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COMPUTATIONAL TECHNIQUES FOR PI-ALGEBRAS

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Introduction

Let R be a (not necessarily associative) algebra over a field of characteristic 0 and let R satisfy a polynomial identity, i.e. R is a PI-algebra. When studying the properties of R, the following natural question arises: How many identities of R are there? Of course, we have to decide how to measure the quantity of the polynomial identities. There are some numerical invariants of R as the codimension and cocharacter sequences and the Hilbert series of the T-ideal of R and they are objects of intensive investigation.

The main purpose of this paper is to survey and present from a unique point of view some recent quantitative results on PI-algebras. Using the powerful technique of the representation theory of the symmetric and general linear groups we present effective computing methods for concrete PI-algebras. In particular, we apply these methods for studying the polynomial identities of some important associative, Lie and Jordan algebras.

The paper is organized as follows. Section 1 gives the necessary background on PI-algebras and representation theory of the symmetric and general linear groups. Sections 2 is devoted to the free Lie and commutative algebras, some other relatively free algebras and to an important reduction of the computations in the case of varieties of unitary algebras. Section 3 studies the polynomial identities of algebras which in some sense are similar to the 2×2 matrix algebra. Section 4 handles the calculation of the codimensions of T-ideals and the Hilbert series of relatively free algebras. For other applications of the presented computational techniques we refer to the surveys [5, 21, 22].

1. The background

1.1. Varieties of algebras

We fix some notation: K is a field of characteristic 0, $X = \{x_1, x_2, \ldots\}$, $F = K\{X\}$ is the absolutely free algebra of infinite rank. The elements of F are polynomials in noncommutative and nonassociative variables and with zero constant terms. Usually we consider the products left-normed; hence $x_1x_2x_3 = (x_1x_2)x_3$.

The element $f(x_1, \ldots, x_n) \in K\{X\}$ is called a *polynomial identity* for the K-algebra R if $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$. The class $\mathbf{U}$ of all algebras satisfying a given system of identities $\{u_i(x_1, \ldots, x_{n_i}) \mid i \in I\}$ is called a *variety of algebras*. The set of all polynomial identities U satisfied by the variety $\mathbf{U}$ (respectively by the algebra R) is a two-sided ideal of $K\{X\}$ called a *T-ideal* and denoted by $T(\mathbf{U})$ (respectively $T(R)$). We use the same letters U and $\mathbf{U}$ respectively for the T-ideals and the related varieties. The algebra $F(\mathbf{U}) = K\{X\}/T(\mathbf{U})$ (with the same set of generators $X = \{x_1, x_2, \ldots\}$) is the *relatively free algebra of* $\mathbf{U}$. We denote by $F_m(\mathbf{U})$ the subalgebra of $F(\mathbf{U})$ generated by the subset $\{x_1, \ldots, x_m\}$. Moreover, for a subspace Q of F we denote by $Q(\mathbf{U})$ the image of Q under the canonical homomorphism $F \to F/U = F(\mathbf{U})$.

1.1.1. Examples. The class of all associative algebras forms a variety defined by the identity $(x_1x_2)x_3 - x_1(x_2x_3)$; the varieties of all Lie and all Jordan algebras are determined by the sets of identities $\{x_1^2, (x_1x_2)x_3 + (x_2x_3)x_1 + (x_3x_1)x_2\}$ and $\{x_1x_2 - x_2x_1, (x_2x_1)(x_1x_1) - (x_2(x_1x_1))x_1\}$, respectively, etc.

We denote by P_n the vector space of all multilinear polynomials in $K\{X\}$, $P_n = \{\sum a_{\sigma d}(x_{\sigma(1)} \ldots)(\ldots x_{\sigma(n)}) \mid a_{\sigma d} \in K\}$, where the summation runs over all permutations σ of the symmetric group $\mathrm{Sym}(n)$ acting on $\{1, \ldots, n\}$ and over all distributions d of brackets. It is well known that every variety $\mathbf{U}$ can be determined by its multilinear identities. Let $P_n(\mathbf{U}) = P_n/(P_n \cap T(\mathbf{U}))$, $n = 1, 2, \ldots$ The *sequence of codimensions* of the variety $\mathbf{U}$ (or of the T-ideal U) is defined by $c_n(\mathbf{U}) = c_n(U) = \dim P_n(\mathbf{U})$, $n = 1, 2, \ldots$ Additionally, the generating function $c(\mathbf{U}, t) = \sum c_n(\mathbf{U})t^n$ and the exponential generating function $\tilde{c}(\mathbf{U}, t) = \sum c_n(\mathbf{U})t^n/n!$ are called the *codimension series* and the *exponential codimension series* of $\mathbf{U}$, respectively.

The relatively free algebra of rank m is a graded vector space, $F_m(\mathbf{U}) = \sum F_m^{(n)}(\mathbf{U})$, where $F_m^{(n)}(\mathbf{U})$ is the homogeneous component of degree n. The Hilbert series of $F_m(\mathbf{U})$ is defined by $H_m(\mathbf{U}, t) = H(F_m(\mathbf{U}), t) = \sum \dim F_m^{(n)}(\mathbf{U})t^n$; $F_m(\mathbf{U})$ has another grading counting the degree in any variable and the corresponding Hilbert series is

$$H(\mathbf{U}, t_1, \ldots, t_m) = \sum \dim F_m^{(n_1, \ldots, n_m)}(\mathbf{U})t_1^{n_1} \ldots t_m^{n_m}.$$

Clearly $H_m(\mathbf{U}, t) = H(\mathbf{U}, t, \ldots, t)$ (with m t's).

1.1.2. Proposition. *Let U be a T-ideal in $K\{X\}$ and let $\tilde{K}$ be an extension of the base field K. Then $\tilde{F}(\tilde{\mathbf{U}}) = \tilde{K} \otimes_K F(\mathbf{U})$ is the relatively free algebra of the variety $\mathbf{U}$ of $\tilde{K}$-algebras defined by the same system U of polynomial identities (assuming that $K\{X\}$ is canonically embedded into $\tilde{K}\{X\} = \tilde{K} \otimes_K K\{X\}$).*

This proposition allows one to extend the base field and, if necessary, consider K algebraically closed.

1.2. Representations of $\mathrm{Sym}(n)$ **and** GL_m

The vector space P_n has a natural structure of a left $\mathrm{Sym}(n)$-module defined by

$$\sigma\colon (x_{i_1}\ldots)(\ldots x_{i_n}) \to (x_{\sigma(i_1)}\ldots)(\ldots x_{\sigma(i_n)}), \qquad \sigma\in\mathrm{Sym}(n),$$

and for a long period this was the main tool for quantitative investigation of the polynomial identities (see [54, 57, 25] for details). It is known that the representations of $\mathrm{Sym}(n)$ are related to the polynomial representations of the general linear group GL_m. The canonical action of GL_m on the m-dimensional vector space spanned by $x_1, \ldots, x_m$ can be extended diagonally to $F_m = K\{x_1, \ldots, x_m\}$ by

$$g\colon (x_{i_1}\ldots)(\ldots x_{i_n}) \to (g(x_{i_1})\ldots)(\ldots g(x_{i_n})), \qquad g\in\mathrm{GL}_m.$$

Although this action was used incidentally before 1980, its systematical application began in [5, 10] (see also [26] for the formalism of the equivalence of the application of $\mathrm{Sym}(n)$ and GL_m).

The irreducible representations of $\mathrm{Sym}(n)$ and GL_m are described by partitions and Young diagrams [31, 65]. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n, $\lambda_1 \geqslant \ldots \geqslant \lambda_r \geqslant 0$, $\lambda_1 + \ldots + \lambda_r = n$, we consider the corresponding Young diagram $[\lambda]$, the number $n = \|\lambda\|$ of boxes of $[\lambda]$ and the related irreducible $\mathrm{Sym}(n)$-module $M(\lambda)$ and GL_m-module $N_m(\lambda)$. In order to obtain generators for the irreducible submodules of P_n and F_m we use the following device:

(i) As a GL_m-module, the homogeneous component $F_m^{(n)}$ is a direct sum of the GL_m-submodules $N_{m,d}^{(n)}$, where d is a fixed distribution of brackets and $N_{m,d}^{(n)} \overset{\varphi}{\cong} A_m^{(n)}$, $A_m = K\langle x_1, \ldots, x_m\rangle$ being the free associative algebra. Here the GL_m-module isomorphism φ deletes the brackets in $N_{m,d}^{(n)}$.

(ii) First, we determine a generator for the irreducible submodules of $A_m^{(n)}$. We define an action of $\mathrm{Sym}(n)$ on $A_m^{(n)}$ by

$$(x_{i_1}\ldots x_{i_n})\varrho^{-1} = x_{i_{\varrho(1)}}\ldots x_{i_{\varrho(n)}}, \qquad \varrho\in\mathrm{Sym}(n).$$

Then every submodule $N_m(\lambda)$ of $A_m^{(n)}$ is generated by a nonzero element

$$f(x_1, \ldots, x_r) = \prod S_{r_i}(x_1, \ldots, x_{r_i}) \sum a_\varrho \varrho, \tag{1}$$

where $a_\varrho \in K$, ϱ runs over $\mathrm{Sym}(n)$, $S_p(x_1, \ldots, x_p) = \sum(\mathrm{sign}\,\sigma)x_{\sigma(1)}\ldots x_{\sigma(p)}$ is the standard polynomial and $r_1, \ldots, r_k$ are the lengths of the columns of the diagram $[\lambda]$.

(iii) An arbitrary irreducible GL_m-submodule $N_m(\lambda)$ of F_m is generated by an element

$$\sum f_d(x_1, \ldots, x_r), \tag{1'}$$

where f_d is of the form (1) and the summation runs over all distributions of brackets. We call (1′) a *standard generator* of $N_m(\lambda)$. It follows from the representation theory of GL_m that the standard generator is uniquely determined up to a multiplicative constant.

(iv) A generator of the irreducible Sym(n)-submodule $M(\lambda)$ of P_n can be obtained by a linearization of a suitable standard generator of $N_m(\lambda)$.

For an arbitrary T-ideal U of F, the subspaces $U \cap P_n$ and $U \cap F_m$ are Sym(n)- and GL_m-modules, respectively. Hence the relatively free algebra $F(U)$ inherits the actions of Sym(n) and GL_m. In particular, the Sym(n)-character sequence $\chi(P_n(U))$, $n = 1, 2, \ldots$, is called the *cocharacter sequence* of U. The following assertion gives the equivalence of the application of the representation theory of Sym(n) and GL_m.

1.2.1. Theorem [5, 10]. (i) *Let Λ and Λ^* be the lattices of submodules of $F_m^{(n)}$ and P_n, respectively (with respect to the sum and intersection of submodules). Then there is a lattice monomorphism ψ: $\Lambda \to \Lambda^*$ such that $\psi(N_m(\lambda)) = M(\lambda)$ and a generator of $M(\lambda)$ is obtained by a linearization of the standard generator of $N_m(\lambda)$. The image of $\psi(\Lambda)$ coincides with $\sum M(\lambda)$, where the diagrams $[\lambda]$ have at most m rows. In particular, ψ is an isomorphism for $m \geqslant n$.*

(ii) *For every variety U, $P_n(U)$ and $F_m^{(n)}(U)$ have the same module structure: If $P_n(U) = \sum k(\lambda)M(\lambda)$, then $F_m^{(n)}(U) = \sum k(\lambda)N_m(\lambda)$.*

It turns out that for concrete computations the representations of GL_m are more convenient than those of Sym(n). For example, the polynomial identities $h(x_1, \ldots, x_n) = \sum x_{\sigma(1)} \ldots x_{\sigma(n)}$ and x_1^n from $A_n^{(n)}$ are equivalent and generate $M(n)$ and $N_n(n)$ respectively but x_1^n is written more compactly. Additionally, the representations of GL_m allow the classical invariant theory to be applied to PI-algebras (see e.g. [26, 50]).

For a multihomogeneous polynomial $f(x_1, \ldots, x_m)$ we denote by

$$f(x_1, y_{11}, \ldots, y_{1n_1} | \ldots | x_m, y_{m1}, \ldots, y_{mn_m})$$

the partial linearization of $f(x_1, \ldots, x_m)$ which equals the component of $f(x_1 + y_{11} + \ldots + y_{1n_1}, \ldots, x_m + y_{m1} + \ldots + y_{mn_m})$ multilinear in y_{ij}. The following result allows one to find standard generators of $N_m(\lambda)$.

1.2.2. Theorem [36]. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition and let $f(x_1, \ldots, x_r)$ be a nonzero polynomial from F_m which is homogeneous of degree λ_i in x_i, $i = 1, \ldots, r$. Then f is a standard generator of $N_m(\lambda)$ if and only if $f(x_1 | \ldots | x_j, x_i | \ldots | x_r) = 0$ for every pair (i, j), $1 \leqslant i < j \leqslant r$.*

In the representation theory of GL_m, the role of characters of GL_m is played by the Schur functions $S_\lambda(t_1, \ldots, t_m)$ [41]. They are symmetric polynomials from $K[t_1, \ldots, t_m]$ and

$$S_\lambda(t_1, \ldots, t_m) = \sum a_i t_1^{i_1} \ldots t_m^{i_m} = H(N_m(\lambda), t_1, \ldots, t_m), \qquad N_m(\lambda) \subset A_m^{(n)}.$$

The coefficient a_i equals the number of semistandard λ-tableaux with content $i = (i_1, \ldots, i_m)$, i.e. the Schur functions can be obtained in a combinatorial way (see Section 1.3). It is known that the Hilbert series of a GL_m-module determines the module uniquely. In particular, $H(F_m(U), t_1, \ldots, t_m) = \sum k(\lambda)S_\lambda(t_1, \ldots, t_m)$ if and only if $F_m(U) = \sum k(\lambda)N_m(\lambda)$.

1.3. The Littlewood–Richardson rule

1.3.1. DEFINITION [41]. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_r)$ be partitions, $\nu_i \geqslant \lambda_i$ and $\|\nu\| = \|\lambda\| + \|\mu\|$.

(i) A *diagram of shape* $[\nu - \lambda]$ is a scheme of boxes obtained from the diagram $[\nu]$ by removing the boxes of the diagram $[\lambda]$. When $\|\lambda\| = 0$, $[\nu - \lambda] = [\nu]$.

(ii) A $[\nu - \lambda]$-*tableau* with *content* μ is the diagram $[\nu - \lambda]$ whose boxes are filled in with μ_1 numbers $1, \ldots, \mu_s$ numbers s.

(iii) A tableau is *semistandard* if its entries do not decrease from left to right in the rows and increase from top to bottom in the columns.

(iv) The sequence $w(T)$ is obtained from a tableau T by listing the entries of T from right to left, consecutively reading the rows from top to bottom (as in Arabic).

(v) The sequence $w = a_1, a_2, \ldots, a_n$ is a *lattice permutation* if it contains the symbols $1, 2, \ldots, s$ and for each $1 \leqslant k \leqslant n$ and $1 \leqslant i \leqslant s-1$, the number i occurs in $a_1, \ldots, a_k$ no less than $i+1$ times.

1.3.2. THEOREM (The Littlewood–Richardson rule). *The following isomorphism of* GL_m-*modules holds*:

$$N_m(\lambda) \otimes_K N_m(\mu) = \sum_\nu c^\nu_{\lambda\mu} N_m(\nu),$$

where $c^\nu_{\lambda\mu}$ *is the number of semistandard tableaux* T *of shape* $[\nu - \lambda]$ *with content* μ, *such that the sequence* $w(T)$ *is a lattice permutation.*

An important role in our concrete computations is played by the following consequence of Theorem 1.3.2.

1.3.3. COROLLARY. $N_m(\lambda_1, \ldots, \lambda_m) \otimes_K N_m(s) \cong \sum N_m(\mu_1, \ldots, \mu_m)$, *where* $\|\mu\| = \|\lambda\| + s$ *and* $\mu_1 \geqslant \lambda_1 \geqslant \mu_2 \geqslant \lambda_2 \geqslant \ldots \geqslant \mu_m \geqslant \lambda_m$.

In particular, for $s = 1$ the corollary coincides with the branching theorem.

1.3.4. *Rule.* The following device allows us to obtain the standard generators of the tensor product of GL_m-modules. It is a combination of Theorems 1.2.2 and 1.3.2. Its preliminary version has been used in [14, 36]. In order to simplify the rule, we consider $N_m(\lambda)$ and $N_m(\mu)$ as submodules of $A_m^{(p)}$ and $A_m^{(q)}$, respectively, where $p = \|\lambda\|$ and $q = \|\mu\|$, and identify $A_m^{(p)} \otimes_K A_m^{(q)}$ with $A_m^{(p+q)}$. Additionally, we fix partitions λ, μ, ν as in the Littlewood–Richardson rule, and $f_\lambda(x_1, \ldots, x_m)$ and $f_\mu(x_1, \ldots, x_m)$ are the standard generators of $N_m(\lambda)$ and $N_m(\mu)$, respectively.

(i) Find all semistandard tableaux $T_\lambda(\alpha)$ and $T_\mu(\beta)$ with contents $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$, respectively, $\alpha_i + \beta_i = \nu_i$.

(ii) For every tableau $T_\lambda(\alpha)$ let

$$f_{\lambda\alpha} = f_\lambda(x_{11}, \ldots, x_{1\lambda_1} | \ldots | x_{m1}, \ldots, x_{m\lambda_m})$$

be the linearization of f_λ, where $x_{ij} = x_k$, k being the (i, j)-entry of $T_\lambda(\alpha)$, and similarly for f_μ and $T_\mu(\beta)$.

(iii) Write $f(x_1, \ldots, x_m) = \sum a_{\alpha\beta} f_{\lambda\alpha} f_{\mu\beta}$ with unknown $a_{\alpha\beta} \in K$.

(iv) Assuming that $f(x_1 | \ldots | x_j, x_i | \ldots | x_m) = 0$ for all pairs (i, j), $1 \leqslant i < j \leqslant m$, obtain a linear homogeneous system for $a_{\alpha\beta}$. Any nonzero solution of this system gives a standard generator for $N_m(\nu) \subset N_m(\lambda) \otimes_K N_m(\mu)$.

By using additional information, this rule can be simplified for concrete cases.

1.3.5. Example. Let $\lambda = (2, 1)$, $\mu = (2)$, $\nu = (3, 1^2)$, $f_\lambda(x_1, x_2) = S_2(x_1, x_2)x_1 = (x_1x_2 - x_2x_1)x_1 = [x_1, x_2]x_1$, $f_\mu(x_1) = x_1^2$. Then

$$T_\lambda(\alpha_1) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad T_\lambda(\alpha_2) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad T'_\lambda(\alpha_3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad T''_\lambda(\alpha_3) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$T_\mu(\beta_1) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \qquad T_\mu(\beta_2) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad T_\mu(\beta_3) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$$

$$f_{\lambda\alpha_1} = f_\lambda(x_1, x_1 | x_2) = 2f_\lambda(x_1, x_2), \quad f_{\mu\beta_1} = f_\mu(x_1, x_3) = x_1x_3 + x_3x_1,$$

$$f_{\lambda\alpha_2} = 2f_\lambda(x_1, x_3), \quad f_{\mu\beta_2} = x_1x_2 + x_2x_1,$$

$$f'_{\lambda\alpha_3} = [x_1, x_3]x_2 + [x_2, x_3]x_1, \quad f''_{\lambda\alpha_3} = [x_1, x_2]x_3 + [x_3, x_2]x_1, \quad f_{\mu\beta_3} = 2x_1^2,$$

$$f(x_1, x_2, x_3) = 2\{a_1[x_1, x_2]x_1(x_1x_3 + x_3x_1) + a_2[x_1, x_3]x_1(x_1x_2 + x_2x_1)$$
$$+ a'_3([x_1, x_3]x_2 + [x_2, x_3]x_1)x_1^2 + a''_3([x_1, x_2]x_3 + [x_3, x_2]x_1)x_1^2\},$$

$$f(x_1 | x_2, x_1 | x_3) = f(x_1, x_1, x_3) = 2(2a_2 + 2a'_3 - a''_3)[x_1, x_3]x_1^3 = 0,$$

$$f(x_1 | x_2 | x_3, x_1) = 2(2a_1 - a'_3 + 2a''_3)[x_1, x_2]x_1^3 = 0,$$

$$f(x_1 | x_2 | x_3, x_2) = 2[x_1, x_2]\{(a_1 + a_2)x_1(x_1x_2 + x_2x_1) + (a'_3 + a''_3)x_2x_1^2\} = 0.$$

Therefore we obtain the system

$$2a_2 + 2a'_3 - a''_3 = 0, \quad 2a_1 - a'_3 + 2a''_3 = 0,$$

$$a_1 + a_2 = 0, \quad a'_3 + a''_3 = 0,$$

with a nonzero solution $a_1 = 1.5$, $a_2 = -1.5$, $a'_3 = 1$, $a''_3 = -1$. Easy calculations show that the standard generator of $N_3(3, 1^2)$ is

$$f(x_1, x_2, x_3) = S_3(x_1, x_2, x_3)x_1^2(-2\mathrm{id}+3(34)+3(35)),$$

where id, (34), (35) $\in$ Sym(5).

The direct product $\mathrm{Sym}(p)\times\mathrm{Sym}(q)$ is canonically embedded into $\mathrm{Sym}(p+q)$, $\mathrm{Sym}(p)$ acting on $\{1, \ldots, p\}$ and $\mathrm{Sym}(q)$ on $\{p+1, \ldots, p+q\}$. Moreover, for a subgroup H of the group G and M being an H-module, we denote by $M\uparrow G$ the G-module induced by M. The Littlewood–Richardson rule has the following interpretation in the language of Sym(n)-representations.

1.3.6. THEOREM. *Let* $\|\lambda\| = p$, $\|\mu\| = q$ *and* $N_m(\lambda)\otimes_K N_m(\mu) \cong \sum c^{\nu}_{\lambda\mu}N_m(\nu)$. *Then*

$$(M(\lambda)\otimes_K M(\mu))\uparrow \mathrm{Sym}(p+q) \cong \sum c^{\nu}_{\lambda\mu}M(\nu),$$

where $M(\lambda)\otimes_K M(\nu)$ *has a structure of* $\mathrm{Sym}(p)\times\mathrm{Sym}(q)$*-module.*

1.4. Other products of modules

In this section we shortly discuss some other products of modules which have applications to PI-algebras.

1.4.1. *The Kronecker product.* Let λ and μ be partitions of n. Then the *Kronecker* (or *inner*) *product* $M(\lambda)\otimes_K M(\mu)$ of $M(\lambda)$ and $M(\mu)$ is a Sym(n)-module with a diagonal action of Sym(n). This product plays an impórtant role for computing the cocharacter sequence of the $k\times k$ matrix algebra (see e.g. [56]).

1.4.2. *Symmetrized tensor powers.* For a GL_m-module N we consider the symmetrized tensor power

$$N^{\overset{s}{\otimes} k} = N\overset{s}{\otimes} \ldots \overset{s}{\otimes} N,$$

identifying the tensors $v_1\otimes\ldots\otimes v_k$ and $v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(k)}$, $\sigma\in\mathrm{Sym}(k)$, $v_i\in N$. A result of Thrall [63] (see also [41], Exercise 5, p. 45) shows that

$$N_m(2)^{\overset{s}{\otimes} k} \cong \sum N_m(2\lambda_1, \ldots, 2\lambda_m). \tag{2}$$

The symmetrized tensor product is a very special case of the general notion of plethysm (see [31, 41]).

1.5. Representations of Lie superalgebras

In this survey we apply the representation theory of Sym(n) and GL_m or equivalently, of the Lie algebra gl_m. This theory works very well when the number of rows of all the Young diagrams appearing in the decomposition $P_n(U) = \sum k(\lambda)M(\lambda)$, $n = 1, 2, \ldots$, is bounded. Generally, the number of rows increases with n and the behaviour of $P_n(U)$ cannot be studied by the representations of a fixed GL_m. Kemer (see [32, 34]) has applied $\mathbf{Z}_2$-graded algebras for the investigation of associative PI-algebras. This has allowed him to obtain important results on polynomial identities. In particular, he has built the structure theory of T-ideals in $K\langle X\rangle$ in the spirit of commutative algebra. A combination of [53] and [32] gives the following theorem whose final version is due to Braun [8].

1.5.1. THEOREM (Razmyslov–Kemer–Braun [53, 32, 8]). *The Jacobson radical of every finitely generated associative PI-algebra (i.e. with nontrivial identity from $K\langle X\rangle$) over an arbitrary field is nilpotent.*

In practice, the application of $\mathbf{Z}_2$-graded algebras involves the representation theory of Lie superalgebras. The formalism of representations of linear Lie superalgebras has been developed by Berele and Regev [7] and they have obtained important quantitative results in the associative case.

2. First applications

2.1. Relatively free algebras

One of the main problems in this paper is the following: How to compute the cocharacter sequence of a T-ideal or, equivalently, to find the multiplicities $k(\lambda)$ of the irreducible GL_m-submodules of $F_m(U)$? A trivial example is the free associative algebra A_m, when the Sym(n)-modules $PA_n \subset A$ and KSym(n) are isomorphic. Therefore, $A_m = \sum(\dim M(\lambda))N_m(\lambda)$. For small n it is possible to use the character table of Sym(n).

2.1.1. EXAMPLE. Let PL_4 be the set of all multilinear elements of degree 4 in the free Lie algebra L. It is well known (see e.g. [4]) that PL_4 has a basis $\{x_4x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} \mid \sigma \in \mathrm{Sym}(3)\}$. Let χ be the character of the Sym(4)-module PL_4 and let $\chi = \sum k(\lambda)\chi_\lambda$. Easy calculations show that $\chi(\mathrm{id}) = \dim PL_4 = 6$, $\chi(12) = \chi(123) = \chi(1234) = 0$, $\chi((12)(34)) = -2$. The character table of Sym(4) is the following:

	id	(12)	(123)	(1234)	(12)(34)
(4)	1	1	1	1	1
(3, 1)	3	1	0	−1	−1
(2^2)	2	0	−1	0	2
(2, 1^2)	3	−1	0	1	−1
(1^4)	1	−1	1	−1	1

Hence we obtain a linear system for $k(\lambda)$:

$$\chi(\mathrm{id}) = k(4)+3k(3,1)+2k(2^2)+3k(2,1^2)+k(1^4) = 6,$$

$$\chi(12) = k(4)+k(3,1)-k(2,1^2)-k(1^4) = 0,$$

$$\chi(123) = k(4)-k(2^2)+k(1^4) = 0,$$

$$\chi(1234) = k(4)-k(3,1)+k(2,1^2)-k(1^4) = 0,$$

$$\chi((12)(34)) = k(4)-k(3,1)+2k(2^2)-k(2,1^2)+k(1^4) = -2.$$

The only solution of the system is $k(4) = k(2^2) = k(1^4) = 0$, $k(3, 1) = k(2, 1^2) = 1$; therefore $PL_4 = M(3, 1)+M(2, 1^2)$. In practice, bearing in mind that we are interested in solutions in nonnegative integers, it suffices to consider only a part of the equations.

2.1.2. *The same example.* The dimension of $M(\lambda)$ equals the number of standard λ-tableaux (i.e. the semistandard tableaux with content $(1, \ldots, 1)$). The hook formula gives another expression for $\dim M(\lambda)$:

$$\dim M(\lambda) = n!/\prod(\lambda_i+\lambda'_j-i-j+1),$$

where λ'_j is the length of the jth column of $[\lambda]$. Applying one of these two expressions for $\dim M(\lambda)$, it is easy to show that $\dim M(3, 1) = \dim M(2, 1^2) = 3$ and hence $\dim PL_4 = \dim M(3, 1)+\dim M(2, 1^2)$. Therefore, in virtue of Theorem 1.2.1 it suffices to obtain in the free Lie algebra nonzero standard generators for $N_m(3, 1)$ and $N_m(2, 1^2)$. But

$$f_{(3,1)} = (x_1x_2-x_2x_1)x_1x_1 \quad \text{and} \quad f_{(2,1^2)} = \sum(\operatorname{sign}\sigma)(x_1x_{\sigma(1)})(x_{\sigma(2)}x_{\sigma(3)})$$

do not vanish in L_m and this gives the desired decomposition.

This method has been used successfully to obtain similar decompositions for $P_n(U)$, $n \geqslant 1$, for the varieties of Lie algebras $U_1 = N_2A \cap AN_2$ determined by the identities $(x_1x_2)(x_3x_4)(x_5x_6)$ and $(x_1x_2x_3)(x_4x_5x_6)$ [10] and $U_2 = [A^2, E, E]$ defined by $(x_1x_2)(x_3x_4)x_5x_6$ [43]. For other applications see [5, 22].

2.1.3. THEOREM [63]. *Let $L_m^{(n)}$ be the homogeneous component of degree n of the free algebra L_m. Then the following* GL_m*-module isomorphism holds:*

$$K+A_m \cong \sum\left((L_m^{(1)})^{\overset{s}{\otimes} p_1}\otimes_K \ldots \otimes_K(L_m^{(r)})^{\overset{s}{\otimes} p_r}\right)$$

where the sum is over all symmetrized tensor powers with $p_i \geqslant 0$.

Proof. It is known that the free associative algebra A_m coincides with the universal enveloping algebra of L_m. Let $\{g_{ij} \mid j = 1, \ldots, d_i\}$ be a multihomogeneous basis of the vector space $L_m^{(i)}$. By the Poincaré–Birkhoff–Witt theorem, $K+A_m$ has a basis $\{\prod_i(\prod_j g_{ij}^{a_{ij}}) \mid a_{ij} \geqslant 0\}$. Hence the Hilbert series of $K+A_m$ equals the Hilbert series of

$$\sum\left((L_m^{(1)})^{\overset{s}{\otimes} p_1}\otimes_K \ldots \otimes_K(L_m^{(r)})^{\overset{s}{\otimes} p_r}\right).$$

Since the Hilbert series determines uniquely the GL_m-module, the desired isomorphism holds.

Since the GL_m-module structure of $A_m^{(n)}$ is known, Theorem 2.1.3 allows one to calculate the structure of $L_m^{(n)}$. This has been done for $n \leqslant 10$ in [63]. For example, $L_m^{(1)} = N_m(1)$, $L_m^{(2)} = N_m(1^2)$, $L_m^{(3)} = N_m(2, 1)$ and $L_m^{(4)} = N_m(3, 1)$

$+N_m(2, 1^2)$. Therefore

$$A_m^{(5)} = (L_m^{(1)})^{\overset{s}{\otimes} 5} + (L_m^{(1)})^{\overset{s}{\otimes} 3} \otimes_K L_m^{(2)} + (L_m^{(1)})^{\overset{s}{\otimes} 2} \otimes_K L_m^{(3)}$$
$$+ L_m^{(1)} \otimes_K \big(L_m^{(4)} + (L_m^{(2)})^{\overset{s}{\otimes} 2}\big) + L_m^{(2)} \otimes_K L_m^{(3)} + L_m^{(5)}$$
$$= N_m(5) + N_m(3) \otimes_K N_m(1^2) + N_m(2) \otimes_K N_m(2, 1)$$
$$+ N_m(1) \otimes_K \big((N_m(3, 1) + N_m(2, 1^2)) + (N_m(2^2) + N_m(1^4))\big)$$
$$+ N_m(1^2) \otimes_K N_m(2, 1) + L_m^{(5)}$$

and bearing in mind that $A_m^{(5)} = N_m(5) + 4N_m(4, 1) + 5N_m(3, 2) + 6N_m(3, 1^2) + 5N_m(2^2, 1) + 4N_m(2, 1^3) + N_m(1^5)$ we establish

$$L_m^{(5)} = N_m(4, 1) + N_m(3, 2) + N_m(3, 1^2) + N_m(2^2, 1) + N_m(2, 1^3).$$

The standard generators of $N_m(\lambda) \subset L_m^{(n)}$ for $n \leqslant 6$ have been obtained in [10, 12]. Another approach to the free Lie algebra is given in [35] (see also [4]).

2.1.4. Example. Let C_m be the free commutative algebra of rank m, i.e. the relatively free algebra $F_m(C)$ of the variety defined by the identity $x_1x_2 - x_2x_1$. It follows from [58] that if $C_m^{(i)}$ has a basis $\{u_{ij} \mid j = 1, \ldots, d_i\}$, $i < n$, then $C_m^{(n)}$ has a basis

$$\{u_{ij}u_{n-i,k} \mid j = 1, \ldots, d_i,\ k = 1, \ldots, d_{n-i},\ i = 1, \ldots, [n/2],$$
$$\text{and if } i = n-i \text{ then } j \leqslant k\}.$$

As in the proof of Theorem 2.1.3,

$$C_m^{(n)} \cong \sum_{i<n-i} C_m^{(i)} \otimes_K C_m^{(n-i)} + \varepsilon C_m^{(n/2)} \overset{s}{\otimes}_K C_m^{(n/2)}, \tag{3}$$

where $\varepsilon = 1$ for n even and $\varepsilon = 0$ for n odd. In particular, $C_m^{(1)} = N_m(1)$,

$$C_m^{(2)} \cong N_m(1) \overset{s}{\otimes}_K N_m(1) \cong N_m(2), \qquad C_m^{(3)} \cong C_m^{(1)} \otimes_K C_m^{(2)} \cong N_m(3) + N_m(2, 1),$$

$$C_m^{(4)} \cong C_m^{(1)} \otimes_K C_m^{(3)} + C_m^{(2)} \overset{s}{\otimes}_K C_m^{(2)} \cong 2N_m(4) + 2N_m(3, 1) + 2N_m(2^2) + N_m(2, 1^2),$$

$$C_m^{(5)} \cong C_m^{(1)} \otimes_K C_m^{(4)} + C_m^{(2)} \otimes_K C_m^{(3)}$$
$$\cong 3N_m(5) + 6N_m(4, 1) + 6N_m(3, 2) + 4N_m(3, 1^2) + 4N_m(2^2, 1) + N_m(2, 1^3).$$

The standard generators of the submodules can be obtained by Rule 1.3.4 or its modification for symmetrized tensor powers. Another method for computing $C_m^{(5)}$ is applied in [46]. Similar formulas can be established for the free anticommutative algebra when in (3) the symmetrized tensor square has to be replaced by the antisymmetrized tensor square.

2.1.5. Example [23]. Let S_2 be the variety of all solvable Jordan algebras of class 2 (i.e. S_2 is defined by $(x_1x_2)(x_3x_4)$). Then the vector space $F(S_2)$ has a basis

$$\{x_{i_1}x_{i_2}\dots x_{i_n} \mid i_1 \geqslant i_2 < i_4 < i_6 < \dots,\ i_3 < i_5 < \dots\}.$$

The Hilbert series of the subspaces of $A_m^{(p)}$ spanned by $\{x_{j_1}x_{j_2}\dots x_{j_p} \mid j_1 \geqslant j_2 < j_3 < \dots < j_p\}$ and $\{x_{k_1}\dots x_{k_p} \mid k_1 < \dots < k_p\}$ coincide with the Hilbert series of $N_m(2, 1^{p-2})$ and $N_m(1^p)$, respectively. Hence $F_m^{(2k+1)}(S_2) \cong N_m(2, 1^{k-1}) \otimes_K N_m(1^k)$ and $F_m^{(2k+2)}(S_2) \cong N_m(2, 1^k) \otimes_K N_m(1^k)$, $k \geqslant 1$, and the Littlewood–Richardson rule allows one to calculate the cocharacter sequence of the variety S_2.

2.2. Unitary algebras

In this section we consider varieties U of unitary algebras only. We denote by $F^\# = K\{X\}^\#$ the absolutely free unitary algebra and by $F^\#(U)$ the free algebra of the variety U. In the associative case it is known that every variety U is determined by its proper (or commutator) multilinear identities [60]. A similar result holds for arbitrary algebras. For $f(x_1, \dots, x_m) \in F^\#$, let $\partial f/\partial x_i$ be the formal derivative in x_i. For a multihomogeneous polynomial, $\partial f/\partial x_i$ equals $f(x_1|\dots|x_i, 1|\dots|x_m)$. We write

$$B_m = \{f \in F_m^\# \mid \partial f/\partial x_i = 0,\ i = 1, \dots, m\} \quad \text{and} \quad \Gamma_n = P_n \cap B_n$$

for the subspace of $F_m^\#$ vanishing under the formal derivations and the space of proper multilinear polynomials of degree n, respectively. It is easy to see that an analogue of Theorem 1.2.1 holds and $\Gamma_n(U)$ and $B_m^{(n)}(U)$ have the same module structure.

2.2.1. Theorem [13, 14, 16]. (i) *Every polynomial from $F_m(U)$ can be uniquely written in the form $\sum b_i(x_1, \dots, x_m)x_{i_1} \dots x_{i_n}$, where $b_i \in B_m(U)$ and $i_1 \leqslant \dots \leqslant i_n$.*

(ii) *Any variety U can be defined by its identities from Γ_n, $n = 2, 3, \dots$*

(iii) $H(U, t_1, \dots, t_m) = H(B_m(U), t_1, \dots, t_m)/\prod_{i=1}^m (1-t_i)$, $H_m(U, t) = H(B_m(U), t)/(1-t)^m$.

(iv) *The following* GL_m*-module isomorphism is valid*:

$$F_m(U) \cong B_m(U) \otimes_K K[x_1, \dots, x_m]^\#,$$

where $K[x_1, \dots, x_m]^\#$ is the ordinary algebra of polynomials in commuting variables.

Since $K[x_1, \dots, x_m]^\# \cong \sum_{n \geqslant 0} N_m(n)$, Theorem 2.2.1 and Corollary 1.3.3 reduce the problem of decomposition of $F_m^\#(U)$ to a similar (but simpler) problem of decomposition of $B_m(U)$. In particular:

2.2.2. Corollary [13]. *Let*

$$F_m^\#(U) = \sum k(\lambda_1, \dots, \lambda_m)N_m(\lambda), \qquad B_m(U) = \sum k_1(\mu_1, \dots, \mu_m)N_m(\mu).$$

Then $k(\lambda_1, \ldots, \lambda_m) = \sum k_1(\mu_1, \ldots, \mu_m)$, *where the summation is over all partitions* $(\mu_1, \ldots, \mu_m)$ *such that* $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \ldots \geqslant \lambda_m \geqslant \mu_m$.

Especially for the free associative algebra $A_m^\#$ there exists a good basis of the space BA_m of proper polynomials in $A_m^\#$.

2.2.3. PROPOSITION [59]. *The vector space* $BA_m \subset A_m^\#$ *has the following basis*: $[x_{i_1}, \ldots, x_{i_p}] \ldots [x_{j_1}, \ldots, x_{j_q}]$, *where* $i_1 > i_2 \leqslant \ldots \leqslant i_p, \ldots, j_1 > j_2 \leqslant \ldots \leqslant j_q$, *and* $[u, v] = uv - vu$, $[u_1, \ldots, u_p] = [[u_1, \ldots, u_{p-1}], u_p]$.

Additional arguments give an expression for the GL_m-module BA_m:

2.2.4. THEOREM [20]. *The following* GL_m*-module isomorphism holds*:

$$BA_m \cong \sum N_m(p_1 - 1, 1) \otimes_K \ldots \otimes_K N_m(p_r - 1, 1),$$

where the summation runs over all integers $p_i \geqslant 2$, $r = 0, 1, \ldots$

In particular, modulo the *T*-ideal of A_m generated by $[x_1, x_2][x_3, x_4]$, the commutators $[x_{i_1}, \ldots, x_{i_p}]$, $i_1 > i_2 \leqslant \ldots \leqslant i_p$, span the module $N_m(p-1, 1)$. Hence Rule 1.3.4, Proposition 2.2.3 and Theorem 2.2.4 allow one to obtain the standard generators of the irreducible submodules of $BA_m^{(n)}$. For $n \leqslant 6$ this has been done in [12], for $n = 7$ in [49] and for $n = 8$ partially in [20].

3. Simulation of 2×2 matrices

3.1. Algebras with good bases

The three-dimensional real vector space $\mathbf{R}^3$ with the usual scalar and vector products enjoys the following properties:

(i) For every basis f_1, f_2, f_3 of $\mathbf{R}^3$ the standard process of orthogonalization gives an orthogonal basis e_1, e_2, e_3.

(ii) Let $g(x_1, x_2, x_3) = (x_{i_1} \times \ldots) \times (\ldots \times x_{i_n})$ be a monomial (with respect to the vector product) of degree d_i in x_i. Then $g(e_1, e_2, e_3) = \varepsilon e_1^{\delta_1} \times e_2^{\delta_2} \times e_3^{\delta_3}$, where $\varepsilon = 0, \pm 1$, $\delta_i = 0, 1$, $\delta_i \equiv d_i \pmod 2$ (and $e_1^0 \times e_2^0 \times e_3^0 = 0$).

It turns out that these simple properties of $\mathbf{R}^3$ play an important role in the investigation of the polynomial identities of 2×2 matrices [10]. Here we give a generalization which works successfully in several different cases.

In virtue of Proposition 1.1.2, we assume that the base field is algebraically closed. We make use of the Zariski topology [30, pp. 36–37]. Let $u_1, \ldots, u_k$ be a fixed basis of a vector space *W*, $K[y]^\# = K[y_{ij} \mid i = 1, \ldots, k,\ j = 1, \ldots, m]^\#$ the polynomial algebra over *K* and *Q* a subset of $K[y]^\#$. The set of all *m*-tuples of vectors $(v_1, \ldots, v_m) \in W^m$, $v_j = \sum_{i=1}^k \xi_{ij} u_i$, $\xi_{ij} \in K$, such that $g(\xi_{ij}) = 0$ for every $g(y_{ij}) \in Q$ is closed in the Zariski topology. Any nonempty open subset is dense in this topology. Hence a polynomial function which vanishes on an open subset vanishes everywhere. In particular, if *R* is a finite-dimensional algebra and

$f(x_1, \ldots, x_m) \in K\{X\}$ and $f(r_1, \ldots, r_m) = 0$ on a nonempty open subset of R^m, then $f(x_1, \ldots, x_m)$ is a polynomial identity for R.

3.1.1. DEFINITION. Let R be a finite-dimensional algebra. We call the basis $r_1, \ldots, r_p$ of R *good* if there exists a $p \times p$ upper-triangular matrix $U = (\alpha_{ij})$, $\alpha_{ii} = 1$, with the following property:

for any two monomials $g_1(x_1, \ldots, x_p)$, $g_2(x_1, \ldots, x_p) \in K\{X\}$ of degree d_j in x_j there exist $\eta_1, \eta_2 \in K$, $(\eta_1, \eta_2) \neq (0, 0)$, η_1, η_2 depending on g_1 and g_2 only, such that

$$\eta_1 g_1(s_1, \ldots, s_p) = \eta_2 g_2(s_1, \ldots, s_p), \text{ where } s_j = \sum_{i=1}^{j} \alpha_{ij} r_i, \; j = 1, \ldots, p.$$

3.1.2. THEOREM. *Let R be a finite-dimensional algebra,* $\dim R = p$ *and let the set of all good bases in R be dense in the Zariski topology in R^p. Then for the variety* $\operatorname{var} R$ *generated by R the* GL_m*-module* $F_m(\operatorname{var} R)$ *is a submodule of* $\sum N_m(\lambda_1, \ldots, \lambda_p)$, *i.e. the multiplicities of the irreducible submodules* $N_m(\lambda)$ *of* $F_m(\operatorname{var} R)$ *equal* 0 *or* 1 *and are zero if* $\lambda_{p+1} \neq 0$.

Proof. Let $F_m(\operatorname{var} R) = \sum k(\lambda) N_m(\lambda)$ and let $\lambda = (\lambda_1, \ldots, \lambda_q)$ be a partition, $\lambda_q \neq 0$. First, assume that $q > p$ and that $f_\lambda(x_1, \ldots, x_q) \in F_m$ is a standard generator of $N_m(\lambda)$. Since arbitrary $u_1, \ldots, u_q \in R$ are linearly dependent and there is a skew symmetry in the q variables of f_λ (see (1) and (1')), $f_\lambda(u_1, \ldots, u_q) = 0$ and f_λ is a polynomial identity for R, i.e. $k(\lambda) = 0$ if $\lambda_{p+1} \neq 0$. Now, let $q \leqslant p$ and let $f'_\lambda, f''_\lambda \in F_m$ be standard generators of two isomorphic copies of $N_m(\lambda)$. Let $r_1, \ldots, r_p$ be a good basis and $s_1, \ldots, s_p$ be the related vectors from Definition 3.1.1. Therefore, for suitable $\beta_{ij} \in K$, $r_1 = s_1$, $r_2 = s_2 + \beta_{12} s_1, \ldots, r_p = s_p + \beta_{1p} s_1 + \ldots + \beta_{p-1,p}\, s_{p-1}$. Bearing in mind the skew symmetry in (1) and (1') we obtain $f'(r_1, \ldots, r_q) = f'(s_1, \ldots, s_q)$, $f''(r_1, \ldots, r_q) = f''(s_1, \ldots, s_q)$. Since f'_λ and f''_λ are multihomogeneous of degree λ_j in x_j, Definition 3.1.1 gives that either $f'_\lambda(s_1, \ldots, s_q) = f''_\lambda(s_1, \ldots, s_q) = 0$ or $f'_\lambda(s_1, \ldots, s_q) \neq 0$ and there exists $v \in K$, $v \neq 0$, such that $f''_\lambda(s_1, \ldots, s_q) - v f'_\lambda(s_1, \ldots, s_q) = 0$. Hence $f_\lambda(x_1, \ldots, x_q) = f''_\lambda(x_1, \ldots, x_q) - v f'_\lambda(x_1, \ldots, x_q)$ is a standard generator of $N_m(\lambda)$ (which does not depend on the choice of the basis $r_1, \ldots, r_p$) and $f_\lambda(r_1, \ldots, r_q) = 0$. Since the set of good bases is dense in R^p this yields that $f_\lambda(x_1, \ldots, x_q)$ is a polynomial identity for R. Hence the two isomorphic copies of $N_m(\lambda)$ are "glued together" in $F_m(\operatorname{var} R)$ and $k(\lambda) \leqslant 1$.

The subvarieties of a variety U form a lattice with respect to the intersection and union. It is distributive if and only if $P_n(U)$, $n = 1, 2, \ldots$, are sums of nonisomorphic irreducible submodules.

3.1.3. COROLLARY. *Under the conditions of Theorem* 3.1.2, *the lattice of subvarieties of* $\operatorname{var} R$ *is distributive.*

A modification of Theorem 3.1.2 works for varieties of unitary algebras. For a unitary algebra R we define

$$Z(R) = \{z \in R \mid zr = rz,\ (zr)s = r(zs) = z(rs) \text{ for all } r, s \in R\}$$

and fix a decomposition $R = Z(R) \oplus S$ into a direct sum of vector spaces.

3.1.4. DEFINITION. Let $\dim R/Z(R) < \infty$ and $R = Z(R) \oplus S$. The basis $r_1, \ldots, r_p$ of S is *good* if it has the property described in Definition 3.1.1.

3.1.5. THEOREM. *Let $R = Z(R) \oplus S$, $\dim S = p$ and let the set of all good bases of S be dense in S^p. Then $B_m(\operatorname{var} R)$ is a submodule of $\sum N_m(\lambda_1, \ldots, \lambda_p)$.*

Proof. Let $f(x_1, \ldots, x_m) \in B_m$ and $u_j = z_j + v_j \in R$, $z_j \in Z(R)$, $v_j \in S$, $j = 1, \ldots, m$. Since we have $\partial f/\partial x_j = 0$ and $z_j \in Z(R)$, it follows that $f(z_1 + v_1, \ldots, z_m + v_m) = f(v_1, \ldots, v_m)$, i.e. $f(x_1, \ldots, x_m)$ is an identity for R if and only if $f(v_1, \ldots, v_m) = 0$ for all $v_j \in S$. The proof is completed by repeating verbatim the arguments of the proof of Theorem 3.1.2.

3.1.6. COROLLARY. *In Theorem* 3.1.5, *the lattice of subvarieties of* $\operatorname{var} R$ *is distributive.*

3.2. The Grassmann algebra

Let V_p be a p-dimensional vector space with a basis $e_1, \ldots, e_p$. The Grassmann (or exterior) algebra $E_p = E(V_p)$ of V_p has a basis $e_{i_1} \ldots e_{i_q}$, $1 \leqslant i_1 < \ldots < i_q \leqslant p$, $q \geqslant 0$, and the multiplication is defined by the associative law and $e_i e_j = -e_j e_i$. The polynomial identities of $E = E_\infty$ have been described in [37] (see also [2]). Here we give an alternative exposition.

3.2.1. THEOREM. (i) *Let $\Gamma_n(\operatorname{var} E)$ be the set of proper multilinear polynomials in $F^{\#}(\operatorname{var} E)$. Then $\Gamma_n(\operatorname{var} E) = M(1^n)$ for n even and $\Gamma_n(\operatorname{var} E) = 0$ for n odd.*
(ii) $P_n(\operatorname{var} E) = \sum_{q=1}^{n} M(q, 1^{n-q})$.

Proof. (i) The algebra E is $\mathbf{Z}_2$-graded, $E = E^0 \oplus E^1$, E^0 and E^1 being spanned by the monomials $e_{i_1} \ldots e_{i_q}$ of even and odd degree, respectively. Since we have $Z(E) = E^0$ and $S = E^1 \ni s = \alpha_1 e_{j_1} + \ldots + \alpha_p e_{j_p}$, $\alpha_i \in E^0$, it follows that $r_{\sigma(1)} \ldots r_{\sigma(n)} = (\operatorname{sign} \sigma) r_1 \ldots r_n$ for arbitrary elements $r_1, \ldots, r_n \in S$. In particular, if $r_i = r_j$, $i \neq j$, then $r_1 \ldots r_n = 0$. For studying the polynomial identities of degree n, without loss of generality we investigate E_p instead of E, p being sufficiently large, and replace the free algebra $K\{X\}^{\#}$ by the free associative algebra $A^{\#} = K\langle X \rangle^{\#}$. In virtue of Theorem 3.1.5, $B_m(\operatorname{var} E_p)$ (and hence $B_m(\operatorname{var} E)$) is a submodule of $\sum N_m(\lambda_1, \ldots, \lambda_m)$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 > 1$. Then the standard generator $f_\lambda(x_1, \ldots, x_m)$ is of degree $\geqslant 2$ in x_1, hence it vanishes on E and therefore $B_m(\operatorname{var} E) \subset \sum N_m(1^n)$. For n even, the standard polynomial $S_n(x_1, \ldots, x_n) \in A_m^{\#}$ is in the space BA_m of proper elements and $S_n(e_1, \ldots, e_n) \neq 0$. For n odd, the only submodule $N_m(1^n)$ of $A_m^{\#}$ is generated by $S_n(x_1, \ldots, x_n)$ which does not belong to BA_m. Hence $B_m(\operatorname{var} E) = \sum N_m(1^{2k})$.

The expression for $\Gamma_n(\operatorname{var} E)$ is obtained immediately, because $\Gamma_n(\operatorname{var} E)$ and $B_m^{(n)}(\operatorname{var} E)$ have the same module structure.

(ii) The assertion follows immediately from (i) and Corollary 2.2.2.

3.3. The Lie algebra sl_2

Let sl_2 be the Lie algebra of all traceless 2×2 matrices with multiplication $[u, v] = uv-vu$. Over an algebraically closed field, sl_2 is isomorphic to the three-dimensional vector space K^3 with the usual vector product

$$(x_1, x_2, x_3)\times(y_1, y_2, y_3) = (x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1).$$

The Killing form of K^3 is proportional to the scalar product $\langle(x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = x_1y_1+x_2y_2+x_3y_3$.

3.3.1. THEOREM [10]. *$F_m(\operatorname{var}\mathrm{sl}_2) = N_m(1)+\sum N_m(\lambda_1, \lambda_2, \lambda_3)$, where the summation runs over all $(\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_2 > 0$ and $\lambda_1 \not\equiv \lambda_2$ or $\lambda_2 \not\equiv \lambda_3 \pmod 2$.*

Proof. We identify the Lie algebras sl_2 and K^3. The basis r_1, r_2, r_3 of K^3 will be *good* if it can be transformed to an orthogonal basis s_1, s_2, s_3 (see Definition 3.1.1 and the very beginning of Section 3.1). Additionally, if $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv \varepsilon \pmod 2$, $\varepsilon = 0,\ 1,$ then $s_1^\varepsilon \times s_2^\varepsilon \times s_3^\varepsilon = 0$ and therefore $f_\lambda(s_1, s_2, s_3) = 0$. In virtue of Theorem 3.1.2, $F_m(\operatorname{var}\mathrm{sl}_2) \subset \sum N_m(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1 \not\equiv \lambda_2$ or $\lambda_2 \not\equiv \lambda_3 \pmod 2$. Since the module $N_m(\lambda_1)$, $\lambda_1 > 1$, does not appear in the decomposition of the free Lie algebra L_m the proof will be completed if we construct standard generators f_λ for all $N_m(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \not\equiv \lambda_2$ or $\lambda_2 \not\equiv \lambda_3 \pmod 2$ and $\lambda_2 > 0$, such that the f_λ are not polynomial identities for sl_2. We refer to [10] for the explicit construction of f_λ.

Actually we have proved that the standard generator $f_\lambda(x_1, x_2, x_3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, is a polynomial identity for sl_2 if and only if $f_\lambda(a_1, a_2, a_3) = 0$ for a basis a_1, a_2, a_3 of sl_2 such that the matrix of the Killing form is diagonal with respect to a_1, a_2, a_3. For example, the following is such a basis:

$$(4)\quad a_1 = -(e_{11}-e_{22})\sqrt{-1}/2, \qquad a_2 = (e_{12}+e_{21})\sqrt{-1}/2, \qquad a_3 = (e_{12}-e_{21})/2$$

with multiplication $a_1a_2 = -a_2a_1 = a_3/2$, $a_2a_3 = -a_3a_2 = a_1/2$, $a_3a_1 = -a_1a_3 = a_2/2$, $a_1^2 = a_2^2 = a_3^2 = -1/4$, hence $[a_1, a_2] = a_3$, $[a_2, a_3] = a_1$, $[a_3, a_1] = a_2$. Therefore we obtain the following consequence.

3.3.2. COROLLARY. *The standard generator $f_\lambda(x_1, x_2, x_3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, is a polynomial identity for sl_2 if and only if $f_\lambda(a_1, a_2, a_3) = 0$, where a_1, a_2, a_3 are defined in (4).*

Razmyslov [52] has discovered a basis for the T-ideal of sl_2 (i.e. a system of generators of the T-ideal) consisting of three identities of degree 5. Later, Filippov [24] has reduced this basis to one identity. Comparing the decompositions

$$P_5(\operatorname{var}\mathrm{sl}_2) = M(4, 1)+M(3, 2)+M(2^2, 1) \quad \text{(see Theorem 3.3.1), and}$$

$$PL_5 = M(4, 1)+M(3, 2)+M(3, 1^2)+M(2^2, 1)+M(2, 1^3)$$

(the multilinear Lie elements of degree 5—this is a consequence of Theorem 2.1.3) we deduce that the Sym(5)-submodule of PL_5 of multilinear identities of degree 5 for sl_2 is isomorphic to $M(3, 1^2)+M(2, 1^3)$. This allows Razmyslov's result to be restated in the following way.

3.3.3. THEOREM [52]. *The elements from the free Lie algebra L*

$$f_{(3,1^2)} = \sum(\operatorname{sign}\sigma)(x_{\sigma(1)}x_{\sigma(2)})(x_{\sigma(3)}x_1)x_1,$$

$$f_{(2,1^3)} = \sum(\operatorname{sign}\sigma)\, x_1x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}$$

form a basis for the polynomial identities of sl_2. (In [52] these polynomials are replaced by the equivalent ones, $(x_1x_2)(x_1x_3)x_1$ and $\sum(\operatorname{sign}\sigma)x_5x_{\sigma(1)}\dots x_{\sigma(4)}$.)

It turns out that almost all identities for sl_2 follow from $(x_1x_2)(x_1x_3)x_1$ and the Lie standard identity gives only the details in lower degrees.

3.3.4. THEOREM [15]. *Let $U \lhd L$ be the T-ideal of all polynomial identities for* sl_2 *and let V be the T-ideal of L generated by* $(x_1x_2)(x_1x_3)x_1$. *Then all homogeneous elements of U of degree ≥ 7 are contained in V and*

$$(U \cap L_m)/(V \cap L_m) = N_m(2, 1^3)+N_m(2^2, 1^2).$$

The proof is based on a result by Nikolaev [45] which asserts that the polynomial identities in three variables for sl_2 follow from $(x_1x_2)(x_1x_3)x_1$, and on the decomposition of $L_m^{(6)}$ and $L_m^{(7)}$ into a sum of irreducible GL_m-modules.

3.4. The 2×2 matrix algebra

The centre of the 2×2 matrix algebra $M_2(K)$ consists of scalar matrices only and, as vector spaces, $M_2(K) = K \oplus \mathrm{sl}_2$. Again the basis r_1, r_2, r_3 of sl_2 will be *good* if it can be transformed to a basis s_1, s_2, s_3 corresponding to a diagonal matrix of the Killing form of sl_2. The proof of the following result makes use of Theorem 3.1.5 and is similar to the proof of Theorem 3.3.1.

3.4.1. THEOREM [10]. $B_m(\operatorname{var} M_2(K)) = K+\sum N_m(\lambda_1, \lambda_2, \lambda_3)$, *where the summation is over all* $\lambda = (\lambda_1, \lambda_2, \lambda_3) \neq (1^3)$ *with* $\lambda_2 > 0$.

As a consequence of Corollary 2.2.2 and Theorem 3.4.1 we obtain immediately

3.4.2. THEOREM [13, 26, 51]. *Let* $F_m(\operatorname{var} M_2(K)) = K+\sum k(\lambda)N_m(\lambda)$, $\lambda = (\lambda_1, \dots, \lambda_r)$. *Then*: (i) $k(\lambda) = 0$ *if* $\lambda_5 \neq 0$; (ii) $k(\lambda_1) = 1$; (iii) $k(\lambda_1, \lambda_2) = (\lambda_1-\lambda_2+1)\lambda_2$ *if* $\lambda_2 > 0$; (iv) $k(\lambda_1, 1, 1, \lambda_4) = (\lambda_1+1)(2-\lambda_4)-1$; (v) $k(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_1-\lambda_2+1)(\lambda_2-\lambda_3+1)(\lambda_3-\lambda_4+1)$ *in all other cases.*

Razmyslov [52] has proved that var $M_2(K)$ can be defined by a system of 9 polynomial identities of degree $\leqslant 6$. In [9] this system has been reduced to 4 identities. Using the fact that Razmyslov's basis for var $M_2(K)$ consists of polynomial identities of degree $\leqslant 6$ (but not the explicit form of the identities) and Theorem 3.4.1, computations in $BA_m^{(5)}$ and $BA_m^{(6)}$ have given a minimal generating set for $T(M_2(K))$.

3.4.3. THEOREM [11]. *The polynomials* $[[x_1, x_2]^2, x_1]$ *and* $S_4(x_1, x_2, x_3, x_4)$ *from the free associative algebra form a basis for the identities of* $M_2(K)$.

Extensively studied objects in ring theory are generic matrix rings and some related rings. Let $\Omega = K[\xi_{ij}^{(r)} \mid i, j = 1, \ldots, k,\ r = 1, 2, \ldots]$ be the polynomial algebra in $\xi_{ij}^{(r)}$. The algebra of $k \times k$ *generic matrices* $R_k(Y)$ is the unitary subalgebra of $M_k(\Omega)$ generated by the matrices $Y = \{y_r = (\xi_{ij}^{(r)}) \mid r = 1, 2, \ldots\}$. It is easy to see that $R_k(Y) \cong F^{\#}(\mathrm{var}\, M_k(K))$. The trace ring $\tilde{R}_k(Y)$ is generated by $R_k(Y)$ and all the traces from $R_k(Y)$. The description of $\tilde{R}_k(y_1, \ldots, y_m)$ as a GL_m-module is given in [26] and explicit computations with $\tilde{R}_2(Y)$ have been done in several papers ([26, 51] etc.). Here we combine the exposition of [51] with the approach of Section 3.1.

3.4.4. THEOREM [51]. *Let* $\tilde{R}_2(Y)$ *be the* 2×2 *trace ring. Then:*

(i) *The following* K*-algebra isomorphism holds:*

$$\tilde{R}_2(Y) \cong K[z_1, z_2, \ldots]^{\#} \otimes_K K\langle X\rangle^{\#}/U,$$

where $U = \{f(x_1, \ldots, x_m) \in K\langle X\rangle^{\#} \mid f(b_1, \ldots, b_m) = 0$ *for all* $b_1, \ldots, b_m \in \mathrm{sl}_2 \subset M_2(K)\}$.

(ii) $K\langle x_1, \ldots, x_m\rangle^{\#}/(K\langle x_1, \ldots, x_m\rangle^{\#} \cap U) = \sum N_m(\lambda_1, \lambda_2, \lambda_3)$, *where the summation is over all partitions* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.

(iii) *As a* GL_m*-module, the centre of* $\tilde{R}_2(y_1, \ldots, y_m)$ *is isomorphic to* $K[z_1, \ldots, z_m]^{\#} \otimes_K \sum N_m(2\mu_1 + \lambda_3, 2\mu_2 + \lambda_3, \lambda_3)$.

Sketch of proof. (i) The Cayley–Hamilton theorem yields that for a 2×2 matrix u,

$$u^2 - (\mathrm{tr}\, u)u + (\mathrm{tr}^2 u - \mathrm{tr}\, u^2)/2 = 0,$$

hence $\mathrm{tr}\, u^2 = 2u^2 - 2(\mathrm{tr}\, u)u + \mathrm{tr}^2 u$. Since $\mathrm{tr}\, uv = \mathrm{tr}\, vu$, the linearization of this equation gives

$$\mathrm{tr}\, uv = uv + vu - (\mathrm{tr}\, u)v - (\mathrm{tr}\, v)u + (\mathrm{tr}\, u)(\mathrm{tr}\, v).$$

Therefore, $\tilde{R}_2(Y)$ is generated by y_i and $\mathrm{tr}\, y_i$, $i = 1, 2, \ldots$ But $y_i = \mathrm{tr}\, y_i/2 + y_i^0$, where y_i^0 is a matrix with zero trace, i.e.

$$\begin{aligned}\tilde{R}_2(Y) &= K\langle y_i^0, \mathrm{tr}\, y_i \mid i = 1, 2, \ldots\rangle^{\#} \\ &\cong K[\mathrm{tr}\, y_i \mid i = 1, 2, \ldots]^{\#} \otimes_K K\langle y_i^0 \mid i = 1, 2, \ldots\rangle^{\#}.\end{aligned}$$

Since the traces of y_i are algebraically independent, $K[\operatorname{tr} y_i \mid i = 1, 2, \dots]^{\#} \cong K[z_1, z_2, \dots]^{\#}$. On the other hand, it is easy to prove that $K\langle y_i^0 \mid i = 1, 2, \dots\rangle^{\#} \cong K\langle X\rangle^{\#}/U$, where U is the set of all weak polynomial identities of the pair $(M_2(K), \mathrm{sl}_2)$ (see [52] for details), i.e.

$$U = \{f(x_1, \dots, x_m) \in K\langle X\rangle^{\#} \mid f(b_1, \dots, b_m) = 0 \text{ for all } b_i \in \mathrm{sl}_2 \subset M_2(K)\}.$$

(ii) Repeating verbatim the arguments of the proof of Theorem 3.3.1 and Corollary 3.3.2 we see that the standard generator $f_\lambda(x_1, x_2, x_3)$ of $A_m/(A_m \cap U)$ is a weak identity for the pair $(M_2(K), \mathrm{sl}_2)$ if and only if $f_\lambda(a_1, a_2, a_3) = 0$, a_1, a_2, a_3 being defined in (4). It follows that $A_m/(A_m \cap U) \subset \sum N_m(\lambda_1, \lambda_2, \lambda_3)$. Since

$$g_\lambda(x_1, x_2, x_3) = S_3^{\lambda_3}(x_1, x_2, x_3) S_2^{\lambda_2 - \lambda_3}(x_1, x_2) x_1^{\lambda_1 - \lambda_2}$$

is a standard generator for $N_m(\lambda_1, \lambda_2, \lambda_3)$ and $g_\lambda(a_1, a_2, a_3) \neq 0$, all $N_m(\lambda_1, \lambda_2, \lambda_3)$ do enter the decomposition of $K\langle y_1^0, \dots, y_m^0\rangle^{\#}$ and this gives the desired result.

(iii) The proof makes use of the fact that, in the notation of (ii), $g_\lambda(a_1, a_2, a_3)$ is a scalar matrix if and only if $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_3$ are even integers.

3.5. The Jordan algebra of a symmetric bilinear form

Let V_p be a vector space of dimension p with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$. Then $G_p = K + V_p$ has a structure of a Jordan algebra with multiplication

$$(\alpha + v)(\beta + w) = (\alpha\beta + \langle v, w\rangle) + (\alpha w + \beta v), \qquad \alpha, \beta \in K,\ v, w \in V_p.$$

For $p > 1$, the G_p form a family of simple Jordan algebras. We call the basis $r_1, \dots, r_p$ of V_p *good* if it can be transformed to an orthogonal basis of V_p. For every orthogonal basis $s_1, \dots, s_p$

$$(s_{j_1} \dots)(\dots s_{j_n}) = \varepsilon s_1^{\delta_1} \dots s_p^{\delta_p},$$

$\varepsilon \in K$, $\delta_i = 0, 1$ and δ_i has the same parity as $\deg_{s_i}(s_{j_1} \dots)(\dots s_{j_n})$. Since the good bases are dense in V_p^p, Theorem 3.1.5 allows us to establish the following result.

3.5.1. THEOREM [16]. (i) *$B_m(\operatorname{var} G_p) = K + \sum N_m(\lambda_1, \dots, \lambda_p)$, where the summation runs over all partitions $(\lambda_1, \dots, \lambda_p)$ such that $\lambda_2 > 0$ and at most one of the integers λ_i is odd.*

(ii) $B_m(\operatorname{var} G_\infty) = [(\sum_{n \geq 0} N_m(2)^{\overset{s}{\otimes} n}) \otimes_K (K + N_m(1))] / \sum_{n \geq 1} N_m(n)$.

(Compare this result with (2)!)

4. Codimensions and Hilbert series

4.1. Reductions

Let N'_m and N''_m be GL_m-submodules of F_m with $N'_m \subset F_m^{(p)}$, $N''_m \subset F_m^{(q)}$, $N'_m \otimes_K N''_m \subset F_m^{(p+q)}$. Assuming $m \geqslant p+q$, define $M' = P_p \cap N'_m$, $M''_m = P_q \cap N''_m$, $M = P_{p+q} \cap N'_m \otimes_K N''_m$. Then the formulas

(5) $\quad H(N'_m \otimes_K N''_m, t_1, \ldots, t_m) = H(N'_m, t_1, \ldots, t_m) H(N''_m, t_1, \ldots, t_m)$,

(6) $\quad \dim M = \dim((M' \otimes_K M'') \uparrow \mathrm{Sym}(p+q)) = \dim M' \cdot \dim M'' \binom{p+q}{p}$

simplify the computing of the Hilbert series of $F_m(U)$ and the codimensions of U. Theorem 2.2.1 (iii) reduces the computing of the Hilbert series of $F_m(U)$ to that of $B_m(U)$. A similar result holds for the codimensions.

4.1.1. Theorem [13, 16, 17]. *Let U be a variety of unitary algebras and let $\gamma_n(U) = \dim \Gamma_n(U)$, $n = 0, 2, 3, \ldots$, be the sequence of the proper codimensions of U. Then $c_n(U)$, $\gamma_n(U)$; $c(U, t) = \sum c_n(U)t^n$, $\gamma(U, t) = \sum \gamma_n(U)t^n$; $\tilde{c}(U, t) = \sum c_n(U)t^n/n!$, $\tilde{\gamma}(U, t) = \sum \gamma_n(U)t^n/n!$ are related by the following equalities*:

(i) $c_n(U) = \sum_{s=0}^{n} \binom{n}{s} \gamma_s(U)$;
(ii) $c(U, t) = \gamma(U, t/(1-t))/(1-t)$;
(iii) $\tilde{c}(U, t) = e^t \tilde{\gamma}(U, t)$.

4.2. Grassmann, matrix and related algebras

The codimension of the Grassmann algebra have been computed in [37]. Here we give an alternative proof.

4.2.1. Theorem [37]. (i) $c_n(E) = 2^{n-1}$, $n \geqslant 1$;
(ii) $c(\mathrm{var}\, E, t) = 1/2 + 1/(2(1-2t))$;
(iii) $\tilde{c}(\mathrm{var}\, E, t) = 1/2 + e^{2t}/2$.

Proof. In virtue of Theorem 3.2.1, $\Gamma_n(\mathrm{var}\, E) = M(1^n)$ for n even and $\Gamma_n(\mathrm{var}\, E) = 0$ for n odd. Since $\dim M(1^n) = 1$, we obtain $\gamma(\mathrm{var}\, E, t) = 1 + t^2 + t^4 + \ldots = 1/(1-t^2)$ and $\tilde{\gamma}(\mathrm{var}\, E, t) = (e^t + e^{-t})/2$ and the result follows immediately from Theorem 4.1.1.

The computing of the codimensions of the $k \times k$ matrices seems to be a very difficult problem. The asymptotic behaviour of $c_n(M_k(K))$ has been established in a series of papers by Regev (see [25, 56] for references). Up till now only for 2×2 matrices has an explicit formula been obtained.

4.2.2. Theorem [51]. (i) $c(\mathrm{var}\, M_2(K), t) = (1 - 2t - (1-4t)^{1/2})/(2t^2) - t^3/(1-t)^4 + 1/(1-t) - 1/(1-2t)$;
(ii) $c_n(M_2(K)) = \binom{2n+2}{n+1}/(n+2) - \binom{n}{3} + 1 - 2^n$;
(iii) [56] $c_n(M_2(K))$ *equals asymptotically* $4^{n+1}/(n(\pi n)^{1/2})$.

Proof. We follow the exposition of [19].

(i) In virtue of Theorem 3.4.1 it suffices to obtain an explicit formula for $g(t) = \sum \dim M(\lambda_1, \lambda_2, \lambda_3)t^{\lambda_1+\lambda_2+\lambda_3}$. By the Littlewood–Richardson rule,

$$\sum N_m(p, p) \otimes_K \sum N_m(q) \cong \sum N_m(\lambda_1, \lambda_2, \lambda_3)$$

and hence, applying (6) as in Theorem 4.1.1 (ii), $g(t) = h(t/(1-t))/(1-t)$, where $h(t) = \sum \dim M(p, p)t^{2p}$. Applying the hook formula for $\dim M(p, p)$ we get $\dim M(p, p) = (2p)!/(p!(p+1)!) = \binom{2p}{p}/(p+1)$ and some calculations show that $h(t) = (1-(1-4t^2)^{1/2})/(2t^2)$, $g(t) = (1-t-(1-2t-3t^2)^{1/2})/(2t^2)$. Finally, we obtain the expression for $c(\operatorname{var} M_2(K), t)$. The assertions (ii) and (iii) are consequences of (i).

Similar considerations allow one to compute the Hilbert series of $F_m(\operatorname{var} M_2(K))$. For $m = 2$ this has been done in [29] (see also [13]). The general case has been handled in [26] (see also [13, 39]). Here we prove the case $m = 2$ only.

4.2.3. Theorem.

$$H(\operatorname{var} M_2(K), t_1, t_2) = (1-t_1)^{-1}(1-t_2)^{-1}(1+t_1t_2(1-t_1t_2)^{-1}(1-t_1)^{-1}(1-t_2)^{-1}).$$

Proof. We make use of Theorem 3.4.1, the Littlewood–Richardson rule and (5):

$$\begin{aligned} H(B_2(\operatorname{var} M_2(K)), t_1, t_2) &= H(\textstyle\sum N_2(\lambda_1, \lambda_2) - \sum_{n \geq 1} N_2(n), t_1, t_2) \\ &= H(\textstyle\sum N_2(p, p) \otimes_K \sum N_2(q) + K - K[x_1, x_2]^*, t_1, t_2) \\ &= (H(\textstyle\sum N_2(p, p), t_1, t_2) - 1)H(K[x_1, x_2]^*, t_1, t_2) + 1 \\ &= (\textstyle\sum (t_1t_2)^p - 1)(1-t_1)^{-1}(1-t_2)^{-1} + 1 \\ &= 1 + t_1t_2(1-t_1t_2)^{-1}(1-t_1)^{-1}(1-t_2)^{-1} \end{aligned}$$

and Theorem 2.2.1 (iii) gives the desired result.

In the same manner one can express the Hilbert series $H(\operatorname{var} \mathrm{sl}_2, t_1, t_2)$ [3, 13].

Developing a complicated technique, including combinatorial methods and analysis (e.g. evaluation of multiple integrals) Regev ([55, 56] and the references there) has established the asymptotic behaviour of the codimensions of the $k \times k$ matrices and of some related algebras. In particular, $c_n(M_k(K))$ equals asymptotically

$$(2\pi)^{(1-k)/2}2^{(1-k^2)/2}1! \ldots (k-1)!\, k^{(k^2+4)/2}n^{(1-k^2)/2}k^{2n+2}$$

(there are monsters not only in the theory of finite simple groups).

Many interesting results have been obtained on the module structure and the Hilbert series of the trace ring $\tilde{R}_k(Y)$ and its centre $\tilde{C}_k(Y)$. It turns out that as a commutative algebra $\tilde{C}_k(Y)$ enjoys a series of interesting properties [28]. In particular,

4.2.4. THEOREM ([39] for $k = 2$, [28, 61]). *$H(\tilde{C}_k(y_1, \ldots, y_m), t_1^{-1}, \ldots, t_m^{-1}) = (-1)^d(t_1 \ldots t_m)^{k^2} H(\tilde{C}_k(y_1, \ldots, y_m), t_1, \ldots, t_m)$, where d is the Krull dimension of $\tilde{C}_k(y_1, \ldots, y_m)$. A similar functional equation holds for $\tilde{R}_k(y_1, \ldots, y_m)$.*

Formanek's proof is based on the investigations of the invariants of $k \times k$ matrices [26]; Teranishi's approach is completely different and applies the Cauchy integral formula to the Molien–Weyl expression for the Hilbert series as a multiple integral. As a consequence Teranishi has evaluated $H(\tilde{C}_k(y_1, y_2), t_1, y_2)$ for $k = 3, 4$ [61, 62].

Kemer [34] has developed the structure theory of the T-ideals in the free associative algebra. An important role in his approach is played by the matrix algebras with entries from the Grassmann algebra. The simplest example is the subalgebra $M_{11} = \begin{bmatrix} E^0 & E^1 \\ E^1 & E^0 \end{bmatrix}$ of $M_2(E)$ consisting of all 2×2 matrices (a_{ij}), $i, j = 1, 2$, such that $a_{11}, a_{22} \in E^0$, $a_{12}, a_{21} \in E^1$ in the canonical grading $E = E^0 + E^1$ (see the proof of Theorem 3.2.1). The polynomial identities of M_{11} are the same as those of the tensor square $E \otimes_K E$. It turns out that in some sense the properties of $E \otimes_K E$ are similar to those of $M_2(K)$. The quantitative results on the polynomial identities of $E \otimes_K E$ obtained by Popov [47] have allowed him to find also a basis for the T-ideal $T(E \otimes_K E)$.

4.2.5. THEOREM [47]. (i) *$\Gamma_n(\operatorname{var} E \otimes_K E) = \sum M(\lambda_1, 2^p, 1^q)$, $n > 0$, where the summation is over all λ_1, p, q such that $\lambda_1 + 2p + q = n$ and $(\lambda_1, 2^p, 1^q) \neq (n), (1^{2k+1})$.*

(ii) *The polynomial identities $[[x_1, x_2], [x_3, x_4], x_5]$ and $[[x_1, x_2]^2, x_1]$ form a basis of the T-ideal $T(E \otimes_K E) \lhd K\langle X \rangle^{\#}$.*

An approach similar to that of Theorem 4.2.2 gives the explicit formula for the codimensions of $E \otimes_K E$.

4.2.6. THEOREM [19]. (i) *$c(\operatorname{var} E \otimes_K E, t) = 1/2 + 1/(2(1-4t)^{1/2}) + t/(1-t)^2 + 1/(1-t) - 1/(1-2t)$;*

(ii) *$c_n(E \otimes_K E) = \binom{2n}{n}/2 + n + 1 - 2^n$, $n > 0$;*

(iii) *$c_n(E \otimes_K E)$ equals asymptotically $4^n/(2(\pi n)^{1/2})$.*

Sketch of proof. It suffices to prove (i) only; then (ii) and (iii) follow immediately. The Littlewood–Richardson rule gives

$$\sum N_m(2^{2p}, 1^{2q}) \otimes_K \sum N_m(n) \cong \sum N_m(\lambda_1, 2^p, 1^q)$$
$$\cong B_m(\operatorname{var} E \otimes_K E) + \sum_{n>0} N_m(n) + \sum_{k>0} N_m(1^{2k+1}).$$

As in the proof of Theorem 4.2.2, explicit computations show that

$$g_1(t) = \sum \dim M(\lambda_1, 2^p, 1^q)t^{\lambda_1+2p+q} = h_1(t/(1-t))/(1-t),$$

where $h_1(t) = \sum \dim M(2^{2p}, 1^{2q})t^{4p+2q}$. The application of the hook formula gives $h_1(t) = (1+(1-4t^2)^{1/2})/2$. Then the proof is completed as in Theorem 4.2.2.

There exist T-ideals of $K\langle X\rangle$ generated by important polynomial identities and which are very close to the T-ideals $T(M_2(K))$ and $T(E\otimes_K E)$. These identities are the standard identity $S_4(x_1, x_2, x_3, x_4)$, the central-by-metabelian identity $[[x_1, x_2], [x_3, x_4], x_5]$ and the Hall identity $[[x_1, x_2]^2, x_3]$ handled by Kemer [33], Popov [48] and Nikolaev [44], respectively. Since these three polynomials belong to ΓA_n, $n = 4, 5$, without loss of generality we consider varieties of unitary algebras.

4.2.7. Theorem. *Let S, C and H be the T-ideals of $K\langle X\rangle^{\#}$ generated by $S_4(x_1, x_2, x_3, x_4)$, $[[x_1, x_2], [x_3, x_4], x_5]$ and $[[x_1, x_2]^2, x_3]$, respectively. Then*:

(i) [33] $(BA_m \cap S)/(BA_m \cap T(M_2(K))) \cong N_m(3, 2)+N_m(3^2)$;

(ii) [48] $(BA_m \cap C)/(BA_m \cap T(E\otimes_K E)) \cong N_m(3, 2)+N_m(3^2)$;

(iii) [44] $(BA_m \cap H)/(BA_m \cap T(M_2(K))) \cong N_m(3, 1^2)+2N_m(2, 1^3)+N_m(3, 1^3)$ $+2N_m(2^2, 1^2)+N_m(2, 1^4)+N_m(2^2, 1^3)+\sum_{k>1} N_m(1^{2k})$.

By the way, in the proofs of (ii) and (iii) the authors have used the decomposition of the Sym(n)-modules ΓA_n for $n \leqslant 8$. As an immediate consequence of Theorems 4.2.7 and 4.1.1 we obtain

4.2.8. Corollary [19]. (i) $c_n(S) = c_n(M_2(K))+5\binom{n}{5}+5\binom{n}{6}$;

(ii) $c_n(C) = c_n(E\otimes_K E)+5\binom{n}{5}+5\binom{n}{6}$;

(iii) $c_n(H) = c_n(M_2(K))+2^{n-1}-1-\binom{n}{2}+14\binom{n}{5}+33\binom{n}{6}+14\binom{n}{7}$, $n > 0$.

The description of $B_m(\operatorname{var} G_\infty)$ for the variety of Jordan algebras generated by G_∞ (see Theorem 3.5.1) gives an interesting formula for the codimensions of G_∞ and G_p. It seems very intriguing that there exists a connection between the asymptotic behaviour of $c_n(G_\infty)$ and the Hermite polynomials (see [16] for details).

4.2.9. Theorem [16]. (i) $c(\operatorname{var} G_\infty, t) = (1+t)\exp(t+t^2/2)+e^t-e^{2t}$; $c_n(G_\infty) = g^{(n+1)}(1)+1-2^n$, *where* $g(t) = \exp((t^2-1)/2)$;

(ii) $c_n(G_\infty) = O((cn)^{n/2})$, *where* $c = \exp(\pi-1)$;

(iii) $\lim_{n\to\infty}(c_n(G_p))^{1/n} = p+1$;

(iv) [18] $c(\operatorname{var} G_2, t) = (t-1+(t+1)/(1-4t^2)^{1/2})/(2t)-t/(1-t)$.

4.3. Products of T-ideals

In this section we deal with unitary associative algebras only. For two algebras P and Q, let $U = T(P)$ and $V = T(Q)$ be the corresponding T-ideals of

$A^{\#} = K\langle X\rangle^{\#}$. For a (P, Q)-bimodule M, it is easy to see that $R = \begin{bmatrix} P & M \\ O & Q \end{bmatrix}$ is a K-algebra and $T(R) \supset W = UV$. Under some additional conditions (see [40, 1, 64]), $T(R) = W$. For example [40] this holds if $P = F^{\#}(U), Q = F^{\#}(V)$, $M = S/(US+SV)$, where $S = \sum_i A^{\#} y_i A^{\#}$ is a free $(A^{\#}, A^{\#})$-bimodule of countable rank. Another example is the algebra $UT_p(K)$ of all $p \times p$ upper-triangular matrices when $T(UT_p(K)) = (T(K))^p = [A^{\#}, A^{\#}]^p$ [42].

We discuss the following problem. How to calculate the numerical invariants of the T-ideal $W = UV$ if we know those of U and V? The following results give the connection between the Hilbert series of the relatively free algebras of U, V and W and their exponential codimension series.

4.3.1. THEOREM [27]. *Let V, U be T-ideals of $A_m^{\#} = K\langle x_1, \dots, x_m\rangle^{\#}$ and let $W = UV$. Then*

$$H(W, t_1, \dots, t_m) = H(U, t_1, \dots, t_m) + H(V, t, \dots, t_m) + (t_1 + \dots + t_m - 1) H(U, t_1, \dots, t_m) H(V, t_1, \dots, t_m). \tag{7}$$

Proof (see e.g. [17]). Since $F_m^{\#}(U) = A_m^{\#}/U$ and

$$H(F_m^{\#}(U), t_1, \dots, t_m) + H(U, t_1, \dots, t_m) = H(A_m^{\#}, t_1, \dots, t_m) = 1/(1-(t_1+\dots+t_m)),$$

the equality (7) is equivalent to

$$H(U, t_1, \dots, t_m) H(V, t_1, \dots, t_m) = H(UV, t_1, \dots, t_m) H(A_m^{\#}, t_1, \dots, t_m).$$

The free algebra $A_m^{\#}$ is a FIR-ring and every homogeneous ideal of $A_m^{\#}$ has a free system of homogeneous generators as a left (or right) $A_m^{\#}$-module. Hence there exist multihomogeneous polynomials $u_1, u_2, \dots$ and $v_1, v_2, \dots$ such that $U = \sum u_i A_m^{\#}$, $V = \sum A_m^{\#} v_j$ and the sums are direct. Therefore

$$H(U, t_1, \dots, t_m) = \sum t^{|u_i|} H(A_m^{\#}, t_1, \dots, t_m),$$

where $t^{|u_i|} = t_1^{p_1} \dots t_m^{p_m}$, $p_s = \deg_{x_s} u_i$, and similarly for $H(V, t_1, \dots, t_m)$. Since $UV = \sum u_i (A_m^{\#})^2 v_j$ and $(A_m^{\#})^2 = A_m^{\#}$, it follows that $UV = \sum u_i A_m^{\#} v_j$, $H(UV, t_1, \dots, t_m) = \sum t^{|u_i|} t^{|v_j|} H(A_m^{\#}, t_1, \dots, t_m)$ and this gives the desired result.

4.3.2. THEOREM [17]. *The exponential codimension series of U, V and $W = UV$ satisfy the equation*

$$\tilde{c}(W, t) = \tilde{c}(U, t) + \tilde{c}(V, t) + (t-1)\tilde{c}(U, t)\tilde{c}(V, t).$$

Proof. Since $c_n(W)$ equals the coefficient of $t_1 \dots t_n$ in $H(A_n^{\#} \cap W, t_1, \dots, t_n)$, Theorem 4.3.1 gives

$$c_n(W) = c_n(U) + c_n(V) + \sum_k \big(n!/(n!\,k!(n-k-1)!)\big) c_k(U) c_{n-k-1}(V) - \sum_k \big(n!/(k!(n-k)!)\big) c_k(U) c_{n-k}(V).$$

Hence

$$\tilde{c}(W, t) = \tilde{c}(U, t) + \tilde{c}(V, t) + (t-1)\sum_n \sum_k c_k(U)c_{n-k}(V)t^n/(k!(n-k)!)$$

and this completes the proof because

$$\sum_n \sum_k c_k(U)c_{n-k}(V)t^n/(k!(n-k)!) = \tilde{c}(U, t)\tilde{c}(V, t).$$

The simplest application of Theorem 4.3.2 is when $W_p = U^p$ and U is the commutator ideal of A^*, i.e. $U = \operatorname{var} K$. In this case the standard notation for W_p is N_pA, W_p is defined by the identity $[x_1, x_2] \ldots [x_{2p-1}, x_{2p}]$ and this variety enjoys many interesting properties [38]. Since $c_n(U) = 1$ and $\tilde{c}(U, t) = e^t$, it is easy to establish the following consequence of Theorem 4.3.2.

4.3.3. Corollary [17]. $\tilde{c}(N_pA, t) = e^t \sum_{k=0}^{p-1}(1+(t-1)e^t)^k$.

Similarly, for the T-ideal $U = T(E)$ Theorem 4.2.1 gives

4.3.4. Corollary [17]. *For the variety $V_p = N_p \operatorname{var} E$ with a T-ideal $V_p = (T(E))^p$ generated by $[x_1, x_2, x_3] \ldots [x_{3p-2}, x_{3p-1}, x_{3p}]$*

$$c(V_p, t) = \sum_{k=0}^{p} f_{kp}(t)e^{2kt},$$

where the $f_{kp}(t)$ are polynomials in t and $f_{pp}(t) = (t-1)^{p-1}/2^p$.

Very often the asymptotic behaviour of the codimension sequence of a T-ideal is more important than the explicit formula. For example, for an associative PI-algebra R Regev [54] has established that $c_n(R) \leqslant d^n$ for a suitable d and this has allowed him to show that a tensor product of PI-algebras is PI again. Another application is due to Kemer [32] who has used the asymptotic behaviour of the cocharacter sequence to give the final form of a result of Razmyslov [53] on the nilpotency of the Jacobson radical of a finitely generated PI-algebra (see Theorem 1.5.1).

An estimate of the asymptotics of the codimension sequence $c_n(U)$ of a variety U is $\limsup (c_n(U))^{1/n}$. Clearly, $1/\limsup (c_n(U))^{1/n}$ equals the radius of convergence of the series $c(U, t)$. It is unknown if $\lim_{n\to\infty}(c_n(U))^{1/n}$ exists for all U but in the few cases when it is explicitly calculated always $\lim_{n\to\infty}(c_n(U))^{1/n}$ does exist and is an integer. We suggest $\lim_{n\to\infty}(c_n(U))^{1/n}$ as a measure of the complexity of U.

4.3.5. Definition [17]. We call the variety U *extremal* if $\lim_{n\to\infty}(c_n(U))^{1/n}$ exists and for any proper subvariety V of U,

$$\lim_{n\to\infty} (c_n(U))^{1/n} > \limsup (c_n(V))^{1/n}.$$

Freely restated, this means that U is more complicated than its subvarieties. The most important examples of extremal varieties are the matrix varieties.

4.3.6. THEOREM [26, 55, 17]. *The varieties of associative algebras* $\operatorname{var} M_p(K)$ *are extremal*: $\lim_{n\to\infty}(c_n(M_p(K)))^{1/n} = p^2$ *and for every proper subvariety* V, $\limsup(c_n(V))^{1/n} \leqslant p^2 - 1$.

It turns out that the varieties of Corollaries 4.3.3 and 4.3.4 are also extremal.

4.3.7. THEOREM [17]. *Let* W_p *and* V_p *be the varieties of associative (not necessarily unitary) algebras defined by the identities* $[x_1, x_2] \ldots [x_{2p-1}, x_{2p}]$ *and* $[x_1, x_2, x_3] \ldots [x_{3p-2}, x_{3p-1}, x_{3p}]$, *respectively. Then* W_p *and* V_p *are extremal*:

(i) $\lim_{n\to\infty}(c_n(W_p))^{1/n} = p$, $\lim_{n\to\infty}(c_n(V_p))^{1/n} = 2p$;

(ii) *For any proper subvarieties* $U_1 \subset W_p$ *and* $U_2 \subset V_p$

$$\limsup(c_n(U_1))^{1/n} \leqslant p-1, \qquad \limsup(c_n(U_2))^{1/n} \leqslant 2p-1.$$

The proof is based on a careful investigation of the cocharacter sequences of the subvarieties of W_p and V_p. In particular, the decomposition of $P_n(W_p)$ into a sum of irreducible $\operatorname{Sym}(n)$-submodules is of the form $P_n(W_p) = \sum k(\lambda)M(\lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_s)$ and λ_{p+1} and s are bounded by constants depending on p only and λ_p is not bounded. On the other hand, for every proper subvariety U_1 of W_p, $P_n(U_1) = \sum k_1(\lambda)M(\lambda)$, where λ_p is bounded by a constant depending on U_1. Then an estimate of the dimensions of the $\operatorname{Sym}(n)$-modules allows one to obtain $\limsup(c_n(U_1))^{1/n} \leqslant p-1$. The bound for $U_2 \subset V_p$ is similar. We refer to [17] for details.

With some modifications the result for W_p holds for Lie algebras as well [17]. In this case W_p has to be replaced by the variety $A^3 \cap N_{p-1}A$ of all Lie algebras which are solvable of class 3 and satisfy the identity $(x_1x_2) \ldots (x_{2p-1}x_{2p})$.

Finally, we shall mention another important example of extremal varieties of Jordan algebras.

4.3.8. THEOREM [36]. *Let* G_p *be the Jordan algebra of a nondegenerate symmetric bilinear form on a vector space of dimension p. Then the varieties* $\operatorname{var} G_p$ *and* $\operatorname{var} G_\infty$ *are extremal*:

(i) *For a proper subvariety* U *of* $\operatorname{var} G_p$

$$\limsup(c_n(U))^{1/n} \leqslant p < \lim_{n\to\infty}(c_n(G_p))^{1/n} = p+1.$$

When U *is a variety of unitary algebras,* $\lim_{n\to\infty}(c_n(U))^{1/n}$ *exists and is an integer.*

(ii) *For a proper subvariety* U *of* $\operatorname{var} G_\infty$

$$\limsup(c_n(U))^{1/n} < \infty = \lim_{n\to\infty}(c_n(G_\infty))^{1/n}.$$

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