# LAPLACIAN OF GRAPHS AND ALGEBRAIC CONNECTIVITY

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## 1. Introduction

In the whole paper, we shall denote by $\mathcal{G}_n$, $n \geqslant 1$ an integer, the class of all nondirected graphs without loops and multiple edges on $n$ vertices. We shall write $G = (V, E)$ where $V$ is the set of vertices and $E$ the set of edges. Usually, the vertices in $V$ will be labelled by $1, \ldots, n$ (for $G \in \mathcal{G}_n$).

Recall that the *Laplacian* of a graph $G = (V, E) \in \mathcal{G}_n$ is the real symmetric matrix $L(G)$ whose quadratic form is given by

$$(1) \qquad (L(G)x, x) = \sum_{\substack{i,k,i<k \\ (i,k)\in E}} (x_i - x_k)^2.$$

Here, and in the sequel, we denote by $(x, y)$ the inner product $\sum_{i=1}^{n} x_i y_i$ of the vectors $x$, $y$ which are considered as (real) column vectors $x = (x_1, \ldots \ldots, x_n)^T$ etc.

$L(G)$ is — since the corresponding quadratic form is a sum of squares — positive-semidefinite and singular since for the vector $e = (1, \ldots, 1)^T$ with $n$ ones,

$$(2) \qquad L(G)e = 0.$$

It is also evident that the Laplacian of the complete graph $K_n \in \mathcal{G}_n$ is the matrix

$$(3) \qquad L(K_n) = nI - J$$

where $I$ is the identity matrix of order $n$ and $J$ is the $n \times n$ matrix whose all entries are ones.

In addition, if $\bar{G}$ is the complement of a graph $G$, then

$$(4) \qquad L(G) + L(\bar{G}) = nI - J.$$

As usual, we shall denote by $A(G)$ the $(0, 1)$-adjacency matrix of $G$. It isobvious that in the case when $G$ is a regular graph of degree $r$,

$$(5) \qquad L(G) = rI - A(G).$$

## 2. Edge and vertex connectivity of graphs

As is well known, the *edge connectivity* $e(G)$ of a finite nondirected graph $G = (V, E)$ is the smallest cardinality of a subset $E_1 \subset E$ having the property that the graph $G_1 = (V, E\backslash E_1)$ is not connected.

Similarly, the *vertex connectivity* $v(G)$ of $G = (V, E)$ is the smallest cardinality of a subset $V_1 \subset V$ having the property that the subgraph $G_1$ generated by $G$ on $V\backslash V_1$ is not connected. In the case of the complete graph $K_n$ on $n$ vertices we set $v(K_n) = n - 1$.

The following important theorem is easily proved:

THEOREM 2.1. *Let* $G = (V, E)$ *be a connected graph,* $V = \{1, \ldots, n\}$. *Let* $A(G) = (a_{ik})$ *be the adjacency matrix of* $G$. *Then*

$$(6) \qquad e(G) = \min_{\substack{W, \varnothing \neq W \subsetneq V}} \sum_{\substack{i \in W \\ k \in V\backslash W}} a_{ik}.$$

*Remark.* Since $L(G) + A(G)$ is a diagonal matrix, we can also write

$$e(G) = \min_{\substack{W, \varnothing \neq W \subsetneq V}} \sum_{\substack{i \in W \\ k \in V\backslash W}} |b_{ik}|$$

where $L(G) = (b_{ik})$.

THEOREM 2.2. *The edge connectivity* $e(G)$ *and vertex connectivity* $v(G)$ *of any graph* $G \in \mathcal{G}_n$ *satisfy*

$$(7) \qquad e(G) \geqslant v(G).$$

*Proof.* This follows from deeper theorems (Whitney, Menger). We shall supply a simple direct proof.

Let $G = (V, E) \in \mathcal{G}_n$. If $G$ is not connected, (7) is fulfilled. Thus let $G$ be connected and let $E_1$ be an edge cut satisfying $|E_1| = e(G)$. By (6), there is a nonvoid proper subset $W$ of $V$ satisfying $(A(G) = (a_{ik}))$

$$e(G) = \sum_{\substack{i \in W \\ k \in V\backslash W}} a_{ik}.$$

Without loss of generality, we can assume that $|W| \leqslant |V\backslash W|$ (otherwise we interchange $W$ and $V\backslash W$). Define $|W| = w$, $e(G) = e$.

*Case* 1: $w = 1$ *and* $|V\backslash W| = e$. Then $e = n - 1$ and, since always $v(G) \leqslant n - 1$, (7) is fulfilled.

In the remaining cases, we shall always find a vertex cut $V_1$ satisfying

(8) $$|V_1| \leqslant e.$$

*Case* 2: $w = 1$ *and* $|V\backslash W| > e$. Let $W = \{z\}$. Then the set $V_1$ of all vertices in $V\backslash W$ adjacent to $z$ satisfies (8).

*Case* 3: $w > 1$ *and* $w > e$. Then the set $V_1$ consisting of those vertices from $W$ which are adjacent to at least one vertex in $V\backslash W$ satisfies (8).

*Case* 4: $w > 1$ *and* $w \leqslant e$. Let $z$ be a vertex in $W$ adjacent to the smallest number of vertices in $V\backslash W$. Let $k$ be this number. If $k = 0$, the set $V_1 = W\backslash\{z\}$ satisfies (8). Let now $k \geqslant 1$. Then, by the definition of $k$,

(9) $$kw \leqslant e.$$

Denote by $U$ the set of those $k$ vertices in $V\backslash W$ which are adjacent to $z$. We shall show that $V_1 = (W\backslash\{z\})\cup U$ satisfies (8).

Clearly, $V_1$ is a vertex cut. Suppose that $|V_1| > e$. Since $|V_1| = w - 1 + k$, it follows from (9) that $w - 1 + k > e \geqslant kw$, i.e. $(w-1)(k-1) < 0$, a contradiction.

In the following theorem, we summarize some properties of $e(G)$ and $v(G)$.

THEOREM 2.3. *Let* $G = (V, E)$, $G_1$, $G_2$ *be graphs. Then:*

(S1)     $e(G) \geqslant 0$, *and* $e(G) = 0$ *if and only if* $G$ *is not connected.*

(S2)     *If* $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ *and* $E_1 \subset E_2$, *then* $e(G_1) \leqslant e(G_2)$.

(S3)     *If* $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $E_1 \subset E$, $E_2 \subset E$ *and* $E_1 \cap E_2 = \varnothing$, *then* $e(G_1) + e(G_2) \leqslant e(G_3)$, *where* $G_3 = (V, E_1 \cup E_2)$.

(S4)     *If* $G_1$ *is obtained from* $G$ *by removing* $k$ *vertices (and incident edges) from* $V$, *then* $v(G_1) \geqslant v(G) - k$.

*Proof.* (S1) and (S2) follow immediately from Thm. 2.1. To prove (S3), let $A(G_s) = (a_{ik}^{(s)})$, $s = 1, 2, 3$, be the adjacency matrices. Since $A(G_3) = A(G_1) + A(G_2)$, we have by Thm. 2.1

$$e(G_3) = \min_{\substack{W \\ i \in W \\ k \in V\backslash W}} \sum a_{ik}^{(3)} = \min_W \left(\sum a_{ik}^{(1)} + \sum a_{ik}^{(2)}\right)$$

$$\geqslant \min_W \sum a_{ik}^{(1)} + \min_W \sum a_{ik}^{(2)} = e(G_1) + e(G_2).$$

(S4) follows immediately from the minimality of $v(G)$.

Let us denote by $K_{p,q}$ the complete (nondirected) bipartite graph with $p + q$ vertices and classes of vertices of size $p$, $q$ respectively. Also, $K_{n,p,q}$ (with $p + q < n$) will denote the graph obtained from the complete graph $K_n$ by removing all $pq$ edges between fixed disjoint subsets $S_1$, $S_2$ of the vertex set of $K_n$ with $p$, $q$ vertices, respectively.

THEOREM 2.4. *We have*

$$v(K_{p,q}) = \min(p, q), \quad e(K_{p,q}) = \min(p, q),$$

$$v(K_{n,p,q}) = n - p - q, \quad e(K_{n,p,q}) = \min(n - p - 1, n - q - 1).$$

*Proof.* Easy.

## 3. Algebraic connectivity

Let $G \in \mathcal{G}_n$, $n \geq 2$. In [4], the second smallest eigenvalue $a(G)$ of the Laplacian $L(G)$ was called the *algebraic connectivity* of the graph $G$.

Almost all the results of Sections 3–5 are based on those of [4].

First, let us formulate a theorem in which $R_n$ is the space of all real column vectors with $n$ coordinates, $(x, y)$ is the inner product $y^T x$ of the vectors $x$, $y$ from $R_n$, $e$ the vector from $R_n$ with all coordinates equal to one and

$$S = \left\{ x = (x_1, \ldots, x_n)^T \in R_n; \ \sum_{i=1}^n x_i = 0, \ \sum_{i=1}^n x_i^2 = 1 \right\}.$$

THEOREM 3.1. *The algebraic connectivity of* $G = (V, E) \in \mathcal{G}_n$ *satisfies*

$$(10) \qquad a(G) = \min_{\substack{x \in S \\ (i,k) \in E \\ i < k}} \sum (x_i - x_k)^2, \quad or$$

$$(11) \qquad a(G) = \min_{x \in S} x^T L(G) x.$$

*Proof.* This follows immediately from the well-known Courant theorem since the smallest eigenvalue of $L(G)$ is zero and the corresponding eigenvector is $e$. Thus $S$ consists of all unit vectors orthogonal to $e$, which means that $a(G)$ as the second smallest eigenvalue is the minimum of $x^T L(G) x$ on $S$.

The following theorem justifies, in view of Thm. 2.2, the term "algebraic connectivity".

THEOREM 3.2. *The algebraic connectivity* $a(G)$ *satisfies the following properties:*

(S'1)   $a(G) \geq 0$, $a(G) = 0$ *if and only if* $G$ *is not connected.*

(S'2)   *If* $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ *and* $E_1 \subset E_2$, *then* $a(G_1) \leq a(G_2)$.

(S'3)   *If* $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ *and* $E_1 \cap E_2 = \varnothing$, *then* $a(G_1) + a(G_2) \leq a(G_3)$ *where* $G_3 = (V, E_1 \cup E_2)$.

(S'4)   *If* $G_1$ *is obtained from* $G$ *by removing* $k$ *vertices (and incident edges), then* $a(G_1) \geq a(G) - k$.

*Proof.* (S'1): The nonnegativity of $a(G)$ being obvious, suppose first that $G$ is not connected; let $G_1 = (V_1, E_1)$ be one its component, let $\hat{G}_1 = (\hat{V}_1, \hat{E}_1)$ be

the graph generated by $G$ on $\hat{V_1} = V\backslash V_1$. Define a vector $y = (y_1, \ldots, y_n)^T$ as follows: for $i \in V_1$, let $y_i = (1/\sqrt{n})\sqrt{\omega_2/\omega_1}$ where $\omega_1 = |V_1|$, $\omega_2 = |\hat{V_1}|$; for $i \in \hat{V_1}$, let $y_i = -(1/\sqrt{n})\sqrt{\omega_1/\omega_2}$. It is easily seen that $y \in S$ and $y^T L(G) y = 0$. Thus $a(G) = 0$.

Conversely, let $a(G) = 0$. Then there exists a vector $y \in S$ such that $y^T L(G) y = 0$. Let $y_k$ be the first nonzero coordinate of $y$; $k$ exists since $y \neq 0$. Let $V_1 = \{i \in \{1, \ldots, n\}; y_i = y_k\}$. Clearly, $\emptyset \neq V_1 \neq V$, and there is no edge in $G$ between any vertex in $V_1$ and any vertex in $V\backslash V_1$ because of $y^T L(G) y = 0$. Thus $G$ is not connected.

(S'2): Follows immediately from (10).

(S'3): Since $E_1 \cap E_2 = \emptyset$, it follows that $L(G_3) = L(G_1) + L(G_2)$. By (11),

$$a(G_3) = \min_{x \in S} \left(x^T L(G_1) x + x^T L(G_2) x\right)$$

$$\geqslant \min_{x \in S} x^T L(G_1) x + \min_{x \in S} x^T L(G_2) x = a(G_1) + a(G_2).$$

(S'4): Let first $k = 1$, $G_1$ arising from $G$ by removing, say, the vertex $n$. Define the graph $\hat{G}$ by completing in $G$ all edges from $n$ to other vertices (if missing). By (S'2),

(12) $$a(\hat{G}) \geqslant a(G).$$

On the other hand, in block form

$$L(\hat{G}) = \begin{bmatrix} L(G_1) + I & -\hat{e} \\ -\hat{e}^T & n-1 \end{bmatrix}$$

where $\hat{e} = (1, \ldots, 1)^T$ with $n-1$ ones.

If $L(G_1)\hat{v} = a(G_1)\hat{v}$, $\hat{v} \in \hat{S}$, then clearly

(13) $$L(\hat{G}) v = (a(G_1) + 1) v$$

for $v = \begin{bmatrix} \hat{v} \\ 0 \end{bmatrix}$.

By (12) and (13), $a(G) \leqslant a(\hat{G}) \leqslant a(G_1) + 1$, so that (S'4) is fulfilled for $k = 1$. The general case follows easily by induction with respect to $k$.

By (3), we obtain

THEOREM 3.3. *For the complete graph* $K_n \in \mathscr{G}_n$, $a(K_n) = n$.

In the following theorem, the *cartesian product* $G_1 \times G_2$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 \times G_2 = (V_1 \times V_2, E)$ where $((u_1, u_2), (v_1, v_2)) \in E$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E_2$, or $(u_1, v_1) \in E_1$ and $u_2 = v_2$.

THEOREM 3.4. *Let* $G_1$, $G_2$ *be graphs. Then* $a(G_1 \times G_2) = \min(a(G_1), a(G_2))$.

*Proof.* Let us order the vertices in $G_1 \times G_2$ lexicographically. One sees easily that

$$L(G_1 \times G_2) = L(G_1) \otimes I_2 + I_1 \otimes L(G_2)$$

where $I_1$, $I_2$ are identity matrices of orders $|V_1|$, $|V_2|$ respectively, and $A \otimes B$, for $B = (b_{ik})$, is the partitioned matrix $(Ab_{ik})$. Therefore, the eigenvalues of $L(G_1 \times G_2)$ have the form $\alpha + \beta$ where $\alpha$ is an eigenvalue of $L(G_1)$ and $\beta$ an eigenvalue of $L(G_2)$. The second smallest eigenvalue $a(G_1 \times G_2)$ is thus the minimum of $a(G_1) + 0$, $0 + a(G_2)$, i.e. $\min(a(G_1), a(G_2))$.

To obtain the value of $a(G)$ for some special graphs, the following lemma will be useful.

LEMMA 3.5. *Let $G \in \mathscr{G}_n$, let $\bar{G}$ be its complement. If $\lambda_1 = 0 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ are the eigenvalues of $L(G)$, then $\lambda'_1 = 0 \leqslant \lambda'_2 \leqslant \ldots \leqslant \lambda'_n$ are the eigenvalues of $L(\bar{G})$ where*

$$(14) \qquad \lambda'_k = n - \lambda_{n+2-k}, \qquad k = 2, \ldots, n.$$

*In addition, the eigenvectors (or eigenspaces) of $L(\bar{G})$ corresponding to $\lambda'_k$ and those of $L(G)$ corresponding to $\lambda_{n+2-k}$ coincide.*

*Proof.* This follows easily from (4).

EXAMPLE 3.6. Let $p$, $n$ be integers, $1 < p \leqslant n$. Let $\hat{K}_{n,p}$ be the graph from $\mathscr{G}_n$ obtained from the complete graph $K_n$ by removing all edges between vertices of a fixed subset with $p$ vertices. We shall show that $L(\hat{K}_{n,p})$ has eigenvalues $0$, $(n-p)_{p-1}$, $(n)_{n-p}$; here and in the sequel $(u)_t$ means the eigenvalue $u$ with multiplicity $t$.

The complement of $\hat{K}_{n,p}$ is the graph $\bar{G}$ from $\mathscr{G}_n$ consisting of the complete graph $K_p$ and $n-p$ isolated vertices. Since $L(\bar{G})$ has eigenvalues $(0)_{n-p+1}$, $(p)_{p-1}$, Lemma 3.5 yields the result.

EXAMPLE 3.7. Let, for $n$, $p$, $q$ positive integers satisfying $p + q < n$, $K_{n,p,q}$ be the graph defined in Section 2. We shall show first that for the complete bipartite graph $K_{p,q} \in \mathscr{G}_{p+q}$ with classes having $p$, $q$ vertices respectively, $L(K_{p,q})$ has eigenvalues $(0)_1$, $(p)_{q-1}$, $(q)_{p-1}$, $(p+q)_1$. This follows again from Lemma 3.5 since the complement $\bar{G}$ of $K_{p,q}$ is the graph from $\mathscr{G}_{p+q}$ with two components $K_p$ and $K_q$ and $L(\bar{G})$ has the spectrum $(0)_2$, $(p)_{p-1}$, $(q)_{q-1}$.

To obtain the spectrum of $L(K_{n,p,q})$, observe that its complement is the graph $\hat{G} \in \mathscr{G}_n$ consisting of $K_{p,q}$ and $n - p - q$ isolated vertices. By the previous result, the spectrum of $L(\hat{G})$ is $(0)_{n-p-q+1}$, $(p)_{q-1}$, $(q)_{p-1}$, $(p+q)_1$. By Lemma 3.5, the spectrum of $L(K_{n,p,q})$ is $(0)_1$, $(n-p-q)_1$, $(n-q)_{p-1}$, $(n-p)_{q-1}$, $(n)_{n-p-q}$.

We can now summarize the results obtained in Examples 3.6 and 3.7.

THEOREM 3.8. *In the notation above, we have*

$$a(K_{p,q}) = \min(p, q) \quad for \quad pq \geqslant 2,$$

$$a(\hat{K}_{n,p}) = n - p,$$

$$a(K_{n,p,q}) = n - p - q.$$

*Remark* 3.9. The eigenspaces corresponding to $a(K_{p,q})$, $a(\hat{K}_{n,p})$ and $a(K_{n,p,q})$ are also easily found using Lemma 3.5. For example, the (unique up to multiplication by a nonzero constant) eigenvector of $L(K_{n,p,q})$ corresponding to $a(K_{n,p,q})$ can be chosen to have all coordinates $q$ on the subset with $p$ vertices, all coordinates $-p$ on the subset with $q$ vertices and all coordinates zero on the remaining $n - p - q$ vertices.

COROLLARY 3.10. *If $G \in \mathcal{G}_n$ contains an independent set of $p \geqslant 2$ vertices (i.e. no two of them are adjacent), then $a(G) \leqslant n - p$.*

*Proof.* This follows from (S'2) of Theorem 3.2 and from Theorem 3.8 since $G$ is contained in the graph $\hat{K}_{n,p}$.

COROLLARY 3.11. *If $G \in \mathcal{G}_n$ is a graph which is not complete, then $a(G) \leqslant n - 2$.*

*Proof.* This follows from Corollary 3.10 for $p = 2$.

THEOREM 3.12. *If $m$ is the minimum valency of a noncomplete graph $G$, then $a(G) \leqslant m$.*

*Proof.* If $G \in \mathcal{G}_n$ is not complete and $w$ is a vertex with valency $m$, then $G$ is contained in the graph $K_{n,1,n-m-1}$ where in the notation of Section 2, $S_1 = \{w\}$ with $p = 1$, $S_2$ consists of all vertices of $G$ which are different from $w$ and are not adjacent to $w$; thus $q = n - m - 1$. By (S'2) of Theorem 3.2 and by Theorem 3.8, $a(G) \leqslant a(K_{n,1,n-m-1}) = m$.

To conclude this section, let us show how algebraic connectivity is related to the singular values of the incidence matrix of the graph.

Let $G = (V, E) \in \mathcal{G}_n$. Let $|E| = m$. Choose an orientation for every edge $(i, k) \in E$; let $\vec{G} = (V, \vec{E})$ be the corresponding directed graph. To $\vec{G}$, we assign the incidence matrix $C = (c_{ik})$ which is an $n \times m$ matrix whose rows correspond to vertices of $\vec{G}$ numbered by $1, \ldots, n$ and whose columns correspond to the directed edges numbered in some way by $1, \ldots, m$. If $e_s \in \vec{E}$, $e_s = (i, j)$, then we set $c_{is} = 1$, $c_{js} = -1$.

It is easily seen that

(15) $$L(G) = CC^T$$

($C^T$ meaning the transposed matrix to $C$), independently of the choice of orientation of edges and of their numbering.

Now, we can also define the $m \times m$ matrix

$$(16) \qquad B(\vec{G}) = C^T C.$$

This matrix which is again symmetric and positive-semidefinite and whose rows and columns correspond to edges of $G$, depends already on the orientation of edges. The change of orientation of one edge results in the multiplication of the corresponding row as well as column of $B$ by $-1$. The change of numeration of edges results, of course, in permuting simultaneously the rows and columns of $B$.

Let us remark that all the off-diagonal entries of $B(\vec{G})$ belong to the set $\{0, 1, -1\}$.

It follows from a well-known theorem from matrix theory that the matrices $CC^T$ and $C^T C$ have the same nonzero eigenvalues including multiplicities. The square roots of these eigenvalues are called *singular values* of the matrix $C$. Thus we obtain:

**Theorem 3.13.** *The algebraic connectivity $a(G)$ of a connected graph $G$ is equal to the smallest positive eigenvalue of the matrix $B(\vec{G})$ as well as to the square of the smallest (positive) singular value of the incidence matrix $C(\vec{G})$ for any orientation $\vec{G}$ of the graph $G$.*

*Remark.* This observation is useful in particular for the case that $G$ is a tree. Then $m = n - 1$ and the matrix $B(\vec{G})$ is positive-definite.

For the sake of completeness, we list in Theorem 3.14 the algebraic connectivities of some special graphs.

**Theorem 3.14.** *Let $P_n \in \mathcal{G}_n$ be the path, $C_n \in \mathcal{G}_n$ the circuit, $L_n \in \mathcal{G}_n$ the ladder ($n$ even), $S_n \in \mathcal{G}_n$ the star, $R_n \in \mathcal{G}_n$ an $m$-dimensional cube ($n = 2^m$), $W_n \in \mathcal{G}_n$ a wheel ($n \geqslant 4$). Then*

$$a(P_n) = 2\left(1 - \cos\frac{\pi}{n}\right), \qquad a(S_n) = 1,$$

$$a(C_n) = 2\left(1 - \cos\frac{2\pi}{n}\right), \qquad a(R_n) = 2,$$

$$a(L_n) = 2\left(1 - \cos\frac{2\pi}{n}\right), \qquad a(W_n) = 1 + 2\left(1 - \cos\frac{2\pi}{n-1}\right).$$

## 4. Relations between $a(G)$, $v(G)$ and $e(G)$

As we know, $e(K_n) = n - 1$, $v(K_n) = n - 1$, $a(K_n) = n$. The relation between $a(G)$ and $v(G)$ in all other cases is given by the following

THEOREM 4.1. *Let* $G$ *be a noncomplete graph. Then* $a(G) \leqslant v(G)$.

*Proof.* Let $G = (V, E)$, and let $V_1$ be a vertex cut with $v(G)$ vertices. Since $G$ is not complete, the graph $G_1$ obtained from $G$ by removing the $v(G)$ vertices from $V_1$ and all incident edges is nonvoid and not connected. By (S'4) of Theorem 3.2, $0 \geqslant a(G) - v(G)$.

COROLLARY 4.2. *Let* $G$ *be a noncomplete graph. Then* $a(G) \leqslant e(G)$.

*Proof.* This follows from Theorems 4.1 and 2.2.

To obtain an estimate of the form $a(G) \geqslant k_n e(G)$ with $k_n$ depending on $n$ only, first recall that an $n \times n$ matrix $A = (a_{ik})$ is called *doubly stochastic* if $A$ is (elementwise) nonnegative and $\sum_i a_{ik} = \sum_k a_{ik} = 1$ for all $i, k = 1, \ldots, n$. For such a matrix $A$, the so-called *measure of irreducibility* $\mu(A)$ was defined in [3] by

$$(17) \qquad \mu(A) = \min_{\substack{M \subset N \\ \emptyset \neq M \neq N}} \sum_{\substack{i \in M \\ k \notin M}} a_{ik}$$

where $N = \{1, \ldots, n\}$.

In the same paper, the following was proved:

THEOREM 4.3 [3]. *Let* $A$ *be a symmetric doubly stochastic matrix with eigenvalues* $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. *Then*

$$(18) \qquad \lambda_1 - \lambda_2 \geqslant 2\left(1 - \cos\frac{\pi}{n}\right)\mu(A),$$

*and the constant* $k_n = 2(1 - \cos(\pi/n))$ *is the best possible.*

*Remark* 4.4. In addition, a simple analysis of the proof yields that if $\mu(A) \neq 0$ then equality in (18) is attained only if the matrix $A$ is—up to simultaneous permutations of rows and columns—a tridiagonal matrix (i.e., $a_{ik} = 0$ if $|i - k| > 1$).

We shall apply Theorem 4.3 and Remark 4.4.

THEOREM 4.5. *For any graph* $G \in \mathcal{G}_n$,

$$(19) \qquad a(G) \geqslant 2\left(1 - \cos\frac{\pi}{n}\right)e(G),$$

*and the constant* $k_n = 2(1 - \cos(\pi/n))$ *is the best possible. The only connected graph in* $\mathcal{G}_n$ *for which equality is attained is the path* $P_n$.

*Proof.* Let $G \in \mathcal{G}_n$. Denote by $M$ the maximum valency among vertices of $G$. The matrix

$$(20) \qquad S = (s_{ij}) = I - \frac{1}{M}L(G)$$

is symmetric, nonnegative and all its row and column sums are equal to one by (2). Thus $S$ is symmetric doubly stochastic. Let $1 = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be its eigenvalues. Then

$$(21) \qquad \lambda_2 = 1 - \frac{1}{M} a(G)$$

by (20).

Further, the measure of irreducibility $\mu(S)$ of $S$ and the edge connectivity $e(G)$ of $G$ are related, by (6) and (17), as follows:

$$\mu(S) = \frac{1}{M} e(G).$$

By (21) and (18),

$$1 - \lambda_2 = \frac{1}{M} a(G) \geqslant 2\left(1 - \cos\frac{\pi}{n}\right) \cdot \frac{1}{M} e(G),$$

which implies (19).

Equality is attained for $P_n$ since $e(P_n) = 1$ and $a(P_n) = 2(1 - \cos(\pi/n))$ by Theorem 3.14. The last assertion follows from Remark 4.4 and the fact that a tridiagonal matrix or its permutation corresponds to a path.

## 5. Decompositions of the set of vertices with respect to $e(G)$ and $a(G)$

Let us return to the edge connectivity $e(G)$ for $G \in \mathscr{G}_n$. As we have seen in (6), there exists a decomposition $V = V_1 \cup V_2$ of the set of vertices $V$ of $G$ for which

$$e(G) = \sum_{\substack{i \in V_1 \\ k \in V_2}} a_{ik}.$$

It follows easily from the minimality of $e(G)$ that—for a connected graph $G$—the subgraphs of $G$ generated by $G$ on $V_1$ as well as on $V_2$ are connected.

In [7], it was proved that a similar result is valid for the algebraic connectivity.

THEOREM 5.1. Let $G = (V, E) \in \mathscr{G}_n$ be a connected graph, and let $u$ be a (real) eigenvector of the Laplacian $L(G)$ corresponding to the second smallest eigenvalue $a(G)$, the algebraic connectivity of $G$. Let $V_1 = \{v_i \in V;\ u_i \geqslant 0\}$ where $u_i$, $i \in \{1, \ldots, n\}$, is the coordinate of $u$ corresponding to the $i$-th vertex in $V$. Then the subgraph $G_1$ of $G$ generated by $G$ on the subset $V_1$ of $V$ is connected.

Remark. The same is, of course, true for the set $V_2$ corresponding to nonpositive coordinates of $u$. If no coordinate of $u$ is zero, we obtain a decomposition $V = V_1 \cup V_2$ similarly to the case of $e(G)$.

## 6. Generalizations

Instead of $\mathscr{G}_n$, we can consider the set of all nondirected complete graphs on $n$ numbered vertices, whose edges are valuated by nonnegative numbers. The previous case is obtained by admitting the numbers zero or one only and by omitting the edges with label zero. If $c_{ik} = c_{ki}$, $i \neq k$, $i, k = 1, \ldots, n$, is the value assigned to the edge $(i, k)$ in such a valuated graph $G_C = (V, E, C)$, we can again define the Laplacian of this graph by

$$x^T L(G_C)x = \sum_{i,k,i<k} c_{ik}(x_i - x_k)^2.$$

As before, we can define the algebraic connectivity $a(G_C)$ as the second smallest eigenvalue of $L(G_C)$, and the edge connectivity $e(G_C)$ as $(N = \{1, \ldots, n\})$

$$\text{(22)} \qquad\qquad e(G_C) = \min_{\substack{\varnothing \neq M \subsetneq N \\ k \notin M}} \sum_{i \in M} c_{ik}.$$

In [5], it was proved that this generalized algebraic connectivity enjoys properties similar to those in Theorem 3.2.

**THEOREM 6.1.** Let $G_C = (V, E, C)$. Then the generalized algebraic connectivity $a(G_C)$ has the following properties:

(i) $a(G_C) \geqslant 0$, and $a(G_C) > 0$ if and only if the graph $\hat{G}_C = (V, E_0)$ where $E_0$ is the set of positively valuated edges of $G_C$, is connected.

(ii) If also $G_D = (V, E, D)$, then $C \leqslant D$ (elementwise) implies $a(G_C) \leqslant a(G_D)$.

(iii) If also $G_D = (V, E, D)$, then for $G_H = (V, E, C+D)$, $a(G_H) \geqslant a(G_C) + a(G_D)$.

(iv) If $\tilde{G}$ is obtained from $G$ as the graph generated by $G$ on a fixed vertex set $V_1$, with the same valuation, then

$$\text{(23)} \qquad\qquad a(\tilde{G}_C) \geqslant a(G_C) - \max_{i \in V_1} \sum_{k \in V \setminus V_1} c_{ik}.$$

*Remark* 6.2. An analogous statement holds for the generalized edge connectivity $e(G_C)$. More precisely, (i)–(iii) hold if we substitute $e(G_C)$ etc. for $a(G_C)$ etc. In (iv), the following holds instead of (23):

$$\text{(24)} \qquad\qquad e(\tilde{G}_C) \geqslant e(G_C) - \sum_{\substack{i \in V_1 \\ k \in V \setminus V_1}} c_{ik}.$$

In [5], a relation between $e(G_C)$ and $a(G_C)$ analogous to (19) was proved:

**THEOREM 6.3.** Let $G_C = (V, E, C)$. Then

$$\text{(25)} \qquad\qquad 2\left(1 - \cos\frac{\pi}{n}\right) e(G_C) \leqslant a(G_C) \leqslant \frac{n}{n-1} e(G_C).$$

The constants on both sides are the best possible.

For a nondirected graph $G = (V, E)$, denote by $\mathscr{C}(G)$ the class of all nonnegative valuations of $G$ with average value on edges equal to one:

$$\mathscr{C}(G) = \left\{ C \geqslant 0; \; c_{ik} = c_{ki}, \; c_{ik} = 0 \text{ if } (i, k) \notin E, \; \sum_{\substack{i,k=1 \\ i<k}}^{n} c_{ik} = |E| \right\}.$$

We can then define the "absolute" edge and algebraic connectivity of a graph $G = (V, E)$ as the numbers

(26)                         $\displaystyle \hat{e}(G) = \max_{C \in \mathscr{C}(G)} e(G_C),$

(27)                         $\displaystyle \hat{a}(G) = \max_{C \in \mathscr{C}(G)} a(G_C).$

Since the previous $(0, 1)$-valuation of $G$ belongs to $\mathscr{C}(G)$, we always have $\hat{e}(G) \geqslant e(G)$, $\hat{a}(G) \geqslant a(G)$.

On the other hand, if $G = (V, E)$ is connected, $|V| = n$, $|E| = m$ and if $T$ is a skeleton in $G$, then the valuation $C$ which assigns to every edge in $T$ the value $m/(n-1)$, and zero to any other edge, belongs to $\mathscr{C}(G)$ and $e(G_C) = m/(n-1)$. Therefore,

(28)                         $\displaystyle \hat{e}(G) \geqslant \frac{m}{n-1}.$

Further, for any $C \in \mathscr{C}(G)$,

$$e(G_C) \leqslant \min_i \sum_k c_{ik} \leqslant \frac{1}{n} \sum_{i,k} c_{ik} = \frac{2m}{n}.$$

Therefore $\hat{e}(G) \leqslant 2m/n$.

Let us summarize:

**THEOREM 6.4.** *Let $G$ be a connected nondirected graph on $n$ vertices with $m$ edges. Then*

(29)                         $\displaystyle \frac{m}{n-1} \leqslant \hat{e}(G) \leqslant \frac{2m}{n}.$

**COROLLARY 6.5.** *For $G \in \mathscr{G}_n$ connected, $1 \leqslant \hat{e}(G) \leqslant n-1$.*

*Proof.* This follows from the estimates $n-1 \leqslant m \leqslant \frac{1}{2}n(n-1)$.

There is an interesting connection with hamiltonian graphs.

**THEOREM 6.6.** *If $G \in \mathscr{G}_n$ with $m$ edges is hamiltonian, then $\hat{e}(G) = 2m/n$. The converse is not true.*

*Proof.* Let $G = (V, E)$ be hamiltonian, $|V| = n$, $|E| = m$. Choose a valuation $C$ in such a way that all edges in the hamiltonian circuit have value $m/n$ and all other edges have value zero. Clearly $C \in \mathscr{C}(G)$ and $e(G_C) = 2m/n$.

By (26) and (29), equality is attained.

To show that the converse is not true, consider the well-known Petersen graph. For this graph $G$, the usual constant valuation $C$ yields $e(G_C) = 2m/n$, so that $\hat{e}(G) = 2m/n$ by (29). However, it is well known that $G$ is not hamiltonian.

In the next lemma, we call two edges of a graph $G \in \mathcal{G}_n$ equivalent if there exists an automorphism of $G$ which maps one of them into the other.

LEMMA 6.7. *In a graph* $G$, *among the valuations* $\mathcal{C}(G)$ *for which the maximum generalized edge connectivity in* (26) *is attained, there is always a valuation for which equivalent edges have the same value. The same is true for the maximum generalized algebraic connectivity in* (27).

*Proof.* We shall prove this for the algebraic connectivity; the proof for the edge connectivity is completely analogous. Observe first that if $A$ is an automorphism of $G$ and $C$ a valuation for which $\hat{a}(G) = a(G_C)$, then for $D = AC$ (in a clear sense), we have again

$$(30) \qquad\qquad a(G_D) = a(G_C).$$

Let $M$ be the valuation of $G$ obtained as the arithmetic mean of all valuations $A_i C$ where $A_i$ runs through all, say $N$, automorphisms of $G$. By (30) and Theorem 6.1 (iii)

$$a(G_M) \geqslant \frac{1}{N} \sum_{i=1}^{N} a(G_{A_i C}) = a(G_C).$$

Also, since $M \in \mathcal{C}(G)$, $\hat{a}(G) \geqslant a(G_M)$; thus $\hat{a}(G) = a(G_M)$.

Since for every automorphism $A$ of $G$

$$AM = \frac{1}{N} \sum_{i=1}^{N} AA_i C = \frac{1}{N} \sum_{j=1}^{N} A_j C = M,$$

equivalent edges have the same valuation in $M$.

COROLLARY 6.8. *If* $G \in \mathcal{G}_n$ *is a graph whose any two edges are equivalent, then* $\hat{a}(G) = a(G)$ *and* $\hat{e}(G) = e(G)$.

We shall not study in detail the absolute algebraic connectivity $\hat{a}(G)$; the reader is referred to a separate paper [8]. It is shown there that in the case that $\hat{a}(G) = a(G_C)$ for $C \in \mathcal{C}(G)$ is a simple eigenvalue of $L(G_C)$ corresponding to the eigenvector $y = (y_i)$, there exists a constant $K$ such that

$$|y_i - y_k| = K$$

whenever $(i, k)$ is a positively valuated (in $C$) edge of $G$, and

$$|y_i - y_k| \leqslant K$$

if $(i, k)$ is an edge of $G$ with value zero in $C$. Also, a method for finding the

absolute algebraic connectivity of a tree is given and it is shown that the result is always a rational number.

We conclude with an extremely interesting approach of Y. Colin de Verdière who defined [1] a new invariant $\mu(G)$ of a graph $G$ as the maximum multiplicity of the second eigenvalue of the Laplacian $L(G_C)$ with $C \in \mathscr{C}(G)$ under the condition that this multiplicity satisfies the so-called transversality hypothesis of Arnold (which means, roughly speaking, that one can vary the valuation $C$ corresponding to the maximum multiplicity in such a way that this multiplicity remains the same). Colin de Verdière shows in [1] that $\mu(G) \leqslant 3$ is necessary and sufficient for a graph $G$ to be planar, and $\mu(G) \leqslant 2$ is necessary and sufficient for $G$ to be outerplanar. He also formulates the conjecture that $\mu(G) \geqslant \chi(G) - 1$ where $\chi(G)$ is the chromatic number of $G$. The positive solution of this conjecture would imply the four-colour theorem.

In [1], it is also shown that if $G$ is linear, i.e. a path, then $\mu(G) = 1$. It may be of interest that the author's result in [2] implies the converse: if $\mu(G) = 1$ then $G$ is linear.

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