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Nearstandardness on a finite set

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Abstract

Let T be a finite set for which $\operatorname{card} T$ is a natural nonstandard number. The linear space $\mathbb{C}^T$ of complex-valued functions on T is nonstandard. For the analysis on $\mathbb{C}^T$ we need a concept of nearstandardness in this space. A version how to introduce such a concept is proposed. Some elementary examples are given.

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Introduction

After E. Nelson's *Radically Elementary Probability Theory* [12] there arises a natural problem to construct a "radically elementary operator theory". What is implied is to bring out prime effects inherent in standard operator theory in infinite-dimensional spaces on the basis of operators in hyperfinite-dimensional spaces. In this paper some simple construction is proposed, namely *the standard filling of a finite set*. In many problems it is possible to obtain relevant results by replacing infinite sets with finite ones, continuous variables with discrete ones, and vice versa. In particular, such a possibility has increased due to Nonstandard Analysis. A finite set T such that $\operatorname{card} T \in \mathbb{N} \setminus {}^{\mathrm{st}}\mathbb{N}$ (this means that $\operatorname{card} T$ is a nonstandard natural number) can be very wide. For instance, T can be a finite increasing sequence of real numbers which includes each standard real number. But in order to develop an analysis on T, we need a nearstandardness for functions, defined on T. A set T with nonstandard cardinality is nonstandard and therefore it does not have a natural nearstandardness structure. We introduce it artificially via suitable maps.

We choose a triplet $(\mathbf{T}, \mathit{\Lambda}, Q)$, where $\mathbf{T}$ is a standard set, $\mathit{\Lambda}$ is a standard σ-algebra, subalgebra of $2^{\mathbf{T}}$, Q is a mapping $2^T \to \mathit{\Lambda}$ such that the sets Qt, $t \in T$, are mutually disjoint, and the standardization ${}^{\mathrm{S}}(\bigcup_{t\in T} Qt)$ of their union coincides with $\mathbf{T}$. Such a triplet $(\mathbf{T}, \mathit{\Lambda}, Q)$ is said to be a *standard filling* of T. Now, let $\mu \in \mathbb{C}^{\mathit{\Lambda}}$ be an arbitrary σ-additive charge on $\mathit{\Lambda}$. Define $\nu = \mathit{\Pi}\mu$ by

$$\nu E = \sum_{t\in E} \mu Qt \quad \text{ for } E \in 2^T.$$

Then ν is an additive charge on 2^T. It is said to be a *standard* or *nearstandard* charge iff μ is one (in the natural sense). If μ has the shadow ${}^\circ\mu$, then ${}^\circ\nu := \mathit{\Pi}({}^\circ\mu)$ is said to be the shadow of ν, since $\operatorname{var}(\nu - {}^\circ\nu) \approx 0$. But for this definition to be correct, we need some additional assumption. Denote by $\mathcal{M}$ and $\mathcal{N}$ the normed spaces of charges on $\mathit{\Lambda}$ and 2^T respectively (with the total variation norm). The set $\operatorname{qker} \mathit{\Pi} := \{\mu \in \mathcal{M} : \|\mu\| \approx 0\}$ is said to be the *quasi-kernel of the inductor* $\mathit{\Pi}$. We require that ${}^{\mathrm{st}}\operatorname{qker} \mathit{\Pi} = \{0\}$, that is, if $\|\mathit{\Pi}\mu\| \approx 0$ and μ is standard, then $\mu = 0$. In natural examples this holds.

Obviously we need a map "inverse" to $\mathit{\Pi}$. But it cannot be unique. Indeed, let $\nu \in \mathcal{N}$. By $(Q\nu)Qt := \nu\{t\}$ we define a charge $\mu = Q\nu$ for $\mathcal{E} \in \mathit{\Lambda}$ such that $\mathcal{E}$ is a union of some Qt's. To define μ of a part of Qt, we have complete freedom. Therefore we associate with the triplet $(\mathbf{T}, \mathit{\Lambda}, Q)$ some *fixed σ-additive measure* $\lambda \in \mathbb{R}_+^{\mathit{\Lambda}}$ such that $\lambda Qt \neq 0$ for $t \in T$, and for any $\mathcal{E} \in \Lambda$ we define

$$\mu\mathcal{E} = (Q\nu)\mathcal{E} := \sum_{t\in\mathcal{E}} \lambda_{\mathcal{E}}(\nu\{t\}),$$

where $\lambda_{\mathcal{E}}(t) := (\lambda Qt)^{-1}\lambda(\mathcal{E} \cap Qt)$ is the "conditional probability of $\mathcal{E}$". We hope that the freedom in the choice of λ makes our construction flexible. Thus the *embedding* $Q : T \to \Lambda$ is extended to a map $Q : \mathcal{N} \to \mathcal{M}$. This Q is *isometric*, ΠQ is the identity $\mathbb{I}_{\mathcal{N}}$, but $P := Q\Pi$ is only a *projector* ($P^2 = P$) $\mathcal{M} \to \mathcal{M}$. It is desirable for P to be a *quasi-unity* on $\mathcal{M}$, which means that $\|P\mu - \mu\| \approx 0$ for standard $\mu \in \mathcal{M}$. In natural examples this holds.

After this first basic step we pass to functions. A function $x \in \mathbb{C}^T$ or a function $\xi \in L_p(\mathbf{T}, \Lambda, \lambda)$ generates a unique charge $\nu_x \in \mathcal{N}$ or $\mu_\xi \in \mathcal{M}$ respectively, where x is the density of ν_x, and ξ that of μ_ξ. A natural way to define $Q : \mathbb{C}^T \to L_p$ and $\Pi : L_p \to \mathbb{C}^T$ is to set $\nu_{Qx} = Q\nu_x$ and $\mu_{\Pi\xi} = \Pi\mu_\xi$. This permits us to introduce the standardness and shadow on $\mathbb{C}^T$. Then automatically ${}^{\mathrm{st}}\mathrm{qker}\, \Pi_{|\mathbb{C}^T} = \{0\}$, and $Q\Pi_{|L_p}$ is a quasi-unity. Once again $Q :\ \mathbb{C}^T \to L_p$ is *isometric* ($\mathbb{C}^T$ is a Hilbert space with a scalar product, induced by $\Pi\lambda$), $\Pi Q = \mathbb{I}_{\mathbb{C}^T}$, $P := Q\Pi$ is a projector $L_p \to L_p$ (orthogonal for $p = 2$), and $P\xi \approx \xi$ for each standard $\xi \in L_p$. Moreover, the operators Π and Q are mutually adjoint: $\Pi^* = Q$, $Q^* = \Pi$.

The next step relates to operators. Denote by $\mathcal{B}(X)$ the algebra of bounded operators in a normed space X. If $A \in \mathcal{B}(\mathbb{C}^T)$ then $\mathbf{A} := QA\Pi \in \mathcal{B}(L_p)$ and if $\mathbf{A} \in \mathcal{B}(L_p)$, then $A := Q\mathbf{A}\Pi \in \mathcal{B}(\mathbb{C}^T)$. The properties of these transformations of operators are inherited from Π, Q, P. Thus we obtain the standardness and nearstandardness on $\mathcal{B}(\mathbb{C}^T)$ (weak, strong, and uniform). We define the shadow of $x \in \mathbb{C}^T$ by ${}^\circ x = \Pi({}^\circ\xi)$ and for $A \in \mathcal{B}(\mathbb{C}^T)$ by ${}^\circ A = \Pi\, {}^\circ\mathbf{A}Q$ where $\xi = Qx$ and $\mathbf{A} = QA\Pi$. An interesting question is the relation between the properties of x and ${}^\circ x$, A and ${}^\circ A$. As to x, we cannot add anything new except, for instance, the theorem on continuous or differentiable shadow. But in the elementary example of discrete differentiation with periodic boundary conditions we show that one can obtain results of the following kind: the spectrum of the shadow coincides with the shadow of the spectrum, the shadow of an eigenvector is an eigenvector of the shadow, and so on. In another publication we prove that this also holds for a nonstandard Sturm–Liouville difference operator. We hope that this activity can be developed.

We call the reader's attention to the fact that in this paper we do not use external maps for the transition of standardness or nearstandardness. Thus the measure of a strictly external set is not defined. Some complementary idea of the content of the paper is given by the list of its sections.

At the end of this paper we use the technique of equipment to generalize the concept of nearstandardness. For instance, a function $x \in \mathbb{C}^T$ which does not have a shadow corresponding to the usual Hilbert norm can have a generalized shadow, which is a distribution. This is useful when we deal with an operator $A \in \mathcal{B}(\mathbb{C}^T)$ which has a "continuous" spectrum. Note that the generalized shadow of an operator $A \in \mathcal{B}(\mathbb{C}^T)$ (we recall that $\dim \mathbb{C}^T = \mathrm{card}\, T \in \mathbb{N}$) can be an unbounded, closed, densely defined operator $L_2 \to L_2$.

For NSA, the reader is invited to consult, for instance, the excellent book [6].

0. Preliminary notes

We (1) use Internal Set Theory (IST)—nonstandard analysis after E. Nelson (2) (see [9], [11], [13]). This theory assigns the property st (read "x is standard" for "st x") to all objects whose existence and uniqueness follow from ZFC (Zermelo–Fraenkel set theory with the axiom of choice). For example, the sets 1, π, $\emptyset$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $C[0,1]$, $L(\mathbb{R})$ are standard. At the same time IST postulates that infinite standard sets contain nonstandard elements. To be more exact, a standard set is infinite if and only if it contains nonstandard elements. The following abbreviations are used:

$$\forall^{\mathrm{st}} x\ p(x) \equiv \forall x\ (\mathrm{st}\, x \Rightarrow p(x)), \quad \exists^{\mathrm{st}} x\ p(x) \equiv \exists x\ (\mathrm{st}\, x \wedge p(x)).$$

A number $x \in \mathbb{R}$ is called *infinitesimal*, written $x \approx 0$, if $\forall^{\mathrm{st}} n \in \mathbb{N}\ (|x| < 1/n)$. Suppose X is a standard set with a standard distance d (i.e., (X, d) is a standard metric space). Points $x, y \in X$ are said to be *infinitely close*, written $x \approx y$, if $d(x, y) \approx 0$. A point $x \in X$ is said to be *nearstandard* if there exists a standard point $y \in X$ such that $x \approx y$. In this case, y is called the *shadow* of x and is denoted by ${}^{\circ}x$.

Let us consider a simple example. A number $x \in \mathbb{R}$ is called *finite* ($|x| \ll \infty$) iff $\exists^{\mathrm{st}} n \in \mathbb{N}\ (|x| < n)$. It is known that *every* finite number x is nearstandard. In other words, it has a shadow ${}^{\circ}x$. ${}^{\circ}x$ is the unique standard number which is infinitely close to x. The notion of nearstandardness is interesting at least because in terms of it one can describe classical notions such as tending to a limit, continuity, compactness. This notion can be generalized to topological spaces. Suppose X is a standard *Hausdorff* topological space (3). A point $x \in X$ is called *nearstandard* if there exists a standard point $y \in X$ such that $x \in U_y$ for any standard neighborhood U_y of y. In this case, the point y, which is unique, is called the *shadow* of x and is denoted by ${}^{\circ}x$ as before. There are different notions of nearstandardness corresponding to different standard topologies on the same standard set X. Consider an example.

0.1. Definitions. Let X be a standard normed space. A point $x \in X$ is said to be *strongly nearstandard* if (4)

(1) This work was accomplished due to a close collaboration with Taras Kudryk and Galyna Chuiko. In particular, they translated the manuscript and corrected many errors. Many details of the constructions presented are the same as in [7], [8]. The author reported these results at the CIMNS (International Colloquium on Non-Standard Mathematics, 18–22 July 1994, Aveiro, Portugal).

(2) This research was stimulated by Nelson's remarkable work [10] and by the wish to have a "radically elementary operator theory". The author is conscious that the problem is beyond his strength.

(3) The following compactness criterion should be mentioned: a standard Hausdorff space is compact if and only if each of its points is nearstandard.

(4) For any set X, we denote by ${}^{\mathrm{st}}X$ the collection of its standard elements: ${}^{\mathrm{st}}X := \{x \in X : \mathrm{st}\, x\}$. Note that $\exists y \in {}^{\mathrm{st}}X \ \equiv\ \exists^{\mathrm{st}} y \in X$, $\forall y \in {}^{\mathrm{st}}X \ \equiv\ \forall^{\mathrm{st}} y \in X$. A collection of the form ${}^{\mathrm{st}}X$ is an external set (see footnote 9).

$$\exists y \in {}^{\mathrm{st}}X \quad \|x - y\| \approx 0. \tag{0.1}$$

It is called *weakly nearstandard* if

$$\exists y \in {}^{\mathrm{st}}X \ \forall f \in {}^{\mathrm{st}}X' \quad f(x) \approx f(y), \tag{0.2}$$

where X' is the space conjugate to X.

Note that strong nearstandardness implies weak nearstandardness (because $\|f\| \ll \infty$ for all $f \in {}^{\mathrm{st}}X'$). Under the former condition, the point y from (0.1) coincides with the point y from (0.2) and is denoted by ${}^\circ x$.

Another concept of nearstandardness ([5]) is also natural. It concerns the case when the elements of a set are maps. Let (Z, d) be a standard metric space and let $E \subset Z$. The *shadow* ${}^\circ E$ of the set E is uniquely defined by the following conditions:

(a) the set ${}^\circ E$ is standard;
(b) $\forall^{\mathrm{st}} z \in Z \ (z \in {}^\circ E \Leftrightarrow \exists e \in E \ z \approx e)$.

To define the shadow of a map it is natural to use its graph.

0.2. $\langle \mathrm{nst} \rangle$ **condition.** Let (X, d_X) and (Y, d_Y) be standard metric spaces, and let (Z, d) be their Cartesian product, $d(z_1, z_2) = (d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2)^{1/2}$ for any $z_i = (x_i, y_i) \in X \times Y$, $i = 1, 2$. A map $F : X \to Y$ is called *nearstandard* (we write $\mathrm{nst}(F)$) if the shadow ${}^\circ(\mathrm{graph}\, F)$ of its graph is a functional graph. This graph is considered as a subset of Z. If $\mathrm{nst}(F)$, then the shadow ${}^\circ F$ is defined by

$$\mathrm{graph}({}^\circ F) = {}^\circ(\mathrm{graph}\, F). \tag{0.3}$$

According to the extensionality principle of IST, a standard set is uniquely defined by its standard elements. Since ${}^\circ F$ and its domain of definition $\mathrm{dom}({}^\circ F)$ are standard, it suffices to know only standard elements of $\mathrm{dom}({}^\circ F)$. Directly from (0.3), we conclude that

$$\forall^{\mathrm{st}} x \in X \quad (x \in \mathrm{dom}({}^\circ F) \Leftrightarrow \exists x_1 \in \mathrm{dom}_{\mathrm{nst}} F \ (x \approx x_1)), \quad \text{where} \tag{0.4}$$

$$\mathrm{dom}_{\mathrm{nst}} F := \{x \in \mathrm{dom}\, F : x \text{ and } F(x) \text{ are nearstandard}\}. \tag{0.5}$$

Since the map ${}^\circ F$ is standard, it suffices to know how it operates at standard elements. By (0.3), we have

$$\forall^{\mathrm{st}} x \in \mathrm{dom}({}^\circ F) \ \forall x_1 \in \mathrm{dom}_{\mathrm{nst}} F \quad (x_1 \approx x \Rightarrow ({}^\circ F)(x) = {}^\circ(F(x_1))). \tag{0.6}$$

Now we introduce the following definition. We say that a map F satisfies the $\langle \mathrm{nst} \rangle$ *condition* if ([6])

$$\forall x_1, x_2 \in \mathrm{dom}_{\mathrm{nst}} F \quad (x_1 \approx x_2 \Rightarrow F(x_1) \approx F(x_2)). \tag{0.7}$$

0.2.1. *A map F is graph-nearstandard* (*in the sense that* ${}^\circ(\mathrm{graph}\, F)$ *is a functional graph*) *if and only if it satisfies the* $\langle \mathrm{nst} \rangle$ *condition.*

([5]) In what follows we call it graph-nearstandardness.

([6]) Let $f : X \to Y$ be a standard mapping, and let $E \subseteq \mathrm{dom}\, f$ be a standard set. In IST, it is proved that f is uniformly continuous on E if and only if $\forall x_1, x_2 \in E \ (x_1 \approx x_2 \Rightarrow f(x_1) \approx f(x_2))$. Note that in (0.7) neither the mapping F nor the collection $\mathrm{dom}_{\mathrm{nst}} F$ are standard. (In the terminology of IST the collection $\mathrm{dom}_{\mathrm{nst}} F$ is an external set.)

◁ According to the definition of the shadow of a set, a point $(x,y) \in {}^{\mathrm{st}}(X \times Y)$ belongs to $^\circ(\operatorname{graph} F)$ if and only if $(x,y) \approx (x_1, F(x_1))$ for some $x_1 \in \operatorname{dom} F$. Let $x_i \in \operatorname{dom}_{\mathrm{nst}} F$, $i = 1,2$, and $x_1 \approx x_2$. Write $x = {}^\circ x_1 = {}^\circ x_2$ and $y_i = {}^\circ(F(x_i))$, $i = 1,2$. Then $(x, y_i) \in {}^{\mathrm{st}}({}^\circ(\operatorname{graph} F))$ and if $^\circ(\operatorname{graph} F)$ is a functional graph, then $y_1 = y_2$. Consequently, $F(x_1) \approx y_1 = y_2 \approx F(x_2)$. Hence F satisfies the $\langle\mathrm{nst}\rangle$ condition because $\approx$ is transitive.

Conversely, let this condition be satisfied. Suppose that $(x, y_i) \in {}^{\mathrm{st}}({}^\circ(\operatorname{graph} F))$ with $i = 1,2$; then $(x, y_i) \approx (x_i, F(x_i))$ for some $x_i \in \operatorname{dom}_{\mathrm{nst}} F$. Since $x_1 \approx x \approx x_2$, we have $F(x_1) \approx F(x_2)$. Therefore $y_1 = {}^\circ(F(x_1)) = {}^\circ(F(x_2)) = y_2$. According to the transfer principle ([7]), if $(x, y_i) \in {}^\circ(\operatorname{graph} F)$, then $y_1 = y_2$ also in the case when (x, y_i) are not supposed to be standard. ▶

0.2.2. Corollary. *The restriction of a nearstandard map is nearstandard.*

0.2.3. Caution. 1. A standard map is not necessarily graph-nearstandard. An example: a standard function $F : \mathbb{R} \to \mathbb{R}$ with discontinuities of the first kind: the set $^\circ(\operatorname{graph} F)$ contains the points $(x, F(x-0))$, $(x, F(x))$, $(x, F(x+0))$ for any $x \in {}^{\mathrm{st}}\operatorname{dom} F$.

2. The notion of shadow $^\circ F$ resembles the notion of standardization $^{\mathrm{S}}F$, but differs from it. Suppose that $\forall^{\mathrm{st}} x \in \operatorname{dom} F$ ($F(x)$ is nearstandard); then $^{\mathrm{S}}F$ is defined as a standard map $\mathcal{G}$ such that $\forall^{\mathrm{st}} x \in \operatorname{dom} \mathcal{G}$ $(\mathcal{G}(x) = {}^\circ(F(x))$. In IST, the existence and uniqueness of such a mapping is proved. If F is standard, then $^{\mathrm{S}}F = F$.

0.2.4. *If a standard map F is graph-nearstandard, then $F \subseteq {}^\circ F$.*

◁ If $x \in {}^{\mathrm{st}}\operatorname{dom} F$, then ([8]) $F(x) \in {}^{\mathrm{st}}Y$. In particular, $x \in \operatorname{dom}_{\mathrm{nst}} F$. Therefore according to (0.6), $\forall^{\mathrm{st}} x \in \operatorname{dom} F$ $(({}^\circ F)(x) = {}^\circ(F(x)) = F(x))$. By the transfer principle, $\forall x \in \operatorname{dom} F$ $(({}^\circ F)(x) = F(x))$. ▶

0.3. $\langle$**nst**$\rangle$ **condition for linear operators.** Let X and Y be standard Banach spaces and let A be a linear operator $X \to Y$. It is easy to see that the $\langle\mathrm{nst}\rangle$ condition for the operator A can be simplified as follows:

$$\forall x \in \operatorname{dom}_{\mathrm{nst}} A \ (x \approx 0 \Rightarrow Ax \approx 0). \tag{0.8}$$

Moreover, formula (0.4) describing the domain of definition of the shadow $^\circ A$ and formula (0.6) describing the rule according to which $^\circ A$ operates are equivalent respectively to the following:

$$^{\mathrm{st}}\operatorname{dom}({}^\circ A) = \{x \in {}^{\mathrm{st}}X : \exists x_1 \in \operatorname{dom}_{\mathrm{nst}} A \ (x \approx x_1)\}, \tag{0.9}$$

$$x \in \operatorname{dom}_{\mathrm{nst}} A \Rightarrow {}^\circ x \in \operatorname{dom}({}^\circ A) \wedge ({}^\circ A){}^\circ x = {}^\circ(Ax). \tag{0.10}$$

Note that the $\langle\mathrm{nst}\rangle$ condition (0.8) is somewhat similar to the closability condition for a linear operator. Therefore, the following is not surprising.

([7]) The transfer principle of IST consists in the following: let $p(x)$ be a sentence about an object x whose formulation *does not* contain (even implicitly) the property "st"= "to be standard". Then $\forall^{\mathrm{st}} x\, p(x) \Leftrightarrow \forall x\, p(x)$. An equivalent form of the transfer principle is $\exists x\, p(x) \Leftrightarrow \exists^{\mathrm{st}} x\, p(x)$. In particular, $\exists! x\, p(x) \wedge p(a) \Rightarrow \mathrm{st}\, a$.

([8]) By the transfer principle, a standard mapping takes standard values at standard points.

0.3.1. *The shadow $^\circ A$ of a nearstandard operator is a closed operator.*

$\lhd$ In IST, it is proved that the shadow of an (internal [9]) set is closed (see, for example, [1]).

0.3.2. Corollary. *A standard closable linear operator is nearstandard and its shadow coincides with its closure.*

$\lhd$ Suppose $(z_n)_{n\in\mathbb{N}}$ is a standard sequence in a standard metric space Z and let $a \in {}^{\mathrm{st}}Z$. It is well known that the equality $\lim_{n\to\infty} z_n = a$ is equivalent to $z_n \approx a$ for all $n \approx +\infty$. Hence, it is clear that for a standard linear operator $A : X \to Y$ the closability condition is equivalent to the $\langle$nst$\rangle$ condition (0.8). It remains to use the fact that the shadow of a standard set coincides with its closure. ▶

0.4. Nearstandardness on $\mathcal{B}(X;Y)$. By $\mathcal{B}(X;Y)$ we denote the Banach space of all bounded linear operators $A \in Y^X$, where X, Y are Banach spaces over the field $\mathbb{C}$ of complex numbers. In the sequel, we assume that X, Y are standard, hence $\mathcal{B}(X;Y)$ is also standard. The cardinality of $\mathcal{B}(X;Y)$ is infinite. Therefore this space contains nonstandard operators as well as standard ones. In particular, for $A \in \mathcal{B}(X;Y)$ it can happen that $\|A\| \approx 0$ or $\|A\| \approx +\infty$.

0.4.1. *Let $A \in \mathcal{B}(X;Y)$; if $\|A\| \ll \infty$, then A is graph-nearstandard, and* [10]

$$\forall x \in {}^{\mathrm{st}}X \quad (x \in \mathrm{dom}(^\circ A) \Leftrightarrow Ax \in {}^{\mathrm{nst}}Y), \tag{0.11}$$

$$\forall x \in {}^{\mathrm{st}}\mathrm{dom}(^\circ A) \quad (^\circ A)x = {}^\circ(Ax), \tag{0.12}$$

$$\forall x \in \mathrm{dom}(^\circ A) \quad \|^\circ Ax\| \le {}^\circ\|A\| \cdot \|x\|. \tag{0.13}$$

$\lhd$ If $\|A\| \ll \infty$, then $\forall x \in \mathrm{dom}\, A$ $(x \approx 0 \Rightarrow Ax \approx 0)$ and, in particular, the $\langle$nst$\rangle$ condition is satisfied. This means that the operator A is nearstandard.

Suppose $x \in {}^{\mathrm{st}}\mathrm{dom}(^\circ A)$. Then (see (0.9)) $x \approx x_1$ for some $x_1 \in X$ such that $Ax_1 \in {}^{\mathrm{nst}}Y$ and (see (0.10)) $(^\circ A)x \approx Ax_1$. We have $Ax_1 \approx Ax$ whenever $\|A\| \ll \infty$. Therefore $Ax \in {}^{\mathrm{nst}}Y$. Moreover, $(^\circ A)x \approx Ax_1 \approx Ax$ and, consequently, $(^\circ A)x = {}^\circ(Ax)$. Conversely, if $x \in {}^{\mathrm{st}}X$ and $Ax \in {}^{\mathrm{nst}}Y$, then according to (0.9), $x \in \mathrm{dom}(^\circ A)$. Finally, if $x \in {}^{\mathrm{st}}\,\mathrm{dom}(^\circ A)$, then [11] $\|(^\circ A)x\| = \|^\circ(Ax)\| = {}^\circ\|Ax\| \le {}^\circ\|A\| \cdot \|x\|$ and by the transfer principle, (0.13) is satisfied. ▶

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(9) We just give a simple explanation. A set that is an element of some standard set is called *internal*. For internal sets (according to the transfer principle) all statements of the standard (= usual) mathematics are true. It should be mentioned that collections that are *parts* of standard sets, generally speaking, do not have such a privilege. For example, $\mathbb{I} := \{x \in \mathbb{R} : x \approx 0\}$. This collection of infinitesimal real numbers is bounded: $\mathbb{I} \subseteq [-1,+1]$ but $\inf \mathbb{I}$ and $\sup \mathbb{I}$ do not exist. The collection $\mathbb{I}$ is an external set, to be more exact, a strictly external set.

(10) Another abbreviation is used here: ${}^{\mathrm{nst}}M := \{m \in M : m \text{ is nearstandard}\}$ for a standard Hausdorff space. Note that the collection ${}^{\mathrm{nst}}M$ is an external set.

(11) A norm defined on a standard normed space is standard. It is well known that a standard function f is continuous at the point $x_0 \in {}^{\mathrm{st}}\mathrm{dom}\, f$ iff $\forall x \in \mathrm{dom}\, f$ $(x \approx x_0 \Rightarrow f(x) \approx f(x_0))$, i.e., $^\circ x = x_0 \Rightarrow {}^\circ[f(x)] = f(x_0)$.

The condition $\|A\| \ll \infty$ is sufficient for a linear operator A to be graph-nearstandard. This shows that the notion of graph-nearstandardness is not sapid enough unless we impose further restrictions. The following observation can serve as a verification.

0.4.2. *There exists an operator* $A \in \mathcal{B}(X;Y)$ *with* $\|A\| \ll \infty$ *such that* $\mathrm{dom}({}^{\circ}A) = \{0\}$.

$\lhd$ Let $X = H$ be a standard Hilbert space, let $(e_n)_{n\in\mathbb{N}}$ be a standard orthonormal basis for H, let $\omega \in \mathbb{N}$, and $\omega \approx +\infty$. Define $Ax := \sum_{k\le\omega}(x|e_k)e_{k+\omega}$ for $x \in H$. Then $A \in \mathcal{B}(H)$ and $\|A\| = 1$. If $x \in {}^{\mathrm{st}}H \setminus \{0\}$, then the number $\sum_{k\le\omega}|(x|e_k)|^2 \approx \|x\|^2$ is not infinitesimal and, consequently ([12]), the vector Ax is not nearstandard. Therefore ${}^{\mathrm{st}}\mathrm{dom}({}^{\circ}A) = \{0\}$. By the transfer principle, $\mathrm{dom}({}^{\circ}A) = \{0\}$. ▶

0.5. Strong and uniform nearstandardness. As before, we denote by X and Y standard Banach spaces. We call an operator $A \in \mathcal{B}(X;Y)$ *strongly nearstandard* if

$$\|A\| \ll \infty \quad \text{and} \tag{0.14}$$

$$\forall x \in {}^{\mathrm{st}}X \quad \|Ax - A_0x\|_Y \approx 0 \tag{0.15}$$

for some $A_0 \in {}^{\mathrm{st}}\mathcal{B}(X;Y)$.

0.5.1. *Let an operator A be strongly nearstandard. By* (0.14) *and* 0.4.1, ${}^{\circ}(\mathrm{graph}\,A)$ *is a functional graph, and the operator A_0 from* (0.15) *coincides with the shadow ${}^{\circ}A$, that is*, ${}^{\circ}(\mathrm{graph}\,A) = \mathrm{graph}\,A_0$.

$\lhd$ If $x \in {}^{\mathrm{st}}X$, then according to (0.15), $Ax \approx A_0x \in {}^{\mathrm{st}}Y$. By (0.11), this means that $x \in \mathrm{dom}({}^{\circ}A)$. Applying the transfer principle, we get $\mathrm{dom}({}^{\circ}A) = X$. From (0.12) and (0.15) it follows that ${}^{\circ}A = A_0$. ▶

0.5.2. *Let $A \in \mathcal{B}(X;Y)$ and let $\|A\| \ll \infty$. For A to be strongly nearstandard it suffices that it is densely defined.*

$\lhd$ Let $\mathrm{Cl}\,\mathrm{dom}({}^{\circ}A) = X$. According to 0.3.1, the operator A is closed. Therefore $\mathrm{dom}({}^{\circ}A)$ is a closed set. Consequently, $\mathrm{dom}({}^{\circ}A) = X$. ▶

0.5.3. Condition (0.14) in the definition of strong nearstandardness is essential. Let $A \in \mathcal{B}(X;Y)$ and suppose there exists an operator $A_0 \in {}^{\mathrm{st}}\mathcal{B}(X;Y)$ such that condition (0.15) holds. If $\|A\| \approx +\infty$, then the set ${}^{\circ}(\mathrm{graph}\,A)$ is not necessarily a functional graph. For example, let H be a standard infinite-dimensional Hilbert space and let H_0 be a subspace of H_0 such that ([13]) $\dim H_0 \in \mathbb{N}$ and ${}^{\mathrm{st}}H \subset H_0$. Consider unit vectors $e, f \in H$ such that $e \in {}^{\mathrm{st}}H$, $f \in H_0^{\perp}$ and define $Ax = \omega(x|f)e$ for $x \in X$, where $\omega \in \mathbb{N}$ and $\omega \approx +\infty$. Set $x := \omega^{-1}f$. Then $x \approx 0$ and $Ax = e$. Since $Ax \in {}^{\mathrm{st}}H$ and $Ax \not\approx 0$, we see that the operator A does not satisfy the $\langle\mathrm{nst}\rangle$ condition. By putting $A_0 = 0$, we find that $A_0 \in {}^{\mathrm{st}}\mathcal{B}(H)$ and that (0.15) is satisfied.

([12]) The following nearstandardness criterion holds in a standard Hilbert space: $\sum_{n\in\mathbb{N}} c_n e_n \in {}^{\mathrm{nst}}H \Leftrightarrow \forall\omega \in \mathbb{N}\ (\omega \approx 0 \Rightarrow \sum_{n>\omega}|c_n|^2 \approx 0)$.

([13]) Existence of such a space is well known. It is proved in IST by means of the idealization principle.

We call an operator $A \in \mathcal{B}(X;Y)$ *uniformly nearstandard* if

$$\|A - A_0\| \approx 0 \tag{0.16}$$

for some $A_0 \in {}^{\mathrm{st}}\mathcal{B}(X;Y)$.

0.5.4. *Uniform nearstandardness implies strong nearstandardness. Moreover, the operator* $A_0 \in {}^{\mathrm{st}}\mathcal{B}(X;Y)$ *from* (0.16) *coincides with the shadow* ${}^\circ A$: ${}^\circ(\operatorname{graph} A) = \operatorname{graph} A_0$.

◁ From (0.16) it follows that $\|A\| \le \|A_0\| + 1$. Since $\|A_0\| \ll \infty$, A satisfies (0.14). Moreover, it follows from (0.16) that (0.15) is satisfied. ▶

If A is uniformly nearstandard, then

$$\|{}^\circ A\| = {}^\circ\|A\|. \tag{0.17}$$

Indeed, from (0.16) it follows that $\|A\| \approx \|A_0\| = \|{}^\circ A\|$.

0.5.5. Suppose A is strongly nearstandard. As follows from (0.13),

$$\|{}^\circ A\| \le {}^\circ\|A\|. \tag{0.18}$$

If an operator A is not uniformly nearstandard, then inequality (0.18) can be strict. For example, let H be a standard infinite-dimensional separable Hilbert space, let $(e_n)_{n\in\mathbb{N}}$ be a standard orthonormal basis for H, $\omega \in \mathbb{N}$, and $\omega \approx +\infty$. Denote by $P = P_\omega$ the orthoprojector of H onto $\operatorname{span}\{e_1, \dots, e_\omega\}$. Then $P \in \mathcal{B}(H)$, $\|P\| = \|\mathbb{I} - P\| = 1$, and (see footnote 12) $\|x - Px\| \approx 0$ for all $x \in {}^{\mathrm{st}}H$. Therefore the operator $\mathbb{I} - P$ (as well as P) is strongly nearstandard, and ${}^\circ(\mathbb{I} - P) = 0$. In particular, $\|{}^\circ(\mathbb{I} - P)\| = 0 < 1 = \|\mathbb{I} - P\|$. P and $\mathbb{I} - P$ are not uniformly nearstandard.

0.5.6. It is easy to see that the (external) set of strongly nearstandard operators $A \in \mathcal{B}(X;Y)$ is a *module* over the ring of *finite* ([14]) complex numbers. Moreover, for operators A, B from this module and for finite $a, b \in \mathbb{C}$, we have

$${}^\circ(aA + bB) = {}^\circ a\,{}^\circ A + {}^\circ b\,{}^\circ B. \tag{0.19}$$

If $A \in \mathcal{B}(X;Y)$, $B \in \mathcal{B}(X;Y)$, and A, B are strongly nearstandard, then the operator BA is strongly nearstandard, and

$${}^\circ(BA) = {}^\circ B\,{}^\circ A. \tag{0.20}$$

Obviously, an analogous statement holds for uniformly nearstandard operators. The next proposition follows easily from (0.20).

0.5.7. *Let* $A \in \mathcal{B}(X;Y)$ *be a bijection, and let* A *and* A^{-1} *be strongly nearstandard. Then* ${}^\circ A$ *is also a bijection and*

$${}^\circ(A^{-1}) = ({}^\circ A)^{-1}. \tag{0.21}$$

0.5.8. It also follows from (0.20) that for a strongly nearstandard projector $P \in \mathcal{N}(X)$ its shadow ${}^\circ P$ is also a *projector*, i.e., $P^2 = P \Rightarrow ({}^\circ P)^2 = {}^\circ P$.

0.5.9. An operator adjoint to a strongly nearstandard one is not necessarily strongly nearstandard. For example, let $(e_n)_{n\in\mathbb{N}}$ be a standard orthonormal basis for a standard

([14]) A number $c \in \mathbb{C}$ is called *finite*, written $|c| \ll \infty$, if $\exists^{\mathrm{st}} n \in \mathbb{N}\ (|c| \le n)$.

Hilbert space H, $\omega \in \mathbb{N}$ and $\omega \approx +\infty$. Define $Ax := (x|e_\omega)e_1$ for $x \in H$. Then A is strongly nearstandard and $°A = 0$ because $\forall^{\mathrm{st}} x \in H\ (x|e_\omega) \approx 0$ ([15]). But $A^*x = (x|e_1)e_\omega$ and $\|A^*\| = 1 \ll \infty$, hence (see 0.4.1) $°(\mathrm{graph}(A^*))$ is a functional graph. Furthermore, since e_ω is a *remote* ([16]) vector, we find $\mathrm{dom}(°(A^*)) = \{0\}$. Thus A^* is not strongly nearstandard.

Nevertheless the following statement holds.

0.5.10. *Let $A \in \mathcal{B}(X;Y)$, and let both A and A^* ($\in \mathcal{B}(Y^*;X^*)$) be strongly nearstandard. Then*

$$°(A^*) = (°A)^*. \tag{0.22}$$

◁ Suppose $x \in {}^{\mathrm{st}}X$ and $f \in {}^{\mathrm{st}}(Y^*)$. By (0.15) and (0.20), we have $f(°Ax) = °[f(Ax)] = °[A^*f(x)] = [°(A^*f)](x) = \{°(A^*)f\}(x)$. Hence (0.22) follows by the transfer principle. ▶

0.5.11. The notion of uniform nearstandardness is more pleasant. In view of the fact that the norms of A and A^* are equal whenever A is a uniformly nearstandard operator ($\in \mathcal{B}(X;Y)$), we see that A^* is then also uniformly nearstandard ($\in \mathcal{B}(Y^*;X^*)$). Moreover (see (0.16)), equality (0.22) is satisfied.

0.5.12. Once more note that the notion of nearstandardness depends on the choice of topology. Supplying the Cartesian product $X \times Y$ of standard Banach spaces X, Y with the weak topology, we can consider a more general notion $°(\mathrm{graph}\,A)$ of the graph shadow for an operator $A \in \mathcal{B}(X;Y)$. There is a more general corresponding nearstandardness notion. Also there is a more general weak nearstandardness notion. Namely, we call an operator $A \in \mathcal{B}(X;Y)$ weakly nearstandard if $\|A\| \ll \infty$ and

$$\forall x \in {}^{\mathrm{st}}X\ \forall f \in {}^{\mathrm{st}}(Y^*) \quad f(Ax) \approx f(A_0x) \tag{0.23}$$

for some operator $A_0 \in {}^{\mathrm{st}}\mathcal{B}(X;Y)$. It is obvious that strong nearstandardness implies weak nearstandardness, and the standard operator A_0 from (0.23) coincides with the shadow $°A$.

Another natural way of generalization of the nearstandardness notion is concerned with the space equipment techniques.

1. Standard filling

The method of finite-dimensional approximation is one of effective means of investigating infinite-dimensional problems. The transition from a nearstandard object to its shadow is a variety of such approximation. The aspiration to infinite precision requires

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([15]) It is well known that for a standard sequence $(a_n)_{n\in\mathbb{N}}$ and for a standard point a in a standard metric space the equality $\lim_{n\to\infty} a_n = a$ is equivalent to the condition $a_\omega \approx a$ for all $\omega \approx +\infty$.

([16]) An element (of a standard space) is called *remote* if it is not nearstandard. Remoteness of a vector e_ω with $\omega \approx +\infty$ is obvious from the nearstandardness criterion in footnote 12.

the use of approximating spaces of a nonstandard natural dimension. There is no natural nearstandardness structure in such spaces. But it may be introduced by means of appropriate maps. One possible variant to realize this plan is presented below.

From now on our constructions are concerned with a fixed set (1) $\mathbb{T}$ such that

$$\operatorname{card}\mathbb{T} \in \mathbb{N}, \quad \operatorname{card}\mathbb{T} \approx +\infty. \tag{1.1}$$

Obviously, the set $\mathbb{T}$ is nonstandard. First of all we introduce the structure of standardness into the algebra (power set) $2^{\mathbb{T}}$ of all subsets of $\mathbb{T}$.

1.1. Definition of a standard filling. We suppose that there exist a standard set $\mathbf{T}$, a standard σ-algebra Λ of sets $\mathcal{E} \in 2^{\mathbf{T}}$, and a mapping $Q : 2^{\mathbb{T}} \to 2^{\mathbf{T}}$ which satisfies the following conditions:

$$\forall t \in \mathbb{T} \quad Qt := Q\{t\} \in \Lambda, \tag{1.2}$$

$$\forall t, s \in \mathbb{T} \quad (t \neq s \Rightarrow Qt \cap Qs = \emptyset), \tag{1.3}$$

$$\forall E \in 2^{\mathbb{T}} \quad QE = \bigcup_{t \in E} Qt, \tag{1.4}$$

$$^{\mathrm{S}}(Q\mathbb{T}) = \mathbf{T}, \tag{1.5}$$

where $^{\mathrm{S}}M$ denotes the standardization (2) of a set M. Note that, by (1.2) and (1.4), the following condition is also satisfied:

$$\forall E \in 2^{\mathbb{T}} \quad QE \in \Lambda. \tag{1.6}$$

1.1.1. If a triple $(\mathbf{T}, \Lambda, Q)$ satisfies conditions (1.2)–(1.6), then we call it a *standard filling* of the finite set $\mathbb{T}$. In this case, the mapping Q is called the *embedding* $2^{\mathbb{T}} \to \Lambda$.

1.1.2. Conditions (1.2)–(1.4) ensure the *injectivity* of the embedding Q. From these conditions it follows that Q preserves the operations $\cap$, $\cup$, $\setminus$. Requirement (1.5) is aimed at the most economical choice of the standard set $\mathbf{T}$. Note, however, that $Q(\mathbb{T} \setminus E)$ is not necessarily equal to $\mathbf{T} \setminus QE$. In particular, $Q\mathbb{T}$ may be different from $\mathbf{T}$.

1.1.3. Let $(\mathbf{T}, \Lambda, Q)$ and $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{Q})$ be two standard fillings of the same finite set $\mathbb{T}$ (see 1.1). These fillings are called *equivalent* if there exists a standard bijection $f : \mathbf{T} \to \widetilde{\mathbf{T}}$ such that f and f^{-1} are measurable with respect to the σ-algebras Λ, $\widetilde{\Lambda}$, and $f(Qt) = \widetilde{Q}t$ for all $t \in \mathbb{T}$.

1.1.4. Example. Let $\omega = \operatorname{card}\mathbb{T}$. Suppose $\mathbb{T}$ is a collection $\{t_0, t_1, \ldots, t_{\omega-1}\}$ of real numbers arranged in increasing order. Let $\varphi \in \mathbb{R}^{\mathbb{R}}$ be a strictly increasing function. Define $Qt_k = [\tau_k, \tau_{k+1}[= \{\tau \in \mathbb{R} : \tau_k \leq \tau < \tau_{k+1}\}$ for $k < \omega$, where $\tau_k := \varphi(t_k)$, $QE = \bigcup_{t \in E} Qt$ for $E \in 2^{\mathbb{T}}$, and $\mathbf{T} = {}^{\mathrm{S}}(Q\mathbb{T})$. Let Λ be the algebra of all *borelian* subsets of the set $\mathbf{T}$, which is considered as a part of the space $\mathbb{R}^1$. Then $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{Q})$ is a standard filling *corresponding to the function* φ. Now take any increasing function $f \in \mathbb{R}^{\mathbb{R}}$ and set $\widetilde{\varphi} := f \circ \varphi$. It is clear

(1) Recall that "set" means "internal set". The same concerns the terms "subset", "algebra", etc.

(2) Suppose that an (external or internal) set M is defined as a part of a standard set M_0 by $M = \{x \in M_0 : p(x)\}$. Then its *standardization* $^{\mathrm{S}}M$ is uniquely defined by the conditions: (1) $^{\mathrm{S}}M$ is a standard set, (2) $\forall^{\mathrm{st}} x \in M_0$ $(x \in {}^{\mathrm{S}}M \Leftrightarrow x \in M \wedge p(x))$.

that the standard filling $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{Q})$ corresponding to $\widetilde{\varphi}$ is equivalent to the standard filling $(\mathbf{T}, \Lambda, Q)$ corresponding to φ.

1.2. Charge spaces. Let $(\mathbf{T}, \Lambda, Q)$ be a fixed standard filling of $\mathbb{T}$. We denote by $\mathcal{N}$ the space of all *charges* (complex-valued additive functions) that are defined on the algebra $2^{\mathbb{T}}$. The set $\mathcal{N}$ is supplied with the structure of a linear space in which addition and multiplication by numbers are defined pointwise, and with the norm defined as follows:

$$\forall \nu \in \mathbb{N} \quad \|\nu\| := (\operatorname{var}\nu)\mathbb{T} = \sum_{t\in\mathbb{T}} |\nu_t|, \quad \nu_t := \nu t. \tag{1.7}$$

Obviously, $\mathcal{N}$ is a Banach space of dimension

$$\dim \mathcal{N} = \operatorname{card}\mathbb{T} \approx +\infty. \tag{1.7′}$$

We denote by $\mathcal{M}$ the set of all σ-additive complex-valued charges defined on the σ-algebra Λ. It is a linear space with the seminorm

$$\forall \mu \in \mathcal{M} \quad \|\mu\|' := (\operatorname{var}\mu)Q\mathbb{T}, \tag{1.8}$$

where

$$\forall \mathcal{E} \in \Lambda \quad (\operatorname{var}\mu)\mathcal{E} := \sup \sum_k |\mu\mathcal{E}_k|;$$

here the sup is taken over all disjoint decompositions $\mathcal{E} = \bigcup_k \mathcal{E}_k$, $\mathcal{E}_k \in \Lambda$.

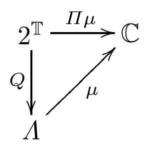
1.2.1. *Inductor.* To each charge $\mu \in \mathcal{M}$ we assign a charge $\nu = \Pi\mu \in \mathcal{N}$ given by

$$\forall E \in 2^{\mathbb{T}} \quad \nu E = \Pi\mu E := \mu Q E; \tag{1.9}$$

in particular,

$$\forall t \in \mathbb{T} \quad (\Pi\mu)_t := \Pi\mu\{t\} = \mu Q t. \tag{1.9′}$$

The commutative diagram below illustrates the situation.



We say that Π is the *inductor* $\mathcal{M} \to \mathcal{N}$. According to definition (1.9), Π is adjoint to the embedding $Q : 2^{\mathbb{T}} \to \Lambda$.

1.2.2. *The inductor Π is a surjection $\mathcal{M} \to \mathcal{N}$ and a contractive map in the sense that*

$$\forall \mu \in \mathcal{M} \quad \|\Pi\mu\| \le \|\mu\|'. \tag{1.10}$$

$\lhd$ It is clear that for each $\nu \in \mathcal{N}$ there exists a $\mu \in \mathcal{M}$ such that

$$\forall t \in \mathbb{T} \quad \mu Q t = \nu_t := \nu\{t\}. \tag{1.11}$$

Moreover,

$$\|\Pi\mu\| = \sum_{t\in\mathbb{T}} |(\Pi\mu)_t| = \sum_{t\in\mathbb{T}} |\mu Q t| \le \sum_{t\in\mathbb{T}} (\operatorname{var}\mu) Q t = \|\mu\|'. \ \blacktriangleright$$

1.2.3. *Quasi-kernel of the inductor.* The set $\operatorname{qker} \Pi$ defined by the formula

$$\operatorname{qker} \Pi := \{\mu \in \mathcal{M} : \forall E \in 2^{\mathbb{T}} \ \Pi\mu \approx 0\} \tag{1.12}$$

is called the *quasi-kernel* of the inductor Π. We say that the inductor $\Pi : \mathcal{M} \to \mathcal{N}$ is *exact* if

$$^{\mathrm{st}}\operatorname{qker} \Pi = \{0\}, \tag{1.13}$$

that is, for all $\mu \in {}^{\mathrm{st}}\mathcal{M}$,

$$\forall E \in 2^{\mathbb{T}} \quad (\mu Q E \approx 0 \Rightarrow (\forall \mathcal{E} \in \Lambda)(\mu\mathcal{E} = 0)). \tag{1.13$'$}$$

1.2.4. Let $(\mathbf{T}, \Lambda, Q)$ and $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{Q})$ be equivalent standard fillings of the same finite set $\mathbb{T}$. Let f be the standard bijection $\mathbf{T} \to \widetilde{\mathbf{T}}$ mentioned in 1.1.3. We denote by $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ the seminormed spaces of σ-additive charges on the σ-algebras Λ and $\widetilde{\Lambda}$ respectively. Define a map $f_* : \mathcal{M} \to \widetilde{\mathcal{M}}$ by the formula

$$\forall \mu \in \mathcal{M} \quad (\widetilde{\mu} = f_*\mu \Leftrightarrow \forall \mathcal{E} \in \Lambda \ (\widetilde{\mu} f(\mathcal{E}) = \mu\mathcal{E})), \tag{1.14}$$

where $f(\mathcal{E}) := \{f(\tau) : \tau \in \mathcal{E}\}$. The map f_* is bijective, and

$$\forall \widetilde{\mu} \in \widetilde{\mathcal{M}} \ \forall \mathcal{M} \in \Lambda \quad (f_*^{-1})\mathcal{E} = \widetilde{\mu} f^{-1}(\mathcal{E}). \tag{1.15}$$

Moreover, since for all $\mu \in \mathcal{M}$ and $\mathcal{E} \in \Lambda$,

$$(\operatorname{var} \mu)\mathcal{E} = (\operatorname{var} f_*\mu) f(\mathcal{E}), \tag{1.16}$$

we see that f_* preserves the seminorm $\|\cdot\|'$:

$$\|\mu\|' = \|f_*\mu\|'. \tag{1.16$'$}$$

To prove (1.16), set $\widetilde{\mathcal{E}} := f(\mathcal{E})$ and $\widetilde{\mu} := f_*\mu$. Since, by (1.14), $\widetilde{\mu}\widetilde{\mathcal{E}} = \mu\mathcal{E}$, we find

$$(\operatorname{var} \mu)\mathcal{E} = \sup \sum_k |\mu\mathcal{E}_k| = \sup \sum_k |\widetilde{\mu}\widetilde{\mathcal{E}}_k| = (\operatorname{var} \widetilde{\mu})\widetilde{\mathcal{E}}.$$

1.2.5. *Let Π and $\widetilde{\Pi}$ be inductors $\mathcal{M} \to \mathcal{N}$ and $\widetilde{\mathcal{M}} \to \mathcal{N}$ respectively associated with the standard fillings from* 1.2.4. *The inductor Π is exact if and only if the inductor $\widetilde{\Pi}$ is exact.*

◁ By the standardness of f, the map f_* is also standard. Therefore,

$$\mu \in {}^{\mathrm{st}}\mathcal{M} \Leftrightarrow f_*\mu \in {}^{\mathrm{st}}\widetilde{\mathcal{M}}. \tag{1.17}$$

Denoting $\widetilde{\mu} := f_*\mu$ we find that $\widetilde{\mu}\widetilde{Q}E = \widetilde{\mu} f(QE) = \mu QE$ for all $E \in 2^{\mathbb{T}}$ and, therefore,

$$\operatorname{qker} \widetilde{\Pi} = f_*(\operatorname{qker} \Pi), \tag{1.18}$$

hence our statement follows. ▶

1.3. Discrete interval. Take a number $h > 0$, $h \approx 0$, and integers m, n, $m < n$. Define

$$a := mh, \ b := nh, \ \overleftarrow{ab} := \{t = kh : m \le k < n\}. \tag{1.19}$$

We say that the set $\overleftarrow{ab}$ is the *discrete interval* with step h and endpoints a, b. The *natural standard filling* of the set $\mathbb{T} = \overleftarrow{ab}$ is defined as follows: put

$$\mathbf{T} := [{}^\circ a, {}^\circ b], \tag{1.20}$$

where for all $\tau \in \mathbb{R}$,

$$(1.21)\qquad {}^\circ\tau \in {}^{\mathrm{st}}\overline{\mathbb{R}} \quad \text{and} \quad {}^\circ\tau \approx \tau \quad \text{if } |\tau| \ll \infty, \qquad {}^\circ\tau = \begin{cases} -\infty & \text{if } \tau \approx -\infty, \\ +\infty & \text{if } \tau \approx +\infty; \end{cases}$$

here $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. We take the σ-algebra of Lebesgue measurable sets $\mathcal{E} \subseteq \mathbf{T}$ for Λ. The embedding $Q : 2^{\mathbb{T}} \to \Lambda$ is defined by

$$(1.22)\qquad \forall E \in 2^{\mathbb{T}} \quad QE = \bigcup_{t \in E} [t, t+h[.$$

Note that $Q\mathbb{T} = [a, b[$. In order to ensure the inclusion $Q\mathbb{T} \subseteq \mathbf{T}$ we require that

$$(1.23)\qquad {}^\circ a \le a, \qquad b \le {}^\circ b.$$

One can achieve this by modifying slightly the integers m and n in (1.19).

Now conditions (1.2)–(1.6) are satisfied so that $(\mathbf{T}, \Lambda, Q)$ is a standard filling for $\overleftarrow{ab}$. It is useful to suppose that the step h has the form

$$(1.24)\qquad h = (\omega_0!)^{-1}, \quad \text{where } \omega_0 \in \mathbb{N} \text{ and } \omega_0 \approx +\infty.$$

1.3.1. LEMMA. *Let condition* (1.24) *be satisfied*; *then each standard rational number is a multiple of the step h.*

◁ Suppose $p \in {}^{\mathrm{st}}\mathbb{Z}$, $q \in {}^{\mathrm{st}}\mathbb{N}$. Then $k := \omega_0!/q \in \mathbb{Z}$. Therefore $p/q = pk/\omega_0! = pkh$ and $pk \in \mathbb{Z}$. ▶

1.3.2. COROLLARY. *Let condition* (1.24) *be satisfied*; *then* $[a, b[\cap {}^{\mathrm{st}}\mathbb{Q} \subset \overleftarrow{ab}$.

1.3.3. *If* (1.24) *is satisfied*, *then the inductor $\Pi : \mathcal{M} \to \mathcal{N}$ corresponding to the natural standard filling is exact.*

◁ Suppose $\mu \in {}^{\mathrm{st}}\mathrm{qker}\,\Pi$, i.e., $\mu \in {}^{\mathrm{st}}\mathcal{M}$ and $\Pi\mu E \approx 0$ for $E \subseteq \overleftarrow{ab}$. We have to prove that $\mu = 0$. By the transfer principle and the regularity of the charges from $\mathcal{M}$, it suffices to show that $\mu[\alpha, \beta] = 0$ for each standard interval $[\alpha, \beta[\subset \mathbf{T}$. To this end, consider numbers $\alpha, \beta \in {}^{\mathrm{st}}\mathbb{R}$, $\alpha < \beta$, and standard sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$ of rational numbers α_n, β_n such that $\alpha_n \nearrow \alpha$ and $\beta_n \nearrow \beta$ as $n \to \infty$. Since μ is σ-additive, we obtain $\mu[\alpha_n, \beta_n[\to \mu[\alpha, \beta[$ as $n \to \infty$. By the nonstandard convergence criterion (see footnote 15 in Section 0),

$$(1.25)\qquad n \approx +\infty \Rightarrow \mu[\alpha_n, \beta_n[\approx \mu[\alpha, \beta[.$$

If $n \in {}^{\mathrm{st}}\mathbb{N}$, then by 1.3.2,

$$\mu[\alpha_n, \beta_n[= \sum_{t \in [\alpha_n, \beta_n[} \mu[t, t+h[= \Pi\mu E,$$

where $E := \{t \in \overleftarrow{ab} : \alpha_n \le t < \beta_n\} \in 2^{\mathbb{T}}$. Consequently, $\mu[\alpha_n, \beta_n[\approx 0$ for standard n. By the Robinson lemma, this holds up to some $n \approx +\infty$. Now (1.25) yields $\mu[\alpha, \beta[\approx 0$. But $\mu[\alpha, \beta[\in {}^{\mathrm{st}}\mathbb{R}$, hence $\mu[\alpha, \beta[= 0$. ▶

1.3.4. REMARK. The inductor $\Pi : \mathcal{M} \to \mathcal{N}$ will still be exact whenever condition (1.24) is replaced by a more "liberal" one. It suffices to require that the step h of the discrete interval has the form

$$\widetilde{h} := \gamma(\omega_0!)^{-1}, \tag{1.26}$$

where $\omega_0 \in \mathbb{N} \setminus {}^{\mathrm{st}}\mathbb{N}$, $\gamma \in {}^{\mathrm{st}}\mathbb{R}$, $\gamma > 0$.

$\lhd$ Define $f(\tau) := \gamma\tau$ for $\tau \in \widetilde{\mathbf{T}}$. The standard linear function f realizes the equivalence between the natural standard fillings $(\mathbf{T}, \Lambda, Q)$ and $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{Q})$. Then $\widetilde{\mathbf{T}} = [\gamma{}^\circ a, \gamma{}^\circ b]$, $\widetilde{\Lambda}$ is the σ-ring of finite Lebesgue measurable sets $\mathcal{E} \subset \mathbf{T}$, and $\widetilde{Q}t = [\gamma t, \gamma(t+h)[$ for all $t \in \mathbb{T}$. By 1.2.5, the inductor $\widetilde{\Pi}$ corresponding to $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{Q})$ is exact. However, $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{Q})$ is the *natural* standard filling of the discrete interval with step $\widetilde{h}$ from (1.26). ▶

1.4. Exact inductors. Further discussion only concerns those standard fillings to which correspond *exact inductors* $\Pi : \mathcal{M} \to \mathcal{N}$. This restriction enables us to introduce the standardness structure in a natural way.

1.4.1. A charge $\nu \in \mathcal{N}$ is called *standard*, written $\nu \in {}^{\mathrm{st}}\mathbb{N}$, if $\nu = \Pi\mu$ for some $\mu \in {}^{\mathrm{st}}\mathcal{M}$. In this case, μ is called the *standardized image* of ν.

Example. Let $\mathbb{T} := \overleftarrow{ab}$ be a discrete interval (see 1.3) and $(\mathbf{T}, \Lambda, Q)$ be its natural standard filling. Define $\nu E := h \operatorname{card} E$ for $E \in 2^{\mathbb{T}}$, where h is the step of the interval $\overleftarrow{ab}$. Then ν is a standard measure for which the standardized image is the usual Lebesgue measure on $\mathbf{T}$.

1.4.2. A charge $\nu \in \mathcal{N}$ is called *nearstandard*, written $\nu \in {}^{\mathrm{nst}}\mathcal{N}$, if $\|\nu - \nu_0\| \approx 0$ for some charge $\nu_\circ \in {}^{\mathrm{st}}\mathcal{N}$. Then ν_0 is said to be the *shadow* of ν *on* $\mathbb{T}$ and is denoted by ${}^\circ\nu$. Since ${}^\circ\nu \in {}^{\mathrm{st}}\mathcal{N}$, we have ${}^\circ\nu = \Pi({}^\bullet\nu)$ for some ${}^\bullet\nu \in {}^{\mathrm{st}}\mathcal{M}$. The charge ${}^\bullet\nu$ is called the *shadow* of ν *on* $\mathbf{T}$.

Example. Let $\mathbb{T} := \overleftarrow{ab}$, and $(\mathbf{T}, \Lambda, Q)$ be its natural standard filling. Suppose that $\nu \in \mathcal{N}$ and $\sum_{t\in\mathbb{T}} |\nu_t - h| \approx 0$; then $\nu \in {}^{\mathrm{nst}}\mathcal{N}$, and ${}^\bullet\nu$ is the ordinary Lebesgue measure on $\mathbf{T}$.

Example. Let $\mathbb{T} := \overleftarrow{ab}$, and $(\mathbf{T}, \Lambda, Q)$ be its natural standard filling. Denote by ν the Kronecker measure on $\mathbb{T}$ concentrated at the point t_0 (i.e., for $E \in 2^{\mathbb{T}}$, $\nu E = 0$ if $t_0 \in \mathbb{T} \setminus E$ and $\nu E = 1$ if $t_0 \in E$). Denote by μ the Dirac measure concentrated at $\tau_0 \in \mathbf{T}$ (i.e., for $\mathcal{E} \in \Lambda$. $\mu\mathcal{E} = 0$ if $\tau_0 \in \mathbf{T} \setminus \mathcal{E}$ and $\mu\mathcal{E} = 1$ if $t_0 \in \mathcal{E}$). Then the norm $\|\nu - \Pi\mu\|$ is either 0, 1, or 2. It is clear that $\|\nu - \Pi\mu\| = 0$ if and only if $\tau_0 \in Qt_0 = [t_0, t_0 + h[$. Therefore the measure ν is nearstandard if and only if $|t_0| \ll \infty$ and ${}^\circ t_0 \in [t_0, t_0 + h]$. In this case, ${}^\bullet\nu = \mu$, the measure ν is standard, and μ is the standardized image of ν.

Note that the map $\nu \mapsto {}^\bullet\nu$ is *not injective*, which can be seen from the preceding examples.

1.4.3. *The standardized image of the charge $\nu \in {}^{\mathrm{st}}\mathcal{N}$ and the shadow of this charge are uniquely defined.*

$\lhd$ Suppose $\|\nu - \Pi\mu_i\| \approx 0$, $\mu_i \in {}^{\mathrm{st}}\mathcal{M}$, $i = 1, 2$. Then $\|\Pi(\mu_1 - \mu_2)\| \approx 0$, hence $\mu_1 - \mu_2 \in {}^{\mathrm{st}}\mathrm{qker}\,\Pi$. By exactness of the inductor Π (see (1.13)), $\mu_1 = \mu_2$. ▶

1.5. Standard measure filling. One gets this from a standard filling by joining to $(\mathbf{T}, \Lambda, Q)$ some fixed standard measure λ defined on the σ-algebra Λ. The measure λ takes nonnegative values, maybe infinite. It is supposed to be σ-additive, finite, or σ-finite. We require that

$$\forall t \in \mathbb{T} \quad 0 < \lambda Q t < \infty. \tag{1.28}$$

In case $\lambda \mathbf{T} = +\infty$, the condition of σ-finiteness is supposed to be satisfied in the following form: there exists a standard sequence $(\mathbf{T}_n)_{n\in\mathbb{N}}$ of sets $\mathbf{T}_n \in \Lambda$ such that

$$\mathbf{T}_1 \subset \mathbf{T}_2 \subset \ldots \subset \bigcup_{n\in\mathbb{N}} \mathbf{T}_n = \mathbf{T}, \quad \forall n \in {}^{\mathrm{st}}\mathbb{N}\ \exists \mathbb{T}_n \in 2^{\mathbb{T}} \quad \mathbf{T}_n = Q\mathbb{T}_n. \tag{1.29}$$

1.5.1. *Let* $(\mathbf{T}, \Lambda, \lambda, Q)$ *be a standard measure filling of a finite set* $\mathbb{T}$*. Define* $\Lambda_n := \{\mathcal{E} \in \Lambda : \ \mathcal{E} \subseteq \mathbf{T}_n\}$*. Let* λ_n *and* Q_n *be the restrictions of the measure* λ *and of the embedding* Q *to* Λ_n*. Then for each* $n \in {}^{\mathrm{st}}\mathbb{N}$ *the collection* $(\mathbf{T}_n, \Lambda_n, \lambda_n, Q_n)$ *is a standard* (*finite*) *measure filling of the finite set* $\mathbb{T}_n$*.*

$\lhd$ Since λ and $(\mathbf{T}_n)_{n\in\mathbb{N}}$ are standard,

$$\forall n \in {}^{\mathrm{st}}\mathbb{N} \quad \lambda \mathbf{T}_n \ll \infty. \ \blacktriangleright \tag{1.30}$$

1.5.2. Define

$$\forall t \in \mathbb{T} \quad \ell_t := \lambda Q t, \qquad \forall E \in 2^{\mathbb{T}} \quad \ell E = \sum_{t\in E} \ell_t. \tag{1.31}$$

We can see that $(\mathbf{T}, \Lambda, \lambda, Q)$ induces the finite measure space $(\mathbb{T}, 2^{\mathbb{T}}, \ell)$. If $\lambda\mathbf{T} < \infty$, we get in fact $\lambda \mathbf{T} \ll \infty$ and, therefore, $\ell\mathbb{T} \ll \infty$. If $\lambda\mathbf{T} = +\infty$, we obtain by (1.30) and (1.31),

$$\forall n \in {}^{\mathrm{st}}\mathbb{N} \quad \ell \mathbb{T}_n \ll \infty. \tag{1.32}$$

1.5.3. The *equivalence* of two standard measure fillings $(\mathbf{T}, \Lambda, \lambda, Q)$ and $(\widetilde{\mathbf{T}}, \widetilde{\Lambda}, \widetilde{\lambda}, \widetilde{Q})$ is defined in the same way as in 1.1.3 with the additional requirement: the standard bijection f preserves the measure, that is, $\widetilde{\lambda} f(\mathcal{E}) = \lambda \mathcal{E}$ for all $\mathcal{E} \in \Lambda$, where $f(\mathcal{E}) = \{f(\tau) : \tau \in \mathcal{E}\}$.

1.5.4. Example. Let $\mathbb{T} := \overleftarrow{ab}$ and $(\mathbf{T}, \Lambda, \lambda, Q)$ be its natural standard filling. We take the usual Lebesgue measure on $\mathbf{T}$ for λ. Then $(\mathbf{T}, \Lambda, \lambda, Q)$ is a standard measure filling of the interval $\overleftarrow{ab}$. This filling is also called *natural*.If $\lambda\mathbf{T} = +\infty$, for example $\mathbf{T} = \mathbb{R}$, we can take the interval $[-n, +n]$ for $\mathbf{T}_n$. Indeed, by Lemma 1.3.1, taking $a \approx -\infty$ and $b \approx +\infty$ we have ${}^{\mathrm{st}}\mathbb{Z} \subseteq \overleftarrow{ab}$, and putting $\mathbb{T}_n := \{t = kh : -n \le k \le n\}$ we have $\mathbf{T}_n = Q\mathbb{T}_n$, where $\mathbb{T}_n \in 2^{\mathbb{T}}$ for all $n \in {}^{\mathrm{st}}\mathbb{N}$.

Note that the ℓ corresponding to the natural standard filling of the discrete interval $\overleftarrow{ab}$ is a counting measure:

$$\forall E \in 2^{\mathbb{T}} \quad \ell E = h \operatorname{card} E. \tag{1.33}$$

1.6. The embedding $\mathcal{N} \to \mathcal{M}$. Hereafter the symbol Q denotes not only the embedding described above of the algebra $2^{\mathbb{T}}$ into the algebra Λ, but also a mapping $\mathcal{N} \to \mathcal{M}$ defined below. First of all we require that

$$\forall \nu \in \mathcal{N}\ \forall E \in 2^{\mathbb{T}} \quad Q\nu Q E = \nu E. \tag{1.34}$$

However, the σ-algebra Λ is not exhausted by sets of the form QE. Again the embedding $Q : 2^{\mathbb{T}} \to \Lambda$ is not surjective. To extend the charge $Q\nu$ to all $\mathcal{E} \in \Lambda$, we use the measure λ. Namely, define

$$\forall \mathcal{E} \in \varLambda \quad Q\nu\mathcal{E} := \sum_{t\in\mathbb{T}} \lambda_{\mathcal{E}}(t)\nu_t, \tag{1.35}$$

where

$$\nu_t := \nu\{t\}, \quad \lambda_{\mathcal{E}}(t) := \lambda(Qt \cap \mathcal{E})\ell_t^{-1}. \tag{1.36}$$

The corresponding diagram is as follows:

$$\begin{array}{ccc} 2^{\mathbb{T}} & \xrightarrow{\ \nu\ } & \mathbb{C} \\ {\scriptstyle Q}\big\downarrow & \overset{Q\nu}{\nearrow} & \vdots\uparrow {\scriptstyle Q\nu=?} \\ Q2^{\mathbb{T}} & \cdots\cdots\to & \varLambda \end{array}$$

1.6.1. *The map* $Q : \mathcal{N} \to \mathcal{M}$ *defined by* (1.35), (1.36) *satisfies* (1.34). *It is isometric in the sense that* (see (1.7), (1.8))

$$\forall \nu \in \mathcal{N} \quad \|Q\nu\|' = \|\nu\|. \tag{1.37}$$

◁ Directly from (1.35) and (1.36), it follows that

$$\lambda_{QE}(t) = \begin{cases} \ell_t & \text{if } t \in E, \\ 0 & \text{if } t \in \mathbb{T} \setminus E, \end{cases} \tag{1.38}$$

which yields (1.34). Since the total variation of the measure $\mathcal{E} \mapsto \lambda_{\mathcal{E}}(t)$ on the set Qt is equal to $\lambda Q_t \cdot \ell_t^{-1} = 1$, the equality (1.37) is satisfied. ▶

1.6.2. Directly from (1.35) and (1.36), it follows that

$$\forall \nu \in \mathcal{N} \quad Q\nu(\mathbf{T} \setminus Q\mathbb{T}) = 0. \tag{1.39}$$

1.6.3. *The inductor* $\varPi : \mathcal{M} \to \mathcal{N}$ *is left inverse to the embedding* $Q : \mathcal{N} \to \mathcal{M}$.

◁ By (1.34) and (1.9),

$$\forall \nu \in \mathcal{N}\ \forall E \in 2^{\mathbb{T}} \quad \varPi Q\nu E = \nu E. \ ▶ \tag{1.40}$$

2. Standardness on $\mathbb{C}^{\mathbb{T}}$

Consider the set $\mathbb{C}^{\mathbb{T}}$ of all functions x such that $\operatorname{dom} x = \mathbb{T}$ and $\operatorname{im} x \subseteq \mathbb{C}$. We treat this set as a linear space with the pointwise operations. Note that $\dim \mathbb{C}^{\mathbb{T}} = \operatorname{card} \mathbb{T} \in \mathbb{N} \setminus {}^{\mathrm{st}}\mathbb{N}$. We denote by $L(\mathbf{T})$ the standard Banach space $L_1(\mathbf{T}, \varLambda, \lambda)$. Put

$$\forall x \in \mathbb{C}^{\mathbb{T}}\ \forall E \in 2^{\mathbb{T}} \quad \ell^x E := \sum_{t\in E} x(t)\ell_t, \tag{2.1}$$

$$\forall \xi \in L(\mathbf{T})\ \forall \mathcal{E} \in \varLambda \quad \lambda^{\xi}\mathcal{E} := \int_{\mathcal{E}} \xi(\tau)\,\lambda(d\tau). \tag{2.2}$$

2.1. The embedding $\mathbb{C}^{\mathbb{T}} \to L(\mathbf{T})$. Recall that the symbol Q denotes some embeddings $2^{\mathbb{T}} \to \varLambda$ and $\mathcal{N} \to \mathcal{M}$ (see (1.4) and (1.35)). In addition, we denote by Q some embedding $\mathbb{C}^{\mathbb{T}} \to L(\mathbf{T})$, which will be defined later in this section. First we note the following.

2.1.1. *The map* $x \mapsto \ell^x$ *is bijective* $\mathbb{C}^{\mathbb{T}} \to \mathcal{N}$, *and the map* $\xi \mapsto \lambda^\xi$ *is bijective* $L(\mathbf{T}) \to \mathcal{M}_\lambda$, *where* $\mathcal{M}_\lambda$ *is the class of charges* $\mu \in \mathcal{M}$ *which are absolutely continuous with respect to the standard measure* λ.

◁ Suppose $\nu \in \mathcal{N}$. Putting $x(t) = \nu_t \ell_t^{-1}$, we find $\ell^x = \nu$. The second assertion follows from the Nikodym–Radon theorem. ▶

Define

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad \|x\|_1 = \ell^{|x|}\mathbb{T}, \qquad \forall \xi \in L(\mathbf{T}) \quad \|\xi\|_1 := \lambda^{|\xi|}\mathbf{T}, \tag{2.3}$$

where $|x|(t) := |x(t)|$ and $|\xi|(\tau) := |\xi(\tau)|$. Then

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad \|x\|_1 = \|\ell^x\|, \qquad \forall \xi \in L(\mathbf{T}) \quad \|\xi\|_1 = \|\lambda^\xi\|, \tag{2.4}$$

where (cf. (1.7) and (1.8))

$$\forall \nu \in \mathcal{N} \quad \|\nu\| = (\operatorname{var}\nu)\mathbb{T}, \qquad \forall \mu \in \mathcal{M} \quad \|\mu\| = (\operatorname{var}\mu)(\mathbf{T}). \tag{2.4$'$}$$

In order to adjust the embedding $Q : \mathbb{C}^{\mathbb{T}} \to L(\mathbf{T})$ to the embedding $Q : \mathcal{N} \to \mathcal{M}$ we require that

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad \lambda^{Qx} = Q(\ell^x). \tag{2.5}$$

2.1.2. *Condition* (2.5) *uniquely defines an embedding* $Q : \mathbb{C}^{\mathbb{T}} \to L(\mathbf{T})$, *namely*

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad Qx(\tau) = \begin{cases} x(t) & \text{if } t \in \mathbb{T},\ \tau \in Qt, \\ 0 & \text{if } \tau \in \mathbf{T} \setminus Q\mathbb{T}\ . \end{cases} \tag{2.6}$$

◁ According to (1.35) and (1.36), the equality $\lambda^\xi = Q(\ell^x)$ means that

$$\forall \mathcal{E} \in \varLambda \quad \sum_{t \in E} \lambda(\mathcal{E} \cap Qt)(\ell^x)_t \ell_t^{-1} = \int_{\mathcal{E}} \xi(\tau)\, \lambda(d\tau).$$

But $(\ell^x)_t = \ell^x\{t\} = x(t)\ell_t$ by (2.1). Therefore

$$\forall t \in \mathbb{T}\ \forall \mathcal{E} \in \varLambda \quad \Big(\mathcal{E} \subset Qt \Rightarrow x(t)\lambda(\mathcal{E}) = \int_{\mathcal{E}} x(\tau)\, \lambda(d\tau), \text{ and } \mathcal{E} \subset \mathbf{T} \setminus Q\mathbb{T} \Rightarrow 0 = \int_{\mathcal{E}} \xi(\tau)\, \lambda(d\tau)\Big).$$

Consequently, it follows from $\lambda^\xi = Q(\ell^x)$ that $\xi(\tau) = x(t)$ if $\tau \in Qt$, and $\xi(\tau) = 0$ if $\tau \in \mathbf{T} \setminus Q\mathbb{T}$. ▶

2.1.3. *The embedding* $Q : \mathbb{C}^{\mathbb{T}} \to L(\mathbf{T})$ *is isometric*:

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad \|Qx\|_1 = \|x\|_1. \tag{2.7}$$

◁ By (2.6), $\|Qx\|_1 = \int_{\mathbf{T}} |Qx(\tau)|\, \lambda(d\tau) = \sum_{t \in E} |x(t)|\ell_t = \|x\|_1$. ▶

2.2. The inductor $\varPi : L(\mathbf{T}) \to \mathbb{C}^{\mathbb{T}}$. We extend the inductor $\varPi$, originally defined on charges, to functions so that it includes the map $L(\mathbf{T}) \to \mathbb{C}^{\mathbb{T}}$ which satisfies the compatibility condition:

$$\forall \xi \in L(\mathbf{T}) \quad \ell^{\varPi\xi} = \varPi(\lambda^\xi). \tag{2.8}$$

2.2.1. *Condition* (2.8) *uniquely defines an inductor* $\Pi : L(\mathbf{T}) \to \mathbb{C}^{\mathbb{T}}$. *This inductor is an averaging operator*:

$$\forall \xi \in L(\mathbf{T})\ \forall t \in \mathbb{T} \quad \Pi\xi(t) = \frac{1}{\ell_t} \int\limits_{Qt} \xi(\tau)\,\lambda(d\tau). \tag{2.9}$$

◁ According to (1.9), (2.1), and (2.2), the equality $\ell^x = \Pi\lambda^\xi$ means that for all $E \in 2^{\mathbb{T}}$,

$$\sum_{t \in E} x(t)\ell_t = \ell^x E = \Pi\lambda^\xi E = \lambda^\xi QE = \int\limits_{QE} \xi(\tau)\,\lambda(d\tau).$$

Putting $E = \{t\}$ here, we obtain $x(t)\ell_t = \int_{Qt} \xi(\tau)\,\lambda(d\tau)$. ▶

2.2.2. *The inductor* $\Pi : L(\mathbf{T}) \to \mathbb{C}^{\mathbb{T}}$ *is a contractive mapping*:

$$\forall \xi \in L(\mathbf{T}) \quad \|\Pi\xi\|_1 \le \|\xi\|_1. \tag{2.10}$$

Again, it is left inverse to the embedding $Q : \mathbb{C}^{\mathbb{T}} \to L(\mathbf{T})$:

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad \Pi Q x = x. \tag{2.11}$$

◁ By (1.10), (2.4), and (2.8), $\|\Pi\xi\|_1 = \|\ell^{\Pi\xi}\| = \|\Pi\lambda^\xi\| \le \|\lambda^\xi\|' \le \|\lambda^\xi\| = \|\xi\|_1$. According to 1.6.2, $\ell^{\Pi Qx} = \Pi\lambda^{Qx} = \Pi Q\ell^x = \ell^x$, and, therefore, (2.11) follows from 2.1.1. ▶

The next proposition follows from the fact that the inductor $\Pi : \mathcal{M} \to \mathcal{N}$ is exact.

2.2.3. *Let* $\xi \in {}^{\mathrm{st}}L(\mathbf{T})$ *and let*

$$\forall E \in 2^{\mathbb{T}} \quad \int\limits_{QE} \xi(\tau)\,\lambda(d\tau) \approx 0; \tag{2.12}$$

then $\xi = 0$.

◁ (2.12) means that $\lambda^\xi \in \operatorname{qker}\Pi$. If $\xi \in {}^{\mathrm{st}}L(\mathbf{T})$, then $\lambda^\xi \in {}^{\mathrm{st}}\mathcal{M}$. Therefore $\lambda^\xi = 0$ whenever (2.12) is true, hence $\xi = 0$ according to 2.1.1. ▶

The *quasi-kernel* of the inductor $\Pi : L(\mathbf{T}) \to \mathbb{C}^{\mathbb{T}}$ is defined as follows:

$$\operatorname{qker}_{L(\mathbf{T})}\Pi := \{\xi \in L(\mathbf{T}) : \|\Pi\xi\|_1 \approx 0\}. \tag{2.13}$$

2.2.4. *The inductor* Π *is exact in the sense that*

$${}^{\mathrm{st}}\operatorname{qker}_{L(\mathbf{T})}\Pi = \{0\}. \tag{2.14}$$

◁ From $\|\Pi\xi\|_1 \approx 0$ it follows that $\sum_{t\in E} |\Pi\xi(t)\ell_t \approx 0$ and, moreover, for all $E \in 2^{\mathbb{T}}$,

$$\left|\sum_{t\in E} \frac{1}{\ell_t} \int\limits_{Qt} \xi(\tau)\,\lambda(d\tau)\ell_t\right| \approx 0, \quad \text{i.e.,} \quad \int\limits_{QE} \xi(\tau)\,\lambda(d\tau) \approx 0.$$

It remains to use 2.2.3. ▶

Formulae (2.5) and (2.8) may be written in the following equivalent form: for all $x \in \mathbb{C}^{\mathbb{T}}$ and $\xi \in L(\mathbf{T})$,

$$\xi = Qx \Leftrightarrow \lambda^\xi = Q\ell^x, \tag{2.15}$$

$$x = \Pi\xi \Leftrightarrow \ell^x = \Pi\lambda^\xi. \tag{2.16}$$

Note that by (2.10) and (2.11),

$$\|x - \Pi\xi\|_1 \le \|Qx - \xi\|_1. \tag{2.17}$$

2.3. Standard and nearstandard functions on $\mathbb{C}^{\mathbb{T}}$; standardized image. Since $L(\mathbf{T})$ is standard, one can introduce the following definition. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called *standard*, written $x \in {}^{\mathrm{st}}\mathbb{C}^{\mathbb{T}}$, if $x = \Pi\xi$ for some $\xi \in {}^{\mathrm{st}}L(\mathbf{T})$. In this case, ξ is called the *standardized image* of x. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called *nearstandard*, written $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$, if $\|x - \Pi\xi\|_1 \approx 0$ for some $\xi \in {}^{\mathrm{st}}L(\mathbf{T})$. In this case, we put ${}^{\bullet}x = \xi$ and ${}^{\circ}x = \Pi\xi$; ${}^{\bullet}x$ is called the *shadow* of x *on* $\mathbf{T}$, and ${}^{\circ}x$ is called the *shadow* of x *on* $\mathbb{T}$. So,

$$\forall x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}} \quad \|x - {}^{\circ}x\|_1 \approx 0, \qquad {}^{\circ}x = \Pi({}^{\bullet}x), \qquad {}^{\bullet}x \in {}^{\mathrm{st}}L(\mathbf{T}). \tag{2.18}$$

2.3.1. *The standardized image of a function $x \in \mathbb{C}^{\mathbb{T}}$ and the shadows ${}^{\circ}x$, ${}^{\bullet}x$ are uniquely defined.*

◁ Suppose $\|x - \Pi\xi_i\| \approx 0$, where $\xi_i \in {}^{\mathrm{st}}L(\mathbf{T})$, $i = 1, 2$. Then $\xi_1 - \xi_2 \in {}^{\mathrm{st}}\mathrm{qker}_{L(\mathbf{T})}\,\Pi$ and $\xi_1 = \xi_2$ by (2.14). ▶

2.3.2. *A function $x \in \mathbb{C}^{\mathbb{T}}$ is standard or nearstandard according as the charge ℓ^x is standard or nearstandard. Let $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$; then*

$$\|\ell^x - \ell^{{}^{\bullet}x}\| \approx 0, \quad \ell^{{}^{\circ}x} = \Pi\ell^{{}^{\bullet}x}. \tag{2.19}$$

2.3.3. *Let $x \in \mathbb{C}^{\mathbb{T}}$ and let $\|x - Q\xi\|_1 \approx 0$ for some $\xi \in {}^{\mathrm{st}}L(\mathbf{T})$; then $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$ and ${}^{\bullet}x = \xi$.*

◁ By (2.17), it follows from $\|Qx - \xi\|_1 \approx 0$ that $\|x - \Pi\xi\|_1 \approx 0$. ▶

2.4. Absolute continuity, integrability. Recall that for a standard charge, the condition of absolute continuity with respect to a standard measure is equivalent to the fact that a set of infinitesimal measure has an infinitesimal charge. In this connection the following definitions are natural (cf. [8]).

2.4.1. A charge $\nu \in \mathcal{N}$ is called *absolutely continuous* (with respect to the measure ℓ) if

$$\forall E \in 2^{\mathbb{T}} \quad (\ell E \approx 0 \Rightarrow \nu E \approx 0). \tag{2.20}$$

From the evident estimate

$$\forall E \in 2^{\mathbb{T}} \quad |\nu E| \le \ell E + \|\nu - \ell\| \tag{2.21}$$

it follows that the following remark is true.

2.4.2. *If $\|\nu - \ell\| \approx 0$, then the charge ν is absolutely continuous.*

Obviously, a more general proposition is also true.

2.4.3. *If $\nu, \nu_1 \in \mathcal{N}$, $\|\nu - \nu_1\| \approx 0$, and the charge ν is absolutely continuous, then the charge ν_1 is also absolutely continuous.*

Let $x \in \mathbb{C}^{\mathbb{T}}$. As before, $|x|$ denotes the function $\mathbb{T} \ni t \mapsto |x(t)|$ and $\ell^{|x|}$ the charge $2^{\mathbb{T}} \ni E \mapsto \sum_{t \in E} |x(t)|\ell_t$.

2.4.4. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called *locally integrable* (w.r.t. ℓ) if

(i) $\forall E \in 2^{\mathbb{T}}$ $(\ell E \ll \infty \Rightarrow \ell^{|x|}E \ll \infty)$,
(ii) the charge $\ell^{|x|}$ is absolutely continuous.

A locally integrable function $x \in \mathbb{C}^{\mathbb{T}}$ is called *integrable* if

(iii) $\ell^{|x|}\mathbb{T} \ll \infty$.

A function $x \in \mathbb{C}^{\mathbb{T}}$ is called *locally summable* (w.r.t. ℓ) if

(iv) $\forall E \in 2^{\mathbb{T}}\ (\ell E \ll \infty \Rightarrow (\forall \omega \in \mathbb{N} \setminus {}^{\mathrm{st}}\mathbb{N}\ \ell^{|x|}E_\omega \approx 0))$,

where

$$\forall n \in \mathbb{N} \quad E_n := \{t \in E : |x(t)| > n\}. \tag{2.22}$$

2.4.5. Theorem (cf. [8]). *A function $x \in \mathbb{C}^{\mathbb{T}}$ is locally integrable if and only if it is locally summable.*

◁ Suppose x is locally integrable. Using the symbol (2.22), we note that the "Chebyshev inequality"

$$\ell^{|x|}E \geq n\ell E_n \tag{2.22'}$$

holds. Therefore if $\ell E \ll \infty$ and $\omega \approx +\infty$, then (by 2.4.4(i)) $\ell E_\omega \approx 0$. Consequently, in virtue of 2.4.4(ii) we conclude that 2.4.4(iv) is satisfied.

Conversely, let a function x be locally summable. For $\varepsilon \in \mathbb{R}_+$ set $\mathbb{N}_\varepsilon := \{n \in \mathbb{N} : \ell^{|x|}E_n < \varepsilon\}$. Suppose $\varepsilon \in {}^{\mathrm{st}}\mathbb{R}_+$. Then, by 2.4.4(iv), the set $\mathbb{N}_\varepsilon$ contains all infinite $n \in \mathbb{N}$. Putting $n_\varepsilon = \min \mathbb{N}_\varepsilon$, by permanence, we conclude that $n_\varepsilon \ll \infty$. Since $\ell^{|x|}E = \ell^{|x|}E_{n_\varepsilon} + \ell^{|x|}(E \setminus E_{n_\varepsilon})$, we have $\ell^{|x|}E \leq \varepsilon + n_\varepsilon \ell E$. Hence, first, $\ell^{|x|}E \ll \infty$ whenever $\ell E \ll \infty$ and, secondly, $\ell^{|x|}E < 2\varepsilon$ whenever $\ell E \approx 0$. Since this holds for any standard $\varepsilon > 0$, it follows from $\ell E \approx 0$ that $\ell^{|x|}E \approx 0$. ▶

Obviously,

$$\forall x \in \mathbb{C}^{\mathbb{T}}\ \forall E \in 2^{\mathbb{T}} \quad |\ell^{|x|}E| \leq \ell E \cdot \max_{t \in E} |x(t)|, \tag{2.23}$$

hence:

2.4.6. *Let $x \in \mathbb{C}^{\mathbb{T}}$ and let $\max_{t\in E} |x(t)| \ll \infty$; then the function x is locally integrable.*

2.4.7. *If $\ell\mathbb{T} \ll \infty$, then each locally integrable function $x \in \mathbb{C}^{\mathbb{T}}$ is integrable.*

◁ From $\ell\mathbb{T} \ll \infty$ and 2.4.4(i) we deduce 2.4.4(iii). ▶

2.4.8. Define

$$\forall s, t \in \mathbb{T} \quad \delta_t(s) = \begin{cases} 0 & \text{if } s \neq t, \\ \ell_t^{-1} & \text{if } s = t. \end{cases} \tag{2.24}$$

Then $\delta_t \in \mathbb{C}^{\mathbb{T}}$ for all $t \in \mathbb{T}$. We say that δ_t is the *discrete Dirac delta* concentrated at the point t since

$$\forall x \in \mathbb{C}^{\mathbb{T}}\ \forall t \in \mathbb{T} \quad \sum_{s \in \mathbb{T}} x(s)\delta_t(s)\ell_s = x(t). \tag{2.25}$$

2.4.9. *The function δ_t is integrable iff $\ell_t \gg 0$. But if $\ell_t \approx 0$, then the function $\ell_t^{1/2}\delta_t$ is integrable (although it takes the infinite value $\ell_t^{-1/2}$ at the point t).*

◁ If $\ell_t \gg 0$, then $\ell^{\delta_t}E = 0$ if $t \in \mathbb{T} \setminus E$ and $\ell^{\delta_t}E = 1$ if $t \in E$. Therefore conditions (i) and (ii) of 2.4.4 are satisfied. ▶

2.4.10. *Suppose $\ell_t \approx 0$ for all $t \in \mathbb{T}$. If $x \in \mathbb{C}^{\mathbb{T}}$ and the charge $\ell^{|x|}$ is absolutely continuous, then the function x is locally integrable.*

$\lhd$ Suppose otherwise; then there exists a set $E \in 2^{\mathbb{T}}$ such that $\ell E \ll \infty$ and $\ell^{|x|}E \approx +\infty$. Divide E into disjoint parts E' and E'' such that $\ell E' \approx \ell E'' \approx \frac{1}{2}\ell E$. Then either $\ell^{|x|}E' \geq \frac{1}{2}\ell^{|x|}E$ or $\ell^{|x|}E'' \geq \frac{1}{2}\ell^{|x|}E$. Using induction, we find sets $E_n \in 2^{\mathbb{T}}$ such that $E \supset E_1 \supset E_2 \supset \ldots$, $\ell E_n \leq 2^{-n}(\ell E + 1)$ and $\ell^{|x|}E_n \geq 2^{-n}\ell^{|x|}E$. Choose $n_0 \in \mathbb{N}$ such that $2^{-n_0}\ell^{|x|}E \approx +\infty$ and $n_0 \approx +\infty$. Then $\ell E_{n_0} \approx 0$ and $\ell^{|x|}E_{n_0} \approx +\infty$. This contradicts the absolute continuity of the charge $\ell^{|x|}$. $\blacktriangleright$

2.5. Some "classical theorems". For the discrete integral

$$2^{\mathbb{T}} \times \mathbb{C}^{\mathbb{T}} \ni (E, x) \mapsto \ell^x E := \sum_{t \in E} x(t)\ell_t \in \mathbb{C}$$

the analogs of classical theorems of integration theory are almost trivial (cf. [12]).

2.5.1. "Nikodym–Radon Theorem". *Let $\nu \in \mathcal{N}$ and suppose that for all $E \in 2^{\mathbb{T}}$, $\ell E \ll \infty \Rightarrow |\nu E| \ll \infty$. Then the charge ν is absolutely continuous iff it may be represented in the form*

$$\forall E \in 2^{\mathbb{T}} \quad \nu E = \ell^x E = \sum_{t \in E} x(t)\ell_t, \tag{2.26}$$

where the function $x \in \mathbb{C}^{\mathbb{T}}$ is locally integrable.

$\lhd$ Put $x(t) = \nu_t \ell_t^{-1}$ for $t \in \mathbb{T}$. Suppose x is locally integrable. If $\ell E \approx 0$, then $|\nu E| \leq \sum_{t \in E} |x(t)|\ell_t = \ell^{|x|}E \approx 0$, i.e., the charge ν is absolutely continuous. Conversely, let ν be absolutely continuous. Since $\ell^{|x|}E = (\operatorname{var}\nu)E$, from $\ell E \ll \infty$ it follows that $\ell^{|x|}E \ll \infty$, and from $\ell E \approx 0$ it follows that $\ell^{|x|}E \approx 0$ because the variation of an absolutely continuous charge is absolutely continuous. $\blacktriangleright$

2.5.2. "Fischer–Riesz Theorem". *Let $x, y \in \mathbb{C}^{\mathbb{T}}$ and let $\|x - y\|_1 \approx 0$. If the function x is integrable, then so is y.*

$\lhd$ Since $|\ell^x E - \ell^y E| \leq \|x - y\|$ for all $E \in 2^{\mathbb{T}}$, we have

$$\|x - y\|_1 \approx 0 \Rightarrow \forall E \in 2^{\mathbb{T}} \ \ell^x E \approx \ell^y E. \tag{2.27}$$

2.5.3. *Quantifier "quasi-everywhere"*. Let $p(t)$ be a sentence with a single free variable $t \in \mathbb{T}$. We say that p is true *quasi-everywhere on the set* $E \in 2^{\mathbb{T}}$ if there exists a set $E_0 \subseteq E$ such that $\ell E_0 \approx 0$ and $p(t)$ is true for all $t \in E \setminus E_0$.

2.5.4. "Lebesgue Theorem". *Let $x, y \in \mathbb{C}^{\mathbb{T}}$ be integrable functions. If $\ell E \ll \infty$ and $x(t) \approx y(t)$ quasi-everywhere on E, then* (1)

$$\ell^x E \approx \ell^y E. \tag{2.28}$$

$\lhd$ It suffices to show that $\ell^x E \approx 0$ whenever x is an integrable function, $\ell E \ll \infty$, and $x(t) \approx 0$ quasi-everywhere on E. Represent E in the form $E_0 \cup [E \setminus E_0]$, where $\ell E_0 \approx 0$

(1) Here the requirement of the existence of an integrable majorant is automatically satisfied.

and $x(t) \approx 0$ for all $t \in E \setminus E_0$. Since $\max_{t\in E} |x(t)| \approx 0$, we obtain

$$|\ell^x E| \le |\ell^x E_0| + |\ell^x(E \setminus E_0)| \le \ell^{|x|} E_0 + \ell E \max_{t\in E\setminus E_0} |x(t)| \approx 0. \ \blacktriangleright$$

2.6. Relation between the "discrete integral" and the ordinary one. It is useful to note the following estimates.

2.6.1. *Let $x \in \mathbb{C}^{\mathbb{T}}$ and $\xi \in L(\mathbf{T})$; then*

$$\forall E \in 2^{\mathbb{T}} \quad \Big|\sum_{t\in E} x(t)\ell_t - \int_{QE} \xi(\tau)\,\lambda(d\tau)\Big| \le \|x - \Pi\xi\|_1, \tag{2.29}$$

$$\forall E \in 2^{\mathbb{T}} \quad \sum_{t\in E} |x(t)|\ell_t \le \int_{QE} |\xi(\tau)|\,\lambda(d\tau) + \|x - \Pi\xi\|_1. \tag{2.30}$$

$\lhd$ From (2.9) it follows that for all $E \in 2^{\mathbb{T}}$ and $\xi \in L(\mathbf{T})$,

$$\sum_{t\in E} \Pi\xi(t)\ell_t = \int_{QE} \xi(\tau)\,\lambda(d\tau). \tag{2.31}$$

Therefore (2.29) is satisfied. Then

$$\sum_{t\in E} |x(t)|\ell_t \le \sum_{t\in E} |\Pi\xi(t)|\ell_t + \sum_{t\in E} |x(t) - \Pi\xi(t)|\ell_t.$$

However,

$$\sum_{t\in E} |\Pi\xi(t)|\ell_t = \sum_{t\in E} \Big|\int_{Qt} \xi(\tau)\,\lambda(d\tau)\Big| \le \int_{QE} |\xi(\tau)|\,\lambda(d\tau).$$

Consequently, (2.30) is satisfied. $\blacktriangleright$

2.6.2. *Each function $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$ is integrable and for all $E \in 2^{\mathbb{T}}$ and $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$,*

$$\int_{QE} {}^{\bullet}x(\tau)\,\lambda(d\tau) \approx \sum_{t\in E} x(t)\ell_t. \tag{2.32}$$

$\lhd$ Suppose $E \in 2^{\mathbb{T}}$ and $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$. Since $\|x - \Pi {}^{\bullet}x\|_1 \approx 0$, in virtue of (2.29) we conclude that (2.32) is satisfied. Suppose $\ell E \approx 0$. Then $\lambda QE = \ell E \approx 0$ and $\int_{QE} {}^{\bullet}x(\tau)\,\lambda(d\tau) \approx 0$ since ${}^{\bullet}x \in {}^{\mathrm{st}}L(\mathbf{T})$. Hence from (2.30) we conclude that $\ell E \approx 0$ implies $\ell^x E \approx 0$, i.e., x is locally integrable. Considering $\int_{\mathbf{T}} |{}^{\bullet}x(\tau)|\,\lambda(d\tau) \ll \infty$ and using (2.30) once again (for $\xi = {}^{\bullet}x$), we obtain $\ell^{|x|}\mathbf{T} \ll \infty$. $\blacktriangleright$

3. The spaces $\mathbb{H}$ and $\mathbf{H}$

The linear space $\mathbb{C}^{\mathbb{T}}$ supplied with the inner product

$$\forall x, y \in \mathbb{C}^{\mathbb{T}} \quad (x, y) := \sum_{t\in\mathbb{T}} x(t)\overline{y(t)}\ell_t \tag{3.1}$$

and with the norm $\|x\| = \|x\|_2 := (x, x)^{1/2}$ is denoted by $\mathbb{H}$. By $\mathbf{H}$ we denote the standard Hilbert space $L_2(\mathbf{T}, \Lambda, \lambda)$ with the inner product

$$\forall \xi, \eta \in \mathbf{H} \quad (\xi, \eta) := \int_{\mathbf{T}} \xi(\tau)\overline{\eta(\tau)}\,\lambda(d\tau) \tag{3.2}$$

and with the norm $\|\xi\| = \|\xi\|_2 = (\xi,\xi)^{1/2}$. Note that the norms $\|\cdot\|_1$ and $\|\cdot\| = \|\cdot\|_2$ are *equivalent* on $\mathbb{H}$ (but not on $\mathbf{H}$) because $\dim \mathbb{H} = \operatorname{card} \mathbb{T} \in \mathbb{N}$. The following estimates are exact:

$$\forall x \in \mathbb{H} \quad \sqrt{\min_{t\in\mathbb{T}} \ell_t}\, \|x\| \le \|x\|_1 \le \sqrt{\ell\mathbb{T}}\, \|x\|. \tag{3.3}$$

Indeed, we have

$$\|x\|_1 := \sum_{t\in\mathbb{T}} |x(t)|\ell_t \le \Big(\sum_{t\in\mathbb{T}} \ell_t\Big)^{1/2} \Big(\sum_{t\in\mathbb{T}} |x(t)|^2\ell_t\Big)^{1/2} = \sqrt{\ell\mathbb{T}}\, \|x\|,$$

where equality is attained for $x = \text{constant}$. Further, we have

$$\|x\|^2 := \sum_{t\in\mathbb{T}} |x(t)|^2\ell_t \le \Big(\sum_{t\in\mathbb{T}} |x(t)|\ell_t^{1/2}\Big)^2 \le \|x\|_1 \max_{t\in\mathbb{T}} \ell_t^{-1};$$

here equality is attained for $x(t) = 0$ except for $t \in \mathbb{T}$ such that ℓ_t is minimal.

3.1. Embedding and inductor. The embedding Q defined before by formula (2.6) as a map $\mathbb{C}^{\mathbb{T}} \to L(\mathbf{T})$ may be interpreted as a map $\mathbb{H} \to \mathbf{H}$.

3.1.1. *This map is isometric* $\mathbb{H} \to \mathbf{H}$.

◁ If $x, y \in \mathbb{H}$, then

$$(Qx, Qy) = \int_{\mathbf{T}} Qx(\tau)\overline{Qy(\tau)}\, \lambda(d\tau) = x(t)\overline{y(t)} \int_{Qt} \lambda(d\tau) = (x, y)$$

by (2.6) and $\int_{Qt} \lambda(d\tau) = \ell_t$. ▶

The inductor Π defined by (2.9) as a map $L(\mathbf{T}) \to \mathbb{C}^{\mathbb{T}}$ may also be interpreted as a map $\mathbf{H} \to \mathbb{H}$. Actually, since $\lambda Qt < \infty$ for all $t \in \mathbb{T}$ (see (1.28)), each L_2 function on Qt is integrable on Qt.

3.1.2. *The inductor Π is a contractive map $\mathbf{H} \to \mathbb{H}$. As before, it is left inverse to the embedding $Q : \mathbb{H} \to \mathbf{H}$.*

◁ Let $\xi \in \mathbf{H}$. Then

$$\begin{aligned}\|\Pi\xi\|^2 &= \sum_{t\in\mathbb{T}} \Big| \frac{1}{\ell_t} \int_{Qt} \xi(\tau)\, \lambda(d\tau) \Big|^2 \ell_t \le \sum_{t\in\mathbb{T}} \frac{1}{\ell_t} \int_{Qt} \lambda(d\tau) \int_{Qt} |\xi(\tau)|^2\, \lambda(d\tau) \\ &= \int_{Q\mathbb{T}} |\xi(\tau)|^2\, \lambda(d\tau) \le \|\xi\|^2. \ \blacktriangleright\end{aligned}$$

3.1.3. *The operators Q and Π are mutually adjoint*:

$$Q^* = \Pi, \quad \Pi^* = Q. \tag{3.4}$$

◁ Let $x \in \mathbb{H}$ and $\xi \in \mathbf{H}$. Then

$$(Qx, \xi) = \int_{\mathbf{T}} Qx(\tau)\overline{\xi(\tau)}\, \lambda(d\tau) = \sum_{t\in\mathbb{T}} x(t) \int_{Qt} \overline{\xi(\tau)}\, \lambda(d\tau) = \sum_{t\in\mathbb{T}} x(t)\overline{\Pi\xi(t)}\ell_t = (x, \Pi\xi). \ \blacktriangleright$$

3.2. Quasi-unity and the orthoprojector P. Let X be an arbitrary standard Banach space. An operator $A \in \mathcal{B}(X)$ is called a *quasi-unity* of the algebra $\mathcal{B}(X)$ if

$$\forall \xi \in {}^{\mathrm{st}}X \quad \|A\xi - \xi\| \approx 0. \tag{3.5}$$

3.2.1. *Let A be a quasi-unity such that $\|A\| \ll \infty$; then*

$$\forall \xi \in {}^{\mathrm{nst}}X \quad \|A\xi - \xi\| \approx 0. \tag{3.5$'$}$$

◁ Let $\xi \in {}^{\mathrm{nst}}X$ and $\eta \in {}^{\mathrm{st}}X$, and suppose $\|\xi - \eta\| \approx 0$ (i.e., η is the shadow of ξ). Then $\|A\xi - \xi\| \le \|A(\xi - \eta)\| + \|A\eta - \eta\| + \|\eta - \xi\| \approx 0$ since $\|A(\xi - \eta)\| \le \|A\| \cdot \|\xi - \eta\| \approx 0$. ▶

3.2.2. Example. Let $(e_n)_{n\in\mathbb{N}}$ be a standard orthonormal basis for a standard Hilbert space $X = H$. Define $P_\omega \xi := \sum_{n\le\omega} (\xi, e_n) e_n$ for $\xi \in H$, where $\omega \approx +\infty$. The orthoprojector P_ω is a quasi-unity of the algebra $\mathcal{B}(H)$. This follows from the nearstandardness criterion in footnote 12 of Section 0.

A more curious example is based on the following reasoning. From the idealization principle of IST it follows that, in a standard Hilbert space H, there exists a *finite-dimensional* subspace H_0 such that ${}^{\mathrm{st}}H \subset H_0$. Suppose P_0 is the orthoprojector $H \to H_0$. Then $P_0 \xi = \xi$ for all $\xi \in {}^{\mathrm{st}}H$ and, consequently, P_0 is a quasi-unity. The next evident remark enables one to vary these examples.

3.2.3. *Let A be a quasi-unity of $\mathcal{B}(X)$ and let $B \in \mathcal{B}(X)$. If $\|A - B\| \approx 0$, then B is also a quasi-unity of the same algebra.*

We return to the Hilbert spaces $\mathbb{H}$ and $\mathbf{H}$. Since $\dim Q\mathbb{H} = \operatorname{card} \mathbb{T} \in \mathbb{N}$, the subspace $Q\mathbb{H}$ is closed in $\mathbf{H}$. Denote by P the orthoprojector $\mathbf{H} \to Q\mathbb{H}$.

3.2.4. *Together with the equality $\varPi Q = \mathbb{I}_{\mathbb{H}}$ noted above, the following equality holds:*

$$Q\varPi = P. \tag{3.6}$$

◁ We use discrete Dirac deltas (see (2.24)). Note that $(\sqrt{\ell_t}\,\delta_t)_{t\in\mathbb{T}}$ is an *orthonormal basis* for the space $\mathbb{H}$. Since the embedding $Q : \mathbb{H} \to \mathbf{H}$ is isometric, we see that $(\sqrt{\ell_t}\, Q\delta_t)_{t\in\mathbb{T}}$ is an orthonormal basis for the subspace $Q\mathbb{H} \subset \mathbf{H}$. Therefore by (3.4),

$$P\xi = \sum_{t\in\mathbb{T}} (\xi, \sqrt{\ell_t}\, Q\delta_t) \sqrt{\ell_t}\, Q\delta_t = Q \sum_{t\in\mathbb{T}} (\varPi\xi, \sqrt{\ell_t}\,\delta_t)\sqrt{\ell_t}\,\delta_t = Q\varPi\xi. \ ▶$$

Throughout the rest we suppose that

3.2.5. The orthoprojector $P : \mathbf{H} \to Q\mathbb{H}$ is a quasi-unity.

3.2.6. *The orthoprojector P corresponding to the discrete interval $\mathbb{T} = \overleftarrow{ab}$ and to its natural standard filling* (see 1.3 and 1.5) *is a quasi-unity in the sense that for all $\xi \in {}^{\mathrm{st}}L_i(\mathbf{T})$, $\|P\xi - \xi\|_i \approx 0$, $i = 1, 2$.*

◁ Suppose $\xi \in {}^{\mathrm{st}}C_0(\mathbf{T})$. Set $\varepsilon = \max\{|P\xi(\tau) - \xi(\tau)| : \tau \in [t, t+h[,\ t \in \mathbb{T}\}$. Since $P\xi(t)$ is a mean value of $\xi(\tau)$ on $[t, t+h[$ and $h \approx 0$, we have $\varepsilon \approx 0$. Since $\operatorname{supp}\xi$ is a standard compact subset of $\mathbf{T}$, we find $\lambda(\operatorname{supp}\xi) \ll \infty$. Therefore

$$\|P\xi - \xi\|_2^2 = \sum_{t\in\mathbb{T}} \int_t^{t+h} |P\xi(\tau) - \xi(\tau)|^2\, d\tau + \int_{\mathbf{T}\setminus Q\mathbb{T}} |\xi(\tau)|^2\, d\tau.$$

As $\xi \in {}^{\mathrm{st}}L_2(\mathbf{T})$, the last integral is infinitesimal. The first sum of integrals does not exceed $\varepsilon^2\lambda(\operatorname{supp}\xi)$, hence $\|P\xi-\xi\|_2 \approx 0$. In the same way we prove that $\|P\xi-\xi\|_1 \approx 0$ whenever $\xi \in L_1(\mathbf{T})$. It remains to take account of the density of $C_0(\mathbf{T})$ in $L_i(\mathbf{T})$, $i = 1, 2$. ▶

3.2.7. Remark. Assumption 3.2.5 is equivalent to

$$\forall \xi \in {}^{\mathrm{st}}\mathbf{H}\ \exists \eta \in Q\mathbb{H} \quad \|\xi - \eta\| \approx 0. \tag{3.7}$$

Actually, $P\xi$ is the element of $Q\mathbf{H}$ nearest to ξ.

$Q\mathbb{H}$ is quite a large part of the space $\mathbb{H}$. Indeed, it is easy to prove from the idealization principle that there exist functions $\xi \in Q\mathbb{H} \setminus \{0\}$ that are orthogonal to ${}^{\mathrm{st}}\mathbf{H}$.

3.2.8. **H**-*nearstandardness.* Recall that

$$x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}} \ \equiv\ (\exists^{\bullet}x \in {}^{\mathrm{st}}L_1(\mathbf{T}))\ (\|x - \mathit{\Pi}\,{}^{\bullet}x\|_1 \approx 0)$$

according to the definition of nearstandardness in Section 2. It is natural to introduce another definition. A function $x \in \mathbf{H}$ is called **H**-*nearstandard*, written $x \in {}^{\mathrm{nst}}\mathbb{H}$, if

$$\exists \xi \in {}^{\mathrm{st}}\mathbf{H} \quad \|x - \mathit{\Pi}\xi\| \approx 0. \tag{3.8}$$

A function $\xi \in {}^{\mathrm{st}}\mathbf{H}$ such that $\|x - \mathit{\Pi}\xi\| \approx 0$ is called the **H**-*shadow* of $x \in \mathbb{H}$. However, ξ may just be called the *shadow* of x because $\xi = {}^{\bullet}x$ whenever $x \in {}^{\mathrm{nst}}\mathbb{H} \cap {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$. Two cases will be taken up separately.

3.2.9. *Let* $\lambda\mathbf{T} \ll \infty$*; then* ${}^{\mathrm{nst}}\mathbb{H} \subseteq {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$ *and the* **H**-*shadow of a function* $x \in {}^{\mathrm{nst}}\mathbb{H}$ *coincides with* ${}^{\bullet}x$*.*

◁ Let $x \in {}^{\mathrm{nst}}\mathbb{H}$ and $\xi \in {}^{\mathrm{st}}\mathbf{H}$, and suppose $\|x - \mathit{\Pi}\xi\| \approx 0$. By (3.3), $\|x - \mathit{\Pi}\xi\|_1 \approx 0$ because $\ell\mathbb{T} \le \lambda\mathbf{T}$. Since

$$\forall \eta \in \mathbb{H} \quad \|\eta\|_1 \le \sqrt{\lambda\mathbf{T}}\,\|\eta\|, \tag{3.9}$$

we conclude that $\xi \in {}^{\mathrm{st}}L(\mathbf{T})$. Therefore $\xi = {}^{\bullet}x$. ▶

3.2.10. *Let* $\lambda\mathbb{T} \approx +\infty$ *and let* $x \in {}^{\mathrm{nst}}\mathbf{H} \cap {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$; *then the* $\mathbb{H}$-*shadow of* x *coincides with* ${}^{\bullet}x$*.*

◁ Suppose $\xi \in {}^{\mathrm{st}}\mathbf{H}$, $\|x - \mathit{\Pi}\xi\| \approx 0$, and $\xi_1 \in {}^{\mathrm{st}}L(\mathbf{T})$, $\|x - \mathit{\Pi}\xi_1\| \approx 0$. For $y \in \mathbb{C}^{\mathbb{T}}$ we set

$$\|y\|_{n,1} = \sum_{t \in \mathbb{T}_n} |y(t)|\ell_t, \quad \|y\|_n = \Big(\sum_{t \in \mathbb{T}_n} |y(t)|^2\ell_t\Big)^{1/2}, \tag{3.10}$$

where $\mathbb{T}_n$ are the same as in 1.5 (see (1.29) and (1.30)). The following exact estimates are satisfied (cf. with (3.3)):

$$\forall y \in \mathbb{C}^{\mathbb{T}} \quad (\min \ell_t)^{1/2}\,\|y\|_n \le \|y\|_{n,1} \le (\ell\mathbb{T}_n)^{1/2}\|y\|_n. \tag{3.11}$$

From these estimates and from the equalities $\ell\mathbb{T}_n = \lambda\mathbf{T}_n$ with $\mathbf{T}_n = Q\mathbb{T}_n$ it follows that

$$\forall n \in {}^{\mathrm{st}}\mathbb{N} \quad \|\mathit{\Pi}\xi - \mathit{\Pi}\xi_1\|_n \approx 0. \tag{3.12}$$

Denote by $(\xi - \xi_1)_n$ the restriction of the function $\xi - \xi_1$ to $\mathbf{T}_n$, followed by extension to $\mathbf{T} \setminus \mathbb{T}_n$ by 0. From (3.12) we conclude that $(\xi - \xi_1)_n \in {}^{\mathrm{st}}\mathrm{qker}_{L(\mathbf{T})}\, \mathit{\Pi}$ for all $n \in {}^{\mathrm{st}}\mathbb{N}$ (see (2.13)). According to (2.14), therefore, $(\xi - \xi_1)_n = 0$ for all $n \in {}^{\mathrm{st}}\mathbb{N}$. Using the transfer principle, we infer that $\xi - \xi_1 = 0$. ▶

3.3. Weak nearstandardness on $\mathbb{H}$. Define the *weak* and *strong quasi-kernel* of the inductor $\mathit{\Pi} : \mathbf{H} \to \mathbb{H}$ by

(3.13) $$\mathrm{qker}_{\mathrm{w}}\, \mathit{\Pi} := \{\xi \in \mathbf{H} : \forall y \in {}^{\mathrm{st}}\mathbb{H}\ (\mathit{\Pi}\xi, y) \approx 0\},$$

(3.14) $$\mathrm{qker}_{\mathrm{s}}\, \mathit{\Pi} := \{\xi \in \mathbf{H} : \|\mathit{\Pi}\xi\| \approx 0\}.$$

Obviously,

(3.15) $$\mathrm{qker}_{\mathrm{s}}\, \mathit{\Pi} \subseteq \mathrm{qker}_{\mathrm{w}}\, \mathit{\Pi}.$$

3.3.1. *The inductor $\mathit{\Pi} : \mathbf{H} \to \mathbb{H}$ is exact in the following sense*:

(3.16) $${}^{\mathrm{st}}\mathrm{qker}_{\mathrm{w}}\, \mathit{\Pi} = \{0\}.$$

◁ Let $\xi \in {}^{\mathrm{st}}\mathrm{qker}_{\mathrm{w}}\, \mathit{\Pi}$ and let $y = \mathit{\Pi}\xi$; then $y \in {}^{\mathrm{st}}\mathbb{H}$. Therefore $\|\mathit{\Pi}\xi\|^2 = (\mathit{\Pi}\xi, y) \approx 0$. Since the embedding Q is an isometry $\mathbb{H} \to \mathbf{H}$, we have $\|P\xi\| = \|Q\mathit{\Pi}\xi\| = \|\mathit{\Pi}\xi\| \approx 0$. Since P is a quasi-unity of $\mathcal{B}(\mathbf{H})$, we get $\xi \approx 0$. Hence $\xi = 0$ because ξ is standard. ▶

3.3.2. A function $x \in \mathbb{H}$ is called *weakly nearstandard*, written $x \in {}^{\mathrm{wst}}\mathbb{H}$, if

(3.17) $$\forall y \in {}^{\mathrm{st}}\mathbb{H} \quad (x - x_0, y) \approx 0.$$

From 3.3.1 it follows that the $x_0 \in {}^{\mathrm{st}}\mathbb{H}$ which satisfies condition (3.17) is *unique*.

3.3.3. Obviously,

(3.18) $${}^{\mathrm{nst}}\mathbb{H} \subset {}^{\mathrm{wst}}\mathbb{H}.$$

Moreover, if $x \in {}^{\mathrm{nst}}\mathbb{H}$, then ${}^{\bullet}x = x_0$, where $x_0 \in {}^{\mathrm{st}}\mathbb{H}$ satisfies (3.17). This permits denoting this function by ${}^{\bullet}x$ and calling it the *shadow* of x, without any danger of confusion.

3.3.4. Example (of a weakly, but not strongly, nearstandard function). Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a standard orthonormal basis for $\mathbf{H}$. Since P is a quasi-unity of $\mathcal{B}(\mathbf{H})$, we find $\forall^{\mathrm{st}} n \in \mathbb{N}\ \|\mathit{\Pi}\varepsilon_n\| = \|P\varepsilon_n\| \approx \|\varepsilon_n\| = 1$. By the Robinson lemma, $\|\mathit{\Pi}\varepsilon_\omega\| \approx 0$ for some $\omega \approx +\infty$. For this ω we define $x := \mathit{\Pi}\varepsilon_\omega$. Suppose $y \in {}^{\mathrm{st}}\mathbb{H}$, that is, $y = \mathit{\Pi}\eta$ for some $\eta \in {}^{\mathrm{st}}\mathbf{H}$. We have $(x, y) = (\mathit{\Pi}\varepsilon_\omega, \mathit{\Pi}\eta) = (\varepsilon_\omega, P\eta) \approx (\varepsilon_\omega, \eta) \approx 0$ because the sequence $((\varepsilon_n, \eta))_{n\in\mathbb{N}}$ is standard and tends to 0 as $n \to \infty$. Therefore $x \in {}^{\mathrm{wst}}\mathbb{H}$ and ${}^{\circ}x := \mathit{\Pi}{}^{\bullet}x = 0$. This yields that $x \in {}^{\mathrm{wst}}\mathbb{H}$ and ${}^{\circ}x := \mathit{\Pi}{}^{\bullet}x = 0$. At the same time, $\|x - {}^{\circ}x\| = \|\mathit{\Pi}\varepsilon_\omega\| \approx 1$. ▶

3.3.5. Remark. *Let $x \in \mathbb{H}$ and let $\xi \in {}^{\mathrm{st}}\mathbf{H}$; then*

(3.19) $$\|x - \mathit{\Pi}\xi\| \approx \|Qx - \xi\|.$$

◁ Since $Q\mathit{\Pi} = P$ is a quasi-unity of $\mathcal{B}(\mathbf{H})$ and Q is an isometry $\mathbb{H} \to \mathbf{H}$, we obtain $\|Qx - \xi\| \approx \|Qx - Q\mathit{\Pi}\xi\| = \|x - \mathit{\Pi}\xi\|$ when $\xi \in {}^{\mathrm{st}}\mathbb{H}$. ▶

3.3.6. Corollary. *A function $x \in \mathbb{H}$ is strongly nearstandard iff*

(3.20) $$\exists \xi \in {}^{\mathrm{st}}\mathbf{H} \quad \|Qx - \xi\| \approx 0.$$

4. Nearstandardness on $\mathcal{B}(\mathbb{H})$

The relations established in Section 3 between elements of the spaces $\mathbb{H}$ and $\mathbf{H}$ will be extended to operators acting in these spaces.

4.1. The embedding $\mathfrak{Q}$ and the inductor $\mathfrak{P}$. To each operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ we assign the operator

$$\mathfrak{Q}\mathbb{A} := Q\mathbb{A}\Pi, \tag{4.1}$$

where Q is the embedding $\mathbb{H} \to \mathbf{H}$, and Π is the inductor $\mathbf{H} \to \mathbb{H}$. Also to each operator $\mathbf{A} \in \mathcal{B}(\mathbf{H})$ there is assigned the operator

$$\mathfrak{P}\mathbf{A} := \Pi\mathbf{A}Q. \tag{4.2}$$

The transformations $\mathfrak{Q}$ and $\mathfrak{P}$ are called respectively the *embedding* $\mathcal{B}(\mathbb{H}) \to \mathcal{B}(\mathbf{H})$ and the *inductor* $\mathcal{B}(\mathbf{H}) \to \mathcal{B}(\mathbb{H})$. Obviously, they are linear.

4.1.1. *The inductor $\mathfrak{P}$ is a contractive transformation which is left inverse to the embedding $\mathfrak{Q}$.*

$\lhd$ Since $\|\Pi\| = \|Q\| = 1$, we have

$$\forall \mathbf{A} \in \mathcal{B}(\mathbf{H}) \quad \|\mathfrak{P}\mathbf{A}\| \le \|\mathbf{A}\|. \tag{4.3}$$

Moreover, since $\Pi Q = \mathbb{I}_{\mathbb{C}^{\mathrm{T}}}$, we get

$$\forall \mathbb{A} \in \mathcal{B}(\mathbb{H}) \quad \mathfrak{P}\mathfrak{Q}\mathbb{A} = \mathbb{A}. \ \blacktriangleright \tag{4.4}$$

4.1.2. *The embedding $\mathfrak{Q}$ is isometric.*

$\lhd$ By the previous statement, $\|\mathbb{A}\| = \|\mathfrak{P}\mathfrak{Q}\mathbb{A}\| \le \|\mathfrak{Q}\mathbb{A}\|$. It is also clear that $\|\mathfrak{Q}\mathbb{A}\| \le \|\mathbb{A}\|$. Therefore,

$$\forall \mathbb{A} \in \mathcal{B}(\mathbb{H}) \quad \|\mathfrak{Q}\mathbb{A}\| = \|\mathbb{A}\|. \ \blacktriangleright \tag{4.5}$$

4.1.3. Let P be the orthoprojector $\mathbf{H} \to Q\mathbb{H}$. Since $Q\Pi = P$, we get

$$\forall \mathbf{A} \in \mathcal{B}(\mathbf{H}) \quad \mathfrak{Q}\mathfrak{P}\mathbf{A} = P\mathbf{A}P. \tag{4.6}$$

Since P is a quasi-unity of the algebra $\mathcal{B}(\mathbf{H})$, we find

$$\forall \xi \in {}^{\mathrm{st}}\mathbf{H} \ \forall \mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H}) \quad \mathfrak{Q}\mathfrak{P}\mathbf{A}\xi \approx \mathbf{A}\xi. \tag{4.7}$$

4.1.4. In view of $Q^* = \Pi$, $\Pi^* = Q$, we obtain for all $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ and $\mathbf{A} \in \mathcal{B}(\mathbf{H})$,

$$(\mathfrak{Q}\mathbb{A})^* = \mathfrak{Q}(\mathbb{A}^*), \quad (\mathfrak{P}\mathbf{A})^* = \mathfrak{P}(\mathbf{A}^*). \tag{4.8}$$

4.2. Exactness of $\mathfrak{P}$. We define the *quasi-kernel* $\operatorname{qker}\mathfrak{P}$ of the inductor $\mathfrak{P}$ by

$$\operatorname{qker}\mathfrak{P} := \{\mathbf{A} \in \mathcal{B}(\mathbf{H}) : \|\mathfrak{P}\mathbf{A}\| \approx 0\}. \tag{4.9}$$

We also introduce the *strong quasi-kernel* $\operatorname{qker}_{\mathrm{s}}\mathfrak{P}$ by putting

$$\operatorname{qker}_{\mathrm{s}}\mathfrak{P} := \{\mathbf{A} \in \mathcal{B}(\mathbf{H}) : \forall x \in {}^{\mathrm{st}}\mathbb{H} \ \|\mathfrak{P}\mathbf{A}x\| \approx 0\}. \tag{4.10}$$

It is obvious that

$$\operatorname{qker}\mathfrak{P} \subseteq \operatorname{qker}_{\mathrm{s}}\mathfrak{P}. \tag{4.11}$$

4.2.1. *The inductor* $\mathfrak{P}$ *is exact in the sense that*

$$ {}^{\mathrm{st}}\mathrm{qker}_{\mathrm{s}}\,\mathfrak{P} = \{0\}. \tag{4.12}$$

◁ Suppose $\mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H})$ and $\|\mathfrak{P}\mathbf{A}x\| \approx 0$ for all $x \in {}^{\mathrm{st}}\mathbb{H}$. Let $\xi \in {}^{\mathrm{st}}\mathbf{H}$. Since $\Pi\xi \in {}^{\mathrm{st}}\mathbb{H}$, we have $\|\mathfrak{P}\mathbf{A}\Pi\xi\| \approx 0$. However, $Q\Pi\xi = P\xi \approx \xi$, hence $\mathfrak{P}\mathbf{A}\Pi\xi = \Pi\mathbf{A}Q\Pi\xi \approx \Pi\mathbf{A}\xi$. Consequently, $\Pi\mathbf{A}\xi \approx 0$ and by the exactness of Π, $\mathbf{A}\xi = 0$. By the transfer principle, $\mathbf{A}\xi = 0$ for all $\xi \in \mathbf{H}$. ▶

Recall that the Hilbert space $\mathbf{H}$ is standard. Thus the algebra $\mathcal{B}(\mathbf{H})$ is also standard. However, the algebra $\mathcal{B}(\mathbb{H})$ is nonstandard because $\dim\mathcal{B}(\mathbb{H}) = (\mathrm{card}\,\mathbb{T})^2 \in \mathbb{N} \setminus {}^{\mathrm{st}}\mathbb{N}$. Therefore, there is no *natural standardness structure* on $\mathcal{B}(\mathbb{H})$. We introduce it using the exactness of the inductor $\mathfrak{P}$.

4.2.2. If

$$\mathbb{A} = \mathfrak{P}\mathbf{A} \tag{4.13}$$

for some $\mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H})$, then we say that the operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is *standard* (with respect to the standard filling $(\mathbf{T}, \Lambda, \lambda, Q)$ of the finite set $\mathbb{T}$) and write $\mathbb{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbb{H})$.

It immediately follows from the exactness of $\mathfrak{P}$ that

4.2.3. *If* $\mathbb{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbb{H})$ *and* $\|\mathbb{A}\| \approx 0$, *then* $\mathbb{A} = 0$. *Moreover*, *the operator* $\mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H})$ *for which* (4.13) *holds is unique. We call it the standardized image of the operator* $\mathbb{A}$.

4.3. Strong and uniform nearstandardness. Similarly to Section 0.5 we introduce the following definitions.

4.3.1. An operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is called *strongly nearstandard*, written $\mathbb{A} \in {}^{\mathrm{sst}}\mathcal{B}(\mathbb{H})$, if $\|\mathbb{A}\| \ll \infty$ and

$$\forall x \in {}^{\mathrm{st}}\mathbb{H} \quad \|(\mathbb{A} - \mathbb{A}_0)x\| \approx 0 \tag{4.14}$$

for some operator $\mathbb{A}_0 \in {}^{\mathrm{st}}\mathcal{B}(\mathbb{H})$. If for some $\mathbb{A}_0 \in {}^{\mathrm{st}}\mathcal{B}(\mathbb{H})$,

$$\|\mathbb{A} - \mathbb{A}_0\| \approx 0, \tag{4.15}$$

then we say that $\mathbb{A}$ is *uniformly nearstandard* and write $\mathbb{A} \in {}^{\mathrm{ust}}\mathcal{B}(\mathbb{H})$.

4.3.2. It is clear that

$${}^{\mathrm{st}}\mathcal{B}(\mathbb{H}) \subset {}^{\mathrm{ust}}\mathcal{B}(\mathbb{H}) \subseteq {}^{\mathrm{sst}}\mathcal{B}(\mathbb{H}). \tag{4.16}$$

Moreover, if $\mathbb{A} \in {}^{\mathrm{ust}}\mathcal{B}(\mathbb{H})$, then the operator $\mathbb{A}_0 \in {}^{\mathrm{st}}\mathcal{B}(\mathbb{H})$ satisfying (4.15) also satisfies (4.14). The operator $\mathbb{A}_0 \in {}^{\mathrm{st}}\mathcal{B}(\mathbb{H})$ satisfying (4.14) is unique. We call it the *shadow* of $\mathbb{A}$ and denote by ${}^\circ\mathbb{A}$. According to 4.2.2, the shadow ${}^\circ\mathbb{A}$ is uniquely represented in the form ${}^\circ\mathbb{A} = \mathfrak{P}({}^\bullet\mathbb{A})$, where ${}^\bullet\mathbb{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H})$. The operator ${}^\bullet\mathbb{A}$ is called the *shadow of* $\mathbb{A}$ *on* $\mathbf{T}$.

4.3.3. *Let* $\mathbb{A} \in {}^{\mathrm{sst}}\mathcal{B}(\mathbb{H})$ *and let* $\mathbf{A} := \mathfrak{Q}\mathbb{A}$; *then the operator* $\mathbf{A}$ *is strongly nearstandard in the sense of Section* 0.5, *and* ${}^\circ\mathbf{A} = {}^\bullet\mathbb{A}$.

◁ Suppose $\xi \in {}^{\mathrm{st}}\mathbf{H}$. Then $\|\mathbf{A}\xi - {}^\bullet\mathbb{A}\xi\| \approx \|Q\mathbb{A}\Pi\xi - P\,{}^\bullet\mathbb{A}\xi\| = \|\mathbb{A}\Pi\xi - \Pi\,{}^\bullet\mathbb{A}\xi\| \approx \|\mathbb{A}\Pi\xi - \Pi\,{}^\bullet\mathbb{A}Q\Pi\xi\| = \|(\mathbb{A} - {}^\circ\mathbb{A})\Pi\xi\| \approx 0$. ▶

4.3.4. *Let* $\mathbf{A} \in \mathcal{B}(\mathbf{H})$ *be an operator that is uniformly nearstandard in the sense of Section* 0.5 *and let* $\mathbb{A} := \mathfrak{P}\mathbf{A}$; *then* $\mathbb{A} \in {}^{\mathrm{ust}}\mathcal{B}(\mathbb{H})$ *and* ${}^\bullet\mathbb{A} = {}^\circ\mathbf{A}$.

◁ We have $\|\mathbb{A} - \mathfrak{P}{}^\circ\mathbf{A}\| = \|\mathfrak{P}(\mathbf{A} - {}^\circ\mathbf{A})\| \le \|\mathbf{A} - {}^\circ\mathbf{A}\| \approx 0$. ▶

4.3.5. Since $\mathfrak{P}\mathfrak{Q} = \mathbb{I}_{\mathcal{B}(\mathbb{H})}$ and $\mathfrak{P}$ is a contractive mapping $\mathcal{B}(\mathbf{H}) \to \mathcal{B}(\mathbb{H})$, we see that

$$\forall \mathbb{A} \in \mathcal{B}(\mathbb{H})\ \forall \mathbf{A} \in \mathcal{B}(\mathbf{H}) \quad \|\mathbb{A} - \mathfrak{P}\mathbf{A}\| \le \|\mathfrak{Q}\mathbb{A} - \mathbf{A}\|. \tag{4.17}$$

Consequently, if $\mathbb{A} \in \mathcal{B}(\mathbb{H})$, $\mathbf{A} \in \mathcal{B}(\mathbf{H})$, and $\|\mathfrak{Q}\mathbb{A} - \mathbf{A}\| \approx 0$, then $\mathbb{A} \in {}^{\mathrm{ust}}\mathcal{B}(\mathbb{H})$ and ${}^\bullet\mathbb{A} = \mathbf{A}$.

4.3.6. Example. Let $n \in \mathbb{N}$, let $e_k, f_k \in \mathbb{H}$ and $\varepsilon_k, \varphi_k \in \mathbf{H}$ for $k \le n$, and let

$$\begin{aligned} &\forall x \in \mathbb{H} \quad \mathbb{A}x := \sum_{k \le n} (x, f_k) e_k, \\ &\forall \xi \in \mathbf{H} \quad \mathbf{A}\xi := \sum_{k \le n} (\xi, \varphi_k) \varepsilon_k; \end{aligned} \tag{4.18}$$

then

$$\begin{aligned} &\forall \xi \in \mathbf{H} \quad (\mathfrak{Q}\mathbb{A})\xi = \sum_{k \le n} (\xi, Qf_k) Qe_k, \\ &\forall x \in \mathbb{H} \quad (\mathfrak{P}\mathbf{A})x = \sum_{k \le n} (x, \Pi\varphi_k) \Pi\varepsilon_k. \end{aligned} \tag{4.19}$$

It is easily shown that for the nearstandardness of $\mathbb{A}$ the following conditions are sufficient. Let $n \in {}^{\mathrm{st}}\mathbb{N}$ and let $e_k \in {}^{\mathrm{nst}}\mathbb{H}$ and $f_k \in {}^{\mathrm{wst}}\mathbb{H}$ for all $k \le n$ (see 3.2.2 and 3.2.8); then $\mathbb{A} \in {}^{\mathrm{sst}}\mathcal{B}(\mathbb{H})$ and

$$\forall x \in \mathbb{H} \quad ({}^\circ\mathbb{A})x = \sum_{k \le n} (x, {}^\circ f_k) {}^\circ e_k. \tag{4.20}$$

If $e_k, f_k \in {}^{\mathrm{nst}}\mathbb{H}$ for all $k \le n$, then $\mathbb{A} \in {}^{\mathrm{ust}}\mathcal{B}(\mathbb{H})$, and as before (4.20) is satisfied.

Note that by the exactness of $\mathfrak{P}$ and by the second formula of (4.19), formula (4.20) is equivalent to

$$\forall \xi \in \mathbf{H} \quad ({}^\bullet\mathbb{A})\xi = \sum_{k \le n} (x, {}^\bullet f_k) {}^\bullet e_k. \tag{4.20$'$}$$

In connection with Example 4.3.6 we introduce the following definition.

4.3.7. An operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is called *weakly nearstandard*, written $\mathbb{A} \in {}^{\mathrm{wst}}\mathcal{B}(\mathbb{H})$, if

$$\forall x, y \in {}^{\mathrm{st}}\mathbb{H} \quad ((\mathbb{A} - \mathbb{A}_0)x, y) \approx 0 \tag{4.21}$$

for some operator $\mathbb{A}_0 \in {}^{\mathrm{st}}\mathcal{B}(\mathbb{H})$. Obviously,

$${}^{\mathrm{sst}}\mathcal{B}(\mathbb{H}) \subset {}^{\mathrm{wst}}\mathcal{B}(\mathbb{H}), \tag{4.22}$$

and if $\mathbb{A} \in {}^{\mathrm{sst}}\mathcal{B}(\mathbb{H})$, then the shadow ${}^\circ\mathbb{A}$ coincides with $\mathbb{A}_0$ for which (4.21) is satisfied. Therefore we can write ${}^\circ\mathbb{A}$ in place of $\mathbb{A}_0$.

Evidently, if $n \in {}^{\mathrm{st}}\mathbb{N}$, $e_k, f_k \in {}^{\mathrm{wst}}\mathbb{H}$ for all $k \le n$, and $\mathbb{A}$ is given by the first formula of (4.19), then $\mathbb{A}$ is a weakly nearstandard operator (i.e., $\mathbb{A} \in {}^{\mathrm{wst}}\mathcal{B}(\mathbb{H})$) and both formulae (4.20) and (4.20$'$) are satisfied.

4.3.8. Remark. *Let* $\mathbb{A} \in \mathcal{B}(\mathbb{H})$, $\mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H})$, $\xi \in {}^{\mathrm{st}}\mathbf{H}$, *and* $x := \Pi\xi$; *then* $\|\mathfrak{Q}\mathbb{A}\xi - \mathbf{A}\xi\| \approx \|\mathbb{A}x - \mathfrak{P}\mathbf{A}x\|$.

◁ By (4.7) we find that $\|\mathfrak{Q}\mathbb{A}\xi - \mathbf{A}\xi\| \approx \|\mathfrak{Q}\mathbb{A}\xi - \mathfrak{Q}\mathfrak{P}\mathbf{A}\xi\| = \|\mathbb{A}\Pi\xi - \Pi\mathbf{A}Q\Pi\xi\| \approx \|\mathbb{A}x - \mathfrak{P}\mathbf{A}x\|$. ▶

4.3.9. Corollary. *Let* $\mathbb{A} \in \mathcal{B}(\mathbb{H})$; *then* $\mathbb{A} \in {}^{\mathrm{sst}}\mathcal{B}(\mathbb{H})$ *iff*

$$\exists \mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H})\ \forall \xi \in {}^{\mathrm{st}}\mathbf{H} \quad \|\mathfrak{Q}\mathbb{A}\xi - \mathbf{A}\xi\| \approx 0.$$

4.4. Graph-nearstandardness. Now we consider nearstandardness concepts concerned with the operator graph (see 0.2, 0.3). Taking into account the injectivity of the embedding $\mathfrak{Q} : \mathcal{B}(\mathbb{H}) \to \mathcal{B}(\mathbf{H})$, we introduce the following definition.

4.4.1. An operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is called *graph-nearstandard* if the shadow $^\circ(\operatorname{graph} \mathfrak{Q}\mathbb{A})$ is the graph of some operator $\mathbb{A}_\bullet : \mathbf{H} \to \mathbf{H}$:

$$^\circ(\operatorname{graph} \mathfrak{Q}\mathbb{A}) = \operatorname{graph}(\mathbb{A}_\bullet). \tag{4.23}$$

4.4.2. Suppose that the operator $\mathbb{A}_\bullet : \mathbf{H} \to \mathbf{H}$ satisfying (4.23) does exist; then it is standard but not necessarily belongs to the algebra $\mathcal{B}(\mathbf{H})$. Of course, it is linear. But it may happen that the set $\operatorname{dom}(\mathbb{A}_\bullet)$ is not dense in $\mathbf{H}$ (cf. Section 0.4.2). On the other hand, if $\mathbb{A}$ is a strongly (and all the more uniformly) nearstandard operator, then it is graph-nearstandard, and its shadow $^\bullet\mathbb{A}$ is equal to the operator $\mathbb{A}_\bullet$ defined by (4.23). Therefore no misunderstanding can arise if we write $^\bullet\mathbb{A}$ in place of $\mathbb{A}_\bullet$. This is clear from 0.5.1.

Now we formulate a graph-nearstandardness criterion.

4.4.3. For $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ define

$$\operatorname{dom}_{\mathrm{nst}} \mathbb{A} = \{x \in {}^{\mathrm{nst}}\mathbb{H} : \mathbb{A}x \in {}^{\mathrm{nst}}\mathbb{H}\}. \tag{4.24}$$

We say that $\mathbb{A}$ satisfies the $\langle \mathrm{nst} \rangle$ *condition* if

$$\forall x \in \operatorname{dom}_{\mathrm{nst}} \mathbb{A} \quad (x \approx 0 \Rightarrow \mathbb{A}x \approx 0). \tag{4.25}$$

4.4.4. *An operator* $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ *is graph-nearstandard iff it satisfies the* $\langle \mathrm{nst} \rangle$ *condition* (4.25). *In this case*, *its shadow* $^\bullet\mathbb{A} := \mathbb{A}_\bullet$ (see (4.23)) *is a closed operator* $\mathbf{H} \to \mathbf{H}$, *and*

$$\forall \xi \in {}^{\mathrm{st}}\mathbf{H} \quad (\xi \in \operatorname{dom}(^\bullet\mathbb{A}) \Leftrightarrow \exists x \in \operatorname{dom}_{\mathrm{nst}} \mathbb{A}\ (\xi \approx {}^\bullet x)), \tag{4.26}$$

$$\forall \xi \in {}^{\mathrm{st}}\operatorname{dom}(^\bullet\mathbb{A})\ \forall x \in \operatorname{dom}_{\mathrm{nst}} \mathbb{A} \quad (\xi = {}^\bullet x \Rightarrow (^\bullet\mathbb{A})\xi = {}^\bullet(\mathbb{A}x)). \tag{4.27}$$

◁ An operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ satisfies the $\langle \mathrm{nst} \rangle$ condition iff $\mathbf{A} := \mathfrak{Q}\mathbb{A}$ (see 0.3) does, i.e.,

$$\forall \xi \in \operatorname{dom}_{\mathrm{nst}} \mathbf{A} \quad (\xi \approx 0 \Rightarrow \mathbf{A}\xi \approx 0). \tag{4.28}$$

In this case, according to (0.11),

$$\forall \xi \in {}^{\mathrm{st}}\operatorname{dom}(^\circ\mathbf{A})\ \forall \xi' \in \operatorname{dom}_{\mathrm{nst}} \mathbf{A} \quad (\xi' \approx \xi \Rightarrow (^\circ\mathbf{A})\xi = {}^\circ(\mathbf{A}\xi')). \tag{4.29}$$

Let $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ be a graph-nearstandard operator so that (4.29) is satisfied. Suppose $x \in \operatorname{dom}_{\mathrm{nst}} \mathbb{A}$ and $x \approx 0$. Setting $\xi := Qx$, we have $\xi \approx 0$. Since $\mathbb{A}x \in {}^{\mathrm{nst}}\mathbb{H}$ and $x = \Pi\xi$, we obtain $\mathbb{A}\Pi\xi \in {}^{\mathrm{nst}}\mathbb{H}$. Therefore $\mathbf{A}\xi = Q\mathbb{A}\Pi\xi \in {}^{\mathrm{nst}}\mathbb{H}$. This means that $\xi \in \operatorname{dom}_{\mathrm{nst}} \mathbf{A}$ and according to (4.28), $\mathbf{A}\xi \approx 0$. Consequently, $\mathbb{A}x = \Pi Q\mathbb{A}\Pi\xi = \Pi\mathbf{A}\xi \approx 0$. We see that (4.25) holds.

Conversely, let $\mathbb{A}$ satisfy condition (4.25). Assume that $\xi \in \operatorname{dom}_{\mathrm{nst}} \mathbf{A}$ and $\xi \approx 0$. Define $x := \Pi\xi$. Since $\|\Pi\| = 1$, we have $x \approx 0$. Moreover, since $Q\mathbb{A}x = \mathbf{A}\xi$, we find

$\mathbb{A}x \in {}^{\mathrm{nst}}\mathbb{H}$ and, by (4.25), $\mathbb{A}x \approx 0$, hence $\mathbf{A}\xi \approx 0$. We see that (4.28) is satisfied, i.e., $\mathbb{A}$ is graph-nearstandard.

That ${}^\bullet\mathbb{A}$ is closed was substantiated in 0.3.1. To verify (4.26), let $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ be a graph-nearstandard operator and let $\xi \in {}^{\mathrm{st}}\mathbf{H}$. Then $\xi \in \mathrm{dom}({}^\bullet\mathbb{A})$ if and only if $\mathbf{A}\xi' := \mathfrak{Q}\mathbb{A}\xi \in {}^{\mathrm{st}}\mathbf{H}$ for some $\xi' \approx \xi$ (cf. (0.9)). Set $\xi' := Qx$, where $x = {}^\circ\xi$. Then $\xi' \approx \xi$ and $\mathbf{A}\xi' \in {}^{\mathrm{nst}}\mathbf{H}$ if and only if $Q\mathbb{A}\Pi Qx \in {}^{\mathrm{nst}}\mathbf{H}$. The latter means that $\mathbb{A}x \in {}^{\mathrm{nst}}\mathbb{H}$.

Finally, we prove (4.27). Suppose $\xi \in {}^{\mathrm{st}}\mathrm{dom}({}^\circ\mathbb{A})$. Then

$$\forall \xi' \in \mathbf{H} \quad (\xi' \approx \xi \wedge \mathbf{A}\xi' \in {}^{\mathrm{nst}}\mathbf{H} \Rightarrow {}^\bullet\mathbb{A}\xi = {}^\circ(\mathbf{A}\xi'))$$

(see (0.10)). Suppose $x \in \mathrm{dom}_{\mathrm{nst}}\,\mathbb{A}$ and ${}^\bullet x = \xi$. Set $\xi' := Qx$. Then $\xi' \in \mathrm{dom}_{\mathrm{nst}}\,\mathbf{A}$ because $\xi' \approx \xi$, $\mathbf{A}\xi' \approx Q\mathbb{A}x$, and $\mathbb{A}x \in {}^{\mathrm{nst}}\mathbb{H}$. Therefore $({}^\bullet\mathbb{A})\xi = {}^\bullet(\mathbf{A}\xi') = {}^\bullet(Q\mathbb{A}x) = {}^\bullet(\mathbb{A}x)$. ▶

4.5. $\mathcal{B}_2$-nearstandardness. Graph-nearstandardness is a broader notion than strong nearstandardness. Now we consider a nearstandardness notion that is narrower than uniform nearstandardness. Denote by $\mathcal{B}_2(\mathbf{H})$ the ideal of Hilbert–Schmidt operators of the algebra $\mathcal{B}(\mathbf{H})$. Recall that it consists of all operators $\mathbf{A} \in \mathcal{B}(\mathbf{H})$ admitting the representation

$$\forall \xi \in \mathbf{H}\ \forall \tau \in \mathbf{T} \quad \mathbf{A}\xi(\tau) = \int_{\mathbf{T}} \mathfrak{A}(\tau,\sigma)\xi(\sigma)\,\lambda(d\sigma), \tag{4.30}$$

where the kernel $\mathfrak{A}$ is quadratically integrable: $\mathfrak{A} \in \mathbf{H} \otimes \mathbf{H}$. The number

$$\|\mathbf{A}\|_2 := \|\mathfrak{A}\| = \Big(\int_{\mathbf{T}^2} |\mathfrak{A}(\tau,\sigma)|^2\,\lambda(d\tau)\,\lambda(d\sigma)\Big)^{1/2} \tag{4.31}$$

is called the *Hilbert–Schmidt norm* of the operator $\mathbf{A}$ defined by (4.30). This norm is *stronger* than the ordinary one:

$$\forall \mathbf{A} \in \mathcal{B}_2(\mathbf{H}) \quad \|\mathbf{A}\| \le \|\mathbf{A}\|_2, \tag{4.32}$$

and satisfies

$$\|\mathbf{A}\|_2 = \Big(\sum_{n\in\mathbb{N}} \|\mathbf{A}\varepsilon_n\|^2\Big)^{1/2} = \Big(\sum_{n,p\in\mathbb{N}} |(\mathbf{A}\varepsilon_n, \widetilde{\varepsilon}_p)|^2\Big)^{1/2}, \tag{4.33}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$, $(\widetilde{\varepsilon}_n)_{n\in\mathbb{N}}$ are arbitrary orthonormal bases of $\mathbf{H}$.

4.5.1. *Matrices.* To each operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ we assign a matrix $\mathcal{A} \in \mathbb{H} \otimes \mathbb{H}$ such that

$$\forall t, s \in \mathbb{T} \quad \mathcal{A}(t,s) := \mathbb{A}\delta_s(t) = (\mathbb{A}\delta_s, \delta_t) \tag{4.34}$$

where $\delta_s(t) = 0$ if $t \ne s$ and $\delta_s(s) = \ell_s^{-1}$ (δ_s is the discrete Dirac delta concentrated at s). The correspondence $\mathbb{A} \mapsto \mathcal{A}$ is *bijective*:

$$\forall x \in \mathbb{H}\ \forall t \in \mathbb{T} \quad \mathbb{A}x(t) = \sum_{s\in\mathbb{T}} \mathcal{A}(t,s)x(s)\ell_s. \tag{4.35}$$

4.5.2. Let $\mathcal{A}$ be the matrix of $\mathbb{A} \in \mathcal{B}(\mathbb{H})$. Define the norm $\|\mathbb{A}\|_2$ by

$$\|\mathbb{A}\|_2 := \|\mathcal{A}\| := \Big(\sum_{t,s\in\mathbb{T}} |\mathcal{A}(t,s)|^2 \ell_t \ell_s\Big)^{1/2}. \tag{4.36}$$

Note that this "Hilbert–Schmidt norm" is stronger than the ordinary one:

$$\forall \mathbb{A} \in \mathcal{B}(\mathbb{H}) \quad \|\mathbb{A}\| \le \|\mathbb{A}\|_2. \tag{4.37}$$

Now we find how the operations $\mathfrak{P}$ and $\mathfrak{Q}$ tell on kernels of operators. First we note

4.5.3. *The mappings* $\mathbf{A} \mapsto \mathfrak{A}$ *and* $\mathbb{A} \mapsto \mathcal{A}$ *established respectively by* (4.30) *and* (4.35) *are isometries* $\mathcal{B}_2(\mathbf{H}) \to \mathbf{H} \otimes \mathbf{H}$ *and* $\mathcal{B}(\mathbb{H}) \to \mathbb{H} \otimes \mathbb{H}$.

4.5.4. *For* $\mathcal{A} \in \mathbb{H} \otimes \mathbb{H}$ *denote by* $Q\mathcal{A}$ *the kernel from* $\mathbf{H} \otimes \mathbf{H}$ *defined by*

$$Q\mathcal{A}(\tau, \sigma) := \begin{cases} \mathcal{A}(t, s) & \text{if } (t, s) \in \mathbb{T}^2, (\tau, \sigma) \in Qt \times Qs, \\ 0 & \text{otherwise}. \end{cases} \tag{4.38}$$

Let $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ *and let* $\mathcal{A}$ *be the matrix of* $\mathbb{A}$. *Then* $\mathfrak{Q}\mathbb{A} \in \mathcal{B}_2(\mathbf{H})$, *and* $Q\mathcal{A}$ *is the kernel of* $\mathfrak{Q}\mathbb{A}$:

$$\forall \xi \in \mathbf{H} \quad \mathfrak{Q}\mathbb{A}\xi(\tau) = \int_{\mathbf{T}} Q\mathcal{A}(\tau, \sigma)\xi(\sigma)\,\lambda(d\sigma). \tag{4.39}$$

The embedding $\mathfrak{Q}$ *is isometric with respect to the norms* $\|\cdot\|_2$.

◁ By (4.39),

$$\mathfrak{Q}\mathbb{A}\xi(\tau) = Q\mathbb{A}\Pi\xi(\tau) = \sum_{s \in \mathbb{T}} Q\mathcal{A}(\tau, s)\frac{1}{\ell_s}\int_{Qs} \xi(\sigma)\,\lambda(d\sigma)\ell_s = \int_{\mathbf{T}} Q\mathcal{A}(\tau, \sigma)\xi(\sigma)\,\lambda(d\sigma).$$

Moreover,

$$\|\mathfrak{Q}\mathbb{A}\|_2^2 = \int_{\mathbf{T}^2} |Q\mathcal{A}(\tau, \sigma)|^2\,\lambda(d\sigma)\,\lambda(d\tau) = \sum_{t,s \in Q\mathbb{T}} |\mathcal{A}(t, s)|^2 \int_{Qt \times Qs} \lambda(d\tau)\,\lambda(d\sigma) = \|\mathbb{A}\|_2^2. \ ▶$$

4.5.5. *Let* $\mathfrak{A} \in \mathbf{H} \otimes \mathbf{H}$ *and let* $\Pi\mathfrak{A}$ *be the matrix defined by*

$$\forall t, s \in \mathbb{T} \ \Pi\mathfrak{A}(t, s) := \frac{1}{\ell_s \ell_t} \int_{Qt \times Qs} \mathfrak{A}(\tau, \sigma)\,\lambda(d\tau)\,\lambda(d\sigma). \tag{4.40}$$

Let $\mathbf{A} \in \mathcal{B}_2(\mathbf{H})$ *be the operator with kernel* $\mathfrak{A}$. *Then the matrix of the operator* $\mathfrak{P}\mathbf{A}$ *equals* $\Pi\mathfrak{A}$:

$$\forall x \in \mathbb{H} \quad \mathfrak{P}\mathbf{A}x(t) = \sum_{s \in \mathbb{T}} \Pi\mathfrak{A}(t, s)x(s)\ell_s. \tag{4.41}$$

The action of the inductor $\mathfrak{P}$ *is contractive with respect to the norms* $\|\cdot\|_2$:

$$\forall \mathbf{A} \in \mathcal{B}_2(\mathbf{H}) \quad \|\mathfrak{P}\mathbf{A}\|_2 \le \|\mathbf{A}\|_2. \tag{4.42}$$

◁ According to (4.34), the operator $\mathfrak{P}\mathbf{A}$ has the matrix with entries

$$(\mathfrak{P}\mathbf{A}\delta_s, \delta_t) = (\mathbf{A}Q\delta_s, Q\delta_t) = \frac{1}{\ell_t \ell_s} \int_{Qt \times Qs} \mathfrak{A}(\tau, \sigma)\,\lambda(d\tau)\,\lambda(d\sigma) = \Pi\mathfrak{A}(t, s).$$

By (4.36), we have

$$\|\mathfrak{P}\mathbf{A}\|_2^2 = \sum_{t,s \in \mathbb{T}} |\Pi\mathfrak{A}(t, s)|^2 \ell_t \ell_s.$$

Therefore,

$$\begin{aligned} \|\mathfrak{P}\mathbf{A}\|_2^2 &= \sum_{t,s \in \mathbb{T}} \left| \frac{1}{\ell_t \ell_s} \int_{Qt \times Qs} \mathfrak{A}(\tau, \sigma)\,\lambda(d\tau)\,\lambda(d\sigma) \right|^2 \ell_t \ell_s \\ &\le \sum_{t,s \in \mathbb{T}} \int_{Qt \times Qs} |\mathfrak{A}(\tau, \sigma)|^2\,\lambda(d\tau)\,\lambda(d\sigma) \le \|\mathfrak{A}\|^2. \ ▶ \end{aligned}$$

Note the following lemmas about quasi-unities.

4.5.6. *Let H be a standard Hilbert space and let $(e_n)_{n\in\mathbb{N}}$ be a standard orthonormal basis for H. Suppose that $A \in \mathcal{B}(H)$ and $\|A\| \ll \infty$. If $Ae_n \approx e_n$ for all $n \in {}^{\mathrm{st}}\mathbb{N}$, then A is a quasi-unity of the algebra $\mathcal{B}(H)$.*

$\lhd$ Let $\xi \in {}^{\mathrm{st}}H$ and $\xi_n := \sum_{k\le n} c_k e_k$ for $n \in \mathbb{N}$, where $c_k := (x, e_k)$. Then $\|A(\xi - \xi_n)\| \le \|A\| \cdot \|\xi - \xi_n\| \approx 0$ as $n \approx +\infty$. For $n \ll \infty$, $A\xi_n - \xi_n = \sum_{k\le n} c_k(Ae_k - e_k) \approx 0$. By the Robinson lemma, $A\xi_{n_0} - \xi_{n_0} \approx 0$ for some $n_0 \approx +\infty$. Therefore $\|A\xi - \xi\| \le \|A(\xi - \xi_{n_0})\| + \|A\xi_{n_0} - \xi_{n_0}\| + \|\xi_{n_0} - \xi\| \approx 0$. $\blacktriangleright$

4.5.7. *Let H_i be a standard Hilbert space, let A_i be a quasi-unity of $\mathcal{B}(H_i)$, and $\|A_i\| \ll \infty$, $i = 1, 2$; then $A := A_1 \otimes A_2$ is a quasi-unity of $\mathcal{B}(H)$, where $H := H_1 \otimes H_2$, and $\|A\| \ll \infty$.*

$\lhd$ Suppose $(e_{in})_{n\in\mathbb{N}}$ is a standard orthonormal basis for H_i, $i = 1, 2$. Define $e_{pq} := e_p \otimes e_q$. Then $(e_{pq})_{p,q\in\mathbb{N}}$ is a standard orthonormal basis for H. Since $Ae_{pq} = A_1 e_{1p} \otimes A_2 e_{2q} \approx e_{1p} \otimes e_{2q}$ for all $p, q \in {}^{\mathrm{st}}\mathbb{N}$, we find, by 4.5.6, that A is a quasi-unity of $\mathcal{B}(H)$. $\blacktriangleright$

4.5.8. *Let $(\mathbf{T}_i, \Lambda_i, \lambda_i, Q_i)$ be a standard measure filling of a finite set $\mathbb{T}_i$, $i = 1, 2$. Define $(\mathbf{T}, \Lambda, \lambda) := (\mathbf{T}_1, \Lambda_1, \lambda_1) \times (\mathbf{T}_2, \Lambda_2, \lambda_2)$, $Q = Q_1 \otimes Q_2$. Then $(\mathbf{T}, \Lambda, \lambda, Q)$ is a standard measure filling of $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$. The corresponding spaces, embeddings and orthoprojectors are related by*

$$\begin{gathered}\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2, \quad \mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2, \quad Q = Q_1 \otimes Q_2, \\ \Pi = \Pi_1 \otimes \Pi_2, \quad P = P_1 \otimes P_2. \end{gathered} \tag{4.43}$$

If P_i is a quasi-unity of $\mathcal{B}(\mathbf{H}_i)$, $i = 1, 2$, then P is a quasi-unity of $\mathcal{B}(\mathbf{H})$, and the inductor Π is exact.

$\lhd$ The first part follows directly from the corresponding definitions. The second part follows from 4.5.7 since the exactness of Π is derived from the fact that P is a quasi-unity. $\blacktriangleright$

4.5.9. Definition. An operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is called $\mathcal{B}_2$-*nearstandard* if

$$\|\mathbb{A} - \mathfrak{P}\mathbf{A}\|_2 \approx 0 \tag{4.44}$$

for some $\mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H})$.

4.5.10. Theorem. *An operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$ is $\mathcal{B}_2$-nearstandard iff*

$$\mathbb{A} \in {}^{\mathrm{ust}}\mathcal{B}(\mathbb{H}), \quad {}^{\bullet}\mathbb{A} \in \mathcal{B}_2(\mathbf{H}), \quad \|\mathbb{A} - {}^{\circ}\mathbb{A}\|_2 \approx 0. \tag{4.45}$$

Under these conditions, the matrix $\mathcal{A}$ of $\mathbb{A}$ is related to the kernel $\mathfrak{A}$ of its shadow ${}^{\bullet}\mathbb{A}$ by

$$\mathfrak{A} = {}^{\bullet}\mathcal{A}, \tag{4.46}$$

i.e.,

$$\int_{\mathbf{T}^2} |Q\mathcal{A}(\tau, \sigma) - \mathfrak{A}(\tau, \sigma)|^2 \,\lambda(d\tau)\,\lambda(d\sigma) \approx 0. \tag{4.46$'$}$$

This is equivalent to the condition

$$\sum_{(t,s)\in\mathbb{T}^2} |\mathcal{A}(t, s) - \Pi\mathfrak{A}(t, s)|^2 \ell_t \ell_s \approx 0. \tag{4.46$''$}$$

$\lhd$ The sufficiency of conditions (4.45) is evident. Suppose $\mathbb{A}\in\mathcal{B}(\mathbb{H})$ is $\mathcal{B}_2$-nearstandard. By (4.37) and (4.41), it follows that $\|\mathbb{A}-\mathfrak{P}\mathbf{A}\|\approx 0$. So the first and third relations of (4.45) hold. According to (4.36) relation (4.44) may be written in the form (4.46″). By 4.5.8, the orthoprojector $P:\mathbf{H}\otimes\mathbf{H}\to Q(\mathbb{H}\otimes\mathbb{H})$ is a quasi-unity of $\mathcal{B}(\mathbf{H}\otimes\mathbf{H})$. Therefore,

$$\forall\mathcal{A}\in\mathbb{H}\otimes\mathbb{H}\;\forall\mathfrak{A}\in{}^{\mathrm{st}}(\mathbf{H}\otimes\mathbf{H})\quad \|Q\mathcal{A}-\mathfrak{A}\|\approx\|\mathcal{A}-\varPi\mathfrak{A}\|. \tag{4.47}$$

Hence (4.46′) is equivalent to (4.46″). Finally, $\|{}^\bullet\mathbb{A}\|_2=\|\mathfrak{A}\|\le\|\mathfrak{A}-Q\mathcal{A}\|+\|\mathcal{A}\|\ll\infty$ because Q is isometric ($Q:\mathbb{H}\otimes\mathbb{H}\to\mathbf{H}\otimes\mathbf{H}$). Thus the second relation of (4.45) is satisfied. ►

4.5.11. Example. Let $\mathbb{T}:=\overleftarrow{ab}$ be a discrete interval and let $(\mathbf{T},\varLambda,\lambda,Q)$ be its natural discrete measure filling (see 1.5.4). We now define "discrete integration". For $x\in\mathbb{H}$ and $t_1,t_2\in\mathbb{T}$ ($t_1<t_2$) put

$$\mathfrak{J}_{t_1}^{t_2}x:=\sum_{t=t_1}^{t_2-h}x(t)h,\quad \mathfrak{J}_{t_2}^{t_1}x:=-\mathfrak{J}_{t_1}^{t_2}x,\quad \mathfrak{J}_{t_1}^{t_1}x=0. \tag{4.48}$$

For $x\in\mathbb{H}$ and $t\in\mathbb{T}$ set

$$\mathbb{I}_a x(t):=\mathfrak{J}_a^t x,\quad \mathbb{I}^b x(t):=\mathfrak{J}_t^b x. \tag{4.49}$$

Then $\mathbb{I}_a,\mathbb{I}^b\in\mathcal{B}(\mathbb{H})$. Let $\mathcal{I}_a$ and $\mathcal{I}^b$ be the matrices corresponding to the operators $\mathbb{I}_a$ and $\mathbb{I}^b$ respectively. Obviously,

$$\begin{aligned}\mathcal{I}_a(t,s)&=\begin{cases}1 & \text{if } a\le s<t\le b-h,\\ 0 & \text{if } a\le t\le s\le b-h,\end{cases}\\ \mathcal{I}^b(t,s)&=\begin{cases}1 & \text{if } a\le t\le s\le b-h,\\ 0 & \text{if } a\le s<t\le b-h.\end{cases}\end{aligned} \tag{4.50}$$

Assume that $-\infty\ll a<b\ll\infty$. It is clear that the operators $\mathbb{I}_a$, $\mathbb{I}^b$ are $\mathcal{B}_2$-nearstandard in this case. Moreover,

$${}^\bullet\mathbb{I}_a=\mathbf{I}_\alpha,\quad {}^\bullet\mathbb{I}^b=\mathbf{I}^\beta, \tag{4.51}$$

where

$$\alpha={}^\circ a,\quad \beta={}^\circ b,\quad \mathbf{I}_\alpha\xi(\tau)=\int_\alpha^\tau\xi(\sigma)\,d\sigma,\quad \mathbf{I}^\beta\xi(\tau)=\int_\tau^\beta\xi(\sigma)\,d\sigma. \tag{4.52}$$

4.5.12. Remark. We say that an operator $\mathbb{A}\in\mathcal{B}(\mathbb{H})$ is *standardly compact* if

$$\forall x\in\mathbb{H}\quad(\|x\|\ll\infty\Rightarrow\mathbb{A}x\in{}^{\mathrm{nst}}\mathbb{H}). \tag{4.53}$$

Each $\mathcal{B}_2$-nearstandard operator is standardly compact.

$\lhd$ Suppose $\mathbb{A}\in\mathcal{B}(\mathbb{H})$ is $\mathcal{B}_2$-nearstandard. Then $\|\mathfrak{Q}\mathbb{A}-{}^\bullet\mathbb{A}\|\approx 0$, where ${}^\bullet\mathbb{A}\in{}^{\mathrm{st}}\mathcal{B}_2(\mathbf{H})$. Since ${}^\bullet\mathbb{A}$ is a standardly compact operator, we have [1]

$$\forall\xi\in\mathbf{H}\quad(\|\xi\|\ll\infty\Rightarrow{}^\bullet\mathbb{A}\xi\in{}^{\mathrm{nst}}\mathbf{H}).$$

Suppose $x\in\mathbb{H}$ and $\|x\|\ll\infty$. Put $\xi:=Qx$. Then $\|\xi\|=\|x\|\ll\infty$ and therefore $Q(\mathbb{A}x)=\mathfrak{Q}\mathbb{A}\xi\approx{}^\bullet\mathbb{A}\xi\in{}^{\mathrm{nst}}\mathbf{H}$. This means that $\mathbb{A}x\in{}^{\mathrm{nst}}\mathbb{H}$. ►

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[1] In IST, it is proved that a standard operator is compact iff it takes each finite vector to a nearstandard one.

5. Discrete Fourier transform

In this chapter we consider an elementary example of graph-nearstandardness. The finite set $\mathbb{T}$ will be a discrete interval:

$$\mathbb{T} := \{t = kh : \ k \in \mathbb{Z}, \ -m \le k \le m\}. \tag{5.1}$$

Assume that the step h is an infinitesimal positive number. Denote the length of $\mathbb{T}$ by $2a$ so that $a := mh$. For a standard measure filling of $\mathbb{T}$ we take its *natural* standard filling (see 1.5.4). Then the Hilbert space $\mathbb{H}$ is the linear space $\mathbb{C}^{\mathbb{T}}$ with the inner product

$$\forall x, y \in \mathbb{H} \quad (x, y) = \sum_{t \in \mathbb{T}} x(t)\overline{y(t)}h, \tag{5.2}$$

and $\mathbf{H} = L_2(\mathbf{T})$, $\mathbf{T} = {}^{\mathrm{S}}[-a, a[$,

$$\forall \xi, \eta \in \mathbf{H} \quad (\xi, \eta) = \int_{\mathbf{T}} \xi(\tau)\overline{\eta(\tau)}\, d\tau, \tag{5.3}$$

where the integral is taken with respect to the standard Lebesgue measure on $\mathbb{R}$. We suppose that the orthoprojector $P : \mathbf{H} \to Q\mathbb{H}$ is a quasi-unity of the algebra $\mathcal{B}(\mathbf{H})$ and the inductor $\mathit{\Pi}$ is *exact* (see 1.2.4, 1.2.5).

5.1. The shift U_θ. Given $\theta \in \mathbb{C}$, define

$$\forall x \in \mathbb{H} \quad U_\theta = \begin{cases} x(t+h) & \text{if } t \in \mathbb{T} \setminus \{a-h\}, \\ \theta x(-a) & \text{if } t = a-h. \end{cases} \tag{5.4}$$

5.1.1. Obviously, $U_\theta \in \mathcal{B}(\mathbb{H})$, and if $\theta \ne 0$ the shift U_θ is *invertible*. Namely, if $\theta \ne 0$, then

$$\forall y \in \mathbb{H} \quad U_\theta^{-1} y(t) = \begin{cases} y(t-h) & \text{if } t \in \mathbb{T} \setminus \{-a\}, \\ \theta^{-1} y(a-h) & \text{if } t = -a. \end{cases} \tag{5.5}$$

and for any $\theta \in \mathbb{C}$,

$$U_\theta^* y(t) = \begin{cases} y(t-h) & \text{if } t > -a, \\ \overline{\theta} y(a-h) & \text{if } t = -a. \end{cases} \tag{5.6}$$

Comparing (5.5) and (5.6), we conclude that

5.1.2. The shift U_θ is unitary iff $|\theta| = 1$. Generally, $U_\alpha^* = U_\beta^{-1}$ when $\alpha\overline{\beta} = 1$. If $\theta = 0$, then U_θ is not invertible because $\ker U_0 = \mathbb{C}\delta_{a-h}$ (δ_{a-h} is the Dirac delta concentrated at $t = a - h$).

It may be proved easily that

5.1.3. *The eigenvalues of the shift U_θ are the ζ-roots of the equation*

$$\zeta^{2m} = \theta. \tag{5.7}$$

If $\theta = 0$, then there is a single eigenvalue $\zeta = 0$, and the operator U_0 is nilpotent. If $\theta \ne 0$, then all eigenvalues are simple, and to the eigenvalue ζ there corresponds the eigenfunction $e(\cdot, \zeta)$:

$$\forall t \in \mathbb{T} \quad e(t, \zeta) = \zeta^{t/h}. \tag{5.8}$$

5.1.4. We introduce the discrete interval $\widehat{\mathbb{T}}$ dual to (5.1) by putting

$$\widehat{h} := \pi/a, \quad \widehat{a} = m\widehat{h}, \quad \widehat{\mathbb{T}} := \{\widehat{t} = k\widehat{h} : k \in \mathbb{Z},\ -\widehat{a} \le \widehat{t} < \widehat{a}\}. \tag{5.9}$$

We label the eigenvalues ζ of U_θ, $\theta \neq 0$, by the points $\widehat{t} \in \widehat{\mathbb{T}}$. Let ϑ be a root of equation (5.7) such that

$$\vartheta = \varrho e^{i\alpha}, \quad \varrho > 0,\ \varrho^{2m} = |\theta|,\ -2\pi/m \le \alpha < 2\pi/m; \tag{5.10}$$

then the numbers

$$\zeta_{\theta,\widehat{t}} := \vartheta e^{ih\widehat{t}} = \varrho e^{i(\alpha + h\widehat{t})}, \quad \widehat{t} \in \widehat{\mathbb{T}}, \tag{5.11}$$

are different eigenvalues of U_θ. To the eigenvalue $\zeta_{\theta,\widehat{t}}$ there corresponds the eigenfunction $e_{\theta,\widehat{t}}$:

$$\forall t \in \mathbb{T} \quad e_{\theta,\widehat{t}}(t) = \vartheta^{t/h} e^{i\widehat{t}t}. \tag{5.12}$$

5.1.5. We introduce the Hilbert space $\widehat{\mathbb{H}}$ dual to $\mathbb{H}$. By definition, it consists of the functions $\widehat{x} \in \mathbb{C}^{\widehat{\mathbb{T}}}$ and is supplied with the inner product

$$\forall \widehat{x}, \widehat{y} \in \widehat{\mathbb{H}} \quad (\widehat{x}, \widehat{y}) := \sum_{\widehat{t} \in \widehat{\mathbb{T}}} \widehat{x}(\widehat{t})\overline{\widehat{y}(\widehat{t})}\,\widehat{h}. \tag{5.13}$$

5.1.6. Define

$$\forall x \in \mathbb{H}\ \forall \widehat{t} \in \widehat{\mathbb{T}} \quad \mathfrak{C}x(\widehat{t}) := \frac{1}{\sqrt{2\pi}} \sum_{t \in \mathbb{T}} x(t) e^{-i\widehat{t}t} h. \tag{5.14}$$

It can be easily checked that the operator $\mathfrak{C}$, which is called the *discrete Fourier transform*, maps $\mathbb{H}$ unitarily onto $\widehat{\mathbb{H}}$. The following inversion formula is true:

$$\forall t \in \mathbb{T} \quad x(t) = \frac{1}{\sqrt{2\pi}} \sum_{\widehat{t} \in \widehat{\mathbb{T}}} \widehat{x}(\widehat{t}) e^{i\widehat{t}t} \widehat{h}, \quad \widehat{x} := \mathfrak{C}x. \tag{5.15}$$

5.1.7. *The transformation $\mathfrak{C}$ diagonalizes the operator U_θ with $\theta = 1$ (i.e., the shift with the periodic boundary condition $x(a) = x(-a)$).*

$\lhd$ Let $\zeta_{\widehat{t}} := \zeta_{1,\widehat{t}}$ and $e_{\widehat{t}} := e_{1,\widehat{t}}$ be respectively the eigenvalues and eigenfunctions of the shift $U = U_1$. According to (5.11) and (5.12),

$$\zeta_{\widehat{t}} = e^{ih\widehat{t}}, \quad e_{\widehat{t}}(t) = e^{i\widehat{t}t}. \tag{5.16}$$

Therefore from (5.15) it follows that

$$x = \frac{1}{\sqrt{2\pi}} \sum_{\widehat{t}} \widehat{x}(\widehat{t}) e_{\widehat{t}} \widehat{h}, \quad Ux = \frac{1}{\sqrt{2\pi}} \sum_{\widehat{t}} \zeta_{\widehat{t}} \widehat{x}(\widehat{t}) e_{\widehat{t}} \widehat{h}. \tag{5.15$'$}$$

We can see that $\mathfrak{C}U\mathfrak{C}^{-1}$ is the multiplication operator by the function $\widehat{t} \mapsto \zeta_{\widehat{t}}$ in the space $\widehat{\mathbb{H}}$. ▶

Now we diagonalize the shift U_θ with arbitrary $\theta \in \mathbb{C} \setminus \{0\}$.

5.1.8. Decomposition of any function $x \in \mathbb{H}$ in the eigenfunctions of U_θ may be written as follows:

$$x = \frac{1}{\sqrt{2\pi}} \sum_{\widehat{t} \in \widehat{\mathbb{T}}} \widehat{x}_\theta(\widehat{t}) e_{\theta,\widehat{t}} \widehat{h}. \tag{5.17}$$

Formula (5.17) implicitly defines the "Fourier transform" $\mathfrak{C}_\theta$, which corresponds to the shift U_θ and diagonalizes it:

$$\mathfrak{C}_\theta x := \widehat{x}_\theta, \quad \mathfrak{C}_\theta U_\theta x = \zeta_{\theta,\widehat{t}}\mathfrak{C}_\theta x. \tag{5.18}$$

$\mathfrak{C}_\theta U_\theta {\mathfrak{C}_\theta}^{-1}$ is the multiplication in $\widehat{\mathbb{H}}$ by the function $\widehat{t} \mapsto \zeta_{\theta,\widehat{t}}$. In order to obtain an explicit expression for $\mathfrak{C}_\theta$ we rewrite (5.17) in the form

$$x(t) = \vartheta^{-t/h} = \frac{1}{\sqrt{2\pi}} \sum_{\widehat{t}\in\widehat{\mathbb{T}}} \widehat{x}_\theta(\widehat{t}) e^{i\widehat{t}t}\widehat{h}.$$

Hence by (5.14),

$$\widehat{x}_\theta(\widehat{t}) = \frac{1}{\sqrt{2\pi}} \sum_{t\in\mathbb{T}} x(t)\vartheta^{-t/h} e^{-i\widehat{t}t} h,$$

i.e.,

$$\mathfrak{C}_\theta x(\widehat{t}) = \widehat{x}_\theta(\widehat{t}) = \frac{1}{\sqrt{2\pi}} \sum_{t\in\mathbb{T}} x(t)\, \overline{\overset{*}{e}_{\theta,\widehat{t}}(t)} h, \tag{5.19}$$

where (see (5.10))

$$\overset{*}{e}_{\theta,\widehat{t}}(t) := \vartheta^{-t/h} e^{i\widehat{t}t} = \varrho^{-2t/h} e_{\theta,\widehat{t}}(t). \tag{5.20}$$

5.1.9. *The functions* $\overset{*}{e}_{\theta,\widehat{t}}$ *are eigenvalues of the operator* U_θ^*, *namely*

$$U_\theta^*\, \overset{*}{e}_{\theta,\widehat{t}} = \overline{\zeta}_{\theta,\widehat{t}}\, \overset{*}{e}_{\theta,\widehat{t}}. \tag{5.21}$$

◁ From (5.17) and (5.19) it follows that

$$\forall x \in \mathbb{H} \quad x = \frac{1}{2\pi} \sum_{\widehat{t}\in\widehat{\mathbb{T}}} (x,\, \overset{*}{e}_{\theta,\widehat{t}}) e_{\theta,\widehat{t}} \widehat{h}. \tag{5.22}$$

Since $\pi\widehat{h}^{-1} = a$ we find

$$\forall \widehat{t}, \widehat{s} \in \widehat{\mathbb{T}} \quad (e_{\theta,\widehat{s}},\, \overset{*}{e}_{\theta,\widehat{t}}) = \begin{cases} 0 & \text{if } \widehat{s} \neq \widehat{t}, \\ 2a & \text{if } \widehat{s} = \widehat{t}. \end{cases} \tag{5.23}$$

Therefore $(e_{\theta,\widehat{s}}, \overline{\zeta}_{\theta,\widehat{t}} \overset{*}{e}_{\theta,\widehat{t}}) = (\zeta_{\theta,\widehat{s}} e_{\theta,\widehat{s}},\, \overset{*}{e}_{\theta,\widehat{t}}) = (U_\theta e_{\theta,\widehat{s}},\, \overset{*}{e}_{\theta,\widehat{t}}) = (e_{\theta,\widehat{s}}, U_\theta^*\, \overset{*}{e}_{\theta,\widehat{t}})$, yielding (5.21). ▶

5.1.10. Remark. If $|\theta| = 1$, then according to (5.10), $\varrho = 1$ and by (5.19) and (5.20),

$$|\theta| = 1 \Rightarrow \overset{*}{e}_{\theta,\widehat{t}} = e_{\theta,\widehat{t}},\ \mathfrak{C}_\theta x(\widehat{t}) = \frac{1}{\sqrt{2\pi}} \sum_{t\in\mathbb{T}} x(t) e_{\theta,\widehat{t}}(t) h. \tag{5.24}$$

In particular, if $|\theta| = 1$ (and only in this case), then $\mathfrak{C}_\theta$ maps $\mathbb{H}$ *unitarily* onto $\widehat{\mathbb{H}}$, i.e., the following Parseval equality holds:

$$|\theta| = 1 \Rightarrow \forall x, y \in \mathbb{H}\ (\mathfrak{C}_\theta x, \mathfrak{C}_\theta y) = (x, y). \tag{5.25}$$

Note also that the shift U_θ is a *normal* operator only in the case when $|\theta| = 1$. This follows from the following formulae, which can be easily be checked:

$$\forall x \in \mathbb{H} \quad U_\theta^* U_\theta x(t) = \begin{cases} |\theta|^2 x(-a) & \text{if } t = -a, \\ x(t) & \text{if } t > -a, \end{cases}$$

$$\forall y \in \mathbb{H} \quad U_\theta U_\theta^* y(t) = \begin{cases} y(t) & \text{if } t < a-h, \\ |\theta|^2 y(a-h) & \text{if } t = a-h. \end{cases} \tag{5.26}$$

5.1.11. Example. According to formulae (5.14) and (5.19), the discrete Fourier transform of the discrete Dirac delta δ_t has the following form: for $\theta = 1$,

$$\widehat{\delta}_t(\widehat{t}\,) = \mathfrak{C}\delta_t(\widehat{t}\,) = \frac{1}{\sqrt{2\pi}} e^{-t\widehat{t}t}, \tag{5.27}$$

and for arbitrary $\theta \in \mathbb{C} \setminus \{0\}$,

$$\widehat{\delta}_{\theta,t}(\widehat{t}\,) = \mathfrak{C}\delta_t(\widehat{t}\,) = \frac{1}{\sqrt{2\pi}} \overset{*}{e}_{\widehat{t}}(t). \tag{5.28}$$

The inversion formulae (5.15) and (5.22) can be viewed as special cases of the Parseval equality

$$(x, \delta_t) = (\widehat{x}, \widehat{\delta}_t) = (\widehat{x}_\theta, \widehat{\delta}_{t,\theta}). \tag{5.29}$$

5.1.12. Denote by $\overset{*}{\mathfrak{C}}_{\theta,\widehat{t}}$ the discrete Fourier transform corresponding to the adjoint shift U_θ^*:

$$\forall y \in \mathbb{H}\ \forall t \in \mathbb{T} \quad \overset{*}{\mathfrak{C}}_{\theta,\widehat{t}}\, y(\widehat{t}\,) := \frac{1}{\sqrt{2\pi}} \sum_{t \in \mathbb{T}} y(t) \, \overset{*}{e}_{\theta,\widehat{t}}(-t) h. \tag{5.30}$$

It is easy to prove that for any $\theta \in \mathbb{C} \setminus \{0\}$ the following Parseval equality holds:

$$(\mathfrak{C}_\theta x, \overset{*}{\mathfrak{C}}_{\theta,\widehat{t}}\, y) = (x, y). \tag{5.31}$$

5.1.13. As already noted, the shift U_0 has a single eigenvalue $\lambda = 0$. To this eigenvalue there corresponds a chain of root elements $\delta_{-a}, \delta_{-a+h}, \dots, \delta_{a-h}$:

$$U_0 \delta_{a-h} = \delta_{a-2h}, \dots, U_0 \delta_{-a+h} = \delta_{-a}, \; U_0 \delta_{-a} = 0.$$

5.2. The operator D_θ. The operator D_θ of "discrete differentiation" corresponding to the boundary condition $x(a) = \theta x(-a)$ is defined by

$$D_\theta = \frac{1}{ih}(U_\theta - \mathbb{I}). \tag{5.32}$$

By the definition (5.4) of the shift U_θ,

$$\forall x \in \mathbb{H}\ \forall t \in \mathbb{T} \quad D_\theta x(t) = \begin{cases} \dfrac{1}{ih}[x(t+h) - x(t)] & \text{if } t < a-h, \\ \dfrac{1}{ih}[\theta x(-a) - x(a-h)] & \text{if } t = a-h. \end{cases} \tag{5.32$'$}$$

The operator D_θ with $\theta = 1$ is denoted by D.

5.2.1. Since $\alpha\overline{\beta} = 1 \Rightarrow U_\alpha^* = U_\beta^{-1}$, from (5.32) we conclude that

$$D_\theta^* = U_{1/\widehat{\theta}}^{-1} D_{1/\widehat{\theta}}, \quad D^* = U^{-1} D, \tag{5.33}$$

i.e.,

$$D^*_\theta y(t)=\begin{cases}\dfrac{1}{ih}[y(-a)-\overline{\theta}y(a-h)] & \text{if } t=-a,\\ \dfrac{1}{ih}[y(t)-y(t-h)] & \text{if } t>-a.\end{cases} \tag{5.33$'$}$$

The operator D_θ is *normal* only in the case $|\theta|=1$ since the shift U_θ is normal in this case. However, $D^*_\theta \neq D_\theta$ for all $\theta\in\mathbb{C}$.

5.2.2. From (5.32) it follows that the eigenvalues of the operator D_θ are

$$\lambda_{\theta,\widehat{t}} := \frac{1}{ih}[\zeta_{\theta,\widehat{t}}-1], \quad \widehat{t}\in\widehat{\mathbb{T}}, \tag{5.34}$$

where $\zeta_{\theta,\widehat{t}}$ are the eigenvalues of the shift U_θ. So (see (5.10) and (5.11)),

$$\lambda_{\theta,\widehat{t}} = \frac{1}{ih}[\vartheta e^{ih\widehat{t}}-1] = \frac{1}{ih}[\varrho e^{i(\alpha+h\widehat{t})}-1], \quad \widehat{t}\in\widehat{\mathbb{T}}, \tag{5.34$'$}$$

where $\varrho>0$, $\varrho^{2m}=\theta$, $-2\pi/m\le\alpha<2\pi/m$. For $\theta\neq 0$ all these eigenvalues are different. For $\theta=0$ there is a single eigenvalue $\lambda=1/h$, having a chain of root elements $\delta_{-a},\dots,\delta_{a-h}$, namely $D_0\delta_{a-h}=\lambda(\delta_{a-h}-\delta_{a-2h}),\dots,D_0\delta_{0a+h}=\lambda(\delta_{-a+h}-\delta_{-a})$, $D_0\delta_{-a}=0$.

5.2.3. If $\theta\neq 0$, then the eigenvalues of D_θ lie on the circle $\mathfrak{S}_\theta=\{\lambda\in\mathbb{C}:|\lambda-i/h|=\varrho/h\}$. The shadow of this circle is nonempty only if

$$|\varrho-1|/h\ll\infty. \tag{5.35}$$

If condition (5.35) is satisfied, then this shadow is the straight line

$$^\circ\mathfrak{S}_\theta=\left\{\lambda\in\mathbb{C}:\operatorname{Im}\lambda={}^\circ\!\left(\frac{1-\varrho}{h}\right)\right\}. \tag{5.36}$$

In particular, if $|\theta|=1$ (i.e., if U_θ is unitary and D_θ is normal), then $^\circ\mathfrak{S}_\theta$ is the *real* axis.

5.2.4. *In case $a\ll\infty$ (recall that $2a$ is the length of the interval $\mathbb{T}$) condition* (5.35) *is equivalent to*

$$0\ll|\theta|\ll\infty. \tag{5.37}$$

$\lhd$ Write $c:=(\varrho-1)/n$ and assume that $|c|\ll\infty$. Then $|\theta|=(1+ch)^{2m}=(1+ch)^{\frac{1}{ch}2ac}\approx e^{2ac}$. Since $|ac|\ll\infty$, we see that $0\ll e^{2ac}\ll\infty$. This means that (5.37) holds.

Assume that $|c|\approx+\infty$. First let $c\approx+\infty$. If $ch\approx 0$, then as before $|\theta|\approx e^{2ac}$, hence $|\theta|\approx+\infty$. If $ch\gg 0$, then $(1+ch)^{2m}\approx+\infty$ and again $|\theta|\approx+\infty$. Finally, let $c\approx-\infty$. If $ch\approx 0$ then $|\theta|\approx e^{2ac}\approx 0$, and if $ch\ll 0$ then $(1+ch)^{2m}\approx 0$, i.e., again $|\theta|\approx 0$. ▶

5.2.5. The operator D_θ has the same eigenfunctions as the shift U_θ, and similarly for D^*_θ and U^*_θ. If $\theta\neq 0$ we have (see (5.12), (5.20), and (5.34))

$$\forall\widehat{t}\in\widehat{\mathbb{T}}\quad D_\theta e_{\theta,\widehat{t}}=\lambda_{\theta,\widehat{t}}e_{\theta,\widehat{t}},\quad D^*_\theta\overset{*}{e}_{\theta,\widehat{t}}=\overline{\lambda}_{\theta,\widehat{t}}\overset{*}{e}_{\theta,\widehat{t}}. \tag{5.38}$$

5.2.6. Remark. If $|\theta| = 1$, then $\widehat{h} := \pi/a$ is the length of the arc of the circle $\mathfrak{S}_\theta$ (see 5.2.3) between the consecutive eigenvalues $\lambda_{\theta,\widehat{t}}$ and $\lambda_{\theta,\widehat{t}+\widehat{h}}$ of the operator D_θ. In fact, the radius of $\mathfrak{S}_\theta$ is equal to $1/h$, and the eigenvalues $\lambda_{\theta,\widehat{t}}$ divide this circle into $2m$ equal parts. Consequently, each such part has length $\pi/(mh) = \widehat{h}$.

If $a \approx +\infty$ then $\widehat{h} \approx 0$ so that consecutive eigenvalues of D_θ are *infinitely close to each other*. We treat such a situation as an analog of a continuous spectrum of an operator which acts in an infinite-dimensional space.

5.3. Discrete Riemann–Lebesgue lemma. We highlight especially the case $|\theta| = 1$ (with the periodic boundary condition $x(a) = x(-a)$). Set $\lambda_{\widehat{t}} := \lambda_{1,\widehat{t}}$ and $e_{\widehat{t}} := e_{1,\widehat{t}}$. By (5.12) and (5.34), we have

$$\lambda_{\widehat{t}} = \frac{1}{ih}(e^{ih\widehat{t}} - 1), \qquad e_{\widehat{t}}(t) = e^{i\widehat{t}t}, \quad t \in \mathbb{T},\ \widehat{t} \in \widehat{\mathbb{T}}. \tag{5.39}$$

From the elementary trigonometric inequality

$$-\pi \le \sigma \le \pi \Rightarrow \frac{2}{\pi}|\sigma| \le |e^{i\sigma} - 1| \le |\sigma| \tag{5.40}$$

it follows that

$$\forall \widehat{t} \in \widehat{\mathbb{T}} \quad \frac{2}{\pi}|\widehat{t}\,| \le |\lambda_{\widehat{t}}| \le |\widehat{t}\,|. \tag{5.41}$$

5.3.1. *Let* $x \in \mathbb{H}$ *and let* $\|Dx\|_1 \ll \infty$; *if* $|\widehat{t}\,| \approx +\infty$, *then*

$$(x, e_{\widehat{t}}) \approx 0. \tag{5.42}$$

◁ From $(Dx, e_{\widehat{t}}) = \lambda_{\widehat{t}}(x, e_{\widehat{t}})$ and $|e_{\widehat{t}}(t)| = 1$ we conclude that $|\lambda_{\widehat{t}}|(x, e_{\widehat{t}}) \le \|Dx\|_1$. Consequently, by (5.41),

$$|(x, e_{\widehat{t}})| \le \frac{\pi\|Dx\|_1}{2|\widehat{t}\,|}. \tag{5.42$'$}$$

In particular, if $|\widehat{t}\,| \approx +\infty$, then (5.42) is satisfied. ►

The following proposition is a discrete analog of the classical Riemann–Lebesgue lemma.

5.3.2. *Let* $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$, *that is*, $\|x - \mathit{\Pi}\xi\|_1 \approx 0$ *for some function* $\xi \in {}^{\mathrm{st}}L_1(\mathbf{T})$; *if* $|\widehat{t}\,| \approx +\infty$, *then* (5.42) *is satisfied.*

First we prove two auxiliary statements.

5.3.3 (Relation between the ordinary derivative and the discrete one). *Let* $\eta \in {}^{\mathrm{st}}C_0^{(2)}(\mathbf{T})$ *and let* $y := \mathit{\Pi}\eta$. *Then*

$$\left\| Dy - \frac{1}{i}\mathit{\Pi}\frac{d\eta}{d\tau} \right\|_1 \approx 0. \tag{5.43}$$

◁ From the definitions of the operators D and Π it follows that

$$Dy(t) - \frac{1}{i}\Pi\eta'(t) = \frac{1}{ih^2}\int\limits_t^{t+h} [\eta(\tau+h) - \eta(\tau)]\,d\tau - \frac{1}{ih}\int\limits_t^{t+h} \eta'(\sigma)\,d\sigma$$

$$= \frac{1}{ih^2}\int\limits_t^{t+h} d\tau \int\limits_\tau^{\tau+h} \eta'(\sigma)\,d\sigma - \frac{1}{ih^2}\int\limits_t^{t+h} d\tau \int\limits_t^{t+h} \eta'(\sigma)\,d\sigma$$

$$= \frac{1}{ih^2}\int\limits_t^{t+h} d\tau \int\limits_t^{t+h} [\eta'(\sigma+\tau-t) - \eta'(\sigma)]\,d\sigma$$

$$= \frac{1}{ih^2}\int\limits_t^{t+h} d\tau \int\limits_t^{t+h} d\sigma \int\limits_\sigma^{\sigma+\tau-t} \eta''(\varrho)\,d\varrho.$$

Hence

(5.43′) $$\forall t \in \mathbb{T} \quad \left| Dy(t) - \frac{1}{i}\Pi\eta'(t)\right| \le \int\limits_t^{t+2h} |\eta''(\varrho)|\,d\varrho.$$

Therefore,

(5.43″) $$\left\| Dy - \frac{1}{i}\Pi\eta'\right\|_1 \le 2h\|\eta''\|_1.$$

Since $\eta'' \in C_0(\mathbf{T})$ is a standard function, we see that $\|\eta''\| \ll \infty$ and from (5.43″) it follows that (5.43) is satisfied. ▶

5.3.4. Density Lemma. *If $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$, then there exists a sequence of functions $y_n \in \mathbb{C}^{\mathbb{T}}$ such that*

(5.44) $$\forall n \in {}^{\mathrm{st}}\mathbb{N} \quad \|Dy_n\|_1 \ll \infty \wedge \|x - y_n\|_1 < 1/n.$$

◁ Define $\xi := {}^{\bullet}x$ so that $\xi \in {}^{\mathrm{st}}L_1(\mathbf{T})$ and $\|x - \pi\xi\|_1 \approx 0$. Construct a standard sequence $\eta_n \in C^{(2)}(\mathbf{T})$ such that $\|\xi - \eta_n\|_1 < 1/(2n)$ for all $n \in \mathbb{N}$. Set $y_n := \Pi\eta_n$. Since Q is an isometry and $Q\Pi = P$ is a quasi-unity (see 3.2.6), we have $\|x - y_n\|_1 = \|Qx - Q\Pi\eta_n\|_1 \approx \|\xi - \eta_n\|_1$ and therefore $\|x - y_n\|_1 < 1/n$. Next we find $\|Dy_n\|_1 \le \left\|Dy_n - \frac{1}{i}\Pi\eta_n'\right\|_1 + \|\Pi\eta_n'\|_1$. For $n \in {}^{\mathrm{st}}\mathbb{N}$ the first summand is infinitesimal by (5.43). Since $\|\Pi\eta_n'\|_1 \le \|\eta_n'\|_1$, we obtain $\forall^{\mathrm{st}} n \in \mathbb{N}\ \|Dy_n\|_1 \ll \infty$. ▶

P r o o f o f 5.3.2. Suppose $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence from Lemma 5.3.4. We have $(x, e_{\widehat{t}}) = (x - y_n, e_{\widehat{t}}) + (y_n, e_{\widehat{t}})$, and according to (5.44) and (5.42′),

$$\forall n \in {}^{\mathrm{st}}\mathbb{N} \quad |(x, e_{\widehat{t}})| \le \frac{1}{n} + \frac{\pi\|Dy_n\|_1}{2|\widehat{t}\,|}.$$

Therefore, for $|\widehat{t}\,| \approx \infty$, we get (5.42). ▶

For later reference it is useful to give *another proof* of 5.3.2. Define

(5.45) $$\forall \widehat{t} \in \widehat{\mathbb{T}}\ \forall \tau \in \mathbf{T} \quad \varepsilon_{\widehat{t}}(\tau) = e^{i\widehat{t}\tau}.$$

Then

$$\forall t \in \mathbb{T} \quad \Pi\varepsilon_{\widehat{t}}(t) = \frac{1}{h}\int\limits_t^{t+h} e^{i\widehat{t}\tau}\,d\tau = \frac{e^{i\widehat{t}h} - 1}{i\widehat{t}h} e^{i\widehat{t}t}$$

and therefore,

$$e_{\hat t} = \gamma_{\hat t} \varPi \varepsilon_{\hat t}, \quad \text{where} \quad \gamma_{\hat t} := \frac{i\hat t h}{e^{i\hat t h} - 1}. \tag{5.46}$$

From (5.40) it follows that

$$\forall \hat t \in \widehat{\mathbb{T}} \quad 1 \le |\gamma_{\hat t}| \le \pi/2. \tag{5.47}$$

Consider an arbitrary function $x \in {}^{\mathrm{nst}}\mathbb{C}^{\mathbb{T}}$. Suppose $\xi \in {}^{\mathrm{st}}L_1(\mathbf{T})$ and $\|Qx - \xi\|_1 \approx 0$. Since Q is isometric and $Q\varPi = P$, $P^* = P$, $PQ = Q$, we find by (5.46) that $(x, e_{\hat t}) = (Qx, Qe_{\hat t}) = \overline{\gamma}_{\hat t}(Qx, P\varepsilon_{\hat t}) = \overline{\gamma}_{\hat t}(Qx, \varepsilon_{\hat t})$, hence

$$(x, e_{\hat t}) \approx \overline{\gamma}_{\hat t}(\xi, \varepsilon_{\hat t}). \tag{5.48}$$

By the ordinary Riemann–Lebesgue lemma and estimate (5.47), for $|\hat t| \approx \infty$ relation (5.42) holds. ▶

5.4. A nearstandardness criterion. This criterion concerns the case of a *finite* discrete interval and is formulated in terms of the discrete Fourier transform. To avoid technical complications we assume that $a \in {}^{\mathrm{st}}\mathbb{R}$ ($2a$ is the length of $\mathbb{T}$).

5.4.1. *Let $a \in {}^{\mathrm{st}}\mathbb{R}$ and let $x \in \mathbb{H}$. Then $x \in {}^{\mathrm{nst}}\mathbb{H}$ iff $\|x\| \ll \infty$ and*

$$n \approx +\infty \Rightarrow \sum_{|\hat t| > n} |(x, e_{\hat t})|^2 \approx 0; \tag{5.49}$$

here $e_{\hat t}(t) = e^{i\hat t t}$, $t \in \mathbb{T}$, $\hat t \in \widehat{\mathbb{T}}$.

◁ Recall that now $\mathbf{H}$ is the standard Hilbert space $L_2(-a, a)$. Suppose $x \in {}^{\mathrm{nst}}\mathbb{H}$, i.e., $Qx \in {}^{\mathrm{nst}}\mathbf{H}$. For $\hat t \in \hat h\mathbb{Z}$ define

$$(Qx)_{\hat t} := \frac{1}{2a} \sum_{|\hat s| > \hat t} (Qx, \varepsilon_{\hat s})\varepsilon_{\hat s}.$$

Since $\left(\frac{1}{\sqrt{2a}}\varepsilon_{\hat t}\right)_{\hat t \in \hat h\mathbb{Z}}$ is a standard orthonormal basis for $\mathbf{H}$, according to the nearstandardness criterion cited above (see footnote 12 in Section 0) we have $(Qx)_{\hat t} \approx 0$ for $|\hat t| \approx +\infty$. However, since $(Qx, \varepsilon_{\hat s}) = (x, \varPi\varepsilon_{\hat s})$, $\varPi\varepsilon_{\hat s} = \gamma_{\hat s}^{-1} e_s$, and $\gamma_{\hat s}^{-1} \ge 2/\pi$, we find that

$$|(Qx)_{\hat t}|^2 = \frac{1}{2a} \sum_{|\hat s| > \hat t} |(Qx, \varepsilon_{\hat s})|^2 \ge \frac{1}{\pi a} \sum_{|\hat s| > \hat t} |(x, e_{\hat s})|^2. \tag{5.50}$$

Consequently, $\|x\| \ll \infty$ and (5.49) is satisfied.

Conversely, let $x \in \mathbb{H}$, $\|x\| \ll \infty$, and suppose (5.49) holds. Note that

$$\forall \hat t \in \widehat{\mathbb{T}} \quad \left| \frac{1}{\overline{\gamma}_{\hat t}} (x, e_{\hat t}) \right| \ll \infty$$

and denote by $(c_{\hat t})_{\hat t \in \hat h\mathbb{Z}}$ the *standard extension* of the sequence

$${}^{\mathrm{st}}(\hat h\mathbb{Z}) \ni \hat t \mapsto {}^{\circ}\!\left(\frac{1}{\overline{\gamma}_{\hat t}} (x, e_{\hat t}) \right) \in \mathbb{C}.$$

Since $|\gamma_{\widehat{t}}| \geq 1$, we get

$$\forall n \in {}^{\mathrm{st}}\mathbb{N} \quad \sum_{|\widehat{t}\,|\leq n} |c_{\widehat{t}}|^2 \leq \sum_{|\widehat{t}\,|\leq n} |(x, e_{\widehat{t}})|^2 + 1 \leq 2a\|x\|^2 + 1.$$

Consequently, the (standard) series $\sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} |c_{\widehat{t}}|^2$ converges, and writing $\xi := \sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} c_{\widehat{t}}\varepsilon_{\widehat{t}}$ we have $\xi \in {}^{\mathrm{st}}\mathbf{H}$. Since

$$\|Qx-\xi\| \leq \Big(\frac{1}{2a}\sum_{|\widehat{t}\,|\leq n} |(Qx-\xi, \varepsilon_{\widehat{t}})|^2\Big)^{1/2} + \Big(\frac{1}{2a}\sum_{|\widehat{t}\,|>n} |(Qx, \xi)|^2\Big)^{1/2} + \Big(\frac{1}{2a}\sum_{|\widehat{t}\,|>n} |(\xi, \varepsilon_{\widehat{t}})|^2\Big)^{1/2},$$

we see that

$$\|Qx-\xi\| \leq \Big(\frac{1}{2a}\sum_{|\widehat{t}\,|\leq n} \Big|\frac{1}{\overline{\gamma}_{\widehat{t}}}(x, e_{\widehat{t}}) - c_{\widehat{t}}\Big|^2\Big)^{1/2} + \Big(\frac{1}{2a}\sum_{|\widehat{t}\,|>n} |(x, e_{\widehat{t}})|^2\Big)^{1/2} + \Big(\frac{1}{2a}\sum_{|\widehat{t}\,|>n} |c_{\widehat{t}}|^2\Big)^{1/2}.$$

For $n \in {}^{\mathrm{st}}\mathbb{N}$ the first summand is infinitesimal by the definition of $c_{\widehat{t}}$. By the Robinson lemma, it is infinitesimal up to some $n \approx +\infty$. For such an n the second summand is infinitesimal by (5.49). The third one is of the same type by the convergence of the standard series $\sum |c_{\widehat{t}}|^2$. ▶

5.4.2 (Shadow formula)**.** *If* $x \in {}^{\mathrm{nst}}\mathbb{H}$, *then*

$$\tag{5.51} {}^{\bullet}x = \frac{1}{2a}\sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} c_{\widehat{t}}\varepsilon_{\widehat{t}} = \frac{1}{2\pi}\sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} c_{\widehat{t}}\varepsilon_{\widehat{t}}\widehat{h},$$

where $(c_{\widehat{t}})_{\widehat{t}\in\widehat{h}\mathbb{Z}}$ *is the standard extension of the sequence* $({}^{\circ}(x, \varepsilon_{\widehat{t}}))_{t\in{}^{\mathrm{st}}(\widehat{h}\mathbb{Z})}$; *it is assumed that* $a \in {}^{\mathrm{st}}\mathbb{R}$.

◁ Since $\big(\frac{1}{\sqrt{2a}}\varepsilon_{\widehat{t}}\big)_{\widehat{t}\in\widehat{h}\mathbb{Z}}$ is an orthonormal basis for $\mathbf{H}$, we have

$${}^{\bullet}x = \frac{1}{2a}\sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} ({}^{\bullet}x, \varepsilon_{\widehat{t}})\varepsilon_{\widehat{t}}.$$

In view of $\|{}^{\bullet}x - Qx\| \approx 0$ we get $({}^{\bullet}x - Qx, \varepsilon_{\widehat{t}}) \approx 0$ for all $\widehat{t} \in \widehat{h}\mathbb{Z}$. As $h \approx 0$ it follows from $\gamma_{\widehat{t}} = i\widehat{t}h(e^{i\widehat{t}h} - 1)^{-1}$ that $\gamma_{\widehat{t}} \approx 1$ whenever $\widehat{t} \in {}^{\mathrm{st}}(\widehat{h}\mathbb{Z})$. Therefore by (5.48),

$$\tag{5.52} \forall \widehat{t} \in {}^{\mathrm{st}}(\widehat{h}\mathbb{Z}) \quad ({}^{\bullet}x, \varepsilon_{\widehat{t}}) \approx (x, e_{\widehat{t}}). \ ▶$$

5.5. Nearstandardness of the shift. First we note that

$$\tag{5.53} \|U_\theta\| = \begin{cases} 1 & \text{if } |\theta| \leq 1, \\ |\theta| & \text{if } |\theta| > 1. \end{cases}$$

Indeed, since

$$\|U_\theta x\|^2 = \sum_{t<a-h} |x(t+h)|^2 h + |\theta x(-a)|^2 h,$$

we have

$$\tag{5.54} \|U_\theta x\|^2 = \|x\|^2 + (|\theta|^2 - 1)|x(-a)|^2 h.$$

So if $|\theta| \leq 1$, then $\|U_\theta x\| \leq \|x\|$, where equality is attained whenever $x(-a) = 0$. And if $|\theta| > 1$, then from (5.54) it follows that $\|U_\theta x\| \leq |\theta| \cdot \|x\|$, where equality is attained whenever $x(t) = 0$ for $t > -a$.

Note also the following formulae connecting U_θ with $U = U_1$:

$$U_\theta = V_\theta U = U + (\theta - 1)W, \tag{5.55}$$

where

$$V_\theta x(t) = \begin{cases} x(t) & \text{if } t < a - h, \\ \theta x(a-h) & \text{if } t = a - h, \end{cases} \tag{5.56}$$

$$Wx(t) = \begin{cases} 0 & \text{if } t < a - h, \\ x(-a) & \text{if } t = a - h. \end{cases} \tag{5.57}$$

5.5.1. THEOREM. *For all $\theta \in \mathbb{C}$ the shift U_θ is graph-nearstandard and has the shadow $^\bullet U_\theta = \mathbb{I}_{\mathbf{H}}$.*

$\lhd$ First let $\theta = 1$. Since $\|U\| = 1 \ll \infty$, the $\langle$nst$\rangle$ condition is trivially satisfied. Suppose $\xi \in {}^{\mathrm{st}}\mathbf{H}$ and $x := \Pi\xi$ so that $x \in {}^{\mathrm{nst}}\mathbb{H}$ and $^\bullet x = \xi$. Applying the nearstandardness criterion to x we conclude that

$$\forall n \approx +\infty \quad \sum_{|\widehat{t}\,|>n} |(x, e_{\widehat{t}})|^2 \approx 0.$$

Since $U^* e_{\widehat{t}} = \zeta_{\widehat{t}} e_{\widehat{t}}$ and $|\zeta_{\widehat{t}}| = 1$, we get $\sum_{|\widehat{t}\,|>n} |(Ux, e_{\widehat{t}})|^2 \approx 0$. Therefore $Ux \in {}^{\mathrm{nst}}\mathbb{H}$. This means that for every $\xi \in {}^{\mathrm{st}}\mathbf{H}$ we have $\xi \in \mathrm{dom}(^\bullet U)$ and $(^\bullet U)\xi = {}^\bullet(Ux)$, where $x := \Pi\xi$. Applying the shadow formula (5.51), we find that

$$^\bullet(Ux) = \frac{1}{2a} \sum_{\widehat{t} \in \widehat{h}\mathbb{Z}} c_{\widehat{t}} \varepsilon_{\widehat{t}},$$

where $(c_{\widehat{t}})_{\widehat{t} \in \widehat{h}\mathbb{Z}}$ is the standard extension of the sequence ${}^{\mathrm{st}}(\widehat{h}\mathbb{Z}) \ni \widehat{t} \mapsto \widetilde{c}_{\widehat{t}} := {}^\circ(Ux, e_{\widehat{t}})$. But since $\zeta_{\widehat{t}} := e^{i\widehat{t}h} \approx 1$ for all $\widehat{t} \in {}^{\mathrm{st}}(\widehat{h}\mathbb{Z})$, we have $\widetilde{c}_{\widehat{t}} \approx (Ux, e_{\widehat{t}}) = \zeta_{\widehat{t}}(x, e_{\widehat{t}}) \approx (x, e_{\widehat{t}})$. Hence by (5.52), $\widetilde{c}_{\widehat{t}} \approx (\xi, \varepsilon_{\widehat{t}})$ for all $\widehat{t} \in {}^{\mathrm{st}}(\widehat{h}\mathbb{Z})$. Since the numbers $\widetilde{c}_{\widehat{t}}$ and $(\xi, \varepsilon_{\widehat{t}})$ are standard, we obtain $\widetilde{c}_{\widehat{t}} = (\xi, \varepsilon_{\widehat{t}})$. Thus

$$\forall \xi \in {}^{\mathrm{st}}\mathbf{H} \quad (^\bullet U)\xi = \frac{1}{2a} \sum_{\widehat{t} \in \widehat{h}\mathbb{Z}} (\xi, \varepsilon_{\widehat{t}})\varepsilon_{\widehat{t}} = \xi.$$

By the transfer principle, $^\bullet U = \mathbb{I}_{\mathbf{H}}$.

Now let $\theta \in \mathbb{C}$ be an arbitrary number. Consider an arbitrary $x \in \mathbb{H}$ such that $x \approx 0$ and $U_\theta x \in {}^{\mathrm{nst}}\mathbb{H}$. Write $y := U_\theta x - Ux$. Since $\|Ux\| = \|x\| \approx 0$, we have $y \approx U_\theta x$ and therefore $y \in {}^{\mathrm{nst}}\mathbb{H}$. But

$$y(t) = \begin{cases} 0 & \text{if } t < a - h, \\ (\theta - 1)x(-a) & \text{if } t = a - h, \end{cases}$$

i.e., $y = c\delta_{a-h}$, where $c := (\theta - 1)x(-a)h$. Such a y is a nearstandard function iff it is infinitesimal. We have verified that $x \approx 0 \wedge U_\theta x \in {}^{\mathrm{nst}}\mathbb{H} \Rightarrow U_\theta x \approx U_\theta x - Ux = y \approx 0$. Therefore the operator U_θ satisfies the $\langle$nst$\rangle$ condition. Thus it is graph-nearstandard. We shall find its shadow.

Suppose $\xi \in {}^{\mathrm{st}}\mathbf{H}$ and $x := \Pi\xi$ so that $x \in {}^{\mathrm{nst}}\mathbb{H}$ and $^\bullet x = \xi$. Set $\eta := \theta U_\theta x - \xi$. Then

$$\|\eta\|^2 = \int_{-a}^{a-h} |QU_\theta x(\tau) - \xi(\tau)|^2 \, d\tau + \int_{a-h}^{a} |\theta x(-a) - \xi(\tau)|^2 \, d\tau.$$

In the first summand, we can replace U_θ by U. Since ${}^\bullet U = \mathbb{I}_{\mathbf{H}}$, this term is infinitesimal: $\|QU\Pi\xi - \xi\| \approx 0$. Suppose in addition that $\xi \in {}^{\mathrm{st}}C_0(\mathbf{T})$. Then $x(-a) = \frac{1}{h}\int_{-a}^{-a+h} \xi(\tau)\,d\tau = 0$ and therefore the second summand is $\int_{a-h}^{a} |\xi(\tau)|^2\,d\tau \approx 0$. This implies that for every $\xi \in {}^{\mathrm{st}}C_0(\mathbf{T})$, $\eta \approx 0$, i.e., $({}^\bullet U_\theta)\xi = \xi$. Taking into account that $C_0(\mathbf{T})$ is a standard subspace which is dense in $\mathbf{H}$, and ${}^\bullet U_\theta$ is a closed standard operator, we conclude that $({}^\bullet U_\theta)\xi = \xi$ for all $\xi \in \mathbf{H}$. ▶

5.6. Nearstandardness of discrete differentiation. In this section we suppose as before that $a \in {}^{\mathrm{st}}\mathbb{R}$ ($2a$ is the length of the discrete interval $\mathbb{T}$).

5.6.1. THEOREM. *Let* $0 \ll \theta \ll \infty$ (see 5.2.3 and 5.2.4). *Then the operator* D_θ *is graph-nearstandard. Its shadow* $\mathbf{D}_\theta := {}^\bullet D_\theta$ *is described by*

(∗) $\operatorname{dom}\mathbf{D}_\theta$ *consists of* $\xi \in \mathbf{H} := L_2(-a,a)$ *that are absolutely continuous and for which*

$$\frac{d\xi}{d\tau} \in \mathbf{H}, \quad \xi(a) = {}^\circ\theta\xi(-a); \tag{5.58}$$

and

$$\forall \xi \in \operatorname{dom}\mathbf{D}_\theta \quad \mathbf{D}_\theta\xi = \frac{1}{i}\frac{d\xi}{d\tau}. \tag{∗∗}$$

◁ First we consider the case $\theta = 1$. To verify that the operator $D = D_1$ satisfies the $\langle\mathrm{nst}\rangle$ condition, we use the decomposition

$$Dx = \frac{1}{2\pi}\sum_{\hat{t}\in\hat{\varepsilon}_{\hat{t}}} \lambda_{\hat{t}}(x, e_{\hat{t}})e_{\hat{t}}\hat{h}. \tag{5.59}$$

According to (5.41), $|\lambda_{\hat{t}}| \le |\hat{t}\,|$ and therefore,

$$\frac{1}{2a}\sum_{|\hat{t}\,|\le n} |\lambda_{\hat{t}}(x, e_{\hat{t}})|^2 \le n^2\|x\|^2.$$

Consequently, if $\|x\| \approx 0$, then for all $n \in {}^{\mathrm{st}}\mathbb{N}$ the left side of the last inequality is infinitesimal. By the Robinson lemma, this is true up to some $n \approx +\infty$. But if $Dx \in {}^{\mathrm{nst}}\mathbb{H}$, then $\sum_{|\hat{t}\,|>n} |\lambda_{\hat{t}}(x, e_{\hat{t}})|^2 \approx 0$ for all $n \approx +\infty$. From $x \approx 0$ and $Dx \in {}^{\mathrm{nst}}\mathbb{H}$ it follows that $Dx \approx 0$, i.e., the graph-nearstandardness of D is proved.

Suppose $\xi \in {}^{\mathrm{st}}\mathrm{dom}\,\mathbf{D}$ and $x := \Pi\xi$ so that $\xi = {}^\bullet x$ and $\mathbf{D}\xi = {}^\bullet(Dx)$. According to the shadow formula,

$$\mathbf{D}\xi = \frac{1}{2\pi}\sum_{\hat{t}\in\hat{\mathbb{T}}} c_{\hat{t}}\varepsilon_{\hat{t}}\hat{h},$$

where $(c_{\hat{t}})_{\hat{t}\in\hat{h}\mathbb{Z}}$ is the standard extension of the mapping ${}^{\mathrm{st}}(\hat{h}\mathbb{Z}) \ni \hat{t} \mapsto {}^\circ(Dx, e_{\hat{t}}) = {}^\circ(\lambda_{\hat{t}}(x, e_{\hat{t}}))$. Since $\lambda_{\hat{t}} = \frac{1}{i\hat{h}}(e^{i\hat{h}t} - 1)$, we find that $\lambda_{\hat{t}} \approx \hat{t}$ for all $\hat{t} \in {}^{\mathrm{st}}(\hat{h}\mathbb{Z})$. By (5.52), ${}^\circ(x, e_{\hat{t}}) = (\xi, \varepsilon_{\hat{t}})$ for all $\hat{t} \in {}^{\mathrm{st}}(\hat{h}\mathbb{Z})$. Therefore,

$$\forall \xi \in \operatorname{dom}\mathbf{D} \quad \mathbf{D}\xi = \frac{1}{2\pi}\sum_{\hat{t}\in\hat{h}\mathbb{Z}} \hat{t}(\xi, \varepsilon_{\hat{t}})\varepsilon_{\hat{t}}\hat{h} \tag{5.60}$$

because the transfer principle lifts the restriction $\xi \in {}^{\mathrm{st}}\mathbf{H}$. From (5.60) it follows that $d\xi/d\tau \in \mathbf{H}$ and $\xi(a) = \xi(-a)$.

Conversely, let $\xi \in \mathbf{H}$ satisfy both these conditions. Then the series $\sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} |\widehat{t}(\xi, \varepsilon_{\widehat{t}})|^2$ converges. Define

$$x := \frac{1}{2\pi} \sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} \overline{\gamma}_{\widehat{t}}(\xi, \varepsilon_{\widehat{t}}) e_{\widehat{t}} \widehat{h}. \tag{5.61}$$

Since $|\gamma_{\widehat{t}}| \le \pi/2$, we have $\sum_{|\widehat{t}|>n} |\overline{\gamma}_{\widehat{t}}(\xi, \varepsilon_{\widehat{t}})|^2 \approx 0$ for all $n \approx +\infty$. By the nearstandardness criterion, $x \in {}^{\mathrm{nst}}\mathbb{H}$. Taking into account that $\gamma_{\widehat{t}} \approx 1$ and $(\xi, \varepsilon_{\widehat{t}}) \in {}^{\mathrm{st}}\mathbb{C}$ for all $\widehat{t} \in {}^{\mathrm{st}}(\widehat{h}\mathbb{Z})$, from the shadow formula it follows that

$$^{\bullet}x = \frac{1}{2\pi} \sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} (\xi, \varepsilon_{\widehat{t}}) \varepsilon_{\widehat{t}} \widehat{h}. \tag{5.61$'$}$$

From (5.61) it follows that

$$Dx = \frac{1}{2\pi} \sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} \lambda_{\widehat{t}} \overline{\gamma}_{\widehat{t}}(\xi, \varepsilon_{\widehat{t}}) e_{\widehat{t}} \widehat{h}. \tag{5.61$''$}$$

As $|\lambda_{\widehat{t}}| \le |\widehat{t}|$ and $|\gamma_{\widehat{t}}| \ll \infty$, we infer from (5.61$''$) that $Dx \in {}^{\mathrm{nst}}\mathbb{H}$. This means that $\xi \in \operatorname{dom} \mathbf{D}$. By the transfer principle, the assumption $\xi \in {}^{\mathrm{st}}\mathbf{H}$ is redundant. Since

$$\mathbf{D}\xi = {}^{\bullet}(Dx) = \frac{1}{2\pi} \sum_{\widehat{t}\in\widehat{h}\mathbb{Z}} \widehat{t}(\xi, \varepsilon_{\widehat{t}}) \varepsilon_{\widehat{t}} \widehat{h},$$

we get $\mathbf{D}\xi = \frac{1}{i}\frac{d\xi}{d\tau}$. Thus for $\theta = 1$, the theorem is proved.

For the general case, we use the decomposition (see (5.22), (5.23), and (5.38))

$$D_\theta x = \frac{1}{2\pi} \sum_{\widehat{t}\in\widehat{\mathbb{T}}} \lambda_{\theta,\widehat{t}}(x, \overset{*}{e}_{\theta,\widehat{t}}) e_{\theta,\widehat{t}} \widehat{h}. \tag{5.62}$$

Denote by Y_θ the operator of multiplication by the function $\mathbb{T} \ni t \mapsto \vartheta^{-t/h}$ in the space $\mathbb{H}$, i.e.,

$$\forall x \in \mathbb{H}\ \forall t \in \mathbb{T} \quad Y_\theta x(t) := \vartheta^{-t/h} x(t). \tag{5.63}$$

According to (5.11) and (5.20), we have

$$e_{\theta,\widehat{t}} = Y_\theta^{-1} e_{\widehat{t}}, \quad \overset{*}{e}_{\theta,\widehat{t}} = Y_\theta^* e_{\widehat{t}}. \tag{5.64}$$

Therefore (5.62) may be written in the form

$$Y_\theta D_\theta x = \frac{1}{2\pi} \sum_{\widehat{t}\in\widehat{\mathbb{T}}} \lambda_{\theta,\widehat{t}}(Y_\theta x, e_{\widehat{t}}) e_{\widehat{t}} \widehat{h}. \tag{5.62$'$}$$

By (5.34), the eigenvalues $\lambda_{\theta,\widehat{t}}$ and $\lambda_{\widehat{t}}$ of the operators D_θ and $D = D_1$ respectively are connected by

$$\lambda_{\theta,\widehat{t}} = \vartheta\lambda_{\widehat{t}} + ib, \quad b := \frac{\vartheta - 1}{h}. \tag{5.65}$$

Consequently, the operators D_θ and D are connected by

$$D_\theta = \vartheta Y_\theta^{-1} D Y_\theta + \frac{1}{i} b \mathbb{I}_{\mathbb{H}}. \tag{5.66}$$

5.6.2. Lemma. *Let $g \in L(-a,a)$ be a standard, essentially bounded function and let*

$$\forall t \in \mathbb{T} \quad f(t) \approx \frac{1}{h} \int_t^{t+h} g(\tau)\, d\tau.$$

Denote by F and G the operators of multiplication by f in $\mathbb{H}$ and by g in $\mathbf{H}$ respectively. Then the operator F is strongly nearstandard and has the shadow ${}^\bullet F = G$.

$\lhd$ Suppose $x \in {}^{\mathrm{nst}}\mathbb{H}$ so that $\|x\| \ll \infty$. Then

$$\|Fx - \mathfrak{P}Gx\|^2 = \sum_{t\in\mathbb{T}} \Big| f(t)x(t) - \frac{1}{h}\int_t^{t+h} g(\tau)x(t)\, d\tau\Big|^2$$

$$\le \sup_{t\in\mathbb{T}} \Big| f(t) - \frac{1}{h}\int_t^{t+h} g(\tau)\, d\tau\Big|^2 \|x\|^2 \approx 0. \ \blacktriangleright$$

5.6.3. Remark. *We have*

$$|\alpha|/h \le 2\pi/a, \quad |b| \ll \infty, \quad \vartheta^{1/h} \approx e^b, \tag{5.67}$$

where $b = (\vartheta - 1)/h$, $\vartheta = \varrho e^{i\alpha}$, $\varrho > 0$, $-2\pi/m \le \alpha < 2\pi/m$.

$\lhd$ The first inequality follows from $mh = a$. The second follows from the fact that

$$b = \frac{\varrho - 1}{h} e^{i\alpha} + \frac{e^{i\alpha} - 1}{h}$$

since $|\varrho - 1|/h \ll \infty$ (see 5.2.4) and

$$\frac{|e^{i\alpha} - 1|}{h} = \frac{|e^{i\alpha} - 1|}{|i\alpha|}\Big|\frac{\alpha}{h}\Big| \approx \Big|\frac{\alpha}{h}\Big| < \frac{2\pi}{a} \ll \infty.$$

Finally, since $\vartheta = 1 + bh$, we obtain $\vartheta^{1/h} = (1 + bh)^{\frac{1}{bh}b} \approx e^b$. $\blacktriangleright$

5.6.4. Corollary. *The operator Y_θ is strongly nearstandard and has the shadow ${}^\bullet Y_\theta$, which is the operator of multiplication (in $\mathbf{H}$) by the (standard) function $\mathbf{T} \ni \tau \mapsto e^{-c\tau}$, where $c := {}^\circ b$.*

$\lhd$ This follows directly from Lemma 5.6.2 since for $t \in \mathbb{T}$,

$$\frac{1}{h}\int_t^{t+h} e^{-c\tau}\, d\tau = e^{-c\tau}\frac{e^{-ch} - 1}{-ch} \approx e^{-ct} \approx e^{-bt} = \vartheta^{t/h}. \ \blacktriangleright$$

Let us return to the proof of Theorem 5.6.1. Suppose $x \approx 0$ and $D_\theta x \in {}^{\mathrm{nst}}\mathbb{H}$. Since the operator Y_θ is strongly nearstandard (in particular, it takes nearstandard vectors to nearstandard ones), we find $Y_\theta D_\theta x \in {}^{\mathrm{nst}}\mathbb{H}$. As $|b| \ll \infty$ and $\vartheta \approx 1$ it follows from (5.66) that $DY_\theta x \in {}^{\mathrm{nst}}\mathbb{H}$. In view of $Y_\theta x \approx 0$ and since by the above the operator D is graph-nearstandard, we have $DY_\theta x \approx 0$. Therefore $D_\theta x = \vartheta Y_\theta^{-1} DY_\theta x + ibx \approx 0$. Thus the operator D_θ satisfies the $\langle\mathrm{nst}\rangle$ condition, hence it is graph-nearstandard. We shall find its shadow $\mathbf{D}_\theta := {}^\bullet D_\theta$.

Suppose $\xi \in \operatorname{dom} \mathbf{D}_\theta$ and $x := \Pi\xi$. Then $\mathbf{D}_\theta \xi = {}^\bullet(D_\theta x) = {}^\bullet\big(\vartheta Y_\theta^{-1} DY_\theta x + \frac{1}{i}bx\big)$. Since Y_θ and Y_θ^{-1} are strongly nearstandard, we find that $Y_\theta x \in {}^{\mathrm{nst}}\mathbb{H}$ and ${}^\bullet(Y_\theta x) = ({}^\bullet Y_\theta)\xi$. Moreover, by (5.61), $\vartheta Y_\theta x = Y_\theta\big(D_\theta x - \frac{1}{i}bx\big) \in {}^{\mathrm{nst}}\mathbb{H}$. Therefore $DY_\theta x \in {}^{\mathrm{nst}}\mathbb{H}$ and $({}^\bullet Y_\theta)\xi \in \operatorname{dom} \mathbf{D}$. Consequently, the function $({}^\bullet Y_\theta)\xi$ is absolutely continuous, $\frac{d}{d\tau}[({}^\bullet Y_\theta)\xi] \in \mathbf{H}$, and

$({}^\bullet Y_\theta)\xi(a) = ({}^\bullet Y_\theta)\xi(-a)$. By Corollary 5.6.4, the last equality means that $e^{-ca}\xi(a) = e^{ca}\xi(-a)$. Since $c \approx b$, we have, according to (5.67), $e^{2ca} \approx e^{2ba} \approx \vartheta^{2a/h} = \vartheta^{2m} = \theta$. Therefore ξ satisfies the boundary condition $\xi(a) = {}^\circ\theta\xi(-a)$ (see (5.58)). We have $\mathbf{D}\eta = \frac{1}{i}\frac{d\eta}{d\tau}$ for all $\eta \in \operatorname{dom}\mathbf{D}$, hence

$$\mathbf{D}_\theta\xi(\tau) = {}^\circ\vartheta e^{c\tau}\frac{1}{i}\frac{d}{d\tau}[e^{-c\tau}\xi(\tau)] + \frac{1}{i}{}^\circ b\xi(\tau) = \frac{1}{i}\frac{d\xi}{d\tau}(\tau)$$

because ${}^\circ\vartheta = 1$ and $c = {}^\circ b$.

Conversely, let conditions (5.58) be satisfied for $\xi \in \mathbf{H}$ and let $x := \Pi\xi$. Since $e^{-ca}\xi(a) = e^{ca}\xi(-a)$ and $\frac{d}{d\tau}({}^\bullet Y_\theta\xi) \in \mathbf{H}$, we get $({}^\bullet Y_\theta)\xi \in \operatorname{dom}\mathbf{D}$. Consequently, $D_\theta x = \vartheta y_\theta^{-1}DY_\theta x + \frac{1}{i}bx \in {}^{\mathrm{nst}}\mathbb{H}$. This means (see (4.26)) that $\xi \in \operatorname{dom}\mathbf{D}_\theta$. ▶

5.7. Case $a \approx +\infty$**.** In this case we shall use a more direct method, which does not employ the discrete Fourier transform.

5.7.1. *For $|\theta| \ll \infty$ the shift U_θ is strongly nearstandard and has the shadow ${}^\bullet U_\theta = \mathbb{I}_{\mathbf{H}}$.*

◁ Since $\|U_\theta\| = |\theta| \ll \infty$ (see (5.53)), we only have to show that

$$\forall\xi \in {}^{\mathrm{st}}\mathbf{H} \quad \|\mathfrak{Q}U_\theta\xi - \xi\| \approx 0.$$

As the space of all smooth, compactly supported functions is dense in $\mathbf{H}$ and the orthoprojector P is a quasi-unity of $\mathcal{B}(\mathbf{H})$ it suffices to prove that

$$\forall\xi \in {}^{\mathrm{st}}C_0^{(2)}(\mathbf{T}) \quad \|QU_\theta\Pi\xi - P\xi\| \approx 0. \tag{5.68}$$

Suppose $\xi \in {}^{\mathrm{st}}C_0^{(2)}(\mathbf{T})$ and $x := \Pi\xi$. Since the embedding $Q : \mathbb{H} \to \mathbf{H}$ is isometric, we have $\|QU_\theta\Pi\xi - P\xi\| = \|U_\theta x - x\|$. From (5.43′) it follows that for all $t \in \mathbb{T}$, $|Dy(t) - \frac{1}{i}\Pi\eta'(t)| \approx 0$ whenever $\eta \in C_0^{(2)}(\mathbf{T})$ and $y := \Pi\eta$. Consequently,

$$\forall t \in \mathbb{T} \quad U_\theta x(t) - x(t) = ihDx(t) = ih\left[\frac{d\xi}{d\tau}(t) + \alpha(t)\right],$$

where $\alpha(t) \approx 0$ for all $t \in \mathbb{T}$ and $\alpha(t) = 0$ outside a finite standard neighbourhood $\mathcal{E}$ of $\operatorname{supp}\xi$. Writing

$$M := \max_{t\in\mathbb{T}\cap\mathcal{E}}\left|\frac{1}{i}\frac{d\xi}{d\tau}(t) + \alpha(t)\right|,$$

we have $\|U_\theta x - x\|^2 \le \sum_{t\in\mathcal{E}}|ihM|^2h = (Mh)^2\lambda(\mathcal{E}) \approx 0$ since $M \ll \infty$ and $\lambda(\mathcal{E}) \ll \infty$. ▶

Now we turn to discrete differentiation. We need the following property of the inductor.

5.7.2. Lemma. *Let $\xi \in {}^{\mathrm{st}}\mathbf{H}$ and let $\Pi\xi(t) \approx 0$ quasi-everywhere on $\mathbb{T}$* (see 2.5.3)*; then $\xi = 0$.*

◁ Let $E_0 \in 2^{\mathbb{T}}$ with $h\operatorname{card}E_0 \approx 0$, and suppose $\Pi\xi(t) \approx 0$ for all $t \in \mathbb{T}\setminus E_0$. Define $\alpha = \max_{t\in\mathbb{T}\setminus E_0}|\Pi\xi(t)|$. Then $\alpha \approx 0$. Since the (Lebesgue) measure of QE_0 equals $h\operatorname{card}E_0 \approx 0$ and the function ξ is standard, we find that for all $E \in 2^{\mathbb{T}}$,

$$\int_{QE}\xi(\tau)\,d\tau \approx \int_{Q(E\setminus E_0)}\xi(\tau)\,d\tau = \sum_{t\in E\setminus E_0}\Pi\xi(t)h \approx 0$$

whenever $h \operatorname{card} E \ll \infty$. However, the set E can be chosen so that QE be an interval with arbitrary given standard endpoints. Therefore $\xi(\tau) = 0$ almost everywhere on $\mathbf{T}$. ▶

5.7.3. Corollary. *Let $\xi \in {}^{\mathrm{st}}\mathbf{H}$ and $\widetilde{\xi} \in {}^{\mathrm{st}}C(\mathbf{T})$, and suppose $\Pi\xi(t) \approx \widetilde{\xi}(t)$ quasi-everywhere on $\mathbb{T}$; then $\xi = \widetilde{\xi}$.*

◁ From $\widetilde{\xi} \in {}^{\mathrm{st}}C(\mathbf{T})$ it follows that $\Pi\widetilde{\xi}(t) \approx \widetilde{\xi}(t)$ for all $t \in \mathbb{T}$. Therefore $\Pi(\xi - \widetilde{\xi})(t) \approx 0$ quasi-everywhere on $\mathbb{T}$. ▶

5.7.4. Theorem. *The operator D_θ is graph-nearstandard. For $a \approx +\infty$, its shadow $\mathbf{D} = {}^\bullet D_\theta$ does not depend on θ ($|\theta| \ll \infty$) and is described by the conditions*

($*$) $\operatorname{dom}\mathbf{D}$ *consists of $\xi \in \mathbf{H}$ such that $d\xi/d\tau \in \mathbf{H}$;*

($**$) $$\forall \xi \in \operatorname{dom}\mathbf{D} \quad \mathbf{D}\xi = \frac{1}{i}\frac{d\xi}{d\tau}.$$

◁ Suppose $x \approx 0$ and $D_\theta x \in {}^{\mathrm{nst}}\mathbb{H}$. Let us show that $D_\theta x \approx 0$. Write $\eta := {}^\bullet(D_\theta x)$. Then $\eta \in {}^{\mathrm{st}}\mathbf{H}$ and $\|QD_\theta x - \eta\| \approx 0$. Consider any $t_0, t \in \mathbb{T}$ such that $0 < t - t_0 \ll \infty$. We have

$$\Big|\int_{t_0}^{t} [QD_\theta x(\tau) - \eta(\tau)]\, d\tau\Big| \le \sqrt{t - t_0}\, \|QD_\theta x - \eta\| \approx 0.$$

Consequently,

$$\int_{t_0}^{t} \eta(\tau)\, d\tau \approx \sum_{s=t_0}^{t-h} \frac{1}{ih}[x(s+h) - x(s)]h = \frac{1}{i}[x(t) - x(t_0)].$$

Since $\|x\| \approx 0$, we get $x(t) \approx 0$ quasi-everywhere on $\mathbb{T}$. So the relation $\int_{t_0}^{t} \eta(\tau)\, d\tau \approx 0$ is satisfied except on infinitesimal ℓ-measure sets of the variables t_0 and t ($\ell_t \equiv h$). As η is standard and the Lebesgue integral is absolutely continuous, we find that $\int_{\tau_0}^{\tau} \eta(\sigma)\, d\sigma = 0$ for all (standard and nonstandard) τ_0 and τ. Therefore $\eta = 0$, i.e., the operator D_θ satisfies the $\langle\mathrm{nst}\rangle$ condition.

Now we prove ($*$) and ($**$). Suppose $\xi \in {}^{\mathrm{st}}\operatorname{dom}\mathbf{D}$, where $\mathbf{D} := {}^\bullet D_\theta$ and $x := \Pi\xi$. Then $\xi = {}^\bullet x$ and $\mathbf{D}\xi = {}^\bullet(D_\theta x)$. Setting $\eta := {}^\bullet(D_\theta x)$, we have $\|D_\theta x - \Pi\eta\| = \|QD_\theta x - P\eta\| \approx \|QD_\theta x - \eta\| \approx 0$. Consider any $t_0, t \in \mathbb{T}$ such that $0 < t - t_0 \ll \infty$. Obviously,

$$\Big|\sum_{s=t_0}^{t-h} [D_\theta x(s) - \Pi\eta(s)]h\Big| \le \sqrt{t - t_0}\, \|D_\theta x - \Pi\eta\| \approx 0.$$

Consequently, for $t < a - h$,

$$x(t) - x(t_0) \approx i \sum_{s=t_0}^{t-h} \Pi\eta(s)h = i\int_{t_0}^{t} \eta(\sigma)\, d\sigma.$$

Therefore,

$$\Pi\xi(t) \approx {}^\circ\Big[x(t_0) + i\int_{{}^\circ t_0}^{t} \eta(\sigma)\, d\sigma\Big], \quad t < a - h. \tag{5.69}$$

Hence we conclude, by 5.7.3, that

$$\xi(\tau) = {}^{\circ}\Big[x(t_0) + i \int_{{}^{\circ}t_0}^{t} \eta(\sigma)\, d\sigma\Big] \tag{5.70}$$

almost everywhere for $\tau \in \mathbb{R}$. But this means that the function ξ is absolutely continuous and $\mathbf{D}\xi = \frac{1}{i}\frac{d\xi}{d\tau} \in \mathbf{H}$.

Conversely, let $\xi \in {}^{\mathrm{st}}\mathbf{H}$ be an absolutely continuous function such that $d\xi/d\tau \in \mathbf{H}$. Write $\eta := \frac{1}{i}\frac{d\xi}{d\tau}$. Now we check that $\xi \in \operatorname{dom}\mathbf{D}$ and $\mathbf{D}\xi = \eta$. According to 4.4.4 one has to verify that $\xi = {}^{\bullet}x$ for some $x \in \operatorname{dom}_{\mathrm{nst}} D_\theta$ and $\eta = {}^{\bullet}(D_\theta x)$. Define $x(t) := \xi(t)$ for $t \in \mathbb{T}$. This is well defined because ξ is continuous. Let us show that $x \in {}^{\mathrm{nst}}\mathbb{H}$ and ${}^{\bullet}x = \xi$. Since $Qx(\tau) = 0$ for $|\tau| > a$ and $\int_{|\tau|>a} |\xi(\tau)|^2\, d\tau \approx 0$, we find

$$\|Qx - \xi\|^2 = \int_{|\sigma|\le a} |Qx(\sigma) - \xi(\sigma)|^2\, d\sigma + \alpha, \quad \text{where } \alpha \approx 0.$$

Therefore

$$\begin{aligned}\|Qx - \xi\|^2 &\approx \sum_{t=-a}^{a-h} \int_t^{t+h} |\xi(t) - \xi(\sigma)|^2\, d\sigma \\ &= \sum_{t=-a}^{a-h} \int_t^{t+h} d\sigma \,\Big|\int_t^{\sigma} \eta(\tau)\, d\tau\Big|^2 \le \sum_{t=-a}^{a-h} \int_t^{t+h} d\sigma\, (\sigma - t) \int_t^{\sigma} |\eta(\tau)|^2\, d\tau \\ &\le \sum_{t=-a}^{a-h} \frac{1}{2} h^2 \int_t^{t+h} |\eta(\tau)|^2\, d\tau \le \frac{1}{2} h^2 \|\eta\|^2 \approx 0\end{aligned}$$

because $\|\eta\| \ll \infty$. So $\|Qx - \xi\| \approx 0$, i.e., ${}^{\bullet}x = \xi$. Set $y := D_\theta x$. It remains to check that ${}^{\bullet}y = \eta$. From $d\xi/d\tau \in \mathbf{H}$ it follows that $|\xi(\tau)| \approx 0$ as $|\tau| \approx +\infty$. By assumption, $|\theta| \ll \infty$. Since

$$\int_{a-h}^{a} \Big|\frac{1}{ih}[\theta\xi(-a) - \xi(a-h)] - \eta(\sigma)\Big|^2\, d\sigma \approx 0,$$

we have

$$\|Qy - \eta\|^2 = \int_{|\sigma|<a} |QD_\theta x(\sigma) - \eta(\sigma)|^2\, d\sigma \approx \sum_{t=-a}^{a-h} \int_t^{t+h} \Big|\frac{1}{ih}[\xi(t+h) - \xi(t)] - \eta(\sigma)\Big|^2\, d\sigma.$$

Thus

$$\begin{aligned}\|Qy - \eta\|^2 &\approx \sum_{t=-a}^{a-h} \int_t^{t+h} \Big|\frac{1}{ih} \int_t^{t+h} \eta(\tau)\, d\tau - \eta(\sigma)\Big|^2\, d\sigma = \sum_{t=-a}^{a-h} \int_t^{t+h} |\Pi\eta(t) - \eta(\sigma)|^2\, d\sigma \\ &= \sum_{t=-a}^{a-h} \int_t^{t+h} |Q\Pi\eta(\sigma) - \eta(\sigma)|^2\, d\sigma = \|P\eta - \eta\|^2 \approx 0\end{aligned}$$

because $\eta \in {}^{\mathrm{st}}\mathbf{H}$. ▶

5.7.5. Remark. The previous method is useful also in case $a \in {}^{\mathrm{st}}\mathbb{R}$. In this case, one must also prove the necessity and sufficiency of the boundary condition $\xi(-a) = {}^{\circ}\theta\xi(-a)$

in addition to $(*)$ and $(**)$. First we prove its necessity. Suppose $\xi \in {}^{\mathrm{st}}\mathrm{dom}\,\mathbf{D}_\theta$, where $\mathbf{D}_\theta := {}^\bullet D_\theta$, $x := \Pi\xi$, and $\eta := {}^\bullet(D_\theta x)$. Then

$$\int_{-a}^{a} \eta(\sigma)\,d\sigma \approx \sum_{t=-a}^{a-h} D_\theta x(t)h = \frac{1}{i}(\theta - 1)x(-a).$$

But, by (5.70),

$$\xi(\tau) \approx x(-a) + i\int_{-a}^{\tau} \eta(\sigma)\,d\sigma. \tag{5.71}$$

In particular, $\xi(a) \approx x(-a) + i\int_{-a}^{a} \eta(\sigma)\,d\sigma \approx \theta x(-a) \approx \theta\xi(-a)$, hence $\xi(a) = {}^\circ\theta\xi(-a)$.

Conversely, let $\xi \in {}^{\mathrm{st}}\mathbf{H}$ be an absolutely continuous function satisfying the boundary condition $\xi(a) = {}^\circ\theta\xi(-a)$ and let $\eta := \frac{1}{i}\frac{d\xi}{d\tau} \in \mathbf{H}$. One has to prove that $\xi = {}^\bullet x$ for some $x \in \mathrm{dom}_{\mathrm{nst}}\, D_\theta$ and that $\eta = {}^\bullet(D_\theta x)$ (for this x). Define x by $x(t) = \xi(t) + \varepsilon t$ for $t \in \mathbb{T}$, where

$$\varepsilon := \frac{\theta - {}^\circ\theta}{{}^\circ\theta \cdot \theta} \cdot \frac{\xi(a)}{a}.$$

Since $0 \ll \theta \ll \infty$, we have $\varepsilon \approx 0$. Arguing as in the proof of Theorem 5.7.4, we find that $x \in {}^{\mathrm{nst}}\mathbb{H}$ and ${}^\bullet x = \xi$. Put

$$y(t) = \begin{cases} \Pi\eta(t) - i\varepsilon & \text{if } t < a-h, \\ \Pi\eta(a-h) & \text{if } t = a-h. \end{cases}$$

Since $\eta \in {}^{\mathrm{st}}\mathbf{H}$, we get ${}^\bullet y = \eta$. Now it suffices to prove that $D_\theta x = y$. Indeed, for $t < a-h$ we have

$$D_\theta x(t) = \frac{1}{ih}[x(t+h) - x(t)] = \frac{1}{ih}[\xi(t+h) - \xi(t) + \varepsilon h] = \Pi\eta(t) - i\varepsilon = y(t),$$

and

$$\begin{aligned} D_\theta x(a-h) &= \frac{1}{ih}[\theta x(-a) - x(a-h)] = \frac{1}{ih}[\theta(\xi(-a) - \varepsilon a) - \xi(a-h)] \\ &= \frac{1}{ih}\left[\theta\left(\frac{\xi(a)}{{}^\circ\theta} - \frac{\theta - {}^\circ\theta}{{}^\circ\theta \cdot \theta} \cdot \frac{\xi(a)}{a}a\right) - \xi(a-h)\right] \\ &= \frac{1}{ih}[\xi(a) - \xi(a-h)] = \Pi\eta(a-h) = y(a-h). \quad \blacktriangleright \end{aligned}$$

6. Application of equipment

To extend the notion of nearstandardness, we use the technique of equipment. By $\mathbb{H}$ and $\mathbf{H}$ we denote the same Hilbert spaces as before. Let us show that every Hilbert equipment

$$\mathbf{H}_- \supset \mathbf{H} \supset \mathbf{H}_+ \tag{6.1}$$

of the space $\mathbf{H}$ naturally induces some equipment of the space $\mathbb{H}$. Recall (see, for example, [2]) that, in the chain (6.1), $\mathbf{H}_+$ is a Hilbert space continuously and densely embedded in $\mathbf{H}$, and $\mathbf{H}_- = \mathbf{H}_+^*$ is the space adjoint to $\mathbf{H}_+$. The inner products and the norms in

$\mathbf{H}_-$ and $\mathbf{H}_+$ are denoted respectively by $(\cdot,\cdot)_-$, $\|\cdot\|_-$ and $(\cdot,\cdot)_+$, $\|\cdot\|_+$. Without loss of generality it can be assumed that

$$\forall\varphi\in\mathbf{H}_+\quad \|\varphi\|\le\|\varphi\|_+,\qquad \forall\xi\in\mathbf{H}\quad \|\xi\|_-\le\|\xi\|. \tag{6.2}$$

For $\varphi\in\mathbf{H}_+$ and $\alpha\in\mathbf{H}$ the value $\alpha(\varphi)$ of the functional α at the element φ is denoted by (φ,α) or $\overline{(\alpha,\varphi)}$. This enables us to treat the pairing of $\mathbf{H}_+$ and $\mathbf{H}_-$ as an extension of the inner product $(\cdot,\cdot)_{\mathbf{H}}$ to all of $\mathbf{H}_+\times\mathbf{H}_-$, which was previously restricted to $\mathbf{H}_+\times\mathbf{H}_+$. Note that

$$\forall\varphi\in\mathbf{H}_+\ \forall\alpha\in\mathbf{H}_-\quad |(\varphi,\alpha)|\le\|\varphi\|_+\|\alpha\|_-. \tag{6.3}$$

6.1. Induced equipment. A negative metric on the linear space $\mathbb{C}^{\mathbb{T}}$ appears naturally. Indeed, since $\mathbf{H}\subset\mathbf{H}_-$, $Qx\in\mathbf{H}$ for all $x\in\mathbb{H}$, and $\ker_{\mathbb{H}}Q=\{0\}$, we can introduce the following definition:

$$\forall x,y\in\mathbb{C}^{\mathbb{T}}\quad (x,y)_-:=(Qx,Qy)_-,\quad \|x\|_-:=\|Qx\|_-. \tag{6.4}$$

There automatically appears also a positive metric on $\mathbb{C}^{\mathbb{T}}$ which is dual with respect to $(\cdot,\cdot)=(\cdot,\cdot)_{\mathbb{H}}$ to the negative one:

$$\forall x\in\mathbb{C}^{\mathbb{T}}\quad \|x\|_+:=\sup_{\|y\|_-=1}|(x,y)|. \tag{6.5}$$

The linear space $\mathbb{C}^{\mathbb{T}}$ with the norm $\|\cdot\|_-$ or $\|\cdot\|_+$ is denoted respectively by $\mathbb{H}_-$ or $\mathbb{H}_+$.

6.1.1. *The following inequalities are satisfied*:

$$\forall x,y\in\mathbb{C}^{\mathbb{T}}\quad |(x,y)|\le\|x\|_+\|y\|_-,\quad \|x\|_-\le\|x\|\le\|x\|_+. \tag{6.6}$$

The embedding $Q:\mathbb{H}_-\to\mathbf{H}_-$ is isometric, and the inductor $\Pi:\ \mathbf{H}_+\to\mathbb{H}_+$ is contractive.

◁ The first inequality in (6.6) follows directly from definition (6.5). Let $x\in\mathbb{C}^{\mathbb{T}}$. Then, by (6.2), $\|x\|_-=\|Qx\|_-\le\|Qx\|=\|x\|$ because Q is an isometric mapping $\mathbb{H}\to\mathbf{H}$. Again by (6.2), we have $\|x\|_+\ge|(x,y)|/\|y\|_-\ge|(x,y)|/\|y\|$. Therefore $\|x\|_+\ge\|x\|$. That Q is an isometric mapping $\mathbb{H}_-\to\mathbf{H}_-$ is assumed in the definition (6.4). Finally, if $\varphi\in\mathbf{H}_+$, $y\in\mathbb{C}^{\mathbb{T}}$, then $|(\Pi\varphi,y)|=|(\varphi,Qy)|\le\|\varphi\|_+\|Qy\|_-=\|\varphi\|_+\|y\|_-$. Therefore,

$$\forall\varphi\in\mathbf{H}_+\quad \|\Pi\varphi\|_+\le\|\varphi\|_+. \tag{6.7}$$ ▶

Of course, the positive norm $\|\cdot\|_+$ on $\mathbb{C}^{\mathbb{T}}$ is generated by the corresponding inner product $(\cdot,\cdot)_+$. The latter may be explicitly described as follows. By the elementary theorem on representation of semilinear forms there exist operators $\mathfrak{I}\in\mathcal{B}(\mathbf{H}_-;\mathbf{H})$ and $\mathcal{I}\in\mathcal{B}(\mathbb{H}_-;\mathbb{H})$ such that

$$\forall\alpha,\beta\in\mathbf{H}_-\quad (\alpha,\beta)_-=(\mathfrak{I}\alpha,\mathfrak{I}\beta), \tag{6.8}$$

$$\forall x,y\in\mathbb{C}^{\mathbb{T}}\quad (x,y)_-=(\mathcal{I}x,\mathcal{I}y). \tag{6.9}$$

Moreover,

$$\forall\xi,\eta\in\mathbf{H}\quad (\mathfrak{I}\xi,\eta)=(\xi,\mathfrak{I}\eta), \tag{6.10}$$

$$\forall x,y\in\mathbb{C}^{\mathbb{T}}\quad (\mathcal{I}x,y)=(x,\mathcal{I}y). \tag{6.11}$$

6.1.2. *We have*

$$\mathfrak{P}\,\mathfrak{I}^2 = \mathcal{I}^2, \quad \mathfrak{Q}\,\mathcal{I}^2 = P\mathfrak{I}^2 P. \tag{6.12}$$

The operator $\mathcal{I}$ *is a bijection* $\mathbb{C}^{\mathbb{T}} \to \mathbb{C}^{\mathbb{T}}$. *The positive inner product on* $\mathbb{C}^{\mathbb{T}}$ *may be expressed in terms of the neutral one* $(\cdot,\cdot) = (\cdot,\cdot)_{\mathbb{H}}$ *as*

$$\forall x, y \in \mathbb{C}^{\mathbb{T}} \quad (x,y)_+ = (\mathcal{I}^{-1}x, \mathcal{I}^{-1}y). \tag{6.13}$$

$\lhd$ Let $x, y \in \mathbb{C}^{\mathbb{T}}$. Then $(\mathcal{I}^2 x, y) = (x,y)_- = (Qx, Qy)_- = (\mathfrak{I}^2 Qx, Qy) = (\mathfrak{P}\mathfrak{I}^2 x, y)$, i.e., $\mathcal{I}^2 = \mathfrak{P}\,\mathfrak{I}^2$. Since $\mathfrak{Q}\mathfrak{P}\mathbf{A} = P\mathbf{A}P$, $\mathbf{A} \in \mathcal{B}(\mathbf{H})$, the second equality in (6.12) is also satisfied. It follows from (6.9) that $\ker \mathcal{I} = \{0\}$. Since $\dim \mathbb{C}^{\mathbb{T}} \in \mathbb{N}$, we have $\operatorname{im} \mathcal{I} = \mathbb{C}^{\mathbb{T}}$. Suppose $y \in \mathbb{C}^{\mathbb{T}}$, and define $f_y(x) = (x,y)$ for $x \in \mathbb{C}^{\mathbb{T}}$; then $f_y(x) = (\mathcal{I}x, \mathcal{I}\mathcal{I}^{-2}y) = (x, \mathcal{I}^{-2}y)_-$. Consequently, $\|f_y\|_{\mathcal{B}(\mathbb{H}_-,\mathbb{C})} = \|\mathcal{I}^{-2}y\|_- = \|\mathcal{I}^{-1}y\|$. But this means that $\|y\|_+ = \|\mathcal{I}^{-1}y\|$. This completes the proof of (6.13). $\blacktriangleright$

6.2. $\mathbf{H}_-$-nearstandardness. In the sequel we assume that the equipment (6.1) of the Hilbert space $\mathbf{H}$ is standard.

Suppose $x \in \mathbb{C}^{\mathbb{T}}$. Since $Qx \in \mathbf{H} \subset \mathbf{H}_-$, Qx may be considered as a functional on $\mathbf{H}_+$. Therefore the condition of strong nearstandardness for Qx reads:

$$\|Qx\|_- \ll \infty \quad \text{and} \quad \forall \varphi \in {}^{\mathrm{st}}\mathbf{H}_+ \quad (\varphi, Qx - \alpha) \approx 0 \tag{6.14}$$

for some $\alpha \in {}^{\mathrm{st}}\mathbf{H}_-$.

6.2.1. Definition. A function $x \in \mathbb{C}^{\mathbb{T}}$ is called *weakly* $\mathbf{H}_-$*-nearstandard* if it satisfies conditions (6.14). The functional $\alpha \in {}^{\mathrm{st}}\mathbf{H}_-$ which satisfies these conditions is called the $\mathbf{H}_-$*-shadow* of x *on* $\mathbf{T}$ and is denoted by ${}^\bullet x$.

It is easy to see that this notation does not cause misunderstandings: the $\mathbf{H}_-$-shadow may just be called the shadow (see below). The condition for Qx, which is considered as a functional on $\mathbf{H}_+$, to be uniformly nearstandard is

$$\|Qx - \alpha\|_- \approx 0 \tag{6.15}$$

for some $\alpha \in {}^{\mathrm{st}}\mathbf{H}$.

6.2.2. Definition. A function $x \in \mathbb{C}^{\mathbb{T}}$ for which (6.15) holds is called *strongly* $\mathbf{H}_-$*-nearstandard*.

6.2.3. *A strongly* $\mathbf{H}_-$*-nearstandard function* $x \in \mathbb{C}^{\mathbb{T}}$ *is weakly* $\mathbf{H}_-$*-nearstandard, and the functional* $\alpha \in {}^{\mathrm{st}}\mathbf{H}_-$ *satisfying condition* (6.15) *coincides with the shadow* ${}^\bullet x$ *of* x. *If* $x \in \mathbb{C}^{\mathbb{T}}$ *is weakly* $\mathbf{H}$*-nearstandard* (see 3.3) (*i.e.*, $\|Qx\| \ll \infty$ *and for all* $\eta \in {}^{\mathrm{st}}\mathbf{H}$, $(x - \Pi\xi_0, \Pi\eta) \approx 0$ *for some* $\xi_0 \in {}^{\mathrm{st}}\mathbf{H}$), *then it is weakly* $\mathbf{H}_-$*-nearstandard, and its* $\mathbf{H}_-$*-shadow* ${}^\bullet x$ *coincides with its* $\mathbf{H}$*-shadow* ξ_0.

$\lhd$ From $(x - \Pi\xi_0, \Pi\eta) \approx 0$ it follows that $(Qx - P\xi_0, \eta) \approx 0$. But for $\xi_0, \eta \in {}^{\mathrm{st}}\mathbf{H}$, $P\xi_0 \approx \xi_0$ and $\|\eta\| \ll \infty$. Then $(Qx - \xi_0, \eta) \approx 0$. We see that (6.14) is satisfied for $\alpha = \xi_0$. $\blacktriangleright$

The following statement is also obvious.

6.2.4. *If* $x \in \mathbb{C}^{\mathbb{T}}$ *is strongly* $\mathbf{H}$*-nearstandard* (*i.e.*, $\|x - \Pi\xi\| \approx 0$ *for some* $\xi \in {}^{\mathrm{st}}\mathbf{H}$), *then it is strongly* $\mathbf{H}_-$*-nearstandard, and its* $\mathbf{H}$*-shadow coincides with its* $\mathbf{H}_-$*-shadow.*

6.3. Example of equipment. A finite set $\mathbb{T}$ will be realized as a "discrete interval"

$$\mathbb{T} = \overleftarrow{ab} := \{a, a+h, \ldots, b-h\}, \tag{6.16}$$

where $a, b, h \in \mathbb{R}$, $a < b$, $h > 0$, and $h \approx 0$. We distinguish the following cases:

$$\text{(i) } a,\, b \in {}^{\mathrm{st}}\mathbb{R}; \quad \text{(ii) } a = 0,\ b \approx +\infty; \quad \text{(iii) } a \approx -\infty,\ b \approx +\infty. \tag{6.17}$$

In all cases we put $\mathbf{T} := {}^{\mathrm{S}}[a,b]$, λ is the standard Lebesgue measure on $\mathbf{T}$, $\mathit{\Lambda}$ is the algebra of all λ-measurable sets $\mathcal{E} \subseteq \mathbf{T}$, and $Qt := [t, t+h[$ for $t \in \mathbb{T}$. Note that in case (i) we have $\mathbf{T} = [a,b]$, in case (ii), $\mathbf{T} = [0, \infty[$, and in case (iii), $\mathbf{T} = \mathbb{R}$. In all cases $\mathbb{H}$ is the linear space $\mathbb{C}^{\mathbb{T}}$ with the inner product $(x,y) = \sum_{t\in\mathbb{T}} x(t)\overline{y(t)}h$, and $\mathbf{H}$ is the standard Hilbert space $L_2(a,b)$ in case (i), $L_2(0,\infty)$ in case (ii), and $L_2(\mathbb{R})$ in case (iii).

6.3.1. *Choice of equipment*. Let p be any standard function on $\mathbf{T}$ which satisfies the conditions

$$\forall \tau \in \mathbf{T} \quad p(\tau) \geq 1, \quad \tau \approx \pm\infty \Rightarrow p(\tau) \approx +\infty, \quad \int_{\mathbf{T}} \frac{d\tau}{p(\tau)} < \infty. \tag{6.18}$$

For $\mathbf{H}_+$ we take the linear space that consists of $\varphi \in \mathbb{C}^{\mathbf{T}}$ which are absolutely continuous and

$$\|\varphi\|_+^2 := \int_{\mathbf{T}} [|\varphi(\tau)|^2 + |\varphi'(\tau)|^2] p(\tau)\, d\tau < \infty. \tag{6.19}$$

Note that $\mathbf{H}_+$ is a standard Hilbert space with inner product

$$(\varphi, \psi)_+ = \int_{\mathbf{T}} [\varphi(\tau)\overline{\psi(\tau)} + \varphi'(\tau)\overline{\psi'(\tau)}] p(\tau)\, d\tau. \tag{6.19$'$}$$

This space is continuously and densely embedded in $\mathbf{H}$, and $\|\varphi\| \leq \|\varphi\|_+$ for all $\varphi \in \mathbf{H}_+$.

6.3.2. *There exists a positive infinitesimal number κ such that*

$$\forall \varphi \in \mathbf{H}_+ \quad \|\varphi - P\varphi\| \leq \kappa \|\varphi\|_+. \tag{6.20}$$

$\lhd$ Let $t \in \mathbb{T}$ and let $\tau \in Qt = [t, t+h[$. Then

$$|\varphi(\tau) - P\varphi(\tau)| = \left| \frac{1}{h} \int_t^{t+h} [\varphi(\tau) - \varphi(\sigma)]\, d\sigma \right| = \left| \frac{1}{h} \int_t^{t+h} d\sigma \int_\sigma^\tau \varphi'(\varrho)\, d\varrho \right| \leq \int_t^{t+h} |\varphi'(\sigma)|\, d\sigma$$

$$\leq \left(\int_t^{t+h} \frac{d\sigma}{p(\sigma)} \right)^{1/2} \left(\int_t^{t+h} |\varphi'(\sigma)|^2 p(\sigma)\, d\sigma \right)^{1/2}.$$

Therefore in case (i),

$$\forall \varphi \in \mathbf{H}_+ \quad \|\varphi - P\varphi\| \leq \sqrt{h}\, \|\varphi\|_+. \tag{6.20$'$}$$

In cases (ii) and (iii) the right hand side is slightly larger:

$$\forall \varphi \in \mathbf{H}_+ \quad \|\varphi - P\varphi\| \leq (\sqrt{h} + \sqrt{\kappa_1}\,) \|\varphi\|_+, \tag{6.20$''$}$$

where

$$\kappa_1 := \sup_{\sigma \in \mathbf{T} \setminus Q\mathbb{T}} \frac{1}{p(\sigma)}. \tag{6.21}$$

The point is that $Q\mathbb{T} \neq \mathbf{T}$ in the last two cases. Since $P\varphi(\tau) = 0$ for all $\tau \in \mathbf{T} \setminus Q\mathbb{T}$, we have

$$\int\limits_{\mathbf{T}\setminus Q\mathbb{T}} |\varphi(\tau) - P\varphi(\tau)|^2\, d\tau = \int\limits_{\mathbf{T}\setminus Q\mathbb{T}} |\varphi(\tau)|^2\, d\tau \le \kappa_1 \|\varphi\|_+^2. \ ▶$$

6.3.3. *Nearstandardness of a discrete Dirac delta.* Under the conditions of this example denote by δ_τ the "ordinary" Dirac delta concentrated at $\tau \in \mathbb{T}$:

$$\forall \varphi \in \mathbf{H}_+ \quad (\varphi, \delta_\tau) := \varphi(\tau). \tag{6.22}$$

Note that $\delta_\tau \in \mathbf{H}_-$ and δ_τ is continuous in τ. To be precise,

$$\forall \tau_1, \tau_2 \in \mathbf{T} \quad \|\delta_{\tau_1} - \delta_{\tau_2}\|_- \le |\tau_1 - \tau_2|^{1/2}. \tag{6.23}$$

Indeed, let $\varphi \in \mathbf{H}_+$ and let $\tau_1 > \tau_2$. Then

$$|(\varphi, \delta_{\tau_1} - \delta_{\tau_2})| = |\varphi(\tau_1) - \varphi(\tau_2)| \le \int\limits_{\tau_1}^{\tau_2} |\varphi'(\tau)|\, d\tau \le \Big(\int\limits_{\tau_1}^{\tau_2} d\tau\Big)^{1/2} \cdot \Big(\int\limits_{\tau_1}^{\tau_2} |\varphi'(\tau)|^2\, d\tau\Big)^{1/2}$$
$$\le (\tau_1 - \tau_2)^{1/2} \|\varphi\|_+.$$

Let $t \in \mathbb{T}$ and $|t| \ll \infty$. Then the discrete Dirac delta δ_t (see 2.4.8) is $\mathbf{H}_-$-nearstandard, and

$$^\bullet(\delta_t) = \delta_{(^\circ t)}, \tag{6.24}$$

to be more exact,

$$\|Q\delta_t - \delta_{(^\circ t)}\|_- \le h^{1/2} + |t - {^\circ t}|^{1/2}. \tag{6.24$'$}$$

Indeed, from the proof of estimate (6.20) it is readily seen that

$$\forall \varphi \in \mathbf{H}_+ \ \forall t \in \mathbb{T} \quad |\varphi(t) - P\varphi(t)| \le h^{1/2} \|\varphi\|_+. \tag{6.25}$$

Suppose $t \in \mathbb{T}$, $|t| \ll \infty$, and $^\circ t =: \tau$. Then $(\varphi, Q\delta_t - \delta_\tau) = \mathit{\Pi}\varphi(t) - \varphi(\tau) = \mathit{\Pi}\varphi(t) - \varphi(t) + \varphi(t) - \varphi(\tau)$ for all $\varphi \in \mathbf{H}_+$. Consequently, by (6.23), we have $|(\varphi, Q\delta_t - \delta_\tau)| \le (h^{1/2} + |t - \tau|^{1/2}) \|\varphi\|_+$.

6.4. $\mathbf{H}_-$-nearstandard operators. To investigate elements of the algebra $\mathcal{B}(\mathbb{H})$ we apply equipment (6.1). Denote by $^\circ\mathcal{O}_+$ and $^-\mathcal{O}_\circ$ the embeddings $\mathbf{H}_+ \to \mathbf{H}$ and $\mathbf{H} \to \mathbf{H}_-$ respectively:

$$\forall \varphi \in \mathbf{H}_+ \quad {^\circ\mathcal{O}_+}\varphi = \varphi, \qquad \forall \xi \in \mathbf{H} \quad {^-\mathcal{O}_\circ}\xi = \xi. \tag{6.25}$$

Note that any operator $\mathbf{A} \in \mathcal{B}(\mathbf{H})$ may be considered as a mapping $\mathbf{H}_+ \to \mathbf{H}_-$ if $\mathbf{A}$ is identified with $^-\mathcal{O}_\circ \mathbf{A} {^\circ\mathcal{O}_+}$. This follows from the embeddings $\mathbf{H}_+ \subset \mathbf{H} \subset \mathbf{H}_-$. From now on we write $\mathbf{A}$ instead of $^-\mathcal{O}_\circ \mathbf{A} {^\circ\mathcal{O}_+}$ since it is always clear what is meant.

6.4.1. *The following inequality holds*:

$$\forall \mathbf{A} \in \mathcal{B}(\mathbf{H}) \quad \|\mathbf{A}\|_{\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)} \le \|\mathbf{A}\|_{\mathcal{B}(\mathbf{H})}. \tag{6.26}$$

◁ If $\varphi \in \mathbf{H}_+$, then

$$\|\mathbf{A}\varphi\|_- = \sup\{|(\mathbf{A}\varphi, \psi)| : \|\psi\|_+ = 1\} \le \sup\{\|\mathbf{A}\|_{\mathcal{B}(\mathbf{H})} \|\varphi\| \cdot \|\psi\| : \|\psi\|_+ = 1\}$$
$$\le \|\mathbf{A}\|_{\mathcal{B}(\mathbf{H})} \|\varphi\|_+. \ ▶$$

Now consider any operator $\mathbb{A} \in \mathcal{B}(\mathbb{H})$. Set $\mathfrak{Q}\mathbb{A} := Q\mathbb{A}\Pi$. The latter may be considered an element of $\mathcal{B}(\mathbf{H})$ as well as of $\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)$.

6.4.2. *The following estimate holds*:

$$\forall \mathbb{A} \in \mathcal{B}(\mathbb{H}) \quad \|\mathfrak{Q}\mathbb{A}\|_{\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)} \le \|\mathbb{A}\|_{\mathcal{B}(\mathbb{H})}. \tag{6.27}$$

$\lhd$ Since $\|Q\| = \|\Pi\| = 1$, we have, by (6.26), $\|\mathfrak{Q}\mathbb{A}\|_{\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)} \le \|Q\mathbb{A}\Pi\|_{\mathcal{B}(\mathbf{H})} \le \|\mathbb{A}\|_{\mathcal{B}(\mathbb{H})}$. ►

6.4.3. Let $\mathbb{A} \in \mathcal{B}(\mathbb{H})$. We say that $\mathbb{A}$ is *weakly*, *strongly*, or *uniformly* $\mathbf{H}_-$*-nearstandard* if respectively

$$\forall \varphi, \psi \in {}^{\mathrm{st}}\mathbf{H}_+ \quad ((\mathfrak{Q}\mathbb{A} - \mathbf{A})\varphi, \psi) \approx 0, \tag{6.28}$$

$$\forall \varphi \in {}^{\mathrm{st}}\mathbf{H}_+ \quad \|(\mathfrak{Q}\mathbb{A} - \mathbf{A})\varphi\|_- \approx 0, \tag{6.29}$$

$$\|\mathfrak{Q}\mathbb{A} - \mathbf{A}\|_{\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)} \approx 0, \tag{6.30}$$

for some operator $\mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)$. In the first and second cases the requirement

$$\|\mathfrak{Q}\mathbb{A}\|_{\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)} \ll \infty \tag{6.31}$$

is included in the definition. In the third case it is automatically satisfied because from (6.30) it follows that $\|\mathfrak{Q}\mathbb{A}\|_{\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)} \le \|\mathbf{A}\|_{\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)} + 1$.

6.4.4. It is clear that uniform $\mathbf{H}_-$-nearstandardness implies strong nearstandardness, and strong $\mathbf{H}_-$-nearstandardness implies weak nearstandardness. Moreover, the operator $\mathbf{A} \in {}^{\mathrm{st}}\mathcal{B}(\mathbf{H}_+;\mathbf{H}_-)$ satisfying (6.30) satisfies (6.29), and the one satisfying (6.29) also satisfies (6.28). This operator is called the *shadow* of the operator $\mathbb{A}$ on $\mathbf{T}$ and is denoted by ${}^\bullet\mathbb{A} = \mathbf{A}$. The *shadow* of $\mathbb{A}$ on $\mathbb{T}$ is the operator ${}^\circ\mathbb{A} = \mathfrak{P}({}^\bullet\mathbb{A}) = \mathfrak{P}\mathbf{A}$.

6.4.5. It is easy to see that our definitions of $\mathbf{H}_-$-nearstandardness are consistent with the definitions of nearstandardness in Section 4.3. Namely, weak, strong, or uniform $\mathbf{H}$-nearstandardness implies the corresponding $\mathbf{H}_-$-nearstandardness, and the shadows coincide.

6.4.6. Example. *Let*

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad \mathbb{A}x := \sum_{i=1}^{n} (x, f_i) e_i, \tag{6.32}$$

where $n \in {}^{\mathrm{st}}\mathbb{N}$, $e_1, \dots, e_n, f_1, \dots, f_n \in \mathbb{C}^{\mathbb{T}}$, *and let all the vectors* e_i, f_i *be weakly* $\mathbf{H}_-$*-nearstandard*; *then the operator* $\mathbb{A}$ *is weakly* $\mathbf{H}_-$*-nearstandard*, *and*

$$\forall \varphi \in \mathbf{H}_+ \quad ({}^\bullet\mathbb{A})\varphi = \sum_{i=1}^{n} (\varphi, {}^\bullet f_i) {}^\bullet e_i. \tag{6.33}$$

Furthermore, *if all the* e_i *are strongly* $\mathbf{H}_-$*-nearstandard*, *then* $\mathbf{A}$ *is strongly* $\mathbf{H}_-$*-nearstandard. And if all the* e_i, f_i *are strongly nearstandard*, *then* $\mathbf{A}$ *is uniformly* $\mathbf{H}_-$*-nearstandard.*

$\lhd$ Since $(\mathfrak{Q}\mathbb{A}\xi, \eta) = (\mathbb{A}\Pi\xi, \Pi\eta)$, we have

$$\forall \xi, \eta \in \mathbf{H} \quad (\mathfrak{Q}\mathbb{A}\xi, \eta) = \sum_{i=1}^{n} (\xi, Qf_i) \cdot (Qe_i, \eta). \tag{6.34}$$

Note that the weak $\mathbf{H}_-$-nearstandardness of a vector f is defined by the condition $(\varphi, Qf - {}^\bullet f) \approx 0$ for all $\varphi \in {}^{\mathrm{st}}\mathbf{H}_+$, and the strong $\mathbf{H}_-$-nearstandardness of f is defined by $\|Qf - {}^\bullet f\| \approx 0$. Therefore our statement follows immediately from the continuity of the form $(\cdot,\cdot)$ and of the arithmetical operations (in the corresponding topologies). ▶

6.5. $\mathbf{H}_-$-nearstandardness of discrete differentiation. The finite set $\mathbb{T}$ will be a discrete interval (see (5.1)), and its standard filling and Hilbert spaces $\mathbb{H}$ and $\mathbf{H}$ will be defined as in Section 5 (see (5.2), (5.3)). We equip the space $\mathbf{H}$ taking for $\mathbf{H}_+$ the Sobolev space $H^2(\mathbf{T})$ consisting of $\varphi \in \mathbf{H}$ such that $\varphi', \varphi'' \in \mathbf{H}$ (differentiation in the sense of distribution theory). This space is supplied with the inner product

$$\forall \varphi, \psi \in \mathbf{H}_+ \quad (\varphi, \psi)_+ = \sum_{j=0}^{2} (\varphi^{(j)}, \psi^{(j)}). \tag{6.35}$$

We need the following remark.

6.5.1. *The "usual" Dirac delta δ_τ and its derivative δ'_τ defined by $(\varphi, \delta_\tau) = \varphi(\tau)$, $(\varphi, \delta'_\tau) = -\varphi'(\tau)$ for $\tau \in \mathbb{T}$ belong to $\mathbf{H}_-$. Moreover, if $\tau \in [b, c] \subseteq \mathbf{T}$, then*

$$\|\delta_\tau\|_-, \|\delta'_\tau\|_- \le (c-b)^{-1/2} + (c-b)^{1/2}; \tag{6.36}$$

in particular, if $c = b + 1$, then

$$\|\delta_\tau\|_-, \|\delta'_\tau\|_- \le 2. \tag{6.36$'$}$$

◁ Let $\sigma \in [b, c]$ be a point such that $\int_b^c \varphi(\tau)\, d\tau = (c - b)\varphi(\sigma)$. Then

$$\varphi(\tau) = \int_\sigma^\tau \varphi'(\tau')\, d\tau' + \frac{1}{c-b} \int_b^c \varphi(\tau)\, d\tau,$$

hence

$$|\varphi(\tau)| \le |\tau - \sigma|^{1/2} \|\varphi'\| + (c-b)^{-1/2} \|\varphi\|. \tag{6.37}$$

Since $\|\varphi\|, \|\varphi'\|, \|\varphi''\| \le \|\varphi\|_+$, (6.36) follows from (6.37). ▶

Consider the case $a \in {}^{\mathrm{st}}\mathbb{R}_+$. Let $\theta \in \mathbb{C}$ and $0 \ll |\theta| \ll \infty$. Denote by $D = D_\theta$ the operator of discrete differentiation defined in Section 5, i.e.,

$$\forall x \in \mathbb{C}^{\mathbb{T}} \quad D_\theta x(t) = \begin{cases} \dfrac{1}{ih}[x(t+h) - x(t)] & \text{if } t < a - h, \\ \dfrac{1}{ih}[\theta x(-a) - x(a-h)] & \text{if } t = a - h. \end{cases} \tag{6.38}$$

Moreover, denote by $\mathbf{D} = \mathbf{D}_\theta$ the operator of "ordinary" differentiation given by

$$\begin{gathered} \operatorname{dom} \mathbf{D} := \{\xi \in \mathbf{H} : \xi' \in \mathbf{H} \wedge \xi(a) = {}^\circ\theta \xi(-a)\}, \\ \forall \xi \in \operatorname{dom} \mathbf{D} \quad \mathbf{D}\xi = \frac{1}{i} \frac{d\xi}{d\tau}. \end{gathered} \tag{6.39}$$

Readjust the equipment of $\mathbf{H}$ by taking for the positive space the space

$$\mathbf{H}_+^\theta := \{\varphi \in \mathbf{H}_+ : \varphi(a) = {}^\circ\theta \varphi(-a)\}, \tag{6.40}$$

with the previous inner product (6.35).

6.5.2. *If*

$$|\theta - {}^\circ\theta|/h \ll \infty, \tag{6.41}$$

then the operator $D = D_\theta$ *is uniformly* $\mathbf{H}^\theta_-$*-nearstandard, and* ${}^\bullet D = \mathbf{D} = \mathbf{D}_\theta$. *To be more exact, there exists a number* C, $0 \ll C \ll \infty$, *such that*

$$\|\mathfrak{Q}D - \mathbf{D}\|_{\mathcal{B}(\mathbf{H}^\theta_+;\mathbf{H}^\theta_-)} \le Ch^{1/2}. \tag{6.42}$$

$\lhd$ Let $\varphi \in \mathbf{H}^\theta_+$, and let us estimate $\gamma := \|(\mathfrak{Q}D - \mathbf{D})\varphi\|_-$. Define $x := \Pi\varphi$ and note that for all $t < a - h$ and $\tau \in [t, t+h[$,

$$\mathfrak{Q}D\varphi(\tau) = QDx(\tau) = \frac{1}{ih}[x(t+h) - x(t)] = \frac{1}{ih^2}\int_t^{t+h}[\varphi(\tau+h) - \varphi(\tau)]\,d\tau$$
$$= \frac{1}{ih^2}\int_t^{t+h} d\tau \int_\tau^{\tau+h}\varphi'(\sigma)\,d\sigma.$$

Let

$$\mathfrak{I} := \Big(\int_{-a}^{a-h}|\mathfrak{Q}D\varphi(\tau) - \mathbf{D}\varphi(\tau)|^2\,d\tau\Big)^{1/2}.$$

Then

$$\mathfrak{I}^2 = \sum_{t<a-h}\int_t^{t+h} d\tau\left|\frac{1}{ih^2}\int_t^{t+h} d\alpha\int_\alpha^{\alpha+h}[\varphi'(\sigma) - \varphi'(\tau)]\,d\sigma\right|^2$$
$$= \sum_{t<a-h}\int_t^{t+h} d\tau\left|\frac{1}{ih^2}\int_t^{t+h} d\alpha\int_\alpha^{\alpha+h} d\sigma\int_\tau^\sigma\varphi''(\beta)\,d\beta\right|^2$$
$$\le \sum_{t<a-h}\int_t^{t+h} d\tau\Big(\int_t^{t+2h}|\varphi''(\beta)|\,d\beta\Big)^2 \le 2h^2\|\varphi\|_+^2.$$

Therefore $\mathfrak{I} \le \sqrt{2}\,h\|\varphi\|_+$. Set

$$\mathfrak{J} := \Big(\int_{a-h}^{a}\Big|\mathfrak{Q}D\varphi(\tau) - \mathbf{D}\varphi(\tau)\Big|^2\,d\tau\Big)^{1/2}.$$

Since $\gamma \le \mathfrak{I} + \mathfrak{J}$, it remains to estimate $\mathfrak{J}$. If $\tau \in [a-h, a]$, then

$$\mathfrak{Q}G\varphi(\tau) = QDx(a-h) = \frac{1}{ih}[\theta x(-a) - x(a-h)] = \frac{1}{ih^2}\Big(\int_{-a}^{-a+h}\theta\varphi(\tau)\,d\tau - \int_{a-h}^{a}\varphi(\tau)\,d\tau\Big).$$

It follows from $\varphi(a) = {}^\circ\theta\varphi(-a)$ that

$$\mathfrak{Q}D\varphi(\tau) = \frac{1}{ih^2}\Big(\int_{-a}^{-a+h}\theta[\varphi(\tau) - \varphi(-a)]\,d\tau + (\theta - {}^\circ\theta)\int_{-a}^{-a+h}\varphi(-a)\,d\tau$$
$$- \int_{a-h}^{a}[\varphi(\tau) - \varphi(a)]\,d\tau\Big)$$
$$= \frac{1}{ih^2}\Big(\theta\int_{-a}^{-a+h} d\tau\int_{-a}^{\tau}\varphi'(\sigma)\,d\sigma + (\theta - {}^\circ\theta)h\varphi(-a) - \int_{a-h}^{a} d\tau\int_a^\tau\varphi'(\sigma)\,d\sigma\Big).$$

Using (6.36′), we get $|\mathfrak{Q}D\varphi(\tau)| \le C_1\|\varphi\|_+$, where $C_1 = 2(|\theta| + 1 + |\theta - {}^\circ\theta|h^{-1}) \ll \infty$. Therefore

$$\mathfrak{J} \le \Big(\int\limits_{a-h}^{a} |\mathfrak{Q}D\varphi(\tau)|^2\,d\tau\Big)^{1/2} + \Big(\int\limits_{a-h}^{a} |\varphi'(\sigma)|^2\,d\sigma\Big)^{1/2} \le C_1h^{1/2}\|\varphi\|_+ + 2h^{1/2}\|\varphi\|_+.$$

Thus $\gamma \le \mathfrak{I} + \mathfrak{J} \le (\sqrt{2}\,h + (C_1+2)h^{1/2})\|\varphi\|_+$. ∎

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