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Delta-convex mappings between Banach spaces and applications

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Abstract

We investigate delta-convex mappings between normed linear spaces. They provide a generalization of functions which are representable as a difference of two convex functions (labelled as δ -convex or d.c. functions) and are considered in many articles. We show that delta-convex mappings have many good differentiability properties of convex functions and the class of them is very stable. For example, the class of locally delta-convex mappings is closed under superpositions and (in some situations) under inverses. Some operators which occur naturally in the theory of integral and differential equations are shown to be delta-convex. As an application of our general results, we show that some "solving operators" of such equations are delta-convex and consequently have good differentiability properties. An implicit function theorem for quasi-differentiable functions is an another application.

0. Introduction and notations

Real functions on R^n which are representable as a difference of two convex functions are labelled as δ -convex ([5], [21]) or d.c. functions ([8], [19]). The class of d.c. functions is the smallest linear space containing all convex functions and it contains also all C^2 -functions. All d.c. functions have many good properties of convex functions (e.g. they have all one-sided directional derivatives at all points and have the second derivative almost everywhere). Further, the class of d.c. functions is very stable, e.g. it forms a lattice and is closed under multiplication and division. These stability facts were generalized in an important article by P. Hartman [18]. He defined d.c. mappings between Euclidean spaces as mappings with d.c. components and proved that the superposition of d.c. mappings is d.c.

It is not difficult to give simple "inner" characterizations of d.c. functions (or mappings) of one variable (cf. [31]). For example, $f: R \rightarrow R$ is d.c. iff it has the right derivative at all points and f'_+ is locally of finite variation. Local finiteness of the convexity gives another characterization of d.c. functions (cf. [31] and Definition 2.1). No "inner" characterization of d.c. functions of many variables is known.

As far as we know, d.c. functions of many variables were at first investigated by A. D. Aleksandrov ([2], [3]) in 1949; he suggested to investigate surfaces which are (locally) graphs of d.c. functions. The main reason was that these surfaces generalize both convex surfaces and classical (very smooth) surfaces, and it is still possible to build a rich geometrical theory on them. It seems that this project was not realized since there was found a more general suitable class of surfaces investigated in [27]. However, there exist at least two subsequent interesting articles on d.c. functions ([23], [39]) by Russian mathematicians.

The notion of d.c. functions on an infinite-dimensional spaces arises quite naturally in [37], where the sets of points of Gâteaux non-differentiability of continuous convex functions on separable Banach spaces are investigated. Since only continuous convex functions are of an interest, d.c. functions are defined as differences of two continuous convex functions.

In recent years d.c. functions were also considered in articles concerning the theory of the non-smooth optimization (e.g. [19], [20]). In the same theory Demjanov, Rubinov and others (e.g. [11], [12], [13]) considered quasidifferen-

tiable functions, which provide a further (slight) generalization of d.c. functions (see Section 6.D).

In the present article we generalize the Hartman's [18] concept of d.c. mappings between Euclidean spaces to the notion of delta-convex mappings between normed linear spaces and build a theory which shows, as we believe, that our generalization is good. However, we obtain also new "finite-dimensional" results concerning d.c. functions and quasidifferentiable functions, e.g. Implicit Function Theorems for d.c. functions (Section 5) and quasidifferentiable functions (Section 6.D). Also Mixing Lemma (4.8) provides a useful tool for recognition of d.c. functions which seems to be new.

If X, Y are infinite-dimensional normed linear spaces, it is by no means obvious which mappings $F: X \rightarrow Y$ provide the "right" generalization of d.c. functions.

The "right" generalization is obvious in the case $Y = R^n$ (all components are d.c.). If $X = R$ then it is possible to define the convexity of F and therefore the statements that F has locally finite convexity and that F'_+ has locally finite variation (which are equivalent one to another, see Theorem 2.3) have the good sense. Consequently it seems that in this case ($X = R$) the "right" generalizations of d.c. functions is provided by mappings (curves) with locally finite convexity. Such curves were investigated and applied in [34].

In the general case, two ways of a generalization of d.c. functions seem to be natural:

D1 (weak definition). A mapping $F: X \rightarrow Y$ is d.c. if $y^* \circ F$ is d.c. function for any $y^* \in Y^*$.

D2 (definition via convex operators). If Y is an ordered normed linear space it is possible to define a convex operator $G: X \rightarrow Y$ as a direct generalization of a convex function (see e.g. [4], [6]). We say that $F: X \rightarrow Y$ is d.c. if $F = G_1 - G_2$ where G_1, G_2 are convex continuous operators.

(Note that Aronszajn's [4] definition of convexoid mappings and Demjanov-Rubinov's [12] definition of quasidifferentiable mappings between Banach spaces are based on the same idea.)

Both these definitions give rather general classes of mappings and provide the canonical generalization of d.c. functions in the case $Y = R^n$ but have the following disadvantages:

(a) In the case $X = R$ both these definitions give classes of d.c. curves which do not coincide with the class of curves with locally finite convexity (see Example 6.1).

(b) Mappings which are d.c. by these definitions lack many good properties of d.c. functions on X . For example, the convex operator $F: l_2 \rightarrow l_2$ defined by $F(x_1, x_2, \dots) = (|x_1|, |x_2|, \dots)$ is Fréchet differentiable at no point.

Note, however, that under some (rather restrictive) additional assumptions convex operators are generically Fréchet differentiable [6].

(c) The stability (e.g. with respect to superpositions) of the corresponding classes of (locally) d.c. mappings is doubtful (cf. [12], where for quasidifferentiable mappings a superposition theorem is proved, but only under some additional assumptions).

Our definition is a slight modification of the weak definition D1 only. Namely we say that $F: X \rightarrow Y$ is *delta-convex* if there exists a continuous convex function f on X such that $y^* \circ F + f$ is a continuous convex function for any $y^* \in Y^*$, $\|y^*\| = 1$. Roughly speaking, a mapping is delta-convex if it is uniformly weakly d.c. (this means that the functions $\{y^* \circ F: \|y^*\| = 1\}$ are in a very natural sense uniformly d.c.).

This definition has none of the disadvantages (a), (b), (c). In fact, we were led to our definition namely by the attempt to give a definition which has not the disadvantage (a). As regards (b), we shall prove that a delta-convex mapping F copies many differentiability properties of the convex function f .

The stability of locally delta-convex mappings with respect to superpositions was proved in [35]. The present proof uses an essentially new characterization of delta-convex mappings (Proposition 1.13) and therefore it is essentially simpler. The method based on this characterization enables us also to prove that under some assumptions the inverse of a delta-convex mapping is delta-convex. Unfortunately, we do not know whether these additional assumptions are necessary (see Problems 1,2). In the finite-dimensional case it is possible to omit them, we prove this by the "projection method" of [37] which works in the finite-dimensional case only. The same holds also for implicitly defined mappings.

The very important point of our article is that some mappings which naturally arise in the theory of integral and differential equations (e.g. the Nemyckii and Hammerstein operators) are under some (not too severe) conditions delta-convex and consequently have many good differentiability properties. We are also able to use our "inverse mapping theorem" and to prove that some "solving operators" for integral or differential equations are delta-convex. Thus we are able to obtain some new stability results.

Notation. The set of all real numbers will be denoted by R and the symbol μ_n stands for the Lebesgue measure on R^n . The subdifferential of a convex function f at a point x is denoted by $\partial f(x)$.

In this paper, all normed linear spaces are real. Unless otherwise specified, the same symbol $\|\cdot\|$ is used for norms of various normed linear spaces that enter the discussion as this does not entail any confusion. The open ball with center x and radius r is denoted by $U(x, r)$. If X, Y are normed linear spaces, then the space of continuous linear operators on X into Y is denoted by $L(X, Y)$. The operator $\text{Id} \in L(X, X)$ is the identity operator.

If X, Z are normed linear spaces, $D \subset X$, $x \in D$, $y \in X$ and if $F: D \rightarrow Z$ is a mapping, then we define the *one-sided directional derivative* of F at x in the direction y as

$$F'(x, y) = \lim_{t \rightarrow 0^+} \frac{F(x+ty) - F(x)}{t}.$$

We shall use the symbol $F: A \rightarrow B$ if a mapping F is defined at all points of A . If $f(x, y)$ is a mapping, then the partial mapping $y \rightarrow f(x, y)$ is denoted by $f(x, \cdot)$.

Let $(X, \rho), (Y, \delta)$ be metric spaces, $F: X \rightarrow Y$ be a mapping and $K > 0$. We say that F is *K-Lipschitz* if for each $x_1, x_2 \in X$

$$\delta(F(x_1), F(x_2)) \leq K\rho(x_1, x_2).$$

The minimal K is denoted by $\text{Lip}(F)$. The mapping F is called *bi-Lipschitz* if both F and F^{-1} are Lipschitz.

If X is a normed linear space and $F: [a, b] \rightarrow X$ is a mapping, then $\int_a^b F$ denotes the variation of F and $\mathbf{K}F$ denotes the convexity of F (see 2.1). The right derivative of F at a is denoted by $F'_+(a)$.

1. Basic properties of delta-convex mappings

DEFINITION 1.1. Let X, Y be normed linear spaces, $A \subset X$ be an open convex set and $F: A \rightarrow Y$ be a mapping. We shall say that F is *delta-convex mapping* (on A) if there exists a continuous convex function f on A such that $f + y^* \circ F$ is a continuous convex function on A for any functional $y^* \in Y^*$, $\|y^*\| = 1$. We shall say that F is *controlled* by f or that F is a delta-convex mapping with a control function f .

Note 1.2. (a) It is possible to omit the assumption that f is a continuous convex function in Definition 1.1 since

$$f = \frac{1}{2} \cdot (f + y^* \circ F) + \frac{1}{2} \cdot (f + (-y^*) \circ F).$$

(b) The definition has a good sense also in the case X is an affine subspace of a normed linear space.

The proof of the following lemma is obvious.

LEMMA 1.3. Let X, Y be normed linear spaces and $A \subset X$ be an open convex set. Then the following assertions hold.

(a) If $F: A \rightarrow Y$ is controlled by f on A and $|a| \leq b$, then aF is controlled by bf on A .

(b) If F_1, \dots, F_n are controlled by f_1, \dots, f_n on A then $F_1 + \dots + F_n$ is controlled by $f_1 + \dots + f_n$ on A .

(c) The system of all delta-convex mappings $F: A \rightarrow Y$ is a linear space which contains all continuous affine maps.

Note 1.4. If $c > 0$ and we write $\|y^*\| \leq c$ instead of $\|y^*\| = 1$ in Definition 1.1, we obtain (by 1.3 (a)) an equivalent definition. Therefore delta-convexity of F depends on the topologies of X, Y only. Consequently it is not necessary to specify norms on product spaces and to consider non-Euclidean finite-dimensional spaces in some cases.

The proof of the following lemma is obvious.

LEMMA 1.5. Let X, Y, Z, T be normed linear spaces, let $A \subset X$ and $B \subset Y$ be open convex sets. Suppose that $F: A \rightarrow Y$ is a delta-convex mapping with a control function f on A and let $G: Z \rightarrow X, H: Y \rightarrow T$ be continuous affine mappings. Then the following assertions hold.

(a) The mapping $H \circ F$ is delta-convex with the control function $\text{Lip}(H) \cdot f$ on A .

(b) If $G(B) \subset A$ then $F \circ G$ is delta-convex with the control function $f \circ G$ on B .

LEMMA 1.6. Let A be an open convex subset of a normed linear space X .

(a) If f is a continuous convex function on A then f is delta-convex on A with a control function f .

(b) A real function on A is delta-convex iff it is a difference of two continuous convex functions.

Proof. The assertion (a) is obvious, (b) follows from (a), 1.3 (c) and 1.1.

LEMMA 1.7. Let X, Y_1, \dots, Y_n be normed linear spaces, $A \subset X$ be an open convex set and $F = (F_1, \dots, F_n): A \rightarrow Y_1 \times \dots \times Y_n$ be a mapping. Then F is delta-convex on A iff all F_1, \dots, F_n are delta-convex on A .

Proof. Let F_1, \dots, F_n be controlled by f_1, \dots, f_n . Consider the maximum-norm on $Y_1 \times \dots \times Y_n$ (cf. 1.4) and choose $y^* \in (Y_1 \times \dots \times Y_n)^*$, $\|y^*\| = 1$. Then $y^* \circ F = y_1^* \circ F_1 + \dots + y_n^* \circ F_n$, where $y_i^* \in Y_i^*$, $\|y_i^*\| \leq 1$. Consequently

$$y^* \circ F + (f_1 + \dots + f_n) = (y_1^* \circ F_1 + f_1) + \dots + (y_n^* \circ F_n + f_n)$$

is a continuous convex function and we have that F is controlled by $f_1 + \dots + f_n$. The converse implication is obvious.

COROLLARY 1.8. Let X be a normed linear space, $A \subset X$ be an open convex set and $F: A \rightarrow \mathbb{R}^n$ be a mapping. Then F is delta-convex iff each component of F is representable as a difference of two continuous convex functions.

The following lemma is well known (see e.g. the proof of Theorem 41.B in [31]).

LEMMA 1.9. Let f be a convex function defined on an open convex subset A of a normed linear space X . Let f be bounded from above on $U(x_0, 2r) \subset A$. Then f is Lipschitz on $U(x_0, r)$.

PROPOSITION 1.10. Every delta-convex mapping is locally Lipschitz.

Proof. Let X, Y be normed linear spaces and $A \subset X$ be open and convex. Let $F: A \rightarrow Y$ be delta-convex with a control function f on A . Let $x_0 \in A$ be arbitrary. Choose $r > 0$ and M such that $f \leq M$ on $U(x_0, 2r) \subset A$, take arbitrary $y^* \in Y^*$ with $\|y^*\| = 1$. Denote $g = y^* \circ F$. Then by Definition 1.1 (used for y^* and $-y^*$) $g = c_1 - f$ and $g = f - c_2$, where c_1, c_2 are continuous convex functions on A . Choose $x_2^* \in \partial c_2(x_0)$. Then for any $x \in U(x_0, 2r)$ the following inequality holds:

$$\begin{aligned} g(x) &= f(x) - f(x_0) - [c_2(x) - c_2(x_0)] + g(x_0) \\ &\leq f(x) - f(x_0) - \langle x - x_0, x_2^* \rangle + g(x_0) \leq M - f(x_0) + 2r \|x_2^*\| + g(x_0). \end{aligned}$$

This implies that g is bounded from above on $U(x_0, 2r)$. Then the same is true for $c_1 = g + f$. By Lemma 1.9, c_1 and f are Lipschitz on $U(x_0, r)$. Consequently g is Lipschitz on $U(x_0, r)$, too. In fact, we have shown that for any $y^* \in Y^*$

$$\sup \left\{ \left\langle \frac{F(x) - F(y)}{\|x - y\|}, y^* \right\rangle : x, y \in U(x_0, r), x \neq y \right\} < \infty.$$

Then all functionals $\frac{F(x) - F(y)}{\|x - y\|}$ on Y^* are uniformly bounded by the standard Banach–Steinhaus Theorem, i.e. F is Lipschitz on $U(x_0, r)$.

In Section 3 we shall prove that delta-convex mappings have some good differentiability properties. Now we prove a proposition which shows that sufficiently smooth mappings on Hilbert spaces are delta-convex. It generalizes the corresponding result for delta-convex functions on R^n from [2].

Let X, Y be normed linear spaces and let $A \subset X$ be an open set. A mapping $F: A \rightarrow Y$ is called to be of the class $C^{1,1}(A)$ if the Fréchet derivative $F'(x)$ exists for any $x \in A$ and the mapping F' is Lipschitz on A .

PROPOSITION 1.11. Let X be a Hilbert space and let F be a mapping from an open convex set $A \subset X$ into a normed linear space Y . If F is of the class $C^{1,1}(A)$ then F is delta-convex on A with a control function $\text{Lip}(F') \cdot \|\cdot\|^2$.

Proof. Denote $L = \text{Lip}(F')$ and choose arbitrary $y^* \in Y^*$ with $\|y^*\| = 1$. We shall prove that the function $y^* \circ F + L \|\cdot\|^2$ has a continuous support at each point $x_0 \in X$ and hence it is a continuous convex function (cf. [31]).

Let $x \in A$ be arbitrary. Then, by the Mean Value Theorem, there exists a point $z = bx + (1-b)x_0$, $b \in (0, 1)$ such that

$$\begin{aligned}
& \langle y^*, F(x) \rangle + L\|x\|^2 - \langle y^*, F(x_0) \rangle - L\|x_0\|^2 \\
&= \langle y^* \circ F'(z), x - x_0 \rangle + L\|x\|^2 - L\|x_0\|^2 \\
&= \langle y^* \circ F'(x_0), x - x_0 \rangle - \langle y^* \circ F'(x_0) - y^* \circ F'(z), x - x_0 \rangle + L\|x\|^2 \\
&\quad - L\|x_0\|^2 \geq \langle y^* \circ F'(x_0), x - x_0 \rangle - L\|z - x_0\| \cdot \|x - x_0\| + L\|x\|^2 \\
&\quad - L\|x_0\|^2 \geq \langle y^* \circ F'(x_0), x - x_0 \rangle + L(-\|x - x_0\|^2 + \|x\|^2 - \|x_0\|^2) \\
&= \langle y^* \circ F'(x_0) + 2Lx_0, x - x_0 \rangle.
\end{aligned}$$

COROLLARY 1.12. *Let X, Y be Hilbert spaces, Z be a normed linear space and let $B: X \times Y \rightarrow Z$ be a continuous bilinear mapping. Then B is delta-convex on $X \times Y$.*

Proof. Since $(B'(x_1, y_1) - B'(x_2, y_2))(h, k) = B(h, y_1 - y_2) + B(x_1 - x_2, k)$, we have

$$B \in C^{1,1}(X \times Y).$$

Now we shall give several characterizations of delta-convex mappings, which are very useful. Moreover, it is not clear whether the alternative definition of delta-convex mappings given by condition (iii) of Proposition 1.13 (assuming continuity of F and f) is not more natural than our Definition 1.1.

PROPOSITION 1.13. *Let X, Y be normed linear spaces, $A \subset X$ be an open convex set, $F: A \rightarrow Y, f: A \rightarrow R$. Then the following assertions are equivalent:*

(i) $y^* \circ F + f$ is convex on A for any y^* from the unit sphere in Y^* ;

$$(ii) \left\| \sum_{i=1}^n \lambda_i F(x_i) - F\left(\sum_{i=1}^n \lambda_i x_i\right) \right\| \leq \sum_{i=1}^n \lambda_i f(x_i) - f\left(\sum_{i=1}^n \lambda_i x_i\right)$$

whenever $x_1, \dots, x_n \in A, \lambda_1 \geq 0, \dots, \lambda_n \geq 0$ and $\sum \lambda_i = 1$;

$$(iii) \|aF(x) + bF(y) - F(ax + by)\| \leq af(x) + bf(y) - f(ax + by)$$

whenever $x \in A, y \in A, a \geq 0, b \geq 0, a + b = 1$;

$$(iv) \left\| \frac{F(z + kv) - F(z)}{k} - \frac{F(z) - F(z - hv)}{h} \right\| \leq \frac{f(z + kv) - f(z)}{k} - \frac{f(z) - f(z - hv)}{h}$$

whenever $z \in A, v \in X, z + kv \in A, z - hv \in A, k > 0, h > 0$.

Moreover, if F is bounded on a ball $B \subset A$ and f upper bounded on B , then these conditions are equivalent to

(v) F is delta-convex with a control function f on A .

Proof. (a) (i) implies that $-y^* \circ F + f$ is convex for any $y^* \in Y^*$ with $\|y^*\| = 1$. Let $x_1, \dots, x_n \in A, \lambda_1, \dots, \lambda_n$ be as in (ii). Then

$$(-y^* \circ F + f)\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i (-y^* \circ F + f)(x_i)$$

and consequently

$$\left\langle \sum_{i=1}^n \lambda_i F(x_i) - F\left(\sum_{i=1}^n \lambda_i x_i\right), y^* \right\rangle \leq \sum_{i=1}^n \lambda_i f(x_i) - f\left(\sum_{i=1}^n \lambda_i x_i\right).$$

Taking the least upper bound of the left-hand side, we get (ii). The implication (ii) \Rightarrow (iii) is trivial.

(b) Suppose that (iii) holds and let v, z, k, h be as in (iv). Put $x = z - hv$, $y = z + kv$. Then $z = ax + by$ with $a = \frac{k}{k+h}$, $b = \frac{h}{k+h}$. Hence by (iii)

$$\begin{aligned} & \left\| \frac{k}{k+h} \cdot F(z-hv) + \frac{h}{k+h} \cdot F(z+kv) - F(z) \right\| \\ & \leq \frac{k}{k+h} \cdot f(z-hv) + \frac{h}{k+h} \cdot f(z+kv) + f(z) \end{aligned}$$

or equivalently

$$\begin{aligned} & \left\| \frac{h}{k+h} \cdot (F(z+kv) - F(z)) - \frac{k}{k+h} \cdot (F(z) - F(z-hv)) \right\| \\ & \leq \frac{h}{k+h} \cdot (f(z+kv) - f(z)) - \frac{k}{k+h} \cdot (f(z) - f(z-hv)). \end{aligned}$$

Multiplying both sides by $(k+h)/kh$ we obtain (iv).

(c) Let $v \in X$, $k > 0$, $h > 0$, $z \in A$, $z + kv \in A$, $z - hv \in A$. If (iv) is true then for any $y^* \in Y^*$ with $\|y^*\| = 1$

$$\begin{aligned} & - \left\langle \frac{F(z+kv) - F(z)}{k} - \frac{F(z) - F(z-hv)}{h}, y^* \right\rangle \\ & \leq \frac{f(z+kv) - f(z)}{k} - \frac{f(z) - f(z-hv)}{h}. \end{aligned}$$

If we denote $g = y^* \circ F + f$, it is possible to write equivalently

$$\frac{g(z+kv) - g(z)}{k} - \frac{g(z) - g(z-hv)}{h} \geq 0.$$

Hence g is a convex function on $A \cap L$ for an arbitrary line L in X . Consequently g is convex on A .

(d) Suppose that our additional condition (involving a ball B) and (i) are satisfied. Since f is clearly convex (use (ii) or Note 1.2) and upper bounded on B , we obtain that f is continuous on A (cf. [31], Theorem 41.C). Hence $y^* \circ F + f$ is bounded on a ball $B_1 \subset B$ and consequently continuous on A for any $y^* \in Y^*$, $\|y^*\| = 1$.

Since the implication (v) \Rightarrow (i) is trivial, the proof is complete.

Note 1.14. In Proposition 1.13 it is obviously possible to assume that F is bounded on B and f is upper bounded on another ball $\tilde{B} \subset A$.

COROLLARY 1.15. *Let X, Y be normed linear spaces and let $A \subset X$ be an open convex set. Suppose that $F_n: A \rightarrow Y$ is a delta-convex mapping on A with a control function f_n for $n = 1, 2, \dots$. Further suppose that*

$$\lim_{n \rightarrow \infty} F_n = F \quad \text{and} \quad \lim_{n \rightarrow \infty} f_n = f$$

where F is bounded on a ball $B \subset A$ and f is finite on A and upper bounded on B . Then F is delta-convex on A with a control function f .

Proof. Fix x, y, a, b as in 1.13 (iii). Use (iii) for F_n and f_n and pass to limits.

We shall need the following simple, probably well-known lemma. Since we were not able to find a reference, we give a sketch of the proof.

LEMMA 1.16. *Let $f: (c, d) \rightarrow \mathbb{R}$ be a continuous function. Suppose that for any $t \in (c, d)$ and $\delta > 0$ there exist $t_1, t_2 \in U(t, \delta) \cap (c, d)$ and $a, b > 0$ such that $a + b = 1$, $t = at_1 + bt_2$ and $f(t) \leq af(t_1) + bf(t_2)$. Then f is convex on (c, d) .*

Proof (geometrically obvious). Choose an arbitrary $\varepsilon > 0$ and put $g(x) = f(x) + \varepsilon x^2$. Suppose that g is not convex on (c, d) . Then there exist points $c < z_1 < z_0 < z_2 < d$ and an affine function h such that $h \geq g$ on $[z_1, z_2]$, $h(z_1) > g(z_1)$, $h(z_2) > g(z_2)$ and $h(z_0) = g(z_0)$. The assumptions of 1.16 and definition of g easily imply that there exist points $t_1, t_2, z_1 < t_1 < z_0 < t_2 < z_2$ and $a, b > 0$, $a + b = 1$ such that $z_0 = at_1 + bt_2$ and $g(z_0) < ag(t_1) + bg(t_2) \leq ah(t_1) + bh(t_2) = h(z_0)$. This is a contradiction which proves the convexity of g . Consequently f is convex as well.

LEMMA 1.17. *Let X, Y be normed linear spaces, $A \subset X$ be an open convex set and $F: A \rightarrow Y, f: A \rightarrow \mathbb{R}$ be continuous mappings. Suppose that for any $x \in A$, $\delta > 0$ and $v \in X$, $\|v\| = 1$ there exist $x_1, x_2 \in U(x, \delta) \cap A$ and $a, b > 0$ such that $a + b = 1$, $x = ax_1 + bx_2$, $x_2 - x_1 = \|x_2 - x_1\| \cdot v$ and*

$$\|aF(x_1) + bF(x_2) - F(x)\| \leq af(x_1) + bf(x_2) - f(x).$$

Then F is delta-convex on A with a control function f .

Proof. Let $y^* \in Y^*$, $\|y^*\| = 1$ be fixed. We are to prove that $y^* \circ F + f = g$ is convex on A . It is sufficient to prove that g is "convex on all lines". To prove this, choose an arbitrary line L parallel to $v \in X$, $\|v\| = 1$ and a point $x \in L \cap A$. Choose an arbitrary $\delta > 0$ and find x_1, x_2, a, b by the assumptions. Then we have $x_1, x_2 \in L \cap A$ and

$$-\langle aF(x_1) + bF(x_2) - F(x), y^* \rangle \leq af(x_1) + bf(x_2) - f(x),$$

which can be written as $g(x) \leq ag(x_1) + bg(x_2)$. Using Lemma 1.16, we easily obtain that g is convex on $L \cap A$.

COROLLARY 1.18. Let X, Y be normed linear spaces, $A \subset X$ be an open convex set and let both $F: A \rightarrow Y, A \rightarrow \mathbb{R}$ be continuous. Then the following assertions are equivalent:

- (i) F is delta-convex on A with a control function f ;
- (ii) $\left\| \frac{1}{2} \cdot (F(x) + F(y)) - F\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{2} (f(x) + f(y)) - f\left(\frac{x+y}{2}\right)$ whenever $x, y \in A$.

Since most of the results concerning delta-convex mappings are of the local nature, the basic class of mappings for us is not the class of delta-convex mappings but the class of locally delta-convex mappings.

DEFINITION 1.19. Let X, Y be normed linear spaces and $A \subset X$ be an open convex set. We say that $F: A \rightarrow Y$ is *locally delta-convex* (on A) if for any $a \in A$ there exists an open convex neighbourhood $U \subset A$ of a such that F is delta-convex on U .

P. Hartman [18] proved that if $A \subset \mathbb{R}^n$ is an open convex set and F is a locally delta-convex function on A , then it is delta-convex on A . Hartman himself notes that his proof works for finite-dimensional domains only (cf. Problem 5). But his proof gives immediately the following theorem.

THEOREM 1.20. Let Y be a normed linear space and $A \subset \mathbb{R}^n, (n \geq 1)$ be an open convex set. Suppose that $F: A \rightarrow Y$ is locally delta-convex on A . Then F is delta-convex on A .

The last result of this section concerns the following natural general question: If a delta-convex mapping has some property, is it controlled by a function with the same property? If this property is Fréchet differentiability at a point x_0 , the answer is negative (see Example 6.3). For strict differentiability we do not know an answer (see Problem 3). The question concerning positive homogeneity, which is important for applications to quasi-differentiable mappings, has simple positive answer.

LEMMA 1.21. Let X, Y be normed linear spaces and let $F: X \rightarrow Y$ be a positively homogenous mapping (i.e. $F(\lambda x) = \lambda F(x)$ whenever $x \in X, \lambda \geq 0$) which is delta-convex on a neighbourhood of the origin 0. Then F is delta-convex on X with a sublinear control function.

Proof. Let F be controlled on the neighbourhood U of 0 by f . Then $F = F'(0, \cdot)$ is controlled on X by the Lipschitz sublinear function $f'(0, \cdot)$. In fact, for any $y^* \in Y^*, \|y^*\| = 1$ we have $y^* \circ F + f = h$, where h is a continuous convex function on U , and consequently $y^* \circ F + f'(0, \cdot) = y^* \circ (F'(0, \cdot)) + f'(0, \cdot) = (y^* \circ F)'(0, \cdot) + f'(0, \cdot) = h'(0, \cdot)$.

2. Delta-convex curves

DEFINITION 2.1. Let X be a normed linear space and $f: [a, b] \rightarrow X$ be a mapping. For every partition $D = \{a = x_0 < x_1 < \dots < x_n = b\}$ of $[a, b]$ we put

$$\mathbf{K}_a^b(f, D) = \sum_{i=1}^{n-1} \left\| \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1}))}{x_i - x_{i-1}} \right\|.$$

The convexity of f on $[a, b]$ we define as

$$\mathbf{K}_a^b f = \sup_a^b \mathbf{K}_a^b(f, D),$$

where the supremum is taken over all partitions D of $[a, b]$.

Note 2.2. For the notion of the convexity see [31], p. 23 and p. 266 (Problem F). Some results concerning mappings $f: [a, b] \rightarrow X$ with finite convexity are proved in [34].

THEOREM 2.3. Let Y be a Banach space and $F: (a, b) \rightarrow Y$ be a continuous mapping. Then the following conditions are equivalent.

- (i) F is a delta-convex mapping;
- (ii) $\mathbf{K}_c^d F < \infty$ whenever $a < c < d < b$;
- (iii) $F'_+(x) = \lim_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h}$ exists for all $x \in (a, b)$ and $\bigvee_c^d F'_+(x) < \infty$ for $a < c < d < b$.

Proof. (i) \Rightarrow (ii): Let F be controlled by a continuous convex function f . Consider a partition $D = \{x_0 = c < x_1 < \dots < x_n = d\}$ of an interval $[c, d] \subset (a, b)$. By 1.13 (iv)

$$\begin{aligned} \mathbf{K}_a^b(F, D) &\leq \sum_{i=1}^{n-1} \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1}))}{x_i - x_{i-1}} \right) \\ &= \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq 2L, \end{aligned}$$

where L is a Lipschitz constant of f on $[c, d]$.

(ii) \Rightarrow (iii): By [34] we have that $F'_+(x)$ exists for every $x \in (a, b)$. Consider a partition $D = \{x_0 = c < x_1 < \dots < x_n = d\}$ of an interval $[c, d] \subset (a, b)$ and put

$$\bigvee_c^d (F'_+, D) = \sum_{i=0}^{n-1} \|F'_+(x_{i+1}) - F'_+(x_i)\|.$$

For an arbitrary $\varepsilon > 0$ find $\delta > 0$ such that $d + \delta < b$, $x_i + \delta < x_{i+1}$ for $i = 0, 1, \dots, n-1$ and

$$\left\| F'_+(x_i) - \frac{F(x_i + \delta) - F(x_i)}{\delta} \right\| < \varepsilon.$$

Then we have

$$\begin{aligned} \bigvee_c^d (F'_+, D) &\leq 2n\varepsilon + \sum_{i=0}^{n-1} \left\| \frac{F(x_{i+1} + \delta) - F(x_{i+1})}{\delta} - \frac{F(x_i + \delta) - F(x_i)}{\delta} \right\| \\ &\leq 2n\varepsilon + \sum_{i=0}^{n-1} \left(\left\| \frac{F(x_{i+1} + \delta) - F(x_{i+1})}{\delta} - \frac{F(x_{i+1}) - F(x_i + \delta)}{x_{i+1} - (x_i + \delta)} \right\| \right. \\ &\quad \left. + \left\| \frac{F(x_{i+1}) - F(x_i + \delta)}{x_{i+1} - (x_i + \delta)} - \frac{F(x_i + \delta) - F(x_i)}{\delta} \right\| \right) \leq 2n\varepsilon + \mathbf{K}_c^{d+\delta} F. \end{aligned}$$

Consequently $\bigvee_c^d F'_+ \leq \mathbf{K}_c^{d+\delta} F < \infty$.

(iii) \Rightarrow (i): Fix $x_0 \in (a, b)$ and consider the functions

$$v(x) = \bigvee_{x_0}^x F'_+ \quad \text{and} \quad f(x) = \int_{x_0}^x v(t) dt$$

(we put, of course, $\bigvee_{x_0}^x F'_+ = -\bigvee_x^{x_0} F'_+$ for $x < x_0$).

We shall prove that F is controlled by f . Clearly $f'(x) = v(x)$ for $x \in M$, where $\lambda((a, b) \setminus M) = 0$. Choose now $y^* \in Y^*$, $\|y^*\| = 1$ and consider the function $h(x) = y^*(F(x)) + f(x)$. Since clearly $(y^* \circ F'_+)(x) = y^*(F'_+(x))$ is locally bounded, we conclude that $y^* \circ F$, and consequently also h , is locally Lipschitz.

Consequently $h(x) = h(x_0) + \int_{x_0}^x h'_+(t) dt$. To prove the convexity of h it is clearly sufficient to prove that h'_+ is non-decreasing on M . Thus suppose that $x_1 < x_2$ are points from M . Then

$$\begin{aligned} h'_+(x_2) - h'_+(x_1) &= y^*(F'_+(x_2) - F'_+(x_1)) + (v(x_2) - v(x_1)) \\ &\geq (v(x_2) - v(x_1)) - \|F'_+(x_2) - F'_+(x_1)\| = \bigvee_{x_1}^{x_2} F'_+ - \|F'_+(x_2) - F'_+(x_1)\| \geq 0. \end{aligned}$$

The proof is complete.

3. Differentiability of delta-convex mappings

A. First derivative. We shall prove that delta-convex mappings have many good differentiability properties of convex functions. We start with the following easy consequence of Theorem 2.3.

PROPOSITION 3.1. *Let X be a normed linear space and let Y be a Banach space. Let $G \subset X$ be an open convex set and let $F: G \rightarrow Y$ be a delta-convex mapping. Then the one-sided directional derivative $F'(a, v)$ exists whenever $a \in G$ and $v \in X$.*

Proof. Consider the mapping $g(t) = F(a+tv)$ and use 2.3 (iii).

Note 3.2. As presented, the proof depends on a result [34]. But it would be possible to use Lemma 3.6 and to copy the proofs of 3.8 and 3.9 below and obtain (independently on [34]) that g has a strict right derivative (cf. 3.4 (b)) at $t = 0$ for each a, v .

It is well known that if a convex function is Fréchet differentiable at a point, then it is also strictly differentiable at this point (Proposition 3.8). Consequently any continuous convex function on an Asplund space is strictly differentiable at all points except those which belong to a first category set. The same holds for delta-convex mappings, too (Theorem 3.10). We shall obtain this result as an immediate consequence of the “convex case” and Proposition 3.7 below which gives a necessary and sufficient condition of “Bolzano–Cauchy type”. This condition can perhaps be of some independent significance.

DEFINITION 3.3. Let X, Y be normed linear spaces. $D \subset X$ and let $F: D \rightarrow Y$ be a mapping. We say that $A \in L(X, Y)$ is a *strict derivative* of F (or a strict derivative with respect to a set $M \subset X$) at a point $a \in D$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1) \quad \|F(y) - F(x) - A(y-x)\| \leq \varepsilon \|y-x\|$$

whenever $x, y \in U(a, \delta)$ (or $x, y \in U(a, \delta) \cap M$, respectively).

Note 3.4. (a) For the definition of the important notion of a strict derivative and some its applications see e.g. [7], [9], [26]. Note that some authors use the name “strong derivative” or “sharp derivative”.

(b) If $X = \mathbb{R}$, $M = [a, b]$, $F: M \rightarrow Y$ and F is strictly differentiable at a with respect to M , we shall say that F is *strictly differentiable from the right* at a and A will be called a *right strict derivative*. This is the only case (except the case $M = X$) which is interesting for us.

(c) The inequality (1) can be rewritten as

$$(1') \quad \left\| \frac{F(y) - F(x)}{\|y-x\|} - A \left(\frac{y-x}{\|y-x\|} \right) \right\| \leq \varepsilon.$$



DEFINITION 3.5. Let X, Y be normed linear spaces, $D \subset X, M \subset X$ and let $F: D \rightarrow Y$ be a mapping. Then for any $\varepsilon > 0$ we define the set $D(F, M, \varepsilon)$ as the set of all points $a \in D$ for which there exists $\delta = \delta(\varepsilon)$ such that

$$(2) \quad \left\| \frac{F(y+kv) - F(y)}{k} - \frac{F(y) - F(y-hv)}{h} \right\| \leq \varepsilon$$

whenever

$$(3) \quad \begin{aligned} v \in X, \quad \|v\| = 1, \quad k > 0, \quad h > 0, \quad y \in U(a, \delta) \cap M, \\ y - hv \in U(a, \delta) \cap M \quad \text{and} \quad y + kv \in U(a, \delta) \cap M. \end{aligned}$$

In the case $M = X$ we put $D(F, M, \varepsilon) = D(F, \varepsilon)$.

LEMMA 3.6. Let Y be a Banach space, $c > 0$ and $F: [0, c] \rightarrow Y$ be a mapping. Then F is strictly differentiable at 0 with respect to $M := [0, c]$ iff $0 \in \bigcap \{D(F, M, \varepsilon): \varepsilon > 0\}$.

Proof. (a) Suppose that F is strictly differentiable at 0 with respect to M and let $A \in Y$ be the vector which corresponds to the strict derivative of F at 0 with respect to M in the canonical identification $L(R, Y) \cong Y$. Let $\varepsilon > 0$ be given and choose $\delta > 0$ which corresponds to $\varepsilon/2$ by Definition 3.3. Let for some v, k, h, y the conditions (3) hold (for $X = R, a = 0$). By (1') we have

$$\left\| \frac{F(y+kv) - F(y)}{k} - A \right\| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \frac{F(y) - F(y-hv)}{h} - A \right\| \leq \frac{\varepsilon}{2}.$$

These inequalities immediately imply (2).

(b) Now suppose that $0 \in \bigcap \{D(F, M, \varepsilon): \varepsilon > 0\}$. At first, we shall prove that the right derivative

$$F'_+(0) = \lim_{x \rightarrow 0^+} \frac{F(x) - F(0)}{x}$$

exists. Let $\varepsilon > 0$ be given. Choose a corresponding $\delta = \delta(\varepsilon)$ by Definition 3.5 and suppose that $0 < x_1 < x_2 < \delta$ are given. By (2)

$$\left\| \frac{F(x_2) - F(x_1)}{x_2 - x_1} - \frac{F(x_1) - F(0)}{x_1} \right\| \leq \varepsilon.$$

Denote

$$B = \frac{F(x_1) - F(0)}{x_1}.$$

Then we have

$$\|F(x_2) - F(x_1) - (x_2 - x_1) \cdot B\| \leq \varepsilon \|x_2 - x_1\|$$

and consequently

$$\begin{aligned} & \|F(x_2) - F(0) - (x_2 - 0) \cdot B\| \\ &= \|F(x_2) - F(x_1) - (x_2 - x_1) \cdot B + F(x_1) - F(0) - x_1 B\| \leq \varepsilon(x_2 - x_1) + 0 \leq \varepsilon x_2. \end{aligned}$$

Therefore

$$(4) \quad \left\| \frac{F(x_2) - F(0)}{x_2} - B \right\| \leq \varepsilon.$$

Thus we have proved that

$$\text{diam} \left\{ \frac{F(x) - F(0)}{x} : 0 < x < \delta \right\} \leq \varepsilon.$$

Since Y is a complete metric space, we conclude that there exists $F'_+(0)$. Moreover

$$\left\| \frac{F(x_2) - F(x_1)}{x_2 - x_1} - F'_+(0) \right\| \leq \left\| \frac{F(x_2) - F(x_1)}{x_2 - x_1} - B \right\| + \|B - F'_+(0)\| \leq 2\varepsilon.$$

Using the condition (1') it is easy to complete the proof.

PROPOSITION 3.7. *Let X be a normed linear space, $D \subset X$, Y be a Banach space and let $F: D \rightarrow Y$ be a mapping. Then F is strictly differentiable at a point $a \in D$ iff F is continuous at a and $a \in \bigcap \{D(F, \varepsilon) : \varepsilon > 0\}$.*

Proof. (a) Suppose that A is a strict derivative of F at a . It is obvious that F is continuous at a . For arbitrary $\varepsilon > 0$ find $\delta > 0$ which corresponds to $\varepsilon/2$ by Definition 3.3. Suppose that for v, y, k, h the conditions (3) hold with $M = X$. By (1') we have

$$\left\| \frac{F(y + kv) - F(y)}{k} - A(v) \right\| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \frac{F(y) - F(y - hv)}{h} - A(v) \right\| \leq \frac{\varepsilon}{2}.$$

Consequently (2) holds and we conclude that $a \in D(F, \varepsilon)$.

(b) Suppose that F is continuous at a and $a \in \bigcap \{D(F, \varepsilon) : \varepsilon > 0\}$. For $v \in X$, $\|v\| = 1$ define the function F_v by the formula

$$F_v(t) = F(a + tv).$$

Clearly $0 \in \bigcap \{D(F_v, [0, 1], \varepsilon) : \varepsilon > 0\}$. By Lemma 3.6 we have that F_v is strictly differentiable at 0 with respect to $[0, 1]$ and consequently F is strictly differentiable at a with respect to the halfline $\{a + tv : t > 0\}$. Moreover, it follows from the proof of Lemma 3.6, that

$$(5) \quad \text{the limit } \lim_{t \rightarrow 0^+} \frac{F(a + tv) - F(a)}{t} = F'(a, v) \text{ is uniform on the sphere } \{v : \|v\| = 1\}.$$

Let now $v_1 \neq v_2$ be given; we shall show that

$$F'(a, v_1 + v_2) = F'(a, v_1) + F'(a, v_2).$$

Suppose on the contrary that

$$(6) \quad \|F'(a, v_1 + v_2) - F'(a, v_1) - F'(a, v_2)\| := 4\omega > 0.$$

Find $\delta > 0$ which corresponds to $\varepsilon = \omega/\|v_1 - v_2\|$ by Definition 3.5. Find now $t > 0$ such that

$$a + 2tv_1, a + 2tv_2, a + t(v_1 + v_2) \in U(a, \delta) \quad \text{and}$$

$$\left\| \frac{F(a + 2tv_1) - F(a)}{2t} - F'(a, v_1) \right\| < \omega,$$

$$\left\| \frac{F(a + 2tv_2) - F(a)}{2t} - F'(a, v_2) \right\| < \omega,$$

$$\left\| \frac{F(a + t(v_1 + v_2)) - F(a)}{t} - F'(a, v_1 + v_2) \right\| < \omega.$$

These inequalities and (6) easily imply

$$\left\| \frac{F(a + 2tv_1) - F(a)}{2t} + \frac{F(a + 2tv_2) - F(a)}{2t} - \frac{F(a + t(v_1 + v_2)) - F(a)}{t} \right\| > \omega.$$

which yields

$$\|(F(a + 2tv_1) - F(a + t(v_1 + v_2))) - (F(a + t(v_1 + v_2)) - F(a + 2tv_2))\| > 2t\omega.$$

Consequently

$$\left\| \frac{F(a + 2tv_1) - F(a + t(v_1 + v_2))}{t\|v_1 - v_2\|} - \frac{F(a + t(v_1 + v_2)) - F(a + 2tv_2)}{t\|v_1 - v_2\|} \right\| > \frac{2\omega}{\|v_1 - v_2\|}.$$

On the other hand, since (3) holds by the assumptions for

$$\varepsilon = \omega/\|v_1 - v_2\|, \quad y = a + t(v_1 + v_2), \quad v = (v_1 - v_2)/\|v_1 - v_2\|,$$

$$h = k = t\|v_1 - v_2\|, \quad M = X,$$

we obtain (2); i.e.

$$\left\| \frac{F(a + 2tv_1) - F(a + t(v_1 + v_2))}{t\|v_1 - v_2\|} - \frac{F(a + t(v_1 + v_2)) - F(a + 2tv_2)}{t\|v_1 - v_2\|} \right\| \leq \frac{\omega}{\|v_1 - v_2\|},$$

and this is a contradiction.

Putting $A(v) := F'(a, v)$ we see that A is positively homogenous and additive, consequently A is linear. Since (5) easily yields

$$\|F(a + h) - F(a) - A(h)\| = o(\|h\|), \quad h \rightarrow 0,$$

we conclude that A is continuous and that A is the Fréchet derivative of F at a .

It remains to prove that A is a strict derivative of F at a . Choose $\varepsilon > 0$ and choose a corresponding $\delta > 0$ by Definition 3.5. Further choose $\delta_1 < \delta$ such that

$$(7) \quad \|F(a+h) - F(a) - A(h)\| < \varepsilon \|h\| \quad \text{whenever } \|h\| < \delta_1.$$

Now put $\delta_2 = \delta_1/4$ and suppose that arbitrary points $x, y, x \neq y$ from $U(a, \delta_2)$ are given. Put $v = (x-y)/\|x-y\|$, $k = \|x-y\|$ and $h = \delta_2 v$. Then we have $\|(y - \delta_2 v) - a\| < \delta_1$. Consequently by (7)

$$\begin{aligned} \|F(y - \delta_2 v) - F(a) - A(y - \delta_2 v - a)\| &< \varepsilon \|y - \delta_2 v - a\| \quad \text{and} \\ \|F(y) - F(a) - A(y - a)\| &< \varepsilon \|y - a\|. \end{aligned}$$

These two inequalities imply

$$\begin{aligned} \|F(y) - F(y - \delta_2 v) - A(\delta_2 v)\| &< \varepsilon (\|y - \delta_2 v - a\| + \|y - a\|) \\ &\leq \varepsilon (\|y - a\| + \delta_2 + \|y - a\|) \leq 3\varepsilon \delta_2. \end{aligned}$$

Consequently

$$\left\| \frac{F(y) - F(y - \delta_2 v)}{\delta_2} - A(v) \right\| \leq 3\varepsilon.$$

Since for our $v, k, h, y, M = X$ the conditions (3) are clearly satisfied, we obtain (2), i.e.

$$\left\| \frac{F(x) - F(y)}{\|x - y\|} - \frac{F(y) - F(y - \delta_2 v)}{\delta_2} \right\| \leq \varepsilon.$$

Therefore

$$\left\| \frac{F(x) - F(y)}{\|x - y\|} - A(v) \right\| \leq 4\varepsilon$$

and hence A is a strict derivative of F at a .

We shall need the following well-known proposition. Since we were not able to find a reference, we add a simple proof based on Proposition 3.7.

PROPOSITION 3.8. *Let G be an open convex subset of a normed linear space X and $a \in G$. Let f be a convex function on G , which is Fréchet differentiable at a . Then f is strictly differentiable at a .*

Proof. Let $f'(a)$ be the Fréchet derivative of f at a and $\varepsilon > 0$. It is sufficient to prove that $a \in D(f, \varepsilon)$. Without any loss of generality we can suppose that $a = 0$, $f'(a) = 0$ and $f(a) = 0$. Find $\omega > 0$ such that

$$(8) \quad |f(y)| < \varepsilon \|y\|/6 \quad \text{for } \|y\| < \omega.$$

Put $\delta = \omega/3$ and suppose that $y, v \in X$ and $k, h > 0$ are such that $\|v\| = 1$, $y \in U(a, \delta)$, $y - hv \in U(a, \delta)$, $y + kv \in U(a, \delta)$. Then $h < 2\delta$, $k < 2\delta$. Consequently the convexity of f implies

$$\left| \frac{f(y + kv) + f(y)}{k} - \frac{f(y) - f(y - hv)}{h} \right| \leq \left| \frac{f(y + 2\delta v) - f(y)}{2\delta} - \frac{f(y) - f(y - 2\delta v)}{2\delta} \right| := T.$$

Since $\|y + 2\delta v\| < \omega$ and $\|y - 2\delta v\| < \omega$, (8) implies

$$T = \frac{1}{2\delta} |f(y + 2\delta v) - 2f(y) + f(y - 2\delta v)| \leq \frac{1}{2\delta} \cdot 4 \cdot \frac{1}{6} \cdot \varepsilon \cdot \omega = \varepsilon.$$

PROPOSITION 3.9. *Let X be a normed linear space and let Y be a Banach space. Let $G \subset X$ be an open convex set, $a \in G$ and let $F: G \rightarrow Y$ be a delta-convex mapping with a control function $f: G \rightarrow \mathbb{R}$. Then the following implications hold.*

- (i) *If f is Fréchet differentiable at a then F is strictly differentiable at a .*
- (ii) *If f is Gateaux differentiable at a then F is Gateaux differentiable at a .*

Proof. (a) Proposition 1.13 (iv) immediately implies that $D(f, \varepsilon) \subset D(F, \varepsilon)$ for $\varepsilon > 0$. This observation, Proposition 3.7 and Proposition 3.8 immediately imply (i).

(b) Let $V \subset X$ be a finite-dimensional space. Since F is controlled by f on the affine subspace $a + V$, we obtain on account of (i) that $F'(a, v)$ exists for any $v \in X$ and $F'(a, \cdot)$ is linear. Proposition 1.10 implies that $F'(a, \cdot)$ is Lipschitz which completes the proof.

THEOREM 3.10. *Let X, Y be Banach spaces. Let $G \subset X$ be an open convex set and let $F: G \rightarrow Y$ be a locally delta-convex mapping. Then the following assertions hold.*

(i) *If X is an Asplund space then F is strictly differentiable at all points of G except those which belong to a first category set.*

(ii) *If X^* is separable then F is strictly differentiable at all points of G except those which belong to an angle small set.*

(iii) *If X is a weak Asplund space then F is Gateaux differentiable at all points of G except those which belong to a first category set.*

(iv) *If X is separable then F is Gateaux differentiable at all points of G except those which belong to a countable union of δ -convex hypersurfaces.*

Proof. To prove (i) and (iii) it is sufficient to use Proposition 3.9 and the well-known fact (cf. [22]) that each locally first category set is a first category set. The separability of X^* and a result of [30] concerning Fréchet differentiability of continuous convex functions on a space with separable dual easily yield (ii). Similarly a result of [37] implies (iv).

B. Second derivative of mappings $F: R^n \rightarrow Y$. In the remaining part of this section Y will be a Banach space.

DEFINITION 3.11. By a *quadratic form* $Q: R^n \rightarrow Y$ we shall mean a mapping of the form

$$Q(x) = Q(x_1, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} x_i x_j A_{ij},$$

where $A_{ij} \in Y$ for $1 \leq i \leq j \leq n$.

DEFINITION 3.12. We shall say that a mapping $F: R^n \rightarrow Y$ has the *second (Peano) derivative* Q at a point $a \in R^n$, if $Q: R^n \rightarrow Y$ is a quadratic form, the Fréchet derivative $F'(a)$ exists and

$$F(x) = F(a) + F'(a)(x-a) + Q(x-a) + o(\|x-a\|^2), \quad x \rightarrow a.$$

The proof of the following lemma is obvious and we shall omit it.

LEMMA 3.13. Let e_i be the i -th vector of the canonical basis of R^n and let vectors $B_{ij} \in Y$, $1 \leq i \leq j \leq n$ be given. Then there exists a unique quadratic form $Q: R^n \rightarrow Y$ such that

$$Q(e_i + e_j) = B_{ij}, \quad 1 \leq i \leq j \leq n.$$

LEMMA 3.14. Let $Q: R^n \rightarrow Y$ be a quadratic form. Then there exists $K > 0$ such that

$$\left\| Q\left(\sum_{i=1}^n \lambda_i s_i\right) - \sum_{i=1}^n \lambda_i Q(s_i) \right\| \leq K \cdot \varepsilon \cdot c,$$

whenever $s_1, \dots, s_n \in R^n$, $\lambda_i > 0, \dots, \lambda_n > 0$, $\sum \lambda_i = 1$, $\|s_i\| \leq c$ for $i = 1, \dots, n$ and $\text{diam} \{s_1, \dots, s_n\} < \varepsilon$.

Proof. Let $Q(x_1, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} x_i x_j A_{ij}$ and $s_i = (s_i^1, \dots, s_i^n)$, $i = 1, \dots, n$.

Then we have

$$\left\| Q\left(\sum_{i=1}^n \lambda_i s_i\right) - \sum_{i=1}^n \lambda_i Q(s_i) \right\| = \left\| \sum_{1 \leq k \leq m \leq n} A_{km} \left(\sum_{\substack{i=1 \\ j=1}}^n \lambda_i \lambda_j s_i^k s_j^m - \sum_{i=1}^n \lambda_i s_i^k s_i^m \right) \right\|.$$

Since for any k, m we have

$$\left| \sum_{i,j=1}^n \lambda_i \lambda_j s_i^k s_j^m - \sum_{i=1}^n \lambda_i s_i^k s_i^m \right| = \left| \sum_{i=1}^n \lambda_i s_i^k \left(\sum_{j=1}^n \lambda_j s_j^m - s_i^m \right) \right| \leq \varepsilon \cdot \sum_{i=1}^n \lambda_i |s_i^k| \leq \varepsilon c,$$

it is sufficient to choose $K = \sum_{1 \leq k \leq m \leq n} \|A_{km}\|$.

It is a well-known fact that each convex function on R^n has the second derivative almost everywhere. This theorem was proved by Buseman and Feller [8] for $n = 2$ and by A. D. Aleksandrov [1] for an arbitrary n . It can be also considered as an immediate consequence of the Mignot's theorem [25] on

the a.e. differentiability of monotone operators $T: R^n \rightarrow R^n$. The Mignot's theorem is an easy consequence of well-known facts concerning Lipschitz mappings (cf. [38]).

The following theorem gives a generalization of Buseman–Feller–Aleksandrov theorem.

Recall that $P \subset Y^*$ is said to be *total* if for any $0 \neq y \in Y$ there exists $p \in P$ such that $\langle y, p \rangle \neq 0$.

THEOREM 3.15. *Let $H \subset R^n$ be an open set and let Y be a Banach space with the Radon–Nikodým property such that Y^* contains a countable total set. Then any locally delta-convex mapping $F: H \rightarrow Y$ has the second derivative at almost all points of H .*

Proof. Without any loss of generality we can suppose that H is convex and F is delta-convex on H . In the first step we shall prove the theorem for $n = 1$. In this case we use Theorem 2.3 (iii) and obtain that $G(x) = F'_+(x)$ has locally finite variation. Since Y has the Radon–Nikodým property, we obtain (cf. [14]) that $G'(x)$ exists a.e. Using Theorem 3.10 we see that it is sufficient to prove that F has the second derivative at all points $a \in H$ at which both $F'(a)$ and $G'(a)$ exist.

We shall prove that under the above conditions the quadratic form

$$Q(x) = \frac{1}{2} \cdot x^2 G'(a)$$

is the second derivative of F at a . We have

$$F'_+(x) = F'(a) + G'(a) \cdot (x-a) + r(x), \quad \text{where } \lim_{x \rightarrow a} \frac{\|r(x)\|}{|x-a|} = 0.$$

Consider the mapping

$$Z(x) = F(x) - F(a) - (x-a)F'(a) - \frac{1}{2}(x-a)^2 G'(a).$$

Then

$$Z(a) = 0 \quad \text{and} \quad Z'_+(x) = F'_+(x) - F'(a) - (x-a)G'(a) = r(x).$$

Let now an arbitrary $\varepsilon > 0$ be given. Find $\delta > 0$ such that $|x-a| < \delta$ implies $\|Z'_+(x)\| = \|r(x)\| \leq \varepsilon|x-a|$. Let y , $0 < |y-a| < \delta$ be given. Using the Mean Value Theorem for right-hand derivatives we obtain

$$\|Z(y)\| = \|Z(y) - Z(a)\| \leq |y-a| \cdot \sup\{\|Z'_+(x)\|: |x-a| \leq |y-a|\} \leq \varepsilon|y-a|^2.$$

Consequently $Z(y) = o(|y-a|^2)$, $y \rightarrow a$.

Now consider the case $n > 1$. Let F be controlled by a continuous' convex function f on H . Let e_i be the i -th vector of the canonical basis of R^n . Let C^* be the set of all vectors of the form $v = e_k$ or $v = (e_i + e_j)/\sqrt{2}$, $1 \leq i \leq j \leq n$,

$1 \leq k \leq n$. Let $C \supset C^*$ be a countable set which is dense in the unit sphere $S = \{x \in R^n: \|x\| = 1\}$. For $v \in C$ denote by D_v the set of all points $a \in H$ at which F has the second derivative in the direction v , i.e. the function $g_{a,v}(t) = F(a+tv)$ of the real variable t has a second derivative $Q_{a,v}$ at 0.

It is not difficult to prove that D_v is a measurable set. In fact, denote by W the set of all points of the strict differentiability of F . Further denote by $A_{p,k,n}$ the set of points $x \in W$ for which

$$\overline{U\left(x, \frac{1}{k}\right)} \subset H \quad \text{and} \quad \text{diam} \left\{ \frac{F(x+tv) - F(x) - t \cdot F'(x, v)}{t^2}; \frac{1}{n} \leq |t| \leq \frac{1}{k} \right\} < \frac{1}{p}$$

for positive integers $p, k, n, k < n$. It is easy to prove that each set $A_{p,k,n}$ is open in W and

$$W \cap D_v = \bigcap_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k+1}^{\infty} A_{p,k,n}.$$

Since $\mu_n(H \setminus W) = 0$, we obtain a measurability of D_v .

Consequently the validity of our theorem for $n = 1$ and the Fubini Theorem imply $\mu_n(H \setminus D_v) = 0$ for any $v \in C$.

Let $P \subset Y^*$ be a countable total set. Using Buseman–Feller–Aleksandrov Theorem we obtain that f has the second derivative a.e. and any function of the form $y^* \circ F, y^* \in P$, has the second derivative a.e. By Theorem 3.10, F has the Fréchet derivative a.e. Consequently it is sufficient to prove that F has the second derivative at each point $a \in R^n$ at which F has the Fréchet derivative, f has the second derivative $Q_f, a \in D_v$ for all $v \in C$ and each function $p \circ F, p \in P$, has the second derivative Q_p at a . Let a be such a point.

Without any loss of generality it is possible to suppose $a = 0, F(a) = 0, f(a) = 0, F'(a) = 0, f'(a) = 0$.

Using Lemma 3.13 it is easy to see that there exists a unique quadratic form $Q: R^n \rightarrow Y$ such that $Q(tv) = Q_{a,v}(t)$ for $v \in C^*$. We shall prove that Q is the second derivative of F at a . Since the second derivative, if it exists, is uniquely determined, we obtain $Q_p(tv) = p \circ Q_{a,v}(t)$ for $v \in C$ and $p \in P$. Since from the same reason $Q_p(tv) = p \circ Q(tv)$ for $v \in C^*$ and $p \in P$, we obtain by Lemma 3.13 that $Q_p = p \circ Q$ and consequently $p \circ Q(tv) = p \circ Q_{a,v}(t)$ for $v \in C, p \in P$. Since P is total, we easily obtain $Q(tv) = Q_{a,v}(t)$ for $v \in C$.

Let an arbitrary $0 < \varepsilon < 1/2$ be given. It is geometrically obvious and easy to prove that there exists a finite set $\tilde{K} \subset C$ with the following property:

- (9) For arbitrary $v \in S$ there exist $s_1, \dots, s_n \in \tilde{K}, \lambda_1, \dots, \lambda_n$ positive and $1/2 < \mu < 1$ such that $\lambda_1 + \dots + \lambda_n = 1, \text{diam} \{s_1, \dots, s_n\} < \varepsilon$ and $\sum \lambda_i s_i = \mu v$.

Now find $0 < \delta < \varepsilon$ such that

$$(10) \quad \|F(tv) - Q(tv)\| \leq \varepsilon t^2 \quad \text{whenever } |t| < \delta \text{ and } v \in \tilde{K},$$

$$(11) \quad \|f(h) - Q_f(h)\| \leq \varepsilon \|h\|^2 \quad \text{whenever } \|h\| < \delta.$$

Let now an arbitrary $h \in R^n$, $0 < \|h\| < \delta/2$, be given. Find $s_1, \dots, s_n, \lambda_1, \dots, \lambda_n, \mu$ which correspond by (9) to $v = h/\|h\|$ and put $z_i = (\|h\|/\mu)s_i$, $i = 1, \dots, n$. Then $\|z_i\| < 2\|h\|$, $i = 1, \dots, n$, $\text{diam}\{z_1, \dots, z_n\} < 2\|h\|\varepsilon$ and $h = \sum_{i=1}^n \lambda_i z_i$. Since F is controlled by f , we have by Proposition 1.13 (ii)

$$\|F(h) - \sum_{i=1}^n \lambda_i F(z_i)\| \leq \sum_{i=1}^n \lambda_i f(z_i) - f(h).$$

Thus we can compute

$$\begin{aligned} \|F(h) - Q(h)\| &\leq \|F(h) - \sum_{i=1}^n \lambda_i F(z_i)\| + \left\| \sum_{i=1}^n \lambda_i F(z_i) - Q(h) \right\| \\ &\leq \left| f(h) - \sum_{i=1}^n \lambda_i f(z_i) \right| + \left\| \sum_{i=1}^n \lambda_i F(z_i) - \sum_{i=1}^n \lambda_i Q(z_i) \right\| \\ &\quad + \left\| \sum_{i=1}^n \lambda_i Q(z_i) - Q(h) \right\| \leq |f(h) - Q_f(h)| + |Q_f(h) - \sum_{i=1}^n \lambda_i Q_f(z_i)| \\ &\quad + \left| \sum_{i=1}^n \lambda_i Q_f(z_i) - \sum_{i=1}^n \lambda_i f(z_i) \right| + \left\| \sum_{i=1}^n \lambda_i F(z_i) - \sum_{i=1}^n \lambda_i Q(z_i) \right\| \\ &\quad + \left\| \sum_{i=1}^n \lambda_i Q(z_i) - Q(h) \right\| := T. \end{aligned}$$

Let $K > 0$, $K_f > 0$ be numbers which correspond to quadratic forms Q , Q_f by Lemma 3.14. Then by Lemma 3.14, (10) and (11) we have

$$T \leq \varepsilon \|h\|^2 + K_f \cdot (2\|h\|) \cdot (2\|h\|\varepsilon) + 4\varepsilon \|h\|^2 + 4\varepsilon \|h\|^2 + K \cdot (2\|h\|) \cdot (2\|h\|\varepsilon) = \varepsilon M \|h\|^2$$

where M does not depend on ε and h . Consequently Q is the second derivative of F at a .

4. Superpositions and inverse mappings

PROPOSITION 4.1. *Let X, Y, Z be normed linear spaces and let $A \subset X$, $B \subset Y$ be open convex sets. Let $F: A \rightarrow B$ be delta-convex on A with a control function f and let $G: B \rightarrow Z$ be delta-convex on B with a control function g . Suppose further that G, g are Lipschitz on B with constants L_G, L_g .*

Then the composite mapping $G \circ F$ is delta-convex on A with a control function $h = g \circ F + (L_G + L_g)f$.

Proof. We shall use Proposition 1.13 (iii). Choose $x, y \in A$ and $a \geq 0, b \geq 0, a+b = 1$. Then the following inequalities hold.

$$\begin{aligned}
& \|aG(F(x)) + bG(F(y)) - G(F(ax+by))\| \\
& \leq \|aG(F(x)) + bG(F(y)) - G(aF(x) + bF(y))\| \\
& \quad + \|G(aF(x) + bF(y)) - G(F(ax+by))\| \\
& \leq ag(F(x)) + bg(F(y)) - g(aF(x) + bF(y)) \\
& \quad + L_G \|aF(x) + bF(y) - F(ax+by)\| \\
& \leq ag(F(x)) + bg(F(y)) - g(F(ax+by)) \\
& \quad + g(F(ax+by)) - g(aF(x) + bF(y)) + L_G \|aF(x) + bF(y) - F(ax+by)\| \\
& \leq ag(F(x)) + bg(F(y)) - g(F(ax+by)) \\
& \quad + (L_G + L_g) \|aF(x) + bF(y) - F(ax+by)\| \leq ah(x) + bh(y) - h(ax+by).
\end{aligned}$$

Since $G \circ F$ and h are obviously continuous, the proof is complete.

As an immediate consequence of Proposition 4.1 and Proposition 1.10 we obtain the following theorem on superpositions of locally delta-convex mappings. Note that this theorem was proved in [35] under some slight additional assumptions (completeness of X, Y and Lipschitz continuity of G). In finite-dimensional case it was proved by P. Hartman [18].

THEOREM 4.2. (Superposition Theorem). *Let X, Y, Z be normed linear spaces and $A \subset X, B \subset Y$ be open convex sets. Let $F: A \rightarrow B$ be locally delta-convex on A and $G: B \rightarrow Z$ be locally delta-convex on B . Then the composite mapping $G \circ F$ is locally delta-convex on A .*

COROLLARY 4.3. *Let T, Z be normed linear spaces and X, Y be Hilbert spaces. Let $G \subset T$ be an open set, $B: X \times Y \rightarrow Z$ be a continuous bilinear mapping and $f: G \rightarrow X, g: G \rightarrow Y$ be locally delta-convex mappings. Then the mapping*

$$h(w) := B(f(w), g(w))$$

is locally delta-convex on G .

Proof. It is sufficient to use 1.12 and 4.2.

Using Proposition 1.13 (iii) we obtain also a short proof of delta-convexity of implicate mappings. Unfortunately, our proof works under rather restrictive additional assumptions only (cf. Problems 1 and 2).

THEOREM 4.4 (delta-convexity of implicate mappings). *Let X, Y, Z be normed linear spaces and $A \subset X, B \subset Y$ be open convex sets. Let $G: A \times B \rightarrow Z$ be a delta-convex mapping with a control function g . Suppose that there are $L \geq 0, c > 0$ such that*

$$|g(x, y) - g(x, \bar{y})| \leq L \|y - \bar{y}\| \quad \text{and} \quad \|G(x, y) - G(x, \bar{y})\| \geq c \|y - \bar{y}\|,$$

whenever $x \in A, y, \bar{y} \in B$.

Let $\varphi: A \rightarrow B$ be a mapping satisfying $G(x, \varphi(x)) = 0$ on A and let $L < c$. Then φ is delta-convex on A with a control function

$$h(x) = \frac{1}{c-L}g(x, \varphi(x)).$$

Proof. We shall use Proposition 1.13 (iii). Let $x, \bar{x} \in A$, $a \geq 0$, $b \geq 0$, $a + b = 1$. Then

$$\begin{aligned} & \|a\varphi(x) + b\varphi(\bar{x}) - \varphi(ax + b\bar{x})\| \\ & \leq \frac{1}{c} \|G(ax + b\bar{x}, a\varphi(x) + b\varphi(\bar{x})) - G(ax + b\bar{x}, \varphi(ax + b\bar{x}))\| \\ & = \frac{1}{c} \|G(ax + b\bar{x}, a\varphi(x) + b\varphi(\bar{x}))\| \\ & = \frac{1}{c} \|G(ax + b\bar{x}, a\varphi(x) + b\varphi(\bar{x})) - aG(x, \varphi(x)) - bG(\bar{x}, \varphi(\bar{x}))\| \\ & \leq \frac{1}{c} [ag(x, \varphi(x)) + bg(\bar{x}, \varphi(\bar{x})) - g(ax + b\bar{x}, a\varphi(x) + b\varphi(\bar{x}))] \\ & = \frac{1}{c} [ag(x, \varphi(x)) + bg(\bar{x}, \varphi(\bar{x})) - g(ax + b\bar{x}, \varphi(ax + b\bar{x}))] \\ & \quad + \frac{1}{c} [g(ax + b\bar{x}, \varphi(ax + b\bar{x})) - g(ax + b\bar{x}, a\varphi(x) + b\varphi(\bar{x}))] \\ & \leq \frac{1}{c} [ag(x, \varphi(x)) + bg(\bar{x}, \varphi(\bar{x})) - g(ax + b\bar{x}, \varphi(ax + b\bar{x}))] \\ & \quad + \frac{1}{c} L \|\varphi(ax + b\bar{x}) - a\varphi(x) - b\varphi(\bar{x})\|. \end{aligned}$$

Hence

$$\|a\varphi(x) + b\varphi(\bar{x}) - \varphi(ax + b\bar{x})\| \leq ah(x) + bh(\bar{x}) - h(ax + b\bar{x}).$$

Thus it is sufficient to prove the continuity of φ . Let $x \in A$. By Proposition 1.10 the function $H = G(\cdot, \varphi(x))$ is Lipschitz with a constant d on an open neighbourhood U of x . Let $y \in U$. Then

$$\begin{aligned} d\|x - y\| & \geq \|G(x, \varphi(x)) - G(y, \varphi(x))\| = \|G(y, \varphi(x))\| \\ & = \|G(y, \varphi(x)) - G(y, \varphi(y))\| \geq c\|\varphi(x) - \varphi(y)\|. \end{aligned}$$

Therefore φ is continuous at x and the proof is complete.

As an immediate consequence we obtain the following theorem.

THEOREM 4.5 (delta-convexity of inverse mappings). *Let Y, X be normed linear spaces and $B \subset Y, A \subset X$ be open convex sets. Let $F: B \rightarrow F(B) \supset A$ be a bijective delta-convex mapping with a control function f . Suppose that f is Lipschitz on B with a Lipschitz constant L, F^{-1} is Lipschitz on A with a constant c^{-1} and $L < c$. Then F^{-1} is delta-convex on A with a control function*

$$h(x) = \frac{1}{c-L} f(F^{-1}(x)).$$

Proof. It suffices to apply Theorem 4.4 with $X, Y, Z = X, A, B, G(x, y) = x - F(y), g(x, y) = f(y)$ and $\varphi(x) = F^{-1}(x)$.

For general bi-Lipschitz delta-convex mappings in the infinite-dimensional case we have the following theorem only (cf. Theorem 5.2 for finite-dimensional case and Problems 1, 2).

THEOREM 4.6. *Let Y be an Asplund-Banach space and let X be a normed linear space. Let $V \subset Y, U \subset X$ be open convex sets and let $F: V \rightarrow U$ be a bijective locally delta-convex mapping on V . Suppose that F^{-1} is locally Lipschitz on U . Then there exists an open dense subset $\tilde{U} \subset U$ such that F^{-1} is locally delta-convex on \tilde{U} .*

Proof. For each $y \in V$ find an open convex neighbourhood W_y of y , a convex continuous function f_y on W_y and $c_y > 0$ such that F is delta-convex on W_y with a control function f_y and F^{-1} is c_y -Lipschitz on $F(W_y)$. Since Y is an Asplund space, we can find a dense subset D_y of W_y such that the Fréchet derivative f'_y exists at all points of D_y . For each $d \in D_y$, choose an open convex neighbourhood $B_{y,d}$ of d such that $B_{y,d} \subset W_y$ and the function

$$f_{y,d} := f_y - f'_y(d)$$

is Lipschitz on $B_{y,d}$ with a constant $L_{y,d} < c_y$ (here we have used the strict differentiability of f_y at d , see Proposition 3.8). Now, it is possible to put

$$\tilde{U} = \bigcup \{F(B_{y,d}): y \in V, d \in D_y\}.$$

In fact, if $x \in \tilde{U}$, we can find $y \in V$ and $d \in D_y$ such that $x \in F(B_{y,d})$. Further choose an open convex neighbourhood A of $x, A \subset F(B_{y,d})$. Since the assumptions of Theorem 4.5 are satisfied with $B = B_{y,d}, L = L_{y,d}, c = c_y, A$, we obtain delta-convexity of F^{-1} on A .

Note 4.7. If Y^* is a separable space, we can use the result of [29] and slightly refine the proof of 4.6 to obtain \tilde{U} for which $U \setminus \tilde{U}$ is σ -porous nowhere dense set.

The following Mixing Lemma is in some situations a useful tool for proofs of delta-convexity (cf. proof of Theorem 5.1 and Example 3). It can be also considered as a generalization of the well-known fact that maximum of finitely many delta-convex functions is delta-convex.

LEMMA 4.8 (Mixing Lemma). *Let X, Y be normed linear spaces and $A \subset X$ be an open convex set. Suppose that $F_i: A \rightarrow Y, i = 1, \dots, m$, be delta-convex mappings on A . Suppose that $F: A \rightarrow Y$ is a continuous mapping such that*

$$F(x) \in \{F_1(x), \dots, F_m(x)\} \quad \text{for any } x \in A.$$

Then F is delta-convex on A .

Proof. Let F_i be controlled by f_i on $A, i = 1, \dots, m$. Let $1 \leq i \leq j \leq m$ be fixed. Since $F_i - F_j$ is obviously controlled by $f_i + f_j$ and $\|\cdot\|_Y$ is controlled by $\|\cdot\|_Y$ which has Lipschitz constant 1, we obtain by Proposition 4.1 that $\|F_i - F_j\|$ is delta-convex on A with the control function $\|F_i - F_j\| + 2(f_i + f_j)$. Put

$$h_{ij} = f_i + f_j + \frac{1}{2}\|F_i - F_j\| \quad \text{and} \quad f = \sum_{i,j=1}^m h_{ij}.$$

We shall prove by Lemma 1.17 that F is controlled by f on A . Choose arbitrary $x \in A, v \in X, \|v\| = 1$ and $\delta > 0$. Continuity of mappings $t \rightarrow F(x + tv)$ and $t \rightarrow F_i(x + tv), i = 1, \dots, m$ easily implies that there exist points $x_1, x_2 \in U(x, \delta)$, indexes $r, s \in \{1, \dots, m\}$ and $a > 0, b > 0$ such that

$$a + b = 1, \quad x = ax_1 + bx_2, \quad x_2 - x_1 = \|x_2 - x_1\| \cdot v \quad \text{and}$$

$$F(x) = F_r(x) = F_s(x), \quad F(x_1) = F_r(x_1), \quad F(x_2) = F_s(x_1).$$

Then the following inequalities hold.

$$\begin{aligned} \|aF(x_1) + bF(x_2) - F(x)\| &= \|aF_r(x_1) + bF_s(x_2) - F(x)\| \\ &= \left\| \frac{1}{2}(aF_r(x_1) + bF_r(x_2) - F_r(x)) + \frac{b}{2}(F_s(x_2) - F_r(x_2)) \right. \\ &\quad \left. + \frac{1}{2}(aF_s(x_1) + bF_s(x_2) - F_s(x)) + \frac{a}{2}(F_r(x_1) - F_s(x_1)) \right\| \\ &\leq \frac{1}{2}\|aF_r(x_1) + bF_r(x_2) - F_r(x)\| + \frac{1}{2}\|aF_s(x_1) + bF_s(x_2) - F_s(x)\| \\ &\quad + \frac{1}{2}(a\|F_r(x_1) - F_s(x_1)\| + b\|F_r(x_2) - F_s(x_2)\| - \|F_r(x) - F_s(x)\|) \\ &\leq \frac{1}{2}(af_r(x_1) + bf_r(x_2) - f_r(x)) + \frac{1}{2}(af_s(x_1) + bf_s(x_2) - f_s(x)) \\ &\quad + \frac{1}{2}(a\|F_r(x_1) - F_s(x_1)\| + b\|F_r(x_2) - F_s(x_2)\| - \|F_r(x) - F_s(x)\|) \\ &= a\left(\frac{1}{2}(f_r + f_s + \|F_r - F_s\|)(x_1)\right) + b\left(\frac{1}{2}(f_r + f_s + \|F_r - F_s\|)(x_2)\right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}(f_r + f_s + \|F_r - F_s\|)(x) \\
& = (af(x_1) + bf(x_2) - f(x)) - (ah(x_1) + bh(x_2) - h(x)) \\
& \leq af(x_1) + bf(x_2) - f(x),
\end{aligned}$$

where $h = \frac{1}{2}f_r + \frac{1}{2}f_s + \sum_{\substack{i,j=1 \\ i \neq r, j \neq s}}^m h_{ij}$.

We have used the convexity of h .

5. Inverse mappings in finite-dimensional case

THEOREM 5.1 (delta-convexity of implicate mappings). *Let X, Z be normed linear spaces and let Y be a finite-dimensional normed linear space. Let $A \subset X, B \subset Y$ be open convex sets, $c > 0$ and let $G: A \times B \rightarrow Z$ be a delta-convex mapping such that $\|G(x, y) - G(x, \bar{y})\| \geq c\|y - \bar{y}\|$ whenever $x \in A, y, \bar{y} \in B$. Let $\varphi: A \rightarrow B$ be a mapping satisfying $G(x, \varphi(x)) = 0$ on A .*

Then φ is locally delta-convex on A .

Proof. It is easy to see that we can suppose that $Y = R^n, n \geq 1$. Further, in the same way as in the proof of Theorem 4.4 we obtain that φ is continuous on A . Suppose that G is delta-convex on $A \times B$ with a control function g .

Now consider the function $\Phi(x, y) = \|G(x, y)\|$. Since the mapping $\|\cdot\|: Z \rightarrow R$ is delta-convex with the control function $\|\cdot\|$ which is 1-Lipschitz, Proposition 4.1 gives that Φ is delta-convex on $A \times B$ with a control function $F(x, y) = \|G(x, y)\| + 2g(x, y)$. Since φ is continuous and we want to prove that φ is locally delta-convex on A , we can suppose without any loss of generality that F is K -Lipschitz on $A \times B$.

Now fix $x \in A$ and put $F_x = F(x, \cdot), g_x = g(x, \cdot)$. Clearly $F_x: B \rightarrow R, g_x: B \rightarrow R$ are convex functions. We shall show that $\partial F_x(\varphi(x))$ contains an open ball with radius c , namely the ball $B_x = U(2g^*, c)$, where $g^* \in R^n$ is an arbitrarily chosen point from $\partial g_x(\varphi(x))$. To prove $B_x \subset \partial F_x(\varphi(x))$ it is sufficient to prove that for arbitrary $h \in B_x$ and $v \in R^n$ the inequality $\langle v, h \rangle \leq F'_x(\varphi(x), v)$ holds. But it always holds, since

$$\begin{aligned}
F'_x(\varphi(x), v) &= 2g'_x(\varphi(x), v) + \lim_{s \rightarrow 0^+} \|G(x, \varphi(x) + sv) - G(x, \varphi(x))\| \cdot s^{-1} \\
&\geq 2\langle v, g^* \rangle + \lim_{s \rightarrow 0^+} \frac{c\|sv\|}{s} = 2\langle v, g^* \rangle + c\|v\|
\end{aligned}$$

and there is $z \in R^n, \|z\| < 1$ for which

$$\langle v, h \rangle = \langle v, 2g^* + cz \rangle = 2\langle v, g^* \rangle + c\langle v, z \rangle \leq 2\langle v, g^* \rangle + c\|v\|.$$

Since F is K -Lipschitz, we have $\partial F_x(\varphi(x)) \subset U(0, K)$ for any $x \in A$. Consequently it is easy to see that there exists a finite number of open balls

$$U\left(z_i, \frac{c}{2}\right) \subset Y^* = R^n, \quad i = 1, \dots, p$$

such that for any $x \in A$ we can choose an index $1 \leq i_x \leq p$ such that

$$U\left(z_{i_x}, \frac{c}{2}\right) \subset B_x \subset \partial F_x(\varphi(x)).$$

Put $A_i = \{x \in A: i_x = i\}$.

Since $A = \bigcup_{i=1}^p A_i$ and φ is continuous, by Mixing Lemma 4.8 it is sufficient to prove that for any $1 \leq i \leq p$ the mapping $\varphi: A_i \rightarrow R^n$ can be extended to a delta-convex mapping on A . To this end it is sufficient to show that all components of φ can be extended to a delta-convex function on A .

To prove this, fix $1 \leq j \leq n$ and denote by e_j the j -th unit vector of the canonical basis of R^n . Put

$$z_i^* = z_i + \frac{c}{4}e_j$$

and for $x \in A_i$ define

$$\begin{aligned} v(x) &= F(x, \varphi(x)) - \langle \varphi(x), z_i \rangle \quad \text{and} \\ v^*(x) &= F(x, \varphi(x)) - \langle \varphi(x), z_i^* \rangle. \end{aligned}$$

Thus we have $v(x) - v^*(x) = \frac{c}{4} \langle \varphi(x), e_j \rangle$ for $x \in A_i$ and therefore it is sufficient to prove that v, v^* are restrictions of convex continuous functions defined on the whole A .

Since $z_i, z_i^* \in \partial F_x(\varphi(x))$ for $x \in A_i$, by the well-known version on the Hahn-Banach Theorem [31] there exist $w_x, w_x^* \in \partial F(x, \varphi(x))$ such that

$$w_x(t, y) = a_x(t) + z_i(y) \quad \text{and} \quad w_x^*(t, y) = a_x^*(t) + z_i^*(y)$$

where $a_x, a_x^* \in X^*$. Obviously $\|a_x\| \leq K, \|a_x^*\| \leq K$. Let $x, \bar{x} \in A_i$. Since $w_x \in \partial F(x, \varphi(x))$, we have

$$F(\bar{x}, \varphi(\bar{x})) - F(x, \varphi(x)) \geq \langle \bar{x} - x, a_x \rangle + \langle \varphi(\bar{x}) - \varphi(x), z_i \rangle.$$

Consequently

$$v(\bar{x}) - v(x) = F(\bar{x}, \varphi(\bar{x})) - F(x, \varphi(x)) - \langle \varphi(\bar{x}), z_i \rangle + \langle \varphi(x), z_i \rangle \geq \langle \bar{x} - x, a_x \rangle.$$

Quite analogously we obtain $v^*(\bar{x}) - v^*(x) \geq \langle \bar{x} - x, a_x^* \rangle$. Consequently we conclude that v, v^* have convex continuous extensions on the whole A (cf. Lemma 1 of [37]) and the proof is over.

As an immediate consequence we obtain the following theorem.

THEOREM 5.2 (delta-convexity of inverse mappings). *Let $n \geq 1$ and $G \subset \mathbb{R}^n$ be an open set. Let $F: G \rightarrow \mathbb{R}^n$ be a bi-Lipschitz mapping which is locally delta-convex on G . Then F^{-1} is locally delta-convex on $F(G)$.*

Proof. The invariance-of-domain theorem implies that $F(G)$ is an open set. Choose a point $x_0 \in F(G)$. Put $y_0 = F^{-1}(x_0)$ and choose an open convex neighbourhood A of x_0 and an open convex neighbourhood B of y_0 such that $F(B) \subset A$ and F is delta-convex on B . Now it is sufficient to use Theorem 5.1 for

$$X = \mathbb{R}^n, \quad Y = \mathbb{R}^n, \quad Z = \mathbb{R}^n, \quad A, \quad B,$$

$$G(x, y) = x - F(y) \quad \text{and} \quad \varphi = F^{-1}|_A.$$

Our theorems say nothing about the existence of the implicate and inverse mappings. They show the stability of the system of locally delta-convex mappings only. Very general conditions which imply the (local) existence of the implicate and inverse mappings can be stated in terms of generalized Clarke's derivatives (Jacobians) (cf. [10], [28]).

We state results which immediately follow from [10], [28], [16] and our results. We shall sketch the proofs only.

DEFINITION 5.3. Let $U \subset \mathbb{R}^n$ ($n \geq 1$) be an open set and let $f: U \rightarrow \mathbb{R}^k$ ($k \geq 1$) be a Lipschitz mapping. Let $N \subset U$ be a set of Lebesgue measure zero which contains all points of (Fréchet) non-differentiability of f and let $a \in A$. Then we put

$$\partial f(a) = \bigcap_{\delta > 0} \overline{\text{co}} \{f'(x): x \in U(a, \delta) \setminus N\}.$$

Note 5.4. In view of [16] the generalized derivative $\partial f(a)$ does not depend on N . Consequently the concepts of [10] and [28] coincide.

DEFINITION 5.5. Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open set and let $f: U \rightarrow \mathbb{R}^p$ be a Lipschitz mapping ($n, k, p \geq 1$). Let $N \subset U$ be a set of Lebesgue measure zero such that for any $z = (z_1, z_2) \in U \setminus N$ the (partial Fréchet) derivative $f'_2(z) = (f(z_1, \cdot))'(z_2)$ exists. Let $a \in A$. Then we put

$$\partial_2 f(a) = \bigcap_{\delta > 0} \overline{\text{co}} \{f'_2(z): z \in U(a, \delta) \setminus N\}.$$

Note 5.6. Slightly changing the proof of [16], we see that the given definition of a generalized partial derivative does not depend on N and coincides with the one given in [28].

Now we can formulate the following results.

PROPOSITION 5.7. *Let $U \subset \mathbb{R}^n$ be an open set, $a \in U$ and let $F: U \rightarrow \mathbb{R}^n$ be locally delta-convex on A . Suppose that $\partial F(a)$ contains surjective linear mappings*

only. Then there exists an open neighbourhood V of a such that $(F|_V)^{-1}$ is locally delta-convex on the open set $F(V)$.

Proof. It is sufficient to use Theorem 5.2 and Clarke's inverse mapping theorem (or Theorem 11.1 of [28]).

Note 5.8. We do not know whether the sufficient condition from Proposition 5.7 is necessary one (cf. Problem 8).

PROPOSITION 5.9. Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open set, $(a, b) \in U$ and let $G: U \rightarrow \mathbb{R}^k$ be a locally delta-convex mapping on U such that $G(a, b) = 0$ and $\partial_2 G(a, b)$ contains surjective mappings only. Then there exist $\delta, \varepsilon > 0$ and a delta-convex mapping $\varphi: U(a, \delta) \rightarrow U(b, \varepsilon)$ such that for $x \in U(a, \delta)$ and $y \in U(b, \varepsilon)$

$$y = \varphi(x) \quad \text{iff} \quad G(x, y) = 0.$$

Proof. It is sufficient to use Theorem 5.1 and the proof of Theorem 12.1 from [28] (cf. Note 5.6). In fact, the uniqueness of φ , which is not explicitly formulated in [28], easily follows from the proof. The validity of the inequality

$$\|G(x, y) - G(x, \bar{y})\| \geq c \|y - \bar{y}\|, \quad c > 0,$$

for x sufficiently near to a and y sufficiently near to b follows by the standard methods from the regularity of $\partial_2 G(a, b)$.

6. Examples and applications

A. Three counterexamples.

EXAMPLE 6.1. There exists a mapping $F: \mathbb{R} \rightarrow l_2$ such that

(i) F is a difference of two Lipschitz convex operators (we consider coordinate-wise ordering of l_2),

(ii) $y^* \circ F$ is delta-convex on \mathbb{R} for any $y^* \in (l_2)^*$,

(iii) F is delta-convex on no neighbourhood of 0.

Sketch of the proof. Let e_i be the i -th vector of the canonical basis of l_2 . Let

$$\begin{aligned} F(t) &= 0 & \text{for } t \leq 0, \\ F\left(\frac{1}{2^i}\right) &= \frac{1}{i2^i} e_i, & i = 1, 2, \dots, \\ F(t) &= \frac{1}{2} e_1 & \text{for } t \geq \frac{1}{2}, \end{aligned}$$

and let F be affine on each interval $\left[\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$, $i = 1, 2, \dots$. It is easy to construct $\frac{2}{i}$ -Lipschitz convex functions G_i, H_i on \mathbb{R} , $i = 1, 2, \dots$, such that

$F = (G_1 - H_1, G_2 - H_2, \dots)$. Then $G = (G_1, G_2, \dots)$, $H = (H_1, H_2, \dots)$ are Lipschitz convex operators and $F = G - H$. A simple computation based on Theorem 2.3 gives (ii) and (iii).

EXAMPLE 6.2. There exists a function f on R^2 which is delta-convex on any line (cf. Note 1.2(b)) and is not delta-convex on R^2 .

Sketch of the proof. Put $x_n = 2^{-n}$, $y_n = x_n^2$ and $z_n = (x_n, y_n)$. It is easy to construct a sequence $(r_n)_{n=1}^\infty$ of positive numbers such that no line in R^2 intersects more than two members of the sequence $(U(z_n, r_n))_{n=1}^\infty$. Put

$$\begin{aligned} f(x) &= r_n - \|x - z_n\| && \text{for } x \in U(z_n, r_n) \quad \text{and} \\ f(x) &= 0 && \text{for } x \notin \bigcup_{n=1}^\infty U(z_n, r_n). \end{aligned}$$

It is easy to see that f is 1-Lipschitz and delta-convex on each line, moreover

$$\mathbf{K}_{t_1}^{t_2} f(a + tv) \leq 8 \quad \text{whenever } a \in R^2, v \in R^2, \|v\| = 1 \text{ and } t_1 < t_2.$$

On the other hand, it is easy to compute that

$$\mathbf{K}_{- \varepsilon}^{\varepsilon} f(t, t^2) = +\infty \quad \text{for each } \varepsilon > 0.$$

Thus Theorem 2.3 implies that $t \rightarrow f(t, t^2)$ is not locally delta-convex on R . Since $t \rightarrow (t, t^2)$ is delta-convex on R , Theorem 4.2 implies that f is not locally delta-convex.

EXAMPLE 6.3. There exists a delta-convex bijection $F: R^2 \rightarrow R^2$ such that F^{-1} is delta-convex and F is differentiable but not strictly differentiable at the origin.

Sketch of the proof. Put

$$\begin{aligned} F(x, y) &= F_1(x, y) := (x, y) && \text{if } x \leq 0, \\ F(x, y) &= F_2(x, y) := (x, y + x^2) && \text{if } x \geq 0 \text{ and } y \geq x^2, \\ F(x, y) &= F_3(x, y) := (x, 2y) && \text{if } x \geq 0 \text{ and } -x^2 \leq y \leq x^2, \\ F(x, y) &= F_4(x, y) := (x, y - x^2) && \text{if } x \geq 0 \text{ and } y \leq -x^2. \end{aligned}$$

It is easy to see that the definition is correct and that F is a continuous bijection of R^2 onto R^2 . Using Mixing Lemma 4.8 we immediately obtain that F is delta-convex on R^2 . It is easy to see that F^{-1} is also a "mixture" of four delta-convex mappings; consequently F^{-1} is delta-convex, too. Obviously $F'((0, 0), v) = v$ and since F is locally Lipschitz (e.g. by 1.10), we conclude that $F'((0, 0)) = \text{Id}$. On the other hand, since for $x > 0$ $F'((x, 0), (0, 1)) = (0, 2)$, F is not strictly differentiable at $(0, 0)$.

Note 6.4. The above example shows that there exists a delta-convex function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ which is differentiable at a point $a \in \mathbb{R}^2$ but it is not strictly differentiable at a . (A similar example is given in [32].) Proposition 3.9 gives that f is controlled by no function which is differentiable at a . In this case, there necessarily exists $v \in \mathbb{R}^2$ such that $f'(\cdot, v)$ is not continuous at a . On the other hand, it is possible to prove that differentiability of a delta-convex function f at a implies continuity of $f'(\cdot, v)$ at a with respect to the ordinary density topology (for definition see, e.g. [24], p. 167) on \mathbb{R}^2 .

B. Nemyckii and Hammerstein operators.

PROPOSITION 6.5 (delta-convexity of Nemyckii operators). *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \leq p < \infty$ and let f, g be real functions on $\Omega \times \mathbb{R}$ satisfying:*

(a) *There exist constants $c_1, c_2 \geq 0$ and functions $w_1, w_2 \in L_1(\Omega)$ such that for almost every $x \in \Omega$*

$$\begin{aligned} |f(x, \cdot)| &\leq w_1(x) + c_1 |\cdot|^p \quad \text{and} \\ |g(x, \cdot)| &\leq w_2(x) + c_2 |\cdot|^p. \end{aligned}$$

(b) *For almost every $x \in \Omega$, the function $g(x, \cdot)$ is delta-convex on \mathbb{R} with the control function $f(x, \cdot)$.*

(c) *For each $t \in \mathbb{R}$ the functions $f(\cdot, t)$, $g(\cdot, t)$ are measurable.*

Then the Nemyckii operator $G: L_p(\Omega) \rightarrow L_1(\Omega)$, defined by

$$G(u)(x) = g(x, u(x)),$$

is delta-convex on $L_p(\Omega)$ with the control function

$$\Phi(u) = \int_{\Omega} f(x, u(x)) dx.$$

Proof. The assumptions imply that the Nemyckii operators $G(u) = g(\cdot, u(\cdot))$ and $F(u) = f(\cdot, u(\cdot))$ act from $L_p(\Omega)$ into $L_1(\Omega)$ and are continuous (see [33]). Hence Φ is continuous, too.

Let $a \geq 0, b \geq 0, a + b = 1$. Then for any $u, v \in L_p(\Omega)$ the following holds.

$$\begin{aligned} \|aG(u) + bG(v) - G(au + bv)\|_{L_1(\Omega)} & \\ &= \int_{\Omega} |ag(x, u(x)) + bg(x, v(x)) - g(x, au(x) + bv(x))| dx \\ &\leq \int_{\Omega} (af(x, u(x)) + bf(x, v(x)) - f(x, au(x) + bv(x))) dx \\ &= a\Phi(u) + b\Phi(v) - \Phi(au + bv). \end{aligned}$$

Hence G is delta-convex with the control function Φ by 1.13(iii).

Note 6.6. The situation for Nemyckii operators acting into $L_q(\Omega)$ with $q > 1$, is not so nice. For example, it is possible to prove that the mapping $u \rightarrow |u|$ is a continuous operator from $L_p((0, 1))$ into $L_2((0, 1))$ for any $p > 2$ but is never locally delta-convex.

EXAMPLE 6.7. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, g be a function on $\Omega \times \mathbb{R}$ and $c > 0$. Suppose that

- (i) $g_x := g(x, \cdot)$ is convex for each $x \in \Omega$,
- (ii) $|(g_x)'_+(t)| \leq c|t|$ for any $x \in \Omega$, $t \in \mathbb{R}$,
- (iii) $g(\cdot, 0) \in L_1(\Omega)$,
- (iv) $g(\cdot, t)$ is measurable for each $t \in \mathbb{R}$.

Then the Nemyckii operator $G: L_2(\Omega) \rightarrow L_1(\Omega)$, $G(u) = g(\cdot, u(\cdot))$ is delta-convex on $L_2(\Omega)$.

In fact, it is sufficient to use Proposition 6.5 for $f = g$.

Using Proposition 3.1 and Theorem 3.10 (i), we obtain the following result.

COROLLARY 6.8. *Let the assumptions of Proposition 6.5 (or of Example 6.7) be satisfied. Then the Nemyckii operator $G: L_p(\Omega) \rightarrow L_1(\Omega)$ has the following properties.*

- (a) G has all one-sided derivatives at each point of $L_p(\Omega)$.
- (b) If $p > 1$ then G is (strictly) Fréchet differentiable at all points of $L_p(\Omega)$ except those which belong to a first category set.

PROPOSITION 6.9. *Let $\Omega' \subset \mathbb{R}^k$, $\Omega \subset \mathbb{R}^n$ be measurable sets, $1 \leq p < \infty$, $1 \leq q < \infty$ and let functions f, g on $\Omega \times \mathbb{R}$ satisfy the conditions (a), (b), (c) from Proposition 6.5. Let $K: \Omega' \times \Omega \rightarrow \mathbb{R}$ be a measurable function with the property:*

There exists $C \geq 0$ such that

- (d) $\int_{\Omega'} |K(x, y)|^q dx \leq C$ for almost every $y \in \Omega$.

Then the Hammerstein operator $H(u) = \int_{\Omega} K(\cdot, y)g(y, u(y))dy$ acts from $L_p(\Omega)$ into $L_q(\Omega')$ and is delta-convex with the control function $\Psi(u) = C^{1/q} \int_{\Omega} f(y, u(y))dy$.

Proof. It is easy to see that $H = T \circ G$ where G is the Nemyckii operator from 6.5 and

$$T(v) = \int_{\Omega} K(\cdot, y)v(y)dy.$$

The assumption (d) implies that T is a bounded linear operator from $L_1(\Omega)$ into $L_q(\Omega')$ with $\|T\| \leq C^{1/q}$ (see [15]. Ch. VI, § 9, Exercise 59). Hence by Lemma 1.5 (a) H is delta-convex with the control function Ψ .

EXAMPLE 6.10. Let $K: (0, 1) \times (0, 1) \rightarrow \mathbb{R}$ be a bounded measurable function and $g: \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary convex Lipschitz function. Then the Hammerstein operator $H(u) = \int_0^1 K(\cdot, y)g(u(y))dy$ is a delta-convex mapping from $L_p((0, 1))$ into $L_q((0, 1))$ for arbitrary $p, q \in [1, +\infty)$.

Let $1 \leq p < \infty$, $q = p$. Denote $M = \sup\{|K(x, y)|: x, y \in (0, 1)\}$. $L = \text{Lip}(g)$ and choose any $\lambda \in \mathbb{R}$ with $|\lambda| < (2ML)^{-1}$. Then the equation $u + \lambda H(u) = v$ is solvable in $L_p((0, 1))$ for any $v \in L_p((0, 1))$ and the mapping $(\text{Id} + \lambda H)^{-1}$ is delta-convex on $L_p((0, 1))$.

Proof. Denote $m = \max\{|g(0)|, 1\}$. Then the Lipschitz property of g and the Hölder inequality imply

$$|g(t)| \leq |g(0) + L|t - 0| \leq m + L|t| \leq (m + L|t|)^p \leq 2^{p-1}(m^p + L^p|t|^p).$$

In addition $\int_0^1 |K(x, y)|^q dx \leq M^q$. Consequently H is delta-convex with the control function $\Psi(u) = M \int_0^1 g(u(y)) dy$ by 6.9.

It is easy to see that $\text{Lip}(H) \leq \|T\| \cdot \text{Lip}(G) \leq ML$, where $T(v) = \int_0^1 K(\cdot, y)v(y) dy$, $G(u) = g(u(\cdot))$, and analogously $\text{Lip}(\Psi) \leq ML$. Our equation can be rewritten in the form $u = v - \lambda H(u)$ and is now solvable by the Banach Contraction Principle.

The mapping $\text{Id} + \lambda H$ is delta-convex with the control function $|\lambda| \cdot \Psi$ and

$$\|(\text{Id} + \lambda H)(u) - (\text{Id} + \lambda H)(\bar{u})\| \geq (1 - |\lambda|ML)\|u - \bar{u}\|.$$

Hence by Theorem 4.5 the mapping $(\text{Id} + \lambda H)^{-1}$ is delta-convex, since $\text{Lip}(|\lambda| \cdot \Psi) \leq |\lambda|ML < (1 - |\lambda|ML)$.

Of course, we obtain that under some conditions the Hammerstein operator H and the solving operator $(\text{Id} + \lambda H)^{-1}$ have good differentiability properties. We formulate the following "stability" result only.

COROLLARY 6.11. *Let the assumptions of Example 6.10 be satisfied and let $p > 1$. Then:*

(a) *The solving operator $S := (\text{Id} + \lambda H)^{-1}$ has all one-sided directional derivatives at all points and is (strictly) Fréchet differentiable at all points of $L_p((0, 1))$ except a first category set.*

(b) *Let u, v_1, \dots, v_n be functions of $L_p((0, 1))$. Define the mapping $V: \mathbb{R}^n \rightarrow L_p((0, 1))$ by*

$$V(t_1, \dots, t_n) = S(u + t_1 v_1 + \dots + t_n v_n).$$

Then V has the second (Peano) derivative at almost all points of \mathbb{R}^n .

C. Weak solution of a differential equation. Consider the following Dirichlet problem:

$$(P) \quad \begin{aligned} -u''(x) + g(u(x)) &= f(x), & x \in J = (a, b) \\ u(a) &= u(b) = 0. \end{aligned}$$

The family

$$W_0^{1,2}(J) = \{u: u \text{ is absolutely continuous on } [a, b], \\ u' \in L_2(J), u(a) = u(b) = 0\}$$

is a (real) Hilbert space with the inner product

$$\langle u, v \rangle = \int_J u'(x)v'(x)dx$$

and the norm $\|u\| = \langle u, u \rangle^{1/2}$ (cf. [17]).

Let $v \in W_0^{1,2}(J)$ and denote $|J| = b - a$.

Then the inequality

$$v^2(x) = \left(\int_a^x v'(y)dy \right)^2 \leq (x-a) \cdot \int_a^x v'^2(y)dy \leq |J| \cdot \|v\|^2$$

implies

$$(12) \quad |v(x)| \leq |J|^{1/2} \cdot \|v\| \quad \text{for } x \in [a, b] \quad \text{and}$$

$$(13) \quad \|v\|_{L_2(J)} \leq |J| \cdot \|v\|.$$

The weak formulation of the problem (P) is following (see [17], p. 26):

Find $u \in W_0^{1,2}(J)$ such that

$$(W) \quad \int_J u'(x)v'(x)dx + \int_J g(u(x))v(x)dx = \int_J f(x)v(x)dx \quad \text{for every } v \in W_0^{1,2}(J).$$

PROPOSITION 6.12. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a delta-convex function such that

$$\frac{1}{2} \cdot \bigvee_{-\infty}^{+\infty} (g'_+) + \text{Lip}(g) < |J|^{-2}.$$

Then for arbitrary right-hand side $f \in L_1(J)$ there exists a unique solution $u \in W_0^{1,2}(J)$ of the problem (W) and the mapping $f \mapsto u$ is a delta-convex mapping from $L_1(J)$ into $W_0^{1,2}(J)$.

Proof. Define a mapping $G: W_0^{1,2}(J) \rightarrow W_0^{1,2}(J)$ and a function $\varphi \in W_0^{1,2}(J)$ by the formulas

$$\langle G(u), v \rangle = \int_J g(u(x))v(x)dx, \quad v \in W_0^{1,2}(J),$$

$$(14) \quad \langle \varphi, v \rangle = \int_J f(x)v(x)dx, \quad v \in W_0^{1,2}(J).$$

Then (W) can be rewritten to the form

$$(E) \quad u + G(u) = \varphi.$$

Let h be a convex function controlling g and satisfying

$$\text{Lip}(h) = \frac{1}{2} \cdot \bigvee_{-\infty}^{+\infty} (g'_+).$$

(Choose $x_0 \in R$ and put $\tilde{h}(x) = \int_{x_0}^x (\bigvee_{x_0}^t (g'_+)) dt$. The proof of (iii) \Rightarrow (i) in Theorem 2.3 shows that \tilde{h} controls g . Further $\bigvee_{x_0}^{+\infty} g = s - i$, where $s = \sup\{\tilde{h}_+(t) : t \in R\} = \lim_{t \rightarrow +\infty} \tilde{h}_+(t)$ and $i = \inf\{\tilde{h}_+(t) : t \in R\} = \lim_{t \rightarrow -\infty} \tilde{h}_+(t)$. It suffices to put $h(t) = \tilde{h}(t) - \frac{s+i}{2} \cdot t$.)

Let $\|v\| = 1$. Then the function

$$\begin{aligned} u \mapsto \langle G(u), v \rangle + |J|^{1/2} \int_J h(u(x)) dx \\ = \int_J |v(x)| \cdot (g(u(x)) \cdot \operatorname{sgn}(v(x)) + h(u(x))) dx \\ + \int_J (|J|^{1/2} - |v(x)|) \cdot h(u(x)) dx \end{aligned}$$

is convex and continuous on $W_0^{1,2}(J)$, since $|v(x)| \leq |J|^{1/2}$ on J and $u \mapsto |v(x)| \cdot (g(u(x)) \cdot \operatorname{sgn}(v(x)) + h(u(x)))$ is convex on $W_0^{1,2}(J)$ for each fixed $x \in J$.

Consequently G is delta-convex on $W_0^{1,2}(J)$ with the control function

$$\Phi(u) = |J|^{1/2} \int_J h(u(x)) dx.$$

By (13),

$$\begin{aligned} |\langle G(u) - G(w), v \rangle| &\leq \int_J |g(u(x)) - g(w(x))| \cdot |v(x)| dx \\ &\leq \operatorname{Lip}(g) \cdot \int_J |u(x) - w(x)| \cdot |v(x)| dx \leq \operatorname{Lip}(g) \cdot \|u - w\|_{L_2(J)} \cdot \|v\|_{L_2(J)} \\ &\leq \operatorname{Lip}(g) \cdot |J|^2 \cdot \|u - w\| \quad \text{for any } \|v\| = 1, \end{aligned}$$

and analogously

$$\begin{aligned} |\Phi(u) - \Phi(w)| &\leq |J|^{1/2} \cdot \int_J |h(u(x)) - h(w(x))| dx \\ &\leq |J|^{1/2} \cdot \operatorname{Lip}(h) \cdot \int_J |u(x) - w(x)| dx \leq |J| \cdot \operatorname{Lip}(h) \cdot \|u - w\|_{L_2(J)} \\ &\leq |J|^2 \cdot \operatorname{Lip}(h) \cdot \|u - w\|. \end{aligned}$$

Consequently $\operatorname{Lip}(G) \leq \operatorname{Lip}(g) \cdot |J|^2$ and $\operatorname{Lip}(\Phi) \leq \operatorname{Lip}(h) \cdot |J|^2$.

The equation (E) is equivalent to $u = \varphi - G(u)$ and is uniquely solvable for any $\varphi \in W_0^{1,2}(J)$ by the Banach Contraction Principle, since $\operatorname{Lip}(G) < 1$. We have

$$\|(\operatorname{Id} + G)(u) - (\operatorname{Id} + G)(w)\| \geq (1 - \operatorname{Lip}(G)) \|u - w\| \geq (1 - \operatorname{Lip}(g) |J|^2) \|u - w\|.$$

Since

$$\text{Lip}(\Phi) \leq \text{Lip}(h)|J|^2 = \frac{1}{2} \bigvee_{-\infty}^{+\infty} (g'_+) \cdot |J|^2 < 1 - \text{Lip}(g) \cdot |J|^2,$$

the mapping $(\text{Id} + G)^{-1}$ is delta-convex by Theorem 4.5. The mapping $S: f \mapsto \varphi$ defined by (14), is a bounded linear operator from $L_1(J)$ into $W_0^{1,2}(J)$, since

$$\|\varphi\| = \sup\{\langle \varphi, v \rangle : \|v\| = 1\} = \sup\left\{\int_J f(x)v(x)dx : \|v\| = 1\right\} \leq |J|^{1/2} \|f\|_{L_1(J)}.$$

The "solving mapping" $f \mapsto u$ for the problem (W) can be written in the form $(\text{Id} + G)^{-1} \circ S$ and is therefore delta-convex on $L_1(J)$.

EXAMPLE 6.13. Let $p > 0$ and f be a p -periodic delta-convex function on \mathbb{R} . Then there exists $\varepsilon_0 > 0$ such that

$$g(x) := \varepsilon f(x)e^{-|x|}$$

satisfies the assumption of Proposition 6.12 for each $0 < \varepsilon < \varepsilon_0$. We omit the simple proof based on the fact that

$$g(x + kp) = g(x) \cdot e^{-kp} \quad \text{for } x > 0 \text{ and } k = 0, 1, 2, \dots$$

In the following example we show ε_0 for some f . We omit an elementary but non-trivial computation.

EXAMPLE 6.14. Let h be a convex Lipschitz function on $[-1, 1]$, $M = \max\{|h(x)| : x \in [-1, 1]\}$ and $L = \text{Lip}(h)$.

If $|\varepsilon| < \frac{1 - e^{-2\pi}}{12(M+L)|J|^2}$ holds, then the function

$$g(x) = \varepsilon e^{-|x|} h(\sin x)$$

satisfies the assumption of Proposition 6.12.

Of course, Proposition 6.12 yields some stability results concerning the dependence of the solution of the problem (W) on f . Nevertheless, we do not obtain generic differentiability of the solving mapping $f \mapsto u$, since $L_1(J)$ is not an Asplund space. Since $|J| < \infty$, we obtain the following result.

COROLLARY 6.15. *Let g satisfy the assumption of Proposition 6.12. Then the restriction of the solving mapping of the problem (W) on $L_2(J)$ is (strictly) Fréchet differentiable at all points of $L_2(J)$ except those which belong to a first category set.*

D. Quasidifferentiable functions and mappings. A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *quasidifferentiable* [11] at $a \in \mathbb{R}^n$ if $f'(a, v)$ exists for each $v \in \mathbb{R}^n$ and the function $f'(a, \cdot)$ is a difference of two sublinear functions.

All delta-convex and all smooth functions are clearly quasidifferentiable.

In [12] the notion of a quasidifferentiable mapping was defined. A mapping $F: X \rightarrow Y$, where X is a normed linear space and Y is a K -space (i.e. a conditionally complete vector lattice provided with a monotonic norm and complete with respect to this norm), is said to be *quasidifferentiable* at $a \in X$ if $F'(a, \cdot)$ is representable as a difference of two continuous sublinear mappings.

If we consider the coordinate-wise ordering on a space R^n , then we obtain on account of 1.8 and 1.21 the following easy fact.

LEMMA 6.16. *Let $A \subset R^k$ be an open set, $a \in A$ and let $F = (F_1, \dots, F_n): A \rightarrow R^n$ be a mapping. Then the following conditions are equivalent.*

- (i) F is quasidifferentiable at a ;
- (ii) F_1, \dots, F_n are quasidifferentiable at a ;
- (iii) $F'(a, \cdot)$ is delta-convex on R^k ;
- (iv) $F'_1(a, \cdot), \dots, F'_n(a, \cdot)$ are delta-convex on R^k .

It is proved in [12] that under some conditions the composition of two quasidifferentiable mappings is quasidifferentiable as well. In particular, these conditions are satisfied in the case of mappings between Euclidean spaces; in this case it is not difficult to obtain the result as an immediate consequence of the Hartman's Superposition Theorem (Theorem 4.2).

Now we shall show that our results concerning delta-convexity of implicate and inverse mappings yield corresponding results concerning quasidifferentiability, which seem to be new.

THEOREM 6.17 (quasidifferentiability of implicate mappings). *Let $A \subset R^k$, $B \subset R^m$ be open sets, $c_0 > 0$, $c_1 > 0$ and let $G: A \times B \rightarrow R^p$ and $\varphi: A \rightarrow B$ satisfy:*

- (i) $G(x, \varphi(x)) = 0$ for any $x \in A$,
- (ii) $c_0 \|y_1 - y_2\| \leq \|G(x, y_1) - G(x, y_2)\| \leq c_1 \|y_1 - y_2\|$ whenever $x \in A$ and $y_1, y_2 \in B$,
- (iii) G is quasidifferentiable at a point $(x_0, \varphi(x_0)) \in A \times B$.

Then φ is quasidifferentiable at x_0 .

Proof. Choose an arbitrary $u \in R^k$. Then for any sequence $t_n \rightarrow 0+$

$$\begin{aligned} \limsup \left\| \frac{1}{t_n} (\varphi(x_0 + t_n u) - \varphi(x_0)) \right\| & \\ & \leq \frac{1}{c_0} \limsup \frac{1}{t_n} \|G(x_0 + t_n u, \varphi(x_0 + t_n u)) - G(x_0 + t_n u, \varphi(x_0))\| \\ & = \frac{1}{c_0} \limsup \frac{1}{t_n} \|G(x_0, \varphi(x_0)) - G(x_0 + t_n u, \varphi(x_0))\| \\ & = \frac{1}{c_0} \|G'((x_0, \varphi(x_0)), (u, 0))\|. \end{aligned}$$

Consequently there exist a subsequence (t'_n) of (t_n) and a vector $v \in R^m$ such that

$$v = \lim_{t'_n} \frac{\varphi(x_0 + t'_n u) - \varphi(x_0)}{t'_n}.$$

Now, let $s_n \rightarrow 0+$ be an arbitrary sequence such that the limit $\lim_{s_n} \frac{\varphi(x_0 + s_n u) - \varphi(x_0)}{s_n}$ exists. Then this limit must equal to v , since

$$\begin{aligned} \left\| \lim_{s_n} \frac{\varphi(x_0 + s_n u) - \varphi(x_0)}{s_n} - v \right\| &= \lim_{s_n} \frac{1}{s_n} \left\| \varphi(x_0 + s_n u) - (\varphi(x_0) + s_n v) \right\| \\ &\leq \frac{1}{c_0} \limsup \frac{1}{s_n} \left\| G(x_0 + s_n u, \varphi(x_0 + s_n u)) - G(x_0 + s_n u, \varphi(x_0) + s_n v) \right\| \\ &= \frac{1}{c_0} \limsup \frac{1}{s_n} \left\| G(x_0, \varphi(x_0)) - G(x_0 + s_n u, \varphi(x_0) + s_n v) \right\| \\ &= \frac{1}{c_0} \left\| G'((x_0, \varphi(x_0)), (u, v)) \right\| \\ &= \frac{1}{c_0} \lim_{t'_n} \frac{1}{t'_n} \left\| G(x_0 + t'_n u, \varphi(x_0) + t'_n v) - G(x_0 + t'_n u, \varphi(x_0 + t'_n u)) \right\| \\ &\leq \frac{c_1}{c_0} \lim_{t'_n} \left\| \frac{\varphi(x_0 + t'_n u) - \varphi(x_0)}{t'_n} - v \right\| = 0. \end{aligned}$$

Consequently $\varphi'(x_0, u)$ exists for any $u \in R^k$. The inequalities above show that $\varphi'(x_0, u) = v$ iff $G'((x_0, \varphi(x_0)), (u, v)) = 0$.

It is easy to prove that the property (ii) implies

$$\left\| G'((x_0, \varphi(x_0)), (u, v_1)) - G'((x_0, \varphi(x_0)), (u, v_2)) \right\| \geq c_0 \|v_1 - v_2\|.$$

Hence it is possible to apply Theorem 5.1 to the delta-convex mapping $(u, v) \rightarrow G'((x_0, \varphi(x_0)), (u, v))$, and obtain that the mapping $\varphi'(x_0, \cdot)$ is delta-convex on R^k . Consequently φ is quasidifferentiable at x_0 .

COROLLARY 6.18. (quasidifferentiability of inverse mappings). *Let $c_0 > 0$, $c_1 > 0$ and let F be a mapping of an open set $A \subset R^k$ onto an open set $B \subset R^m$ with the property*

$$c_0 \|x_1 - x_2\| \leq \|F(x_1) - F(x_2)\| \leq c_1 \|x_1 - x_2\|, \quad \text{whenever } x_1, x_2 \in A.$$

If F is quasidifferentiable at a point $x_0 \in A$, then F^{-1} is quasidifferentiable at $F(x_0)$.

Similarly as in Section 5 it is possible to obtain the following propositions which are analogous to Propositions 5.7 and 5.9.

PROPOSITION 6.19. *Let $U \subset R^n$ be an open set, $a \in U$ and let $F: U \rightarrow R^n$ be quasidifferentiable at a (or at each point of U). Suppose that $\partial F(a)$ contains*

surjective linear mappings only. Then there exists an open neighbourhood V of a such that $(F|_V)^{-1}$ is quasidifferentiable at $F(a)$ (or at each point of the open set $F(V)$, respectively).

PROPOSITION 6.20. *Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open set, $(a, b) \in U$ and let $G: U \rightarrow \mathbb{R}^k$ be a mapping which is quasidifferentiable at (a, b) (or at each point of U). Suppose that $G(a, b) = 0$ and $\partial_2 G(a, b)$ contains surjective mappings only. Then there exist $\delta, \varepsilon > 0$ and a mapping $\varphi: U(a, \delta) \rightarrow U(b, \varepsilon)$ quasidifferentiable at a (or at each point of $U(a, \delta)$ respectively) such that for $x \in U(a, \delta)$ and $y \in U(b, \varepsilon)$*

$$y = \varphi(x) \quad \text{iff} \quad G(x, y) = 0.$$

7. Some open problems

PROBLEM 1. *Let X, Y be Banach spaces, $G \subset X, H \subset Y$ be open sets and let $F: G \rightarrow H$ be a bi-Lipschitz bijection which is locally delta-convex on G . Is then F^{-1} locally delta-convex on H ?*

Note. If $X = Y = \mathbb{R}^n$, Theorem 5.2 gives a positive answer. We do not know an answer either in the case $X = Y = l_2$.

PROBLEM 2. *Let X, Y be Banach spaces, $G \subset X$ be an open set and $a \in G$. Let $F: G \rightarrow Y$ be a delta-convex mapping which has an invertible strict derivative at a . Does there exist an open neighbourhood U of a such that $(F|_U)^{-1}$ is locally delta-convex on $F(U)$?*

Note. The inverse mapping theorem for a strict differentiable mappings ([7], [9], [26]) implies that a positive solution of Problem 1 gives immediately a positive solution of Problem 2. Also the case $X = Y = l_2$ is open.

PROBLEM 3. *Let X, Y be Banach spaces, $G \subset X$ be an open set and $a \in G$. Let $F: G \rightarrow Y$ be a delta-convex mapping which is strictly differentiable at a . Does there exist a function f which controls F on G and which is strictly differentiable at a ?*

Note. Theorem 4.5 easily implies (cf. the proof of Theorem 4.6) that a positive solution of Problem 3 gives a positive solution of Problem 2. Also the case $X = \mathbb{R}^2, Y = \mathbb{R}$ is open. It is easy to see that the case $X = \mathbb{R}, Y$ arbitrary, has a positive answer.

PROBLEM 4. *Let X, Y be Banach spaces, $G \subset X$ be an open set and let $F: G \rightarrow Y$ be a continuous mapping. Let there exist a convex function f (which is not a priori continuous) such that $f + y^* \circ F$ is convex on G for each $y^* \in Y^*, \|y^*\| = 1$. Is then F delta-convex on G ?*

PROBLEM 5. Let X, Y be Banach spaces and $A \subset X$ be an open convex set. Let $F: A \rightarrow Y$ be locally delta-convex on A . Is then F delta-convex on A ?

Note. If $X = \mathbb{R}^n$, the answer is positive (Theorem 1.20). We do not know a solution also in the case $X = l_2, Y = \mathbb{R}$, but we conjecture that the answer is negative whenever X is infinite-dimensional.

PROBLEM 6. Let X, Y be Banach spaces, $A \subset X$ be an open convex set and $F: A \rightarrow Y$ be a mapping. Suppose that $F \circ f$ is delta-convex on $(0, 1)$ whenever $f: (0, 1) \rightarrow A$ is delta-convex. Is then F locally delta-convex on A ?

Note. Also the case $X = \mathbb{R}^2, Y = \mathbb{R}$ is open. Compare with Example 6.2.

PROBLEM 7. Let X, Y be Banach spaces, $A \subset X$ be an open convex set and $F: A \rightarrow Y$ be a Lipschitz mapping. Suppose that there are $a \geq 0, b \geq 0$ such that

$$\mathbf{K}_0^1 F \circ f \leq a \mathbf{K}_0^1 f + b \cdot \text{Lip}(f)$$

whenever $f: [0, 1] \rightarrow A$ be a Lipschitz mapping. Is then F delta-convex on A ?

Note. It is possible to prove that if F is delta-convex on A , then it satisfies the condition from Problem 7. For a motivation of this problem see [36]. Also the case $X = \mathbb{R}^2, Y = \mathbb{R}$ is open.

PROBLEM 8. Let G, H be open subsets of $\mathbb{R}^n, a \in G$ and let $F: G \rightarrow H$ be a bijection. Suppose that F is locally delta-convex on G and F^{-1} is locally delta-convex on H . Does the Clarke's generalized derivative $\partial F(a)$ contain surjective linear mappings only?

Note. Also the case $n = 2$ is open. For the case $n = 1$, the answer is positive.

PROBLEM 9. Let X, Y be separable Hilbert spaces, $G \subset X$ be an open set and let $F: X \rightarrow Y$ be a delta-convex mapping. Does there exist a point $a \in G$ such that the mapping

$$F_v(t) = F(a + tv)$$

has the second Peano derivative at 0 for each $v \in X, \|v\| = 1$?

Note. For $X = \mathbb{R}^n$ the answer is positive (Theorem 3.15). We do not know an answer even in the case $X = l_2, Y = \mathbb{R}$.

PROBLEM 10. Let X, Y be Banach spaces, $A \subset X$ be an open convex set and $\emptyset \neq M \subset A$. We shall say that $F: M \rightarrow Y$ is delta-convex on M (with respect to A) if there exists a continuous convex function f on A such that for each $y^* \in Y^*, \|y^*\| = 1$, there exists a continuous convex function g_{y^*} on A such that $y^* \circ F = g_{y^*} - f$ on M .

Is it possible to extend each delta-convex mapping on M to a delta-convex mapping on A ?

Note. It is a problem (slightly modified) from [34], p. 557, where it is shown that a positive answer would have an interesting application to an investigation of singularities of convex functions.

The answer is positive if $X = R$ or $Y = R^n$ [34] and open even in the case $X = R^2$, $Y = l_2$.

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