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KAZIMIERZ GŁAZEK

Independence with respect to family of mappings in abstract algebras

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CONTENTS

Dedicated to
Profesor Edward Marczewski
on the 40th anniversary
of the publication of his first paper

Introduction

In 1958 E. Marczewski introduced a general notion of independence (called also "algebraic independence"; see [11], [13] and [16]), which contained as special cases majority of independence notions used in various branches of mathematics. In particular; it included linear independence (of vectors, points and numbers), algebraic independence of numbers, independence of polynomials and, more generally, of continuous functions, set theoretical independence, logical independence, etc.

However, there are independence notions which are not covered by this scheme, although they have much in common with it, such as linear independence in abelian groups. Some weaker notions than algebraic independence (this notion we shall call $\boldsymbol{M}$-independence) were developed. J. Schmidt introduced in [20] the "independence-in-itself" (which we shall call here $\boldsymbol{S}$-independence), S. Świerczkowski dealt (in [21]; for the sake of only one particular theorem) with "weak independence" (it appears in our paper as $\boldsymbol{S}_0$-independence). Further G. Graetzer used such a "weak independence" (in [8]) to include the linear independence in abelian groups (for subsets which do not contain the zero element). This notion will be called here $\boldsymbol{G}$-independence.

As a common way of defining all this notions E. Marczewski proposed in [17] a notion of independence with respect to a family $\boldsymbol{Q}$ of mappings, and called if $\boldsymbol{Q}$-independence. We shall introduce in this paper two kinds of $\boldsymbol{Q}$-independence: $\boldsymbol{A}_1$-independence and $\boldsymbol{R}$-independence.

The general properties of the $\boldsymbol{Q}$-independence were investigated in [17] — we complete (in § 2) this list with a number of simple remarks. In § 3 we give a certain (quite natural) necessary and sufficient condition for a family $\boldsymbol{J}$ of subsets of a fixed algebra in order that there exists a family $\boldsymbol{Q}$, such that the family of $\boldsymbol{Q}$-independent subsets overlaps with $\boldsymbol{J}$. In an algebra $\mathfrak{A}$ every family of mappings may be extended in a unique way to a maximal family giving the same independence. The set of all such maximal families for a given algebra forms a Boolean algebra anti-isomorphic to the algebra of $\boldsymbol{Q}$-independence subsets (Theorem 1).

It is of great importence in investigating the $\boldsymbol{M}$-independence, that the family of $\boldsymbol{M}$-independence subsets is hereditary and of finite character. Also the families of $\boldsymbol{Q}$-independent subsets are hereditary whenever $\boldsymbol{Q} = \boldsymbol{S}_0, \boldsymbol{S}, \boldsymbol{G}, \boldsymbol{A}_1$, and they are of finite character for $\boldsymbol{Q} = \boldsymbol{G}, \boldsymbol{A}_1, \boldsymbol{R}$. E. Marczewski in [17] found some sufficient conditions for the family $\boldsymbol{Ind}(\boldsymbol{Q})$ to possess any of these two properties (see (iv) and (v) in § 2). Using maximal families of mappings it is easy to formulate necessary and sufficient conditions for the family of $\boldsymbol{Q}$-independent subsets to be hereditary, and of finite character, respectively (Theorem 2 and 3). Hence, we obtain a connection between these properties, for families of $\boldsymbol{Q}$-independent subsets (Corollary 6), which seems to be interesting.

It is important also to define a family of mappings for a given algebra in such a manner as to make the notion of independence with respect to this family equivalent to C-independence i. e. to the independence defined by the algebraic closure (Corollary 4 comp. also Theorem 10 in § 6).

In § 4 we get the results analogous to respective theorems concerning $\boldsymbol{M}$-bases of different powers (under some assumptions on the family $\boldsymbol{Q}$, which are fulfilled in particular for the family $\boldsymbol{Q} = \boldsymbol{R}$ of all injective mappings; Theorems 4, 5 and 8). Among others we prove under some restrictions on $\boldsymbol{Q}$, that the powers of all $\boldsymbol{Q}$-bases are finite and form an arithmetical progresion, whenever there exist two $\boldsymbol{Q}$-bases of different powers (Theorem 8). The assumptions of $\boldsymbol{Q}$ are of this kind, that as a consequence of them we get an isomorphism between subalgebras generated by $\boldsymbol{Q}$-independent subsets of the same cardinality (Theorem 7), and furthermore that $\boldsymbol{Q}$-independence is stronger than $\boldsymbol{C}$-independence (Theorem 6). Under those assumptions a $\boldsymbol{Q}$-independent set of generators (a $\boldsymbol{Q}$-basis) is simultaneously a minimal set of generators (corollary of Theorem 6) and a maximal $\boldsymbol{Q}$-independent set (Corollary 12), which is not true, in general. It is worth adding that there may exist simultaneously finite and infinite $\boldsymbol{G}$-bases.

In § 5 we prove a theorem on exchange of $\boldsymbol{Q}$-independent subsets under some natural conditions on the family $\boldsymbol{Q}$ (Theorem 9).

In Chapter II we construct a certain family of mappings for which the $\boldsymbol{Q}$-independence of at least two-element subsets coincides with the C-independence (§ 6, Theorem 10). Since this family is contained in the family $\boldsymbol{S}_0$ we conclude that in v^{**}-algebras the notions of $\boldsymbol{M}$-, $\boldsymbol{S}$-, $\boldsymbol{S}_0$- and C-independence coincide for subsets containing at least two elements (Corollary 20). In particular, it is true also for linear and affine spaces (Corollary 21). In § 7 we also characterize the $\boldsymbol{G}$-independence in mentioned algebras (Corollary 21 and 22).

In abelian groups $\boldsymbol{G}$-independence is equivalent to linear independence for subsets not containing the zero element (G. Graetzer [8], see also [17]). In Chapter III we give a characterization of $\boldsymbol{S}_0$- and $\boldsymbol{S}$-independence in abelian groups (Theorems 12 and 13 of § 8). A connection of $\boldsymbol{S}_0$- and $\boldsymbol{G}$-independence is formulate in the Corollary 23, and seems to be of some interest. However, the $\boldsymbol{A}_1$-independence is not interesting in abelian groups, because every subset is $\boldsymbol{A}_1$-independent and the same holds in all so-called weakly commutative algebras (see (xi) of § 2).

In § 9 we generalize the results concerning $\boldsymbol{S}$-, $\boldsymbol{S}_0$ and $\boldsymbol{G}$-independence in abelian groups to quasi-linear algebras defined in [3] (Theorem 14 and Corollary 28, Theorems 15 and 16). In this algebras $\boldsymbol{Ind}(\boldsymbol{R}) = \boldsymbol{Ind}(\boldsymbol{M})$.

In Chapter IV we consider some reducts of Boolean algebras, called here regular. We obtain in this way a generalization of Marczewski's Theorems on $\boldsymbol{M}$-independence in Boolean algebras and some of their reducts (Theorem 19), and we come to conclusion that $\boldsymbol{S}$-, $\boldsymbol{S}_0$- and $\boldsymbol{M}$-independence coincide for subsets with at least two elements in regular reducts (Theorem 20). Finally, we characterize also the $\boldsymbol{G}$-independence (Theorem 21) and $\boldsymbol{A}_1$-independence (Theorem 22) in those reducts. Moreover in the same reduct the $\boldsymbol{R}$-independence is equivalent to $\boldsymbol{M}$-independence (Theorem 23).

We pose also some problems, the answer to which is not known to the author (Problems 1-6).

This investigation and a part of the theorems obtained here arose from questions posed to me by Professor E. Marczewski in the years 1967 and 1968(*).

I wish to express my sincere thanks to Professor Marczewski for his patient guidance of my work. I wish to thank him, and also Dr S. Fajtlowicz, for numerous discussions and remarks which greatly influenced this paper.

I. INDEPENDENCE WITH RESPECT TO A GIVEN FAMILY OF MAPPINGS (GENERAL PROPERTIES)

§ 1. Notations and main definitions

In this paper we adopt the definitions and notations given by Professor E. Marczewski in [13] and [17]. By an *algebra* $\mathfrak{A}$ we shall mean a pair $\mathfrak{A} = (A; \boldsymbol{F})$, where A is a non-empty set (we call it the support of $\mathfrak{A}$) and $\boldsymbol{F}$ is a class of *fundamental operations* consisting of A-valued functions of several variables running over A. We denote by $\boldsymbol{A}$ the class of all *algebraic operations* i. e. the smallest class containing *trivial operations*

$$e_k^{(n)}(x_1, \ldots, x_n) = x_k \quad (k = 1, 2, \ldots, n;\, n = 1, 2, \ldots)$$

(*) This paper is a doctoral thesis which was presented to the Faculty of Mathematics, Physics and Chemistry of the Wrocław University at 27th May 1969.

and closed under the composition with the fundamental operations. We write $\boldsymbol{A}(\mathfrak{A})$ to make clear that we consider the algebraic operations of the algebra $\mathfrak{A}$. Two algebras $\mathfrak{A}_1 = (A; \boldsymbol{F}_1)$ and $\mathfrak{A}_2 = (A; \boldsymbol{F}_2)$ with the same support and having the same class of algebraic operations will be treated here as identical; if $\boldsymbol{A}(\mathfrak{A}_1) \subset \boldsymbol{A}(\mathfrak{A}_2)$ then $\mathfrak{A}_1$ is called a *reduct* of $\mathfrak{A}_2$. By $\boldsymbol{A}^{(n)}$ (or $\boldsymbol{A}^{(n)}(\mathfrak{A})$) we denote the subclass of the class $\boldsymbol{A}$ of all n-ary algebraic operations. By the *algebra of n-ary algebraic operations* of a given algebra $\mathfrak{A}$ we mean $\mathfrak{A}^{(n)} = (\boldsymbol{A}^{(n)}; \hat{\boldsymbol{F}})$ with fundamental operations $\hat{f} \in \hat{\boldsymbol{F}}$ induced by fundamental operations $f \in \boldsymbol{F}$ of algebra $\mathfrak{A}$ in the following way: if $f \in \boldsymbol{F}$ is m-ary then $\hat{f} \in \hat{\boldsymbol{F}}$ is defined by the formula:

$$\big(\hat{f}(f_1, \ldots, f_m)\big)(x_1, \ldots, x_n) = f\big(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\big),$$

where $x_1, \ldots, x_n \in A$ oraz $f_1, \ldots, f_m \in \boldsymbol{A}^{(n)}$.

By $\boldsymbol{A}^{(0)} = \boldsymbol{C}(\emptyset)$ we denote the class of *constant algebraic operations* of algebra $\mathfrak{A}$ as well as their values. Further for $T \subset A$ let $\boldsymbol{C}(T)$ be the smallest set containing T such that the values of any algebraic operation of $\mathfrak{A}$ run over $C(T)$ whenever the variables are in $\boldsymbol{C}(T)$. The pair $\big(\boldsymbol{C}(T); \boldsymbol{F} \mid \boldsymbol{C}(T)\big)$ forms a *subalgebra* of $\mathfrak{A}$ i. e. $\boldsymbol{C}(T)$ is closed under restrictions of operations from $\boldsymbol{F}$ to $\boldsymbol{C}(T)$. If $\boldsymbol{C}(\emptyset)$ is non-empty it is a least subalgebra. The operator C which carries the set T into $\boldsymbol{C}(T)$ is extensive, monotone and idempotent so it is a *generalized closure operator* (comp. [1], p. 49 and [15]). This operator has an additional property — it is of finite character i. e. for any $T \subset A$ the set $\boldsymbol{C}(T)$ is the sum of $\boldsymbol{C}(F)$ where F runs over all finite subsets of T. We shall call $\boldsymbol{C}$ an *algebraic closure*.

Suppose there is a generalized closure operator $\boldsymbol{D}$ of finite character on the family 2^A of all subsets of A. Then the algebra $\mathfrak{A} = (A; \boldsymbol{A})$ with all algebraic operations $f \in \boldsymbol{A}$ satisfying the condition

$$f(a_1, \ldots, a_n) \in \boldsymbol{D}(a_1, \ldots, a_n),$$

yields the realization of the given generalized closure $\boldsymbol{D}$, i. e. $\boldsymbol{D}(T) = \boldsymbol{C}(T)$ for any $T \subset A$.

A set $I \subset A$ is called *C-independent* if $a \notin \boldsymbol{C}(I \setminus \{a\})$ for any $a \in I$. The family of all $\boldsymbol{C}$-independent set of algebra $\mathfrak{A}$ will be denoted by $\boldsymbol{C}$-$\boldsymbol{Ind}(\mathfrak{A})$ (or shortly $\boldsymbol{C}$-$\boldsymbol{Ind}$).

A set $I \subset A$ is called *$\boldsymbol{M}$-independent* if for any system of different elements $a_1, \ldots a_n \in I$ and for any pair operations $f, g \in \boldsymbol{A}^{(n)}$ the equality

$$f(a_1, \ldots a_n) = g(a_1, \ldots, a_n) \tag{1}$$

implies $f = g$ in $\mathfrak{A}$. The last notion was called simply "independence" in earlier papers (see [13]). The family of all $\boldsymbol{M}$-independent set of algebra $\mathfrak{A}$ will be denoted by $\boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{M})$ (or shortly $\boldsymbol{Ind}(\boldsymbol{M})$).

A family $\boldsymbol{J} \subset 2^A$ is said to be of *finite character* if for subset $T \subset A$, $T \in \boldsymbol{J}$ whenever any finite subset $F \subset T$ is in $\boldsymbol{J}$. A family $\boldsymbol{J}$ is *hereditary* if each subset of a member of $\boldsymbol{J}$ is in $\boldsymbol{J}$ too. For instance the families $\boldsymbol{Ind}(\boldsymbol{M})$ and C-$\boldsymbol{Ind}$ are of finite character and hereditary. Also the family of C-closed subsets of $\mathfrak{A}$ is of finite character (but it is not hereditary).

Finally let $\boldsymbol{M}(A)$ (or shortly $\boldsymbol{M}$) denote the family of all mappings $p\colon T \to A$ from any $T \subset A$, i. e. $\boldsymbol{M}(A) = \{p\colon p \in A^T, T \subset A\}$. Further $\boldsymbol{H}(A)$ (shortly $\boldsymbol{H}$) will denote the set of such mappings $p\colon T \to A$ (for $T \subset A$) which posses an extension to a homomorphisms $\bar{p}\colon \boldsymbol{C}(T) \to A$.

A more detailed discussion of the above notions may be fond in the cited literature first of all in papers of Marczewski [13] and [16].

§ 2. Notions of independence defined by families of mappings (Q-independence)

Professor E. Marczewski observed that several independence notions weaker than M-independence fall under a common scheme (together with M-independence), see [16], p. 173. He proposed in [17] the following definition:

Let $\boldsymbol{Q} \subset \boldsymbol{M}(A)$. A set $I \subset A$ will be said *independent with respect to the family* $\boldsymbol{Q}$ or, shortly, $\boldsymbol{Q}$*-independent* (in the algebra $\mathfrak{A}$), if

$$\boldsymbol{Q} \cap A^I \subset \boldsymbol{H}(\mathfrak{A}).$$

We use then notation $I \in \boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{Q})$ or $I \in \boldsymbol{Ind}(\boldsymbol{Q})$.

Let an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ be given. It is know from [13] and [17]:

THEOREM (Marczewski [17]). *The following conditions are equivalent:*

(a) $$I \in \boldsymbol{Ind}(\boldsymbol{Q}),$$

(b) *for every* $p \in \boldsymbol{Q} \cap A^I$ *if* $a_1, \ldots, a_{m+n} \in I, f \in \boldsymbol{A}^{(m)}, g \in \boldsymbol{A}^{(n)}$ $(m, n = 1, 2, \ldots)$ *and* $f(a_1, \ldots, a_m) = g(a_{m+1}, \ldots, a_{m+n})$, *then*

$$f(p(a_1), \ldots, p(a_m)) = g(p(a_{m+1}), \ldots, p(a_{m+n})),$$

(c) *for every* $p \in \boldsymbol{Q} \cap A^I$ *if* $a_1, \ldots, a_n \in I$, $f, g \in \boldsymbol{A}^{(n)}$ $(n = 1, 2, \ldots)$ *and* (1), *then*

(2) $$f(p(a_1), \ldots, p(a_n)) = g(p(a_1), \ldots, p(a_n)),$$

(d) *for every* $p \in \boldsymbol{Q} \cap A^I$ *if* $a_1, \ldots, a_n$ *are different elements of* I, $f, g \in \boldsymbol{A}^{(n)}$ $(n = 1, 2, \ldots)$ *and* (1), *then* (2).

It is worthwhile to remark that

(i) *For every* $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \subset \boldsymbol{M}$ *the following conditions are equivalent*:

(e) $$\boldsymbol{Ind}(\boldsymbol{Q}_1) \subset \boldsymbol{Ind}(\boldsymbol{Q}_2),$$

(f) *for every* $T \subset A$ *if* $A^T \cap \boldsymbol{Q}_1 \subset \boldsymbol{H}$ *then* $A^T \cap \boldsymbol{Q}_2 \subset \boldsymbol{H}$,

(g) $$\boldsymbol{Ind}\,\boldsymbol{Q}_1 = \boldsymbol{Ind}(\boldsymbol{Q}_1 \cup \boldsymbol{Q}_2).$$

From the equivalence of (e) and (f) we conclude immediately

(ii) ([17]). *If* $\boldsymbol{Q}_1 \subset \boldsymbol{Q}_2 \subset \boldsymbol{M}$, *then* $\boldsymbol{Ind}(\boldsymbol{Q}_2) \subset \boldsymbol{Ind}(\boldsymbol{Q}_1)$.

Recall for completeness the following properties (see [17]):

(iii) *For every* $\boldsymbol{Q} \subset \boldsymbol{M}$

$$\boldsymbol{Ind}(\boldsymbol{M}) \subset \boldsymbol{Ind}(\boldsymbol{Q}) \subset \boldsymbol{Ind}(\boldsymbol{H}) = 2^A.$$

(iv) *Let* $\boldsymbol{Q} \subset \boldsymbol{M}$ *and let for every* $U \subset T \subset A$ *and every* $p \in \boldsymbol{Q} \cap A^U$ *there exists* $q \in \boldsymbol{Q} \cap A^T$ *such that* $q \mid U = p$. *Then the family* $\boldsymbol{Ind}(\boldsymbol{Q})$ *is hereditary.*

(v) *Let* $\boldsymbol{Q} \subset \boldsymbol{M}$ *and let for every* $U \subset T \subset A$ *be* $q \mid U \in \boldsymbol{Q}$ *for arbitrary* $q \in \boldsymbol{Q} \cap A^T$. *Then the family* $\boldsymbol{Ind}(\boldsymbol{Q})$ *has finite character.*

It is easy to check, that

(vi) *For an arbitrary set of families* $\boldsymbol{Q}_t \subset \boldsymbol{M}$ $(t \in T)$ *we have*

$$\bigcap_{t\in T} \boldsymbol{Ind}(\boldsymbol{Q}_t) = \boldsymbol{Ind}\Big(\bigcup_{t\in T} \boldsymbol{Q}_t\Big), \qquad \bigcup_{t\in T} \boldsymbol{Ind}(\boldsymbol{Q}_t) \subset \boldsymbol{Ind}\Big(\bigcap_{t\in T} \boldsymbol{Q}_t\Big).$$

Since

(vii) $I \in \boldsymbol{Ind}(\boldsymbol{Q})$ *if and only if* $I \in \boldsymbol{Ind}(\boldsymbol{Q} \cup \boldsymbol{H})$,

we may suppose for convenience, without loss of generality, that $\boldsymbol{H} \subset \boldsymbol{Q}$.

Observe

(viii) *If* $c \in C(\emptyset) \cap I \neq \emptyset$ *and there exists* $p \in \boldsymbol{Q} \cap A^I$ *such that* $p(c) \neq c$ *then* $I \notin \boldsymbol{Ind}(\boldsymbol{Q})$.

In fact, we have $e_1^{(1)}(c) = c(c)$ and $e_1^{(1)}\big(p(c)\big) \neq c\big(p(c)\big)$, where $c(x)$ stands for the constant algebraic operation with value c. Thus I cannot be $\boldsymbol{Q}$-independent. ∎

If we put $\boldsymbol{Q} = \boldsymbol{M}$, we obtain M-independence introduced and called "independence" by E. Marczewski in 1958 (see [11] and [13]).

If we put $\boldsymbol{Q} = \boldsymbol{S} = \{p\colon p \in C(T)^T, T \subset A\}$ we obtain *S-independence* introduced by J. Schmidt in [20].

However, if we put $\boldsymbol{Q} = \boldsymbol{S}_0 = \{p\colon p \in T^T, T \subset A\}$, we obtain $\boldsymbol{S}_0$*-independence* introduced by S. Świerczkowski in [21].

Another notions of independence may be obtained putting $\boldsymbol{Q} = \boldsymbol{A}_1 = \{f \mid T; f \in \boldsymbol{A}^{(1)}, T \subset A\}$.

From (ii) we conclude

(ix) $\boldsymbol{Ind}(\boldsymbol{M}) \subset \boldsymbol{Ind}(\boldsymbol{S}) \subset \boldsymbol{Ind}(\boldsymbol{S}_0)$ ([17]), *and* $\boldsymbol{Ind}(\boldsymbol{S}) \subset \boldsymbol{Ind}(\boldsymbol{A}_1)$.

It is easy to see, that

(x) *A single-point set* $\{a\}$ *is* $\boldsymbol{S}$*-independent iff it is* $\boldsymbol{A}_1$*-independence.*

We shall call an algebra *weakly commutative*, if any algebraic operation of one variable is a homomorphism i. e. for any $f \in \boldsymbol{A}^{(n)}$, $g \in \boldsymbol{A}^{(1)}$ the equality

$$f\big(g(x_1), \ldots, g(x_n)\big) = g\big(f(x_1, \ldots, x_n)\big)$$

holds for arbitrary $x_1, \ldots, x_n \in A$.

Such algebras are, for instance, linear spaces, abelian groups (more generally so-called commutative algebras; [10], p. 32), and idempotent algebras. There is $|C(\emptyset)| \leqslant 1$ in weakly commutative algebras.

Moreover

(xi) *In the algebra $\mathfrak{A}$ every subset of A is A_1-independent iff $\mathfrak{A}$ is weakly commutative.*

The sufficiency, in fact, is clear. Let now every subset of $\mathfrak{A}$ be A_1-independent. In virtue of A_1-independence of A, the equality $x_0 = f(x_1, \ldots, x_n)$, where $x_0, x_1, \ldots, x_n \in A$, $f \in A^{(n)}$, implies $g(x_0) = f(g(x_1), \ldots, g(x_n))$ for every $g \in A^{(1)}$. Thus g is homomorphism of $\mathfrak{A}$ into itself. ∎

From (viii) get easily

(xii) *If* $I \cap C(\emptyset) \neq \emptyset$ *and* $|I| \geqslant 2$ $(|C(\emptyset)| \geqslant 2)$, *then* $I \notin \boldsymbol{Ind}(\boldsymbol{S}_0)$ $(I \notin \boldsymbol{Ind}(\boldsymbol{A}_1)$, resp.). ∎

For a discussion one more notion of independence, we define a certain family of mappings.

A mapping $p\colon T \to A$, where $T \subset A$, is called *diminishing* if for every $f, g \in A^{(1)}$ and for each $a \in T$ the equality $f(a) = g(a)$ implies $f(p(a)) = g(p(a))$.

Evidently, if I is Q-independent, then every mapping $p \in Q \cap A^I$ is diminishing. Also we have

(xiii) ([17]). *$\{a\} \in \boldsymbol{Ind}(\boldsymbol{Q})$ iff every mapping $p\colon a \to A$ belonging to Q is diminishing.*

Hence we conclude

(xiv) *If there are no M-self-dependent elements in $\mathfrak{A}$ (in particular this is in the case of idempotent algebras) then every one-point set is Q-independent, for every $Q \subset M$. In an algebra with only one constant c $(C(\emptyset) = \{c\})$ the set $\{c\}$ is S-independent; moreover, if there are no other than c M-self-dependent elements, then every one-point set is S-independent.*

It is easy to see, that.

(xv) *If the algebra $\mathfrak{A}$ contains only one algebraic constant c then mappings $p_1, p_2\colon T \to C(T)$ defined below:*

$$p_1(x) = \begin{cases} a_0, & \text{if } x = a_0, \\ c, & \text{if } x \neq a_0, \end{cases} \qquad p_2(x) = \begin{cases} c, & \text{if } x = a_0, \\ x, & \text{if } x \neq a_0. \end{cases}$$

(for a fixed $a_0 \in C(T)$) are diminishing.

(xvi) *In the arbitrary algebra $\mathfrak{A}$ the diminishing mapping $p\colon T \to A$ $(T \subset A)$ preserves every algebraic constant. If, moreover, $T \subset A$ does not contain non-constant M-self-dependent elements then every mapping $T \to A$ $(T \subset A)$ preserving all algebraic constants (in T) is diminishing.*

Put $\boldsymbol{Q} = \boldsymbol{G}$ — the family of all diminishing mappings, then the $\boldsymbol{Q}$-independence becomes the *$\boldsymbol{G}$-independence* introduced by G. Graetzer in [8] (see also [17].

From (xvi) we conclude that in algebras without $\boldsymbol{M}$-self-dependent elements (and so without constants) we have $\boldsymbol{Ind}(\boldsymbol{G}) = \boldsymbol{Ind}(\boldsymbol{M})$.

Bearing in mind (iv) and (v) we easily obtain

(xvii) (comp. [17]). *The families $\boldsymbol{Ind}(\boldsymbol{S}_0)$, $\boldsymbol{Ind}(\boldsymbol{S})$, $\boldsymbol{Ind}(\boldsymbol{G})$ and $\boldsymbol{Ind}(\boldsymbol{A}_1)$ are hereditary, and the families $\boldsymbol{Ind}(\boldsymbol{G})$ and $\boldsymbol{Ind}(A_1)$ are of finite character.* ■

Now we prove

(xviii) *A subset $I \subset A$ is $\boldsymbol{G}$-independent in the algebra iff for every subset $\boldsymbol{B} \subset \boldsymbol{C}(\emptyset)$ the set $I \cup B$ is $\boldsymbol{G}$-independent in $\mathfrak{A}$.*

Indeed, let I be a $\boldsymbol{G}$-independent set, $B \subset \boldsymbol{C}(\emptyset)$ and

$$f(a_1, \ldots, a_k, c_1, \ldots, c_m) = g(a_1, \ldots, a_k, c_1, \ldots, c_m)$$

where $a_1, \ldots, a_k \in I$, $c_1, \ldots, c_m \in B$, $f, g \in \boldsymbol{A}^{(k+m)}$. Define the algebraic operation $f_0, g_0 \in \boldsymbol{A}^{(k)}$ in the following manner:

$$f_0(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, c_1, \ldots, c_m),$$
$$g_0(x_1, \ldots, x_k) = g(x_1, \ldots, x_k, c_1, \ldots, c_m)$$

for every $x_1, \ldots, x_k \in A$. From $\boldsymbol{G}$-independence of I we get

$$f_0\big(p(a_1), \ldots, p(a_k)\big) = g_0\big(p(a_1), \ldots, p(a_k)\big)$$

for every diminishing mapping p. Hence because of the definition of f_0 and g_0 and in view of (xvi) we get

$$f\big(p(a_1), \ldots, p(a_k), p(c_1), \ldots, p(c_m)\big) = g\big(p(a_1), \ldots, p(a_k), p(c_1), \ldots, p(c_m)\big)$$

for any diminishing mapping $p\colon I \cup B \to A$. Thus $I \cup B$ is $\boldsymbol{G}$-independent.

The convers implication is obvious (put $B = \emptyset$). ■

In particular, we conclude, that $\boldsymbol{G}$-independence of a subset $I \subset A$ doesn't imply it's $\boldsymbol{C}$-independence. This result was obtained by another way by Graetzer (see [8], p. 233).

Finally, let us define *$\boldsymbol{R}$-independence* putting for $\boldsymbol{Q}$ the family $\boldsymbol{R}$ of all injective mappings.

Note that (in view of (v)) that the family $\boldsymbol{Ind}(\boldsymbol{R})$ is of finite character. At once (viii) implies that an algebraic constant cannot belong to an $\boldsymbol{R}$-independent set whenever the support of $\mathfrak{A}$ has at least two elements.

Further we have

(xix) *If $I \in \boldsymbol{Ind}(\boldsymbol{M})$, $J \in \boldsymbol{Ind}(\boldsymbol{R})$ in the algebra $\mathfrak{A}$ and $|I| \geqslant |J|$ then $J \in \boldsymbol{Ind}(\boldsymbol{M})$.*

In fact, it suffices to consider the case when $|I| = |J|$ because of hereditarity of $\boldsymbol{Ind}(\boldsymbol{M})$. Let us suppose $a_1, \ldots, a_n$ be different elements from J and $f, g \in A^{(n)}(\mathfrak{A})$, $p \in A^J \cap \boldsymbol{R}$ and (1) holds. Considering the bijective mapping $q\colon J \to I$ and making use of $\boldsymbol{R}$-independence of J we get

$$f\big(q(a_1), \ldots, q(a_n)\big) = g\big(q(a_1), \ldots, q(a_n)\big),$$

where $q(a_i) \in I$ $(i = 1, \ldots n)$. From $\boldsymbol{M}$-independence of I the last equality gives

$$f\big((p \circ q^{-1})(q(a_1)), \ldots, (p \circ q^{-1})(q(a_n))\big) = g\big((p \circ q^{-1})(q(a_1)), \ldots, (p \circ q^{-1})(q(a_n))\big),$$

which leads to (2), whence J is $\boldsymbol{R}$-independent. ∎

§ 3. Maximal families of mappings for given independence

In consideration of this paragraph the algebra $\mathfrak{A} = (A; \boldsymbol{F})$ be fixed (except Corollaries 4 and 5).

From (vi) it follows easily, that for every family $\boldsymbol{Q} \subset \boldsymbol{M}(A) = \bigcup_{T \in A} A^T$ $(= \boldsymbol{M})$ there exists the greatest family $\bar{\boldsymbol{Q}}$ of mappings such that

$$\boldsymbol{Ind}(\bar{\boldsymbol{Q}}) = \boldsymbol{Ind}(\boldsymbol{Q}).$$

A family Q such that $\bar{\boldsymbol{Q}} = \boldsymbol{Q}$ shortly will be called *maximal.*

Now we prove

THEOREM 1. *For every family $\boldsymbol{J}$ of subsets of A such that*

$$\boldsymbol{Ind}(\boldsymbol{M}) \subset \boldsymbol{J} \subset 2^A \tag{3}$$

there exists a family of mappings $\boldsymbol{Q} \subset \boldsymbol{M}$ satisfying the equality

$$\boldsymbol{Ind}(\boldsymbol{Q}) = \boldsymbol{J}. \tag{4}$$

Moreover the mapping $\boldsymbol{Q} \to \boldsymbol{Ind}(\boldsymbol{Q})$ is an anti-isomorphism of the algebra $\mathfrak{A}$ of all maximal families $\boldsymbol{Q} \subset \boldsymbol{M}$ of mappings defined on $\mathfrak{A}$ with set-theoretic join and meet and complementation defined by the equality

$$\boldsymbol{Q}' = (\boldsymbol{M} \setminus \boldsymbol{Q}) \cup \boldsymbol{H}, \tag{5}$$

and the Boolean algebra of all subsets of $2^A \setminus \boldsymbol{Ind}(\boldsymbol{M})$.

Proof. Following the idea of S. Fajtlowicz of the proof of Corollary 3 (which will be a consequence of our Theorem) we define for $\boldsymbol{J}$ satisfying (3) a family $\boldsymbol{Q}$ of mappings by putting

$$A^T \cap \boldsymbol{Q} = A^T, \quad \text{if } T \notin \boldsymbol{J},$$

and

$$A^T \cap \boldsymbol{Q} = \boldsymbol{H} \cap A^T, \quad \text{if } T \in \boldsymbol{J}.$$

Hence we obtain that $\boldsymbol{J} \subset \boldsymbol{Ind}(\boldsymbol{Q})$. Now if $T \notin \boldsymbol{J}$ then T is an $\boldsymbol{M}$-dependent set (from (3)). Thus there exists a mapping $p\colon T \to A$ which is not extendible to a homomorphism from $C(T)$ into A. This mapping belongs, in view of our definition, to the family $\boldsymbol{Q}$ and so T is $\boldsymbol{Q}$-dependent. This ends the proof of first part of Theorem 1. (It is worth to remark that the above constructed family $\boldsymbol{Q}$ is maximal).

It is clear that the family $\boldsymbol{Q}'$ define by (5) is also maximal whenever a family $\boldsymbol{Q}$ is maximal. Taking under consideration (vi) we observe that to every family of subsets $\boldsymbol{J}$ with property (3) there coresponds a one-to-one manner a family $\boldsymbol{Q}$ giving the equality (4). It is easy to verify that the mapping $\boldsymbol{Q} \to \boldsymbol{Ind}(\boldsymbol{Q})$ gives the mentioned isomorphism, because the equivalence

$$\boldsymbol{Q}_1 \subset \boldsymbol{Q}_2 \Leftrightarrow \boldsymbol{Ind}(\boldsymbol{Q}_1) \supset \boldsymbol{Ind}(\boldsymbol{Q}_2)$$

holds for maximal families $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$. ∎

It is worth to observe that the second part of Theorem 1 mey be deduced from more general considerations dealing with certain mappings of direct sums of complete Boolean algebras $\{A_v\}$, indexed by a set V, into 2^V.

Finally observe that the conditions

$$\begin{aligned} &\boldsymbol{H} \cap \boldsymbol{Q} = \emptyset, \\ &A^T \cap \boldsymbol{Q} = \emptyset, \quad \text{if } T \in \boldsymbol{J}, \\ &|A^T \cap \boldsymbol{Q} \cap (\boldsymbol{M} \setminus \boldsymbol{H})| = 1, \quad \text{if } T \notin \boldsymbol{J}, \end{aligned}$$

determine (not uniquely) minimal families (of mappings) satisfying (4) (where $\boldsymbol{J}$ satisfies (3)).

From the second part of Theorem 1 we deduce

COROLLARY 1. *The set $\mathfrak{Q}$ of all maximal families $\boldsymbol{Q} \subset \boldsymbol{M}$ of mappings defined on $\mathfrak{A}$ with set-theoretic join and meet operations and complementation $\boldsymbol{Q} \to \boldsymbol{Q}'$ defined by* (5) *is a complete atomic Boolean algebra. A family $\boldsymbol{Q} \in \mathfrak{Q}$ is an atom of this algebra iff there exists a unique set $T \notin \boldsymbol{Ind}(\boldsymbol{M})$, such that $A^T \subset \boldsymbol{Q}$.* ∎

However from the first part of Theorem 1 we deduce simply corollaries.

COROLLARY 2 (S. Fajtlowicz). *For arbitrary family $\boldsymbol{J} \subset 2^A$ there exists a subfamily $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ and an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ such that* (4) *holds.*

Indeed, it suffices to observe that $\boldsymbol{Ind}(\boldsymbol{M}) = \emptyset$ in the functionally complete algebra on the set A. ∎

Putting in Theorem 1 $\boldsymbol{J} = \boldsymbol{C}\text{-}\boldsymbol{Ind}$ we get an answer to a question raised to me by Professor E. Marczewski, and obtained earlier in another way (see Chapter II, § 6, remark to Theorem 10);

COROLLARY 3. *For every algebra $\mathfrak{A} = (A; \boldsymbol{F})$ there exists a family of mappings $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ such that*

(6) $$\boldsymbol{C}-\boldsymbol{Ind} = \boldsymbol{Ind}(\boldsymbol{Q}).\ ∎$$

If $\boldsymbol{D}$ is a generalized closure operator of finite character on 2^A, deduced from Corollary 3 we get a corollary for the algebra which yields the realization of this closure operator (see § 1).

COROLLARY 4. (S. Fajtlowicz). *For every generalized closure operator $\boldsymbol{D}$ of finite character given on 2^A, there exists an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ and a family of mappings $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ such that for any $T \subset A$ the subalgebra generated by T is equal to $\boldsymbol{D}(T)$ and the equality* (6) *holds.* ∎

Let $\boldsymbol{Q} \subset \boldsymbol{M}(A)$. By an analogy to the property v^* investigated by several author ([18] and [24]) we shall say that *an algebra $\mathfrak{A}$ has the property v_Q^** if the following conditions are satisfied

(*) *every Q-dependent element of A is an algebraic constant,*

(**) *if the set $\{a_1, \ldots, a_n\}$ $(n \geqslant 1)$ is Q-independent and the set $\{a_1, \ldots \ldots, a_n, a_{n+1}\}$ is Q-dependent then $a_{n+1} \in C(\{a_1, \ldots, a_n\})$.*

An algebra $\mathfrak{A}$ satisfies (*) and (**) iff the equality (6) holds for $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ and the $\boldsymbol{C}$-closure has the exchange property (comp. [24], p. 235):

(***) *if $a \notin \boldsymbol{C}(T)$ and $a \in \boldsymbol{C}(T \cup \{b\})$, then $b \in \boldsymbol{C}(T \cup \{a\})$ $(T \subset A)$.*

Using the last corollary we get

COROLLARY 5. *For every generalized closure operator $\boldsymbol{D}$ of finite character given on 2^A and fulfilling (***) there exist an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ and a family of mappings $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ such that the algebraic closure is equal $\boldsymbol{D}$ and the algebra $\mathfrak{A}$ has the property v_Q^*.* ∎

It is worth to remark, that from the definition of the operation $\boldsymbol{Q} \to \bar{\boldsymbol{Q}}$ we have immediately the following properties:

$$A^T \cap \bar{\boldsymbol{Q}} \subset \boldsymbol{H} \Leftrightarrow A^T \cap \boldsymbol{Q} \subset \boldsymbol{H}, \tag{7}$$

$$\boldsymbol{H} \cap A^T \subsetneqq A^T \subset \bar{\boldsymbol{Q}} \Leftrightarrow (\boldsymbol{Q} \cap A^T) \setminus \boldsymbol{H} \neq \emptyset \tag{8}$$

for $T \subset A$.

Thus this operation is uniquely determined by the family $\boldsymbol{H}$. Moreover as a direct consequence (7) and (8) we get:

$$\bar{\emptyset} = \boldsymbol{H} = \bar{\boldsymbol{H}}, \tag{9}$$

$$\boldsymbol{Q} \subset \bar{\boldsymbol{Q}}, \tag{10}$$

$$\bar{\boldsymbol{Q}} = \overline{(\bar{\boldsymbol{Q}})}, \tag{11}$$

$$\overline{\bigcup_{i \in J} \boldsymbol{Q}_i} = \bigcup_{i \in J} \bar{\boldsymbol{Q}}_i, \tag{12}$$

$$\overline{\bigcap_{i \in J} \boldsymbol{Q}_i} = \bigcap_{i \in J} \bar{\boldsymbol{Q}}_i, \tag{13}$$

Therefore the operation $\boldsymbol{Q} \setminus \boldsymbol{H} \to \bar{\boldsymbol{Q}} \setminus \boldsymbol{H}$, where $\boldsymbol{Q} \subset \boldsymbol{M}(A)$, is a topological closure on $\boldsymbol{M} \setminus \boldsymbol{H}$ (as defined by Kuratowski). However the topology obtained by this way is not interesting, since it is even not a T_1-topology, on generality. If it is the fact then there would be $\boldsymbol{Q} \setminus \boldsymbol{H} = \bar{\boldsymbol{Q}} \setminus \boldsymbol{H}$ for every $\boldsymbol{Q} \subset \boldsymbol{M}$, and hence the algebra $\mathfrak{A} = (A; \boldsymbol{F})$ would have to be strongly homogeneous i. e. one of the two-element algebras defined by Post (see [14]) or the trivial algebra.

It would be interesting to know the answer for the following (probably difficult)

PROBLEM 1. *In a fixed set A the generalized closure $\boldsymbol{D}\colon 2^A \to 2^A$ is given. Let $\boldsymbol{H} \subset \boldsymbol{M}(A)$ and further let the familly $\boldsymbol{H}$ fulfills following conditions:*

(a) *if $p \in \boldsymbol{H} \cap A^U$ $q \in \boldsymbol{H} \cap A^T$ and $q(T) \subset U$, then $p \circ q \in \boldsymbol{H} \cap A^T$,*

(b) *$e_T \in \boldsymbol{H} \cap A^T$, where e_T is an identical mapping on T.*

Moreover for arbitrary $T \subset A$ let this family be associated with the operator $\boldsymbol{D}$ by the following way:

(c) *for every $p \in \boldsymbol{H} \cap A^{\boldsymbol{D}(T)}$*

$$p\big(\boldsymbol{D}(T)\big) \subset \boldsymbol{D}\big(p(T)\big),$$

(d) *for every $p \in \boldsymbol{H} \cap A^T$ there exists $\bar{p} \in \boldsymbol{H} \cap A^{\boldsymbol{D}(T)}$ such that $\bar{p}|_T = p$.*

Does (or: under which additional assumptions) there exist an algebra $\mathfrak{A}$ with a support A such that $\boldsymbol{H}(\mathfrak{A}) = \boldsymbol{H}$ and $\boldsymbol{D} = \boldsymbol{C}$ (where $\boldsymbol{C}$ is the algebraic closure)?

(It is worth to note, that from conditions (a) and (b) it follows easily that: if $p \in \boldsymbol{H} \cap A^U$ and $T \subset U$ then $p|_T \in \boldsymbol{H} \cap A^T$.)

In the second paragraph we quote certain conditions obtained by E. Marczewski in [17] sufficient for the family $\boldsymbol{Ind}(\boldsymbol{Q})$ to be hereditary or of finite character (see (iv) and (v)). Using the maximal families of mappings we present now necessary and sufficient conditions for hereditary and finite character of the family $\boldsymbol{Ind}(\boldsymbol{Q})$.

Taking into account (8) and the definition of $\boldsymbol{Q}$-independence it is easy to verify that following holds

THEOREM 2. *For the family $\boldsymbol{Q} \subset \boldsymbol{M}$ the following conditions are equivalent:*

(α) *The family $\boldsymbol{Ind}(\boldsymbol{Q})$ is hereditary.*

(β) *For every subset $T \subset A$ if $A^T \cap \boldsymbol{Q} \subset \boldsymbol{H}$ then $A^U \cap \boldsymbol{Q} \subset \boldsymbol{H}$ for an arbitrary subset $U \subset T$.*

(γ) *If $U \notin \boldsymbol{Ind}(\boldsymbol{M})$ and $A^U \subset \bar{\boldsymbol{Q}}$ then $A^T \subset \bar{\boldsymbol{Q}}$ for every $T \supset U$.*

(δ) *If $A^T \subset \bar{\boldsymbol{Q}}'$ for $T \subset A$, then $A^U \subset \bar{\boldsymbol{Q}}'$ for every subset U of T (where $\bar{\boldsymbol{Q}}'$ is defined by (5)).*

We have also the following equivalences.

THEOREM 3. *For the family $\boldsymbol{Q} \subset \boldsymbol{M}$ the following conditions are equivalent:*

(ε) *The family $\boldsymbol{Ind}(\boldsymbol{Q})$ is of finite character.*

(ζ) *For every subset $T \subset A$, $\boldsymbol{Q} \cap A^T \subset \boldsymbol{H}$ whenever for all finite subsets $U \subset T$ the following holds $\boldsymbol{Q} \cap A^U \subset \boldsymbol{H}$.*

(η) *If $A^T \subset \overline{Q}$ for $T \notin \boldsymbol{Ind}(M)$ $(T \subset A)$, then there exists a finite subset $U \subset T$ such that $A^U \subset \overline{Q}$ and $U \notin \boldsymbol{Ind}(M)$.*

Proof. The equivalence of conditions (ε) and (ζ) follows easily from the definitions of Q-independence and finite character of $\boldsymbol{Ind}(Q)$ (cp. § 1). It is sufficient to establish the equivalence of conditions (ε) and (η).

If the family $\boldsymbol{Ind}(Q)$ is of finite character and if for $T \subset A$ we have $T \notin \boldsymbol{Ind}(M)$ and $A^T \subset \overline{Q}$, then $T \notin \boldsymbol{Ind}(Q) = \boldsymbol{Ind}(\overline{Q})$. Thus there exists a finite subset $U \subset T$ such that $U \notin \boldsymbol{Ind}(Q)$. Therefore, from (iii), we get $U \notin \boldsymbol{Ind}(M)$. Finally, in virtue of construction of the family $\overline{Q}$ (see (8)), we have $A^U \subset \overline{Q}$.

Conversely, let $U \in \boldsymbol{Ind}(Q)$ for every finite subset $U \subset T \subset A$. Then, from the definition of the family $\overline{Q}$, we have $A^U \cap \overline{Q} \subset \boldsymbol{H}$ for every finite subset $U \subset T$. Hence either $U \in \boldsymbol{Ind}(M)$ or $A^U \not\subset \overline{Q}$. Therefore by contraposition of condition (η) we get either $T \in \boldsymbol{Ind}(M)$ or $A^T \not\subset \overline{Q}$, which yields $T \in \boldsymbol{Ind}(\overline{Q}) = \boldsymbol{Ind}(Q)$. ■

It is easy to remark, that the condition (η) for the family $\overline{Q}'$ is following from the condition (δ). For, if $U \in \boldsymbol{Ind}(M)$ for every finite subset $U \subset T$, then also $T \in \boldsymbol{Ind}(M)$ because the family $\boldsymbol{Ind}(M)$ is of finite character. We have therefore

COROLLARY 6. *If the family $\boldsymbol{Ind}(Q)$ is hereditary, then the family $\boldsymbol{Ind}(\overline{Q}')$ is of finite character.* ■

Taking into account (8) and equivalence of conditions (α) and (γ), and equivalence of conditions (ε) and (η) for the family $\boldsymbol{Ind}(Q)$ we get

COROLLARY 7. *If the families $\boldsymbol{Ind}(Q_i)$ for $i \in J$ are hereditary (or of finite character), then the families*

$$\boldsymbol{Ind}\Big(\bigcup_{i \in J} Q_i\Big) \quad \text{and} \quad \boldsymbol{Ind}\Big(\bigcap_{i \in J} \overline{Q}_i\Big)$$

are also hereditary (or of finite character, resp.). ■

§ 4. Q-independent sets of generators (Q-bases)

Now we shall prove a certain results for Q-independence, which are analogous to respective theorems for M-independence concerning M-bases of different cardinality.

In this paragraph firstly we shall try to find for an algebra $\mathfrak{A}$ a necessary and sufficient condition (under certain additional conditions about the family Q) in order that $\mathfrak{A}$ has an n-element Q-basis. This result will be therefore the generalization of the Theorem 1 in [6] well-known for the M-independence.

To this aim we introduce certain relation in the set of n-ary algebraic operations, which, under the later specified assumptions about the family Q, will be the equivalence relation.

Namely, we shall say that the operations $f, g \in A^{(n)}(\mathfrak{A})$ are *equivalent with respect to the family* $Q \subset M$ (in short: are *in relation* $\sim_Q$; then we write $f \sim_Q g$, or, if there could be no misunderstanding, simply: $f \sim g$) if the following conditions are fulfielled:

(α) *there exists* Q*-independent subset* I *of the algebra* $\mathfrak{A}$ *and there exist different elements* $a_1, \ldots, a_n \in I$ *such that the equality*

$$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n) \tag{1}$$

holds,

(β) *for every at least* n*-element subset* $T \subset A$ *if the equality*

$$f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n) \tag{1'}$$

holds for arbitrary different elements $b_1, \ldots, b_n \in T$, *then for all* $p \in A^T \cap Q$

$$f(p(b_1), \ldots, p(b_n)) = g(p(b_1), \ldots, p(b_n)) \tag{2'}$$

is valid.

Obviously, if for every $f, g \in A^{(n)}$ and certain different $a_1, \ldots, a_n \in A$ the equality (1) implies $f \sim_Q g$, then $\{a_1, \ldots, a_n\} \in \boldsymbol{Ind}(Q)$.

In considerations of this paragraph we shall assume about the family Q that for certain (or all) n this family Q satisfies the condition

(γ_n) *for every at least* n*-element subset* $T \subset A$ *and for every different* $a_1, \ldots, a_n \in T$ *and different* $b_1, \ldots, b_n \in A$ *there exists a mapping* $p \in A^T \cap Q$ *such that*

$$p(a_i) = b_i \quad \textit{for } i = 1, 2, \ldots, n. \tag{14}$$

It is worth to note, that in our considerations it is always sufficient to take a weaker condition (γ_n') which is obtained from (γ_n) if we restrict ourselves to subsets $T \in \boldsymbol{Ind}(Q)$.

The condition (γ_n) can be often replace for appropriately chosed at least n-element set $I \in \boldsymbol{Ind}(Q)$ by the still weaker condition

($\gamma_{n,I}$) *for every different* $a_1, \ldots, a_n \in I$ *and for arbitrary different* $b_1, \ldots, b_n \in A$ *there exists* $p \in A^I \cap Q$ *satisfying* (14).

Note, that the condition (γ_n) is fulfilled if the family of mappings Q contains a family $\boldsymbol{R}$ consisting of all injective mappings. Since for many algebras (e. g. for linear spaces, abelian groups and Boolean algebras) $\boldsymbol{R}$-independence coincides with $\boldsymbol{M}$-independence, hence will be worthy to give some simple examples of algebras for which it is not so.

Consider an algebra $(\{a, b\}; f)$, where f is unary operation defined by putting $f(x) \neq x$. Taking into account that $f(f(x)) = x$, one easy verifies that the set $\{a, b\}$ is $\boldsymbol{R}$-independent, this set is not $\boldsymbol{M}$-independent as it is $\boldsymbol{C}$-dependent.

One can give also an example of an algebra $\mathfrak{A}$ such that for every n there exists an $\boldsymbol{R}$-independent n-element set (which we shall assume in Theorem 8) and $\boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{R}) \neq \boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{M})$.

For example in algebra $(A: \{l_n; n = 2, 3, \ldots\})$, where A is an infinite set and the operations l_n are defined in the following way:

$$l_n(x_1, \ldots, x_n) = \begin{cases} x_1 & \text{if } x_1, \ldots, x_n \text{ are different,} \\ x_n & \text{if } x_1, \ldots, x_n \text{ are not different,} \end{cases}$$

every subset of the set A is $\boldsymbol{R}$-independent whereas only one-element sets are $\boldsymbol{M}$-independent.

From the definition of the relation $\sim_Q$ we get immediately

LEMMA 1. *If the family $\boldsymbol{Q} \subset \boldsymbol{M}$ satisfied the condition (γ_n)*(¹) *and $f \sim_Q g$, then equality* (1) *is satisfied for every different $a_1, \ldots, a_n \in A$.* ∎

We shall prove now the fundamental lemma:

LEMMA 2. *If the family $\boldsymbol{Q}$ satisfies (γ_n), then the relation $\sim_Q$ is a congruence relation in the algebra $\mathfrak{A}^{(n)}$.*

Indeed, even without any assumption about the family $\boldsymbol{Q}$ this relation is evidently reflexive and symmetric. If $f \sim g$ and $g \sim h$, then there exist different elements $a_1, \ldots, a_n$ and different elements $b_1, \ldots, b_n$ from the $\boldsymbol{Q}$-independent subsets I_1 and I_2, respectively, such that the equalities (1) and

$$g(b_1, \ldots, b_n) = h(b_1, \ldots, b_n)$$

hold. If now for any elements $c_1, \ldots, c_n \in T \subset A$ the equality

$$f(c_1, \ldots, c_n) = h(c_1, \ldots, c_n),$$

holds, then, in virtue of Lemma 1, we have $f(c_1, \ldots, c_n) = g(c_1, \ldots, c_n) = h(c_1, \ldots, c_n)$, hence by (β) we get

$$f\big(p(c_1), \ldots, p(c_n)\big) = g\big(p(c_1), \ldots, p(c_n)\big) = h\big(p(c_1), \ldots, p(c_n)\big)$$

for every $p \in A^T \cap \boldsymbol{Q}$. We have the proof then that $\sim$ is an equivalence relation. Further, let $f_i, g_i \in A^{(n)}(\mathfrak{A})$ $(i = 1, 2, \ldots, m)$, $\hat{f} \in A^{(m)}(\mathfrak{A}^{(n)})$ and let $f_i \sim g_i$. Then, by virtue of Lemma 1, there exist different elements $a_1, \ldots, a_n$ belonging to a certain $\boldsymbol{Q}$-independent set I, such that

$$f_i(a_1, \ldots, a_n) = g_i(a_1, \ldots, a_n) \quad \text{for every } i = 1, \ldots, m.$$

We have now

$$\begin{aligned}\big(\hat{f}(f_1, \ldots, f_m)\big)(a_1, \ldots, a_n) &= f\big(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)\big) \\ &= f\big(g_1(a_1, \ldots, a_n), \ldots, g_m(a_1, \ldots, a_n)\big) \\ &= \big(\hat{f}(g_1, \ldots, g_m)\big)(a_1, \ldots, a_n).\end{aligned}$$

Finally, if for any different $b_1, \ldots, b_n \in T \subset A$ the equality

$$\big(\hat{f}(f_1, \ldots, f_m)\big)(b_1, \ldots, b_n) = \big(\hat{f}(g_1, \ldots, g_m)\big)(b_1, \ldots, b_n)$$

(¹) Here the condition (γ_n) can be replaced by the condition $(\gamma_{n,I})$, where I is a $\boldsymbol{Q}$-independent set of condition (α).

holds, then taking into consideration Lemma 1, also $f_i(b_1, \ldots, b_n) = g_i(b_1, \ldots, b_n)$ for $i = 1, \ldots, m$. Hence by (β) for every $p \in A^T \cap Q$ we have $f_i(p(b_1), \ldots, p(b_n)) = g_i(p(b_1), \ldots, p(b_n))$, and we get

$$\big(\hat{f}(f_1, \ldots, f_m)\big)\big(p(b_1), \ldots, p(b_n)\big) = \big(\hat{f}(g_1, \ldots, g_m)\big)\big(p(b_1), \ldots, p(b_n)\big).$$

We have proved therefore the equivalence $\hat{f}(f_1, \ldots, f_m) \sim \hat{f}(g_1, \ldots, g_m)$. Thus $\sim$ is a congruence relation in $\mathfrak{A}^{(n)}$. ∎

We shall denote by $\mathfrak{A}^{(n)}/\sim$ a factor algebra obtained from $\mathfrak{A}^{(n)}$ by dividing by this congruence relation. The coset of algebraic operation $f \in \boldsymbol{A}^{(n)}$ with respect to the relation $\sim$ will be denoted by $[f]_\sim$, or shortly $[f]$ (if it will not lead to misunderstanding); and the by symbol $\tilde{g}$ — m-ary operation in the algebra $\mathfrak{A}^{(n)}/\sim$ induced by the operation $\hat{g} \in \boldsymbol{A}^{(m)}$ (in $\mathfrak{A}^{(n)}$).

We shall say that the subset I of the set A is a *Q-basis* of an algebra $\mathfrak{A}$, if I is Q-independent set of generators of $\mathfrak{A}$.

Obviously, $\boldsymbol{S}$-basis is simultanously the $\boldsymbol{M}$-independent set, hence is also the $\boldsymbol{M}$-basis.

We shall prove the theorem, which is a generalization of the Theorem 1 of [6]. To this aim we put an additional condition on the family Q:

(δ_n) *If $U = \{a_1, \ldots, a_n\} \subset A$, $b_1, \ldots, b_n$ are different elements of an arbitrary subsets $T \subset A$, a mapping $p \in A^U \cap Q$ satisfies* (14) *and $q \in A^T \cap Q$, then the superposition $q \circ p$ belongs to $A^U \cap Q$.*

Frequently, if ($\gamma_{n,I}$) holds for $I = \{a_1, \ldots, a_n\} \in \boldsymbol{Ind}(Q)$, then taking $U = I$ the condition (δ_n) can be replaced by a weaker condition, which we shall denote in the following by ($\delta_{n,I}$).

One should note also that the conditions (γ_n) and (δ_n) are satisfied if for example, $Q = \boldsymbol{R}$.

Let us verify the following

LEMMA 3. *If a family Q defined on the algebra $\mathfrak{A}$ satisfies the conditions* (γ_n) *and* (δ_n)(²), *then the condition* (β), *occuring in the definition of $\sim_Q$, follows from* (α).

In fact, if for $a_1, \ldots, a_n \in I \in \boldsymbol{Ind}(Q)$ the equality (1) holds, and for some different $b_1, \ldots, b_n \in T \subset A$ the equality (1′) is satisfied, then in virtue of (γ_n) and (δ_n) there exists a mapping $p \in A^I \cap Q$ fulfilling (14), such that $q \circ p \in A^I \cap Q$ for every $q \in A^T \cap Q$. Therefore, in view of Q-independence of the set I, we get

$$\begin{aligned} f\big(q(b_1), \ldots, q(b_n)\big) &= f\big((q \circ p)(a_1), \ldots, (q \circ p)(a_n)\big) \\ &= g\big((q \circ p)(a_1), \ldots, (q \circ p)(a_n)\big) = g\big(q(b_1), \ldots, q(b_n)\big). \end{aligned}$$

Thus the condition (β) is satisfied. ∎

(²) Assumptions (γ_n) and (δ_n) can be replaced by weaker conditions ($\gamma_{n,I}$) and ($\delta_{n,I}$), where I is a Q-independent set of condition (α).

Due to Lemma 3 and condition (γ_n) we have immediately

COROLLARY 8. *Let $f, g \in \boldsymbol{A}^{(n)}$, the family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ satisfies the conditions (γ_n) and (δ_n), and let $I = \{a_1, \ldots, a_n\} \in \boldsymbol{Ind}(\boldsymbol{Q})$*[3]. *Then $f \sim g$ iff the equality* (1) *is satisfied.* ■

It is worth to remark, that, by the assumptions (γ_n) and (δ_n) about the family $\boldsymbol{Q}$ defined on the algebra $\mathfrak{A}$, the fact that $\sim_{\boldsymbol{Q}}$ is the congruence relation in the algebra $\mathfrak{A}^{(n)}$ can be deduced from the following simple observation.

Let $\boldsymbol{X} \subset A^n$ and let a relation $\sim_{\boldsymbol{X}}$ on $\boldsymbol{A}^{(n)}$ be defined by taking

$f \sim_{\boldsymbol{X}} g$ iff for any n-tuple elements $a_1, \ldots, a_n$ the equality (1) holds.

Then $\sim_{\boldsymbol{X}}$ is the congruence relation in $\mathfrak{A}^{(n)}$.

Indeed, from the conditions (γ_n) and (δ_n) satisfied by $\boldsymbol{Q}$ (or from $(\gamma_{n,I})$ and $(\delta_{n,I})$ if in the algebra there exists an appropriate set I) it follows easily, by Lemma 3, that the relation $\sim_{\boldsymbol{Q}}$ coincides with $\sim_{\boldsymbol{X}}$ for $\boldsymbol{X}$ being the set of sequences with n different elements.

Taking into account Lemmas 2 and 3 we have

THEOREM 4. *Let a family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ defined on an algebra $\mathfrak{A}$ satisfies the conditions (γ_n) and (δ_n)*[4]. *Then $\mathfrak{A}$ has an n-element Q-basis if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^{(n)}$.*

Proof. Let $I = \{a_1, \ldots, a_n\}$ be a $\boldsymbol{Q}$-basis of $\mathfrak{A}$ and let $a \in A$. Then there is an $f_a \in \boldsymbol{A}^{(n)}$, such that

$$a = f_a(a_1, \ldots, a_n). \tag{15}$$

If $a = f(a_1, \ldots, a_n)$ for some $f \in \boldsymbol{A}^{(n)}$, then $f \sim f_a$ by Corollary 8. Thus the element $a \in A$ determines uniquely an element of $\mathfrak{A}^{(n)}/\sim$. Define a mapping $h\colon A \to \boldsymbol{A}^{(n)}/\sim$ by $h(a) = [f_a]$, where f_a satisfies (15). If $h(a) = h(b)$, then in consequence of the quoted Corollary we have $f_a(a_1, \ldots, a_n) = f_b(a_1, \ldots, a_n)$, whence $a = b$. It means that h is a injection of A into $\boldsymbol{A}^{(n)}/\sim$. Obviously, it is even a surjection: if $f \in \boldsymbol{A}^{(n)}$, then $f(a_1, \ldots, a_n) = b \in A$, and so there is $b \in A$, such that $h(b) = [f]$. Now let us prove that h is homomorphism. For arbitrary $x_1, \ldots, x_p \in A$ there exist $f_1, \ldots, f_p \in \boldsymbol{A}^{(n)}$ such that $x_i = f_i(a_1, \ldots, a_n)$, in other words $h(x_i) = [f_i]$ for $i = 1, \ldots, p$. From the definition of h and Lemma 2 we obtain

$$\begin{aligned} h\big(f(x_1, \ldots, x_p)\big) &= h\Big(f\big(f_1(a_1, \ldots, a_n), \ldots, f_p(a_1, \ldots, a_n)\big)\Big) \\ &= h\big(\hat{f}(f_1, \ldots, f_p)(a_1, \ldots, a_n)\big) = [\hat{f}(f_1, \ldots, f_p)] \\ &= \tilde{f}([f_1], \ldots, [f_p]) = \tilde{f}\big(h(x_1), \ldots, h(x_p)\big) \end{aligned}$$

for every $f \in \boldsymbol{A}^{(p)}$. Thus $\mathfrak{A} \simeq \mathfrak{A}^{(n)}/\sim$.

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[3] Instead of (γ_n) and (δ_n) we can make weaker assumptions $(\gamma_{n,I})$ and $(\delta_{n,I})$.

[4] See footnote [3].

Conversely, let $\mathfrak{A}$ be isomorphic to $\mathfrak{A}^{(n)}/\sim$, and $t\colon \boldsymbol{A}^{(n)}/\sim \to A$ yields this isomorphism. The trivial operation $e_1^{(n)}, \ldots, e_n^{(n)}$ form an $\boldsymbol{M}$-basis of algebra $\mathfrak{A}^{(n)}$, and so the elements $[e_1^{(n)}], \ldots, [e_n^{(n)}]$ form a generating system of $\mathfrak{A}^{(n)}/\sim$. It is clear that $e_i^{(n)}$ is not $\sim$-equivalent with any operation $e_j^{(n)}$ for $i \neq j$. For suppose

$$e_i^{(n)}(b_1, \ldots, b_n) = e_j^{(n)}(b_1, \ldots, b_n), \tag{16}$$

for some $b_1, \ldots, b_n \in A$, then $b_i = b_j$, so there are no sets $I \in \boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{Q})$, such that for any different $b_1, \ldots, b_n \in I$ the equality (16) holds. In consequence, the condition (α) does not holds. Thus $[e_1^{(n)}], \ldots, [e_n^{(n)}]$ are different elements of $\boldsymbol{A}^{(n)}/\sim$. Now we shaw that the set $J = \{t[e_i^{(n)}];\ i = 1, \ldots, n\}$ is $\boldsymbol{Q}$-independent. Let f, g be arbitrary n-ary algebraic operations. If $f(t[e_1^{(n)}], \ldots, t[e_n^{(n)}]) = g(t[e_1^{(n)}], \ldots, t[e_n^{(n)}])$, then $\tilde{f}([e_1^{(n)}], \ldots, [e_n^{(n)}]) = \tilde{g}([e_1^{(n)}], \ldots, [e_n^{(n)}])$ thus $\hat{f}(e_1^{(n)}, \ldots, e_n^{(n)}) \sim \hat{g}(e_1^{(n)}, \ldots, e_n^{(n)})$. Hence we obtain easily $f \sim g$. Taking account of (β) we observe $J \in \boldsymbol{Ind}(\boldsymbol{Q})$. Therefore J is an n-element $\boldsymbol{Q}$-basis of the algebra $\mathfrak{A}$. ■

COROLLARY 9. *If the family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ defined on an algebra $\mathfrak{A}$ contains the family $\boldsymbol{R}$ of all injective mappings and satisfies (δ_n), then $\mathfrak{A}$ has an n-element $\boldsymbol{Q}$-basis if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^{(n)}/\sim_{\boldsymbol{Q}}$.* ■

COROLLARY 10. *$\mathfrak{A}$ has an n-element $\boldsymbol{R}$-basis iff $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^{(n)}/\sim_{\boldsymbol{R}}$.* ■

Now we prove a theorem analogous to Theorem 2 of [6].

THEOREM 5. *If $\mathfrak{A}$ has an n-element $\boldsymbol{Q}$-basis and an m-element $\boldsymbol{Q}$-basis, and the family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ satisfies (γ_k) and (δ_k) for $k = n, m$*(5)*, then there exist algebraic operations $f_1, \ldots, f_n \in \boldsymbol{A}^{(m)}$ and $g_1, \ldots, g_m \in \boldsymbol{A}^{(n)}$ such that*

$$\hat{f}_i(g_1, \ldots, g_m) \sim e_i^{(n)} \quad \text{in } \mathfrak{A}^{(n)} (i = 1, \ldots, n), \tag{17}$$

$$\hat{g}_j(f_1, \ldots, f_n) \sim e_j^{(m)} \quad \text{in } \mathfrak{A}^{(m)} (j = 1, \ldots, m). \tag{18}$$

Proof. Let $I_1 = \{a_1, \ldots, a_n\}$ and $I_2 = \{b_1, \ldots, b_m\}$ are $\boldsymbol{Q}$-bases of $\mathfrak{A}$. Then there are $f_1, \ldots, f_n \in \boldsymbol{A}^{(m)}$ and $g_1, \ldots, g_m \in \boldsymbol{A}^{(n)}$, such that

$$a_i = f_i(b_1, \ldots, b_m), \quad i = 1, \ldots, n, \tag{19}$$

$$b_j = g_j(a_1, \ldots, a_n), \quad j = 1, \ldots, m. \tag{20}$$

We get

$$f_i\big(g_1(a_1, \ldots, a_n), \ldots, g_m(a_1, \ldots, a_n)\big) = a_i, \quad i = 1, \ldots, n, \tag{21}$$

$$g_j\big(f_1(b_1, \ldots, b_m), \ldots, f_n(b_1, \ldots, b_m)\big) = b_j, \quad j = 1, \ldots, m. \tag{22}$$

Whence we obtain (17) and (18) in virtue of Lemma 3. ■

(5) The theorem remains valid when the algebra $\mathfrak{A}$ and the family $\boldsymbol{Q}$ satisfy the conditions (γ_{n,I_1}), (δ_{n,I_1}) and (γ_{m,I_2}), (δ_{m,I_2}) where I_1, I_2 are n-, and m-element $\boldsymbol{Q}$-bases, respectively.

In the special cases of $\boldsymbol{Q} = \boldsymbol{M}$ or $\boldsymbol{Q} = \boldsymbol{R}$ the assumptions of Theorem 6 are fulfilled. Since every $\boldsymbol{S}$-basis is also an $\boldsymbol{M}$-basis we obtain a theorem for $\boldsymbol{S}$-independence like the preceding theorem.

Now we prove

LEMMA 4. *Let I be a $\boldsymbol{Q}$-independent subset of an algebra $\mathfrak{A}$, and let $b, a_1, \ldots, a_n$ be different elements, all of them, probably except b, belonging to I. It the family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ satisfies (γ_n)*(⁶)*, then $a_1 \notin C(\{a_2, \ldots, a_n\})$.*

Indeed, if there would exist an operation $f \in A^{(n-1)}$ such that

$$a_1 = f(a_2, \ldots, a_n),$$

then considering the mapping p defined by

$$p(a_1) = b, \quad p(a_i) = a_i, \quad i = 2, \ldots, n,$$

which belongs to $A^I \cap \boldsymbol{Q}$ in view (γ_n), we would deduced (using of $\boldsymbol{Q}$-independence of I) $a_1 = b$ countrary to our assumptions. ∎

Hence we obtain immediately

COROLLARY 11. *If a family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ satisfies (γ_{n+1}) and $I = \{a_1, \ldots, a_n\}$ is a $\boldsymbol{Q}$-basis of an algebra $\mathfrak{A}$ with $|A| \geqslant n+2$, then I is a maximal $\boldsymbol{Q}$-independent set.* ∎

COROLLARY 12. *If a family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ satisfies (γ_n) for every n, and I is an arbitrary $\boldsymbol{Q}$-basis of an algebra $\mathfrak{A}$ with an infinite support A, then I is a maximal $\boldsymbol{Q}$-independent set.* ∎

Using Lemma 4 we have also

THEOREM 6. *Let $\mathfrak{A}$ be algebra with an infinite support A, and let a family $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ satisfies (γ_n) for every n. Then $\boldsymbol{Ind}(\boldsymbol{Q}) \subset \boldsymbol{C}$-$\boldsymbol{Ind}$.* ∎

Proof. Suppose, on the contrary, that $I \in \boldsymbol{Ind}(\boldsymbol{Q})$ and I is C-dependent set. Thus there exists an element $a \in I$, such that $a \in C(I \setminus \{a\})$. Since the operator C is of finite character (comp. § 1) there is a finite subset $\{a_1, \ldots, a_n\} \subset I \setminus \{a\}$, such that $a \in C(\{a_1, \ldots, a_n\})$, which is imposible because of Lemma 4, for an infinite set A. ∎

In particular

COROLLARY 13. *If $\mathfrak{A}$ is an algebra with an infinite support and $\boldsymbol{R} \subset \boldsymbol{Q}$, then $\boldsymbol{Ind}(\boldsymbol{Q}) \subset \boldsymbol{C}$-$\boldsymbol{Ind}$, in the algbera $\mathfrak{A}$.* ∎

Let us observe, that if $\boldsymbol{Ind}(\boldsymbol{Q}) \subset \boldsymbol{C}$-$\boldsymbol{Ind}$ in the algebra $\mathfrak{A}$, then every $\boldsymbol{Q}$-basis is a minimal generating system of $\mathfrak{A}$. Thus if $\boldsymbol{Q}$, defined on $\mathfrak{A}$, satisfies (γ_n) for every n, then every $\boldsymbol{Q}$-basis is a minimal generating system and a maximal $\boldsymbol{Q}$-independent set. This is not true for arbitrary family $\boldsymbol{Q}$, for instance: an arbitrary algebraic constant may be added to any $\boldsymbol{G}$-basis.

(⁶) We can assume $(\gamma_{n,I})$ instead of (γ_n).

From Theorem 1.3 (iv) of [13] and the implication: if a family Q satisfies (γ_n) for all n, then a Q-basis is a minimal generating system, we get

COROLLARY 14. *Let a family $Q \subset M(A)$ satisfies the condition (γ_n) for all n, and let an algebra $\mathfrak{A}$ has an finite Q-basis. Then all Q-bases of $\mathfrak{A}$ are of the same cardinality.* ∎

It is worth to remark, that is not true for G-independence. Consider an algebra $\mathfrak{A}$ in which every element from its infinite support A is an algebraic constant. Then, in virtue of (xviii) and Theorem 3(iii) of [17], every subset of A is a G-basis of $\mathfrak{A}$, so there exist a finite and an infinite G-bases. Note that S. Fajtlowicz has shown an algebra $\mathfrak{A}$ without algebraic constants with the above property (this is answers to problem P 603 possed by G. Graetzer in [8]).

Now we prove a theorem analogous to Theorem 2.4 (iii) of [13] using a similar method.

THEOREM 7. (a). *If a family $Q \subset M(A)$ satisfies the condition (γ_n) and sets $I_1 = \{a_1, \ldots, a_n\}$, $I_2 = \{b_1, \ldots, b_n\}$ are Q-independent in the algebra $\mathfrak{A}$, then $C(I_1) \simeq C(I_2)$.*

(b). *If a family $Q \subset M(A)$ contains the family R of all injective mappings defined on the subsets of the algebra $\mathfrak{A}$, and Q-independent subsets I_1 and I_2 have the same cardinality, then the subalgebras generated by I_1, I_2 are isomorphic.*

Indeed, to prove (a) — by virtue of (γ_n), and to prove (b) — by virtue of $R \subset Q$, there exists an injective mapping $p\colon I_1 \xrightarrow[\text{onto}]{} I_2$ belonging to Q, such that $p^{-1} \in A^{I_2} \cap Q$. By Q-independence of the sets I_1 and I_2 the mapping p extends to homomorphism $h\colon C(I_1) \to A$, and the mapping p^{-1} extends to homomorphism $h^*\colon C(I_2) \to A$. We have therefore $h^*(h(x)) = x$ for every $x \in I_1$. Hence we get easily, that h is a bijection and so — the required isomorphism. ∎

It should be noted, that for a given set A and an arbitrary sequence of cardinal numbers smaller than the power A there exist an algebra $\mathfrak{A}$ and a family Q of mappings defined on A, such that this sequence consists of cardinal numbers being the powers of Q-bases of the algebra $\mathfrak{A}$. In fact, we can take for $\mathfrak{A}$ the algebra functionally complete on the set A; what follows easily from the Corollary 2 and from remark, that in the functionally complete algebra every element generates the whole algebra. The above effect was not posible for the M-independence.

It is known, that if the algebra $\mathfrak{A}$ has two M-bases of different powers, then the powers of all M-bases of this algebra are finite and form an arithmetic progresion (see [13], p. 59, [6], p. 159 and [7], pp. 353-4). It is easy to see also that the same holds for S-independence. Making use of the idea of unpublished proof of this theorem (see Corollary 15) due to J. Dudek (in a similar way S. Świerczkowski gets certain weaker result in [22]) we shall prove now an analogous theorem for Q-independence under the same type of assumptions about the family Q as in Theorems 4 and 5. To this aim we shall prove three lemmas.

Let the sets $I_1 = \{a_1, \dots, a_n\}$, $I_2 = \{b_1, \dots, b_m\}$ and $I_3 = \{c_1, \dots, c_{m+r}\}$ be Q-bases of the algebra $\mathfrak{A}$. Further, let $f_1, \dots, f_n, g_1, \dots, g_m$ be algebraic operations satisfying (19) and (20). Let us denote by J the set consisting of the following elements $f_1(c_1, \dots, c_m), \dots, f_n(c_1, \dots, c_m), c_{m+1}, \dots, c_{m+r}$. We shall prove

LEMMA 5. *If a family satisfies conditions* (γ_m) *and* (γ_{m+1})(7), *and besides* $|A| \geqslant m+2$, *then* $|J| = n+r$.

Indeed, if the following equality holds

$$f_i(c_1, \dots, c_m) = f_j(c_1, \dots, c_m) \quad \text{for } i \neq j,$$

then, in virtue of (γ_m), we would have

$$a_i = f_i(b_1, \dots, b_m) = f_j(b_1, \dots, b_m) = a_j$$

in the contrary to our assumption, that the elements $a_1, \dots, a_n$ are different. Taking into account the assumptions $|A| \geqslant m+2$ and (γ_{m+1}) we conclude that the equality

$$f_i(c_1, \dots, c_m) = c_{m+k} \quad \text{for } 0 < k \leqslant r$$

is ruled out due to Lemma 4. Therefore all elements numbered in J are different. ■

LEMMA 6. *If the algebra* $\mathfrak{A}$ *possesses* $n+r$*-element* Q*-independent subset* $I = \{d_1, \dots, d_{n+r}\}$, $|A| \geqslant n+2$ *and a family* $Q \subset M(A)$ *fulfills the conditions* (γ_{n+k}), (γ_{m+k}) *for* $k = 0, 1, r$ *and* (δ_{n+k}) *for* $k = 0, r$(8), *then the above defined set* J *is* Q*-independent.*

Proof. On the basis of (γ_n) and (γ_{n+1}) similarly as in Lemma 5 we conclude that the elements

$$g_1(d_1, \dots, d_n), \dots, g_m(d_1, \dots, d_n), d_{n+1}, \dots, d_{n+r}$$

are different. Taking into account now (γ_{m+r}), we deduce, that there exists a mapping $p \in A^{I_3} \cap Q$ defined by equalities

$$p(c_i) = g_i(d_1, \dots, d_n) \quad \text{for } i = 1, \dots, m,$$
$$p(c_{m+j}) = d_{n+j} \quad \text{for } j = 1, \dots, r,$$

therefore from the equality

$$F_1\big(f_1(c_1, \dots, c_m), \dots, f_n(c_1, \dots, c_m), c_{m+1}, \dots, c_{m+r}\big)$$
$$= F_2\big(f_1(c_1, \dots, c_m), \dots, f_n(c_1, \dots, c_m), c_{m+1}, \dots, c_{m+r}\big)$$

(7) We can assume here (γ_{m,I_3}) and (γ_{m+1,I_3}).

(8) We can assume $(\gamma_{n+k,I})$ and $(\gamma_{m+k,I})$ for $k = 0, 1, r$, (γ_{n,I_1}) and $(\delta_{n+r,I})$, (δ_{n,I_1}).

for certain $F_1, F_2 \in A^{(n+r)}$, by virtue of Q-independence of the set I_3, and (17) of Theorem 5 it follows

$$F_1(d_1, \ldots, d_{n+r}) = F_2(d_1, \ldots, d_{n+r}).$$

Taking into consideration Lemma 5, (γ_{n+r}) and (δ_{n+r}) there exists a mapping $p_1 \in A^I \cap Q$, such that

$$p_1(d_i) = f_i(c_1, \ldots, c_m) \quad \text{for } i = 1, \ldots, n,$$
$$p_1(d_{n+j}) = c_{m+j} \quad \text{for } j = 1, \ldots, r,$$

and $q \circ p_1 \in A^I \cap Q$, for every $q \in A^I \cap Q$. Making use now of the Q-independence of the set I, we have:

$$\begin{aligned} & F_1\big(q(f_1(c_1, \ldots, c_m)), \ldots, q(f_n(c_1, \ldots, c_m)), q(c_{m+1}), \ldots, q(c_{m+r})\big) \\ &= F_1\big((q \circ p_1)(d_1), \ldots, (q \circ p_1)(d_{n+r})\big) = F_2\big((q \circ p_1)(d_1), \ldots, (q \circ p_1)(d_{n+r})\big) \\ &= F_2\big(q(f_1(c_1, \ldots, c_m)), \ldots, q(f_n(c_1, \ldots, c_m)), q(c_{m+1}), \ldots, q(c_{m+r})\big) \end{aligned}$$

for every $q \in A^J \cap Q$. Therefore $J \in \boldsymbol{Ind}(Q)$. ■

From Lemmas 5 and 6 we easily get

LEMMA 7. *Let an algebra $\mathfrak{A}$ has Q-bases of n-, m-, and $(m+r)$ -element, and has an $(n+r)$-element Q-independent set, and besides $|A| \geqslant \max(m, n) + 2$. Moreover let a family Q satisfies the conditions (γ_{m+j}), (γ_{n+j}) for $j = 0, 1, r$, (δ_m) and (δ_{n+k}) for $k = 0, r$*(9)*. Then the algebra $\mathfrak{A}$ possesses also $(n+r)$-element Q-basis.*

Indeed, if the sets $I_1 = \{a_1, \ldots, a_n\}$, $I_2 = \{b_1, \ldots, b_m\}$ and $I_3 = \{c_1, \ldots, c_{m+r}\}$ are Q-bases of the algebra $\mathfrak{A}$, then, in virtue of Lemmas 5 and 6, the $(n+r)$-element set $J = \{f_1(c_1, \ldots, c_m), \ldots, f_n(c_1, \ldots, c_m), c_{m+1}, \ldots, c_{m+r}\}$ is Q-independent. And so, it is sufficient to show that $C(J) = A$. In view of the assumptions (γ_m) and (δ_m), and (18) of Theorem 5, we have

$$\big(g_j(f_1, \ldots, f_n)\big)(c_1, \ldots, c_m) = c_j$$

for every $j = 1, \ldots, m$. Therefore $c_j \in C(J)$, and hence $A = C(I_3) = C(J)$, so J is a Q-basis. ■

Then taking $I_3 = I_1$, we immediately get

COROLLARY 15. *Let an algebra $\mathfrak{A}$ possesses Q-bases of m- and n-elements, where $n = m+r$ for certain $r > 0$, and has $(n+r)$-element Q-independent set, and besides let $|A| \geqslant n+2$. Moreover let the family $Q \subset \boldsymbol{M}(A)$ satisfies the conditions (γ_{m+j}) for $j = 0, 1, r, r+1, 2r$ and (δ_{m+j}) for $j = 0, r, 2r$*(10)*. Then $\mathfrak{A}$ possesses also an $(n+r)$-element Q-basis.* ■

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(9) We can assume $(\gamma_{n+k, I})$ and (γ_{m+k, I_2}) for $k = 0, 1, r$, (γ_{n, I_1}), (γ_{m, I_2}) and $(\delta_{n+r, I})$, (δ_{n, I_1}), (δ_{m, I_2}), where I is an $n+r$-element Q-independent set.

(10) We can assume (γ_{m+k, I_1}), $(\gamma_{m+r+k, I})$ for $k = 0, 1, r$, (γ_{m, I_2}) and $(\delta_{m+2r, I})$, (δ_{m+r, I_1}), (δ_{m, I_2}), where I is an $(m+2r)$-element Q-independent set.

From Lemmas 5-7 follows easily

COROLLARY 16. *If an algebra* $\mathfrak{A}$ *has two finite* $\boldsymbol{Q}$*-bases of less than* $(k-2)$ *elements, besides* $|A| \geqslant k$, *and for all* $n < k$ *there exists an* n*-element* $\boldsymbol{Q}$*-independent set and the conditions* (γ_n) *and* (δ_n) *hold, then powers of* $\boldsymbol{Q}$*-bases, smaller than* k, *form an arithmetic progression.* ∎

We get also

THEOREM 8. *If an algebra* $\mathfrak{A}$ *has two* $\boldsymbol{Q}$*-bases of different powers and for every* n (11) *there exists an* n*-element set* $\boldsymbol{Q}$*-independent with respect to a family* $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ *satisfying* (γ_n), (δ_n) *for every* n, *then all powers of* $\boldsymbol{Q}$*-bases are finite and form an arithmetic progression.*

In fact, finitness of $\boldsymbol{Q}$-bases follows from Corollary 14 and the remaining part of thesis we get from Lemma 7 (using also Corollary 15). ∎

As the family $\boldsymbol{R}$ fulfills the assumptions (γ_n) and (δ_n) for every n, hence we have

COROLLARY 17. *If an algebra has two* $\boldsymbol{R}$*-bases of different powers and for every* n (11) *there exists an* n*-element* $\boldsymbol{R}$*-independent set, then all powers of* $\boldsymbol{R}$*-bases are finite and form an arithmetic progression.* ∎

It should be noted, that the above given theorem on the arithmetic progression does not hold for $\boldsymbol{G}$-independence. Since, by (xviii) it is easy to construct an approprite counter-example.

It would be interesting to know the answer for the following

PROBLEM 2. *What additional conditions can be put directly on the family* $\boldsymbol{Q}$ *to eliminate in Theorem 8 the assumption, that for every* n (11) *there exists an* n*-element* $\boldsymbol{Q}$*-independent set in an algebra* $\mathfrak{A}$*?*

§ 5. Exchange of Q-independent sets

We shall prove now the theorem on exchange of $\boldsymbol{Q}$-independent sets (when the family $\boldsymbol{Q}$ satisfies certain conditions). The theorem on exchange of $\boldsymbol{M}$-independent sets proved by E. Marczewski in [13] (p. 58, Theorem 2.4 (ii)) plays an fundamental role in investigations of $\boldsymbol{M}$-independence. The idea of the proof of this theorem will be partially employed in the proof of the following

THEOREM 9. *Let* $\boldsymbol{Q} \subset \boldsymbol{M}(A)$ *be a family of mappings defined on an algebra* $\mathfrak{A} = (A; \boldsymbol{F})$ *and having the properties:*

(α) $\boldsymbol{H} \subset \boldsymbol{Q}$,

(β) *if* $p \in A^T \cap \boldsymbol{Q}$ *and* $U \subset T$, *then* $p|_U \in \boldsymbol{Q} \cap A^U$,

(11) It is sufficient to assume that $n \geqslant n_0$, where n_0 is the minimal power of $\boldsymbol{Q}$-basis of the algebra $\mathfrak{A}$.

(γ) *if* $T_1 \cap T_2 = \emptyset$, $p_i \in A^{T_i} \cap Q$ *and* $p(x) = p_i(x)$, *for* $x \in T_i$ $(i = 1, 2)$, *then* $p \in A^{T_1 \cup T_2} \cap Q$.

Then we have $I_1 \cup J \in \boldsymbol{Ind}(Q)$, *whenever* $I_0 \cup J \in \boldsymbol{Ind}(Q)$, $I_1 \in \boldsymbol{Ind}(Q)$, $I_0 \cap J = \emptyset$ *and* $C(I_1) = C(I_0)$.

In other words: *Then the theorem on exchange of Q-independent sets holds.*

Proof. Let $I_1, I_0 \cup J \in \boldsymbol{Ind}(Q)$, $C(I_0) = C(I_1)$ and $I_0 \cap J = \emptyset$. One should prove, that every mapping $p \in A^{J \cup I_1} \cap Q$ extends to a homomorphism $C(J \cup I_1)$ into A. Obviously, for $J = \emptyset$ the thesis is fulfilled. Let us take $J \neq \emptyset$. For an arbitrary mapping $p \in A^{(J \cup I_1)} \cap Q$, in view of ($\beta$), we have $p|_{I_1} \in A^{I_1} \cap Q$ and $p|_J \in A^J \cap Q$. Since I_1 is a Q-independent set hence $p|_{I_1}$ extends to a homomorphism h_1: $C(I_1) = C(I_0) \to A$. Taking into account (α) and (β) we infer that $h_1|_{C(I_0) \setminus J} \in Q$. From ($\gamma$) it follows now, that the mapping q: $C(I_0) \cup J \to A$ defined by equalities

$$q(x) = \begin{cases} h_1(x) & \text{for } x \in C(I_0) \setminus J, \\ p(x) & \text{for } x \in J \end{cases}$$

belongs also to Q. In view of (β) we conclude that the mapping $q|_{I_0 \cup J}$ belongs to Q, and because $I_0 \cup J$ is Q-independent set, hence this mapping extends to a homomorphism h: $C(I_0 \cup J) \to A$. Taking into consideration, that the sets I_0 and J are separate, we have $h_1|_{I_0} = h|_{I_0}$. Therefore the homomorphisms h and h_1 coincide also on the subalgebra $C(I_0)$. Since $h_1|_{I_1} = p|_{I_1} = q|_{I_1}$, hence $h(x) = p(x)$ for $x \in I_1 \cup J$. We have proved therefore, that $p \in H$, which implies that set $I_1 \cup J$ is Q-independent. ■

It is worth to note, that the condition (β) of our theorem is, due to (v), sufficient for the family $\boldsymbol{Ind}(Q)$ to be of finite character. But from the condition (γ) the sufficient condition (formulated by E. Marczewski, see (iv)) for hereditary of the family $\boldsymbol{Ind}(Q)$ follows easily.

Moreover, let us notice that the conditions (β) and (γ), apart from the family $\boldsymbol{M}$ of all mappings, are satisfied by the family Q_* consisting of mappings, which are constant outside of a (some) finite set, then however $\boldsymbol{Ind}(Q_*) = \boldsymbol{Ind}(\boldsymbol{M})$, as well as the family Q_1 consisting of a certain mapping p: $A \to A$ and all its restrictions $p|_T$ where $T \subset A$; in this case obviously $\boldsymbol{Ind}(Q_1) \neq \boldsymbol{Ind}(\boldsymbol{M})$.

Let us denote by $\boldsymbol{H}_*$ the minimal family of mappings containing $\boldsymbol{H}$ and satisfying the conditions (β) and (γ). Enclosing to the family $\boldsymbol{H}_*$ all mappings obtained from $\boldsymbol{H}$ by employing the condition (γ) (condition (β) does not give anything new) does not imply yet the equality $\boldsymbol{Ind}(\boldsymbol{H}_*) = \boldsymbol{Ind}(\boldsymbol{M})$. The functionally complete algebra is the simplest example indicating that, as we have for it $\boldsymbol{Ind}(\boldsymbol{M}) = \emptyset$ and $\boldsymbol{Ind}(\boldsymbol{H}_*) = 2^A$. We shall show a less degenerate example. In the ring of integers $\mathfrak{Z} = (Z; +, \cdot)$, considering as an abstract algebra with the ordinary operations "+"

and " · ", every set (in particular a single element) is $\boldsymbol{M}$-independent. But every single element (and every two-element set containing 0) is $\boldsymbol{H}_*$-independent, because the only mappings of $A^{(a)} \cap \boldsymbol{H}_*$ are mappings $p_1(a) = a$ and $p_2(a) = 0$.

II, VARIOUS NOTIONS OF INDEPENDENCE IN v^{**}-ALGEBRAS AND LINEAR SPACES

§ 6. Construction of some family of mappings

In § 3 we have shown, that for every established algebra $\mathfrak{A}$ there exists a family of mappings $\boldsymbol{Q}$, such that the independence with respect to this family is exactly the $\boldsymbol{C}$-independence (Corollary 3). Moreover, in § 4 we gave some conditions, such that for the family $\boldsymbol{Q}$ fulfilling these conditions the independence with respect to $\boldsymbol{Q}$ is stronger than the $\boldsymbol{C}$-independence (Theorem 6).

Now, for established algebra $\mathfrak{A} = (A; \boldsymbol{F})$ we shall construct the family $\boldsymbol{Q}_C \subset \boldsymbol{M}(A)$, such that $\boldsymbol{Q}_C \subset \boldsymbol{S}_0 \cup \boldsymbol{H}$ and the $\boldsymbol{Q}_C$-independence coincide with the $\boldsymbol{C}$-independence for subsets with at least two elements.

Let $\boldsymbol{P}_T$ denotes the family of all mappings $p_{(a,b)}\colon T \to A$ defined for all $a, b \in T$ in the following way:

$$p_{(a,b)} = \begin{cases} x & \text{for } x = a \text{ or } x \notin \boldsymbol{C}(T \setminus \{x\}), \\ b & \text{in other cases.} \end{cases} \tag{23}$$

Let us define the family $\boldsymbol{Q}_C$ taking for every $T \subset A$

$$A^T \cap \boldsymbol{Q}_C = (A_T \cap \boldsymbol{H}) \cup \boldsymbol{P}_T.$$

THEOREM 10. *A subset I of A with at least two elements is $\boldsymbol{C}$-independent if and only if it is $\boldsymbol{Q}_C$-independent. Moreover every one-element set is $\boldsymbol{Q}_C$-independent.*

Proof. From the definition of mappings $p_{(a,b)}$ one can easy see, that every one-element set is $\boldsymbol{Q}_C$-independent.

Let us remark, that from our definitions of families $\boldsymbol{Q}_C$ and $\boldsymbol{P}_T$ follows, that we must only show, that

$$I \in \boldsymbol{C}\text{-}\boldsymbol{Ind} \quad \text{iff} \quad \boldsymbol{P}_I \subset \boldsymbol{H} \cap A^I$$

for every set I with at least two elements.

Let I be $\boldsymbol{C}$-independent. Then $x \notin \boldsymbol{C}(I \setminus \{x\})$ for every $x \in I$. So from the definition of $p_{(a,b)}$ we have $\boldsymbol{P}_I \subset H \cap A^I$.

Now, let I has at least two elements and let I be C-dependent. Then there exist $a_0, a_1, \ldots, a_n \in I$, $a_0 \neq a_i$ for $i = 1, \ldots, n$, and there exists algebraic operation $f \in A^{(n)}$ (for $n \geqslant 1$), such that

$$a_0 = f(a_1, \ldots, a_n).$$

Of course, we can assume, that $a_i \in C(I \setminus \{a_i\})$ for $i = 2, \ldots, k$ and $a_i \notin C(I \setminus \{a_i\})$ for $i = k+1, \ldots, n$. If I is Q_C-independent, then both $p_{(a_0, a_1)}$ and $p_{(a_1, a_1)}$ can be extended to homomorphisms of $C(I)$ into A. Taking into account, that

$$\begin{aligned} p_{(a_0,a_1)}(a_1) &= a_1 = p_{(a_1,a_1)}(a_1), \\ p_{(a_0,a_1)}(a_i) &= a_1 = p_{(a_1,a_1)}(a_i) \qquad \text{for } i = 2, \ldots, k, \\ p_{(a_0,a_1)}(a_i) &= a_i = p_{(a_1,a_1)}(a_i) \qquad \text{for } i = k+1, \ldots, n, \end{aligned}$$

we have

$$f\big(p_{(a_0,a_1)}(a_1), \ldots, p_{(a_0,a_1)}(a_n)\big) = f\big(p_{(a_1,a_1)}(a_1), \ldots, p_{(a_1,a_1)}(a_n)\big).$$

So we would have $a_0 = a_1$, what contradicts the assumption $a_1 \in I \setminus \{a_0\}$. Therefore we proved, that I is Q_C-independent. ■

Remark. Taking P_T^* as the family of all mappings $p_{(a,b)}$ defined by (23) for every $a, b \in A$ and defined Q_C^* in such a way, that for every $T \subset A$:

$$A^T \cap Q_C^* = (A^T \cap H) \cup P_T^*,$$

we can obtain the constructive proof of Corollary 3. In this case we have for an established algebra

$$C\text{-}\boldsymbol{Ind} = \boldsymbol{Ind}(Q_C^*).$$

Indeed, as in the proof of Theorem 10, one can easy see, that C-$\boldsymbol{Ind} \subset \boldsymbol{Ind}(Q_C^*)$. The converse inclusion is obvious for algebras with only one element. As in the proof of Theorem 10 we state, that if I has at least two elements and is Q_C^*-independent, then I is also C-independent. Now, let $|A| \geqslant 2$, $I = \{c\}$, and let I be C-dependent. Then $c \in C(\emptyset)$ and $e_1^{(1)}(c) = c(c)$, where $c(x)$ denotes the constant algebraic operation with value c. Taking into consideration the mapping $p_{(a,a)}$ for $a \neq c$, we have $a = e_1^{(1)}\big(p_{(a,a)}(c)\big) \neq c\big(p_{(a,a)}(c)\big) = c$. So $\{c\}$ is Q_C^*-dependent. ■

From (vii), (ii), the definition of the family P_T, and Theorem 10 we have the following immediate.

Corollary 18. *In any algebra $\mathfrak{A}$, every S_0-independent subset of A, which does not consist of only one element being an algebraic constant, is C-independent.* ■

In algebra, in which $C(\emptyset) = \emptyset$, we can also consider the family P_T^0 of mappings $p_c \colon T \to C(T)$ defined for every $c \in C(\emptyset)$ in the following way:

$$p_c(x) = \begin{cases} x & \text{for } x \notin C(T \setminus \{x\}), \\ c & \text{for } x \in C(T \setminus \{x\}). \end{cases}$$

Let $Q_0 = \bigcup_{T \subset A} P_T^0$.

Similarly to Theorem 10 we can prove

THEOREM 11. *If $\mathfrak{A}$ has at least two algebraical constants, then $\boldsymbol{C}$-$\boldsymbol{Ind}(\mathfrak{A}) = \boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{Q}_0)$.* ■

From the last two theorems we get

COROLLARY 19 ([20], p. 489). *In any algebra, every $\boldsymbol{S}$-independent subset, which has at least two elements, is $\boldsymbol{C}$-independent. Moreover, if an algebra has at least two algebraical constants or has no algebraical constant, then every $\boldsymbol{S}$-independent subset is $\boldsymbol{C}$-independent.*

Indeed. The case $C(\emptyset) = \emptyset$ is obvious. The first part of this Corollary, in view of (ix), easily follows from Corollary 18. The second part follows from Theorem 11, and the definition of mappings p_c (or from (viii)). ■

Thesis of Corollary 19 cannot be strengthen. In fact, in every algebra with only one algebraic constant, this constant forms a one-element $\boldsymbol{S}$-independent set (in virtue by (xiv)).

§ 7. Corollaries concerning v^{**}-algebras and linear spaces

An algebra $\mathfrak{A}$ is called a *v^{**}-algebra* (see [19]), if $\boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{M}) = \boldsymbol{C}$-$\boldsymbol{Ind}(\mathfrak{A})$.

For such algebras, in virtue of Theorem 10, we have a simple

COROLLARY 20. *In the v^{**}-algebras, except one-element sets consisting of an algebraic constant, the following properties are equivalent*

(a) *$\boldsymbol{M}$-independence,*

(b) *$\boldsymbol{S}$-independence,*

(c) *$\boldsymbol{S}_0$-independence,*

(d) *$\boldsymbol{C}$-independence.*

Indeed, from (ix) we have the implications (a) ⇒ (b) ⇒ (c). The implication (c) ⇒ (d) is a consequence of Corollary 18. The lacking implication (d) ⇒ (a) is a consequence of the definition of v^{**}-algebras. ■

For these algebras we get

$$\boldsymbol{Ind}(\boldsymbol{M}) = \boldsymbol{Ind}(\boldsymbol{S}) \setminus \{C(\emptyset)\} = \boldsymbol{Ind}(\boldsymbol{S}_0) \setminus \{\{c\}\colon c \in C(\emptyset)\} = \boldsymbol{C}\text{-}\boldsymbol{Ind}$$

Taking into account the definition of *v^{**}-algebras*, (xvi) and (xviii) we obtain

COROLLARY 21. *In an v^{**}-algebra a set I is $\boldsymbol{G}$-independent iff $I \setminus C(\emptyset)$ is $\boldsymbol{M}$-independent.* ■

In particular, the theses of Corollaries 20 and 21 will be satisfied in the so-called v^*-algebras (see [18]). This class contain linear spaces, and so-called *affine spaces* over an arbitrary field K, i. e. algebras $\mathfrak{A}$ in which

A is a linear space over K and the algebraic operations have the form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i x_i$$

where $\lambda_i \in K$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^{n} \lambda_i = 1$. Obviously, $C(\emptyset) = \emptyset$ for an affine space.

From Corollaries 20 and 21, and (xi) we conclude immediately

Corollary 22. (a). *If $\mathfrak{A}$ is a linear space, then*

$$\boldsymbol{Ind}(\boldsymbol{M}) \cup \{\{0\}\} = \boldsymbol{Ind}(\boldsymbol{S}) = \boldsymbol{Ind}(\boldsymbol{S}_0) = \boldsymbol{S}\text{-}\boldsymbol{Ind} \cup \{\{0\}\},$$

and

$$I \in \boldsymbol{Ind}(\boldsymbol{G}) \Leftrightarrow (I \setminus \{0\}) \in \boldsymbol{Ind}(\boldsymbol{M}); \quad \boldsymbol{Ind}(\boldsymbol{A}_1) = 2^A.$$

(b). *If $\mathfrak{A}$ is an affine space, then*

$$\boldsymbol{Ind}(\boldsymbol{M}) = \boldsymbol{Ind}(\boldsymbol{S}) = \boldsymbol{Ind}(\boldsymbol{S}_0) = \boldsymbol{Ind}(\boldsymbol{G}) = \boldsymbol{C}\text{-}\boldsymbol{Ind}, \ \boldsymbol{Ind}(\boldsymbol{A}_1) = 2^A. \ \blacksquare$$

It would be interesting to know the answer to the following

Problem 3. *For which algebras the $\boldsymbol{S}_0$-independence is equivalent to the $\boldsymbol{S}$-independence for subsets consisting of at least two elements?*

Just we proved that this equivalence holds for v^{**}-algebras. It is true for torsion-free abelian groups (see § 8, Corollary 26) as well as for Boolean algebras and some their reducts (see § 11). We describe yet an example, due to S. Fajtlowicz, of an algebra $\mathfrak{A}$ for which

$$\boldsymbol{Ind}(\boldsymbol{S}) = \boldsymbol{Ind}(\boldsymbol{S}_0).$$

Let $\mathfrak{A}$ be an algebra with $|A| \geqslant 3$, in which the set of algebraic operations consists of operations $f(x_1, \ldots, x_n)$ with the property, that there exists an index $k(f)$, $1 \leqslant k(f) \leqslant n$, such that $f(x_1, \ldots, x_n) = x_{k(f)}$ whenever $|\{x_1, \ldots, x_n\}| \leqslant 2$. It is easy to see that this class of operations is closed under composition. In this algebra $\mathfrak{A}$ all at most two-element subsets of A are $\boldsymbol{Q}$-independent with respect to an arbitrary family $\boldsymbol{Q}$ of mappings, meanwhile the subsets consisting of at least three elements are not $\boldsymbol{S}_0$-independent (the more they are not $\boldsymbol{S}$-independent). The reason is that the mapping $p\colon I \to A$ $(I \subset A)$ defined by

$$p(x) = \begin{cases} a & \text{for } x = a \text{ or } x = b, \\ c & \text{in the other cases,} \end{cases}$$

where a, b, c are different elements of I, has no extension to a homomorphism of $C(I)$ into A. Indeed, we may choose an operation $f \in \boldsymbol{A}^{(3)}$ with $k(f) = 3$, and such that $f(x_1, x_2, x_3) = x_1$ for different x_1, x_2, x_3. Then $f(a, b, c) = a$ and $F(p(a), p(b), p(c)) = f(a, a, c) = c \neq p(a)$. $\blacksquare$

Obviously, this algebra is not v^{**}-algebra, for the set $\{a, b, c\}$ is $\boldsymbol{C}$-independent but not $\boldsymbol{M}$-independent (it is even not $\boldsymbol{S}_0$-independent), and is neither a Boolean algebras nor an abelian group.

It is worth to remark, that Theorem 6 of § 4 for the v^{**}-algebras with an infinite support yields

$$(24)\qquad \boldsymbol{Ind}(\boldsymbol{R}) = \boldsymbol{Ind}(\boldsymbol{M}) = \boldsymbol{C}\text{-}\boldsymbol{Ind}$$

directly from the definition of v^{**}-algebras.

The assumption on the support is necessary, which may be seen in the two-element algebra $(\{a, b\}; f)$ considered in § 4 — it is obviously v^{**}-algebras, but $\boldsymbol{Ind}(\boldsymbol{R}) \neq \boldsymbol{Ind}(\boldsymbol{M})$. However, in consequence of Theorem 17, the equalities (24) holds in all linear spaces (without any assumption on the support).

III. THE INDEPENDENCE NOTIONS IN ABELIAN GROUPS AND QUASI-LINEAR ALGEBRAS

§ 8. S_0- and S-independence in abelian groups

In this paragraph we consider abelian groups as abstract algebras with fundamental operations: a binary, $x+y$, and an unary, $-x$. In a group $\mathfrak{G} = (G; +, -)$ the zero element 0 is the only algebraic constant. It is well known (see [17] and [8]), that G-independence coincides with the linear-independence for subsets not containing 0 in abelian groups. However the notion of $\boldsymbol{A}_1$-independence becomes is not interesting, since it is easily seen (cp. (xi)), that every subset of abelian group G is $\boldsymbol{A}_1$-independence. Now we shall investigate $\boldsymbol{S}$- and $\boldsymbol{S}_0$-independence in abelian groups. Theorems which will be proved in this paragraph are special cases of the respective theorems of the following paragraph but to fascilate the understanding they will be proved directly.

LEMMA 8. *If I is an $\boldsymbol{S}_0$-independent subset of an abelian group $\mathfrak{G}$, then all elements of I has the same finite order or all of them has the infinite order.*

Proof. From Marczewski's Theorem (§ 2) it follows, that in an abelian group $\mathfrak{G}$ the $\boldsymbol{S}_0$-independence of a subset I of G is equivalent to the following property:

For every $p\colon I \to I$ we have

(α) *for every different $a_1, \ldots, a_n \in I$ and arbitrary integers $k_1, \ldots, k_n \in Z$ the equality*

$$(25)\qquad \sum_{i=1}^{n} k_i a_i = 0$$

implies

$$(26)\qquad \sum_{i=1}^{n} k_i p(a_i) = 0.$$

Suppose $I \in \boldsymbol{Ind}(\boldsymbol{S}_0)$, and I contains at least two elements a and b. Considering the mappings $p_a\colon I \to I$ and $p_b\colon I \to I$ defined by equalities

$$p_a(x) = \begin{cases} b & \text{for } x = a, \\ x & \text{for } x \neq a, \end{cases} \qquad p_b(x) = \begin{cases} a & \text{for } x = b, \\ x & \text{for } x \neq b, \end{cases}$$

we infer, that if a has a finite order, then an arbitrary element $b \in I$ has the same order. ∎

Obviously (by (ix)), Lemma 8 is true also for $\boldsymbol{S}$-independence.

We prove

THEOREM 12. *A subset I of an abelian group $\mathfrak{G}$ is $\boldsymbol{S}_0$-independent if and only if for every different $a_1, \ldots, a_n \in I$ and arbitrary $k_1, \ldots, k_n \in Z$ the equality* (25) *implies*

$$(27) \qquad k_i(a-b) = 0 \quad \textit{for } i = 1, \ldots, n \textit{ and arbitrary } a, b \in I.$$

Moreover all elements of $I \in \boldsymbol{Ind}(\boldsymbol{S}_0)$ has the same order.

Proof. Theorem is valid by (xiii) for an one-element subset I. Thus let I be an $\boldsymbol{S}_0$-independent subset containing at least two elements. The $\boldsymbol{S}_0$-independence of I is equivalent to (α) for every $p\colon I \to I$. Let $a \neq b$ be elements of I. If in (25) $k_i = 0$ for $i = 1, \ldots, n$, then equalities (27) hold. Consider mappings $q_j\colon I \to I$ ($j = 0, 1, \ldots, n$) defined below

$$q_0(x) = b, \qquad q_j(x) = \begin{cases} a & \text{for } x = a_j, \\ b & \text{for } x \neq a_j, \end{cases}$$

for every $x \in I$ and $j = 1, \ldots, n$.

From (25) and $\boldsymbol{S}_0$-independence of I we infer

$$mb = 0, \qquad k_j a + (m - k_j) b = 0,$$

where $m = \sum_{i=1}^{n} k_i$. This yields (27).

Conversely, let for different $a_1, \ldots, a_n \in I$ and some $k_i \in Z$ ($i = 1, \ldots, n$) the equality (25) holds. From (27) we have $k_i(a_i - p(a_i)) = 0$ for every $p\colon I \to I$, $i = 1, \ldots, n$. Therefore, by (25), we get

$$0 = -\sum_{i=1}^{n} k_i\big(a_i - p(a_i)\big) = -\sum_{i=1}^{n} k_i a_i + \sum_{i=1}^{n} k_i p(a_i) = \sum_{i=1}^{n} k_i p(a_i).$$

In consequence the condition (α) holds for every $p\colon I \to I$. The second part of our thesis was proven in Lemma 8. ∎

The thesis of Theorem 12 suggests (as was remarked by S. Fajtlowicz) a connection between $\boldsymbol{S}_0$- and $\boldsymbol{G}$-independence in abelian groups. A confirmation of this is our

COROLLARY 23. *A subset I is $\boldsymbol{S}_0$-independent if and only if the set $I - I = \{a - b\colon a, b \in I\}$ is $\boldsymbol{G}$-independent and all elements have a finite order or all of them have an infinite order.*

Proof. Let I be $\boldsymbol{S}_0$-independent and let for some $a_i, b_i \in I$ ($i = 1, \ldots, n$)

the equality

$$\sum_{i=1}^{n} k_i(a_i - b_i) = 0 \tag{28}$$

holds. From Theorem 12 we conclude that all elements of I has the same finite order or their orders are infinite. The algebraic operation $\sum_{i=1}^{n} k_i(x_i - -y_i)$ may be considered as a function of n variables running over $I-I$ as well as a function of $2n$ variables running over I. Let us consider the mappings $p_i\colon I \to I$ $(i = 1, \ldots, n)$ defined by formulas

$$p_i(a_j) = b_j \quad \text{for } j \neq i \ (i \text{ is fixed}),$$
$$p_i(x) = x \quad \text{if } x \neq a_j \text{ for all } j \neq i.$$

Using $\boldsymbol{S}_0$-independence we conclude from (28) that

$$k_i(a_i - b_i) = 0 \quad \text{for every } i = 1, \ldots, n.$$

So we proved the $\boldsymbol{G}$-independence of the set $I-I$.

Conversely, let the set $I-I$ be $\boldsymbol{G}$-independent and all elements of I has the same finite order or all of them have an infinite order, and moreover let (25) holds for every different elements $a_1, \ldots, a_n$. Consider the mappings $q_{(i,a,b)} \in A^{I-I}$ defined by the formulas

$$q_{(i,a,b)}(a_i) = a - b,$$
$$q_{(i,a,b)}(x) = 0 \quad \text{if } x \neq a_i,$$

for arbitrary $a, b \in I$ and $i = 1, \ldots, n$. These mappings are diminishing because the equality $kx = 0$ implies the equalities $kq_{(i,a,b)}(x) = 0$ for every $x \in I-I$, $a, b \in I$ and $i = 1, \ldots, n$, in consequence of the assumptions on the orders of elements from I. By $\boldsymbol{G}$-independence of the set $I-I$ we conclude, that (25) implies (27). Therefore $I \in \boldsymbol{Ind}(\boldsymbol{S}_0)$ by Theorem 12. ■

Considering the mapping $p\colon I \to \boldsymbol{C}(I)$ defined by

$$p(x) = \begin{cases} a & \text{if } x = a_j, \\ 0 & \text{if } x \neq a_j \end{cases}$$

for $a \in I$ with infinite order, we get a completion of Lemma 8:

LEMMA 9. *If I is an $\boldsymbol{S}$-independent subset of an abelian group $\mathfrak{G}$ and (25) holds for some different elements $a_1, \ldots, a_n$ of I, where at least one k_i does not vanish, then all elements of I has the same finite order.* ■

Now we prove

THEOREM 13. *In an abelian group $\mathfrak{G}$ the following conditions are equivalent (I being arbitrary subset of G):*

(a) *I is an S-independent subset,*

(b) *all elements of I have the same finite order or all of them are infinite orders, and $I \in \mathbf{Ind}(G)$,*

(c) *for every different $a_1, \ldots, a_n \in I$ and arbitrary integers $k_1, \ldots, k_n$ the equality* (25) *implies*

$$k_i a = 0, \quad i = 1, \ldots, n \tag{29}$$

for arbitrary $a \in I$,

(d) *the condition* (c) *holds for arbitrary $a \in C(I)$.*

Proof. It is easy to see, that $\boldsymbol{S}$-independence of a subset I in an abelian group $\mathfrak{G}$ is equivalent to the condition (α) for every $p\colon I \to \boldsymbol{C}(I)$.

Let I be an arbitrary $\boldsymbol{S}$-independent subsed of G. Then Lemma 8 applies to I. Now, if $k_i = 0$ for $i = 1, \ldots, n$ in (25), then obviously $k_i a_i = 0$. If there exists $k_i \neq 0$, then all elements of I have the same finite order by Lemma 9. Using (α) for $p\colon I \to C(I)$ we obtain

$$k_i a_i = 0 \quad \text{for every } i = 1, \ldots, n \tag{30}$$

from (25) in the same way as in the proof of Theorem 12 putting $a = a_i$ and $b = \mathbf{0}$. We proved (a) $\Rightarrow$ (b).

Now let I satisfies the condition (b) and let (25) holds. If $k_i = 0$ for every $i = 1, \ldots, n$, then the equalities (29) are fulfilled. However, if there is some $k_{i_0} \neq 0$, then (30) implies that a_{i_0} has a finite order m (common to all elements of I). We conclude from (30), that $m | k_i$ for every $i = 1, \ldots, n$, and this implies immediately (29) for $a \in I$. Thus (b) $\Rightarrow$ (c).

Further, if I satisfies the condition (c), and (25) holds for some different $a_1, \ldots, a_n \in I$ and some integers $k_1, \ldots, k_n$, then (29) holds for $a \in I$ and $i = 1, \ldots, n$. Since every element a of $C(I)$ is of the form

$$a = \sum_{j=1}^{m} l_j b_j \quad \text{for some} \quad b_j \in I,\ l_j \in Z \ (j = 1, \ldots, m)$$

we get

$$k_i a = \sum_{j=1}^{m} l_j k_i b_j = 0,$$

and so the condition (d) is fulfilled.

The last implication (d) $\Rightarrow$ (a) is clear since $p(a_i) \in \boldsymbol{C}(I)$ and so $k_i p(a_i) = 0$ follows from (25) by (d). Thus for every $p\colon I \to C(I)$ the condition (α) is satisfied and whence $I \in \boldsymbol{Ind}(\boldsymbol{S})$. ■

The notion of $\boldsymbol{S}_0$-independence is essentially more general than the notion of $\boldsymbol{S}$-independence in abelian groups, as the following example shows. Let $\mathfrak{G}$ be the direct product of the cyclic group of order 2 and the cyclic group of order 4 i. e. $G = Z_2 \oplus Z_4$. Further let e_i denote the generator of Z_{2i} for $i = 1, 2$. Then it is easy to verify that the set $\{e_1 + e_2, e_2\}$ is $\boldsymbol{S}_0$-independent but not $\boldsymbol{S}$-independent, for $2(e_1 + e_2) - 2e_2 = 0$ but $2e_2 \neq 0$, and therefore the condition (c) of Theorem 13 does not hold. ■

From the condition (b) and (viii) we obtain immediately

Corollary 24. *A subset* $I \in \boldsymbol{Ind}(\boldsymbol{S})$ *of an abelian group is linearly independent whenever* $I \neq \{0\}$. ∎

From Theorem 13 (or from (xiv)) it follows easily, that every one-element subset of an abelian group is $\boldsymbol{S}$-independent. Even more

Corollary 25. *Every one-element subset of an arbitrary group is* $\boldsymbol{S}$*-independent.*

Indeed, the cyclic subgroups generated by an element are abelian and, since every $\boldsymbol{S}$-independent subset of a certain subalgebra is $\boldsymbol{S}$-independent in the whole algebra, we may apply Theorem 13, and we obtain our corollary from (b). ∎

From the condition (α) and Lemma 9 we get easily

Corollary 26. *In a torsion-free abelian group*

$$\boldsymbol{Ind}(\boldsymbol{S}) = \boldsymbol{Ind}(\boldsymbol{S}_0) = \boldsymbol{Ind}(\boldsymbol{M}) \cup \{0\}$$

and

$$I \in \boldsymbol{Ind}(\boldsymbol{G}) \Leftrightarrow (I \setminus \{0\}) \in \boldsymbol{Ind}(\boldsymbol{M}). \quad ∎$$

Since abelian groups are quasi-linear algebras (see § 9), we get $\boldsymbol{Ind}(\boldsymbol{R}) = \boldsymbol{Ind}(\boldsymbol{M})$ in every abelian group, by Theorem 17 of § 9.

§ 9. The $\boldsymbol{S}$-, $\boldsymbol{S}_0$-, $\boldsymbol{G}$-, and $\boldsymbol{R}$-independence in quasi-linear algebras

We shall investigate the notions of $\boldsymbol{S}$-, $\boldsymbol{S}_0$-, $\boldsymbol{G}$-, and $\boldsymbol{R}$-independence in quasi-linear algebras developed as a generalization of abelian groups and linear spaces.

An algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is called *quasi-linear* if the following conditions hold

(a) *A is a subset of a certain abelian group G,*

(b) *for every algebraic operation* $f \in \boldsymbol{A}^{(n)}$ $(n = 1, 2, \ldots)$ *there exist unary (not necessary algebraic) operations* $f_1, \ldots, f_n$ *defined on* A, *such that*

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j), \tag{31}$$

where the summation is the group-operation in G,

(c) *there exists an injective algebraic operation* $q \in \boldsymbol{A}^{(1)}$, *such that the binary operation* $r(x, y) = q(x) - q(y)$ *is algebraic.*

The condition (c) implies that the zero element of G is an algebraic constant in $\mathfrak{A}$.

It was proved in [3], that the class of quasi-linear algebras coincides with the class so-called separable variables algebras introduced by E. Marczewski, in [9].

An algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is called a *separable variables algebra* if for every $k = 1, 2, \ldots$, and for every pair $f, g \in \boldsymbol{A}^{(n)}$ $(n > k)$ there exist algebraic operations $f_0 \in \boldsymbol{A}^{(k)}$ and $g_0 \in \boldsymbol{A}^{(n-k)}$ such that the equality

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

is equivalent to

$$f_0(x_1, \ldots, x_k) = g_0(x_{k+1}, \ldots, x_n).$$

The only separable variables algebras in the class of groups (more general, in the class of n-groups; see [4]) are the abelian groups. In [2] there was investigated a certain property of $\boldsymbol{M}$-independence, which is fulfilled in separable variables algebras.

In our considerations we shall use the following lemma proved in [3]:

LEMMA 10 ([3], p. 163). *Let $\mathfrak{A} = (A; \boldsymbol{F})$ be a quasi-linear algebra. If $g \in \boldsymbol{A}^{(n)}, f \in \boldsymbol{A}^{(n)}$ and f has the form* (31), *and $b = f(0, 0, \ldots, 0)$, then*

$$g\big(f(x_1, \ldots, x_n)\big) = \sum_{j=1}^{n} g\big(f_j(x_j) - f_j(0) + b\big) - (n-1)g(b).$$

This lemma we shall use in a case $g = q$ (where q is the operation from the condition (c) of the definition of quasi-linear algebras).

Taking into account this Lemma and Marczewski's Theorem from the § 2, we have immediately

LEMMA 11. *Let $\mathfrak{A}$ be a quasi-linear algebra and let $\boldsymbol{Q} \subset \boldsymbol{M}(A)$. Then a subset I of A is $\boldsymbol{Q}$-independent iff one of the following conditions is satisfied:*

(β) *for every $p \in A^I \cap \boldsymbol{Q}$, every different $a_1, \ldots, a_n \in I$ and for arbitrary $f, g \in \boldsymbol{A}^{(n)}$ of the form*

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j), \qquad g(x_1, \ldots, x_n) = \sum_{j=1}^{n} g_j(x_j), \tag{31'}$$

where f_j, g_j $(j = 1, \ldots, n)$ are certain unary operations, the equality

$$\sum_{j=1}^{n} f_j(a_j) = \sum_{j=1}^{n} g_j(a_j) \tag{32}$$

implies

$$\sum_{j=1}^{n} f_j\big(p(a_j)\big) = \sum_{j=1}^{n} g_j\big(p(a_j)\big). \tag{33}$$

(γ) *for every $p \in A^I \cap \boldsymbol{Q}$, every different $a_1, \ldots, a_n \in I$, and for arbitrary $f, g \in \boldsymbol{A}^{(n)}$ of the form* (31′) *the equality*

$$\sum_{j=1}^{n} q\big(f_j(a_j) - f_j(0) + b_1\big) - (n-1)q(b_1) = \sum_{j=1}^{n} q\big(g_j(a_j) - g_j(0) + b_2\big) - (n-1)q(b_2)$$

implies

$$\sum_{j=1}^{n} q\big(f_j(p(a_j)) - f_j(0) + b_1\big) - (n-1)q(b_1) = \sum_{j=1}^{n} q\big(g_j(p(a_j)) - g_j(0) + b_2\big) - (n-1)q(b_2),$$

where $b_1 = \sum_{j=1}^{n} f_j(0)$ *and* $b_2 = \sum_{j=1}^{n} g_j(0)$.

(δ) *For every* $p \in A^I \cap \boldsymbol{Q}$, *every different* $a_1, \ldots, a_n \in I$, *and arbitrary* $f, g \in \boldsymbol{A}^{(n)}$ *of the form* (31′) *the equality*

$$\sum_{j=1}^{n} \big[q\big(f_j(a_j) - f_j(0) + b_1\big) - q\big(g_j(a_j) - g_j(0) + b_2\big)\big] - (n-1)[q(b_1) - q(b_2)] = 0$$

implies

$$\sum_{j=1}^{n} \big[q\big(f_j(p(a_j)) - f_j(0) + b_1\big) - q\big(g_j(p(a_j)) - g_j(0) + b_2\big)\big] - (n-1)[q(b_1) - q(b_2)] = 0. \ \blacksquare$$

These lemmas take a simpler form if in the quasi-linear algebra $\mathfrak{A}$ is only one algebraic constant 0. For in this case we have $b_1 = b_2 = 0 = q(b_1) = q(b_2)$. We get

COROLLARY 27. *Let* $\mathfrak{A}$ *be quasi-linear algebra with only one algebraic constant, and let* $\boldsymbol{Q} \subset \boldsymbol{M}(A)$. *Then a subset* I *of* A *is* $\boldsymbol{Q}$*-independent iff*

(γ') *for every* $p \in A^I \cap \boldsymbol{Q}$, *every different* $a_1, \ldots, a_n \in I$, *and for arbitrary* $f, g \in \boldsymbol{A}^{(n)}$ *of the form* (31′) *the equality*

$$\sum_{j=1}^{n} q\big(f_j(a_j) - f_j(0)\big) = \sum_{j=1}^{n} q\big(g_j(a_j) - g_j(0)\big) \tag{32′}$$

implies

$$\sum_{j=1}^{n} q\big(f_j(p(a_j)) - f_j(0)\big) = \sum_{j=1}^{n} q\big(g_j(p(a_j)) - g_j(0)\big). \ \blacksquare$$

We shall prove now the following

THEOREM 14. *Subset* I *of a quasi-linear algebra* $\mathfrak{A}$ *is* $\boldsymbol{S}$*-independent* ($\boldsymbol{M}$*-independent*) *if and only if for every different* $a_1, \ldots, a_n \in I$, *and arbitrary algebraic operations* $f, g \in \boldsymbol{A}^{(n)}$ *of the form* (31′), *the equality* (32) *implies*

$$f_j(x) - f_j(0) = g_j(x) - g_j(0) \quad (j = 1, \ldots, n) \tag{34}$$

for every $x \in \boldsymbol{C}(I)$ (*or* $x \in A$, *respectively*), *and*

$$\sum_{j=1}^{n} f_j(0) = \sum_{j=1}^{n} g_j(0). \tag{35}$$

Proof. Let I be a $\boldsymbol{S}$-independent ($\boldsymbol{M}$-independent) subset of $\mathfrak{A}$. We define the mapings $p_{(j,x)}\colon I \to C(I)$ $(p_{(j,x)}\colon I \to A)$ for $j = 1, \ldots, n$ and every $x \in C(I)$ $(x \in A)$:

$$p_{(j,x)}(y) = \begin{cases} x & \text{for } y = a_j, \\ 0 & \text{for } y \neq a_j. \end{cases}$$

From $\boldsymbol{S}$-independence ($\boldsymbol{M}$-independence) using the condition of Lemma 11 for $\boldsymbol{Q} = \boldsymbol{S}$ $(\boldsymbol{Q} = \boldsymbol{M})$, we get

$$q\big(f_j(x) - f_j(0) + b_1\big) = q\big(g_j(x) - g_j(0) + b_2\big).$$

Taking into consideration the injectivity of the operation q we have

$$f_j(x) - f_j(0) + b_1 = g_j(x) - g_j(0) + b_2$$

for any $x \in C(I)$ $(x \in A$, resp.). Putting $x = 0$ in this equality we get $b_1 = b_2$. Therefore for any $x \in C(I)$ $(x \in A)$ and $j = 1, \ldots, n$ we obtain the equalities (34) and (35).

Conversely, let the equalities (34) and (35) follow from (32), for any $x \in C(I)$ $(x \in A)$ and $j = 1, \ldots, n$. Hence the equalities

$$f_j\big(p(a_j)\big) - f_j(0) = g_j\big(p(a_j)\big) - g_j(0), \quad j = 1, \ldots, n.$$

Summing up these equalities and using (35), we get (33). Therefore, in virtue of the condition (β) of Lemma 11 for $\boldsymbol{Q} = \boldsymbol{S}$ $(\boldsymbol{Q} = \boldsymbol{M})$, the subset I is $\boldsymbol{S}$-independent ($\boldsymbol{M}$-independent), which completes the proof. ∎

It is worth to note, that if $C(\emptyset) = \{0\}$ for a quasi-linear algebra $\mathfrak{A}$, then (35) is always fulfilled.

Let us consider now quasi-linear algebras $\mathfrak{A}$, in which the algebraic operations have the form

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j) + a, \tag{36}$$

where $a \in C(\emptyset)$, and $f_1, \ldots, f_n$ are certain endomorphisms of the group G.

Let us recall the following well-known

THEOREM ([3], p. 163). *Let* $\mathfrak{A} = (A; \boldsymbol{F})$ *be a quasi-linear algebra (a separable variables algebra) satisfying one of the following conditions:*

(d) *A is a finite set,*

(e) *$g(A) = A$ for each non-constant unary algebraic operation g.*

Then A is an abelian group and each operation $f \in \boldsymbol{A}^{(n)}$ is of the form (36) *(where f_j are endomorphisms of A, $j = 1, \ldots, n$).*

In particular, algebras satisfying the assumptions of this theorem are arbitrary reducts of finite abelian groups or of linear spaces, in which there exists an algebraic operation q of the condition (c). It is also worthy to note, that the following lemma (which is a generalization of Lemma 1 of [5]) holds.

LEMMA 12. *If $\mathfrak{A}$ is a quasi-linear algebra, in which $+$ is an algebraic operation, then every algebraic operation has the form* (36), *where f_j are endomorphisms of semi-group $(A; +)$ and $a \in C(\emptyset)$.*

Proof. The idea of this proof is similar to that of the Lemma 1 of [5]. Any algebraic operation in a quasi-linear algebra has the form:

$$f(x_1, \dots x_n) = \sum_{j=1}^{n} f_j'(x_j),$$

where f_j' are certain (not necessaly algebraic) unary operations. Let us take $f_j(x) = f_j'(x) - f_j'(0)$ and denote $f(0, \dots, 0) = a$. Then

$$f(x_1, \dots, x_n) = \sum_{j=1}^{n} \big(f_j'(x_j) - f_j'(0)\big) + a = \sum_{j=1}^{n} f_j(x_j) + a.$$

Obviously, $f_j(0) = 0$, and $f_j(x) + a$ is an algebraic operation in $\mathfrak{A}$. Since the algebra $\mathfrak{A}$ is a quasi-linear algebra, then, by virtue of (b), there exist unary operations h and g, such that

$$f_j(x_1 + x_2) + a = h(x_1) + g(x_2).$$

We have therefore $f_j(x_1) + a = h(x_1) + g(0)$, $f_j(x_2) + a = h(0) + g(x_2)$. Having regard to the fact that $h(0) + g(0) = a$, we get

$$f_j(x_j + x_2) + 2a = h(x_1) + g(x_2) + a = f_j(x_1) + f_j(x_2) + 2a.$$

Thus f_j is an endomorphism of the semigroup $(A; +)$. ∎

From Theorem 14 it follows the corollary being a generalization of Theorem 13.

COROLLARY 28. *If in a quasi-linear algebra $\mathfrak{A}$ every algebraic operation has the form* (36), *then a subset I of A is* **S***-independent* (**M***-independent*) *iff for every different $a_1, \dots, a_n$ of I, and arbitrary algebraic operations*

$$f(x_1, \dots, x_n) = \sum_{j=1}^{n} f_j(x_j) + a, \quad g(x_1, \dots, x_n) = \sum_{j=1}^{n} g_j(x_j) + b,$$

where f_j and g_j $(j = 1, \dots, n)$ are endomorphisms of G, and $a, b \in C(\emptyset)$, from equality

$$\sum_{j=1}^{n} f_j(a_j) + a = \sum_{j=1}^{n} g_j(a_j) + b$$

it follows $a = b$ and for every $j = 1, \dots, n$

$$f_j(x) = g_j(x) \quad \textit{for any } x \in C(I) (x \in A, \textit{ resp.}).$$

Indeed, it follows easily from Theorem 14 because f_j and g_j are endomorphisms of the group G, and so $f_j(0) = 0 = g_j(0)$, and because $a = f(0, \dots, 0)$, $b = g(0, \dots, 0)$. ∎

Let $\mathfrak{A}$ be a quasi-linear algebra, in which every algebraic operation has the form (36). We shall denote by $\boldsymbol{E}(\mathfrak{A})$ the set of those endomorphisms of the group G, by means of which every algebraic operation in the algebra $\mathfrak{A}$ can be expressed in the form given by the formula (36).

From Corollary 28 it follows immediately

COROLLARY 29. *Let $\mathfrak{A}$ be a quasi-linear algebra, in which every algebraic operation has the form* (36), *and any two endomorphisms of $\boldsymbol{E}(\mathfrak{A})$ are commutative. Then the element $a \in A$ forms an S-independent subset iff for arbitrary operations $h_1, h_2 \in \boldsymbol{A}^{(1)}$ of the form*

$$h_1(x) = f(x)+b_1, \quad h_2(x) = g(x)+b_2 \quad (f, g \in \boldsymbol{E}(\mathfrak{A}), b_1, b_2 \in \boldsymbol{C}(\emptyset))$$

the equality

$$f(a)+b_1 = g(a)+b_2$$

implies

$$b_1 = b_2 \quad \text{and} \quad f(c) = g(c) \quad \text{for every } c \in C(\emptyset). \ \blacksquare$$

It is worth to note, that the assumptions of this corollary are e. g. satisfied in abelian groups and for arbitrary reducts of linear space, in which there exists an algebraic operation q fulfilling the condition (c). Every one-element set in an arbitrary algebra $\mathfrak{A}$ is $\boldsymbol{S}_0$-independent and $\boldsymbol{G}$-independent, as follows from (xiii). If $\mathfrak{A}$ is an algebra satisfying the assumptions of Corollary 29 and $\boldsymbol{C}(\emptyset) = \{0\}$, then every subset of $\mathfrak{A}$ is $\boldsymbol{A}_1$-independent. Therefrom by (x) and (xi) (or basing directly on Corollary 29) we have:

COROLLARY 30. *Let $\mathfrak{A}$ be a quasi-linear algebra with only one algebraic constant, in which every algebraic operation has the form* (36) *and every two endomorphisms of $\boldsymbol{E}(\mathfrak{A})$ are commutative. Then every one-element subset of $\mathfrak{A}$ is $\boldsymbol{S}$-independent.* $\blacksquare$

We shall characterize now $\boldsymbol{G}$-independent sets in quasi-linear algebras with only one algebraic constant. This easy result is a generalization of the respective result for abelian groups (see [17]).

THEOREM 15. *Let $\mathfrak{A}$ be a quasi-linear algebra with only one algebraic constant. Then a subset I of A is $\boldsymbol{G}$-independent if and only if for every two algebraic operations $f, g \in \boldsymbol{A}^{(n)}$ of the form* (31′) *and every different $a_1, \ldots, \ldots, a_n \in I$ the equality* (32) *implies*

$$f_j(a_j)-f_j(0) = g_j(a_j)-g_j(0) \quad \text{for } j = 1, \ldots, n.$$

In fact, it suffices to observe, that, in view of (xv), the mappings $p_j\colon T \to A$ defined by the formulas

$$p_j(x) = \begin{cases} 0 & \text{for } x \neq a_j, \\ a_j & \text{for } x = a_j \end{cases}$$

for every $T \subset A$, and $a_j \in T$ $(j = 1, \ldots, n)$, are diminishing in every quasi-linear algebra with only one constant 0. In a similar way as in Theorem 14 using this mappings, we prove our theorem. ∎

In the special case of quasi-linear algebras with only one constant, and with algebraic operations of the form (36), a subset $I \subset A$ is $\boldsymbol{G}$-independent iff the equality (32) (where f_j, g_j are endomorphisms of the group G) implies $f_j(a_j) = g_j(a_j)$ for $j = 1, \ldots, n$.

Up to now there was not given a characterization of G-independent subsets (Problem 4) and $\boldsymbol{S}_0$-independent subsets (Problem 5) in quasi-linear algebras with more than one algebraic constant.

We have only some partial results proceeding the solution of Problem 5, which consist of a generalization of Theorem 12 and Corollary 23. Using Corollary 27 for $\boldsymbol{Q} = \boldsymbol{S}_0$ and a method similar to that which occurs in the proof of Theorem 12, we have

THEOREM 16. *Let $\mathfrak{A}$ be a quasi-linear algebra with only one algebraic constant* 0. *A subset I of A is S_0-independent if and only if for every algebraic operations $f, g \in \boldsymbol{A}^{(n)}$ of the form* (31′) *from the equality* (32′) *for any $a_1, \ldots, a_n \in I$ it follows*

$$q\big(f_j(a)-f_j(0)\big)-q\big(f_j(b)-f_j(0)\big) = q\big(g_j(a)-g_j(0)\big)-q\big(g_j(b)-g_j(0)\big)$$

for arbitrary $a, b \in I$ $(j = 1, \ldots, n)$. ∎

From Theorem 15 and 16 we get

COROLLARY 31. *Let $\mathfrak{A}$ be a quasi-linear algebra with only one algebraic constant* 0. *A subset $I \subset A$ is $\boldsymbol{S}_0$-independent if and only if the set $q(I)-q(I) = \{q(a)-q(b)\colon a, b \in I\}$ is $\boldsymbol{G}$-independent and the following condition holds*:

(+) *if $f\big(q(a)-q(b)\big) = 0$ for some $a, b \in I, f \in \boldsymbol{A}^{(1)}$, then for every $c, d \in I$*

$$f\big(q(c)-q(d)\big) = 0.$$

The proof of this Corollary is similar to the proof of Corollary 23 making use of the fact, that $q(x)-q(y)$ is an algebraic operation in the quasi-linear algebra $\mathfrak{A}$ and that q is an injection. The condition (+) allows to that the mappings $q_{(i,a,b)}\colon q(I)-q(I) \to A$ defined as in the proof of the quoted Corollary (instead $a-b$ we put $q(a)-q(b)$) are diminishing. ∎

Finally, for $\boldsymbol{R}$-independence we have:

THEOREM 17. $\boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{M}) = \boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{R})$ *in every quasi-linear algebra* $\mathfrak{A}$.

Proof. By (i) of § 2 and by Theorem 14, it suffices to show that in non-one-element algebra for $a_1, \ldots, a_n \in I \in \boldsymbol{Ind}(\boldsymbol{R})$ and $f, g \in \boldsymbol{A}^{(n)}$ of the form (31′), the equality (32) implies (34) and (35) for every $x \in A$. Observe that, by (viii), the set I can not contain the element 0. Consider the mapping $p_{10} \in A^I \cap \boldsymbol{R}$ defined by equalities

$$p_{10}(a_n) = 0, \quad p_{10}(x) = x \quad \text{for } x \neq a_n.$$

By $\boldsymbol{R}$-independence of the set I, from (32) we get

$$\sum_{j=1}^{n-1} f_j(a_j)+f_n(0) = \sum_{j=1}^{n-1} g_j(a_j)+g_n(0). \tag{37}$$

The operations $\sum_{j=1}^{n-1} f_j(x_j)+f_n(0)$ and $\sum_{j=1}^{n-1} g_j(x_j)+g_n(0)$ are algebraic operations in $\mathfrak{A}$ since $0 \in \boldsymbol{C}(\emptyset)$. Consider now the mapping $p_{11} \in A^I \cap \boldsymbol{R}$ defined in the folowing way:

$$p_{11}(a_{n-1}) = 0, \quad p_{11}(x) = x \quad \text{for } x \neq a_{n-1}.$$

By $\boldsymbol{R}$-independence of I we obtain from (37):

$$\sum_{j=1}^{n-2} f_j(a_j)+f_{n-1}(0)+f_n(0) = \sum_{j=1}^{n-2} g_j(a_j)+g_{n-1}(0)+g_n(0).$$

After $n-1$ steps we obtain the equality

$$f_1(a_1) + \sum_{j=2}^{n} f_j(0) = g_1(a_1) + \sum_{j=2}^{n} g_j(0). \tag{38}$$

Taking $p_{1n}(a_1) = 0$ and $p_{1n}(x) = x$ for $x \neq a_1$, and using $\boldsymbol{R}$-independence of I, we get (35) from (38). Taking for $x \in A$ a mapping $p_{1x} \in A^I \cap \boldsymbol{R}$ defined by

$$p_{1x}(a_1) = x,$$
$$p_{1x}(y) = y, \quad \text{if} \quad y \neq a_1, x,$$
$$p_{1x}(x) = a_1, \quad \text{if} \quad x \in I$$

we get in the same way

$$f_1(x) + \sum_{j=2}^{n} f_j(0) = f_1(x) + \sum_{j=1}^{n} f_j(0) - f_1(0)$$
$$= g_1(x) + \sum_{j=1}^{n} g_j(0) - g_1(0) = g_1(x) + \sum_{j=2}^{n} g_j(0).$$

Whence, by (35), we have

$$f_1(x) - f_1(0) = g_1(x) - g_1(0)$$

for every $x \in A$.

Similarly considering a sequence of $n-1$ mappings $p_{jk} \in A^I \cap \boldsymbol{R}$ for $k = 0, 1, \ldots, n-j-1, n-j+1, \ldots, n-1$ defined by

$$p_{jk}(a_{n-k}) = 0, \quad p_{jk}(x) = x \quad \text{for} \quad x \neq a_{n-k}, x \in I,$$

and mappings p_{jx} defined by equalities

$$p_{jx}(a_j) = x, \quad p_{jx}(x) = a_j \quad \text{and} \quad p_{jx}(y) = y \quad \text{if } y \neq a_j, x,$$

for every $x \in A$, we get the equalities (34) for every $x \in A$, which ends the proof of our theorem. ■

Taking into account, that abelian groups and linear spaces are quasi-linear algebras we conclude from Theorem 16 immediately

COROLLARY 32. *If* $\mathfrak{A}$ *is an abelian group or a linear space, then*

(39) $$\boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{R}) = \boldsymbol{Ind}(\mathfrak{A}, \boldsymbol{M}). \blacksquare$$

It would be interesting to characterize the algebras with the property (39) (Problem 6). It is worth to add, that this equality is fulfilled in Boolean algebras too (see Theorem 23).

IV. VARIOUS NOTIONS OF INDEPENDENCE IN BOOLEAN ALGEBRAS AND SOME THEIR REDUCTS

§ 10. Additional notations, and some known results

In this chapter we will deal with the Boolean algebra $\mathfrak{B} = (B; \cup, \cap, ', 0, 1) = (B; \cup, ') = (B; \cap, ')$ and its reducts $\mathfrak{B}_1 = (B; \cup, \setminus) = B; \cup, \dot{-}) = (B; \dot{-}, \setminus) = (B; \cup, \cap, \dot{-}, \setminus, 0)$, $\mathfrak{B}_2 = (B; \cup, \cap)$ and $\mathfrak{B}_3 = (B; \setminus) = (B; \cap, \setminus, 0)$ where $\cup, \cap, ', \setminus, \dot{-}$ are denoting operations of join, meet, complement ation, subtraction and symmetric subtraction respectively. It is clear, that the algebras $\mathfrak{B}_2$ and $\mathfrak{B}_3$ are also reducts of $\mathfrak{B}_1$.

Using notation $x = x^1$ and $x' = x^0$ for $x \in B$ we can define the atom with respect to the elements $x_1, \ldots, x_n \in B$ indexed by the sequence $(i_1, \ldots, i_n)$, where each $i_k = 0$ or 1, by the equality

(40) $$A_{(i_1, \ldots, i_n)}(x_1, \ldots, x_n) = \bigcap_{k=1}^{n} x_k^{i_k}.$$

When $i_1, \ldots, i_n$ are fixed, and $x_1, \ldots, x_n$ are variable, then the atom can be treaten as the n-ary operation in the set B.

Denoting by 2^n the set of all n-tuples of the numbers 0 and 1, we can define Boolean polynomials as follows:

(41) $$A_J(x_1, \ldots, x_n) = \bigcup_{(i_1 \ldots i_n) \in J} A_{(i_1, \ldots, i_n)}(x_1, \ldots, x_n)$$

for $\emptyset \neq J \subset 2^n$, and

(42) $$A_\emptyset(x_1, \ldots, x_n) = 0.$$

So we have (see [12]) for $J, J_1, J_2 \subset 2^n$:

(43) $$A_J(i_1, \ldots, i_n) = \begin{cases} 0 & \text{for } (i_1, \ldots, i_n) \notin J, \\ 1 & \text{for } (i_1, \ldots, i_n) \in J, \end{cases}$$

(44) $$A_{J_1} \cup A_{J_2} = A_{J_1 \cup J_2}$$

(45) $$A_{J_1} \cap A_{J_2} = A_{J_1 \cap J_2}$$

(46) $$A_{J_1} \setminus A_{J_2} = A_{J_1 \setminus J_2}$$

(47) $$A_{J_1} \dot{-} A_{J_2} = A_{J_1 \dot{-} J_2}$$

(48) $$(A_J)' = A_{J'} \text{ (where } J' = 2^n \setminus J),$$

(49) $$A_{J_1} = A_{J_2} \text{ iff } J_1 = J_2.$$

We recall the following well-known description of algebraic operations in considered algebras.

THEOREM (E. Marczewski, [12]). (a) *Operations A_J form exactly the set of all algebraic operations in the algebra $\mathfrak{B}$. 0 and 1 are the only algebraic constants in $\mathfrak{B}$.*

(b) *Any algebraic operation in $\mathfrak{B}_2$ is an algebraic operation in $\mathfrak{B}_1$, and any algebraic operation in $\mathfrak{B}_1$ is algebraic in $\mathfrak{B}$.*

(c) *A function of n variables is an algebraic operation in $\mathfrak{B}_1$ iff it is of the form A_J, for J not constaining of the n-tuple $(0, \ldots, 0)$. 0 is the only algebraical constant in $\mathfrak{B}_1$.*

(d) *The n-ary function is an algebraic operation in $\mathfrak{B}_2$ iff it is of the form A_J, for non-empty set J not containning the sequence $(0, \ldots, 0)$, and for which the following condition holds:*
if $(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_n) \in J$, then $(i_1, \ldots, i_{k-1}, 1, i_{k+1}, \ldots, i_n) \in J$. In particular, we have always $(1, \ldots, 1) \in J$.

(e) *If $(i_1, \ldots, i_n)$ is the non-constant sequence, then $A_{(i_1, \ldots, i_n)}$ is the symmetric subtraction of two different algebraic operations in $\mathfrak{B}_2$.*

Let us consider the algebra $\mathfrak{B}_3 = (B; \setminus) = (B; \cap, \setminus, 0)$. We have

(f) *0 is the only algebraic constant in $\mathfrak{B}_3$, and $(0, \ldots, 0) \notin J$ for each $A_J \in A^{(n)}(\mathfrak{B}_3)$. The atom (40) is an algebraic operation in $\mathfrak{B}_3$ if and only if not all terms $i_1, \ldots, i_n$ are equal 0.*

Proof of (f). The first part of (f) is the consequence of the facts that $\mathfrak{B}_3$ is a reduct of $\mathfrak{B}_1$ and $0 = x \setminus x$. When (40) is an atom for which non every index i_k is equal 0, then there exist $m \geqslant 1$ and the permutation $(k_1, \ldots, k_n)$ of the set $\{1, \ldots, n\}$, such that $i_{k_1} = i_{k_2} = \ldots = i_{k_m} = 1$ and $i_{k_{m+1}} = \ldots = i_{k_n} = 0$. Then we have

$$A_{(i_1, \ldots, i_n)}(a_1, \ldots, a_n) = \Big(\ldots \Big(\Big(\bigcap_{j=1}^{m} a_{k_j}\Big) \setminus a_{k_{m+1}}\Big) \setminus \ldots\Big) \setminus a_{k_n}.$$

So $A_{(i_1, \ldots, i_n)} \in A^{(n)}(\mathfrak{B}_3)$. ∎

Let us also note, that the algebra $\mathfrak{B}_3$ is an essential reduct of $\mathfrak{B}_1$ because the join operation can not be defined with the aid of the meet and subtraction.

§ 11. Various notions of independence in regular reducts of Boolean algebra

We shall study now the various notions of independence in some reducts of Boolean algebras $\mathfrak{B}$ among which algebras $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ are particular cases.

For any reduct $\mathfrak{B}_r$ of the Boolean algebra $\mathfrak{B}$ we shall introduce the following notation:

$$S_r^{(n)} = \bigcup \{J\colon A_J \in \boldsymbol{A}^{(n)}(\mathfrak{B}_r)\},$$
$$T_r^{(n)} = S_r^{(n)} \setminus \bigcap \{J\colon A_J \in \boldsymbol{A}^{(n)}(\mathfrak{B}_r)\}$$

and

$$T_r = \bigcup_{n=1}^{\infty} T_r^{(n)}.$$

In particular, for the reducts $\mathfrak{B}_1$ and $\mathfrak{B}_3$: $S_1^{(n)} = 2^n \setminus \{(0, \ldots, 0)\} = S_3^{(n)} = T_1^{(n)} = T_3^{(n)}$, and for the reduct $\mathfrak{B}_2$: $S_2^{(n)} = S_1^{(n)}$, $T_2^{(n)} = 2^n \setminus \setminus \{(0, \ldots, 0), (1, \ldots, 1)\}$ (in virtue of (c), (d) and (f) of the previous paragraph).

We shall call $\mathfrak{B}_r$ to be a *regular reduct of the Boolean algebra* $\mathfrak{B}$, if it has the following two properties:

(I) $T_r \neq \emptyset$,

(II) for each sequence $(i_1, \ldots, i_n) \in T_r$ there exist two different operations A_{J_1} and A_{J_2} algebraic in $\mathfrak{B}_r$, such that

$$A_{(i_1, \ldots, i_n)} = A_{J_1} \dot{-} A_{J_2}. \tag{50}$$

It is worth to remark, that the condition (a) is equivalent with existence in $\mathfrak{B}_r$ of two different algebraic operation which essentialy depend on the same variables.

By C_r we shall denote the algebraic closure in $\mathfrak{B}_r$.

From the quoted in § 10 Theorem of Marczewski it follows, that algebras $\mathfrak{B}$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are regular reducts, and from (f) (considering the symmetric subtraction $A_{(i_1, \ldots, i_n)} \dot{-} 0$), we also deduce, that $\mathfrak{B}_3$ is a regular reduct of the Boolean algebra $\mathfrak{B}$.

Firstly we can state the following

THEOREM 18. *Let* $\mathfrak{B}$ *be the fixed Boolean algebra. Then* $\boldsymbol{Ind}(\mathfrak{B}_1, \boldsymbol{Q}) = \boldsymbol{Ind}(\mathfrak{B}_3, \boldsymbol{Q})$ *for* $\boldsymbol{Q} \subset \boldsymbol{M}(B)$.

Proof. It is clear, that $\boldsymbol{Ind}(\mathfrak{B}_1, \boldsymbol{Q}) \subset \boldsymbol{Ind}(\mathfrak{B}_3, \boldsymbol{Q})$, because $\mathfrak{B}_3$ is a reduct of $\mathfrak{B}_1$. Now suppose that $I \notin \boldsymbol{Ind}(\mathfrak{B}_1, \boldsymbol{Q})$. Then there exist algebraic operations A_{J_1} and A_{J_2} in $\mathfrak{B}_1$, the mapping $p \in \boldsymbol{Q} \cap A^I$, and elements $a_1, \ldots, a_n \in I$, such that

$$A_{J_1}(a_1, \ldots, a_n) = A_{J_2}(a_1, \ldots, a_n)$$

and

$$A_{J_1}(p(a_1), \ldots, p(a_n)) \neq A_{J_2}(p(a_1), \ldots, p(a_n)).$$

So we have, from (47), in the same time $A_{J_1 \dot{-} J_2}(a_1, \ldots, a_n) = 0$ and $A_{J_1 \dot{-} J_2}(p(a_1), \ldots, p(a_n)) \neq 0$. Then there exists the atom $A_{(i_1, \ldots, i_n)}$, where $(i_1, \ldots, i_n) \in J_{1 \dot{-} J_2}$, such that

$$A_{(i_1, \ldots, i_n)}(a_1, \ldots, a_n) = 0 \tag{51}$$

and

$$A_{(i_1, \ldots, i_n)}(p(a_1), \ldots, p(a_n)) \neq 0.$$

Since, by (c) not all indices $i_1, \ldots, i_n$ are equal to 0, hence from (f) the atom $A_{(i_1, \ldots, i_n)}$ is an algebraic operation in $\mathfrak{B}_3$. So $I \notin \boldsymbol{Ind}(\mathfrak{B}_3, Q)$. ■

In the paper [12] the $\boldsymbol{M}$-independence in algebras $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ had been investigated. The common generalization of this results is given in the following theorem.

THEOREM 19. *A subset $I \subset B$ is $\boldsymbol{M}$-independent in the regular reduct $\mathfrak{B}_r$, if and only if,*

$$A_{(i_1, \ldots, i_n)}(a_1, \ldots, a_n) \neq 0$$

for each sequence $(i_1, \ldots, i_n) \in T_r$, and for every different $a_1, \ldots, a_n \in I$.

Proof. The idea of this proof is implicite contained in the proofs of Theorems 4(i), 4(ii) and 4(iii) of [12].

Suppose, that there exist different elements $a_1, \ldots, a_n$ of I and sequence $(i_1, \ldots, i_n) \in T_r$, such that

$$A_{(i_1, \ldots, i_n)}(a_1, \ldots, a_n) = 0.$$

Taking into account, that $\mathfrak{B}_r$ is a regular reduct, we have two different algebraic operations A_{J_1} and A_{J_2} for which (50) holds. Therefore

$$A_{J_1}(a_1, \ldots, a_n) = A_{J_2}(a_1, \ldots, a_n), \tag{52}$$

and so I is $\boldsymbol{M}$-dependent in $\mathfrak{B}_r$.

Now suppose, that every atom $A_{(i_1, \ldots, i_n)}$ with respect to elements belonging to I is non-void for $(i_1, \ldots, i_n) \in T_r$. If for any different $a_1, \ldots, a_n \in I$, and $A_{J_1}, A_{J_2} \in \boldsymbol{A}^{(n)}(\mathfrak{B}_r)$ the equality (52) holds, then, from (47), we have

$$A_{J_1 \dot{-} J_2}(a_1, \ldots, a_n) = 0.$$

Let us see, that if $(i_1, \ldots, i_n) \notin T_r$, then for every $A_{J_1}, A_{J_2} \in \boldsymbol{A}^{(n)}(\mathfrak{B}_r)$ also $(i_1, \ldots, i_n) \notin J_1 \dot{-} J_2$. Because for every non-empty subset $J \subset T_r$

$$A_J(a_1, \ldots, a_n) \neq 0.$$

So we obtain $J_1 \dot{-} J_2 = \emptyset$, $J_1 = J_2$, and from (49) the set I is $\boldsymbol{M}$-independent. ■

In the following theorem we shall give the characterization of S- and S_0-independence in regular reducts of the Boolean algebra.

THEOREM 20. *Let $\mathfrak{B}_r$ be a regular reduct of the Boolean algebra $\mathfrak{B}$. Then for every subset $I \subset B$ the following conditions are equivalent:*

(α) *I is M-independent or $I = \{c\}$ where $c \in C_r(\emptyset)$,*

(β) *I is S-independent or $I = \{c\}$ where c is an algebraic constant in $\mathfrak{B}_r$, such that $c' \in C_r(\emptyset)$,*

(γ) *I is S_0-independent.*

Proof. It is easy to see, that every subset containing only one element a is M-independent (so also S-independent) in $\mathfrak{B}_r$, whenever $a \notin C_r(\emptyset)$. Now, if $c \in C_r(\emptyset)$, then $\{c\}$ is M-dependent in $\mathfrak{B}_r$, but it is S-independent in the case $c' \notin C_r(\emptyset)$ (see § 2 (xiv)). However each one-element subset is S_0-independent (cp. § 2 (xiii)).

Now, let I be a set with at least two elements. For such I, by (ix) of § 2, we have implications $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$. So, by Theorem 19 it is sufficient to show, that if I is S_0-independent, then for every sequence $(i_1, \ldots, i_n) \in T_r$ and each $a_1, \ldots, a_n \in I$ the atom $A_{(i_1,\ldots,i_n)}(a_1, \ldots, a_n)$ is non-void.

From quoted in § 2 Theorem of Marczewski and from (47), the S_0-independence of the subset I in the algebra $\mathfrak{B}_r$ is equivalent with the following property:

for every mapping $p\colon I \to I$, each different elements $a_1, \ldots, a_n \in I$, and for every $A_{J_1}, A_{J_2} \in A^{(n)}(\mathfrak{B}_r)$ the equality

$$A_{J_1 \dot{-} J_2}(a_1, \ldots, a_n) = 0 \tag{53}$$

implies

$$A_{J_1 \dot{-} J_2}\big(p(a_1), \ldots, p(a_n)\big) = 0\,. \tag{54}$$

Let I be the S_0-independent set. Assume *a contrario*, that I is M-dependent. So from Theorem 19 there exist different elements $a_1, \ldots, a_n \in I$ and sequence $(i_1, \ldots, i_n) \in T_r$, such that (51) holds. By the assumption of regularity of $\mathfrak{B}_r$, and from (47), there exist two different algebraic operations A_{J_1}, A_{J_2}, such that

$$A_{J_1 \dot{-} J_2}(a_1, \ldots, a_n) = A_{(i_1,\ldots,i_n)}(a_1, \ldots, a_n) = 0\,.$$

Because $|I| \geqslant 2$, without loosing generality we can assume that $n \geqslant 2$. Let us consider two mappings $p_1, p_2\colon I \to I$ defined as follows:

$$p_1(x) = \begin{cases} a_1 & \text{for } x = a_j \text{ if } i_j = i_1, \\ a_2 & \text{in other cases,} \end{cases} \tag{55}$$

$$p_2(x) = \begin{cases} a_2 & \text{for } x = a_j \text{ if } i_j = i_1, \\ a_1 & \text{in other cases.} \end{cases} \tag{56}$$

Taking into account S_0-independence of I, we have

$$A_{(i_1,\ldots,i_n)}(p_i(a_1),\ldots,p_i(a_n)) = 0 \quad (i = 1, 2).$$

Thus $a_1 \cap a_2 = 0 = a_2 \cap a_1'$. Hence we have $a_1 \cap a_2 = a_1$ and $a_1 \cap a_2 = a_2$ what contradicts the assumption that a_1 and a_2 are different. So $I \in \boldsymbol{Ind}(\boldsymbol{M})$. ∎

Because $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ fulfil the assumptions of Theorem 20, we obtain the following corollaries

COROLLARY 33. *In the Boolean algebra* $\mathfrak{B}$:

$$\boldsymbol{Ind}(\boldsymbol{M}) = \boldsymbol{Ind}(\boldsymbol{S}) = \boldsymbol{Ind}(\boldsymbol{S}_0)\setminus\{\{0\},\{1\}\}. \ \blacksquare$$

COROLLARY 34. *In the algebra* $\mathfrak{B}_1$ (*and* $\mathfrak{B}_3$):

$$\boldsymbol{Ind}(\boldsymbol{M}) = \boldsymbol{Ind}(\boldsymbol{S})\setminus\{\{0\}\} = \boldsymbol{Ind}(\boldsymbol{S}_0)\setminus\{\{0\}\}. \ \blacksquare$$

COROLLARY 35. *In the algebra* $\mathfrak{B}_2$:

$$\boldsymbol{Ind}(\boldsymbol{M}) = \boldsymbol{Ind}(\boldsymbol{S}) = \boldsymbol{Ind}(\boldsymbol{S}_0). \ \blacksquare$$

In a regular reduct $\mathfrak{B}_r$ the notions of $\boldsymbol{M}$-, $\boldsymbol{S}$- and $\boldsymbol{S}_0$-independence coincide for sets with at least two elements. From Corollaries 33-35 and Theorem 4(vi) of [12] it follows that this notions coincide in the algebras $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ (on the same set B) for infinite sets.

Analogously to Theorem 4(iv) of [12], from Theorems 19 and 20 we can obtain the following

COROLLARY 36. *In the family of all subsets of a given set* X, *treaten as a regular reduct* $\mathfrak{B}_r$ *of the Boolean algebra* 2^X, *in which* $C_r(\emptyset) = \emptyset$ *there exist* n $\boldsymbol{M}$-*independent* ($\boldsymbol{S}$-*independent*, $\boldsymbol{S}_0$-*independent*) *sets if and only if* $|X| \geqslant |T_r^{(n)}|$. ∎

Now let us characterize the $\boldsymbol{G}$-independence in regular reducts.

THEOREM 21. *In the regular reduct* $\mathfrak{B}_r$ *of the Boolean algebra* $\mathfrak{B}$, *subset* $I \subset B$ *is* $\boldsymbol{G}$-*independent if and only if* $I\setminus C_r(\emptyset)$ *is an* $\boldsymbol{M}$-*independent set.*

Proof. From (xviii) and (ii) it is sufficent to show, that if $I \cap C_r(\emptyset) = \emptyset$ and I is $\boldsymbol{G}$-independent, then I is $\boldsymbol{M}$-independent. Because in $\mathfrak{B}_r$ there are no non-constant self-dependent elements, then we can take into account sets with at least two elements.

Let I be a $\boldsymbol{G}$-independent set, with at least two elements and with no an algebraic constant. Let us assume *a contrario*, that I is $\boldsymbol{M}$-dependent. Then there exist, by Theorem 19, different elements $a_1, \ldots, a_n \in I$ ($n \geqslant 2$) and sequence $(i_1, \ldots, i_n) \in T_r$, such that the equality (51) is fulfilled. As in the proof of previous theorem using the mappings p_1 and p_2 defined by (55) and (56), which, by (xv) of § 2 are diminishing, we obtain the contradiction. So $I \in \boldsymbol{Ind}(\boldsymbol{M})$. ∎

As an easy corollary from Theorem 21 we have

COROLLARY 37. *In a regular reduct* $\mathfrak{B}_r$ *of the Boolean algebra of all subsets of* X, *there exist* n $\boldsymbol{G}$-*independent sets iff* $|X| \geqslant T_r^{(n)}| - |C_r(\emptyset)|$. ∎

Because $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ are regular reducts, from Theorem 21 we obtain

COROLLARY 38. 1) *In the algebra* $\mathfrak{B}$:

$$I \in \boldsymbol{Ind}(G) \Leftrightarrow I \setminus \{0, 1\} \in \boldsymbol{Ind}(M).$$

2) *In the algebra* $\mathfrak{B}_1$ (*and* $\mathfrak{B}_3$):

$$I \in \boldsymbol{Ind}(G) \Leftrightarrow I \setminus \{0\} \in \boldsymbol{Ind}(M).$$

3) *In the algebra* $\mathfrak{B}_2$:

$$\boldsymbol{Ind}(G) = \boldsymbol{Ind}(M). \ \blacksquare$$

Let us characterize A_1-independence in regular reducts of Boolean algebra.

THEOREM 22. *In a regular reduct* $\mathfrak{B}_r$ *of the Boolean algebra* $\mathfrak{B}$, *the subset* $I \subset B$ *is* A_1-*independent if and only if the following conditions hold*:

(δ) *if* $(1, \ldots, 1) \in T_r$ *and* $1 \in C_r(\emptyset)$, *then*

(57) $$A_{(1,\ldots,1)}(a_1, \ldots, a_n) \neq 0$$

for every different $a_1, \ldots, a_n \in I$,

(ζ) *if* $(0, \ldots, 0) \in T_r$ *and* $0 \in C_r(\emptyset)$, *then*

(58) $$A_{(0,\ldots,0)}(a_1, \ldots, a_n) \neq 0$$

for every different $a_1, \ldots, a_n \in I$,

(η) *if the operation* $x \to x'$ *is an algebraic one in* $\mathfrak{B}_r$, $(i_1, \ldots, i_n) \in T_r$ *and for some* $a_1, \ldots, a_n \in I$ *the equality*

(59) $$A_{(i_1,\ldots,i_n)}(a_1, \ldots, a_n) = 0$$

holds, then

(60) $$A_{(1-i_1,\ldots,1-i_n)}(a_1, \ldots, a_n) = 0.$$

Proof. Taking into account Theorem of Marczewski (see § 2) and (47) we infer, that the A_1-independence of the subset I of B is equivalent with the following condition:

for every unary operation $p \in \boldsymbol{A}^{(1)}(\mathfrak{B}_r)$, *each different* $a_1, \ldots, a_n \in I$, *and for every* $A_{J_1}, A_{J_2} \in \boldsymbol{A}^{(n)}(\mathfrak{B}_r)$, *the equality* (53) *implies* (54).

Let the set I fulfil (δ), (ζ) and (η). Obviously if for every $a_1, \ldots, a_n \in I$ and each sequence $(i_1, \ldots, i_n) \in T_r$ the inequality

$$A_{(i_1,\ldots,i_n)}(a_1, \ldots, a_n) \neq 0$$

holds, then, in view of Theorem 19, the set I is M-independent, and so, by (ii) of § 2, it is A_1-independent. Let now for some $a_1, \ldots, a_n$ of I, and $A_{J_1}, A_{J_2} \in \boldsymbol{A}^{(n)}(\mathfrak{B}_r)$ the equality (53) holds. If the operation $p(x) = x'$ is algebraic in $\mathfrak{B}_r$, then, in virtue of (η), p preserves (53). The conditiosn (δ) and (ζ) imply, that if (53) holds and $1 \in C_r(\emptyset)$ or $0 \in C_r(\emptyset)$, then $(1,\dots,1) \notin J_1 \dotdiv J_2$ or $(0,\dots,0) \notin J_1 \dotdiv J_2$, respectively. It gives us that, the mappings $x \to 1$ (if $1 \in C_r(\emptyset)$) and $x \to 0$ (if $0 \in C_r(\emptyset)$) preserve (53). So for every unary operation p algebraic in $\mathfrak{B}_r$ from the equality (53) follows (54), what proves A_1-independence of I.

Conversely, let I be A_1-independent. If $1 \in C_r(\emptyset)$, $(1,\dots,1) \in T_r$ and $A_{(1,\dots,1)}(a_1,\dots,a_n) = 0$ for some $a_1,\dots,a_n \in I$, then there exist algebraic operations A_{J_1}, A_{J_2} in $\mathfrak{B}_r$, such that the equalities (50) and (53) hold. Taking the mapping $p(x) = 1$ we state, that the set I can not be A_1-independent. In the same way one can prove, that (ζ) is also the necessary condition for A_1-independence of I. Now, let $p(x) = x'$ be algebraic operation in $\mathfrak{B}_r$ and for some sequence $(i_1,\dots,i_n) \in T_r$ and some $a_1,\dots,a_n \in I$ the equality (59) holds, and in the same time

$$A_{(1-i_1,\dots,1-i_n)}(a_1,\dots,a_n) \neq 0,$$

then considered mapping $p(x) = x'$ leads to a contradiction with the assumption of A_1-independence of the set I. Therefore, the conditions (δ), (ζ) and (η) are necessary for the A_1-independence. ∎

Let us see, that if the operation $x \to x'$ is algebraic in $\mathfrak{B}_r$ and $1 \in C_r(\emptyset)$, then the conditions (δ) and (η) imply (ζ). Using the part (a) of Marczewski's Theorem, recalled in § 10, we have:

COROLLARY 39. *A subset I of B is A_1-independent in the Boolean algebra $\mathfrak{B}$, if and only if, both the following conditions hold:*

(δ') *for every $a_1,\dots,a_n \in I$ the equality* (57) *holds,*

(η') *if some $a_1,\dots,a_n \in I$ fulfil* (59), *then also* (60). ∎

From this we have next corollary, that to the A_1-independent set, in the Boolean algebra $\mathfrak{B}$, two disjoint elements (in particular an element a and its complement a) can not belong.

It is worth also to remark, that two-element set in a Boolean algebra is A_1-independent iff it is M-independent. However one can to give an example of Boolean algebra and three-element A_1-independent set, which is M-dependent. Let $B = 2^{\{1,\dots,6\}}$, and $I = \{a, b, c\}$, where $a = \{1,2,3\}$, $b = \{2,3,4\}$, $c = \{3,4,5\}$. Then I has the required properties, because $a \cap b' \cap c = \emptyset = a' \cap b \cap c'$, and all others atoms with respect to the sets a, b, c are non-empty. ∎

From the parts (c) and (e) of the theorem quoted in § 10, one can easily see, that for algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$, the conditions (δ), (ζ) and (η) always hold. So, we have the following corollary, which could be also obtained from (xi) of § 2.

COROLLARY 40. *In the algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$ (and $\mathfrak{B}_3$) every subset of B is A_1-independent.* ■

Finally, let us look on the $\boldsymbol{R}$-independence in regular reducts of Boolean algebra.

THEOREM 23. *The $\boldsymbol{R}$-independence coincides with the $\boldsymbol{M}$-independence for every regular reduct $\mathfrak{B}_r$ of the Boolean algebra.*

Proof. Firstly, let us see, that in any algebra an one-element set is $\boldsymbol{M}$-independent iff it is $\boldsymbol{R}$-independent, and that the $\boldsymbol{R}$-independence is hereditary with respect to finite subsets. Now, taking into consideration Theorem 19 and (ii) it is sufficient to show, that if I is $\boldsymbol{R}$-independent in $\mathfrak{B}_r$, then for every sequence $(i_1, \ldots, i_n) \in T_r$ and each different $a_1, \ldots, a_n \in I$ the atom $A_{(i_1,\ldots,i_n)}(a_1, \ldots, a_n)$ is non-void.

Let us assume, that, contrary, for some $a_1, \ldots, a_n \in I (n \geqslant 2)$ the equality (59) holds. So, from the regularity of $\mathfrak{B}_r$, there exist two different algebraic operations A_{J_1} and A_{J_2}, such that (50) holds. From (59) we get (53). In view of the $\boldsymbol{R}$-independence of I, for every injective mapping $p\colon I \to B$ the equality (54) will hold.

If $a_j = 0$ and $i_j = 0$, or $a_j = 1$ and $i_j = 0$ occur in the considered atom, then we can treat it as a new atom with respect to $n-1$ elements among 0 or 1 does not appear, respectively. So, we can assume, that if $a_j = 0$ then $i_j = 1$, and $i_j = 0$ for $a_j = 1$. Obviously, any permutation of the set B, which exchanges the elements 0 and 1, restricted to I is injective. Therefore, if 0 or 1 appears among elements $a_1, \ldots, a_n$, then, by the $\boldsymbol{R}$-independence of I, from (59) we can obtain a new, equal to 0, atom with respect to elements of I being not equal 0 and 1. So, without loosing generality, we can assume, that no element $a_1, \ldots, a_n$ is equal 0 or 1.

Let us consider the permutation p of the set B, which exchanges 0 with a_n, when $i_n = 0$, or exchanges 1 with a_n, when $i_n = 1$. Then from (59), by the $\boldsymbol{R}$-independence of I, we get

$$A_{(i_1,\ldots,i_{n-1},i_n)}\big(p(a_1), \ldots, p(a_n)\big) = A_{(i_1,\ldots,i_{n-1})}(a_1, \ldots, a_{n-1}) = 0.$$

Using the same argumentation $n-1$ times, we will obtain $a_1^{i_1} = 0$, hence $a_1 = 0$ or 1. This contradicts our assumption on the elements $a_1, \ldots, a_n$. Therefore we proved, that $I \in \boldsymbol{Ind}(\boldsymbol{M})$. ■

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INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY.