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BOGDAN BOJARSKI redaktor
ANDRZEJ BIAŁYNICKI-BIRUŁA, ZBIGNIEW CIESIELSKI,
JERZY ŁOŚ, WIKTOR MAREK, ZBIGNIEW SEMADENI

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S. J. ALDERSLEY, G. W. HORNDESKI

and

G. MESS

**The construction of concomitants
from an almost complex structure**

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1. Introduction

An almost complex structure on a real $2n$ -dimensional C^∞ manifold M is defined to be a smooth $(1, 1)$ tensor field J which is such that for every point $p \in M$, J_p , defines a complex structure on the tangent space $T_p M$. Locally this entails that the components of J must satisfy

$$(1.1) \quad J_b^a J_c^b = -\delta_c^a,$$

where repeated lower case Latin indices are summed from 1 to $2n$. M , together with a fixed almost complex structure, J , is called an almost complex manifold, and denoted by the pair (M, J) . In what follows it is assumed that the integer n is fixed at some value greater than 1.

The objects of interest in this paper are tensorial concomitants of an almost complex structure, and linear connection valued concomitants of an almost complex structure. For convenience in defining these objects we set

$$AC_{2n} := \{\text{the set of all almost complex manifolds } (M, J) \text{ where } \dim M = 2n\};$$

$$T_{2n}^k(\alpha, \beta) := \{\text{the set of all pairs } (M, T) \text{ where } M \text{ is a real } C^\infty \text{ manifold of dimension } 2n \text{ and } T \text{ is a class } C^k \text{ tensor field of type } (\alpha, \beta) \text{ on } M\};$$

$$LC_{2n}^k := \{\text{the set of all pairs } (M, \nabla) \text{ where } M \text{ is a real } C^\infty \text{ manifold of dimension } 2n \text{ and } \nabla \text{ is a class } C^k \text{ linear connection on } M\}.$$

DEFINITION 1.1. A map \mathcal{F} from the set AC_{2n} into the set $T_{2n}^k(\alpha, \beta)$ is called a *class C^k type (α, β) tensorial concomitant of a $2n$ -dimensional almost complex structure* if

(i) for every $(M, J) \in AC_{2n}$ the underlying manifold of the pair $\mathcal{F}(M, J)$ is M and we write

$$\mathcal{F}(M, J) = (M, F(J));$$

and

(ii) there exist real valued C^k functions $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ of real variables $V_b^a; V_{bc_1}^a; \dots; V_{bc_1 \dots c_\gamma}^a$; that is,

$$F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha} = F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(V_b^a; V_{bc_1}^a; \dots; V_{bc_1 \dots c_\gamma}^a),$$

which are such that if $(M, J) \in AC_{2n}$, and x is a chart of M then the x -components of $F(J)$ are given by

$$[F(J)]_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha} = F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(J_b^a; J_{b,c_1}^a; \dots; J_{b,c_1 \dots c_\gamma}^a),$$

where a comma denotes a partial derivative with respect to the chart x . ■

At this time it should be noted that the functions $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ associated with the tensorial concomitant \mathcal{F} are *not* unique. This is essentially due to equation (1.1) and the identities which can be obtained from it by differentiation with respect to local coordinates. Thus when we say that two tensorial concomitants of an almost complex structure are equal we mean that they are equal as maps from AC_{2n} into $T_{2n}^k(\alpha, \beta)$.

The set of all class C^k type (α, β) tensorial concomitants of a $2n$ -dimensional almost complex structure will be denoted by $\mathcal{F}_{2n}^k(\alpha, \beta)$. This set admits the structure of a real vector space when addition and scalar multiplication are defined in the obvious way. The zero element of this vector space is denoted by 0 and is such that if $(M, J) \in AC_{2n}$ then $0(M, J)$ is the zero tensor field of type (α, β) on M .

If $\mathcal{F} \in \mathcal{F}_{2n}^k(\alpha, \beta)$ then we shall usually express the real valued functions $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ which locally determine \mathcal{F} in terms of $J_b^a; J_{b,c_1}^a; \dots; J_{b,c_1 \dots c_\gamma}^a$ and *not* in terms of the real variables $V_b^a; V_{bc_1}^a; \dots; V_{bc_1 \dots c_\gamma}^a$, where J_b^a serves to denote the components of an arbitrary almost complex structure with respect to an arbitrary chart x . If $\mathcal{F} \in \mathcal{F}_{2n}^k(\alpha, \beta)$ then we shall say that \mathcal{F} is of γ th order if it is possible to choose the class C^k functions $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ so that the highest occurring derivative of J_b^a appearing in them is of γ th order.

A class C^k linear connection valued concomitant of a $2n$ -dimensional almost complex structure is a map \mathcal{C} from the set AC_{2n} into the set LC_{2n}^k which satisfies the obvious generalizations of conditions (i) and (ii) in Definition 1.1.

Remark 1.1. The definitions of a tensorial concomitant and of a linear connection valued concomitant of an almost complex structure adopted above represent slight extensions of the definition of a differentiable concomitant given by Nijenhuis [8]. ■

Remark 1.2. Let (M, J) and (\tilde{M}, \tilde{J}) be almost complex manifolds of dimension $2n$. A diffeomorphism φ of M into \tilde{M} is said to be an automorphism of (M, J) into (\tilde{M}, \tilde{J}) if for every $p \in M$ and $V \in T_p M$, $\varphi_*(J_p V) = \tilde{J}_{\varphi(p)}(\varphi_* V)$. (Note that we do not require the range of φ to be \tilde{M} .) Following Epstein [2] we define a natural tensor of class C^k and type (α, β) on $2n$ -dimensional almost complex manifolds to be a map \mathcal{F} from AC_{2n} into $T_{2n}^k(\alpha, \beta)$ which is such that if $\varphi: (M, J) \rightarrow (\tilde{M}, \tilde{J})$ is an automorphism then $\varphi_*(F(J)) = F(\tilde{J})|_{\varphi(M)}$. It is easily seen that if $\mathcal{F} \in \mathcal{F}_{2n}^k(\alpha, \beta)$ then \mathcal{F} is a natural tensor of class C^k and type (α, β) on $2n$ -dimensional almost complex

manifolds. It is not known whether the converse of this statement is valid. ■

When studying pseudo-Riemannian spaces one finds that there is no shortage of tensorial concomitants of the metric tensor. In fact it is easily seen that for any choice of the non-negative integers α, β, γ (with $\gamma \geq 3$) the real vector space of smooth, type (α, β) , γ th order tensorial concomitants of an n -dimensional pseudo-Riemannian metric is infinite dimensional. Moreover, not all elements of this vector space can locally be expressed as rational functions in the derivatives of the metric tensor. In spite of these complications the structure of tensorial concomitants of a pseudo-Riemannian metric tensor is fairly well understood due to the classical replacement theorem (cf., Thomas [10]). This theorem tells us that if \mathcal{F} is a type (α, β) , γ th order tensorial concomitant of a pseudo-Riemannian metric which locally is defined by the functions

$$T_{J_1 \dots J_\beta}^{i_1 \dots i_\alpha} = T_{J_1 \dots J_\beta}^{i_1 \dots i_\alpha}(g_{rs}; g_{rs,t_1}; g_{rs,t_1 t_2}; \dots; g_{rs,t_1 \dots t_\mu})$$

then

$$T_{J_1 \dots J_\beta}^{i_1 \dots i_\alpha} = T_{J_1 \dots J_\beta}^{i_1 \dots i_\alpha}(g_{rs}; 0; g_{rst_1 t_2}; \dots; g_{rst_1 \dots t_\mu})$$

where for $\mu = 2, \dots, \gamma$, $g_{rst_1 \dots t_\mu}$ denotes the functions which locally define the μ th metric normal tensorial concomitants of a pseudo-Riemannian metric, which are class C^∞ , type $(0, \mu + 2)$, μ th order tensorial concomitants. (The formal definition of the metric normal tensorial concomitants is given on page 104 of reference [10]. One important property of these concomitants is that they are polynomials in the derivatives of the metric tensor.) Due to the existence of the metric normal tensorial concomitants it is clear how one can proceed to construct type (α, β) , γ th order tensorial concomitants a pseudo-Riemannian metric.

One of the objectives of this paper is to determine the form of the functions which locally define the value of a tensorial concomitant of an almost complex structure. To this end we shall demonstrate in the next section that if $k \geq \max(0, \beta - \alpha)$ then

(i) the vector space $\mathcal{F}_{2n}^k(\alpha, \beta)$ is zero dimensional if $\alpha > \beta$, and finite dimensional otherwise; and

(ii) the functions which locally define a γ th order element of $\mathcal{F}_{2n}^k(\alpha, \beta)$ can be chosen to be polynomials in $J_{b,c_1}^a; \dots; J_{b,c_1 \dots c_\gamma}^a$ with coefficients which are zeroth order elements of $\mathcal{F}_{2n}^k(\tau, \tau)$ and in fact polynomial in J_b^a .

The above two results clearly illustrate that the functional form of tensorial concomitants of an almost complex structure is far simpler than that of tensorial concomitants of a pseudo-Riemannian metric. However, this does not mean that it is easier to construct tensorial concomitants of an almost complex structure than it is to construct tensorial concomitants of a pseudo-Riemannian metric—in fact, just the opposite is true!

In the third section of this paper we shall present examples of how the second result mentioned above can be employed to construct tensorial concomitants of an almost complex structure. At that time it will be shown that

(i) if $\mathcal{F} \in \mathcal{T}_{2n}^1(0, 1)$ then $\mathcal{F} = 0$;

(ii) if $\mathcal{F} \in \mathcal{T}_{2n}^1(1, 2)$ then the functions F_{ij}^h which locally define \mathcal{F} can be chosen to be

$$F_{ij}^h = \begin{cases} 0 & \text{if } \mathcal{F} \text{ is of zeroth order, otherwise,} \\ \alpha N_{ij}^h + \beta J_i^l N_{lj}^h, & \alpha, \beta \in \mathbf{R}, \end{cases}$$

where N_{ij}^h denotes the functions which define the Nijenhuis tensorial concomitant [7] and are given by

$$(1.2) \quad N_{ij}^h = J_i^m J_{j,m}^h - J_j^m J_{i,m}^h + J_m^h J_{i,j}^m - J_m^h J_{j,i}^m;$$

and

(iii) if $\mathcal{F} \in \mathcal{T}_{2n}^2(0, 2)$ then the functions F_{ij} which locally define \mathcal{F} can be chosen to be

$$F_{ij} = \begin{cases} 0 & \text{if } \mathcal{F} \text{ is of zeroth order, otherwise,} \\ \alpha N_{is}^r N_{jr}^s + \beta J_i^l N_{ls}^r N_{jr}^s, & \alpha, \beta \in \mathbf{R}. \end{cases}$$

The key behind proving the replacement theorem for tensorial concomitants of a pseudo-Riemannian metric is the fact that it is possible to construct a linear connection valued concomitant of a pseudo-Riemannian metric; viz., the Levi-Civita connection valued concomitant. In the last section of this paper we shall demonstrate that it is impossible to construct a linear connection valued concomitant of an almost complex structure. This result does not imply that there does not exist a "replacement theorem" for tensorial concomitants of an almost complex structure. However, since $\mathcal{T}_{2n}^k(\alpha, \beta)$ is always a finite dimensional vector space, when $k \geq \max(0, \beta - \alpha)$, a replacement theorem would not actually prove to be of value in the study of tensorial concomitants of an almost complex structure. What we really need to know is, what is a basis for $\mathcal{T}_{2n}^k(\alpha, \beta)$ when $k \geq \max(0, \beta - \alpha)$? Such bases have been presented above for $(\alpha, \beta) = (0, 1)$, $(1, 2)$ and $(0, 2)$.

In concluding this introduction we would like to point out that although it is impossible to construct a linear connection valued concomitant of an almost complex structure it is still possible to construct "differential operators" which act on tensorial concomitants of an almost complex structure to produce other tensorial concomitants of an almost complex structure of higher order. One such operator is Walker's [11] torsional derivative, and another such operator is Horndeski's [4] S-derivative.

2. The local form of concomitants of an almost complex structure

We begin this section with

PROPOSITION 2.1. Suppose that \mathcal{F} is an element of $\mathcal{T}_{2n}^k(\alpha, \beta)$.

- (i) If $\alpha > \beta$ and $k \geq 0$ then $\mathcal{F} = 0$;
- (ii) If $\alpha = \beta$ and $k \geq 0$ then \mathcal{F} is of zeroth order; and
- (iii) If $\alpha < \beta$ and $k \geq \beta - \alpha$ then the functions $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ which locally determine \mathcal{F} can be taken to have the following form

$$(2.1) \quad F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha} = \sum_{\mu=1}^{\beta-\alpha} \sum_{\substack{\omega_1 + \dots + \omega_\mu = \beta - \alpha \\ 0 < \omega_1 \leq \dots \leq \omega_\mu}} \eta_{j_1 \dots j_\beta i_1 \dots i_\mu}^{i_1 \dots i_\alpha k_1 \dots k_\mu r_1 \dots r_{\omega_1} \dots s_1 \dots s_{\omega_\mu}} \times \\ \times J_{k_1 r_1 \dots r_{\omega_1}}^{i_1} \dots J_{k_\mu s_1 \dots s_{\omega_\mu}}^{i_\mu}$$

where the functions $\eta_{b_1 \dots b_\beta}^{a_1 \dots a_\alpha}$ are class C^k functions of only J_b^a .

Proof. Let $(M, J) \in AC_{2n}$ and let x be a chart of M . For every $t \in \mathbf{R}^+$ we define a new chart \bar{x} by $x^i = t\bar{x}^i$. Since \mathcal{F} is a tensorial concomitant the x and \bar{x} components of $F(J)$ must be related by

$$F_{k_1 \dots k_\beta}^{h_1 \dots h_\alpha}(J_b^a; J_{b,c_1}^a; \dots; J_{b,c_1 \dots c_\gamma}^a) \frac{\partial \bar{x}^{i_1}}{\partial x^{h_1}} \dots \frac{\partial \bar{x}^{i_\alpha}}{\partial x^{h_\alpha}} \frac{\partial x^{k_1}}{\partial \bar{x}^{j_1}} \dots \frac{\partial x^{k_\beta}}{\partial \bar{x}^{j_\beta}} \\ = F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(\bar{J}_b^a; \bar{J}_{b,c_1}^a; \dots; \bar{J}_{b,c_1 \dots c_\gamma}^a).$$

Using the fact that $x^i = t\bar{x}^i$ the above equation reduces to

$$(2.2) \quad t^{\beta-\alpha} F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(J_b^a; J_{b,c_1}^a; \dots; J_{b,c_1 \dots c_\gamma}^a) = F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(J_b^a; tJ_{b,c_1}^a; \dots; t^\gamma J_{b,c_1 \dots c_\gamma}^a).$$

If $\alpha > \beta$ and $k \geq 0$ we multiply equation (2.2) by $t^{\alpha-\beta}$ and take the limit as $t \rightarrow 0^+$ to find that $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha} = 0$. (Note that the expression $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(J_b^a; 0; \dots; 0)$ makes sense since it corresponds to evaluating $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ at a complex structure in complex coordinates.) Thus $\mathcal{F} = 0$ if $\alpha > \beta$ and $k \geq 0$.

If $\alpha = \beta$ and $k \geq 0$ then upon taking the limit as $t \rightarrow 0^+$ in equation (2.2) we obtain

$$F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha} = F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(J_b^a; 0; \dots; 0).$$

Hence if $\alpha = \beta$ and $k \geq 0$ then \mathcal{F} is of zeroth order.

Lastly, if $\alpha < \beta$ and $k \geq \beta - \alpha$ then we differentiate equation (2.2), $(\beta - \alpha)$ times with respect to t and take the limit as $t \rightarrow 0^+$. (In order to evaluate the derivative of the right-hand side of equation (2.2) with respect to t we use the chain rule along with the fact that $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ is of class C^k in the real variables

$V_b^a; \dots; V_{b,c_1 \dots c_\gamma}^a$.) The end result of this calculation is that $F_{J_1 \dots J_\beta}^{i_1 \dots i_\alpha}$ can be expressed as in equation (2.1). If the original order γ of \mathcal{F} is less than $\beta - \alpha$ then those terms involving $J_{b,c_1 \dots c_\mu}^a$ for $\mu > \gamma$ can be dropped from the expression for $F_{J_1 \dots J_\beta}^{i_1 \dots i_\alpha}$. ■

Remark 2.1. Without loss of generality we can assume that the functions $\eta_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}$ appearing in equation (2.1) have various symmetries. For example, $\eta_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha r_1 \dots r_{\beta-\alpha}}$ can be assumed to be symmetric in the indices $r_1, \dots, r_{\beta-\alpha}$ since these indices sum into $J_{i_1 r_1 \dots r_{\beta-\alpha}}^k$. Another useful symmetry is that if $\omega_1 = \omega_2$ (say) then we can assume that

$$\eta_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha l_1 l_2 r_1 \dots r_{\omega_1} s_1 \dots s_{\omega_2}} = \eta_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha l_2 l_1 s_1 \dots s_{\omega_2} r_1 \dots r_{\omega_1}}. \quad \blacksquare$$

Remark 2.2. If $\mathcal{F} \in \mathcal{F}_{2n}^k(\alpha, \beta)$ where $k \geq \beta - \alpha > 0$ then Proposition 2.1 implies that $F(J) = 0$ whenever $(M, J) \in AC_{2n}$ is a complex manifold. ■

Employing an argument similar to the one used to prove Proposition 2.1 we obtain

PROPOSITION 2.2. *If \mathcal{C} is a class C^1 linear connection valued concomitant of a $2n$ -dimensional almost complex structure then the functions C_{jk}^i which locally determine \mathcal{C} can be taken to be of the form*

$$(2.3) \quad C_{jk}^i = \eta_{jka}^{ibc} J_{b,c}^a$$

where η_{jka}^{ibc} is a class C^1 function of only J_b^a . ■

We shall now proceed to determine the form of the coefficient functions $\eta_{b_1 \dots b_\tau}^{a_1 \dots a_\tau}$ appearing in equations (2.1) and (2.3). This will be accomplished by means of two propositions. The first proposition will tell us that we can choose the functions $\eta_{b_1 \dots b_\tau}^{a_1 \dots a_\tau}$ so that they determine a zeroth order element of $\mathcal{F}_{2n}^k(\tau, \tau)$. The second proposition will tell us that any zeroth order element of $\mathcal{F}_{2n}^k(\tau, \tau)$ is locally determined by a function which is a polynomial in J_b^a .

PROPOSITION 2.3. *The class C^k functions $\eta_{b_1 \dots b_\tau}^{a_1 \dots a_\tau} = \eta_{b_1 \dots b_\tau}^{a_1 \dots a_\tau}(J_m^l)$ appearing in equations (2.1) and (2.3) can be chosen so that they determine a zeroth order element of $\mathcal{F}_{2n}^k(\tau, \tau)$.*

Proof. In order to prove that a collection of real valued functions $\psi_{b_1 \dots b_\tau}^{a_1 \dots a_\tau} = \psi_{b_1 \dots b_\tau}^{a_1 \dots a_\tau}(J_s^r)$ defines a type (τ, τ) tensorial concomitant of a $2n$ -dimensional almost complex structure we must show that given a $2n \times 2n$ matrix of constants $[J_s^r]$ which are such that $J_s^r J_t^s = -\delta_t^r$, and any $2n \times 2n$ matrix $[A_m^l] \in GL(2n, \mathbb{R})$ with inverse $[\bar{A}_m^l]$ then

$$(2.4) \quad \psi_{b_1 \dots b_\tau}^{a_1 \dots a_\tau}(\bar{A}_i^r J_m^l A_s^m) = \psi_{f_1 \dots f_\tau}^{e_1 \dots e_\tau}(J_s^r) \bar{A}_{e_1}^{a_1} \dots \bar{A}_{e_\tau}^{a_\tau} A_{b_1}^{f_1} \dots A_{b_\tau}^{f_\tau}.$$

However, given any such matrix $[J_s^r]$ there exists a matrix $[g_m^l] \in GL(2n; \mathbb{R})$,

with inverse $[\bar{g}_m^l]$ which is such that $J_s^r = \bar{g}_s^r \bar{J}_m^l g_s^m$, where

$$(2.5) \quad [\bar{J}_m^l] := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

and I is the $n \times n$ identity matrix (cf. page 114 of Kobayashi and Nomizu [6]). Now suppose that we can prove that for every $[B_m^l] \in GL(2n, \mathbf{R})$ with inverse $[\bar{B}_m^l]$

$$(2.6) \quad \psi_{b_1 \dots b_\tau}^{a_1 \dots a_\tau} (\bar{B}_t^l \bar{J}_m^l B_s^m) = \psi_{f_1 \dots f_\tau}^{e_1 \dots e_\tau} (\bar{J}_s^r) \bar{B}_{e_1}^{a_1} \dots \bar{B}_{e_\tau}^{a_\tau} B_{b_1}^{f_1} \dots B_{b_\tau}^{f_\tau}.$$

This would imply that equation (2.4) holds, as can easily be seen by examining the equations which result by choosing $B_m^l = g_p^l A_m^p$ and $B_m^l = g_m^l$ in equation (2.6).

Thus in order to prove the proposition we must show that the functions $\eta_{b_1 \dots b_\tau}^{a_1 \dots a_\tau}$ appearing in equations (2.1) and (2.3) can be chosen so as to satisfy equation (2.6). This can be done as follows.

Let x denote the standard chart of \mathbf{R}^{2n} and consider the almost complex structure J which is defined on a neighborhood of the origin 0 by

$$(2.7) \quad J_k^l = \bar{g}_p^l \bar{J}_q^p g_k^q,$$

where $[g_k^q]$ is an invertible matrix of smooth real valued functions with inverse $[\bar{g}_p^l]$, and such that $g_k^k(0) = \delta_k^k$.

We desire to examine the effect of a linear coordinate transformation on the functions appearing in equation (2.1) when these functions are evaluated at the almost complex structure given in equation (2.7). In order to do this we first have to determine an expression for the derivatives of J_k^l with respect to the chart x and then determine the effect of a linear coordinate transformation on these derivatives of J .

Upon differentiating equation (2.7) μ -times we find that

$$(2.8) \quad J_{k, r_1 \dots r_\mu}^l = \bar{g}_{p, r_1 \dots r_\mu}^l \bar{J}_q^p g_k^q + \bar{g}_p^l \bar{J}_q^p g_{k, r_1 \dots r_\mu}^q + \{\mu - 1\}$$

where $\{\mu - 1\} := \{\text{terms involving derivatives of order less than or equal to } \mu - 1\}$. Since $\bar{g}_e^e g_f^e = \delta_f^e$, we have

$$\bar{g}_{p, r_1 \dots r_\mu}^l = -\bar{g}_e^l \bar{g}_p^e g_{f, r_1 \dots r_\mu}^f + \{\mu - 1\}$$

and hence equation (2.8) can be rewritten as

$$(2.9) \quad J_{k, r_1 \dots r_\mu}^l = -\bar{g}_e^l \bar{g}_p^e g_{f, r_1 \dots r_\mu}^f \bar{J}_q^p g_k^q + \bar{g}_p^l \bar{J}_q^p g_{k, r_1 \dots r_\mu}^q + \{\mu - 1\}.$$

Let $[B_m^l]$ be any element of $GL(2n, \mathbf{R})$ with inverse $[\bar{B}_m^l]$. We define another chart \bar{x} for \mathbf{R}^{2n} by $\bar{x}^i := \bar{B}_j^i x^j$. The \bar{x} components of J are given by

$$(2.10) \quad \bar{J}_b^a = \bar{B}_t^a J_k^l B_b^k$$

and thus

$$(2.11) \quad \bar{J}_{b,c_1 \dots c_\mu}^a = J_{k,r_1 \dots r_\mu}^l \bar{B}_l^a B_b^k B_{c_1}^{r_1} \dots B_{c_\mu}^{r_\mu}.$$

Equations (2.9) and (2.11) combine to give us

$$(2.12) \quad \bar{J}_{b,c_1 \dots c_\mu}^a = [-\bar{g}_e^l \bar{g}_p^f g_{f,r_1 \dots r_\mu}^e \bar{J}_q^p g_k^q + \bar{g}_p^l \bar{J}_q^p g_{k,r_1 \dots r_\mu}^q] \times \\ \times \bar{B}_l^a B_b^k B_{c_1}^{r_1} \dots B_{c_\mu}^{r_\mu} + \{\mu - 1\}.$$

With these preliminaries disposed of we are now ready to evaluate the functions presented in equation (2.1) for the almost complex structure J given in equation (2.7) and to determine the effect of a linear coordinate transformation on these functions.

Since \mathcal{F} is a tensorial concomitant we know that the barred and unbarred components of $F(J)$ are related by

$$(2.13) \quad F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(\bar{J}_b^a; \dots; \bar{J}_{b,c_1 \dots c_\gamma}^a) \\ = F_{v_1 \dots v_\beta}^{u_1 \dots u_\alpha}(J_b^a; \dots; J_{b,c_1 \dots c_\gamma}^a) \bar{B}_{u_1}^{i_1} \dots \bar{B}_{u_\alpha}^{i_\alpha} B_{j_1}^{v_1} \dots B_{j_\beta}^{v_\beta}.$$

We employ equations (2.1), (2.9) and (2.12) to evaluate equation (2.13). If we only keep track of the terms which involve derivatives of g_b^a of order $(\beta - \alpha)$ we find that equation (2.13) implies that

$$(2.14) \quad \eta_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha b c_1 \dots c_\beta - \alpha}(\bar{J}_z^y) [\bar{g}_p^l \bar{J}_q^p g_{k,r_1 \dots r_{\beta-\alpha}}^q - \bar{g}_v^l \bar{g}_p^f g_{f,r_1 \dots r_{\beta-\alpha}}^e \bar{J}_q^p g_k^q] \times \\ \times \bar{B}_l^a B_b^k B_{c_1}^{r_1} \dots B_{c_{\beta-\alpha}}^{r_{\beta-\alpha}} = \eta_{v_1 \dots v_\beta}^{u_1 \dots u_\alpha k r_1 \dots r_{\beta-\alpha}}(J_z^y) [\bar{g}_p^l \bar{J}_q^p g_{k,r_1 \dots r_{\beta-\alpha}}^q + \\ + \bar{g}_e^l \bar{g}_p^f g_{f,r_1 \dots r_{\beta-\alpha}}^e \bar{J}_q^p g_k^q] \bar{B}_{u_1}^{i_1} \dots \bar{B}_{u_\alpha}^{i_\alpha} B_{j_1}^{v_1} \dots B_{j_\beta}^{v_\beta} + \{\beta - \alpha - 1\}.$$

The functions $g_{w,m_1 \dots m_{\beta-\alpha}}^l$ appearing in equation (2.14) are essentially arbitrary and hence we can differentiate this equation with respect to them. Upon doing so, and evaluating the result at $O \in R^{2n}$ we find (noting remark 2.1, equations (2.7), (2.10) and the fact that $g_b^a(0) = \delta_b^a$) that

$$\eta_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha b c_1 \dots c_\beta - \alpha}(\bar{J}_z^y(0)) [\bar{J}_q^a(0) \bar{B}_l^q B_b^w - \bar{J}_l^p(0) \bar{B}_l^a B_p^w] B_{c_1}^{m_1} \dots B_{c_{\beta-\alpha}}^{m_{\beta-\alpha}} \\ = [\eta_{v_1 \dots v_\beta}^{u_1 \dots u_\alpha w m_1 \dots m_{\beta-\alpha}}(\bar{J}_z^y) \bar{J}_l^i - \eta_{v_1 \dots v_\beta}^{u_1 \dots u_\alpha k m_1 \dots m_{\beta-\alpha}}(\bar{J}_z^y) \bar{J}_k^w] \times \\ \times \bar{B}_{u_1}^{i_1} \dots \bar{B}_{u_\alpha}^{i_\alpha} B_{j_1}^{v_1} \dots B_{j_\beta}^{v_\beta},$$

where $\bar{J}_z^y(0) = \bar{B}_a^y \bar{J}_b^a B_z^b$. When this equation is multiplied by \bar{J}_s^i we discover that the functions

$$(2.15) \quad \hat{\eta}_{v_1 \dots v_\beta}^{u_1 \dots u_\alpha w m_1 \dots m_{\beta-\alpha}} = \eta_{v_1 \dots v_\beta}^{u_1 \dots u_\alpha w m_1 \dots m_{\beta-\alpha}}(J_z^y) + \eta_{v_1 \dots v_\beta}^{u_1 \dots u_\alpha k m_1 \dots m_{\beta-\alpha}}(J_z^y) J_k^w J_s^i$$

satisfy equation (2.6) and hence determine a zeroth order element of $\mathcal{F}_{2n}^k(\beta + 1, \beta + 1)$.

The term involving $J_{k,r_1 \dots r_{\beta-\alpha}}^l$ in equation (2.1) is

$$(2.16) \quad \begin{aligned} & \eta_{j_1 \dots j_{\beta^l}}^{i_1 \dots i_{\alpha} k r_1 \dots r_{\beta-\alpha}} J_{k,r_1 \dots r_{\beta-\alpha}}^l \\ &= \frac{1}{2} \eta_{j_1 \dots j_{\beta^l}}^{i_1 \dots i_{\alpha} k r_1 \dots r_{\beta-\alpha}} J_{k,r_1 \dots r_{\beta-\alpha}}^l + \frac{1}{2} \eta_{j_1 \dots j_{\beta^l}}^{i_1 \dots i_{\alpha} a r_1 \dots r_{\beta-\alpha}} J_{a,r_1 \dots r_{\beta-\alpha}}^b. \end{aligned}$$

Upon repeatedly differentiating equation (1.1) we find that

$$(2.17) \quad J_{a,r_1 \dots r_{\beta-\alpha}}^b = J_a^b J_a^k J_{k,r_1 \dots r_{\beta-\alpha}}^l + \{\beta - \alpha - 1\}.$$

Thus we can use equations (2.15)–(2.17) to conclude that

$$\eta_{j_1 \dots j_{\beta^l}}^{i_1 \dots i_{\alpha} k r_1 \dots r_{\beta-\alpha}} J_{k,r_1 \dots r_{\beta-\alpha}}^l = \frac{1}{2} \hat{\eta}_{j_1 \dots j_{\beta^l}}^{i_1 \dots i_{\alpha} k r_1 \dots r_{\beta-\alpha}} J_{k,r_1 \dots r_{\beta-\alpha}}^l + \{\beta - \alpha - 1\}.$$

Consequently we could have originally chosen the functions $F_{j_1 \dots j_{\beta}}^{i_1 \dots i_{\alpha}}$ appearing in equation (2.1) so that the coefficients $\eta_{j_1 \dots j_{\beta^l}}^{i_1 \dots i_{\alpha} k r_1 \dots r_{\beta-\alpha}}$ define a zeroth order element of $\mathcal{F}_{2n}^k(\beta+1, \beta+1)$.

At this time one should note that the argument presented above implies that the coefficients η_{jka}^{ibc} appearing in equation (2.3) can be chosen so as to define a zeroth order element of $\mathcal{F}_{2n}^1(3, 3)$.

We shall now sketch how the remainder of the proof of this proposition can be completed using induction.

Suppose that we have shown that for $\mu = 1, \dots, v-1$, the coefficients $\eta_{j_1 \dots j_{\beta^l} i_1 \dots i_{\mu}}^{i_1 \dots i_{\alpha} k_1 \dots k_{\mu}}$ appearing in equation (2.1) can be chosen so as to define zeroth order elements of $\mathcal{F}_{2n}^k(\beta+\mu, \beta+\mu)$. We shall now prove that under this assumption the coefficients $\eta_{j_1 \dots j_{\beta^l} i_1 \dots i_{\nu}}^{i_1 \dots i_{\alpha} k_1 \dots k_{\nu}}$ can be chosen so as to define zeroth order elements of $\mathcal{F}_{2n}^k(\beta+\nu, \beta+\nu)$. To this end we consider the almost complex structure J defined on a neighborhood of $0 \in \mathbb{R}^{2n}$ by equation (2.7). Employing equation (2.1), (2.11) and the assumed tensoriality of $\eta_{j_1 \dots j_{\beta^l} i_1 \dots i_{\mu}}^{i_1 \dots i_{\alpha} k_1 \dots k_{\mu}}$ (for $\mu = 1, \dots, v-1$) we see that equation (2.13) implies that in the present case

$$(2.18) \quad \begin{aligned} & \sum_{\mu=v}^{\beta-\alpha} \sum_{\substack{\omega_1 + \dots + \omega_{\mu} = \beta - \alpha \\ 0 < \omega_1 \leq \dots \leq \omega_{\mu}}} \eta_{j_1 \dots j_{\beta^l} a_1 \dots a_{\mu}}^{i_1 \dots i_{\alpha} b_1 \dots b_{\mu} c_1 \dots c_{\omega_1} \dots d_1 \dots d_{\omega_{\mu}}} (\bar{J}_z^y) \bar{J}_{b_1, c_1 \dots c_{\omega_1}}^{a_1} \dots \bar{J}_{b_{\mu}, d_1 \dots d_{\omega_{\mu}}}^{a_{\mu}} \\ &= \left\{ \sum_{\mu=v}^{\beta-\alpha} \sum_{\substack{\omega_1 + \dots + \omega_{\mu} = \beta - \alpha \\ 0 < \omega_1 \leq \dots \leq \omega_{\mu}}} \eta_{v_1 \dots v_{\beta^l} i_1 \dots i_{\mu}}^{u_1 \dots u_{\alpha} k_1 \dots k_{\mu} r_1 \dots r_{\omega_1} \dots s_1 \dots s_{\omega_{\mu}}} (J_z^y) J_{k_1, r_1 \dots r_{\omega_1}}^{l_1} J_{k_{\mu}, s_1 \dots s_{\omega_{\mu}}}^{l_{\mu}} \right\} \times \\ & \quad \times \bar{B}_{v_1}^{i_1} \dots \bar{B}_{v_{\alpha}}^{i_{\alpha}} B_{j_1}^{v_1} \dots B_{j_{\beta}}^{v_{\beta}}. \end{aligned}$$

Let us choose v positive integers π_1, \dots, π_v which are such that $\pi_1 + \dots + \pi_v = \beta - \alpha$ and $\pi_1 \leq \dots \leq \pi_v$. We now use equations (2.9) and (2.12) to replace the derivatives of J_k^l and \bar{J}_b^a in equation (2.18) (however, we need only keep track of those terms which involve $g_{b_1, r_1 \dots r_{\pi_1}}^{a_1} \dots g_{b_v, s_1 \dots s_{\pi_v}}^{a_v}$). Upon differentiating the resulting equation with respect to

$$g_{w_1, h_1 \dots h_{\pi_1}}, \dots, g_{w_v, m_1 \dots m_{\pi_v}}$$

and evaluating at $0 \in \mathbb{R}^{2n}$ we find, through use of equation (2.6), that

$$(2.19) \quad \hat{\eta}_{j_1 \dots j_{\beta} i_1 \dots i_v}^{i_1 \dots i_{\alpha} k_1 \dots k_v r_1 \dots r_{\pi_1} \dots s_1 \dots s_{\pi_v}} \\ := \eta_{j_1 \dots j_{\beta} b_1 \dots b_v}^{i_1 \dots i_{\alpha} a_1 \dots a_v r_1 \dots r_{\pi_1} \dots s_1 \dots s_{\pi_v}} (\delta_{a_1}^{k_1} \delta_{i_1}^{b_1} + J_{a_1}^{k_1} J_{i_1}^{b_1}) \dots (\delta_{a_v}^{k_v} \delta_{i_v}^{b_v} + J_{a_v}^{k_v} J_{i_v}^{b_v})$$

defines a zeroth order element of $\mathcal{F}_{2n}^k(\beta + v, \beta + v)$ for each admissible choice of the positive integers π_1, \dots, π_v .

The term involving $J_{k_1, r_1 \dots r_{\pi_1}}^{i_1} \dots J_{k_v, s_1 \dots s_{\pi_v}}^{i_v}$ in equation (2.1) is

$$\eta_{j_1 \dots j_{\beta} i_1 \dots i_v}^{i_1 \dots i_{\alpha} k_1 \dots k_v r_1 \dots r_{\pi_1} \dots s_1 \dots s_{\pi_v}} J_{k_1, r_1 \dots r_{\pi_1}}^{i_1} \dots J_{k_v, s_1 \dots s_{\pi_v}}^{i_v}$$

Employing the obvious generalization of equation (2.17) we see that this term can be rewritten as

$$\frac{1}{2^v} \hat{\eta}_{j_1 \dots j_{\beta} i_1 \dots i_v}^{i_1 \dots i_{\alpha} k_1 \dots k_v r_1 \dots r_{\pi_1} \dots s_1 \dots s_{\pi_v}} J_{k_1, r_1 \dots r_{\pi_1}}^{i_1} \dots J_{k_v, s_1 \dots s_{\pi_v}}^{i_v} + \dots$$

where “...” denotes terms such as those appearing within the curly brackets on the right-hand side of equation (2.18) with $v+1 \leq \mu \leq \beta - \alpha$. Consequently under our present assumptions we see that we could have originally chosen the functions $F_{j_1 \dots j_{\beta}}^{i_1 \dots i_{\alpha}}$ appearing in equation (2.1) so that the coefficients $\eta_{j_1 \dots j_{\beta} i_1 \dots i_{\mu}}^{i_1 \dots i_{\alpha} k_1 \dots k_{\mu}}$ define zeroth order elements of $\mathcal{F}_{2n}^k(\beta + \mu, \beta + \mu)$ for $1 \leq \mu \leq v$.

Due to our earlier work we know that our inductive hypothesis is valid for $v = 2$. Hence our proof of this proposition is complete. ■

Propositions 2.1, 2.2 and 2.3 have shown us that the problem of building suitably differentiable tensorial concomitants and connection valued concomitants of an almost complex structure essentially reduces to the problem of constructing zeroth order type (α, α) tensorial concomitants. This problem is resolved in

PROPOSITION 2.4. *If the functions $\eta_{j_1 \dots j_{\alpha}}^{i_1 \dots i_{\alpha}} = \eta_{j_1 \dots j_{\alpha}}^{i_1 \dots i_{\alpha}}(J_{\beta}^{\alpha})$ define a zeroth order tensorial concomitant of a $2n$ -dimensional almost complex structure then they must be a real linear combination of terms of the form $F_{j_1}^{i_1} F_{j_2}^{i_2} \dots F_{j_{\alpha}}^{i_{\alpha}}$, where for all $\mu = 1, \dots, \alpha$, $F_{\mu}^{i_{\mu}} = \delta_{j_{\mu}}^{i_{\mu}}$ or $J_{j_{\mu}}^{i_{\mu}}$.*

Proof. Let $V := \mathbb{R}^{2n}$ and let $P(\alpha)$ denote the representation of the permutation group of α objects in $GL(V^{\otimes \alpha})$ which permutes the factors. We define subgroups $N(\alpha)$ and $G(\alpha)$ of $GL(V^{\otimes \alpha})$ by

$$N(\alpha) := \{A^{\otimes \alpha} \mid A \in GL(V) \text{ and } A \circ J = J \circ A\}$$

and

$$G(\alpha) := \{\tau \in GL(V^{\otimes \alpha}) \mid \tau = \pm \pi \circ 1_V^{\otimes \beta} \otimes \tilde{J}^{\otimes (\alpha - \beta)}, \beta = 0, 1, \dots, \alpha, \text{ and } \pi \in P(\alpha)\}$$

where 1_V denotes the identity element of $GL(V)$ and \tilde{J} is the automorphism of V whose matrix with respect to the standard basis $\{e_{i,i=1}^{2n}\}$ of V is given by equation (2.5).

If H is a subgroup of $GL(V^{\otimes n})$ we let \bar{H} denote the subalgebra of $\text{End}(V^{\otimes n})$ generated by real linear combinations of elements of H .

For any subset S of $\text{End}(V^{\otimes n})$ we define

$$C(S) := \{\tau \in \text{End}(V^{\otimes n}) \mid \tau \circ \zeta = \zeta \circ \tau, \text{ for every } \zeta \in S\}.$$

If S is a subgroup of $GL(V^{\otimes n})$ then $C(S) = C(\bar{S})$.

The real numbers $\eta_{j_1 \dots j_n}^{i_1 \dots i_n}(\tilde{J}_n)$ are the components of an element of $\text{End}(V^{\otimes n})$ with respect to the standard basis for $V^{\otimes n}$ determined by the basis $\{e_{i,i=1}^{2n}\}$ for V . Moreover this element of $\text{End}(V^{\otimes n})$ commutes with any element of $N(\alpha)$ due to equation (2.6). Thus in order to complete the proof of this proposition all we need do is show that $C(N(\alpha)) = \overline{G(\alpha)}$. This can be done as follows.

$G(\alpha)$ forms a finite group under composition which contains $P(\alpha)$ as a subgroup. From Maschke's Theorem (Weyl [12], p. 101) it follows that $\overline{G(\alpha)}$ is fully reducible and by Theorem 3.5D on page 95 in Weyl [12], we have $C(C(\overline{G(\alpha)})) = \overline{G(\alpha)}$. To prove that $C(N(\alpha)) = \overline{G(\alpha)}$ we shall show that $C(\overline{G(\alpha)}) = N(\alpha)$, and hence $C(N(\alpha)) = C(\overline{N(\alpha)}) = \overline{G(\alpha)}$ as required. We now determine $C(\overline{G(\alpha)})$.

Due to the obvious fact that $\overline{N(\alpha)} \subset C(\overline{G(\alpha)})$, we concentrate on showing that any element of $C(\overline{G(\alpha)})$ is in $\overline{N(\alpha)}$. To this end we construct a basis for $\text{End}(V^{\otimes n})$ which is particularly well adapted to our problem.

Define the sets M^+ and M^- by

$$M^+ := \{A \in \text{End}(V) \mid A \circ \tilde{J} = \tilde{J} \circ A\},$$

and

$$M^- := \{A \in \text{End}(V) \mid A \circ \tilde{J} = -\tilde{J} \circ A\}.$$

It is easily seen that (i) $\dim M^+ = \dim M^- = 2n^2$; (ii) $\text{End}(V) = M^+ \oplus M^-$; and (iii) M^+ and M^- possess bases which consist of automorphisms of V .

We now let $B := \{A_\lambda\}_{\lambda=1}^{4n^2}$ be a basis for $\text{End}(V)$ which is such that for every $\lambda = 1, \dots, 4n^2$, $A_\lambda \in GL(V)$ with $A_\lambda \in M^+$ for $1 \leq \lambda \leq 2n^2$ and $A_\lambda \in M^-$ for $2n^2 + 1 \leq \lambda \leq 4n^2$. We set $B^{\otimes n}$ equal to the basis for $\text{End}(V^{\otimes n})$ obtained by taking tensor products of the elements of B .

Due to the fact that $P(\alpha) \subset G(\alpha)$, we see that $C(\overline{G(\alpha)}) \subset C(\overline{P(\alpha)})$. It is well known (cf. Weyl [12], p. 130) that $C(\overline{P(\alpha)})$ consists of the so-called bi-symmetric endomorphisms of $V^{\otimes n}$. Now $\tau \in \text{End}(V^{\otimes n})$ is bi-symmetric if (and

only if)

$$(2.20) \quad \tau = \sum_{\lambda_1, \dots, \lambda_\alpha = 1}^{4n^2} \tau^{\lambda_1 \dots \lambda_\alpha} A_{\lambda_1} \otimes \dots \otimes A_{\lambda_\alpha}$$

where $\tau^{\lambda_1 \dots \lambda_\alpha}$ is symmetric in the indices $\lambda_1, \dots, \lambda_\alpha$. (This fact is valid for any basis of $\text{End}(V^{\otimes \alpha})$ determined by a basis for $\text{End}(V)$.)

The endomorphism $\bar{J} \otimes 1_{V^{\otimes(\alpha-1)}} \in G(\alpha)$ and hence must commute with every element τ of $C(\overline{G(\alpha)})$. Using the basis $B^{\otimes \alpha}$ and the fact that

$$\tau \circ \bar{J} \otimes 1_{V^{\otimes(\alpha-1)}} = \bar{J} \otimes 1_{V^{\otimes(\alpha-1)}} \circ \tau,$$

we deduce from equation (2.20) that

$$\sum_{\lambda_1, \dots, \lambda_\alpha = 1}^{4n^2} \tau^{\lambda_1 \dots \lambda_\alpha} (A_{\lambda_1} \circ \bar{J} - \bar{J} \circ A_{\lambda_1}) \otimes A_{\lambda_2} \otimes \dots \otimes A_{\lambda_\alpha} = 0.$$

By the linear independence of the A_λ 's we must have

$$\sum_{\lambda_1 = 1}^{4n^2} \tau^{\lambda_1 \dots \lambda_\alpha} (A_{\lambda_1} \circ \bar{J} - \bar{J} \circ A_{\lambda_1}) = 0$$

and hence due to our choice of basis we see that

$$\sum_{\lambda_1 = 2n^2 + 1}^{4n^2} \tau^{\lambda_1 \dots \lambda_\alpha} A_{\lambda_1} = 0.$$

Again, linear independence of the basis implies that $\tau^{\lambda_1 \dots \lambda_\alpha} = 0$ for $\lambda_1 = 2n^2 + 1, \dots, 4n^2$. The symmetry of $\tau^{\lambda_1 \dots \lambda_\alpha}$ in its indices now implies that $\tau^{\lambda_1 \dots \lambda_\alpha} = 0$ whenever one or more of its indices takes a value in the set $\{2n^2 + 1, \dots, 4n^2\}$. Consequently if $\tau \in C(\overline{G(\alpha)})$ we may write

$$(2.21) \quad \tau = \sum_{\lambda_1, \dots, \lambda_\alpha = 1}^{2n^2} \tau^{\lambda_1 \dots \lambda_\alpha} A_{\lambda_1} \otimes \dots \otimes A_{\lambda_\alpha}$$

where $\tau^{\lambda_1 \dots \lambda_\alpha}$ is symmetric in $\lambda_1, \dots, \lambda_\alpha$. Conversely any totally symmetric collection of real numbers $\tau^{\lambda_1 \dots \lambda_\alpha}$ ($\lambda_1, \dots, \lambda_\alpha = 1, \dots, 2n^2$) defines an element of $C(\overline{G(\alpha)})$ by means of equation (2.21). In fact the above work shows that the set

$$\bar{B}^{\otimes \alpha} := \{(A_{\lambda_1} \otimes \dots \otimes A_{\lambda_\alpha}) \mid 1 \leq \lambda_\beta \leq 2n^2, \beta = 1, \dots, \alpha\}$$

forms a basis for $C(\overline{G(\alpha)})$, where round brackets about a collection of indices denotes symmetrization over those indices.

To finish our proof that $N(\alpha) = C(\overline{G(\alpha)})$ we shall now demonstrate that demanding that $\tau \in C(\overline{G(\alpha)})$ and $\tau \notin N(\alpha)$ leads to a contradiction. By definition $N(\alpha)$ is spanned by elements of the form $A^{\otimes \alpha}$, $A \in \text{GL}(V)$ with

$A \circ \bar{J} = \bar{J} \circ A$. Therefore there exist real numbers b^λ such that $A = \sum_{\lambda=1}^{2n^2} b^\lambda A_\lambda$, and $\overline{N(\alpha)}$ is spanned by linear combinations of the form

$$A \otimes \dots \otimes A = \sum_{\lambda_1, \dots, \lambda_\alpha=1}^{2n^2} b^{\lambda_1} \dots b^{\lambda_\alpha} A_{\lambda_1} \otimes \dots \otimes A_{\lambda_\alpha}$$

where

$$(2.22) \quad \det \left(\sum_{\lambda=1}^{2n^2} b^\lambda A_\lambda \right) \neq 0.$$

If $\tau \in C(\overline{G(\alpha)})$ and $\tau \notin \overline{N(\alpha)}$ it suffices to assume that $\tau \in \overline{N(\alpha)}^\perp$, where the inner product in question is determined by the demand that the basis $\bar{B}^{\otimes \alpha}$ of $C(\overline{G(\alpha)})$ is orthonormal. Thus we can use equation (2.21) to deduce that there exist real numbers $\tau^{\lambda_1 \dots \lambda_\alpha} = \tau^{(\lambda_1 \dots \lambda_\alpha)}$ which are such that

$$(2.23) \quad \sum_{\lambda_1, \dots, \lambda_\alpha=1}^{2n^2} \tau^{\lambda_1 \dots \lambda_\alpha} b^{\lambda_1} \dots b^{\lambda_\alpha} = 0$$

for all b^λ which satisfy equation (2.22). Due to the principle of the irrelevance of algebraic inequalities (Weyl [12], p. 4) equation (2.23) implies that $\tau^{\lambda_1 \dots \lambda_\alpha} = 0$, and hence $\tau = 0 \in \overline{N(\alpha)}$ contrary to our original assumptions. Consequently $\overline{N(\alpha)} = C(\overline{G(\alpha)})$ and our proof is complete. ■

Remark 2.3. Note that in stating Proposition 2.4 we have not required the functions $\eta_{j_1 \dots j_\alpha}^{i_1 \dots i_\alpha}$ to even be continuous, and yet we have succeeded in proving that they must be polynomial in the J_a^a 's and hence are of class C^ω . ■

Remark 2.4. In general the set of all distinct terms of the form $F_{j_1}^{i_1} \dots F_{j_\alpha}^{i_\alpha}$ in Proposition 2.4 is not linearly independent. For example, when $\alpha = 4$ and $n = 2$ we have the identity $0 = \delta_{j_1 j_2 j_3 j_4}^{i_1 i_2 i_3 i_4} J_a^b$ to contend with, where $\delta_{j_1 \dots j_5}^{i_1 \dots i_5}$ is the 5×5 generalized Kronecker delta. ■

Let $\mathcal{F}_{2n}(\alpha, \alpha)$ denote the real vector space of tensorial concomitants described in Proposition 2.4. The dimension of this vector space is less than or equal to $\alpha! 2^\alpha$. Thus we can now employ Propositions 2.1 and 2.3 to deduce

PROPOSITION 2.5. *If $k \geq \max(0, \beta - \alpha)$ then the vector space $\mathcal{F}_{2n}^k(\alpha, \beta)$ is finite dimensional and any element of this vector space is of order less than or equal to $\max(0, \beta - \alpha)$.*

In the next section we shall show how the theory developed above can be used to determine bases for the vector spaces $\mathcal{F}_{2n}^k(\alpha, \beta)$ when $k \geq \beta - \alpha > 0$.



3. Illustrative examples

The purpose of this section is to show, by means of examples, how the previous theory may be implemented to construct all of the elements of $\mathcal{F}_{2n}^k(\alpha, \beta)$ when $k \geq \max(0, \beta - \alpha)$. In particular we shall be concerned with the vector spaces $\mathcal{F}_{2n}^1(0, 1)$, $\mathcal{F}_{2n}^1(1, 2)$ and $\mathcal{F}_{2n}^2(0, 2)$.

EXAMPLE 1. *Constructing the elements of $\mathcal{F}_{2n}^1(0, 1)$.*

Propositions 2.1 and 2.3 tell us that if $\mathcal{F} \in \mathcal{F}_{2n}^1(0, 1)$ then the functions F_j which locally define \mathcal{F} can be taken to have the form

$$(3.1) \quad F_j = \eta_{ja}^{bc} J_{b,c}^a$$

where η_{ja}^{bc} defines a zeroth order element of $\mathcal{F}_{2n}^1(2, 2)$.

Horndeski [4] has established the following replacement theorem for first order tensorial concomitants of an almost complex structure.

PROPOSITION 3.1. *If the functions $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha} = F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(J_b^a; J_{b,c}^a)$ locally define a tensorial concomitant of a 2n-dimensional almost complex structure then $F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha} = F_{j_1 \dots j_\beta}^{i_1 \dots i_\alpha}(J_b^a; -\frac{1}{4}J_i^a N_{bc}^i)$ where N_{bc}^i is defined in equation (1.2). ■*

In view of this result we see that equation (3.1) implies that

$$(3.2) \quad F_j = -\frac{1}{4} \eta_{ja}^{bc} J_i^a N_{bc}^i.$$

Due to Proposition 2.4 we know that

$$(3.3) \quad \eta_{ja}^{bc} = \alpha_1 \delta_j^b \delta_a^c + \alpha_2 \delta_j^c \delta_a^b + \alpha_3 J_j^b \delta_a^c + \alpha_4 J_j^c \delta_a^b + \alpha_5 \delta_j^b J_a^c + \\ + \alpha_6 \delta_j^c J_a^b + \alpha_7 J_j^b J_a^c + \alpha_8 J_j^c J_a^b,$$

where $\alpha_1, \dots, \alpha_8$ are real constants. Upon combining equations (3.2) and (3.3) noting that

$$(3.4) \quad N_{ab}^a = J_m^i N_{ib}^m = 0$$

we find that $F_j = 0$. Consequently the vector space $\mathcal{F}_{2n}^1(0, 1)$ is zero dimensional. ■

EXAMPLE 2. *Constructing the elements of $\mathcal{F}_{2n}^1(1, 2)$.*

Propositions 2.1, 2.3 and 3.1 imply that if $\mathcal{F} \in \mathcal{F}_{2n}^1(1, 2)$ then the functions F_{jk}^i which locally define \mathcal{F} can be taken to have the form

$$(3.5) \quad F_{jk}^i = \eta_{jka}^{ibc} N_{bc}^a$$

where η_{jka}^{ibc} defines a zeroth order element of $\mathcal{F}_{2n}^1(3, 3)$. Proposition 2.4 shows us how to express the coefficients η_{jka}^{ibc} appearing in equation (3.5) in terms of J_s^i . Employing this expression for η_{jka}^{ibc} along with equation (3.4) and the

identities

$$(3.6) \quad J_a^c N_{cj}^i = -J_c^i N_{aj}^c, \quad J_i^c N_{cj}^k = -J_j^c N_{ci}^k$$

we find that equation (3.5) becomes

$$F_{jk}^i = \alpha N_{jk}^i + \beta J_1^i N_{jk}^i,$$

where α and β are real constants. Consequently the vector space $\mathcal{F}_{2n}^1(1, 2)$ is 2-dimensional, and any element in it must be of class C^∞ and of first order. ■

Remark 3.1. At this time we could use Propositions 2.2, 2.3 and 2.4 to show that there does not exist a class C^1 linear connection valued concomitant of an almost complex structure. However, the argument would be a bit cumbersome (cf., pages 96–103 of Aldersley [1]). In the next section we shall present a simple explanation of why such a linear connection valued concomitant does not exist. ■

So far the examples which we have considered have been quite elementary. We shall now conclude this section with a more formidable example.

EXAMPLE 3. *Constructing the elements of $\mathcal{F}_{2n}^2(0, 2)$.*

Proposition 2.1 and 2.3 tell us that if $\mathcal{F} \in \mathcal{F}_{2n}^2(0, 2)$ then the functions F_{jk} which locally define \mathcal{F} can be taken to have the form

$$(3.7) \quad F_{jk} = \eta_{jka}^{ibc} J_{i,bc}^a + \eta_{jkab}^{cdef} J_{c,d}^a J_{e,f}^b$$

where η_{jka}^{ibc} and η_{jkab}^{cdef} define zeroth order elements of $\mathcal{F}_{2n}^2(3, 3)$ and $\mathcal{F}_{2n}^2(4, 4)$ respectively. Due to Remark 2.1 we can choose the η 's to have the following symmetries

$$\eta_{jka}^{ibc} = \eta_{jka}^{icb} \quad \text{and} \quad \eta_{jkab}^{cdef} = \eta_{jkba}^{fedc}.$$

Thus we can use Proposition 2.4 to conclude that there exists 24 real constants $\alpha_1, \dots, \alpha_{24}$ which are such that

$$(3.8) \quad \begin{aligned} \eta_{jka}^{ibc} = & \alpha_1 \delta_j^i \delta_k^b \delta_a^c + \alpha_1 \delta_j^j \delta_k^c \delta_a^b + \alpha_2 \delta_j^b \delta_k^c \delta_a^i + \alpha_2 \delta_j^c \delta_k^b \delta_a^i + \alpha_3 \delta_j^c \delta_k^i \delta_a^b + \\ & + \alpha_3 \delta_j^b \delta_k^i \delta_a^c + \alpha_4 J_j^i \delta_k^b \delta_a^c + \alpha_4 J_j^j \delta_k^c \delta_a^b + \alpha_5 J_j^b \delta_k^c \delta_a^i + \alpha_5 J_j^c \delta_k^b \delta_a^i + \\ & + \alpha_6 J_j^c \delta_k^i \delta_a^b + \alpha_6 J_j^b \delta_k^c \delta_a^i + \alpha_7 \delta_j^i J_k^b \delta_a^c + \alpha_7 \delta_j^j J_k^c \delta_a^b + \alpha_8 \delta_j^b J_k^c \delta_a^i + \\ & + \alpha_8 \delta_j^c J_k^b \delta_a^i + \alpha_9 \delta_j^c J_k^i \delta_a^b + \alpha_9 \delta_j^b J_k^i \delta_a^c + \alpha_{10} \delta_j^j \delta_k^b J_a^c + \alpha_{10} \delta_j^i \delta_k^c J_a^b + \\ & + \alpha_{11} \delta_j^b \delta_k^c J_a^i + \alpha_{11} \delta_j^c \delta_k^b J_a^i + \alpha_{12} \delta_j^c \delta_k^i J_a^b + \alpha_{12} \delta_j^b \delta_k^i J_a^c + \\ & + \alpha_{13} J_j^j J_k^b \delta_a^c + \alpha_{13} J_j^i J_k^c \delta_a^b + \alpha_{14} J_j^b J_k^c \delta_a^i + \alpha_{14} J_j^c J_k^b \delta_a^i + \\ & + \alpha_{15} J_j^c J_k^i \delta_a^b + \alpha_{15} J_j^b J_k^i \delta_a^c + \alpha_{16} J_j^j \delta_k^b J_a^c + \alpha_{16} J_j^i \delta_k^c J_a^b + \end{aligned}$$

$$\begin{aligned}
& +\alpha_{17} J_j^b \delta_k^c J_a^i + \alpha_{17} J_j^c \delta_k^b J_a^i + \alpha_{18} J_j^c \delta_k^i J_a^b + \alpha_{18} J_j^b \delta_k^i J_a^c + \\
& +\alpha_{19} \delta_j^i J_k^b J_a^c + \alpha_{19} \delta_j^i J_k^c J_a^b + \alpha_{20} \delta_j^b J_k^c J_a^i + \alpha_{20} \delta_j^c J_k^b J_a^i + \\
& +\alpha_{21} \delta_j^c J_k^i J_a^b + \alpha_{21} \delta_j^b J_k^i J_a^c + \alpha_{22} J_j^i J_k^b J_a^c + \alpha_{22} J_j^i J_k^c J_a^b + \\
& +\alpha_{23} J_j^b J_k^c J_a^i + \alpha_{23} J_j^c J_k^b J_a^i + \alpha_{24} J_j^c J_k^i J_a^b + \alpha_{24} J_j^b J_k^i J_a^c.
\end{aligned}$$

There exists a similar expression for η_{jka}^{def} involving 192 real constants and 384 terms.

As one might expect the coefficients η_{jka}^{ibc} and η_{jka}^{def} appearing in equation (3.7) are not independent and they must satisfy various relations if F_{jk} is to define a tensorial concomitant. These relations can be found in the following way. Let x be a chart of $(M, J) \in AC_{2n}$ at a point $p \in M$ and such that $x(p) = 0 \in \mathbb{R}^{2n}$. We now define another chart \bar{x} on a neighborhood of p by $\bar{x}^i := B_j^i x^j + \frac{1}{2} B_{jk}^i x^j x^k + \frac{1}{6} B_{jkl}^i x^j x^k x^l$ where the real constants B_j^i , B_{jk}^i and B_{jkl}^i are such that $\det(B_j^i) \neq 0$, $B_{jk}^i = B_{(jk)}^i$ and $B_{jkl}^i = B_{(jkl)}^i$. Upon examining the effect of this coordinate transformation on $F(J)$ at p we obtain an identity in B_j^i , B_{jk}^i and B_{jkl}^i . When this identity is differentiated with respect to B_{uvw}^h and B_{vw}^h and evaluated for $B_j^i = \delta_j^i$, $B_{jk}^i = 0$ and $B_{jkl}^i = 0$ we obtain

$$(3.9) \quad \eta_{jka}^{(uvw)} J_h^a = \eta_{jkh}^{a(uv)} J_a^w$$

and

$$(3.10) \quad \frac{1}{2} \eta_{jka}^{buw} J_{b,h}^a + \eta_{jka}^{(vw)b} J_{h,b}^a - \eta_{jkh}^{ab(v)} J_{a,b}^w + \eta_{jkab}^{cd(uw)} J_h^b J_{c,d}^a - \eta_{jka}^{bcd(v)} J_d^w J_{b,c}^a = 0.$$

(A third identity could be found by differentiating with respect to B_w^h but this identity is not required for what follows. For more information on this approach to deriving identities satisfied by tensorial concomitants see, for example, Horndeski [3].)

Equation (3.9) implies that the constants $\alpha_1, \dots, \alpha_{24}$ appearing in equation (3.8) must satisfy

$$(3.11) \quad \left\{ \begin{array}{l} \alpha_1 + \alpha_3 = -\alpha_{16} - \alpha_{21}; \quad \alpha_{13} + \alpha_{15} = \alpha_{18} + \alpha_{19}; \quad \alpha_{16} + \alpha_{18} = -\alpha_1 + \alpha_{15}; \\ \alpha_{19} + \alpha_{21} = -\alpha_3 + \alpha_{13} \\ \text{and} \\ \alpha_4 + \alpha_6 = \alpha_{10} - \alpha_{24}; \quad \alpha_7 + \alpha_9 = \alpha_{12} - \alpha_{22}; \quad \alpha_{10} + \alpha_{12} = \alpha_4 + \alpha_9; \\ \alpha_{22} + \alpha_{24} = -\alpha_6 - \alpha_7. \end{array} \right.$$

Due to equation (3.11) and the identities which can be obtained by differentiating equation (1.1) (such as $J_i^a J_{a,bc}^i =$ first order terms) we find that

the term $\eta_{jka}^{ibc} J_{i,bc}^a$ appearing in equation (3.7) can be expressed as

$$\begin{aligned} \eta_{jka}^{ibc} J_{i,bc}^a &= 2(\alpha_1 + \alpha_{16})(J_{j,ka}^a - J_{k,ja}^a - J_{b,ca}^a J_j^b J_k^c + J_{b,ca}^a J_k^b J_j^c) + \\ &\quad + 2(\alpha_4 - \alpha_{10}) J_j^l (J_{l,ka}^a - J_{k,la}^a - J_{b,ca}^a J_l^b J_k^c + J_{b,ca}^a J_k^b J_l^c) + \\ &\quad + \{\text{first order terms}\}. \end{aligned}$$

Thus at first sight it appears as though there may exist a second order element of $\mathcal{F}_{2n}^2(0, 2)$. Unfortunately the analysis of equation (3.10) (which is quite lengthy) shows that this is not the case and that we can take η_{jka}^{ibc} equal to zero in equation (3.7). Consequently the replacement theorem, Proposition 3.1, can now be employed to deduce that

$$F_{jk} = \frac{1}{16} \eta_{jkab}^{cdef} J_i^a N_{cd}^l J_m^b N_{ef}^m,$$

which, due to Proposition 2.4, and equations (3.4) and (3.6), reduces to

$$(3.12) \quad F_{jk} = \alpha N_{js}^r N_{kr}^s + \beta J_j^l N_{ls}^r N_{kr}^s$$

where α and β are real constants.

In summary we have shown that the vector space $\mathcal{F}_{2n}^2(0, 2)$ is 2-dimensional and if $\mathcal{F} \in \mathcal{F}_{2n}^2(0, 2)$ then the functions F_{jk} which locally define \mathcal{F} can be chosen to be

$$F_{jk} = \begin{cases} 0 & \text{if } \mathcal{F} \text{ is of zeroth order, otherwise,} \\ \alpha N_{js}^r N_{kr}^s + \beta J_j^l N_{ls}^r N_{kr}^s, & \alpha, \beta \in \mathbf{R}. \end{cases}$$

The concomitant $\mathcal{S} \in \mathcal{F}_{2n}^2(0, 2)$ with local components given by $S_{jk} := N_{js}^r N_{kr}^s$ has been previously investigated by Horndeski [4], [5]. His work has shown that $S(J)$ is a symmetric, Hermitian, (0, 2) tensor field, and that there exist almost complex structures J for which $S(J)$ is a degenerate (0, 2) tensor field. As a result of these facts and equation (3.12) we can now conclude that there do not exist elements of $\mathcal{F}_{2n}^k(0, 2)$ (for $k \geq 2$) which are pseudo-Riemannian metric valued or symplectic structure valued. ■

In the past there was some interest shown in the elements of $\mathcal{F}_{2n}^\infty(1, 3)$ (see e.g., Ślebodziński [9], and Willmore [13]). Due to Propositions 2.1 and 2.3 we know that if $\mathcal{F} \in \mathcal{F}_{2n}^2(1, 3)$ then the functions F_{ijk}^h which locally define \mathcal{F} can be taken to have the form

$$(3.13) \quad F_{ijk}^h = \eta_{ijka}^{hbcd} J_{b,cd}^a + \eta_{ijkab}^{hdef} J_{c,d}^a J_{e,f}^b$$

where η_{ijka}^{hbcd} and η_{ijkab}^{hdef} define zeroth order elements of $\mathcal{F}_{2n}^2(4, 4)$ and $\mathcal{F}_{2n}^2(5, 5)$ respectively. We presently *suspect* that an argument similar to the one employed in Example 3 can be used to prove that η_{ijka}^{hbcd} can be set equal

to zero in equation (3.13). If this is the case then any element of $\mathcal{F}_{2n}^2(1, 3)$ would have to be of first order and the computation of its explicit form would be routine.

4. Connection valued concomitants of an almost complex structure

The primary purpose of this section is to prove the following

PROPOSITION 4.1. *There does not exist a class C^1 linear connection valued concomitant of a $2n$ -dimensional almost complex structure.*

Proof. The proof is by contradiction. Suppose that \mathcal{C} is a class C^1 linear connection valued concomitant of a $2n$ -dimensional almost complex structure. Without loss of generality we can assume that \mathcal{C} is symmetric linear connection valued. Proposition 2.2 tells us that the functions C_{jk}^i which locally define \mathcal{C} can be expressed as

$$(4.1) \quad C_{jk}^i = \eta_{jka}^{ibc} J_{b,c}^a$$

where the real valued functions $\eta_{jka}^{ibc} = \eta_{jka}^{ibc}(J_q^p)$ are of class C^1 , and such that $\eta_{jka}^{ibc} = \eta_{kja}^{ibc}$.

We now choose \tilde{J} to be the complex structure on \mathbf{R}^{2n} whose components with respect to the standard chart x of \mathbf{R}^{2n} are presented in equation (2.5). Due to equation (4.1) we see that the x -components of $C(\tilde{J})$ vanish.

Let \bar{x} be another chart of \mathbf{R}^{2n} which is defined on a neighbourhood of $0 \in \mathbf{R}^{2n}$ by

$$(4.2) \quad \bar{x}^i = x^i - \frac{1}{2} B_{jk}^i x^j x^k$$

where $B_{jk}^i = B_{kj}^i$ are real constants. Using the transformation law for a linear connection we find that the \bar{x} components of $C(\tilde{J})$ at 0 are

$$(4.3) \quad \overline{[C(\tilde{J})]_{jk}^i} = B_{jk}^i.$$

However, equation (4.1) tells us that

$$\overline{[C(\tilde{J})]_{jk}^i} = \eta_{jka}^{ibc} (\bar{J}_q^p) \bar{J}_{b,c}^a$$

and hence we may use equation (4.2) and the transformation law for \bar{J}_q^p to conclude that at 0

$$(4.4) \quad \overline{[C(\tilde{J})]_{jk}^i} = \eta_{jka}^{ibc} (\bar{J}_q^p) (\bar{J}_i^a B_{bc}^i - \bar{J}_b^i B_{ic}^a).$$

Upon combining equations (4.3) and (4.4) we discover that for all collections of real numbers $B_{jk}^i = B_{kj}^i$ we must have

$$(4.5) \quad B_{jk}^i = \eta_{jka}^{ibc} (\tilde{J}_q^p) (\tilde{J}_1^a B_{bc}^i - \tilde{J}_b^i B_{ic}^a).$$

Let S denote the vector space which consists of all collections of real numbers $[B_{jk}^i]$ which are such that $B_{jk}^i = B_{kj}^i$. The dimension of S is $4n^3 + 2n^2$. Let M^- denote the vector space of $2n \times 2n$ matrices which anticommute with $[\tilde{J}_s^t]$; i.e., if $[A_b^a] \in M^-$ then $\tilde{J}_t^i A_s^i = -A_t^i \tilde{J}_s^i$. As mentioned earlier M^- is of dimension $2n^2$. We now set $W := M^- \oplus \dots \oplus M^-$ ($2n$ -factors). Evidently the dimension of W is $4n^3$, which is less than the dimension of S .

We define a map L from S into W by

$$[B_{jk}^i] \mapsto \underset{1}{M} \oplus \dots \oplus \underset{2n}{M}$$

where for every $c = 1, \dots, 2n$,

$$M_c^a := \tilde{J}_t^i B_{bc}^i - \tilde{J}_b^i B_{ic}^a.$$

L is easily seen to be well-defined and linear.

Let T denote the map from W into S given by

$$\underset{1}{M} \oplus \dots \oplus \underset{2n}{M} \mapsto [\eta_{jka}^{ibc} M_c^a].$$

T is also linear.

Due to equation (4.5) $T \circ L$ is the identity on S and hence S and W are isomorphic. This is absurd since $\dim W < \dim S$. Thus our original assumption that there exists a class C^1 linear connection valued concomitant of a $2n$ -dimensional almost complex structure is false. ■

One important aspect of the above proof is that it does not require the use of Propositions 2.3 and 2.4. As a result of this fact Proposition 4.1 can be extended to some other structures defined by $(1, 1)$ tensor fields on a manifold. We shall now devote the remainder of this section to a discussion of such extensions.

Let M be a real C^∞ m -dimensional manifold. An *almost product structure* on M is defined by a smooth $(1, 1)$ tensor field P ($\neq 1$, the identity tensor) which is such that $P^2 = 1$. An *almost tangent structure* on M is defined by a smooth $(1, 1)$ tensor field Q which is such that $Q^2 = 0$, $\text{rank } Q = n$, and $m = 2n$. Lastly an F -structure on M is defined by a smooth $(1, 1)$ tensor field F ($\neq 0$) which is such that $F^3 = -F$.

Let K denote any one of the $(1, 1)$ tensor fields introduced in the previous paragraph. Employing an argument similar to the one used to prove Proposition 2.1 we can show that if \mathcal{C} is a class C^1 linear connection

valued concomitant of K then the functions C_{jk}^i which locally define \mathcal{C} can be expressed as

$$(4.6) \quad C_{jk}^i = \eta_{jka}^{ibc} K_{b,c}^a$$

where $\eta_{jka}^{ibc} = \eta_{jka}^{ibc}(K_s^r)$ is of class C^1 .

If K is taken to be an almost product structure then we can prove that \mathcal{C} does not exist by essentially repeating the proof of Proposition 4.1 with $[\tilde{J}_s^r]$ replaced by

$$[\tilde{P}_s^r] := \begin{bmatrix} I_a & 0 \\ 0 & -I_{m-a} \end{bmatrix}.$$

where I_a and I_{m-a} denote the identity matrices of order a and $(m-a)$ respectively and a is an integer such that $1 \leq a < m$.

If K is chosen to be an almost tangent structure then we can prove that \mathcal{C} does not exist by replacing $[\tilde{J}_s^r]$ by

$$[\tilde{Q}_s^r] := \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}$$

in the proof of Proposition 4.1. Once again we find that the dimension of W is less than the dimension of S and hence $T \circ L$ cannot be the identity map.

Lastly, if K is taken to be an F structure then without loss of generality we can assume that $\det K = 0$ (for otherwise K would define an almost complex structure and we know that \mathcal{C} does not exist in that case). We now replace $[\tilde{J}_s^r]$ in the proof of Proposition 4.1 by

$$[\tilde{F}_s^r] = \begin{bmatrix} 0_{1 \times 1} & & 0_{1 \times 2n} & \\ & 0_{n \times n} & & I_n \\ 0_{2n \times 1} & & & \\ & -I_n & & 0_{n \times n} \end{bmatrix}.$$

For this choice of \tilde{F} the map L has a kernel and hence \mathcal{C} cannot exist.

To recapitulate the above work we have

PROPOSITION 4.2. *There does not exist a class C^1 linear connection valued concomitant of either an almost product structure or an almost tangent structure or an F -structure. ■*

In concluding this paper we would like to point out that there does exist a structure involving $(1, 1)$ tensor fields on a manifold which gives rise to a class C^∞ linear connection valued concomitant; viz., the almost quaternionic structure of the first kind (cf., Yano and Ako [14]). In keeping with Proposition 2.2 this connection valued concomitant is of first order.

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