U}$ and $\Phi_1 \sim \Phi_2$, then* $\deg(\Phi_1, U, 0) = \deg(\Phi_2, U, 0)$.

(5.3) *If U is a symmetric neighbourhood of the origin, and Φ is a convex-valued vector field on $\bar{U}$ such that*

$$\Phi(-x) = -\Phi(x) \quad \textit{for every } x \in U,$$

then $\deg(\Phi, U, 0)$ *is odd.*

(5.4) *Let $\Phi = I - \varphi$ be a convex-valued vector field on $\bar{U}$ and let E_1 be a linear subspace of E such that $\overline{\varphi(\bar{U})} \subset E_1$, then* $\deg(I - \varphi, U, 0) = \deg(I - \varphi_{|\overline{U \cap E_1}}, U \cap E_1, 0)$.

III. Multi-valued boundary value problems

In this chapter we shall give some applications of the Leray–Schauder degree to convex-valued boundary value problems.

It will be divided into two parts. The first part is devoted to the notion and properties of the topological degree of admissible boundary value problems. The second part is devoted to the existence theorems for admissible boundary value problems.

In what follows we will denote by E_1 and E_2 two Banach spaces and by $B \subset E_1$ an open ball with the centre at the zero point of E_1 and some radius r.

1. The degree of the boundary value problem. Let $\operatorname{dom} L$ be a linear subspace of E_1 and let $L: \operatorname{dom} L \to E_2$ be a linear (not necessarily continuous) mapping such that $\operatorname{Im} L = E_2$ and $\operatorname{Ker} L$ is a finite dimensional space. Moreover, let $\varphi: E_1 \to E_2$ be a convex-valued mapping and let $l: E_1 \to \operatorname{Ker} L$ be a completely continuous single-valued mapping.

With such a tree of mappings as (L, φ, l) we associate the following boundary value problem

$$(1.1) \qquad \begin{aligned} L(x) &\in \varphi(x), \\ l(x) &= 0, \end{aligned}$$

which we will call the (L, φ, l)-problem.

Each point $x \in \operatorname{dom} L$ satisfying equations (1.1) is called a solution of problem (1.1).

(1.2) Definition. The problem (L, φ, l) is called an *admissible boundary value problem* (written A-BVP) provided the mapping $\varphi: E_1 \to E_2$ is L-completely continuous.

From (II.2.2) (i) and (II.2.4) we obtain the following

(1.3) Proposition. *Let (L, φ, l) be an* A-BVP, *let $P: E_1 \to E_1$ be a linear continuous projection onto* $\operatorname{Ker} L$ *and let $T: E_2 \to E_1$ be a right inverse to L such that* $\operatorname{Im} P \circ T = 0$. *Then:*

(1.3.1) *for every ball $B \subset E_1$ the mapping $\psi_{(\varphi,l)}: \bar{B} \to E_1$ given by*

$$\psi_{(\varphi,l)}(x) = P(x) + l(x) + T \circ \varphi(x) \quad \textit{for each } x \in \bar{B}$$

is convex-valued and compact,

(1.3.2) *a point $x \in \operatorname{dom} L$ is a solution of the problem (L, φ, l) iff $x \in (P + l + T \circ \varphi)(x)$.*

(1.4) Definition. Let (L, φ, l) be an A-BVP, let $B \subset E_1$ be an open ball with the centre at zero such that the problem (L, φ, l) has no solutions on the boundary ∂B of B and let $\psi_{(\varphi,l)}: \bar{B} \to E_1$ be a convex-valued compact mapping given in (1.3.1).

The *degree* $D[(L, \varphi, l), B]$ *of the problem* (L, φ, l) we define by putting

$$D[(L, \varphi, l), B] = \deg(I - \psi_{(\varphi,l)}, B, 0),$$

where $\deg(I - \psi_{(\varphi,l)}, B, 0)$ denote the Leray–Schauder degree for convex-valued vector fields.

(1.5) PROPOSITION. *Definition* (1.4) *does not depend upon the choice of the linear projection* $P: E_1 \to E_1$ *and the right inverse* $T: E_2 \to E_1$ *to* L *corresponding to* P.

Proof. Let $P_0, P_1: E_1 \to E_1$ be two linear continuous projections onto $\operatorname{Ker} L$ and let $T_0, T_1: E_2 \to E_1$ be two right inverses to L such that $\operatorname{Im} P_i \circ T_i = 0$ $(i = 0, 1)$. It follows from (II.2.1.2) that, for every $t \in [0, 1]$, $P_t = (1-t) \cdot P_0 + t \cdot P_1$ is a continuous linear projection onto $\operatorname{Ker} L$.

Now, let $T_t: E_2 \to E_1$, $t \in [0, 1]$, be a right inverse to L such that $\operatorname{Im} P_t \circ T_t = 0$ and let $\chi: [0, 1] \times \bar{B} \to E_1$ be a convex-valued mapping given by

$$\chi(t, x) = P_t(x) + l(x) + T_t \circ \varphi(x),$$

where $B \subset E_1$ is the ball given in (1.4).

It follows from (II.2.1.4) (iii) that the mapping χ is compact. Moreover, by (1.3.2), for every $x \in \partial B$ and $t \in [0, 1]$, $x \notin \chi(t, x)$ because the problem (L, φ, l) has no solutions on the boundary ∂B of B.

Hence, by the homotopy property of the Leray–Schauder degree (comp. (II.4.2)), we obtain

$$D[(L, \varphi, l), B] = \deg(I - \psi^i_{(\varphi,l)}, B, 0) \quad (i = 0, 1),$$

where

$$\psi^i_{(\varphi,l)}(x) = P_i(x) + l(x) + T_i \circ \varphi(x) \quad \text{for} \quad x \in \bar{B} \quad (i = 0, 1).$$

The proof is completed.

(1.6) DEFINITION. Two A-BVP's, (L, φ_0, l_0) and (L, φ_1, l_1) are called *homotopic on a ball* $B \subset E_1$, written $(L, \varphi_0, l_0) \underset{B}{\sim} (L, \varphi_1, l_1)$, if there exists a family (L, φ_t, l_t), $t \in [0, 1]$, of A-BVP's such that the following conditions are satisfied:

(i) no point $x \in \partial B$ is a solution of the problem (L, φ_t, l_t) for any $t \in [0, 1]$;

(ii) the mapping $H: [0, 1] \times \bar{B} \to E_2$ given by $H(t, x) = \varphi_t(x)$ is L-compact;

(iii) the mapping $h: [0, 1] \times \bar{B} \to \operatorname{Ker} L$ given by $h(t, x) = l_t(x)$ is compact.

Now we can formulate the basic properties of the degree of an A-BVP.

(1.7) PROPOSITION. *Let* (L, φ, l) *be an* A-BVP *and let* $B \subset E_1$ *be a ball such that the problem* (L, φ, l) *has no solutions on the boundary* ∂B *of* B. *Then*:

(1.7.1) *if* $D[(L, \varphi, l), B] \neq 0$, *there exists at least one solution of the problem* (L, φ, l);

(1.7.2) *if* $\varphi(-x) = -\varphi(x)$ *and* $l(-x) = -l(x)$ *for every* $x \in B$, *then* $D[(L, \varphi, l), B]$ *is odd*;

(1.7.3) *if two* A-BVP's, (L, φ_0, l_0) *and* (L, φ, l), *are homotopic on a ball* $B \subset E_1$, *then* $D[(L, \varphi_0, l_0), B] = D[(L, \varphi, l), B]$;

(1.7.4) *if* $(L, \varphi_0, l_0) \underset{B}{\simeq} (L, \varphi, l)$ *and* $\varphi_0(x) = \{0\}$ *for every* $x \in \bar{B}$, *then* $D[(L, \varphi, l), B] = \deg(l_{|B \cap \operatorname{Ker} L}, B \cap \operatorname{Ker} L, 0)$.

All these results are consequences of the definition of the degree $D[(L, \varphi, l), B]$ and of the corresponding properties of the Leray–Schauder degree for convex-valued vector fields (comp. (II.5.1)–(II.5.4)).

(1.8) PROPOSITION. *Let* (L, φ_0, l) *and* (L, φ_1, l) *be two* A-BVP's *and let* (L, φ, l_t), $t \in [0, 1]$, *be a family of problems (non-necessarily* A-BVP's) *such that for* $i = 0, 1$, $\varphi_i(x) \subset \varphi(x)$ *for each* $x \in E_1$. *Assume, moreover, that there exists a ball* $B \subset E_1$ *such that no problem* (L, φ, l_t), $t \in [0, 1]$, *has solutions on the boundary* ∂B *of* B. *Then*:

(1.8.1) $D[(L, \varphi_0, l_t), B] = D[(L, \varphi_1, l_t), B]$ *for every* $t \in [0, 1]$;

(1.8.2.) *if the mapping* h: $[0, 1] \times \bar{B} \to E_2$ *given by*

$$h(t, x) = l_t(x) \quad \text{for every } t \in [0, 1] \text{ and } x \in \bar{B}$$

is compact, then $D[(L, \varphi_0, l_0), B] = D[(L, \varphi_0, l_1), B]$.

Proof. To prove (1.8.1) we assume, for instance, $t = 0$. Since the set $\varphi(x)$ is convex and $\varphi_i(x) \subset \varphi(x)$ for every $x \in \bar{B}$, $i = 0, 1$, then $s \cdot \varphi_0(x) + (1-s) \cdot \varphi_1(x) \subset \varphi(x)$ for every $s \in [0, 1]$ and $x \in \bar{B}$.

Now, it follows from our assumption that the A-BVP $(L, s \cdot \varphi_0 + (1-s) \cdot \varphi_1, l_0)$ does not have a solution on the boundary ∂B of B for any $s \in [0, 1]$; so $(L, \varphi_0, l_0) \underset{B}{\simeq} (L, \varphi_1, l_0)$. Therefore, from (1.7.3) we obtain (1.8.1).

By analogy, we see that in the case of (1.8.2), for $t \in [0, 1]$ the A-BVP (L, φ_0, l_t) does not have a solution on the boundary ∂B of B. Therefore $(L, \varrho_0, l_0) \underset{B}{\simeq} (L, \varphi_0, l_1)$.

The proof is complete.

2. Existence theorems. In this section we will study sufficient conditions, which asssure the existence of the solutions of admissible boundary value problems.

(2.1) THEOREM (conditions of Rothe's type). *Let* (L, φ, l) *be an* A-BVP. *Assume, moreover, that there exists a ball* $B \subset E_1$ *with the centre at zero, a convex set* $K \subset E_2$, $0 \in K$ *and a real number* $r > 0$, *such that the following conditions are satisfied*:

(i) *if* $x \in \partial B \cap \operatorname{dom} L$, $x \notin \operatorname{Ker} L$ *and* $l(x) = 0$, *then* $L(x) \notin K$ *and* $\varphi(x) \subset K$;

(ii) *if* $x \in \partial B \cap \operatorname{Ker} L$, *then* $\|l(x)\| \geq r$;

(iii) $\deg(l_{|\bar{B}\cap \operatorname{Ker} L}, B \cap \operatorname{Ker} L, 0) \neq 0$.

Then, the problem (L, φ, l) *has at least one solution.*

Proof. Let us define convex-valued L-completely continuous mappings $\varphi_t\colon E_1 \to E_2$, $t \in [0, 1]$, by putting

$$\varphi_t(x) = t \cdot \varphi(x) \quad \text{for every } x \in E_1.$$

By (i)–(ii), the problem (L, φ_t, l) does not have a solution on the boundary of B for any $t \in [0, 1]$. Therefore,

$$(L, \varphi, l) \underset{B}{\sim} (L, \varphi_0, l).$$

Now, in view of assumption (iii) and (1.7.4), (1.7.1) we obtain (2.1). The proof is complete.

(2.2) THEOREM (conditions of Lasota and Opial's type). *Let* (L, φ, l), (L, φ_1, l_1) *and* (L, φ_2, l_2) *be three* A-BVP's *such that the following conditions are satisfied*:

(i) $\varphi_1(r \cdot x) = r \cdot \varphi_1(x)$ *and* $l_1(r \cdot x) = r \cdot l(x)$ *for all* $x \in E_1$ *and* $r \in R$;

(ii) $x = 0$ *is a unique solution of the problem* (L, φ_1, l_1);

(iii) *the mapping* φ_2 *is L-bounded and* l_2 *is bounded*;

(iv) $\varphi(x) \subset \varphi_1(x) + \varphi_2(x)$ *and* $l(x) = l_1(x) + l_2(x)$ *for* $x \in E_1$.

Then:

(2.2.1) *the set of all solutions of the problem* $(L, \varphi_1 + \varphi_2, l_1 + l_2)$ *is bounded,*

(2.2.2) *the problem* $(L, \varphi, l_1 + l_2)$ *has at least one solution.*

Proof. Let us consider the following two families of A-BVP's $(L, \varphi_1 + t \cdot \varphi_2, l_1 + t \cdot l_2)$ and $(L, t \cdot (\varphi_1 + \varphi_2) + (1-t) \cdot \varphi, l_1 + l_2)$, where $t \in [0, 1]$.

First, we will show that there exists an open ball $B \subset E_1$ containing the set of all solutions of the problems

$$(L, \varphi_1 + t \cdot \varphi_2, l_1 + t \cdot l_2) \quad \text{with } t \in [0, 1].$$

Suppose that the above statement is not true, i.e. there exist sequences $\{x_n\} \subset E_1$ and $\{t_n\} \subset [0, 1]$ such that $\lim_n \|x_n\| = \infty$ and each point x_n $(n = 1, 2, \ldots)$ is a solution of the problem

$$(L, \varphi_1 + t_n \cdot \varphi_2, l_1 + t_n \cdot l_2).$$

Let us put $\|x_n\| = r_n$.

By assumption (i) and (iii) we have:

$$l_1(x_n/r_n) + t_n/r_n \cdot l_2(x_n) = 0 \quad \text{and} \quad \lim_n t_n/r_n \cdot l_2(x_n) = 0.$$

Moreover, if $P\colon E_1 \to E_1$ is a continuous linear projection onto Ker L and $T\colon E_2 \to E_1$ is a right inverse to L such that Im $P \circ T = 0$, then by (1.3.2) and by assumptions (i), (iii) we get

$$x_n/r_n = P(x_n/r_n) + T(y_n) + t_n/r_n \cdot T(z_n)$$

for some $y_n \in \varphi_1(x_n/r_n)$ and $z_n \in \varphi_2(x_n)$, and

$$\lim_n t_n/r_n \cdot T(z_n) = 0.$$

On account of appropriate assumptions the sequence $\{x_n/r_n\}$ contains a subsequence $\{u_i\} = \{x_{n(i)}/r_{n(i)}\}$ convergent to a point $u \in E_1$. Since the mapping $T \circ \varphi_1$ is u.s.c., so above equalities imply

$$u \in (P + T \circ \varphi_1)(u)$$

and

$$l_1(u) = 0, \quad \text{where } \|u\| = 1.$$

Thus, it follows from (1.3.2) that the point $u \neq 0$ is a solution of the problem (L, φ_1, l_1), which contradicts the assumption (ii).

Therefore, there exists an open ball $B \subset E_1$ containing the set of all solutions of the problems

$$(L, \varphi_1 + t \cdot \varphi_2, l_1 + t \cdot l_2) \quad \text{for} \quad t \in [0, 1].$$

So we obtain (2.2.1). Moreover,

$$(L, \varphi_1, l_1) \underset{B}{\sim} (L, \varphi_1 + \varphi_2, l_1 + l_2).$$

Now, in virtue of assumption (iv) we have

$$t \cdot (\varphi_1 + \varphi_2)(x) + (1-t) \cdot \varphi(x) \subset \varphi_1(x) + \varphi_2(x)$$

and therefore

$$(L, \varphi_1 + \varphi_2, l_1 + l_2) \underset{B}{\sim} (L, \varphi, l_1 + l_2).$$

Hence, by (1.7.3)–(1.7.2), the degree $D[(L, \varphi, l_1 + l_2), B]$ is odd; so by (1.7.1) we obtain (1.2.2).

The proof is complete.

(2.3) THEOREM. *Let (L, φ, l) be an* A-BVP *and let (L, φ_1, l) be a problem (not necessarily an* A-BVP*) such that the following conditions are satisfied:*

(i) $\varphi(x) \subset \varphi_1(x)$ *and* $0 \in \varphi_1(x)$ *for all* $x \in E_1$;

(ii) *there exists a ball $B \subset E_1$ containing the set of all solutions of the problem (L, φ_1, l);*

(iii) $\deg(l_{|B \cap \operatorname{Ker} L}, B \cap \operatorname{Ker} L, 0) \neq 0$.

Then, the problem (L, φ, l) has at least one solution.

Proof. By applying (1.8.1) to the problems (L, φ, l) and (L, φ_0, l) with $\varphi_0(x) = \{0\}$ for every $x \in E_1$, we obtain

$$D[(L, \varphi_1, l), B] = D[(L, \varphi_0, l), B].$$

Now, in virtue of (1.7.4), $D[(L, \varphi_0, l), B] = \deg(l_{|\bar{B} \cap \mathrm{Ker} L}, B \cap \mathrm{Ker}\, L, 0)$. Hence, from (1.7.1) and (iii) we obtain (2.3).

(2.4) THEOREM (conditions of Browder's type). *Let (L, φ, l) be an* A-BVP, *let $P: E_1 \to \mathrm{Ker}\, L$ be a linear continuous projection onto* Ker L *and let (L, φ_1, l) be a problem (non-necessarily an* A-BVP) *such that the following conditions are satisfied*:

(i) $\varphi(x) \subset \varphi_1(x)$ *and* $0 \in \varphi_1(x)$ *for all* $x \in E_1$;

(ii) *there exists a real number $r > 0$ such that for each point $x \in E_1$ and constant mapping c_x: $E_1 \to \mathrm{Ker}\, L$ given by $c_x(y) = P(x)$ for every $y \in E_1$, there exists an integer $n = n(x)$ such that*

$$(c_x + T_P \circ \varphi_1)^n(x) \subset \{u \in E_1: \|u\| < r\};$$

(iii) $\deg(l_{|\bar{B} \cap \mathrm{Ker} L}, B \cap \mathrm{Ker}\, L, 0) \neq 0$, *where* $B = \{u \in E_1: \|u\| < r\}$.

Then the problem (L, φ, l) has at least one solution.

Proof. It follows from (1.3.2) that each solution $x \in \mathrm{dom}\, L$ of the problem (L, φ_1, l) satisfies the following

$$x \in (P + T_P \circ \varphi_1)(x).$$

Therefore, by (ii) the set all solutions of the problem (L, φ_1, l) is contained in the ball B. Now, in virtue of (2.3) we obtain (2.4). The proof is complete.

Remark. We can consider the boundary value problems with non-convex multi-valued mapping φ: $E_1 \to E_2$ (comp. (1.1)). In this case it is required that the composition $T \circ \varphi$: $E_1 \to E_1$ of φ and a right inverse T: $E_2 \to E_1$ to the linear mapping L: dom $L \to E_2$ should be *admissible multi-valued mapping*; see Górniewicz [37] and Bryszewski [14].

IV. Boundary value problems for ordinary differential equations

Let us denote by $(C^{k-1}, |\cdot|_{k-1})$ the Banach space of all C^{k-1}-mappings of the form x: $[0, a] \to R^n$ with

$$|x|_{k-1} = \max\{|x|_0, |x'|_0, \ldots, |x^{(k-1)}|_0\}.$$

In this chapter we will study the existence questions for the following boundary value problem

$$(1) \qquad \begin{aligned} & x^{(k)}(t) \in F(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) \quad \text{a.e. on } [0, a], \\ & l(x) = 0, \end{aligned}$$

where F: $[0, a] \times R^{n \cdot k} \to R^n$ admits a convex-valued, weakly compact selector (comp. (II.4.1)) and l: $C^{k-1} \to R^{n \cdot k}$ is a completely continuous single-valued mapping.

A C^{k-1}-mapping x: $[0, a] \to R^n$ with the absolutely continuous derivative $x^{(k-1)}$: $[0, a] \to R^n$ satisfying equations (1), is called a *solution of problem* (1).

In what follows the problem (1), where F and l are two mappings as above, will be called the *(F, l)-problem.*

Moreover, in this chapter, we will denote by $\operatorname{dom} L \subset C^{k-1}$ a linear subspace of all functions from C^{k-1} whose $(k-1)$th derivative is absolutely continuous and by L: $\operatorname{dom} L \to \mathscr{L}^1$, where $\mathscr{L}^1 = \mathscr{L}^1((0, a), R^n)$, a differential operator given by putting

$$L(x)(t) = x^{(k)}(t) \qquad \text{a.e. on } (0, a).$$

1. Admissible boundary value problems associated with problem (1). First, we introduce some mappings. Let τ be a real number from $[0, a]$ and let P_τ: $C^{k-1} \to C^{k-1}$ be a linear continuous projection onto $\operatorname{Ker} L$ given by putting

$$P_\tau(x)(t) = x(\tau) + x'(\tau) \cdot t + \ldots + x^{(k-1)}(\tau) \cdot t^{k-1} \qquad \text{for} \qquad t \in [0, a].$$

Moreover, let us consider for each $j = 0, 1, \ldots, k-1$ the following operators:

$$T_0\colon \mathscr{L}^1 \to C, \qquad C = C^0, \qquad T_0(y)(t) = \int_\tau^t y(s)\,ds \qquad \text{for every } t \in [0, a]$$

and

$$T_j\colon C^{j-1} \to C^j, \quad T_j(y)(t) = \int_\tau^t y(s)\,ds \quad \text{for every } t \in [0, a] \text{ and } j = 1, \ldots, k-1.$$

We define an operator T_τ: $\mathscr{L}^1 \to C^{k-1}$ by putting

$$T_\tau = T_{k-1} \circ \ldots \circ T_1 \circ T_0.$$

It is easy to see that T_τ is a right inverse to L such that $\operatorname{Im} P_\tau \circ T_\tau = 0$.

Now we will prove the following lemma.

(1.1) *Let F_1, F_2, F_3: $[0, a] \times R^{n \cdot k} \to R^n$ be three multi-valued mappings and let l: $C^{k-1} \to R^{n \cdot k}$ be a single-valued completely continuous mapping. Assume, moreover, that the following conditions are satisfied:*

(i) *F_3 is convex-valued and integrably bounded;*

(ii) *F_1 and F_2 admits a convex-valued, weakly compact selector and*

$$F_j(t, z) \subset F_3(t, z) \quad \textit{for} \quad (t, z) \in [0, a] \times R^{n \cdot k},\ j = 1, 2;$$

(iii) *there exists a point $\tau \in [0, a]$ such that the set*

$$\{(x(\tau), x'(\tau), \ldots, x^{(k-1)}(\tau)) \in R^{n \cdot k}\colon x \text{ is a solution of } (F_3, l)\}$$

is bounded in $R^{n \cdot k}$.

Then, there exist convex-valued L-compact mappings $\varphi_1, \varphi_2\colon C^{k-1} \to \mathscr{L}^1$ *and an open ball* $B \subset C^{k-1}$, *with the centre at the zero point, such that:*

(1.1.1) *the set* A_0 *of all solutions of the problem* (F_3, l) *is contained in* B,

(1.1.2) *if a point* $x \in B$ *is a solution of the* A-BVP (L, φ_j, l), $j = 1, 2$, *then the point* x *is also a solution of the problem* (F_j, l),

(1.1.3) $(L, \varphi_1, l) \underset{B}{\sim} (L, \varphi_2, l)$,

(1.1.4) *if* $m\colon [0, a] \to R_+$ *is an integrable function such that* $|F_3(t, z)| \leqslant m(t)$ *for all* $(t, z) \in [0, a] \times R^{n \cdot k}$, *then, for every* $y \in \operatorname{Im} \varphi_j$ $(j = 1, 2)$, $|y(t)| \leqslant m(t)$ *for every* $t \in [0, a]$.

Proof. Let $\psi\colon C^{k-1} \to \mathscr{L}^1$ be defined by

$$\psi(x) = \{y \in \mathscr{L}^1\colon\ y(t) \in F_3(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) \text{ a.e. on } [0, a]\}.$$

Since each solution of the problem (F_3, l) has the form $x = P_\tau(x) + T_\tau(y)$ for some $y \in \psi(x)$, then by (i) and (iii) the set A_0 of all solutions of the problem (F_3, l) is bounded. Let $B \subset C^{k-1}$ be an open ball with the centre at zero such that

$$\operatorname{conv}(\operatorname{Im} T \circ \psi) + P(A_0) \subset B.$$

It is obvious that $A_0 \subset B$, and so we have (1.1.1).

By (I.4.2) we can see that the set

$$X = \operatorname{conv}(\operatorname{Im} T_\tau \circ \psi) + P_\tau(\bar{B})$$

is compact.

Let $D\colon C^{k-1} \to C([0, a]; R^{n \cdot k})$ be a linear continuous mapping given by the formula

$$D(x)(t) = (x(t), x'(t), \ldots, x^{(k-1)}(t)) \quad \text{for every } t \in [0, a]$$

and let $\psi_j\colon D(X) \to \mathscr{L}^1$ be a convex-valued, weakly compact selector for F_j $(j = 1, 2)$, i.e.

$$(*) \qquad \{y(t)\colon\ y \in \psi_j(x)\} \subset F_j(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) \quad \text{for} \quad x \in X.$$

Since by (I.4.2) the set $\overline{\operatorname{Im} T_0 \circ \psi_j}$ is compact, so in virtue of (II.1.8) the mapping $T_0 \circ \psi_j\colon D(X) \to C^{k-1}$ is compact. Hence $\psi_j\colon D(X) \to \mathscr{L}^1$ $(j = 1, 2)$ are two convex-valued L-compact mappings.

It follows from (II.2.5) that there exist convex-valued L-compact mappings

$$\tilde{\psi}_j\colon C([0, a]; R^{n \cdot k}) \to \mathscr{L}^1 \ (j = 1, 2)$$

such that

$$(**) \qquad \tilde{\psi}_j(z) = \psi_j(z) \text{ for } z \in D(X) \quad \text{and} \quad \operatorname{Im} \tilde{\psi}_j \subset \operatorname{conv} \operatorname{Im} \psi_j.$$

Now we define the mapping φ_j: $C^{k-1} \to \mathscr{L}^1$ $(j = 1, 2)$ by putting

$$\varphi_j(x) = \tilde{\psi}_j(D(x)) \quad \text{for every } x \in C^{k-1}.$$

Note that

$$(***) \qquad \operatorname{Im} \varphi_j \subset \operatorname{conv} \operatorname{Im} \psi \quad (j = 1, 2).$$

Next, we will be interested in some properties of the mappings $\varphi_j (j = 1, 2)$.

1° Let a point $x \in B$ be a solution of the A-BVP (L, φ_j, l), $j = 1, 2$. Then

$$x \in (P_\tau + T_\tau \circ \varphi_j)(x) \quad \text{and} \quad l(x) = 0.$$

Since by $(***)$ $x \in P_\tau(x) + \operatorname{conv} \operatorname{Im} T_\tau \circ \psi$, we have $x \in X$. Now, by $(**)$, we get $\varphi_j(x) = \psi_j(D(x))$.

Hence, the point x is a solution of the problem (F_j, l), $j = 1, 2$. The proof (1.1.2) is complete.

2° Let us consider the following family of A-BVP's:

$$(L, t \cdot \varphi_1 + (1-t) \cdot \varphi_2, l), \quad t \in [0, 1].$$

If a point $x \in \bar{B}$ is a solution of the problem $(L, t \cdot \varphi_1 + (1-t) \cdot \varphi_2, l)$ for some $t \in [0, 1]$, then $x \in (P_\tau + t \cdot T_\tau \circ \varphi_1 + (1-t) \cdot T_\tau \circ \varphi_2)(x)$ and so by $(***)$ $x \in X$.

Now, by conditions (ii) and $(**)$, we can see that the point x is a solution of the problem (F_3, l) and therefore $x \in B$. Hence, $(L, \varphi_1, l) \underset{B}{\sim} (L, \varphi_2, l)$. The proof (1.1.3) is complete.

3° Since $\operatorname{Im} \varphi_j \subset \operatorname{conv} \operatorname{Im} \psi_j$, assertion (1.1.4) is a simple consequence of assumption (ii). The proof is complete.

2. Existence theorems. We start with the following

(2.1) THEOREM. *Suppose some multi-valued mappings F_1, F_2: $[0, a] \times R^{n \cdot k} \to R^n$ and single-valued mappings l_1, l_2: $C^{k-1} \to R^{n \cdot k}$ satisfy the following conditions:*

(i) *F_2 is an integrably bounded mapping which admits a convex-valued, weakly compact selector;*

(ii) *$F_1(t, z) = \{0\}$ for every $(t, z) \in [0, a] \times R^{n \cdot k}$;*

(iii) *l_1 is a linear continuous mapping;*

(iv) *l_2 is a bounded and continuous mapping;*

(v) *problem (F_1, l_1) has at most one solution.*

Then problem $(F_2, l_1 + l_2)$ has at least one solution.

Proof. Let m: $[0, a] \to R_+$ be an integrable function such that $|F_2(t, z)| \leqslant m(t)$ for every $(t, z) \in [0, a] \times R^{n \cdot k}$ and let ψ: $C^{k-1} \to \mathscr{L}^1$ be a convex-valued, weakly compact mapping given by

$$\psi(x) = \{y \in \mathscr{L}^1 : |y(t)| \leqslant m(t) \text{ a.e. on } [0, a]\} \quad \text{for} \quad x \in C^{k-1}.$$

It is easy to see that the mapping ψ is L-bounded, i.e., $T_\tau \circ \psi\colon C^{k-1} \to C^{k-1}$ is bounded.

Now, in virtue of (III.2.2.1) to the problems (L, ψ, l_1+l_2) and (L, ψ_1, l_1) with $\psi_1(x) = \{0\}$ for every $x \in C^{k-1}$, the set A_0 of all solutions of the problem (L, ψ, l_1+l_2) is bounded in C^{k-1}. We can see that the set of all solutions of the problem (F_3, l_1+l_2), where $F_3(t, z) = \{u \in R^n\colon |u| \leqslant m(t)\}$ for every $(t, z) \in [0, a] \times R^{n \cdot k}$, is equal to the set A_0.

Therefore, in virtue of (1.1) applied to the problems (F_2, l_1+l_2) and (F_3, l_1+l_2), there exist a ball $B \subset C^{k-1}$ and a convex-valued L-compact mapping $\varphi\colon C^{k-1} \to \mathscr{L}^1$ such that: $A_0 \subset B$, $\varphi(x) \subset \psi(x)$ for every $x \in C^{k-1}$ and each solution of the A-BVP (L, φ, l_1+l_2), belonging to B, is also a solution of the problem (F_2, l_1+l_2).

Since $\varphi(x) \subset \psi(x)$ for every $x \in C^{k-1}$, the set of all solutions of the problem (L, φ, l_1+l_2) is contained in A_0 and so each solution of (L, φ, l_1+l_2) is a solution of the problem (F_2, l_1+l_2).

Now again, in virtue of (III.2.2) applied to the A-BVP (L, φ, l_1+l_2), (L, ψ_1, l_1) and (L, ψ, l_2), there exists at least one solution of (L, φ, l_1+l_2). The proof is complete.

(2.2) THEOREM. *Let $F_1, F_2\colon [0, a] \times R^{n \cdot k} \to R^n$ be two multi-valued mappings and let $l\colon C^{k-1} \to R^{n \cdot k}$ be a single-valued completely continuous mapping. Assume, moreover, that the following conditions are satisfied:*

(i) *F_1 admits a convex-valued, weakly compact selector;*

(ii) *F_2 is a convex-valued integrably bounded mapping such that*

$$0 \in F_2(t, z) \quad \text{and} \quad F_1(t, z) \subset F_2(t, z) \quad \text{for every } (t, z);$$

(iii) *there exists a point $\tau \in [0, a]$ such that the set $\{(x(\tau), x'(\tau), \ldots, x^{(k-1)}(\tau))\colon x$ is a solution of $(F_2, l)\}$ is bounded in $R^{n \cdot k}$.*

Then:

(2.2.1) *there exists a ball $B \subset C^{k-1}$ with the centre at zero containing the set of all solutions of the problem (F_2, l)*

(2.2.2) *if $\tilde{l}\colon C^{k-1} \to C^{k-1}$ is given by*

$$\tilde{l}(x) = a_0 + a_1 \cdot t + \ldots + a_{k-1} \cdot t^{k-1} \quad \text{iff} \quad l(x) = (a_0, a_1, \ldots, a_{k-1})$$

and $\deg(\tilde{l}_{|B \cap \operatorname{Ker} L}, B \cap \operatorname{Ker} L, 0) \neq 0$, *then problem (F_1, l) has at least one solution,*

(2.2.3) *if l is a linear mapping, then the problem (F_1, l) has at least one solution.*

Proof. From (1.1.1) we obtain (2.2.1). Moreover, by (1.1) there exist convex-valued L-compact mappings

$$\varphi, \varphi_1\colon C^{k-1} \to \mathscr{L}^1$$

such that each solution of the A-BVP $(L, \varphi, \tilde{l})$, belonging to B, is a solution of the problem (F_1, l), $\varphi_1(x) = \{0\}$ for every $x \in C^{k-1}$ and $(L, \varphi_1, \tilde{l}) \underset{B}{\sim} (L, \varphi, \tilde{l})$. Now, in virtue of (III.1.7.4) we have

$$(*) \qquad D[(L, \varphi, \tilde{l}), B] = \deg(\tilde{l}_{|B \cap \operatorname{Ker} L}, B \cap \operatorname{Ker} L, 0)$$

and therefore by (III.1.7.1) we obtain (2.2.2).

If l is a linear mapping, then by equality $(*)$ and (III.1.7.2) the degree $D[(L, \varphi, \tilde{l}), B]$ is odd and therefore from (III.1.7.1) we obtain (2.2.3). The proof is complete.

3. First order problems. In this section we will assume that a multi-valued mapping F: $[0, a] \times R^n \to R^n$ and two single-valued continuous mappings l_1, l_2: $C \to R^n$ satisfy the following conditions:

(i) the mapping F admits a convex-valued weakly compact selector;

(ii) there exist two single-valued integrable functions p, q: $[0, a] \to R_+$ such that

$$|F(t, x)| \leqslant p(t) + q(t) \cdot |x| \quad \text{for every } (t, x) \in [0, a] \times R^n;$$

(iii) l_2 *is bounded*;

(iv) *the mapping* l_1 *has one of the following form*:

(iv$_1$) $l_1(x) = x(0)$ *(Cauchy initial condition)*

or

(iv$_2$) $l_1(x) = x(0) + \lambda \cdot x(a)$ *(Floquet boundary condition)*

or

(iv$_3$) $l_1(x) = (x_1(t_1), \ldots, x_n(t_n))$ *(Nicoletti boundary condition) where* $x(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot))$, $t_1, \ldots, t_n \in [0, a]$, *or*

(iv$_4$) l_1 *is a nonlinear completely continuous mapping.*

(3.1) Proposition. *Let the mappings* F, l_1, l_2 *be as above. Assume, moreover, that the set of all solutions of the system*

$$(3.1.1) \qquad \begin{gathered} |x'(t)| \leqslant p(t) + q(t) \cdot |x(t)| \quad \text{a.e. on } [0, a], \\ l_1(x) + l_2(x) = 0 \end{gathered}$$

is contained in a ball with centre at zero and radius r. *Then, if a multi-valued mapping* F_1: $[0, a] \times R^n \to R^n$ *is given by*

$$(3.1.2) \qquad F_1(t, x) = \begin{cases} F(t, x) & \text{for } |x| \leqslant 2r, \\ F(t, x \cdot 2r/|x|) & \text{for } |x| \geqslant 2r, \end{cases}$$

the set of all solutions of the problem $(F, l_1 + l_2)$ *is equal to the set of all solutions of the problem* $(F_1, l_1 + l_2)$.

Proof. Let a point $x \in C$ be a solution of the problem $(F_1, l_1 + l_2)$. Then, by assumption (ii), we have

$$|x'(t)| \leqslant p(t) + q(t) \cdot |x(t)| \quad \text{a.e. on } [0, a].$$

Therefore, the point x is a solution of system (3.1.1). Hence, $|x|_0 < r$, and so the point x is a solution of the problem $(F, l_1 + l_2)$.

The inverse inclusion is obvious. The proof is complete.

(3.2) THEOREM. *Let the mappings F, l_1 and l_2 be as above. Then, we have:*

(3.2.1) *if $l_1(x) = x(0)$, then the problem $(F, l_1 + l_2)$ has at least one solution,*

(3.2.2) *if $l_1(x) = x(0) + \lambda \cdot x(a)$ and $\int_0^a q(s)\,ds < \sqrt{\pi^2 + \log^2 \lambda}$, then the problem $(F, l_1 + l_2)$ has at least one solution,*

(3.2.2) *if $l_1(x) = x(0) + \lambda \cdot x(a)$ and $\int_0^a q(s)\,ds < \sqrt{\pi^2 + \log^2 \lambda}$, then the problem $(F, l_1 + l_2)$ has at least one solution,*

Proof. Let the mapping l_1 have one of the forms (iv_1)–(iv_3). First we show that the set of all solutions of system (3.1.1) is bounded. For this purpose let us define two convex-valued mappings $\varphi_1, \varphi_2 \colon C \to \mathscr{L}^1$ by putting

$$\varphi_1(x) = \{y \in \mathscr{L}^1 \colon |y(t)| \leqslant q(t) \cdot |x(t)| \text{ a.e. on } (0, a)\}$$

and

$$\varphi_2(x) = \{y \in \mathscr{L}^1 \colon |y(t)| \leqslant p(t) \text{ a.e. on } (0, a)\}.$$

It follows from (II.3.6) that φ_1, φ_2 are L-completely continuous mappings. Recall that in our case $L = \frac{d}{dt}$.

In virtue of inequality (I.5.1)–(I.5.3), the A-BVP (L, φ_1, l_1) with condition l_1 has only the zero solution. Therefore, by (III.2.2.1), the set of all solutions of the problem $(L, \varphi_1 + \varphi_2, l_1 + l_2)$ is bounded. Hence, the set of all solutions of system (3.1.1) is also bounded.

Now, it follows from (3.1) that it is sufficient state the existence of a solution of the problem $(F_1, l_1 + l_2)$, where F_1 is given in (3.1.2) for F. Since by (2.1) the problem $(F_1, l_1 + l_2)$ has at least one solution, the proof is complete.

Making use of (II.4.3) we get the following theorem.

(3.3) THEOREM. *Suppose that a convex-valued mapping $F \colon [0, a] \times R^n \to R^n$ satisfies the following conditions:*

(i_1) *for every $t \in [0, a]$, $F(t, \cdot)$ is a u.s.c. mapping from R^n into R^n;*

(i_2) *for every $x \in R^n$, $F(\cdot, x)$ is a measurable mapping from $[0, a]$ into R^n;*

(ii) *there exist two integrable functions $p, q \colon [0, a] \to R_+$ such that*

$$|F(t, x)| \leqslant p(t) + q(t) \cdot |x| \quad \text{for} \quad (t, x) \in [0, a] \times R^n.$$

Then (3.2.1), (3.2.2) *and* (3.2.3) *are true.*

Remark. In particular, if $l_2\colon C \to R^n$ is a constant mapping and the convex-valued mapping F satisfies assumptions (i_1), (i_2) and (ii) of Theorem (3.3), then:

1° from (3.2.1) we obtain Casting's [17] and Pliś's [84] result: there exists at least one solution of the Cauchy initial value problem $(F, l_1 + l_2)$,

2° from (3.2.3) we obtain Lasota and Opial's result (see [67], [68]): there exists at least one solution of the Nicolletti boundary value problem $(F, l_1 + l_2)$.

Making use of (II.4.2), we obtain the following theorem.

(3.4) THEOREM. *Suppose that a multi-valued mapping* $F\colon [0, a] \times R^n \to R^n$ *satisfies the following conditions:*

(i_1) *for every* $t \in [0, a]$, $F(t, \cdot)$ *is a continuous mapping from* R^n *into* $(\mathscr{B}(R^n), d)$;

(i_2) *for every* $x \in R^n$, $F(\cdot, x)$ *is a measurable mapping from* $[0, a]$ *into* R^n;

(ii) *there exist two integrable functions* $p, q\colon [0, a] \to R_+$ *such that*

$$|F(t, x)| \leqslant p(t) + q(t) \cdot |x| \quad \textit{for} \quad (t, x) \in [0, a] \times R^n.$$

Then, (3.2.1), (3.2.2) *and* (3.2.3) *are true.*

Remark. In particular, if an integrably bounded mapping $F\colon [0, a] \times R^n \to R^n$ satisfies assumptions (i_1)–(i_2) of Theorem (3.4), then, from (3.2.1) we obtain Filippov's [32], Antosiewicz and Cellina's [4], and Kaczyński and Olech's [52] result: there exists at least one solution of the Cauchy initial value problem (F, l_1).

Making use of (II.4.4) we obtain the following theorem.

(3.5) THEOREM. *Suppose that a multi-valued mapping* $F\colon [0, a] \times R^n \to R^n$ *satisfies the following conditions:*

(i) F *is lower semicontinuous;*

(ii) *there exist two integrable functions* $p, q\colon [0, a] \to R_+$ *such that*

$$|F(t, x)| \leqslant p(t) + q(t) \cdot |x| \quad \textit{for} \quad (t, x) \in [0, a] \times R^n.$$

Then, (3.2.1), (3.2.2) *and* (3.2.3) *are true.*

Remark. In particular, if a lower continuous mapping $F\colon [0, a] \times R^n \to R^n$ is integrably bounded, then, from (3.2.1), we obtain Bressan's [11] and Łojasiewicz's (Jr) [70] result: there exists at least one solution of the Cauchy initial value problem (F, l_1).

(3.6) THEOREM. *Let* $F\colon [0, a] \times R^n \to R^n$ *be a multi-valued mapping satisfying assumption* (i) *and assumption* (ii) *with some essentially bounded functions* $p, q\colon [0, a] \to R_+$. *Let* $b = \operatorname{ess\,sup}_{t \in [0,a]} p(t)$, $c = \operatorname{ess\,sup}_{t \in [0,a]} q(t)$ *and* *let l: $C \to R^n$ be a single-valued completely continuous mapping satisfying the following conditions:*

(iii$_1$) *there exists a point $\tau \in [0, a]$ such that the set*

$$K = \{x(\tau) \in R^n \colon l(x) = 0\} \quad \text{is bounded in } R^n;$$

(iii$_2$) *there exists an open ball $B_r = \{x \in C \colon |x|_0 < r$, with $r > 2 \cdot |K| + (b/c + |K|) \cdot e^{c \cdot |K|}$, such that*

$$\deg(l_{|B_r \cap \operatorname{Ker} L}, B_r \cap \operatorname{Ker} L, 0) \neq 0.$$

Then, problem (F, l) has at least one solution.

Proof. First, let us note that the set of all solutions of the problem

$$|x'(t)| \leqslant p(t) + q(t) \cdot |x(t)| \quad \text{a.e. on } [0, a],$$
$$l(x) = 0$$

is contained in the ball B_r given in (iii$_2$) (see [89]). By using (3.1) we deduce that the set of all solutions of the problem (F, l) is equal to the set of all solutions of the problem (F_1, l) with F_1 given in (3.1.2) for F.

Now we can see that by (2.2.2) the problem (F_1, l) has at least one solution. The proof is complete.

4. Second order problems. We start with the following theorem.

(4.1) Theorem. *Let l, l_0: $C^1 \to R^n$ be two continuous and bounded mappings and let F: $[0, a] \times R^{2n} \to R^n$ be an integrably bounded mapping which admits a convex-valued, weakly compact selector. Then, there exists at least one solution of the following problem*

$$x''(t) \in F(t, x(t), x'(t)) \quad \text{a.e. on } [0, a],$$
$$x(0) = l_0(x) \quad \text{and} \quad x(a) = l(x).$$

Proof. Putting $l_1(x) = (x(0), x(a))$ and $l_2(x) = (l_0(x), l(x))$ for every $x \in C^1$ and applying (2.1) to the problem $(F, l_1 - l_2)$, we obtain (4.1).

In what follows we will assume that a multi-valued mapping F: $[0, a] \times R^{2n} \to R^n$ and a single-valued continuous mapping l: $C^1 \to R^{2n}$ satisfy the following conditions:

(i) the mapping F admits a convex-valued, weakly compact selector;

(ii) there exist two essentially bounded functions p, q: $[0, a] \to R_+$ such that

$$|F(t, x, y)| \leqslant p(t) + q(t) \cdot |x| \quad \text{for every } (t, x, y) \in [0, a] \times R^n \times R^n;$$

(iii) there are two real numbers $M > 0$ and $\tau > 0$ such that

$$F(t, x, y) \subset \{u \in R^n \colon u \cdot x > 0\} \quad \text{for all } t \in [0, a], |x| > M$$

and $|y| < \tau$ ($u \cdot x$ denotes the scalar product of u and x);

(iv) the mapping l has the following form:

$l(x) = (x(0), x(a))$ (Picard boundary condition).

Let us put $A = [0, a] \times \{x \in R^n \colon |x| > M\} \times \{y \in R^n \colon |y| < \tau\}$, where the numbers M and τ are given in (iii).

We will need the following

(4.2) PROPOSITION ([89]). *Let the functions p, q and the set A be as above, let $F_1 \colon [0, a] \times R^{2n} \to R^n$ be a convex-valued mapping given by*

$$F_1(t, x, y) = \begin{cases} \{u \in R^n \colon |u| \leqslant p(t) + q(t) \cdot |x|\} & \text{for } (t, x, y) \notin A, \\ \{u \in R^n \colon |u| \leqslant p(t) + q(t) \cdot |x|,\ u \cdot x > 0\} \cup \{0\} & \text{for } (t, x, y) \in A \end{cases}$$

and let the mapping l have the form (iv). *Then, the set of all solutions of the problem (F_1, l) is bounded.*

(4.3) LEMMA. *Let the mapping F_1 be the same as that given in* (4.2) *and let $B_r \subset C^1$ be an open ball with centre at zero and radius $r > M$ containing the set of all solutions of the problem (F_1, l). Then, if the multi-valued mapping $F_2 \colon [0, a] \times R^{2n} \to R^n$ is given by*

$$F_2(t, x, y) = \begin{cases} F(t, x, y) & \text{if } |x| \leqslant r, \\ F(t, x \cdot r/|x|, y) & \text{if } |x| \geqslant r, \end{cases} \tag{4.3.1}$$

the set of all solutions of the problem (F, l) is equal to the set of all solutions of the problem (F_2, l).

Proof. Let a function $x \in C^1$ be a solution of the problem (F, l). Since, by assumptions (ii)–(iii), $F(t, z) \subset F_1(t, z)$ for $(t, z) \in [0, a] \times R^{2n}$, x is a solution of the problem (F_1, l). Since by our assumptions $|x|_0 < r$, so x is a solution of the problem (F_2, l).

Now let a function $x \in C^1$ be a solution of the problem (F_2, l). Then, by (4.3.1) and by assumption (ii), we can see that x is a solution of the problem (F_1, l) and so $|x|_0 < r$. Therefore x is a solution of the problem (F, l). The proof is complete.

(4.4) THEOREM. *Suppose that a multi-valued mapping $F \colon [0, a] \times R^{2n} \to R^n$ and a continuous single-valued mapping $l \colon C^1 \to R^{2n}$ satisfy assumptions* (i)–(iv). *Then the problem (F, l) has at least one solution.*

Proof. Let $F_2 \colon [0, a] \times R^{2n} \to R^n$ be a multi-valued mapping given in (4.3.1) for F. Then it follows from (4.3) that the set of all solutions of the problem (F, l) is equal to the set of all solutions of problem (F_2, l).

Let us define a convex-valued mapping $F_1 \colon [0, a] \times R^{2n} \to R^n$ by

$$F_1(t, x, y) = \begin{cases} \{u \in R^n \colon |u| \leqslant p(t) + q(t) \cdot r\} & \text{for } (t, x, y) \notin A, \\ \{u \in R^n \colon |u| \leqslant p(t) + q(t) \cdot r,\ u \cdot x > 0\} \cup \{0\} & \text{for } (t, x, y) \in A, \end{cases}$$

where the real number r is given in (4.3) and the set A is given in (4.2).

It is easy to see that $F_2(t, x, y) \subset F_1(t, x, y)$ for every $(t, x, y) \in [0, a] \times \times R^n \times R^n$ and that by (4.2) the set of all solutions of the problem (F_1, l) is bounded.

Now, in virtue of (2.2.3) applied to the problems (F_1, l) and (F_2, l) there exists at least one solution of the problem (F_2, l). The proof is complete.

In particular, making use of (II.4.2), (II.4.3) and (II.4.4) respectively we obtain the following theorems, which simply result from (4.4).

(4.5) THEOREM. *Suppose that a multi-valued mapping* $F\colon [0, a] \times R^{2n} \to R^n$ *satisfies conditions* (ii), (iii) *and moreover:*

(i_1) *for every* $t \in [0, a]$, $F(t, \cdot)$ *is a continuous mapping from* R^{2n} *into* $(\mathscr{B}(R^n), d)$;

(i_2) *for every* $z \in R^{2n}$, $F(\cdot, z)$ *is a measurable mapping from* $[0, a]$ *into* R^n.

Then, if a single-valued mapping $l\colon C^1 \to R^{2n}$ *has the form* (iv), *the problem* (F, l) *has at least one solution.*

(4.6) THEOREM. *Suppose that a convex-valued mapping* $F\colon [0, a] \times R^{2n} \to R^n$ *satisfies conditions* (ii), (iii) *and moreover:*

(i_1) *for every* $t \in [0, a]$, $F(t, \cdot)$ *is an u.s.c. mapping from* R^{2n} *into* R^n;

(i_2) *for every* $z \in R^{2n}$, $F(\cdot, z)$ *is a measurable mapping from* $[0, a]$ *into* R^n.

Then, if a single-valued mapping $l\colon C^1 \to R^{2n}$ *has the form* (iv), *the problem* (F, l) *has at least one solution.*

(4.7) THEOREM. *Suppose that a multi-valued mapping* $F\colon [0, a] \times R^{2n} \to R^n$ *is lower semicontinuous and satisfies conditions* (ii)–(iii). *Then, if a single-valued mapping* $l\colon C^1 \to R^{2n}$ *has the form* (iv), *the problem* (F, l) *has at least one solution.*

V. Boundary value problems for some hyperbolic partial differential equations

In this chapter we will study the existence questions for two boundary value problems:

1° the multi-valued Darboux problem and

2° the multi-valued problem with nonlinear boundary conditions.

1. Multi-valued Darboux problem. Let Δ be the Cartesian product $[0, a] \times [0, a]$ and let $F\colon \Delta \times R^n \to R^n$ be a convex-valued mapping satisfying the following conditions:

(i) for each $u \in R^n$, $F(\cdot, u)$ is a measurable mapping from Δ into R^n;
(ii) for each $(x, y) \in \Delta$, $F(x, y, \cdot)$ is a u.s.c. mapping from R^n into R^n;
(iii) there are two integrable functions $p, q\colon \Delta \to R_+$ such that

$$|F(x, y, u)| \leqslant p(x, y) + q(x, y) \cdot |u| \quad \text{for each } (x, y) \in \Delta,\ u \in R^n.$$

Let $C_*([0, a]; R^n)$ denote the space of all absolutely continuous functions from $[0, a]$ into R^n.

For the mapping F and for two finite dimensional continuous bounded mappings $l, \bar{l}\colon C(\Delta; R^n) \to C_*([0, a]; R^n)$, such that $l(u)(0) = \bar{l}(u)(0)$ for every u $C(\Delta; R^n)$, we formulate the following Darboux problem:

$$(1.1) \qquad \begin{aligned} &u_{xy}(x, y) \in F(x, y, u(x, y)) \quad \text{a.e. on } \Delta, \\ &u(0, \cdot) = \bar{l}(u) \quad \text{and} \quad u(\cdot, 0) = l(u). \end{aligned}$$

A function $u\colon \Delta \to R^n$ satisfying the above equations and such that u_{xy} exists almost everywhere on Δ and u_{xy} is an integrable function is called a solution of problem (1.1).

(1.2) THEOREM. *Problem* (1.1) *has at least one solution.*

Proof. Let the mapping F, l and $\bar{l}$ be as in (1.1) and let Y_1, Y_2 be two finite dimensional subspaces of $C_*([0, a]; R^n)$ such that $\operatorname{Im} l \subset Y_1$ and $\operatorname{Im} \bar{l} \subset Y_2$.

We introduce the following functional spaces:

$$Y = \{f \oplus g \in C(\Delta; R^n)\colon f \in Y_1,\ g \in Y_2 \text{ and } (f \oplus g)(x, y) = f(x) + g(y) \text{ for } (x, y) \in \Delta\},$$

$$E_0 = \{u \in C(\Delta; R^n)\colon u(x, 0) = 0 \text{ and } u(0, y) = 0 \text{ for } x, y \in [0, a]\}$$

and $E_1 = E_0 + Y$.

Let $\operatorname{dom} L \subset E_1$ be a linear subspace of all functions $u \in E_1$ whose derivative u_{xy} is an integrable function and let $L\colon \operatorname{dom} L \to \mathscr{L}^1(\Delta; R^n)$ be a linear mapping given by

$$L(u) = u_{xy}.$$

It is clear that $\operatorname{Ker} L = Y$ and $\operatorname{Im} L = \mathscr{L}^1(\Delta; R^n)$.

Next, we define the convex-valued mappings $\varphi, \varphi_1, \varphi_2\colon E_1 \to \mathscr{L}^1(\Delta; R^n)$ given respectively by the following formulas:

$$\begin{aligned} \varphi(u) &= \{w \in \mathscr{L}^1(\Delta; R^n)\colon w(x, y) \in F(x, y, u(x, y)) \text{ a.e. on } \Delta\}, \\ \varphi_1(u) &= \{w \in \mathscr{L}^1(\Delta; R^n)\colon |w(x, y)| \leqslant q(x, y) \cdot |u(x, y)| \text{ a.e. on } \Delta\}, \\ \varphi_2(u) &= \{w \in \mathscr{L}^1(\Delta; R^n)\colon |w(x, y)| \leqslant p(x, y) \text{ a.e. on } \Delta\}, \end{aligned}$$

where p and q are given in assumption (iii).

Moreover, let $l_1, l_2\colon E_1 \to Y$ be two single-valued continuous mappings given by

$$l_1(u) = u(\cdot, 0) + u(0, \cdot) \quad \text{and} \quad l_2(u) = l(u) \oplus \bar{l}(u) \quad \text{for} \quad u \in E_1.$$

Since the right inverse $T\colon \mathscr{L}^1(\Delta; R^n) \to E_1$ to L has the form

$$T(w)(x, y) = \int_0^x \int_0^y w(s, t)\, ds\, dt,$$

by (II.3.8) the problems $(L, \varphi, l_1 - l_2)$, (L, φ_1, l_1) and (L, φ_2, l_2) are A-BVP's.

From (I.5.5), the problem (L, φ_1, l_1) has only the zero solution. Therefore, in virtue of (III.2.2), the problem $(L, \varphi, l_1 - l_2)$ has at least one solution. The proof is complete.

Remark. In particular, when the mapping $F\colon \Delta \times R^n \to R^n$ satisfies assumption (iii) with $q(x, y) = 0$ for every $(x, y) \in \Delta$ and $l(u) = 0 = \bar{l}(u)$ for every $u \in C(\Delta; R^n)$, we obtain from Theorem (1.2) Lasota's result [62].

2. A multi-valued problem with nonlinear boundary conditions. Let $F\colon \Delta \times R^n \to R^n$ be a convex-valued mapping satisfying the following conditions:

(i) for each $u \in R^n$, $F(\cdot, u)$ is a measurable mapping from Δ into R^n;

(ii) for each $(x, y) \in \Delta$, $F(x, y, \cdot)$ is a u.s.c. mapping from R^n into R^n;

(iii) there exists an integrable function $p\colon \Delta \to R_+$ such that

$$|F(x, y, u)| \leqslant p(x, y) \quad \text{for each } (x, y) \in \Delta \text{ and } u \in R^n.$$

Let $C_*([0, a]; R^1)$ denote the space of all absolutely continuous functions from $[0, a]$ into R^1 and let l_j, $\bar{l}_j\colon C(\Delta; R^n) \to C_*([0, a]; R^1)$ $(j = 1, 2, \ldots, n)$ be finite dimensional continuous bounded mappings such that, for some $(s_j, t_j) \in \Delta$, $l(u)(s_j) = l(u)(t_j)$ $(j = 1, 2, \ldots, n)$ for every u.

For the mappings F, l_j and $\bar{l}_j$ $(j = 1, 2, \ldots, n)$ we formulate the following nonlinear boundary value problem:

$$u_{xy}(x, y) \in F(x, y, u(x, y)) \quad \text{a.e. on } \Delta$$

(2.1) $$u_j(\cdot, t_j) = l_j(u) \quad \text{and} \quad u_j(s_j, \cdot) = \bar{l}_j(u),$$

where $j = 1, 2, \ldots, n$ and $u = (u_1, \ldots, u_n)$.

A function $u\colon \Delta \to R^n$ satisfying the above equations such that $u_{xy}(x, y)$ exists almost everywhere and u_{xy} is an integrable function is called a solution of problem (2.1).

(2.2) Theorem. *Problem* (2.1) *has at least one solution.*

Proof. Let the mapping F, l_j and $\bar{l}_j$, $j = 1, 2, \ldots, n$, be as in (2.1) and let Y_j, $\tilde{Y}_j \subset C_*([0, a]; R^1)$ be finite dimensional subspaces such that $\operatorname{Im} l_j \subset Y_j$ and $\operatorname{Im} \bar{l}_j \subset \tilde{Y}_j$.

We introduce the following functional spaces:

$$Y = \{u \in C(\Delta; R^n)\colon u(x, y) = (f_1(x) + \bar{f}_1(y), \ldots, f_n(x) + \bar{f}_n(y)) \text{ for } (x, y) \in \Delta, f_j \in Y_j, \bar{f}_j \in \tilde{Y}_j\ (j = 1, \ldots, n)\},$$

$E_0 = \{u \in C(\Delta; R^n)$: $u(x, y) = (u_1(x, y), \ldots, u_n(x, y))$, $u_i(x, t_j) = 0$ and $u_j(s_j, y) = 0$ for $x, y \in [0, a]$, $j = 1, \ldots, n$, where (s_j, t_j) are given in (2.1)$\}$ and

$$E_1 = E_0 + Y.$$

Let $\operatorname{dom} L \subset E_1$ be a linear subspace of all functions $u \in E_1$ whose derivative u_{xy} is an integrable function and let L: $\operatorname{dom} L \to \mathscr{L}^1(\Delta; R^n)$ be a linear mapping given by

$$L(u) = u_{xy}.$$

It is clear that $\operatorname{Ker} L = Y$ and $\operatorname{Im} L = \mathscr{L}^1(\Delta; R^n)$.

Next, we define the convex-valued mappings φ, φ_1, φ_2: $E_1 \to \mathscr{L}^1(\Delta; R^n)$ given respectively by the formulas:

$$\varphi(u) = \{w \in \mathscr{L}^1(\Delta; R^n)\colon\ w(x, y) \in F(x, y, u(x, y)) \text{ a.e. on } \Delta\},$$

$$\varphi_1(u) = \{0\},$$

and

$$\varphi_2(u) = \{w \in \mathscr{L}^1(\Delta; R^n)\colon\ |w(x, y)| \leqslant p(x, y) \text{ a.e. on } \Delta\},$$

where p is given in (iii).

Moreover, let l_0, l: $E_1 \to Y$ be two single-valued continuous mappings such that for each function $u \in E_1$ of the form $u(x, y) = (u_1(x, y), \ldots, u_n(x, y))$ for $(x, y) \in \Delta$ we have:

$$l_0(u)(x, y) = (u_1(x, t_1) + u_1(s_1, y), \ldots, u_n(x, t_n) + u_n(s_n, y))$$

and

$$l(u)(x, y) = (l_1(u)(x) + \bar{l}_1(u)(y), \ldots, l_n(u)(x) + \bar{l}_n(u)(y)).$$

Since a right inverse T: $\mathscr{L}^1(\Delta; R^n) \to E_1$ to L has the form

$$T(w)(x, y) = \int_0^x \int_0^y w(s, t)\, ds\, dt \quad \text{for} \quad (x, y) \in \Delta,$$

by (II.3.8) the problems $(L, \varphi, l_0 - l)$, (L, φ_1, l_0) and (L, φ_2, l) are A-BVP's.

It is easy to see that the problem (L, φ_1, l_0) has only the zero solution. Therefore, in virtue of (III.2.2), the problem $(L, \varphi, l_0 - l)$ has at least one solution. The proof is complete.

Remark. Theorems (1.2) and (2.2) remain also true for the multi-valued non-convex mapping F: $\Delta \times R^n \to R^n$, which admits a convex-valued, weakly compact selector (comp. (II.4.1)).

VI. Boundary value problems for elliptic partial differential equations

In this chapter we will study the existence question for the multi-valued general boundary value problem.

1. Basic function spaces. Let $G \subset R^n$ be a bounded open domain whose boundary ∂G is an C^∞-manifold. In what follows we will consider real-valued functions of the following type:
$u\colon G \to R^1$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a function $u\colon G \to R^1$ the symbol

$$D^\alpha u = \frac{D^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n}$$

will denote the partial derivative of u of the order $|\alpha| = \alpha_1 + \ldots + \alpha_n$ (if it exists).

Let $C^m(G)$ be a space of all functions u from G into R^1 which are continuous together with derivatives $D^\alpha u$, $|\alpha| \leqslant m$, and let

$$\tilde{C}^m_p(G) = \{u \in C^m(G)\colon [\sum_{|\alpha| \leqslant m} \int_G |D^\alpha u(x)|^p dx]^{1/p} < \infty\}$$

for $1 \leqslant p < \infty$. In the space $\tilde{C}^m_p(G)$ we define the norm as follows

$$\|u\|_{m,p} = [\sum_{|\alpha| \leqslant m} \int_G |Du(x)|^p dx]^{1/p}.$$

By $H_{m,p}(G)$ we will denote the Sobolev space which is the completion of $\tilde{C}^m_p(G)$ with respect to the norm $\|\cdot\|_{m,p}$.

By $C^m_0(G)$ we will denote the space of all functions $u \in C^m(G)$ which have a compact support in G. The completion space of all the functions in $C^m_0(G)$ with respect to the norm $\|\cdot\|_{m,2}$ will be denoted by $H^m_0(G)$.

Let u, $v\colon G \to R^1$ be two integrable functions. We say that the function v is the *α-th weak derivative of u*, if for every $f \in C^\infty_0(G) = \bigcap_{m=0}^{\infty} C^m_0(G)$,

$$\int_G u(x)\, D^\alpha f(x)\, dx = (-1)^{|\alpha|} \int_G v(x) f(x)\, dx.$$

Then we write $\tilde{D}^\alpha(u) = v$.

The following three facts are well known (see, for instance, [1, 33, 79]).

(1.1) *The embedding $\tilde{J}$: $H_{m,p}(G) \to H_{m-1,p}(G)$, given by $\tilde{J}(u) = u$, is a completely continuous mapping.*

(1.2) $H_{m,p}(G) = \{u \in \mathscr{L}^p(G)\colon \tilde{D}^\alpha(u) \in \mathscr{L}^p(G), |\alpha| \leqslant m\}$.

(1.3) *Let $|\alpha| \leqslant m$. The operator $\tilde{D}^\alpha$: $H_{m,p}(G) \to \mathscr{L}^p(G)$ is a continuous extension of the operator D^α: $\tilde{C}^m_p(G) \to C^0(G)$.*

Let $C^m(\bar{G})$ be a space of all functions u from G into R^1 which are uniformly continuous together with derivatives $D^\alpha(u)$ for $|\alpha| \leqslant m$. In the space $C^m(\bar{G})$ we define the norm by putting:

$$|u|_m = \sum_{|\alpha| \leqslant m} \sup_{x \in G} |D^\alpha(u)(x)|.$$

Let $C^{m+\mu}(\bar{G})$, $0 < \mu < 1$, be the Hölder space with the norm

$$|u|_{m+\mu} = |u|_m + \sum_{|\alpha|=m} \sup\left\{\frac{|D^\alpha(u)(x) - D^\alpha(u)(y)|}{|x-y|^\mu} \colon x, y \in G, x \neq y\right\}.$$

Note the following (see, for instance, [1], [79]).

(1.4) *The embedding* i: $C^{m+\mu}(\bar{G}) \to C^m(\bar{G})$, *given by* $i(u) = u$, *is a completely continuous mapping.*

From the Sobolev embedding theorem (see, for instance, [1], [79]) we instantly obtain the following.

(1.5) *Let* $p > n$. *Then, for* $\mu = n/p$, *the mapping* j: $H_{m,p}(G) \to C^{m-1+\mu}(\bar{G})$ *given as follows*:

$$[j(\tilde{u}) = u] \Leftrightarrow [u \in C^{m-1+\mu}(\bar{G}) \text{ and } u(x) = \tilde{u}(x) \text{ a.e. on } G]$$

is correctly defined and it is a continuous mapping.

2. The general boundary value problem. Let us introduce the following differential operator

$$\sum_{|\alpha| \leqslant m} a_\alpha(\cdot) D^\alpha = \sum_{k=0}^{m} \sum_{\alpha_1 + \ldots + \alpha_n = k} a_{\alpha_1 \ldots \alpha_n}(\cdot) D_1^{\alpha_1} \circ \ldots \circ D_n^{\alpha_n}, \tag{2.1}$$

where the coefficients $a_\alpha(\cdot)$ are functions from $\bar{G}$ into R^1.

The operator (2.1) is called *elliptic* if for all $x \in G$ and $\xi = (\xi_1, \ldots, \xi_n)$ from $R^n \setminus \{0\}$ we have

$$\sum_{\alpha_1 + \ldots + \alpha_n = m} a_{\alpha_1 \ldots \alpha_n}(x) \cdot \xi^{\alpha_1} \cdot \ldots \cdot \xi^{\alpha_n} \neq 0.$$

Let A_p: $H_{m,p}(G) \to \mathscr{L}^p(G)$ be an elliptic operator given by

$$A_p(u)(x) = \sum_{|\alpha| \leqslant m} a_\alpha(x) \tilde{D}^\alpha(u)(x) \quad \text{for every } x \in G,$$

where

$$a(\cdot) \in \bigcap_{m=0}^{\infty} C^m(\bar{G}) = C^\infty(\bar{G}).$$

Let B_k: $C^{m-1}(\bar{G}) \to C^0(\bar{G})$, $k = 1, 2, \ldots, k_0$, be differential operators given by

$$B_k(u)(x) = \sum_{|\alpha| \leqslant m_k} b_\alpha^k(x) D^\alpha(u)(x) \quad \text{for every } x \in \bar{G},$$

where $m_k < m$, $b_\alpha^k(\cdot) \in C^\infty(\bar{G})$ for $k = 1, 2, \ldots, k_0$.

Recall the following a priori estimates for A_p (see [79]).

(2.2) PROPOSITION. *Let* $p > n$, *let* $\operatorname{Ker} A_p = 0$ *and let* $j\colon H_{m,p}(G) \to C^{m-1+\mu}(\bar{G})$ *be the embedding given in* (1.5). *Then there exists a constant* $M > 0$ *such that, for every* $u \in \{v \in H_{m,p}(G)\colon (B_k \circ j(v))_{\partial G} = 0,\ k = 1, 2, \ldots, k_0\}$,

$$\|u\|_{m,p} \leqslant M \cdot \|A_p(u)\|_p.$$

For a convex-valued mapping $F\colon \bar{G} \times R^2 \to R^1$ and for the boundary differential operators B_k ($k = 1, 2, \ldots, k_0$) we formulate the following boundary value problem

$$(2.3)\qquad \begin{aligned} &u \in C^{m-1}(\bar{G}),\\ &A_p(u)(x) \in F(x, u(x), D^\beta(u)(x)) \quad \text{a.e. on } G,\ |\beta| < m,\ p > n,\\ &B_k(u)(x) = 0 \quad \text{for} \quad x \in \partial G,\ k = 1, 2, \ldots, k_0. \end{aligned}$$

(2.4) THEOREM. *Suppose that a convex-valued mapping* $F\colon \bar{G} \times R^2 \to R^1$ *satisfies the following conditions*:

(i) *for each* $x \in \bar{G}$, $F(x, \cdot)$ *is a u.s.c. mapping from* R^2 *into* R^1;

(ii) *for each* $v \in R^2$, $F(\cdot, v)$ *is a measurable mapping from* $\bar{G}$ *into* R^1;

(iii) *there exists a function* $f \in \mathscr{L}^p(\bar{G})$ *with* $p > n$, *such that*

$$|F(x, v)| \leqslant f(x) \quad \text{for} \quad x \in \bar{G} \text{ and } v \in R^2.$$

Assume moreover that $\operatorname{Im} A_p = \mathscr{L}^p(G)$ *and* $\operatorname{Ker} A_p = 0$. *Then problem* (2.3) *has at least one solution.*

First, we prove the following lemma.

(2.5) LEMMA. *Suppose that all assumptions of* (2.4) *are satisfied. Then there exists an* A-BVP (L, φ, l) *the set of all solutions of which is equal to the set of all solutions of the problem* (2.3).

Proof. Let us put

$$E_1 = (C^{m-1}(\bar{G}), |\cdot|_{m-1}) \quad \text{and} \quad \operatorname{dom} L = \{u \in C^{m-1}(\bar{G})\colon u \in H_{m,p}(G)\}.$$

Now, let us specify the following mappings:

$$\psi\colon E_1 \to \mathscr{L}^p(G),\ p > n, \quad \text{and} \quad L\colon \operatorname{dom} L \to \mathscr{L}^p(G)$$

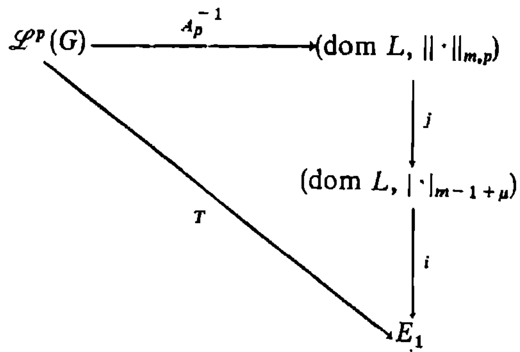
given respectively by

$$\psi(u) = \{w \in \mathscr{L}^p(G)\colon w(x) \in F(x, u(x), D^\beta(u)(x)) \text{ a.e. on } G\} \quad \text{for every } u \in E_1$$

and

$$L(u) = A_p(u) \quad \text{for every } u \in \operatorname{dom} L.$$

Let $T\colon \mathscr{L}^p(G) \to E_1$ be a right inverse to L. We consider the comutative diagram



Since by the Banach theorem the mapping A_p^{-1} is continuous, in virtue of (1.5) and (1.4) T is a completely continuous mapping. Let D: $E_1 \to C(\bar{G}; R^2)$ be a linear continuous mapping given by

$$D(u)(x) = \big(u(x), D^{\beta}(u)(x)\big) \quad \text{for every } x \in \bar{G}$$

and let φ: $E_1 \to \mathscr{L}^p(G)$ be given by putting

$$\varphi = \psi \circ D.$$

It follows from (II.3.6) that the composition $T \circ \varphi$: $E_1 \to E_1$ of φ and T is a completely continuous mapping.

It is easy to see that a single-valued mapping l: $E_1 \to \operatorname{Ker} L$ is a constant mapping, $l(u) = 0$ for each $u \in E_1$, and the set of all solutions of problem (2.3) is equal to the set of all solutions of the A-BVP (L, φ, l). The proof is complete.

Proof of (2.4). Let us put $r = \int_G f(x)\,dx$, where the function f is given in (2.4) (iii), let $B_r \subset \mathscr{L}^p(G)$ be a ball with centre at zero and radius r and let (L, φ, l) be the A-BVP given in (2.5). Then, in virtue of our assumptions we get

$$\varphi(u) \subset B_r \quad \text{for} \quad u \in E_1.$$

Let us put $r_1 = 2Mr$, where the constant $M > 0$ is given in (2.2). From (2.2) we obtain

$$\|L(u)\|_p > r \quad \text{for each } \|u\|_{m,p} = r_1.$$

Now, in virtue of (III.2.1), the problem (L, φ, l) has at least one solution. The proof is complete.

Remark. Theorem (2.4) remains also true for the multi-valued non-convex mapping F: $\bar{G} \times R^2 \to R^1$, which admits a convex-valued, weakly compact selector (comp. (II.4.1)).

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