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ON SOME MINIMAX SEQUENTIAL DECISION PROBLEMS WITH PARTIAL OBSERVABILITY

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1. Introduction

The paper presents a description and examples of sequential minimax decision problems in models with controlled partial observability. The latter may be interpreted as a control of the performed experiment. A chosen control (observability) and a stopping time form the observation strategy fully characterizing the sample space.

The importance of taking into account observation strategies as essential elements of general statistical models has been discussed by Pleszczyńska and Dąbrowska in [3]. The sequential decision problems investigated here may also be considered as a special type of a sequential experimental design. A general model occurring in a sequential experimental design and some related problems have been presented by Heckendorff in [2].

In the sequential decision problems analysed in the paper the goal is to find an observation strategy and a decision rule (referring to the unknown "true" distribution) such that the mean cost of the experiment is the minimax one and simultaneously the mean loss is sufficiently small. Some asymptotical properties of minimax solutions are investigated.

2. Sequential experiments with partial observability

A description will be presented of sequential experiments with partial observability and related minimax decision problems. The following example is a good introduction to the subject. It will be analysed in detail in Section 3.

Assume that $\{(X_n, Y_n)\}$ is a sequence of i.i.d. bivariate random variables. A distribution of (X_1, Y_1) is known to belong to the parametrized family $\{Q_\vartheta\colon \vartheta \in \Theta\}$. Let $(\Omega, \mathscr{F}, \mathscr{P})$ denote a statistical space referring to the situation, where

$$\Omega = R^2 \times R^2 \times \ldots, \qquad \mathcal{F} = \mathcal{B}(R^2)\otimes\mathcal{B}(R^2)\otimes\ldots,$$

$$\mathcal{P} = \{P_\vartheta: P_\vartheta = Q_\vartheta\otimes Q_\vartheta\otimes\ldots, \vartheta\in\Theta\}.$$

Assume that any realization of the sequence $\{(X_n, Y_n)\}$ can be observed sequentially according to an observation strategy $\lambda = (u, \tau^u)$; $u = \{B_n\}$ is a sequence of events $B_n \in \mathcal{F}$, τ^u is a stopping time with respect to the appropriate sequence of σ-fields of observable events. The given λ specifies the way observations are taken, namely for any $\omega = \{(x_n, y_n)\}\in\Omega$

$$X_i(\omega) = x_i \quad \text{is observed iff} \quad 1 \leqslant i \leqslant \tau^u(\omega),$$

$$Y_i(\omega) = y_i \quad \text{is observed iff} \quad \omega\in B_i,\ 1 \leqslant i \leqslant \tau^u(\omega).$$

This way of taking observations may correspond to a situation where the cost of observation of the second coordinate is relatively high in comparison with the first one.

The σ-field of observable events at the nth moment has the following form:

$$\mathcal{F}_n^u = \sigma\big(X_1, 1_{B_1}, 1_{B_1}(Y_1), \ldots, X_n, 1_{B_n}, 1_{B_n}(Y_n)\big), \qquad n = 1, 2, \ldots$$

According to the sequential way of taking observations it is natural to assume that the observation strategy λ is such that:

$$\tau^u \in \mathcal{T}^u \subset \{\tau: \Omega \to N |\ \tau \text{ is a stopping time w.r.t. } \{\mathcal{F}_n^u\}\},$$

$$u\in U \subset \{\{B_n\}: B_1\in\sigma(X_1), B_n\in\sigma(\mathcal{F}_{n-1}^u, X_n), n > 1\}.$$

Thus the decision whether to take observation Y_n is based on observation X_n and previous observations till the nth moment, $n = 1, 2, \ldots$

The above sequential experiment with partial observability is an example of the observations characterized by the following objects:

1. A statistical space $(\Omega, \mathcal{F}, \mathcal{P})$ representing a model of a random phenomenon under investigation, where Ω denotes the set of realizations and $\mathcal{P} = \{P_\vartheta: \vartheta\in\Theta\}$ is a parametrized family of distributions on the σ-field $\mathcal{F}$ containing the "true" distribution.

2. A sequence $\{\mathcal{F}_n\}$ of increasing σ-subfields of $\mathcal{F}$ corresponding to full observability of any realization $\omega\in\Omega$ at each nth moment.

3. A set of controls U such that any $u\in U$ specifies the way of observation of any $\omega\in\Omega$ at any nth step of the experiment.

4. Sequences $\{\mathcal{F}_n^u\}$, $u\in U$, of increasing σ-subfields of $\mathcal{F}$, such that $\mathcal{F}_n^u \subset \mathcal{F}_n$ for any $u\in U$, $n = 1, 2, \ldots$ $\mathcal{F}_n^u$ is interpreted as the σ-field of observable events at the nth moment under the control u.

5. Sets of stopping times $\mathcal{T}^u$, $u\in U$, such that

$$\mathcal{T}^u \subset \{\tau: \Omega \to N \,|\, \tau \text{ is a stopping time w.r.t. } \{\mathcal{F}_n^u\}\}.$$

Any observation strategy $\lambda = (u, \tau^u)$, from the set $\Lambda = \bigcup_{u \in U} \{u\} \times \mathcal{T}^u$, fully characterizes the experiment performed. The choice of the "best" observation strategy refers to some optimization problem.

Let us suppose that there are given:

6. A decision space $(\boldsymbol{D}, \mathcal{G})$ with the set of decisions $\boldsymbol{D}$ and the σ-field $\mathcal{G}$ of subsets of $\boldsymbol{D}$.

7. Sets of decision rules Δ_λ referring to λ, $\lambda \in \Lambda$. In a nonrandomized case

$$\Delta_\lambda \subset \{\delta \colon d = \sum_{n=1}^{\infty} \delta_n 1_{\{\tau^u = n\}}, \ \delta_n \colon \Omega \to \boldsymbol{D}, \delta_n \text{ is } \mathcal{F}_n^u \text{ measurable, } n \geqslant 1\}.$$

8. Sequences of loss functions $\{L_n^u\}$, $\{\tilde{L}_n^u\}$, $u \in U$, such that

$$L_n^u \colon \Theta \times \Omega \times \boldsymbol{D} \to R^+, \qquad \tilde{L}_n^u \colon \Theta \times \Omega \times \boldsymbol{D} \to R^+,$$

$L_n^u(\vartheta, \cdot, \cdot)$, $\tilde{L}_n^u(\vartheta, \cdot, \cdot)$ are $\mathcal{F}_n^u \times \mathcal{D}$ measurable.

Let R_1, R_2 denote risk functions, i.e.,

$$R_1(\vartheta, e) = E_\vartheta L_{\tau^u}^u(\vartheta, \cdot, \delta(\cdot)), \qquad R_2(\vartheta, e) = E_\vartheta \tilde{L}_{\tau^u}^u(\vartheta, \cdot, \delta(\cdot)),$$

where $e = (\lambda, \delta)$, $\delta \in \Delta_\lambda$, $\lambda = (u, \tau^u) \in \Lambda$.

There are three possible different formulations of minimax sequential decision problems concerning the above risks. One may look for an observation strategy and a decision rule minimizing the supremum over $\vartheta \in \Theta$ of either the linear combination of the values of R_1, R_2 or the value of one of these functions under some restrictions imposed on the other one. In the latter case a minimax solution $e^0 = (\lambda^0, \delta^0)$, $\delta^0 \in \Delta_{\lambda^0}$, $\lambda^0 = (u^0, \tau^0) \in \Lambda$, is defined such that

$$\sup_{\vartheta \in \Theta} R_i(\vartheta, e^0) = \inf_{e \in E_\eta} \sup_{\vartheta \in \Theta} R_i(\vartheta, e),$$

where $E_\eta = \{e = (\lambda, \delta) \colon \sup_{\vartheta \in \Theta} R_j(\vartheta, e) \leqslant \eta\}$, η being the given positive number, $i \neq j$; $i, j \in \{1, 2\}$.

3. Asymptotical results on some minimax solutions

In this section some asymptotical behaviour of minimax solutions to particular sequential decision problems will be investigated.

Let us consider the model of the sequential experiments described at the beginning of Section 2, with the following additional assumptions:

A1. Distributions Q_ϑ, $\vartheta \in \Theta = (0, 1)$, are such that

$$P_\vartheta(\{Y_i = 1\}) = \vartheta, \qquad P_\vartheta(\{Y_i = 0\}) = 1 - \vartheta,$$

$$\left.\begin{aligned} P_\vartheta(\{X_i \in A\} \mid \{Y_i = 1\}) &= \int_A f_1(x)\,dx, \\ P_\vartheta(\{X_i \in A\} \mid \{Y_i = 0\}) &= \int_A f_2(x)\,dx, \end{aligned}\right. \qquad i = 1, 2, \ldots, A \in \mathcal{B}(R^1),$$

where f_1, f_2 are density functions not proportional over sets of positive measure.

A.2. At any nth step, Y_n is observed iff $X_n \in A$, where A is a chosen nonempty Borel set.
Then $U = \{u = \{B_n\}: B_n = X_n^{-1}(A), A \in \mathscr{B}(R^1), A \neq \emptyset\}$.

A3. For any $u \in U$, $\mathscr{T}^u$ is the set of stopping times with respect to $\{\sigma(I_{B_1}(Y_1), \ldots, I_{B_n}(Y_n))\}$ and with finite expectation for any $\vartheta \in \Theta$.

A4. $(\Theta, \mathscr{B}(\Theta))$ is the decision space.

A5. $\Delta_\lambda = \{\delta: \delta = \sum_{n=1}^{\infty} I_{\{\tau^u = n\}} \delta_n; \delta_n \in \Delta(u, n), E_\vartheta \delta = \vartheta, \vartheta \in \Theta\}$, where $\lambda = (u, \tau^u)$, $\Delta(u, n)$ is the set of estimators of ϑ, $\vartheta \in \Theta$, based on $I_{B_1}(Y_1), \ldots, I_{B_n}(Y_n)$, $n = 1, 2, \ldots$

A6. The cost of observation of any element of the sequence $\{(X_n, Y_n)\}$ is equal to c_1 if (x, y) is observed and $y = 1$, and is equal to c_2 if only x is observed and $y = 0$, where c_1, c_2 are given positive constants.
Then, under the following simplified notation,

$$L_n^u(\vartheta, \omega, d) := L_n^u(\vartheta, \omega) = \sum_{k=1}^{n} c_1 I_{A \times \{1\}}(X_k, Y_k) + c_2 I_{\bar{A} \times \{0\}}(X_k, Y_k)$$

A7. a) $\tilde{L}_n^u(\vartheta, \omega, d) := \tilde{L}(\vartheta, d, u) = (d - \vartheta)^2 I(\vartheta, u)$, $d, \vartheta \in \Theta$, $u \in U$, where $I(\vartheta, u)$ is Fisher's information for any $I_{B_k}(Y_k)$, $k = 1, 2, \ldots$

b) $\tilde{L}_n^u(\vartheta, \omega, d) := \tilde{L}(\vartheta, d, u) = (d - \vartheta)^2$, $d, \vartheta \in \Theta$, $u \in U$.

In the light of A6, A7 the values of R_1, R_2 represent the mean cost of the experiment and the mean loss, respectively, associated with an observation strategy λ and a decision rule δ.

Let us introduce, for simplicity, the following notation:

$$R_1(\vartheta, \lambda) = E_\vartheta L_{\tau^u}^u(\vartheta, \lambda),$$

$$R_2(\vartheta, \delta, u) = E_\vartheta \tilde{L}(\vartheta, \delta, u), \qquad \vartheta \in \Theta, \delta \in \Delta_\lambda, \lambda = (u, \tau^u) \in \Lambda,$$

$$\hat{R}_1(\eta) = \inf_{\lambda \in \Lambda_\eta} \sup_{\vartheta \in \Theta} R_1(\vartheta, \lambda),$$

$$R_1^* = \inf_{\lambda \in \Lambda_\eta^*} \sup_{\vartheta \in \Theta} R_1(\vartheta, \lambda),$$

where η is any positive number and

$$\Lambda_\eta = \{\lambda = (u, \tau^u) \in \Lambda: \exists\, \delta \in \Delta_\lambda \sup_{\vartheta \in \Theta} R_2(\vartheta, \delta, u) \leqslant \eta\},$$

$$\Lambda_\eta^* = \{\lambda = (u, \tau^u) \in \Lambda_\eta: \exists\, n \in N\, P_\vartheta(\{\tau^u = n\}) = 1, \vartheta \in \Theta\}.$$

Proposition 1. *Under assumptions* A1–A6, A7 a) *the following equalities hold*:

$$\lim_{\eta \to 0^+} \frac{\hat{R}_1(\eta)}{R_1^*(\eta)} = 1, \tag{1}$$

$$\hat{R}_1(\eta) = R_1^*(\eta), \qquad \text{if } \frac{1}{\eta} = \operatorname{Ent}\left(\frac{1}{\eta}\right), \ \eta > 0. \tag{2}$$

Proof. Let $\vartheta \in \Theta$, $\delta \in \Delta_\lambda$, $\lambda = (u, \tau^u) \in \Lambda$. The form of the sequences of loss functions $\{L_n^u\}$, $\{\tilde{L}_n^u\}$ implies that

$$R_1(\vartheta, \lambda) = r(\vartheta, A) E_\vartheta \tau^u, \tag{3}$$

$$R_2(\vartheta, \delta, u) = I(\vartheta, u) E_\vartheta (\delta - \vartheta)^2 \tag{4}$$

where $A \in \mathscr{B}(R^1)$ is such that $B_n = X_n^{-1}(A)$, $u = \{B_n\}$, and

$$r(\vartheta, A) = c_1 \vartheta \int_A f_1(x)\,dx + c_2 (1 - \vartheta) \int_{\bar{A}} f_2(x)\,dx.$$

Assumptions A1–A5 imply that the generalized Cramér–Rao inequality is fulfilled for the decision rule δ. Hence, in the light of (4), it is easy to obtain the inequality

$$R_2(\vartheta, \delta, u) \geqslant (E_\vartheta \tau^u)^{-1}. \tag{5}$$

If in addition τ^u is a deterministic stopping time which is equal to n and

$$\delta = \frac{1}{n} \sum_{k=1}^{n} I_{B_k}(Y_k) \left(\int_A f_1(x)\,dx\right)^{-1},$$

then (5) changes into equality.

From (3) and (5) it is easy to conclude that, for any $\lambda = (u, \tau^u) \in \Lambda_\eta$, $\lambda^* = (u, n) \in \Lambda_\eta^*$, $(n \geqslant 1/\eta)$, we have

$$R_1(\vartheta, \lambda) \geqslant r(\vartheta, A) \frac{1}{\eta}, \qquad R_1(\vartheta, \lambda^*) = r(\vartheta, A) \frac{1}{n}.$$

Hence the following relations hold:

$$R_1^*(\eta) = r(\vartheta^*, A^*) \left(\operatorname{Ent}\left(\frac{1}{\eta}\right) + 1 \right), \tag{6}$$

$$\hat{R}_1(\eta) \geqslant r(\vartheta^*, A^*) \frac{1}{\eta}, \tag{7}$$

where $\vartheta^* \in \Theta$, $A^* \in \mathscr{B}(R^1)$ are such as in the classical minimax discrimination problem, i.e., $r(\vartheta^*, A^*) = \min_{A \in \mathscr{B}(R^1)} \max_{\vartheta \in \Theta} r(\vartheta, A)$. Relations (6) and (7) complete the proof as $R_1^*(\eta) \geqslant \hat{R}_1(\eta)$. ∎

In the sequel the following lemma will be useful.

LEMMA. *Let* $\{Z_n\}$ *be a sequence of i.i.d. random variables with 0–1 distribution* p_ϑ *such that* $p_\vartheta(\{1\}) = \vartheta$, $p_\vartheta(\{0\}) = 1-\vartheta$, $\vartheta \in \Theta = \langle a, b\rangle$, $0 < a < b < 1$. *For any positive* ε *and* μ *such that* $\varepsilon < a < b < 1-\varepsilon$, *let* τ_μ *be a stopping time defined as follows*:

$$\tau_\mu = \inf\left\{n \geqslant 1:\ n \geqslant \frac{\hat{\vartheta}_n(1-\hat{\vartheta}_n)}{\mu} I_{C_n} + \frac{1}{\mu} I_{\bar{C}_n}\right\}, \tag{8}$$

where $\hat{\vartheta}_n = \frac{1}{n}\sum_{k=1}^{n} Z_k$, $C_n = \{\hat{\vartheta}_n > \varepsilon,\ 1-\hat{\vartheta}_n > \varepsilon\}$, $n = 1, 2, \ldots$ *Let* $\beta(\mu) = \sup_{\vartheta\in\Theta} \frac{\mu E_\vartheta \tau_\mu}{\vartheta(1-\vartheta)}$. *Under the above assumptions the following relations are fulfilled*:

L1. $\lim_{\mu\to 0} \beta(\mu) = 1$,

L2. $\sup_{\vartheta\in\Theta} E_\vartheta |\hat{\vartheta}_{\tau_\mu} - \vartheta|^2 \leqslant \mu\beta(\mu)\delta(\mu)$, *where* $\lim_{\mu\to 0}\delta(\mu) = 1$.

Proof. It is easy to present τ_μ in the following form:

$$\tau_\mu = \inf\left\{n \geqslant 1:\ n \geqslant \frac{\vartheta(1-\vartheta)}{\mu} + \frac{W_n}{\mu} + \frac{V_n}{\mu}\right\}, \tag{9}$$

where

$$W_n = (\hat{\vartheta}_n - \vartheta)(1-\hat{\vartheta}_n - \vartheta), \tag{10}$$

$$V_n = [1-\hat{\vartheta}_n(1-\hat{\vartheta}_n)]\, I_{\bar{C}_n}, \qquad n = 1, 2, \ldots \tag{11}$$

Hence, in the light of (9), for any $\vartheta \in \Theta$ the following three inequalities are fulfilled:

$$\tau_\mu \leqslant 1 + \frac{\vartheta(1-\vartheta)}{\mu} + \frac{W_{\tau_\mu}}{\mu} + \frac{V_{\tau_\mu}}{\mu},$$

$$\frac{\mu E_\vartheta \tau_\mu}{\vartheta(1-\vartheta)} \leqslant \frac{\mu}{\vartheta(1-\vartheta)} + 1 + \frac{E_\vartheta |W_{\tau_\mu}|}{\vartheta(1-\vartheta)} + \frac{E_\vartheta |V_{\tau_\mu}|}{\vartheta(1-\vartheta)},$$

$$\frac{\mu E_\vartheta \tau_\mu}{\vartheta(1-\vartheta)} \geqslant 1 - \frac{E_\vartheta |W_{\tau_\mu}|}{\vartheta(1-\vartheta)} - \frac{E_\vartheta |V_{\tau_\mu}|}{\vartheta(1-\vartheta)}.$$

Thus in order to prove L1 it suffices to show that $\sup_{\vartheta\in\Theta} E_\vartheta |W_{\tau_\mu}|$ and $\sup_{\vartheta\in\Theta} E_\vartheta |V_{\tau_\mu}|$ converge to 0 as $\mu \to 0$. This would be achieved if one could prove that

$$\sup_{\vartheta\in\Theta} E_\vartheta |\hat{\vartheta}_{\tau_\mu} - \vartheta|^2 \to 0 \qquad \text{as} \quad \mu \to 0, \tag{12}$$

$$\sup_{\vartheta\in\Theta} P_\vartheta(\bar{C}_{\tau_\mu}) \to 0 \qquad \text{as} \quad \mu \to 0. \tag{13}$$

It is easy to see that

$$\tau_\mu \leqslant \frac{1}{\mu}+1,$$

$$\frac{1}{\tau_\mu} \leqslant \mu \frac{1}{\hat{\vartheta}_{\tau_\mu}(1-\hat{\vartheta}_{\tau_\mu}) 1_{C_{\tau_\mu}} + 1_{\bar{C}_{\tau_\mu}}} \leqslant \frac{\mu}{\varepsilon^2}. \tag{14}$$

Hence $E_\vartheta\left(\frac{1}{\tau_\mu}\right)^4$, $E_\vartheta \tau_\mu$ exist. Now, using Schwartz' inequality and Wald's second equality, it is easy to obtain

$$E_\vartheta |\hat{\vartheta}_{\tau_\mu} - \vartheta|^2 = E_\vartheta \left(\frac{1}{\tau_\mu} \sum_{k=1}^{\tau_\mu} (Z_k - \vartheta)\right)^2 \leqslant \sqrt{E_\vartheta \left(\frac{1}{\tau_\mu}\right)^4} E_\vartheta \tau_\mu \vartheta(1-\vartheta) \sqrt{1+\frac{1}{E_\vartheta \tau_\mu}}. \tag{15}$$

(12) is the consequence of (14) and (15). Elementary considerations allows to show that there exists a constant C such that

$$P_\vartheta(\bar{C}_{\tau_\mu}) \leqslant C E_\vartheta |\hat{\vartheta}_{\tau_\mu} - \vartheta|, \qquad \vartheta \in \Theta.$$

Thus, in the light of (12), (13) is fulfilled, and the proof of L1 is complete.

To prove L2 let us note that (15) and the definition of $\beta(\mu)$ imply the following inequality:

$$E_\vartheta |\hat{\vartheta}_{\tau_\mu} - \vartheta|^2 \leqslant \frac{\beta(\mu)}{\mu} \sqrt{E_\vartheta \left(\frac{1}{\tau_\mu}\right)^4} \vartheta^2 (1-\vartheta)^2 \sqrt{1+\frac{1}{E_\vartheta \tau_\mu}}. \tag{16}$$

Formulas (9), (10), (11) guarantee that

$$V_{\tau_\mu} \geqslant 0,$$

$$|W_{\tau_\mu}| \leqslant |\hat{\vartheta}_{\tau_\mu} - \vartheta|,$$

$$\mu \tau_\mu \geqslant \vartheta(1-\vartheta) - |\hat{\vartheta}_{\tau_\mu} - \vartheta|.$$

Hence, if we take into account (14), the following inequality is fulfilled:

$$\frac{\vartheta(1-\vartheta)}{\mu \tau_\mu} \leqslant 1 + \frac{|\hat{\vartheta}_{\tau_\mu} - \vartheta|}{\varepsilon^2}. \tag{17}$$

As a consequence of (16) one may obtain

$$E_\vartheta |\hat{\vartheta}_{\tau_\mu} - \vartheta|^2 \leqslant \mu \beta(\mu) \left[\sup_{\vartheta \in \Theta} E_\vartheta \left(\frac{\vartheta(1-\vartheta)}{\mu \tau_\mu}\right)^4\right]^{1/2} \left(1+\frac{1}{E_\vartheta \tau_\mu}\right)^{1/2}. \tag{18}$$

Relations (17), (18) guarantee L2.

PROPOSITION 2. *Let* $\Theta = \langle a, b \rangle$, *where* $0 < a < b < 1$, $b > 0.5$, *and let* A1–A4, A6, A7b) *and* A5 *without the assumption* $E_\vartheta \delta = \vartheta$ *be fulfilled. Then*

$$\lim_{\eta \to 0} \frac{\hat{R}_1(\eta)}{R_1^*(\eta)} < 1.$$

Proof. For any $\lambda = (u, n) \in \Delta_\eta^*$, $\delta \in \Delta(u, n)$ and $\vartheta \in \Theta$ the following inequalities hold:

$$\eta \geqslant E_\vartheta(\delta - \vartheta)^2 \geqslant \frac{1}{I(\vartheta, u)\, n}.$$

Hence

$$n \geqslant \frac{1}{\eta} \sup_{\vartheta \in \Theta} \frac{1}{I(\vartheta, u)}. \tag{19}$$

Let us denote $\gamma = \int_A f_1(x)\,dx$, $\bar{\gamma} = \int_{\bar{A}} f_2(x)\,dx$.

As

$$\frac{1}{I(\vartheta, u)} = \frac{\vartheta(1-\vartheta\gamma)}{\gamma},$$

in the light of (19) we have

$$R_1^*(\eta) \geqslant \inf_{u \in U} \left\{ \sup_{\vartheta \in \Theta} \left[c_1\, \vartheta + c_2(1-\vartheta)\frac{\bar{\gamma}}{\gamma} \right] \sup_{\vartheta \in \Theta} \frac{\vartheta(1-\vartheta\gamma)}{\eta} \right\}$$

$$\geqslant \frac{1}{\eta} c_1\, b \inf_{0 \leqslant \gamma \leqslant 1} \sup_{\vartheta \in \Theta} \vartheta(1-\vartheta\gamma) \geqslant \frac{c_1\, b}{\eta} \sup_{\vartheta \in \Theta} \vartheta(1-\vartheta). \tag{20}$$

Let $\lambda_\mu = (u^0, \tau_\mu)$, where $u^0 = \{B_n^0\}$, $B_n^0 = X_n^{-1}(R^1)$, and let τ_μ be a stopping time given by (8). In the light of the lemma it suffices to take $\mu > 0$ sufficiently small in order to obtain $\lambda_\mu \in \Lambda_\eta$.
Hence

$$\hat{R}_1(\eta) \leqslant \sup_{\vartheta \in \Theta} R_1(\vartheta, \lambda_\mu) = \sup_{\vartheta \in \Theta} [c_1\, \vartheta E_\vartheta \tau_\mu],$$

$$\hat{R}_1(\eta) \leqslant \frac{c_1}{\mu} \sup_{\vartheta \in \Theta} \left[\frac{\mu E_\vartheta \tau_\mu}{\vartheta(1-\vartheta)} \right] \sup_{\vartheta \in \Theta} [\vartheta^2(1-\vartheta)]$$

$$= \frac{c_1}{\mu} \beta(\mu) \sup_{\vartheta \in \Theta} \vartheta^2(1-\vartheta). \tag{21}$$

(20) and (21) imply the inequality

$$\frac{\hat{R}_1(\eta)}{R_1^*(\eta)} \leqslant \beta(\mu) \frac{\eta}{\mu} \alpha,$$

where

$$\alpha = \frac{\sup_{a \leqslant \vartheta \leqslant b} \vartheta^2(1-\vartheta)}{b \sup_{a \leqslant \vartheta \leqslant b} \vartheta(1-\vartheta)}.$$

In order to complete the proof it suffices to note that $\alpha < 1$ if $b > 0.5$ and that in the light of the lemma there exists a $\mu(\eta)$ such that $\lim_{\eta \to 0} \frac{\eta}{\mu(\eta)} = 1$ and $E_\vartheta |\hat{\vartheta}_{\tau_{\mu(\eta)}} - \vartheta|^2 \leqslant \eta$, $\vartheta \in \Theta$. ∎

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