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WROŃSKI'S FACTORIZATION OF POLYNOMIALS

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In the middle of Wroński's papers [8], [9], [10], [11] one can come across solutions of important problems such as finding, for any polynomial with complex coefficients, the factor which corresponds to roots of modulus less than 1.

1. Background

Józef Maria Hoene-Wroński (1778–1853) was a universal philosopher and scientist. He also knew all languages of culture, Polish, French, Latin, Greek, Hebraic, Arabic, Aramaic, though not English.

His aim was a complete "Réforme du savoir Humain" including both the theory of spontaneous locomotion and the art of governing.

However, his industrial speculations were not bought by the government, nor was his mathematical work accepted by the Academy.

He was therefore compelled to extract (painfully, having even to go to court) money from a banker to publish his philosophical theories. Unfortunately, the finiteness of the banker's fortune and the malevolence of the banker's wife led to a delay of more than 30 years in the publication of his work, apart from a small "Canon des Logarithmes".

Wroński summarizes his object in his "Prolégomènes du Messianisme":

> *"L'objet de cet ouvrage est de fonder péremptoirement la vérité sur la terre, de réaliser ainsi la philosophie absolue, d'accomplir la religion, de réformer les sciences, d'expliquer l'histoire, de découvrir le but suprême des Etats, de fixer les fins absolues de l'homme et de dévoiler les destinées des nations"* [10, p. 10].

In fact, according to his own terms,

> *It was with much grief that Hoene-Wroński was forced to leave his grave philosophical tasks to indulge in the Réforme des Mathématiques ... Math-*

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ematical questions, however difficult, are only a secondary object, a sort of hobby in the middle of his high philosophical thoughts. [10, p. 25].

He was forced to offer his mathematical laws as a proof of the absolute truth of his philosophical and religious Messianic doctrine (that he borrowed from Towiański), the main weakness of philosophy being that Popper's criterion of *falsification* cannot be applied to its statements, leaving place for much *illusions* which Wroński proposes to dispel.

Though restricting his field of research, he was nevertheless able to attain the *Supreme Law of mathematics, which contains all known mathematics as a very special case and extends indefinitely beyond what is known.* Moreover, his *law would also comprise all formulas and all methods which can be obtained in the future of Science* [10, p. 32].

Wroński almost instantly met with the outright hostility of the "*savants sur brevêt*" belonging to this "*born enemy of truth*", that is to say, to the Académie des Sciences de Paris. No doubt Wroński's clear-cut opinion would nowadays be totally reversed, now that algorithmics and combinatorics are so well received in this noble assembly [5]; he would no more write that *the only aim of this corporation is exploitation of Man, consequently exploitation of Heads of State, using the imposing authority of Science* [10, p. 4].

Thus, instead of devoting his full attention to solving the following *rigorous system of equations* [11, p. 6]:

"*Let α be the anarchy degree, δ the degree of despotism. Then one has the following precise relations*:

$$\alpha = \left\{\frac{m+n}{m}\cdot\frac{m+n}{n}\right\}^{p-r} \times \left(\frac{m}{n}\right)^{p+r}, \tag{1.1}$$

$$\delta = \left\{\frac{m+n}{m}\cdot\frac{m+n}{n}\right\}^{r-p} \times \left(\frac{n}{m}\right)^{p+r}, \tag{1.2}$$

where m represents the numerical influence of the national party, p the standard deviation of the philosophy of this party from true religion, n the influence of the moral party and r the deviation of religion from true philosophy".

Wroński had to write such trivial things as the *Résolution Générale des Equations* (*de tout degré*) which we shall examine in detail in Section 2. For a survey of his mathematical work, we refer to [1].

"*We do not need to emphasize how painful such a pedestrian task must be to a man who, in the innermost recesses of his retreat, has spent his life scrutinizing and discovering creation laws, as well as the final destiny of rational beings*" [10, p. 14].

2. Universal resolution of equations

From 1810 on, Wroński made great use of determinants, this at a time where the theory of these objects was not much developed. Moreover, in the work of classics such as Cramer, Bézout, Vandermonde, ..., theory of determinants stems from elimination of variables in systems of linear equations, and involves only determinants of "quantities", i.e. (real) numbers (see the first volume of [7]).

Weighing the respective merits of these pioneers, the scale is turning towards Wroński who used "sommes combinatoires" which he denoted by the Hebrew letter "sin" or "shin", more general even than those determinants which have been called "Wrońskians". Indeed, Wroński considers determinants of linear functionals, not restricting himself and the reader to the case of consecutive derivatives. Moreover, he obtains relations between "compound" determinants which were later and independently developed by such people as Sylvester, Bazin, ...

As for symmetric functions, he repeatedly claimed that the complete symmetric functions (which he denotes by "aleph": ℵ) are more fundamental than the elementary ones. To get the "universal factorization of polynomials", he furthermore generalized the aleph functions into a family which includes the Schur functions indexed by partitions of the type $1 \ldots 1q \ldots q$. He defined his new functions by simple recursions, here placing more emphasis on algorithms than on determinants, having in mind to provide anybody *non prévenu ni aveuglé* with an efficient tool to attack all problems of mathematics (and physics). This is perhaps why he missed general Schur functions, which had to wait thirty more years to come into existence at the hands of Cauchy and Jacobi, and many more years to be christened "Schur functions" (by Littlewood and Richardson, in 1934).

From different places in the voluminous work [10], suplemented by [8] and [9], one can extract the mathematical properties that we are going to rewrite in the following.

Given a finite set $\mathbf{A}$ of indeterminates, the associated aleph functions (which we shall, departing from Wroński's philosophy, denote by $S_j(\mathbf{A})$) are defined through the generating function (using an extra formal variable z):

$$1/\prod_{a\in\mathbf{A}}(1-za)=\sum_{0}^{\infty}z^j S_j(\mathbf{A}). \tag{2.1}$$

One can asociate with such a formal series the infinite matrix $\mathbf{S}=[S_{k-h}(\mathbf{A})]_{h,k\geqslant 1}$. Products and inversions of series correspond to products and inversions of matrices. The advantage of matrices, apart from figuring in Bourbaki's treatise on linear algebra, contrary to the scanty appearance of symmetric functions and the almost total absence of aleph functions, is that anybody can now think of considering minors of these matrices — that is to say, precisely Schur functions.

More precisely, given any J in $\mathbf{N}^p$, one defines the *Schur Function of index* J to be

(2.2) $$S_J(\mathbf{A}) = |S_{j_k+k-h}(\mathbf{A})|_{1 \leqslant h,k \leqslant p}.$$

Elimination theory, as well as rational (Padé) approximation, make great use of the Schur Functions associated with a rational function with poles and zeroes:

(2.3) $$\prod_{x \in \mathbf{X}} (1-zx)/\prod_{a \in \mathbf{A}} (1-za) = \sum_0^\infty z^j S_j(\mathbf{A}-\mathbf{X})$$

instead of a function having only poles: $1/\prod_{a \in \mathbf{A}}(1-za)$. They are defined as in 2.2, using the aleph functions $S_j(\mathbf{A}-\mathbf{X})$ instead of $S_j(\mathbf{A})$, and denoted by $S_j(\mathbf{A}-\mathbf{X})$. The fundamental property, in that case, is that for J "large" enough, $S_J(\mathbf{A}-\mathbf{X})$ factorizes into a product $S_{J'}(\mathbf{A}) \cdot S_{J''}(\mathbf{X}) \cdot \prod_{a \in \mathbf{A}, x \in \mathbf{X}}(a-x)$; thus one can express, up to the factor $\prod_{a \in \mathbf{A}, x \in \mathbf{X}}(a-x)$ which is called the "Résultante" of $\mathbf{A}$ and $\mathbf{X}$, any Schur function of $\mathbf{A}$ or $\mathbf{B}$ as a determinant in the coefficients of the series $\prod_{x \in \mathbf{X}}(1-zx)/\prod_{a \in \mathbf{A}}(1-za)$ (see [3]).

In short, though slightly incorrectly, one can say that from the knowledge of $\mathbf{A}-\mathbf{X}$ one can recover separately both $\mathbf{A}$ and $\mathbf{X}$.

The problem considered by Wroński was exactly of the same type. Given the union of two finite sets (with multiplicity) of complex numbers $\mathbf{A}$, $\mathbf{B}$, such that $|a| > 1$ for every $a \in \mathbf{A}$ and $|b| < 1$ for every $b \in \mathbf{B}$, Wroński wanted to recover $\mathbf{A}$ and $\mathbf{B}$ separately. More precisely, given the polynomial $\prod_{a \in \mathbf{A}}(x-a) \cdot \prod_{b \in \mathbf{B}}(x-b)$, Wroński claimed to be able to get its "universal" factorization, i.e. to produce separately both polynomials $\prod_{a \in \mathbf{A}}(x-a)$ and $\prod_{b \in \mathbf{B}}(x-b)$. He took care to emphasize that his solution was transcendental and not algebraic, in other words that he had to go to infinity to attain an *exact* factorization, but could nevertheless provide an *approximate* factorization before proceeding to the limit.

In fact his methods extended that of Bernoulli, which dealt only with the case where the cardinal of $\mathbf{A}$ was 1. Any symmetric function having a "leading term" furnished in that case a solution: take for example the power sum $\psi_k = a^k + \sum b^k$ which, of course, can be expressed in terms of the coefficients of the polynomial $(x-a) \cdot \prod_{b \in \mathbf{B}}(x-b)$. Even the most severe *rapporteur* will admit that $x-a$ is the limit, as $k \to \infty$, of $(x\psi_k - \psi_{k+1})/\psi_k$. The same benevolence should apply to Wroński's work, at least posthumously.

We write $\mathbf{A}+\mathbf{B}$ for the (disjoint) union of $\mathbf{A}$ and $\mathbf{B}$, and define accordingly the associated aleph and Schur functions through the formal series $1/\prod_{a \in \mathbf{A}}(1-za) \cdot \prod_{b \in \mathbf{B}}(1-zb) = \sum z^j S_j(\mathbf{A}+\mathbf{B})$. The following proposition (with some restrictions on I and J) is due to Wroński.

PROPOSITION 2.4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be (multi) sets of complex numbers*: $|a| > 1$, $|b| < 1$, $\operatorname{card}(\mathbf{A}) = m$, $\operatorname{card}(\mathbf{B}) = n$. *Let* $I \in \mathbf{N}^n$, $J \in \mathbf{N}^m$, k *an integer; denote*

$(J_1+k, \ldots, J_m+k)$ by $J(k)$, $(I_1, \ldots, I_n, J_1+k, \ldots, J_m+k)$ by $IJ(k)$. Then the quotient of Schur functions $S_{IJ(k)}(\mathbf{A}+\mathbf{B})/S_{J(k)}(\mathbf{A}+\mathbf{B})$ tends to $S_I(\mathbf{B})$ as k tends to infinity.

Proof. We have here to use the fact that Schur functions can also be expressed as the quotient of a determinant of powers of the variables (when these variables are all different; it is not difficult to pass from this generic case to any special one) by the Vandermonde determinant (*Jacobi–Trudi relation*, see [3, ch. I, 3]), i.e. that

$$S_{IJ(k)}(\mathbf{A}+\mathbf{B})\cdot\Delta(\mathbf{A}+\mathbf{B}) = \begin{vmatrix} a^{I_1} & \ldots & a^{I_n+n-1} & a^{J_1+n+k} & \ldots & a^{J_m+n+k+m-1} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ b^{I_1} & \ldots & b^{I_n+n-1} & b^{J_1+n+k} & \ldots & b^{J_m+n+k+m-1} \end{vmatrix} \begin{matrix} \}\, a\in\mathbf{A} \\ \\ \}\, b\in\mathbf{B} \end{matrix}$$

the special case of this determinant for $I=(0, \ldots, 0)$ being equal to $S_{J(k)}(\mathbf{A}+\mathbf{B})\cdot\Delta(\mathbf{A}+\mathbf{B})$, the even more special case $I=(0, \ldots, 0)$, $J=(0, \ldots, 0)$, $k=0$, being the *Vandermonde* determinant of the set $\mathbf{A}+\mathbf{B}$ (i.e. the product of differences, two by two). When k is large enough, this determinant is not very far from

$$\begin{vmatrix} a^{I_1} & \ldots & a^{I_n+n-1} & a^{J_1+n+k} & \ldots & a^{J_m+n+k+m-1} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ b^{I_1} & \ldots & b^{I_n+n-1} & 0 & \ldots & 0 \end{vmatrix} \begin{matrix} \}\, a\in\mathbf{A} \\ \\ \}\, b\in\mathbf{B} \end{matrix}$$

which factorizes, and thus $S_{IJ(k)}(\mathbf{A}+\mathbf{B})/S_{J(k)}(\mathbf{B})$ is not very far from $|b^{I_1}\ldots b^{I_n+n-1}|/|b^0\ldots b^{n-1}|$, which in turn is equal to $S_I(\mathbf{B})$ as wanted. ■

Proposition 2.4 allows us to exhibit any Schur function of $\mathbf{B}$ as a limit of the quotient of two Schur functions of $\mathbf{A}+\mathbf{B}$. Recall now that the coefficients of the polynomial $\prod_{b\in\mathbf{B}}(x-b)$ are special Schur functions (case $I=0\ldots0\,1\ldots1$, denoted by $0^{n-p}1^p$, $0\leqslant p\leqslant n$).

COROLLARY 2.5. *Under the hypothesis of* 2.4, *the polynomial $\prod_{b\in\mathbf{B}}(x-b)$ is the limit of*

$$\Big(\sum_{0\leqslant p\leqslant n}(-1)^p x^{n-p} S_{1^pJ(k)}(\mathbf{A}+\mathbf{B})\Big)/S_{J(k)}(\mathbf{A}+\mathbf{B}).$$

One can give a little more compact expression by making use of 2.3, in the case that the set $\mathbf{X}=\{x\}$ has only one element, $\mathbf{A}+\mathbf{B}$ replacing $\mathbf{A}$. Let us denote by $S_{1^nJ}(\mathbf{A}+\mathbf{B}-x,\ \mathbf{A}+\mathbf{B})$ the determinant obtained from $S_{1^nJ}(\mathbf{A}+\mathbf{B})$ by replacing each aleph function $S_k(\mathbf{A}+\mathbf{B})$ by $S_k(\mathbf{A}+\mathbf{B}-x)$ in the first n columns. Using the fact that $S_k(\mathbf{A}+\mathbf{B}-x)=S_k(\mathbf{A}+\mathbf{B})-xS_{k-1}(\mathbf{A}+\mathbf{B})$, one thus obtains the equivalent form of 2.5:

COROLLARY 2.5′. *Under the hypothesis of* 2.4, *the polynomial $\prod_{b\in\mathbf{B}}(x-b)$ is the limit of*

$$(-1)^n S_{1^nJ(k)}(\mathbf{A}+\mathbf{B}-x,\ \mathbf{A}+\mathbf{B})/S_{J(k)}(\mathbf{A}+\mathbf{B}).$$

Jacobi's approach [2] to the same problem is rather different. As above, denote by ψ_k and φ_k, for any integer k, the kth power sums: $\psi_k = \sum_{a\in \mathbf{A}} a^k + \sum_{b\in \mathbf{B}} b^k$, and $\varphi_k = \sum_{a\in \mathbf{A}} a^k$. Writing $\prod (x-a) = x^m - x^{m-1} S_1(\mathbf{A}) + x^{m-2} S_{11}(\mathbf{A}) - \ldots$, we have for any k the system of equations

$$(2.6)\qquad \begin{cases} x^m & -x^{m-1} S_1(\mathbf{A}) & +x^{m-2} S_{11}(\mathbf{A}) & - \ldots = 0 \\ \varphi_k & -\varphi_{k-1} S_1(\mathbf{A}) & +\varphi_{k-2} S_{11}(\mathbf{A}) & - \ldots = 0 \\ \cdot & \cdot & \cdot & \ldots \\ \varphi_{k+m-1} & -\varphi_{k+m-2} S_1(\mathbf{A}) & +\varphi_{k+m-3} S_{11}(\mathbf{A}) & - \ldots = 0. \end{cases}$$

We deduce from this system, by eliminating the "unknowns" $S_1(\mathbf{A})$, $S_{11}(\mathbf{A}), \ldots$, a determinant which is a polynomial of degree m in x vanishing for $x =$ any a in $\mathbf{A}$. Thus, this polynomial, whose coefficients are determinants in the φ_p, is, up to a factor, the polynomial $\prod (x-a)$.

Now, if k is large enough, ψ_p provides a good approximation to φ_p and thus, $\prod (x-a)$ can be approximated by a determinant whose coefficients are determinants in the power sums ψ_p. In fact, up to a factor, these determinants are equal to the same Schur functions as in 2.5′ (for $J = 0, \ldots, 0$) and the method of Jacobi is not very different from the first step in the method of Wroński.

We have defined the Schur functions $S_H(\mathbf{A}+\mathbf{B})$ as determinants in the aleph functions; they can also be expressed as determinants in the elementary functions, i.e. the coefficients of the polynomial $\prod_{a\in\mathbf{A}}(x-a)\cdot\prod_{b\in\mathbf{B}}(x-b)$ [6, ch. I, 3]. As was already noticed by Wroński, this is not at all suited to the use of computers; Wroński rightly did not stop at this point, but produced an *algorithm*, which we shall examine in the next section, to generate the required functions through elementary steps.

3. Wroński's algorithm

Recall that the *normalized difference* of two polynomials $P = p_0 x^n + p_1 x^{n-1} + \ldots$ and $Q = q_0 x^n + q_1 x^{n-1} + \ldots$ is the difference $q_0 P - p_0 Q$ and is denoted by $^*P - {}^*Q$ in accordance with the requirements of modern computer science.

Given a polynomial P of degree N, Wroński first takes the successive remainder $\mathcal{R}_{k,1}$ of the powers of x:

$$\mathcal{R}_{k,1} \equiv x^{N+k} \bmod P.$$

His next steps are now only normalized differences:

$$(3.1)\qquad \begin{aligned} \mathcal{R}_{k,2} &= {}^*\mathcal{R}_{k,1} - {}^*\mathcal{R}_{k+1,1}, \\ &\cdots\cdots\cdots\cdots \\ \mathcal{R}_{k,j+1} &= {}^*\mathcal{R}_{k,j} - {}^*\mathcal{R}_{k+1,j}, \qquad k \geqslant 0, j \geqslant 1. \end{aligned}$$

We assume here that no irregular drop in degrees happens (otherwise, one has to take more care of the definition of normalized differences).

PROPOSITION 3.2 (Wroński). *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be (multi) sets of complex numbers:* $|a| > 1$, $|b| < 1$, $\operatorname{card}(\mathbf{A}) = m$, $\operatorname{card}(\mathbf{B}) = n$, *and let* P *be the polynomial* $\prod(x-a)\cdot\prod(x-b)$. *Then, denoting by* $\mathscr{C}$ *the coefficient of the leading term, the normalized polynomial* $\mathscr{R}_{k,m}/\mathscr{C}$ *tends to* $\prod(x-b)$ *for* k *tending to infinity.*

Proof. We are saved if we can recognize in $\mathscr{R}_{k,m}$ one of the Schur functions considered in 2.5′. In fact, the abbé Fontaine des Bertins (see [7, t. I, p. 11]) comes to the rescue with what has been named much later *Plücker's Relations*: given any matrix, denote by $[ij\ldots k]$ the minor formed by its first rows and columns $i, j, \ldots, k$; then one has the following three-term relations, for each choice of $1, \ldots, p$; $\underline{i}, \underline{j}, \underline{h}, \underline{k}$:

$$[1\ldots p\underline{i}\underline{j}]\cdot[1\ldots p\underline{h}\underline{k}]+[1\ldots p\underline{i}\underline{h}]\cdot[1\ldots p\underline{k}\underline{j}]+[1\ldots p\underline{i}\underline{k}]\cdot[1\ldots p\underline{j}\underline{h}] = 0, \tag{3.3}$$

i.e. we have to take a fixed set $\{1, \ldots, p\}$ of columns and another set of four columns $\{\underline{i}, \underline{j}, \underline{h}, \underline{k}\}$ that we decompose in all possible manners into two pairs to get the three terms of the relation.

Let us now take the following matrix of aleph functions of $\mathbf{A}+\mathbf{B}-x$ for the first columns, and aleph functions of $\mathbf{A}+\mathbf{B}$ for the last ones (for any positive integers k, p, q):

$$\begin{vmatrix} S_0(\mathbf{A}+\mathbf{B}-x) & S_1(\mathbf{A}+\mathbf{B}-x) & \ldots & S_p(\mathbf{A}+\mathbf{B}-x) & S_{k+p}(\mathbf{A}+\mathbf{B}) & \ldots & S_{k+p+q}(\mathbf{A}+\mathbf{B}) \\ 0 & S_0(\mathbf{A}+\mathbf{B}-x) & \ldots & S_{p-1}(\mathbf{A}+\mathbf{B}-x) & S_{k+p-1}(\mathbf{A}+\mathbf{B}) & \ldots & S_{k+p+q-1}(\mathbf{A}+\mathbf{B}) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \end{vmatrix}$$

Then the relation

$$\begin{aligned} 0 = {} & [1\ldots \underline{p}\ k+1+p\ldots \underline{k+p+q}]\cdot[\underline{0}\ldots p-1\ k+1+p\ldots \underline{k+p+q}] \\ & -[1\ldots \underline{p}\,\underline{k+p}\ldots k+p+q-1]\cdot[\underline{0}\ldots p-1\ k+1+p\ldots \underline{k+p+q}] \\ & +[1\ldots p-1\ \underline{k+p}\ldots \underline{k+p+q}]\cdot[\underline{0}\ldots \underline{p}\ k+1+p\ldots k+p+q-1] \end{aligned}$$

produced by the choice of the set of fixed columns: $\{1, \ldots, p-1, k+1+p, \ldots, k+p+q-1\}$ and of the set that we cut in two: $\{\underline{0}, \underline{p}, \underline{k+p}, \underline{k+p+q}\}$ is nothing else but

$$\begin{aligned} & S_{1^p(k+1)^q}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B})\cdot S_{0^pk^q}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B}) \\ & \quad -S_{1^pk^q}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B})\cdot S_{0^p(k+1)^q}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B}) \\ & \quad +S_{1^{p-1}(k+1)^{q+1}}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B})\cdot S_{0^{p+1}k^{q-1}}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B}) = 0. \end{aligned}$$

Notice that $S_{0^pk^q}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B}) = S_{k^q}(\mathbf{A}+\mathbf{B})$ is a scalar independent of x. Thus, the Schur functions $S_{1^pk}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B})$ satisfy Wroński's relations; checking the starting point, i.e. that the remainder of x^{m+n+k} modulo $\prod(x-a)\prod(x-b)$ is equal to $S_{1^{m+n-1}(k+1)}(\mathbf{A}+\mathbf{B}-x, \mathbf{A}+\mathbf{B})$ up to a factor, we can conclude the proof. ■

Remark. The coefficients of the successive remainders in the euclidian division of two polynomials are also Schur functions indexed by partitions of the type $1 \ldots 1\ k \ldots k$ (see [4]).

That division is related to factorization is rather clear in the case where $\mathrm{card}(\mathbf{A}) = 1$, i.e. when the polynomial has only one root of modulus greater than 1. Indeed, from the definition, $x^{n+1+k} \equiv \mathscr{R}_{k,1}(x)$; thus, for all b in $\mathbf{B}$, the polynomial $\mathscr{R}_{k,1}(x)$ almost vanishes: $\mathscr{R}_{k,1}(b) = b^{n+1+k} \simeq 0$; since $\mathrm{card}(\mathbf{B}) = n =$ degree of $\mathscr{R}_{k,1}(x)$, this polynomial is not "far" from being $\prod(x-b)$ up to a scalar.

EXAMPLE. Let P be the polynomial $(x-1/10)^2(x+5)(x-6)$. Then

$$x^4 \equiv \mathscr{R}_{0,1} = 1.2x^3 - 30.21x^2 + 6.01x - 0.3$$
$$x^5 \equiv \mathscr{R}_{1,1} = -28.77x^3 - 30.242x^2 + 6.912x - 0.36$$
$$x^6 \equiv \mathscr{R}_{2,1} = -64.766x^3 + 876.056x^2 - 173.2677x + 8.631.$$

Therefore, $\mathscr{R}_{0,2} = {}^*\mathscr{R}_{0,1} - {}^*\mathscr{R}_{1,1} = {}^*(x^2 - 0.198688x + 0.010009)$, $\mathscr{R}_{1,2} = {}^*\mathscr{R}_{1,1} - {}^*\mathscr{R}_{2,1} = {}^*(x^2 - 0.2000006x + 0.0099991)$ are already from the start good approximations of the factor $x^2 - 0.2x + 0.01$.

On the other hand, Bernoulli's method would not allow an easy extraction of the root 6, then of the root -5, to get the required factor. Indeed, the power sums take the values: ... $\psi_4 = 1\,921.0002$, $\psi_5 = 4\,641.00002, \ldots, \psi_8 = 2\,070\,241$, $\psi_9 = 8\,124\,571$ and neither ψ_5/ψ_4 $(= 2.416)$, nor even ψ_9/ψ_8 $(= 3.92)$ can be considered as a sufficiently good approximation of the greatest root $(= 6)$ to permit proceeding to the research of -5.

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