MATHEMATICAL CONTROL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 14
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

DYNAMIC FAMILY OF MULTICOMMODITY INVENTORY PROBLEMS

RYSZARDA REMPAŁA

Institute of Mathematics, Polish Academy of Sciences, Warszawa, Poland

1. Introduction

A generalization of the Arrow and Karlin [1] model is considered. It is assumed that n commodities are produced. These commodities are identical with respect to production cost and different with respect to holding cost. The demand functions are positive and initial inventories are zero.

For a fixed planning interval $[0, T]$ and given demand functions an optimization problem is investigated. Some qualitative properties of optimal solutions are given.

Next the endpoint T of the planning interval $[0, T]$ and the demand functions are treated as dynamical parameters. The family of multicommodity inventory problems is considered.

The main problem which arises here is the following: what information about dynamical parameters is needed in order to obtain a solution which is optimal on a certain time interval $[0, a)$ for all problems in the family? In this direction we establish the existence of a horizon in the sense of Modigliani and Hohn [5] and Blikle and Łoś [3].

The same problem for one commodity model was investigated in the above-mentioned papers [5], [3] and for one commodity model with backlogging in Z. Lieber's paper [4]. The problem for two commodities was treated in the author's paper [6].

In this paper the general n-commodities case is considered. In Section 2 the single optimization problem is described and all technical assumptions are introduced. The problem is formulated as a control problem with state space constraints. Control and state variables are, respectively, the rate of production and the level of inventory. In Theorem 1 some necessary optimality conditions obtained by the maximum principle are formulated. In Section 3 several properties of adjoint functions as well as of admissible solutions satisfying the maximum principle are given.

Some of these properties have a very clear interpretation. For instance, Proposition 1 tells that if the optimal level of the inventory of a commodity is zero then also the optimal levels of inventories with greater holding costs are zero. Section 4 is rather of an auxiliary character. Lemmas 3 and 4 are used in the proof of the main result, i.e., of Theorem 2, which is given in Section 6. Proposition 2 of Section 5 is also used in the proof of Theorem 2. Moreover, Proposition 2 has a very clear interpretation. Namely, it states that in every sufficiently large time interval there exists a point at which the levels of the optimal inventories of all commodities are equal to zero. The last section contains the definition of a horizon and the proof of Theorem 2.

2. Multicommodity inventory problem

The problem is formulated as an optimal control problem.

Let the vector of inventories $Y(t) = [Y_1(t), \ldots, Y_n(t)]$ be the state variable and let the rate of production $u(t) = [u_1(t), \ldots, u_n(t)]$ be the control variable. The state space constraints and the control space constraints are:

$$Y_i(t) \geqslant 0, \quad u_i(t) \geqslant 0, \quad i = 1, 2, \ldots, n. \tag{1}$$

Let $r(t) = [r_1(t), \ldots, r_n(t)]$, the rate of demand, be a continuous and strictly positive function. The differential equations governing the behaviour of the inventory are

$$\dot{Y}(t) = u(t) - r(t); \quad Y(0) = 0. \tag{2}$$

The cost functional is given by the formula

$$F(u; r, T) = \int_0^T \{c(a_1 u_1(t) + \ldots + a_n u_n(t)) + h_1 Y_1(t) + \ldots \\ \ldots + h_n Y_n(t)\} dt, \tag{3}$$

where $c(\cdot)$ is assumed to be increasing, strictly convex and twice continuously differentiable; a_i and h_i are some positive constants.

For fixed r and T the problem is to schedule the production plan $u(t)$ so as to minimize the cost functional F under the constraints (1) and (2).

The main problem of this paper is to give some property of the family of problems (1)–(3) indexed by T and r. For this purpose we first fix the function r and number T.

It is easy to see that without any loss of generality we may assume $a_1 = a_2 = \ldots = a_n = 1$ in (1)–(3). This will be done in the sequel.

In this paper we assume that $h_1 < h_2 < \ldots < h_n$ and $Y_i(0) = 0$.

Let us now suppose that the function u is in $L^2(0, T; R^n)$ and Y belongs to $H^1(0, T; R^n)$. By the maximum principle (cf. Bensoussan et al. [2]) the following theorem may easily be obtained.

THEOREM 1. *If* $u \in L^2(0, T; R^n)$ *is an optimal solution of problem* (1)–(3), *then there exist functions* λ_i, $i = 1, 2, \ldots, n$, *such that*

(i) λ_i *is non-decreasing and right-continuous,*

(ii) λ_i *is constant in any time interval on which*

$$Y_i(t) = \int_0^t \big(u_i(\tau) - r_i(\tau)\big)\, d\tau > 0,$$

(iii)
$$\max_{w_1, \ldots, w_n \geqslant 0} [v_1(t) w_1 + \ldots + v_n(t) w_n - c(w_1 + \ldots + w_n)] = v_1(t) u_1(t) + \ldots + v_n(t) u_n(t) - c\big(u_1(t) + \ldots + u_n(t)\big)$$

where $v_i(t) = h_i t - \lambda_i(t)$, $i = 1, 2, \ldots, n$.

In the following we will consider the set M of functions $u \in L^2$ (not necessarily optimal) satisfying conditions (1)–(2) and such that there exist functions λ_i (and so also v_i) satisfying (i)–(iii) of Theorem 1.

3. Properties of elements of M and related functions

Let $u \in M$, Y be given by (2) and let $v(t)$ satisfy (i)–(iii) of Theorem 1. It is easy to obtain the following

COROLLARIES. I. *At every point* $t \in (0, T)$ *the function* r_i *has both one-side limits and*

$$v_i(t-) \geqslant v_i(t+) = v_i(t).$$

II. *If* $v_i(t_0) < \max\big(v_1(t_0), \ldots, v_n(t_0)\big)$ *then* $u_i = 0$ *in* $(t_0, t_0, +\varepsilon)$ *for some* $\varepsilon > 0$.

III. $u_1 + \ldots + u_n = \partial\big(\max(v_1, \ldots, v_n)\big)$ *where*

$$\partial(z) = \begin{cases} 0 & \text{for} \quad z < c'(0), \\ (c')^{-1}(z) & \text{for} \quad z \geqslant c'(0). \end{cases}$$

In the sequel $\max(v_1, \ldots, v_n)$ will be denoted by $v_{\max}$.

For the proof of II and III let W denote the set of vectors $w = (w_1, \ldots, w_n)$ which maximize the left-hand side of (iii). Let $I = \{i\colon v_i(t_0) = v_{\max}(t_0)\}$. Statement II follows from the easy observation that, for every $\bar{w} \in W$, if $i \notin I$ then $\bar{w}_i = 0$. By this remark it is sufficient to consider the expression

$$\max_{w_i\geqslant 0, i\in I}\Big\{v_{\max}(t_0)\cdot\sum_{i\in I} w_i - c\Big(\sum_{i\in I} w_i\Big)\Big\}$$

instead of the left-hand side of (iii). From this III follows immediately.

LEMMA 1. *If* $j > i$ *then* $v_j \geqslant v_i$ *on* $[0, T)$.

For the proof let us suppose that there exists a $t_1 \in [0, T)$ with $v_j(t_1) < v_i(t_1)$. Then by I this inequality holds in $[t_1, t_1+\varepsilon)$ for some $\varepsilon > 0$. Thus, by II, $u_j = 0$ in $(t_1, t_1+\varepsilon)$ and

$$Y_j(t_1) = Y_j(t) + \int_{t_1}^{t} r_j(s)ds > 0$$

because $r_j > 0$ and $Y_j \geqslant 0$. Let $s = \sup\{t < t_1;\ Y_j(t) = 0\}$. Thus $Y_j(t) > 0$ on $(s, t_1]$ and, by (ii) and (i), $v_j(t) = h_j t + a_j$ in $[s, t_1]$, and for some constant a_j the hypothesis $v_j(t_1) < v_i(t_1)$ implies also that $v_j(t) < v_i(t)$ for $t \in [s, t_1]$. Indeed, if $v_j(t_0) \geqslant v_i(t_0)$ for some $t_0 \in [s, t_1)$, then

$$\begin{aligned} v_i(t_1) &= h_i t_1 - \lambda_i(t_1) + v_i(t_0) - v_i(t_0)\\ &= v_i(t_0) + h_i(t_1 - t_0) - \lambda_i(t_1) + \lambda_i(t_0) \leqslant v_i(t_0) + \\ &\qquad\qquad + h_i(t_1 - t_0) < v_j(t_0) + h_j(t_1 - t_0) = v_j(t_1). \end{aligned}$$

Theorefore, by II, $u_j(t) = 0$ on $[s, t_1]$ and

$$Y_j(t_1) = Y_j(s) - \int_{s}^{t_1} r_j(s)ds < 0$$

because $Y_j(s) = 0$ and $r_j > 0$. This contradicts (1) and so the lemma is proved.

COROLLARIES. *Let* $n \geqslant j > i \geqslant 1$.

IV. *If* $\lambda_j =$ *constant on* $[t_1, t_2)$ *then* $v_j > v_i$ *on* (t_1, t_2).

V. *If* $Y_i(t_0) = 0$ *for some* $t_0 \in [0, T)$ *then* $v_i(t_0) = v_{i+1}(t_0) = \ldots$ $\ldots = v_n(t_0) = v_{\max}(t_0) \geqslant c'(0)$.

VI. *If* $v_j(t_0) = v_i(t_0)$ *for* $t_0 \in [0, T)$ *then* $Y_j(t_0) = 0$, $v_i(t_0) = v_{i+1}(t_0) = \ldots$ $\ldots = v_n(t_0)$ *and so* $Y_{i+1}(t_0) = \ldots = Y_n(t_0) = 0$.

Proofs. IV: Note that, by Lemma 1, $v_j(t_1) \geqslant v_i(t_1)$ and moreover, for $t \in (t_1, t_2)$,

$$\begin{aligned} v_j(t) &= h_j(t - t_1) + v_j(t_1) > h_i(t - t_1) + v_i(t_1)\\ &= h_i(t - t_1) + h_i t_1 - \lambda_i(t_1) = h_i t - \lambda_i(t_1) \geqslant h_i t - \lambda_i(t) = v_i(t). \end{aligned}$$

V: Let us suppose the contrary: $v_i(t_0) < v_k(t_0) \leqslant v_{\max}(t_0)$ for some $k > i$. By II and I, $u_i = 0$ on $(t_0, t_0+\varepsilon)$ for an $\varepsilon > 0$. Hence, for $t \in (t_0, t_0+\varepsilon)$, $Y_i(t) = 0 - \int_{t_0}^{t} r_i(s)ds < 0$, which contradicts (1).

VI: If $Y_j(t_0) > 0$ then, by (ii) of Theorem 1 and Corollary IV, $v_j(t_0) > v_i(t_0)$. This proves the first statement of VI.

To prove the second one, let us suppose $v_j(t_0) = v_i(t_0) < v_k(t_0)$ for some $k > i$. Then, by I, II, $v_j < v_k$ in $(t_0, t_0+\varepsilon)$ for an $\varepsilon > 0$ and $u_j = 0$ in this interval. Thus, for $t \in (t_0, t_0+\varepsilon)$, we have $Y_j(t) = 0 - \int_{t_0}^{t} r_j(t) < 0$. This contradiction proves that $v_i(t_0) = v_{i+1}(t_0) = \ldots = v_n(t_0)$. Now the last equality $Y_{i+1}(t_0) = \ldots = Y_n(t_0) = 0$ follows form the first part of this corollary.

A very interesting property of $Y(t)$ follows at once from Corollaries V and VI.

PROPOSITION 1. *If* $Y_i(t_0) = 0$ *for* $t_0 \in [0, T)$ *then*

$$Y_{i+1}(t_0) = Y_{i+2}(t_0) = \ldots = Y_n(t_0) = 0.$$

This property means that if the inventory of the ith commodity is zero then the inventories with greater holding costs are also equal to zero.

The next lemma will be of a more analytical character.

LEMMA 2. *The functions* $v_i(t)$ *are continuous on* $[0, T)$.

Proof. In the proof we will consider three cases.

(a) Let $t_0 \in (0, T)$ and $Y_i(t_0) > 0$. By (ii) of Theorem 1, v_i is linear in a neighbourhood of t_0 and so it is continuous.

(b) Let $Y_i(t_0) = 0$ for $i = 1, 2, \ldots, n$ and some $t_0 \in (0, T)$. By III, $\partial\big(v_n(t)\big) = u_1(t) + \ldots + u_n(t)$ and thus

$$\sum_{i=1}^{n} Y_i(t) = \int_0^t \Big\{\partial\big(v_n(s)\big) - \sum_{i=1}^{n} r_i(s)\Big\}\, ds.$$

The sum attains zero at t_0, and so $v_n(t_0) \geqslant c'(0)$ and moreover

$$r_1(t_0) + \ldots + r_n(t_0) \leqslant \partial\big(v_n(t_0)\big) \leqslant \partial\big(v_n(t_0-)\big) \leqslant r_1(t_0) + \ldots + r_n(t_0).$$

Therefore $v_n(t_0) = v_n(t_0-)$. Hence, by Corollary V, $v_i(t_0) = v_n(t_0) = v_n(t_0-) \geqslant v_i(t_0-)$ and so, by I, $v_i(t_0) = v_i(t_0-)$ for all i.

(c) Let $Y_i(t_0) = 0$ and $Y_{i-1}(t_0) > 0$ for some $i \in \{2, 3, \ldots, n\}$ and $t_0 \in (0, T)$. By Proposition 1, $Y_j(t_0) = 0$ for $j > i$ and $Y_j(t_0) > 0$ for $j < i$.

Then, by Lemma 1 and Corollaries V, IV, $v_n(t_0) = \ldots = v_i(t_0) \geqslant c'(0)$ and $v_i(t) \geqslant v_{i-1}(t) > v_{i-j}(t)$ in $(t_0-\varepsilon, t_0+\varepsilon)$ for $j = 2, \ldots, i-1$ and some

$\varepsilon > 0$. Hence $u_1 = \ldots = u_{i-2} = 0$ in $(t_0-\varepsilon, t_0+\varepsilon)$. From this and III it follows that

$$(*)\qquad \partial\big(v_n(t)\big) = u_n + \ldots + u_{i-1} \quad \text{in} \quad (t_0-\varepsilon, t_0+\varepsilon).$$

Let us now suppose that t_0 is a discontinuity point of $v_n(t)$. This means that

$$v_n(t_0-) > v_n(t_0+) = v_n(t_0) = v_i(t_0) \geqslant v_{i-1}(t_0).$$

Since $v_{i-1}(t)$ is linear in a neighbourhood of t_0, $v_n > v_{i-1}$ in $(t_0-\eta, t_0)$ for some η, and so $u_{i-1} = 0$ in this interval. The function $Y_n + Y_{n-1} + \ldots + Y_i$ attains zero at t_0, and so by $(*)$ and the above remarks

$$(**)\qquad \partial\big(v_n(t_0-)\big) = (u_n + \ldots + u_i)\,(t_0-) \leqslant r_n(t_0) + \ldots + r_i(t_0).$$

On the other hand, the inequality $\partial\big(v_n(t_0)\big) < \sum_i^n r_j(t_0)$ cannot hold because it implies the same inequality in $(t_0, t_0+\varepsilon_1)$ for an $\varepsilon_1 > 0$, and so we would have

$$u_n + \ldots + u_i \leqslant u_n + \ldots + u_i + u_{i-1} = \partial(v_n) < r_n + \ldots + r_i$$

in $(t_0, t_0+\varepsilon_2)$; $\varepsilon_2 = \min(\varepsilon, \varepsilon_1)$. This is impossible because $(Y_n + \ldots + Y_i)\,(t_0) = 0$. From this and $(**)$ it follows that $v_n(t_0) = v_n(t_0-)$ and so as in (b), $v_j(t_0-) = v_j(t_0)$ for $j = n, n-1, \ldots, i$. Moreover, the functions $v_{i-1}, \ldots, v_1$ are continuous at t_0 because they are linear in a neighbourhood of t_0.

4. Relations between the elements of M

Let $u, \tilde{u} \in M$ and $Y, \tilde{Y}, v, \tilde{v}$ be corresponding inventories and adjoint functions.

LEMMA 3. *Let $i = 2, \ldots, n$ and $t_0 \in [0, T)$. If $v_{i-1}(t_0) = v_i(t_0)$ and $v_{i-1}(t_0) \leqslant \tilde{v}_{i-1}(t_0)$ and $\tilde{Y}_i(t) = \tilde{Y}_{i+1}(t) = \ldots = \tilde{Y}_n(t) = 0$ for some $t > t_0$, then $\tilde{v}_{i-1}(t_0) = \tilde{v}_i(t_0)$.*

Remark. Note that by Proposition 1 for $t \in (0, T)$ one may put $\tilde{Y}_i(t) = 0$ instead of $\tilde{Y}_i(t) = \tilde{Y}_{i+1}(t) = \ldots = \tilde{Y}_n(t) = 0$.

Proof of Lemma 3. The proof will be by induction on i.

Let $i = n$. Hypothesis $v_{n-1}(t_0) = v_n(t_0)$ gives, by VI, $Y_n(t_0) = 0$, and so

$$(*)\qquad r_n(t_0) \leqslant \partial\big(v_n(t_0)\big) = \partial\big(v_{n-1}(t_0)\big).$$

In fact, if $r_n(t_0) > \partial\big(v_n(t_0)\big)$ then, for an $\varepsilon > 0$,

$$Y_n(t_0+\varepsilon) = 0 + \int\limits_{t_0}^{t_0+\varepsilon} \big(u_n(t) - r_n(t)\big)\,dt < 0$$

since $\partial(v_n) = u_1 + \ldots + u_n \geqslant u_n$ and $r_n > 0$. For the proof of the lemma let us assume for a moment that

$$(**) \qquad \tilde{v}_n(t_0) > \tilde{v}_{n-1}(t_0).$$

Hence, by hypothesis and $(*)$, $\tilde{v}_n(t_0) > \tilde{v}_{n-1}(t_0) \geqslant v_{n-1}(t_0) = v_n(t_0) \geqslant c'\big(r_n(t_1)\big)$. This gives $\tilde{u}_1 = \ldots = \tilde{u}_{n-1} = 0$ in $(t_0-\varepsilon, t_0+\varepsilon)$ for an $\varepsilon > 0$, and so $\tilde{Y}_n(t_0) > 0$ since $\tilde{Y}_n(t_0) = 0$ would imply $\partial\big(\tilde{v}_n(t_0)\big) = r_n(t_0)$.

Let $t_0' = \inf\{t > t_0;\ \tilde{Y}_n(t) = 0\}$. By the hypothesis of the lemma t_0' is well defined. On the interval (t_0, t_0'), $\tilde{v}_n(t) = h_n t + \tilde{a}_n$ for some $\tilde{a}_n$, and so $\tilde{v}_n > v_n$ in this interval. Indeed, from $\tilde{v}_n(t_0) > v_n(t_0)$ we get $h_n t_0 + \tilde{a}_n > h_n t_0 - \lambda_n(t_0)$, and thus $\tilde{a}_n > -\lambda_n(t_0) \geqslant -\lambda_n(t)$ for $t \in (t_0, t_0')$.

Moreover, by IV, $\tilde{v}_n(t) > \tilde{v}_{n-1}(t)$ for $t \in (t_0, t_0')$. Therefore in the interval (t_0, t_0')

$$\tilde{u}_n = \partial(\tilde{v}_n) > \partial(v_n) = u_1 + \ldots + u_n \geqslant u_n;$$

hence

$$0 = \tilde{Y}_n(t_0') = \tilde{Y}_n(t_0) + \int\limits_{t_0}^{t_0'} \big(\tilde{u}_n(s) - r_n(s)\big)\,ds$$

$$> Y_n(t_0) + \int\limits_{t_0}^{t_0'} \big(u_n(s) - r_n(s)\big)\,ds \geqslant 0.$$

This contradiction proves the lemma for $i = n$.

Assume now that the lemma is true for $i = n, n-1, \ldots, j+1$. We will prove it for $i = j$. The hypothesis $v_{j-1}(t_0) = v_j(t_0)$ gives $v_{j-1}(t_0) = \ldots = v_n(t_0)$ and $Y_j(t_0) = \ldots = Y_n(t_0) = 0$. This implies

$$({*}_{*}{*}) \qquad r_j(t_0) + \ldots + r_n(t_0) \leqslant \partial\big(v_{j-1}(t_0)\big) = \partial\big(v_j(t_0)\big).$$

Indeed, if $r_j(t_0) + \ldots + r_n(t_0) > \partial\big(v_{j-1}(t_0)\big)$ then $r_j(t) + \ldots + r_n(t) > \partial\big(v_n(t)\big) = u_1(t) + \ldots + u_n(t) \geqslant u_j(t) + \ldots + u_n(t)$ in a neighbourhood of t_0, which is impossible since $Y_i(t_0) = \ldots = Y_n(t_0) = 0$ and $r_j(t_0) > 0$.

As in the case $i = n$, the proof will be carried out by contradiction. Let us suppose

$$\tilde{v}_{j-1}(t_0) < \tilde{v}_j(t_0).$$

Then

$$\tilde{v}_j(t_0) > \tilde{v}_{j-1}(t_0) \geqslant v_{j-1}(t_0) = v_j(t_0) \geqslant c'\big(r_j(t_0) + \ldots + r_n(t_0)\big),$$

and so $\tilde{u}_1 = \tilde{u}_2 = \ldots = \tilde{u}_{j-1} = 0$ in $(t_0 - \varepsilon, t_0 + \varepsilon)$ for an $\varepsilon > 0$. Moreover,

$$(*4) \qquad \tilde{Y}_j(t_0) > Y_j(t_0) = 0.$$

Indeed, if $\tilde{Y}_j(t_0) = 0$ then $\partial(\tilde{v}_j(t_0)) = \partial(\tilde{v}_n(t_0)) = r_j(t_0) + \ldots + r_n(t_0)$, because $\partial(\tilde{v}_n) = \tilde{u}_j + \ldots + \tilde{u}_n$ in $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Let $t_0' = \inf\{t > t_0;\ \tilde{Y}_j(t) = 0\}$. Then

$$(*5) \qquad \tilde{v}_j = h_j t + \tilde{a}_j \text{ for an } \tilde{a}_j \text{ and } \tilde{v}_j > \tilde{v}_{j-1} \text{ and } \tilde{v}_j > v_j \text{ on } (t_0, t_0') \text{ as in the case } i = n.$$

Thus by the induction hypothesis on the interval $[t_0, t_0')$ if $v_j = v_{j+1}$ then $\tilde{v}_j = \tilde{v}_{j+1}$, because we have $\tilde{v}_j(t) > v_j(t)$ for $t \in [t_0, t_0')$ and $\tilde{Y}_{j+1}(t_0') = \ldots = \tilde{Y}_n(t_0') = 0$.

In particular, for $t = t_0$,

$$(*6) \qquad v_{j-1}(t_0) = v_j(t_0), \text{ so } v_j(t_0) = v_{j+1}(t_0) = \ldots = v_n(t_0) \text{ and thus } \tilde{v}_j(t_0) = \tilde{v}_{j+1}(t_0) = \ldots = \tilde{v}_n(t_0) \text{ and } \tilde{Y}_{j+k}(t_0) = Y_{j+k}(t_0) = 0 \text{ for } k = 1, 2, \ldots, n-j.$$

Let $\bar{t} = \sup_t \{t_0 \leqslant t \leqslant t_0';\ v_j(t) = v_{j+1}(t).\}$ Then

$$(*7) \qquad v_j(\bar{t}) = v_{j+1}(\bar{t}) = \ldots = v_n(\bar{t}), \quad \tilde{v}_j(t) = \tilde{v}_{j+1}(t) = \ldots = \tilde{v}_n(t) \text{ and } Y_{j+1}(\bar{t}) = \ldots = Y_n(\bar{t}) = \tilde{Y}_{j+1}(\bar{t}) = \ldots = \tilde{Y}_n(\bar{t}) = 0.$$

Let $X = \{t \in (t_0, t_0');\ v_j(t) < v_{j+1}(t)\}$. The set X may be written as $\bigcup (a_i, b_i) \cup (\bar{t}, t_0')$ where $(a_i, b_i) \subset (t_0, \bar{t})$ with $v_j(a_i) = v_{j+1}(a_i)$, $v_j(b_i) = v_{j+1}(b_i)$ and $v_j(t) < v_{j+1}(t)$ for $t \in (a_i, b_i)$. By the induction hypothesis

$$(*8) \qquad \int_{a_i}^{b_i} \big(u_{j+1}(s) + \ldots + u_n(s)\big) ds = \int_{a_i}^{b_i} \big(\tilde{u}_{j+1}(s) + \ldots + \tilde{u}_n(s)\big) ds = \int_{a_i}^{b_i} \big(r_{j+1}(s) + \ldots + r_n(s)\big) ds.$$

From Corollary II

$$(*9) \qquad 0 = u_j \leqslant \tilde{u}_j \quad \text{on} \quad \bigcup_i (a_i, b_i) \cup (\bar{t}, t_0).$$

On the set $[t_0, t_0') \setminus X$ we have $v_j(t) = v_{j+1}(t) = \ldots = v_n(t)$ so $\tilde{v}_j(t) = \tilde{v}_{j+1}(t) = \ldots = \tilde{v}_n(t)$ and by $(*5)$

$$(*10) \qquad \tilde{v}_n(t) > v_n(t) \quad \text{for} \quad t \in [t_0, t_0') \setminus X.$$

By (*5) we have, moreover, $\tilde{u}_1 + \ldots + \tilde{u}_{j-1} = 0$ on (t_0, t_0'). Thus, using (*6)–(*10), we obtain

$$\int_{t_0}^{\bar{t}} \big(\tilde{u}_j(s) - u_j(s)\big)ds = \int_{t_0}^{\bar{t}} \big(\tilde{u}_j(s) + \ldots + \tilde{u}_n(s) - u_j(s) - \ldots - u_n(s)\big)ds$$

$$= \int_{\bigcup_i (a_i, b_i)} \big(\tilde{u}_j(s) - u_j(s)\big)ds + \int_{[(t_0, \bar{t})\setminus \bigcup_i (a_i, b_i)]} \Big(\partial\big(\tilde{v}_n(s)\big) - \partial(v_n)(s)\big) +$$

$$+ u_{j-1}(s) + \ldots + u_1(s)\Big)ds \geqslant 0.$$

Finally, from this and (*4), (*9) it follows that

$$0 = \tilde{Y}_j(t_0') = \tilde{Y}_j(t_0) + \int_{t_0}^{\bar{t}} \big(\tilde{u}_j(s) - r_j(s)\big)ds + \int_{\bar{t}}^{t_0'} \big(\tilde{u}_j(s) - r_j(s)\big)ds$$

$$> Y_j(t_0) + \int_{t_0}^{\bar{t}} \big(u_j(s) - r_j(s)\big)ds + \int_{\bar{t}}^{t_0'} \big(u_j(s) - r_j(s)\big)ds \geqslant 0.$$

This contradiction proves the lemma.

LEMMA 4. *If $\tilde{v}_1(t_0) > v_1(t_0)$ for some $t_0 \in [0, T)$, then $\tilde{v}_1(t) > v_1(t)$ on $[t_0, T)$, $\tilde{v}_1(T-) > v_1(T-)$ and $\tilde{Y}_1(t) + \ldots + \tilde{Y}_n(t) > 0$ for $t \in (t_0, T]$.*

Proof. Let us suppose that there exists a $t' > t_0$ such that $t' \in (t_0, T]$ and $\tilde{v}_1(t') = v_1(t')$ if $t' \in (t_0, T)$ or $\tilde{v}_1(t'-) = v_1(t'-)$ if $t' = T$. Let

$$t_0' = \sup\{t \leqslant t_0;\ \tilde{v}_1(t) = v_1(t)\},$$
$$t_0'' = \inf\{t \geqslant t_0;\ \tilde{v}_1(t-) = v_1(t-)\}.$$

If the first set is empty then we put $t_0' = 0$.

Note that

(i) $\tilde{v}_1(t) > v_1(t)$ on (t_0', t_0'');

(ii) $Y_1(t_0') = Y_2(t_0') = \ldots = Y_n(t_0') = 0$.

If $t_0' = 0$ the last equality follows from the assumption. If $t_0' > 0$ then $\tilde{v}_1(t_0') = \tilde{v}_1(t_0')$ implies $Y_1(t_0') = 0$. Indeed, if $Y_1(t_0') > 0$ then

$$\lambda_1(t) = \lambda_1(t_0') = \tilde{\lambda}_1(t_0') = -\tilde{v}_1(t_0') + h_1 t_0' \leqslant \tilde{\lambda}_1(t) \quad \text{in} \quad (t_0', t_0' + \eta)$$

for some $\eta > 0$, and thus $\tilde{v}_1(t) \leqslant v_1(t)$ for $t \in (t_0', t_0' + \eta)$, which contradicts (i).

It may be noted, moreover, that

(iii) $\tilde{Y}_1(t_0'') = \tilde{Y}_2(t_0'') = \ldots = \tilde{Y}_n(t_0'') = 0$.

To prove this let us observe that $\tilde{Y}_1(t) > 0$ cannot hold in any left-hand neighbourhood of t_0'', because if $\tilde{Y}_1(t) > 0$ in $(t_0'' - \eta, t_0'')$ for some

$\eta > 0$ then $\tilde{v}_1(t_0'' - \eta) > v_1(t_0'' - \eta)$ implies

$$\tilde{\lambda}_1(t_0'' -) = \tilde{\lambda}_1(t_0'' - \eta) < \lambda_1(t_0'' - \eta) \leqslant \lambda_1(t_0'' -),$$

which contradicts the definition of t_0''. Therefore there exists a sequence $t_n \to t_0''$, $t_n < t_0''$ such that $\tilde{Y}_1(t_n) = 0$ and, by Proposition 1, $\tilde{Y}_j(t_n) = 0$ for $j = 1, 2, \ldots, n$. This gives equality (iii).

(iv) If $v_1 = v_2$ on the interval $[t_0', t_0'')$ then $\tilde{v}_1 = \tilde{v}_2$ on this interval. This follows from (i), (iii) and Lemma 3.

(v) $0 = u_1 \leqslant \tilde{u}_1$ on $\{t;\ v_1 < v_2\} \subset (t_0', t_0'')$.

(vi) The set $\{t \in (t_0', t_0'');\ v_1 = v_2\}$ has positive measure, because $Y_1(t_0') = 0$ and $r > 0$.

(vii) $v_1(t_0') = v_2(t_0') = \ldots = v_n(t_0')$ because $Y_1(t_0') = 0$. Thus, by (iv), $\tilde{v}_1(t_0') = \tilde{v}_2(t_0') = \ldots = \tilde{v}_n(t_0')$ and $\tilde{Y}_2(t_0') = \ldots = \tilde{Y}_n(t_0') = 0$.

(viii) Let $\bar{t} = \sup\{t;\ t \in [t_0', t_0'');\ v_1(t) = v_2(t_0)\}$. Thus $Y_2(\bar{t}) = \ldots$ $\ldots = Y_n(\bar{t}) = 0$. By (iv) also $\tilde{v}_1(\bar{t}) = \ldots = \tilde{v}_n(\bar{t})$; hence $\tilde{Y}_2(\bar{t}) = \ldots = \tilde{Y}_n(\bar{t}) = 0$.

Therefore, as in the proof of Lemma 3, we obtain:

$$\text{(ix)} \quad \int_{t_0'}^{t_0''} \big(\tilde{u}_1(s) - u_1(s)\big)\,ds = \int_{t_0'}^{\bar{t}} \{\tilde{u}_1(s) + \ldots + \tilde{u}_n(s) - u_1(s) - \ldots$$

$$\ldots - u_n(s)\}\,ds + \int_{\bar{t}}^{t_0''} \big(\tilde{u}_1(s) - u_1(s)\big)\,ds = \int_{\{t \in (t_0', \bar{t});\ v_2 > v_1\}} \big(\tilde{u}_1(s) - u_1(s)\big)\,ds +$$

$$+ \int_{\{t;\ v_1 = v_2\}} \Big(\partial\big(\tilde{v}_1(s)\big) - \partial\big(v_1(s)\big)\Big)\,ds + \int_{\bar{t}}^{t_0''} \big(\tilde{u}_1(s) - u_1(s)\big)\,ds > 0.$$

The last inequality results from the following facts:

(a) $Y_i(t_0') = \tilde{Y}_i(t_0')$, $Y_i(\bar{t}) = \tilde{Y}_i(\bar{t}) = 0$ for $i = 2, 3, \ldots, n$.

(b) $\{t \in (t_0', \bar{t});\ v_2(t) > v_1(t)\} = \bigcup_i (a_i, b_i)$, with $v_1(a_i) = v_2(a_i)$; $v_1(b_i) = v_2(b_i)$ and $v_2(t) > v_1(t)$ for $t \in (a_i, b_i)$. Therefore, by Lemma 3,

$$\int_{a_i}^{b_i} \big(u_2(s) + \ldots + u_n(s)\big)\,ds = \int_{a_i}^{b_i} \big(\tilde{u}_2(s) + \ldots + \tilde{u}_n(s)\big)\,ds = \int_{a_i}^{b_i} \big(r_2(s) + \ldots$$
$$\ldots + r_n(s)\big)\,ds.$$

(c) By (vi) the set $\{t \in (t_0', \bar{t});\ v_1 = v_2\}$ has positive measure. On this set $\tilde{v}_1(t) > v_1(t) = v_2(t) \geqslant \sigma'(0)$; thus $\partial\big(\tilde{v}_1(t)\big) > \partial\big(v_1(t)\big)$.

(d) On every (a_i, b_i) and on $(\bar{t}, t_0'')$ we have $0 = u_1 \leqslant \tilde{u}_1$.

From (ix) we get

$$\text{(x)}\qquad 0 = \tilde{Y}_1(t_0'') = \tilde{Y}_1(t_0') + \int_{t_0'}^{t_0''} \big(\tilde{u}_1(s) - r_1(s)\big)\,ds$$

$$> Y_1(t_0') + \int_{t_0'}^{t_0''} \big(u_1(s) - r_1(s)\big)\,ds = Y_1(t_0'') \geqslant 0.$$

This contradiction proves the first part of the lemma.

For the second part it is sufficient to observe that putting $\tilde{Y}_i(t_0''') = 0$ for some $t_0''' \in (t_0', T]$ and $i = 1, 2, \ldots, n$ one may obtain the contradiction $\tilde{Y}_1(t_0''') > Y_1(t_0''')$. The proof may be carried out as in (i)–(x) by putting t_0''' instead of t_0''. (In this case the observation (iii) follows at once from the assumption $\tilde{Y}_i(t_0''') = 0$.)

5. Optimal solution of (1)–(3)

So far we have considered the set M of functions $u(\cdot)$ which satisfy (1) and (2) and for which there exist functions $\lambda(t)$ and $v(t)$ satisfying conditions (i)–(iii) of Theorem 1. (In Z. Lieber's paper [4] such functions are called extrapolation.)

It is clear of course that an optimal solution of (1)–(3) belongs to M. Moreover, it is not difficult to see that an optimal solution $u(t)$ of (1)–(3) has to satisfy the terminal conditions

$$\text{(4)}\qquad Y_i(T) = \int_0^T \big(u_i(s) - r_i(s)\big)\,ds = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

PROPOSITION 2. *Let nonnegative constants b, B be such that $b \leqslant r_1(t) + \ldots + r_n(t) \leqslant B$ for $t \in [0, T]$. Let u be an optimal solution of* (1)–(3) *and let Y be the corresponding optimal inventory. Then, in any interval $[t_1, t_2] \subset [0, T]$ with $t_2 - t_1 \geqslant \big(c'(B) - c'(b)\big)/h_1$, there exists a point t_0 such that*

$$Y_1(t_0) = Y_2(t_0) = \ldots = Y_n(t_0) = 0.$$

Proof. The proof will be given by contradiction. Let us suppose that $|t_1 - t_2| \geqslant c'(B) - c'(b)$ and $Y_1(t) > 0$ for $t \in [t_1, t_2]$. Let

$$t_1' = \sup\{t < t_1;\ Y_1(t) = 0\},$$

$$t_2' = \inf\{t > t_1;\ Y_1(t) = 0\}.$$

By conditions (2) and (4), ($Y_1(0) = 0$, $Y_1(T) = 0$) the points t_1', t_2' are well defined and, by Proposition 1,

$$0 = Y_1(t_1') = \ldots = Y_n(t_1') = Y_1(t_2') = \ldots = Y_n(t_2').$$

In the interval (t_1', t_2'), $v_1(t) = h_1 t + a_1$ for some a_1. Since $Y_1(t_1') = 0$, $v_1(t_1') = \ldots = v_n(t_2') \geqslant c'(0)$ and thus $v_1(t) > c'(0)$ on (t_1', t_2').

The function $Y_1 + \ldots + Y_n$ attains zero at the points t_2' and t_1'; so we have

$$\partial\big(v_1(t_1')\big) = \partial\big((v_n(t_1')\big) \geqslant r_1(t_1') + \ldots + r_n(t_1') \geqslant b$$

and

$$\partial\big(v_1(t_2'-)\big) \leqslant \partial\big(v_n(t_2'-)\big) \leqslant r_1(t_2') + \ldots + r_n(t_2) \leqslant B.$$

Therefore

$$\begin{aligned} c'(B) - c'(b) &\geqslant c'\big(r_1(t_2') + \ldots + r_n(t_2')\big) - c'\big(r_1(t_1') + \ldots + r_n(t_1')\big) \\ &\geqslant v_1(t_2'-) - v_1(t_1') = h_1(t_2' - t_1') > h_1(t_2 - t_1), \end{aligned}$$

which contradicts the hypothesis and proves the proposition.

6. Horizon in dynamic family

So far we have dealt with one fixed problem (1)–(3).

Now let us assume that a family of problems (1)–(3) is given. It is known that demand r is a continuous positive vector function defined in $[0, +\infty)$ such that $0 \leqslant b \leqslant r_1(t) + \ldots + r_n(t) \leqslant B$ for some known constants b, B (which are independent of t and r). The class of such demand functions will be denoted by R. A family F of problems (1)–(3) indexed by positive numbers T and functions $r \in R$ will be called a *dynamic family with dynamic parameters T and r*. Let $u_{T,r}$ be the optimal solution of (1)–(3) for parameters T and r.

In the following an important property of optimal solutions of problems from F will be given. For this purpose the following definition of a horizon due to Blikle and Łoś [3] will be adopted.

DEFINITION. The number $H \geqslant 0$ is called a *horizon* for the dynamic family F if, for all parameters T, T^*, for all parameters $r, r^* \in R$ such that $H < T < T^*$ and $r = r^*$ on $[0, T)$, and for all $u_{T,r}$, there exists a u_{T^*,r^*} such that

$$u_{T,r}(t) = u_{T^*,r^*}(t) \quad \text{for} \quad t \in [0, T-H).$$

Remarks. (a) By definition, if H is a horizon then any number $H_1 \geqslant H$ is also a horizon.

(b) Note that if a horizon H is known then for T^* sufficiently large an optimal solution u_{T^*,r^*} on a subinterval $[0, T-H)$ may be obtained independently of the shape of the function r^* on the interval $[T, T^*)$. It is sufficient to know the demand only on the subinterval $[0, T)$.

(c) The existence of a horizon for one commodity problem follows at once from Proposition 2 and the Optimality Principle. This fact was proved in a different way in [3]. We will now prove a similar theorem in our more complicated multicommodity problem.

THEOREM 2. *Any number* $H \geqslant (c'(B)-c'(b))/h_1$ *is a horizon for the dynamic family* F.

Proof. Let us consider H, T, T^* such that $\frac{1}{h_1}(c'(B)-c'(b)) \leqslant H < T < T^*$ and r, r^* such that $r = r^*$ on the interval $[0, T)$. For simplicity, put $u^* = u_{T^*,r^*}$ and $u = u_{T,r}$. Similarly, for the corresponding inventories, put $Y^* = Y_{T^*,r^*}$ and $Y = Y_{T,r}$. Since Y^* and Y are optimal inventories, by (4), $Y(T) = Y^*(T^*) = 0$. Moreover, by Proposition 2, there exist $t_1, t_2 \in [T-H, T]$ such that $Y^*(t_1) = Y(t_2) = 0$.

(a) If $t_1 = t_2$ then, by the Optimality Principle, the function

$$u^{**}(t) = \begin{cases} u(t) & \text{on} \quad [0, t_1), \\ u^*(t) & \text{on} \quad [t_1, T^*) \end{cases}$$

is the optimal solution (1)–(3) for parameters T^* and r^*. This proves the theorem in this case.

(b) Let us note that $Y(T) = 0$. So if $Y_1^*(T) = 0$ then, by Prop. 1, $Y^*(T) = 0$, and putting $t_1 = t_2 = T$ we have case (a).

(c) Therefore let us suppose $Y_1^*(T) > 0$ and let us take $t_2 = T$ and $t_1 \in [T-H, T)$. It is clear that $u, u^* \in M(T, r)$. Let v, v^* be the corresponding adjoint variables defined by the conditions of Theorem 1 for u and u^*. By Lemma 4 $v_1(t) > v_1^*(t)$ cannot hold for any $t \in [0, T)$ because $Y(T) = 0$. Thus $v_1(t) \leqslant v_1^*(t)$ on $[0, T)$. Moreover, from the equality $Y^*(t_1) = 0$ and Lemma 4 we obtain $v_1(t) = v_1^*(t)$ on $[0, t_1]$. Because $Y^*(t_1) = Y(T) = 0$, and $v_1 = v_1^*$ on $[0, t_1]$, by Lemma 3 we get

$$\{t \in [0, t_1];\ v_1(t) = v_n(t)\} = \{t \in [0, t_1];\ v_1^*(t) = v_n^*(t)\} \stackrel{\text{def}}{=} X.$$

Note that $t = 0$ and $t = t_1$ belong to X. For $t = 0$ it follows at once from the assumption that $Y(0) = Y^*(0) = 0$. Let us consider the case $t = t_1$.

By $Y^*(t_1) = 0$ we have $v_1^*(t_1) = v_2^*(t_1) = \ldots = v_n^*(t_1)$. Thus $v_1^*(t_1) = v_1(t_1)$ and $Y(T) = 0$ implies, by Lemma 3, $v_1(t_1) = v_2(t_1) = \ldots = v_n(t_1) = v_1^*(t_1) = \ldots = v_n^*(t_1)$, and so $t_1 \in X$. Hence $Y_1^*(t_1) = Y_2^*(t_1) = \ldots = Y_n^*(t_1) = Y_2(t_1) = \ldots = Y_n(t_1) = 0$.

The set $[0, t_1] \setminus X$ may be written as $\bigcup_i (a_i, b_i)$ with

$$v_1(a_i) = v_2(a_i) = v_1^*(a_i) = v_2^*(a_i), \qquad v_1(b_i) = v_2(b_i) = v_1^*(b_i) = v_2^*(b_i);$$

$$v_2 > v_1, \quad v_2^* > v_1^* \quad \text{on} \quad (a_i, b_i).$$

Thus

$$\int_{a_i}^{b_i} \big(u_2(s)+\ldots+u_n(s)\big)ds = \int_{a_i}^{b_i} \big(u_2^*(s)+\ldots+u_n^*(s)\big)ds,$$

$$\int_0^{t_1} \big(u_2(s)+\ldots+u_n(s)\big)ds = \int_0^{t_1} \big(u_2^*(s)+\ldots+u_n^*(s)\big)ds.$$

Hence

$$\int_X \big(u_2(s)+\ldots+u_n(s)\big)ds = \int_X u_2^*(s)+\ldots+u_n^*(s)\,ds.$$

On every interval (a_i, b_i) we have $u_1 = u_1^* = 0$ because $v_2 > v_1$ and $v_2^* > v_1^*$. Therefore

$$\int_0^{t_1} u_1^*(s)\,ds = \int_X \hat{c}\big(v_n^*(s)\big) - u_2^*(s) - \ldots - u_n^*(s)$$

$$= \int_X \Big(\hat{c}\big(v_n(s)\big) - u_2(s) - \ldots - u_n(s)\Big)ds = \int_0^{t_1} u_1(s)\,ds.$$

By this we conclude that $Y_1^*(t_1) = 0$ implies $Y_1(t_1) = 0$. Thus, by (a), we have proved Theorem 2.

References

[1] K. J. Arrow and S. Karlin, *Production over time with increasing marginal costs*, Studies in the Mathematical Theory of Inventory and Production, Stanford University Press, 1958.

[2] A. Bensoussan, E. G. Hurst, Jr. and B. Naslund, *Management Applications of Modern Control Theory*, North-Holland, 1974.

[3] A. Blikle and J. Łoś, *Horizon in dynamic programs with continuous time*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 15 (1967), 513–519.

[4] Z. Lieber, *An extension to Modigliani and Hohn's planning horizon results*, Management Sci. 20 (1973), 319–330.

[5] F. Modigliani and F. Hohn, *Production planning over time and nature of the expectations and planning horizon*, Econometrica 23 (1955), 46–66.

[6] R. Rempała, *A dynamic programming problem for two-commodity inventory model*, in: New Trends in Dynamic System Theory and Economics, Academic Press, New York 1979, 269–280.