

**THE PRODUCT SPACE APPROACH IN
THE STATE SPACE THEORY OF
LINEAR TIME-INVARIANT DIFFERENTIAL SYSTEMS
WITH DELAYS IN STATE AND CONTROL VARIABLES**

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1. INTRODUCTION

In the past ten years the use of the product space $M^p = \mathbf{R}^n \times L^p(-h, 0; \mathbf{R}^n)$ ($1 \leq p < \infty$, $n \geq 1$, $h > 0$) has been broadly popularized in the control theory of delay systems of the "retarded type". The object of this paper is to review recent advances which show the usefulness of the product space in the state-space formulation of systems and control problems with delays in the state and control variables.

The first set of results ended the controversy around the alleged limited use of the product space for systems with delays in the state variable. We now know that everything that can be done in the framework of continuous functions can be done in the product space framework. In fact we even can do more.

A second unexpected result showed that systems with delays in the derivative of the state variable (for instance, neutral type) can also be recast in the product space framework.

Finally, a third set of results showed how structural operators can be used to transform a system with delays in the control variable into one without delays in that variable.

No proofs are presented in this paper which is intended as a guided tour to the above-mentioned developments. However, all constructions and pertinent references will be given for further detailed study. This material will also be found with greater details in the forthcoming monograph of Bensoussan–Delfour–Mitter [6].

Notation

\mathbf{R} will denote the field of all real numbers. Given $-\infty \leq a < b \leq +\infty$ and a Banach space X , $L^p(a, b; X)$ will denote the vector space of all equivalence classes of Lebesgue measurable functions $[a, b] \cap \mathbf{R} \rightarrow X$ which are p -integrable ($1 \leq p < \infty$) or essentially bounded ($p = \infty$).

2. HOMOGENEOUS LINEAR TIME-INVARIANT SYSTEMS

We briefly review the available theory in the framework of continuous functions for systems with finite and infinite memory. We describe the development of the same theory in the product space framework for systems with delays in the state variable. Finally, we show how the product space framework can also accommodate systems with delays in the derivative of the state variable; for instance, this includes systems of the neutral type.

2.1. Continuous functions framework

Systems with delays in the state variable are usually modelled by the differential system

$$(2.1) \quad \begin{cases} \frac{dx}{dt}(t) = Lx_t, & t \geq 0, \\ x(\theta) = \varphi(\theta), & -h \leq \theta \leq 0. \end{cases}$$

The solution $x(t)$ at time t is a vector in \mathbf{R}^n , $n \geq 1$. The real number h , $0 \leq h \leq +\infty$, is the length of the memory which is finite when h is finite and infinite when h is $+\infty$. The map L is linear and continuous from $C_0(-h, 0; \mathbf{R}^n)$ into \mathbf{R}^n , where $C_0(-h, 0; \mathbf{R}^n)$ is the Banach space of all bounded continuous functions from $[-h, 0] \cap \mathbf{R}$ into \mathbf{R}^n . The function φ in $C_0(-h, 0; \mathbf{R}^n)$ is the initial function. At each $t \geq 0$, x_t is an element of $C_0(-h, 0; \mathbf{R}^n)$ constructed from φ and $x: [0, \infty[\rightarrow \mathbf{R}^n$ as follows:

$$(2.2) \quad x_t(\theta) = \begin{cases} x(t+\theta), & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta < 0. \end{cases}$$

For systems with finite memory, it is well known (cf. J. K. Hale [38], [39], [40]) that (2.1) has a unique solution x in $C^1(0, \infty; \mathbf{R}^n)$, the space of all continuous functions on $[0, \infty[$ into \mathbf{R}^n with a continuous derivative. At each time $t \geq 0$, the linear map $\varphi \rightarrow x_t$ defines a continuous linear operator $S(t): C_0(-h, 0; \mathbf{R}^n)$ into $C_0(-h, 0; \mathbf{R}^n)$. It is also well known that the family $\{S(t): t \geq 0\}$ forms a strongly continuous semi-

group of transformations of $C_0(-h, 0; \mathbf{R}^n)$ of class C_0 (cf. Hille–Phillips [45]). Its infinitesimal generator A and its domain $\mathcal{D}(A)$ are of the following form :

$$(2.3) \quad (A\varphi)(\theta) = \begin{cases} L\varphi, & \theta = 0, \\ \frac{d\varphi}{d\theta}(\theta), & \theta \neq 0, \end{cases} \quad \mathcal{D}(A) = \left\{ \varphi \in C_0^1(-h, 0; \mathbf{R}^n) : L\varphi = \frac{d\varphi}{d\theta}(0) \right\}.$$

For systems with infinite memory (that is, $h = +\infty$) the above results do not hold since the continuous functions are not uniformly continuous on an unbounded time interval. However, in 1972 Barbu–Grossman [4] showed that the above results can be recovered provided that $C_0(-\infty, 0; \mathbf{R}^n)$ is replaced by $C_l(-\infty, 0; \mathbf{R}^n)$, the space of all bounded continuous functions for which the limit exists at $-\infty$:

$$(2.4) \quad C_l(-\infty, 0; \mathbf{R}^n) = \{ \varphi \in C_0(-\infty, 0; \mathbf{R}^n) : \lim_{\theta \rightarrow -\infty} \varphi(\theta) \text{ exists} \}.$$

So it is possible to write an evolution equation of the form

$$(2.5) \quad \begin{cases} \frac{dx_t}{dt} = Ax_t, & t \geq 0, \\ x_0 = \varphi \in \mathcal{D}(A). \end{cases}$$

For the non-homogeneous Cauchy problem

$$(2.6) \quad \begin{cases} \frac{dx_t}{dt} = Ax_t + f(t), & t \geq 0, \\ x_0 = \varphi \in \mathcal{D}(A), \end{cases}$$

restrictions must be placed on the function $f: [0, \infty[\rightarrow C_0(-h, 0; \mathbf{R}^n)$, for instance

$$f \in L_{loc}^1(0, \infty; \mathcal{D}(A)) \quad \text{or} \quad f \in C^1(0, \infty; C_0(-h, 0; \mathbf{R}^n)).$$

Of course, such restrictions can be relaxed by going to the integral representation of solutions,

$$(2.7) \quad x_t = S(t)\varphi + \int_0^t S(t-s)f(s)ds,$$

which is valid for all φ in $C_0(-h, 0; \mathbf{R}^n)$ and f in $L_{loc}^1(0, \infty; C_0(-h, 0; \mathbf{R}^n))$.

Unfortunately, we cannot naturally deal with the non-homogeneous equation

$$(2.8) \quad \begin{cases} \frac{dx}{dt}(t) = Lx_t + f^0(t), & t \geq 0, \\ x_0 = \varphi \end{cases}$$

in this framework since it would require the substitution of the following function f into equation (3.7):

$$(2.9) \quad \tilde{f}(t)(\theta) = \begin{cases} f^0(t), & \theta = 0, \\ 0, & \theta \neq 0. \end{cases}$$

The function \tilde{f} does not take its values $\tilde{f}(t)$ in $C_0(-h, 0; \mathbf{R}^n)$. This simplest form of the non-homogeneous problem was the basic motivation to search for a larger space which could at least accommodate functions f of the type (2.9).

2.2. Product space framework

The basic idea behind the introduction of the product space $M^p = \mathbf{R}^n \times L^p(-h, 0; \mathbf{R}^n)$, $1 \leq p < \infty$, is the observation that the initial point $x(0) = \varphi(0)$ where the solution x starts can be disconnected from the piece of initial function φ that we need to make sense of the right-hand side of the differential equation (2.1):

$$(2.10) \quad \begin{cases} \frac{dx}{dt}(t) = Lx_t, & t \geq 0, \\ x(0) = \varphi^0, & x(\theta) = \varphi^1(\theta), & \varphi = (\varphi^0, \varphi^1) \in M^p, \end{cases}$$

where now

$$(2.11) \quad x_t(\theta) = \begin{cases} x(t+\theta), & t+\theta \geq 0, \\ \varphi^1(t+\theta), & t+\theta < 0. \end{cases}$$

The difficulty lies in the fact that, in general, the function x_t only belongs to $L^p(-h, 0; \mathbf{R}^n)$ so that Lx_t does not make sense for a continuous linear map

$$(2.12) \quad L: C_0(-h, 0; \mathbf{R}^n) \rightarrow \mathbf{R}^n.$$

In 1969 Borisovič–Turbabin [10] promoted such an approach for systems with finite memory under three hypotheses, α), β) and γ), on the map L . Their hypotheses essentially said that for all $T > 0$ and all functions y in $L^p(-h, T; \mathbf{R}^n)$ the map

$$(2.13) \quad t \rightarrow (\mathcal{L}y)(t) = Ly_t: [0, T] \rightarrow \mathbf{R}^n$$

$(y_t(\theta) = y(t+\theta))$ belongs to $L^1(0, T; \mathbf{R}^n)$ and the map

$$(2.14) \quad y \rightarrow \mathcal{L}y: L^p(-h, T; \mathbf{R}^n) \rightarrow L^1(0, T; \mathbf{R}^n)$$

is linear and continuous.

Remark 2.1. Several authors also independently introduced the product space (cf. M. Artola [1], [2], [3], Coleman–Mizel [14], [15], Delfour–Mitter [34], [35]).

For systems with infinite memory Delfour-Mitter [33] in 1972 considered L 's of the form

$$(2.15) \quad L\varphi = \sum_{i=0}^N A_i \varphi(\theta_i) + \int_{-\infty}^0 A_{01}(\theta) \varphi(\theta) d\theta,$$

where $a > 0$ is a finite real and $\{\theta_i\}_{i=0}^N, N \geq 0$, are a sequence of real numbers such that

$$(2.16) \quad -\infty < -a = \theta_N < \dots < \theta_{i+1} < \theta_i < \dots < \theta_0 = 0;$$

the entries of the matrix function A_{01} were assumed to belong to $L^\infty(-\infty, 0; \mathbf{R}^n)$. In 1974, R. K. Miller [64] studied similar systems with $N = 1$ and the entries of A_{01} in $L^1(-\infty, 0; \mathbf{R}^n)$. In 1975-76, Burns-Herdman [11], [12] used the same hypotheses α), β) and γ) as Borisovič-Turbabin [10] and an additional one δ) to extend the previous results.

For systems with finite or infinite memory the linear map

$$(2.17) \quad \varphi^0, \varphi^1 \rightarrow x(t), x_t: M^p \rightarrow M^p$$

defines a continuous linear operator also denoted $S(t): M^p \rightarrow M^p$. It was also shown that the family $\{S(t): t \geq 0\}$ forms a strongly continuous semigroup of class C_0 . Its infinitesimal generator A and its domain $\mathcal{D}(A)$ become:

$$(2.18) \quad A(\varphi^0, \varphi^1) = (L\varphi^1, D\varphi^1), \quad \mathcal{D}(A) = \{(\varphi^0, \varphi^1): \varphi^1 \in W^{1,p}(-h, 0; \mathbf{R}^n), \varphi^0 = \varphi^1(0)\},$$

where $D\varphi^1$ denotes the derivative of φ^1 and $W^{1,p}(-h, 0; \mathbf{R}^n)$ is the Sobolev space of all functions in $L^p(-h, 0; \mathbf{R}^n)$ with a distributional derivative in $L^p(-h, 0; \mathbf{R}^n)$.

Of course, an integral formula also exists:

$$(2.19) \quad (x(t), x_t) = S(t) (\varphi^0, \varphi^1) + \int_0^t S(t-s) f(s) ds.$$

But now equation (2.19) covers the non-homogeneous problem (2.8) by picking

$$(2.20) \quad f(s) = (f^0(s), 0) \quad \text{in} \quad M^p.$$

For some time it was not quite clear whether the three conditions α), β) and γ) were restrictive or not. Was it more restrictive to use the product space than the space of continuous functions? In 1977 R. Vinter [77] showed that for an arbitrary linear operator L with domain $\mathcal{D}(L)$ such that $W^{1,1}(-h, 0; \mathbf{R}^n) \subset \mathcal{D}(L) \subset C_0(-h, 0; \mathbf{R}^n)$ the operator A is the infinitesimal generator of a C_0 -semigroup on M^2 . This result indicated that for systems with finite memory the three hypotheses of Borisovič-Turbabin

were redundant. At that time R. Vinter and J. Zabczyk raised the question of the characterization of class of such linear maps L . In view of the form of the generator A , the largest family of L 's was the class \mathcal{L} of all continuous linear maps

$$(2.21) \quad L: W^{1,p}(-h, 0; \mathbf{R}^n) \rightarrow \mathbf{R}^n.$$

In 1978, M. Delfour [24] proved that the above conjecture was true not only for systems with finite history but also for the ones with infinite history. In particular, he showed that conditions α), β), γ) and δ) of Burns-Herdman [12] were also redundant.

Given the continuous linear map (2.21), there exist $n \times n$ matrices A_1 and A_2 with entries in $L^q(-h, 0; \mathbf{R})$, $p^{-1} + q^{-1} = 1$, $1 \leq p < \infty$, such that

$$(2.22) \quad L\varphi = \int_{-h}^0 [A_1(\theta)\varphi(\theta) + A_2(\theta)D\varphi(\theta)]d\theta.$$

Formally system (2.8) becomes

$$(2.23)$$

$$\begin{cases} \frac{dx}{dt}(t) = \int_{-h}^0 [A_1(\theta)x(t+\theta) + A_2(\theta)\frac{\partial x}{\partial \theta}(t+\theta)] + f(t), & t \geq 0, \\ x(\theta) = \varphi(\theta), & \theta \in [-h, 0] \cap \mathbf{R}. \end{cases}$$

Then noting that

$$\frac{\partial x}{\partial \theta}(t+\theta) = \frac{\partial x}{\partial t}(t+\theta)$$

and integrating (2.23) from 0 to t we obtain the integral equation

$$(2.24)$$

$$x(t) = \varphi(0) + \int_0^t ds \int_{-h}^0 d\theta [A_1(\theta)x(s+\theta) + A_2(\theta)\frac{dx}{ds}(s+\theta)] + \int_0^t f(s)ds.$$

Change the order of integration in the term in A_2 in order to eliminate the derivative of x :

$$(2.25) \quad \begin{cases} x(t) = \varphi(0) + \int_0^t ds \int_{-h}^0 d\theta A_1(\theta)x(s+\theta) + \int_{-h}^0 d\theta A_2(\theta) [x(t+\theta) - x(\theta)] + \int_0^t ds f(s), \\ x(\theta) = \varphi(\theta), & \theta \in [-h, 0] \cap \mathbf{R}. \end{cases}$$

Using (2.25), φ^0, φ^1 in M^p and f in $L^1_{loc}(0, \infty; \mathbf{R}^n)$, we finally consider the following equation for $x: [0, \infty[\rightarrow \mathbf{R}^n$:

$$(2.26) \quad x(t) = \varphi^0 + \int_{-h}^0 d\theta A_1(\theta) \int_0^t ds \left\{ \begin{array}{l} x(s+\theta), \quad s+\theta \geq 0 \\ \varphi^1(s+\theta), \quad s+\theta < 0 \end{array} \right\} + \\ + \int_{-h}^0 d\theta A_2(\theta) \left\{ \begin{array}{l} x(t+\theta) - \varphi^1(\theta), \quad t+\theta \geq 0 \\ \varphi^1(t+\theta) - \varphi^1(\theta), \quad \text{otherwise} \end{array} \right\} + \int_0^t f(s) ds.$$

The following theorem summarizes the main results for equations (2.23) and (2.26).

THEOREM 2.1. *Let $1 \leq p < \infty$, $(\varphi^0, \varphi^1) \in M^p$, $f \in L^1_{loc}(0, \infty; \mathbf{R}^n)$ and A_1, A_2 be $n \times n$ matrices with entries in $L^q(-h, 0; \mathbf{R})$, $q^{-1} + p^{-1} = 1$.*

(i) *There exists a unique continuous solution x in $C(0, \infty; \mathbf{R}^n)$ to equation (2.26). Moreover, for each $T > 0$, there exists a constant $c(T) > 0$ such that*

$$(2.27) \quad \|x\|_{C(0,T;\mathbf{R}^n)} \leq c(T) (\|(\varphi^0, \varphi^1)\|_{M^p} + \|f\|_{L^1(0,T;\mathbf{R}^n)}).$$

(ii) *If $\varphi \in W^{1,p}(-h, 0; \mathbf{R}^n)$ and $f \in C(0, \infty; \mathbf{R}^n)$, then there exists a unique solution x in $C^1(0, \infty; \mathbf{R}^n)$ to (2.23) which coincides with the solution of (2.26) for $(\varphi^0, \varphi^1) = (\varphi(0), \varphi)$.*

(iii) *The family of linear maps $S(t): M \rightarrow M^p$ defined as*

$$(2.28) \quad S(t) (\varphi^0, \varphi^1) = (x(t), x_t)$$

defines a strongly continuous semigroup of linear bounded transformations of M^p of class C_0 . ■

In general, the solution of (2.26) is not differentiable. Theorem 2.1 has an interesting consequence. For all p , $1 \leq p < \infty$ and all h , $0 < h \leq +\infty$, the injection

$$W^{1,p}(-h, 0; \mathbf{R}^n) \rightarrow C_0(-h, 0; \mathbf{R}^n)$$

is linear and continuous. So the restriction to $W^{1,p}(-h, 0; \mathbf{R}^n)$ of any continuous linear map $L: C_0(-h, 0; \mathbf{R}^n) \rightarrow \mathbf{R}^n$ is continuous on $W^{1,p}(-h, 0; \mathbf{R}^n)$ and Theorem 2.1 applies. As a result for systems with both finite or infinite memories the three hypotheses of Borisovič–Turbabin [10] and the four hypotheses of Burns–Herdman [12] are redundant.

The situation where $p = 2$ (M^2 is a Hilbert space) is of special interest for the study of stability in the style of R. Datko [16], the characterization of the adjoint semigroup and the linear quadratic optimal control problem.

2.3. Extension of the product space approach to delay in the derivative (neutral type)

In the previous section we have seen that the theory covers some systems with delays in the derivative of x ,

$$(2.29) \quad \frac{dx}{dt}(t) = \int_{-h}^0 \left[A_1(\theta)x(t+\theta) + A_2(\theta)\frac{dx}{dt}(t+\theta) \right] d\theta.$$

However, it falls short of dealing with systems of the form

$$(2.30) \quad \frac{dx}{dt}(t) = \sum_{i=1}^N C_i \frac{dx}{dt}(t+\theta_i) + Lx_t.$$

In the framework of continuous functions, equation (2.30) is rewritten as follows:

$$(2.31) \quad \frac{d}{dt} \left[x(t) - \sum_{i=1}^N C_i x(t+\theta_i) \right] = Lx_t.$$

Given two continuous linear maps

$$L \text{ and } M: C_0(-h, 0; \mathbf{R}^n) \rightarrow \mathbf{R}^n,$$

a general model for (2.31) would be

$$(2.32) \quad \frac{d}{dt}(Mx_t) = Lx_t, \quad t \geq 0, \quad x_0 = \varphi,$$

with

$$(2.33) \quad M\varphi = \varphi(0) - \sum_{i=1}^N C_i \varphi(\theta_i),$$

for example (2.31). For a complete theory with initial function φ in $C_0(-h, 0; \mathbf{R}^n)$ and h finite see J. K. Hale [40].

In 1979 Burns-Herdman-Stech [13] found the way to recast this problem in the product space framework. An easy way to understand their construction is to introduce a new variable $y: [0, \infty[\rightarrow \mathbf{R}^n$ and break equation (2.32) into two:

$$(2.34) \quad \begin{cases} Mx_t = y(t), & t \geq 0, \\ \frac{dy}{dt}(t) = Lx_t, & t \geq 0 \end{cases}$$

with initial conditions

$$(2.35) \quad y(0) = \varphi^0, \quad x_0 = \varphi^1.$$

THEOREM 2.2. *Let L and M be continuous linear maps from $C_0(-h, 0; \mathbf{R}^n)$ into \mathbf{R}^n . Assume that M is atomic at 0 (cf. J. K. Hale [40]).*

(i) *There exist a unique pair y, x in $C(0, \infty; \mathbf{R}^n) \times L^p_{loc}(0, \infty; \mathbf{R}^n)$, solution to system (2.34)–(2.35). Moreover for all $T > 0$, there exists a constant $c(T)$ such that*

$$(2.36) \quad \|y\|_{C_0(0,T;\mathbf{R}^n)} + \|x\|_{L^p(0,T;\mathbf{R}^n)} \leq c(T) \|(\varphi^0, \varphi^1)\|_{M^p}.$$

(ii) *The family of linear maps $\{S(t): M^p \rightarrow M^p: 0 \leq t\}$ defined as*

$$(2.37) \quad S(t) (\varphi^0, \varphi^1) = (y(t), x_t)$$

defines a strongly continuous semigroup of linear bounded transformations of class C_0 on M^p . Its infinitesimal generator A and its domain $\mathcal{D}(A)$ are given by

$$(2.38) \quad \begin{aligned} A(\varphi^0, \varphi^1) &= (L\varphi^1, D\varphi^1), \\ \mathcal{D}(A) &= \{(\varphi^0, \varphi^1): \varphi^1 \in W^{1,p}(-h, 0; \mathbf{R}^n), \varphi^0 = M\varphi^1\}. \blacksquare \end{aligned}$$

In the situation of Theorem 2.2 the integral formula still holds:

$$(2.39) \quad (y(t), x_t) = S(t) (\varphi^0, \varphi^1) + \int_0^t S(t-s)f(s)ds.$$

So non-homogeneous problems of the form

$$(2.40) \quad \begin{cases} Mx_t = y(t), \\ \frac{dy}{dt}(t) = Lx_t + f^0(t) \end{cases}$$

are included in (2.39) by picking $f(t) = (f^0(t), 0)$. However, it is not clear whether non-homogeneous problems of the form

$$(2.41) \quad \begin{cases} Mx_t = y(t) + g(t), \\ \frac{dy}{dt}(t) = Lx_t + f^0(t), \quad (y(0), x_0) = (\varphi^0, \varphi^1) \end{cases}$$

can be accommodated here. It is not too difficult to show that the solution of (2.41) can be written in the form

$$(2.42) \quad (y(t), x_t) = S(t) (\varphi^0, \varphi^1) + \int_0^t S(t-s)(f^0(s), 0)ds + A \int_0^t S(t-s)(g(s), 0)ds.$$

The non-homogeneous problem (2.41) includes as a special case Volterra integral equations of the form

$$(2.43) \quad \begin{cases} x(t) - \int_{-h}^0 A(\theta)x(t+\theta) = g(t), & t \geq 0, \\ x_0 = \varphi^1 \in L^p(-h, 0; \mathbf{R}^n). \end{cases}$$

To see this pick

$$\varphi^0 = 0, \quad L = 0, \quad f^0 = 0 \quad \text{and} \quad M\varphi = \varphi(0) - \int_{-h}^0 A(\theta)\varphi(\theta)d\theta$$

in (2.41). In that special situation (2.42) reduces to

$$(2.44) \quad (0, x_t) = S(t)(0, \varphi^1) + (0, D) \int_0^t S(t-s)(g(s), 0)ds.$$

The non-homogeneous Volterra integral equation has also been studied by O. Dieckmann [37].

3. LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM REVISITED

Consider the control system

$$(3.1) \quad \begin{cases} \frac{dx}{dt}(t) = Lx_t + Bu(t), & t \geq 0, \\ (x(0), v_0) = (\varphi^0, \varphi^1) \in M^2, \end{cases}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ ($m \geq 1$ an integer), B an $n \times m$ matrix and $L: C_0(-h, 0; \mathbf{R}^n) \rightarrow \mathbf{R}^n$ is a continuous linear map.

The state space theory of Section 2 makes it possible to represent the state $\tilde{x}(t) = (x(t), x_t)$ of system (3.1) in the form

$$(3.2) \quad \tilde{x}(t) = S(t)(\varphi^0, \varphi^1) + \int_0^t S(t-s)\tilde{B}u(s)ds, \quad t \geq 0,$$

where $\tilde{B}: \mathbf{R}^m \rightarrow M^2$ is defined as $\tilde{B}v = (Bv, 0)$. For smooth initial conditions (3.2) is equivalent to the evolution equation

$$(3.2a) \quad \frac{d\tilde{x}}{dt}(t) = A\tilde{x}(t) + \tilde{B}u(t), \quad \tilde{x}(0) = \varphi \in \mathcal{D}(A).$$

Associate with the solution of (3.1) the quadratic cost function

$$(3.3) \quad J(u, \varphi) = \int_0^T [Qx(t) \cdot x(t) + Nu(t) \cdot u(t)]dt,$$

where “.” denotes the inner product in \mathbf{R}^n or \mathbf{R}^m and Q (resp. N) is an $n \times n$ (resp. $m \times m$) positive semi-definite (resp. definite) matrix. The cost function (3.3) can be rewritten in terms of $\tilde{w}(t)$ by introducing the new operator $\tilde{Q}: M^2 \rightarrow M^2$ defined as

$$(3.4) \quad \tilde{Q}(\varphi^0, \varphi^1) = (Q\varphi^0, 0).$$

Finally

$$(3.5) \quad J(u, \varphi) = \int_0^T [(\tilde{Q}\tilde{w}(t), \tilde{w}(t)) + Nu(t) \cdot u(t)] dt.$$

So the minimization problem

$$(3.6) \quad \text{Inf}\{J(u, \varphi): u \in L^2(0, T; \mathbf{R}^m)\}$$

is the same for (3.3) subject to (3.1) that for (3.5) subject to (3.2).

It is well known that the optimal control u^* is fully characterized by the following optimality system:

$$(3.7) \quad u^*(t) = -N^{-1}\tilde{B}^*p(t),$$

where p is the adjoint variable

$$(3.8) \quad p(t) = \int_t^T S(r-t)^* \tilde{Q}\tilde{w}(r) dr, \quad 0 \leq t \leq T,$$

or equivalently

$$(3.8a) \quad \frac{dp}{dt} + A^*p + \tilde{Q}\tilde{w} = 0, \quad p(T) = 0.$$

The optimality system (3.2) with $u = u^*$, (3.7) and (3.8) can be decoupled. There exists a family $\{P(t): 0 \leq t \leq T\}$ of continuous linear transformations of M^2 such that

$$(3.9) \quad p(t) = P(t)\tilde{w}(t),$$

and P is the solution of the operator Riccati differential equation

$$(3.10) \quad \begin{cases} \frac{dP}{dt} + A^*P + PA - P\tilde{R}P + \tilde{Q} = 0 & \text{in } \mathcal{L}(\mathcal{D}(A), \mathcal{D}(A)'), \\ P(T) = 0, \quad \tilde{R} = \tilde{B}N^{-1}\tilde{B}^*. \end{cases}$$

The above theory is quite standard, but it can be further improved by introducing the structural operator (Section 3.1). This approach will be the key to the transformation of the systems with delays in control into systems without delays in control (Section 4).

3.1. Structural operators

The right-hand side of equation (3.1) can be rewritten in order to explicitly show its dependence on the initial function φ^1 and the solution $x(t)$ for $t \geq 0$:

$$(3.11) \quad \begin{cases} \frac{dx}{dt}(t) = L(x\chi_{[0,\infty[})_t + Bu(t) + L(\varphi^1\chi_{[-h,0]})_t, & t \geq 0, \\ x(0) = \varphi^0, \end{cases}$$

where χ_I is the characteristic function of the interval I . For $t > h$, the term in φ^1 is zero; for t in $[0, h]$ it is equal to

$$(3.12) \quad L(\varphi^1\chi_{[-h,0]})_t = \int_{-h}^0 d\eta(\theta)\varphi^1(t+\theta)\chi_{[-h,0]}(t+\theta) = \int_{-h}^{-t} d\eta(\theta)\varphi^1(t+\theta),$$

where η is an $n \times n$ matrix of functions of bounded variation associated with the map L :

$$(3.13) \quad L\varphi = \int_{-h}^0 d\eta(\theta)\varphi(\theta), \quad \forall \varphi \in C_0(-h, 0; \mathbf{R}^n).$$

This suggests the introduction of the function

$$(3.14) \quad (H\varphi^1)(\alpha) = \int_{-h}^{\alpha} d\eta(\theta)\varphi^1(\theta - \alpha), \quad \alpha \in [-h, 0].$$

It can be shown that this induces a continuous linear map $H: L^2(-h, 0; \mathbf{R}^n) \rightarrow L^2(-h, 0; \mathbf{R}^n)$. Finally, the structural operator $F: M^2 \rightarrow M^2$ is defined as the continuous linear map

$$(3.15) \quad F(\varphi^0, \varphi^1) = (\varphi^0, H\varphi^1).$$

With the above definitions, equation (3.11) can be rewritten in the form

$$(3.16) \quad \begin{cases} \frac{dx}{dt}(t) + L(x\chi_{[0,\infty[})_t + Bu(t) + (H\varphi^1)(-t)\chi_{[0,h]}(t), \\ x(0) = \varphi^0. \end{cases}$$

From this it is readily seen that the real initial condition is $F(\varphi^0, \varphi^1) = (\varphi^0, H\varphi^1)$ and that the real state at time t is

$$(3.17) \quad \hat{x}(t) = F\tilde{x}(t) = F(x(t), x_t) = (x(t), Hx_t).$$

So the natural thing to do is to construct an equation for $\hat{x}(t)$.

3.2. Transposed system

Associate with the map L and its representation (3.13), the "transposed" map

$$(3.18) \quad L^T \psi = \int_{-h}^0 d\eta(\theta)^T \psi(\theta), \quad \psi \in C_0(-h, 0; \mathbf{R}^n).$$

The transposed system associated with system (2.10) is given by the following differential equation:

$$(3.19) \quad \begin{cases} \frac{dz}{dt}(t) + L^T z_t, & t \geq 0, \\ (z(0), z_0) = (\psi^0, \psi^1) \in M^2. \end{cases}$$

As in Section 2.2 we can associate with the solution of (3.19) a strongly continuous semigroup of class C_0 :

$$(3.20) \quad \begin{cases} S^T(t)(\psi^0, \psi^1) = (z(t), z_t), \\ A^T(\psi^0, \psi^1) = (L^T \psi^1, D\psi^1), \quad \mathcal{D}(A^T) = \mathcal{D}(A). \end{cases}$$

THEOREM 3.1 (Intertwining theorem).

$$(i) \quad \forall t \geq 0, \quad FS(t) = S^T(t)^* F,$$

$$(ii) \quad F\mathcal{D}(A) \subset \mathcal{D}(A^{T*}), \quad A^{T*}F\varphi = FA\varphi, \quad \forall \varphi \in \mathcal{D}(A),$$

where $S^T(t)^*$ and A^{T*} are the adjoint semigroup and infinitesimal generator associated with $S^T(t)$ and A^T . ■

3.3. Equation for $\hat{x}(t)$ and factorization of $P(t)$

Recall equation (3.2):

$$(3.21) \quad \tilde{x}(t) = (x(t), x_t) = S(t)(\varphi^0, \varphi^1) + \int_0^t S(t-s)\tilde{B}u(s)ds.$$

Apply F to both sides of (3.21) to get $\hat{x}(t)$:

$$(3.22) \quad \hat{x}(t) = FS(t)(\varphi^0, \varphi^1) + \int_0^t FS(t-s)\tilde{B}u(s)ds.$$

Now use the intertwining theorem to obtain

$$(3.23) \quad \hat{x}(t) = S^T(t)^* F\varphi + \int_0^t S^T(t-s)^* F\tilde{B}u(s)ds, \quad \varphi = (\varphi^0, \varphi^1).$$

But $F\tilde{B} = \tilde{B}$ and finally

$$(3.24) \quad \hat{x}(t) = S^T(t)^* F\varphi + \int_0^t S^T(t-s)^* \tilde{B}u(s)ds$$

or in differential form:

$$(3.24a) \quad \frac{d\hat{x}}{dt}(t) = A^T \hat{x}(t) + \tilde{B}u(t), \quad \hat{x}(0) = F\varphi, \quad \varphi \in \mathcal{D}(A).$$

Again the cost function (3.3) can be rewritten in terms of $\hat{x}(t)$

$$(3.25) \quad J(u, \varphi) = \int_0^T [(\tilde{Q}\hat{x}(t), \hat{x}(t)) + Nu(t) \cdot u(t)] dt.$$

The solution u^* of the minimization problem is completely characterized by the new optimality system

$$(3.26) \quad u^*(t) = -N^{-1} \tilde{B}^* q(t),$$

where q is the adjoint variable

$$(3.27) \quad q(t) = \int_t^T S^T(r-t) \tilde{Q} \hat{x}(r) dr, \quad 0 \leq t \leq T,$$

or equivalently,

$$(3.27a) \quad \frac{dq}{dt} + A^T q + \tilde{Q} \hat{x} = 0, \quad q(T) = 0.$$

Again the optimality system in \hat{x} and q can be decoupled by the introduction of the family $\{\Pi(t): 0 \leq t \leq T\}$ of continuous linear transformations of M^2 which are solution of the operator Riccati equation

$$(3.28) \quad \begin{cases} \frac{d\Pi}{dt} + A^T \Pi + \Pi(A^T)^* - \Pi \tilde{R} \Pi + \tilde{Q} = 0 & \text{in } \mathcal{L}(\mathcal{D}(A^T), \mathcal{D}(A^T)'), \\ \Pi(T) = 0. \end{cases}$$

The variables \hat{x} and q are related as follows:

$$(3.29) \quad q(t) = \Pi(t) \hat{x}(t).$$

But from the intertwining theorem:

$$(3.30) \quad P(t) \tilde{x}(t) = p(t) = F^* q(t) = F^* \Pi(t) \hat{x}(t) = F^* \Pi(t) F \tilde{x}(t).$$

In fact using invariant embedding we can show that for all t in $[0, T]$, $P(t) = F^* \Pi(t) F$, providing a factorization of $P(t)$. For additional detail see Delfour–Lee–Manitius [27].

Remark 3.1. More applications of the structural operators can be found in Delfour–Manitius [30], [31]. For results on completeness of eigenfunctions see A. Manitius [59], [60] and on F -controllability and F -observability see A. Manitius [57], [59], [60] and Delfour–Manitius [32]. Additional results on properties of $\text{Im} F$ and $\text{Ker} F$ can be found in Z. Bartosiewicz [5].

**4. LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM
WITH DELAYS IN THE CONTROL VARIABLE**

Consider system (3.1) but now with delays in the control variable u :

$$(4.1) \quad \frac{dx}{dt}(t) = Lx_t + Bu_t, \quad t \geq 0, \quad (x(0), x_0, u_0) = (\varphi^0, \varphi^1, w),$$

where $u_t: [-h, 0] \rightarrow \mathbf{R}^m$ is defined as

$$(4.2) \quad u_t(\theta) = \begin{cases} u(t + \theta), & t + \theta \geq 0, \\ w(t + \theta), & \text{otherwise} \end{cases}$$

and $B: C_0(-h, 0; \mathbf{R}^m) \rightarrow \mathbf{R}^n$ is a continuous linear map. As in Section 3.1 we separate in the right-hand side of (4.1) the terms in φ^1 and w from those in $x(t)$ and $u(t)$ for $t \geq 0$:

$$(4.3) \quad \frac{dx}{dt}(t) = L(x\chi_{[0, \infty[})_t + B(u\chi_{[0, \infty[})_t + L(\varphi^1\chi_{[-h, 0]})_t + B(w\chi_{[-h, 0]})_t.$$

We introduce the structural operators $H: L^2(-h, 0; \mathbf{R}^n) \rightarrow L^2(-h, 0; \mathbf{R}^n)$ and $K: L^2(-h, 0; \mathbf{R}^m) \rightarrow L^2(-h, 0; \mathbf{R}^n)$ defined as follows:

$$(4.4) \quad (H\varphi^1)(a) = \int_{-h}^a d\eta(\theta)\varphi^1(\theta - a), \quad (Kw)(a) = \int_{-h}^a d\beta(\theta)w(\theta - a),$$

where β is an $n \times m$ matrix of functions of bounded variation on $[-h, 0]$ such that

$$(4.5) \quad Bw = \int_{-h}^0 d\beta(\theta)w(\theta), \quad \forall w \in C_0(-h, 0; \mathbf{R}^m).$$

4.1. The transformation of Vinter and Kwong

The above remarks essentially led Vinter-Kwong [79] to the introduction of the state variable

$$(4.6) \quad x(t) = (\hat{x}(t), Hx_t + Ku_t)$$

since the true initial condition to (4.3) is $(\varphi^0, H\varphi^1 + Kw)$:

$$(4.7) \quad \begin{cases} \frac{dx}{dt}(t) = L(x\chi_{[0, \infty[})_t + B(u\chi_{[0, \infty[})_t + (H\varphi^1 + Kw)(-t)\chi_{[0, h]}(t), \\ x(0) = \varphi^0. \end{cases}$$

They studied the special case where B is of the form

$$(4.8) \quad Bw = B_0w(0) + \int_{-h}^0 B_1(\theta)w(\theta)d\theta, \quad \forall w \in C_0(-h, 0; \mathbf{R}^m),$$

where B_0 is an $n \times m$ matrix and B_1 is an $n \times m$ matrix of $L^2(-h, 0; \mathbf{R})$ -functions. They showed that $x(t)$ is given by the integral formula

$$(4.9) \quad \hat{x}(t) = S^T(t)^*(\varphi^0, H\varphi^1 + Kw) + \int_0^t S^T(t-s)^*(B^T)^*u(s)ds,$$

where

$$(4.10) \quad \begin{cases} B^T(\varphi^0, \varphi^1) = B_0^T\varphi^0 + \int_{-h}^0 B_1(\theta)^T\varphi^1(\theta)d\theta, & \forall(\varphi^0, \varphi^1) \in M^2, \\ (B^T)^*v = (B_0v, B_1(\cdot)v) \in M^2. \end{cases}$$

In differential form

$$(4.9a) \quad \frac{d\hat{x}}{dt}(t) = (A^T)^*\hat{x}(t) + (B^T)^*u(t), \quad \hat{x}(0) = (\varphi^0, H\varphi^1 + Kw) \in \mathcal{D}(A^T).$$

Again associate with system (4.1) the cost function (3.3) to be minimized. Transform (3.3) into (3.25) and repeat the analysis of Section 3.3. The minimizing control is completely characterized by the following optimality system:

$$(4.11) \quad u^*(t) = -N^{-1}B^Tq(t),$$

where q is the adjoint variable

$$(4.12) \quad q(t) = \int_t^T S^T(r-t)\tilde{Q}\hat{x}(r)dr, \quad 0 \leq t \leq T$$

and \hat{x} is given by (4.9) with $u = u^*$. There exists a family $\{\Pi(t): 0 \leq t \leq T\}$ of continuous linear transformations of M^2 which are the solution of the operator Riccati differential equation

$$(4.13) \quad \begin{cases} \frac{d\Pi}{dt} + A^T\Pi + \Pi(A^T)^* - \Pi R \Pi + \tilde{Q} = 0 & \text{in } \mathcal{L}(\mathcal{D}(A^{T*}), \mathcal{D}(A^{T*})'), \\ \Pi(T) = 0, & \tilde{R} = (B^T)^*N^{-1}B^T, \end{cases}$$

and for all t

$$(4.14) \quad q(t) = \Pi(t)\hat{x}(t)$$

which defines the feedback law

$$(4.15) \quad u^*(t) = -N^{-1}B^T\Pi(t)\hat{x}(t) = -N^{-1}B^T\Pi(t)(x(t), Hx_t + Ku_t^*).$$

4.2. Extension to pure delays in the control variable

We now go back to system (4.1) but for a general continuous linear map $B: \mathcal{C}_0(-h, 0; \mathbf{R}^m) \rightarrow \mathbf{R}^n$. The transposed B is defined in terms of the representation (4.5) as

$$(4.16) \quad B^T\varphi = \int_{-h}^0 d\beta(\theta)^T\varphi(\theta).$$

All constructions up to equation (4.7) can be repeated. However, the adjoint operator $(B^T)^*$ is no longer a continuous linear map from \mathbf{R}^m into M^2 , but from \mathbf{R}^m into $C_0(-h, 0; \mathbf{R}^n)'$, the topological dual of $C_0(-h, 0; \mathbf{R}^n)$; by restricting B^T to $\mathcal{D}(A^T)$, $(B^T)^*$ can also be viewed as a continuous operator from \mathbf{R}^m into $\mathcal{D}(A^T)'$, the topological dual of $\mathcal{D}(A^T)$. As a result formulas (4.9) and (4.9a) do not make sense and they must be reinterpreted in an appropriate weak sense.

4.2.1. Smoother transposed semigroup. Let V be the domain $\mathcal{D}(A^T)$ endowed with the $H^1(-h, 0; \mathbf{R}^n)$ -topology ($H^m = W^{1,m}$) and consider the semigroup

$$(4.17) \quad S_V^T(t) = S^T(t)|_V: V \rightarrow V;$$

its infinitesimal generator becomes

$$(4.18) \quad \begin{cases} A_V^T(\psi^0, \psi^1) = (L^T \psi^1, D\psi^1), \\ \mathcal{D}(A_V^T) = \{(\psi^0, \psi^1): \psi^1 \in H^1(-h, 0; \mathbf{R}^n), \psi^0 = \psi^1(0), D\psi^1(0) = L\psi^1\}. \end{cases}$$

Let $j: V \rightarrow M^2$, $j(\varphi) = (\varphi(0), \varphi)$ be the injection of V into M^2 . Fix $T > 0$. By construction of \hat{x} , it belongs to $C_0(0, T; M^2)$ and it can be shown that for all ψ in V :

$$(4.19) \quad (j\psi, \hat{x}(t)) = (S^T(t)j\psi, (\varphi^0, H\varphi^1 + Kw)) + \int_0^t B^T S_V^T(t-s) \psi \cdot u(s) ds$$

or

$$(4.20) \quad j^* \hat{x}(t) = S_V^T(t)^* j^* (\varphi^0, H\varphi^1 + Kw) + \int_0^t S_V^T(t-s)^* (B^T)^* u(s) ds.$$

So $j^* \hat{x}$ belongs to $C_0(0, T; V')$. In differential form

$$(4.20a) \quad \begin{cases} \frac{d\hat{x}}{dt}(t) = (A^T)^* \hat{x}(t) + (B^T)^* u(t) \in L^2(0, T; V'), \\ \hat{x}(0) = (\varphi^0, H\varphi^1 + Kw). \end{cases}$$

Remark 4.1. At this juncture it is very tempting to say that the previous techniques for the linear-quadratic optimal control problem apply. This is not true since (4.20) would force us to use the adjoint system

$$q(t) = \int_t^T S_V^T(r-t) \hat{Q} \hat{x}(r) dr$$

which does not make sense since $S_V^T(t)$ is continuous from V to V and $\hat{Q} \hat{x}$ does not belong to $L^2(0, T; V)$.

4.2.2. Isomorphisms and transposition. To get around the difficulties raised in Remark 4.1 we shall use J.-L. Lions' method of transposition and invariant embedding techniques typical of parabolic partial differential equations (cf. J.-L. Lions [56]). In the decoupling process we shall use initial data ξ in V' . So for ξ in V' and f in $L^2(0, T; V')$ we consider

$$(4.21) \quad y(t) = S_V^T(t)^* \xi + \int_0^t S_V^T(t-s)^* f(s) ds.$$

The map A_V^T is continuous from $\mathcal{D}(A_V^T)$ into V . The solution (4.21) defines an isomorphism

$$(4.22) \quad y \rightarrow \frac{dy}{dt} - (A_V^T)^* y, \quad y(0): \mathcal{V}(0, T; V', V') \rightarrow L^2(0, T; V') \times V',$$

where

$$(4.23) \quad \mathcal{V}(0, T; X, Y) = \left\{ y \in C(0, T; X) : \frac{dy}{dt} - (A_V^T)^* y \in L^2(0, T; Y) \right\}.$$

Similarly for the adjoint system, the map

$$(4.24) \quad q \rightarrow - \left(\frac{dq}{dt} + A^T q \right), \quad q(T): \mathcal{V}(0, T; V, V) \rightarrow L^2(0, T; V) \times V$$

is an isomorphism.

In fact for delay systems we get more. Let

$$(4.25) \quad W = \mathbf{R}^n \times H^1(-h, 0; \mathbf{R}^n).$$

Then isomorphism (4.24) can be extended to the following one:

$$(4.26) \quad q \rightarrow - \left(\frac{dq}{dt} + A^T q \right), \quad q(T): \mathcal{V}(0, T; V, W) \rightarrow L^2(0, T; W) \times V.$$

Now by transposition of (4.26) we obtain more information on the smoothness of (4.22). In fact (4.22) becomes an isomorphism

$$(4.27) \quad y \rightarrow \frac{dy}{dt} - (A_V^T)^* y, \quad y(0): \mathcal{V}(0, T; V', V') \cap L^2(0, T; W') \rightarrow L^2(0, T; V') \times V'.$$

Those results have been announced in M. C. Delfour [25].

4.2.3. Solution of the control problem. The product structure of $W = \mathbf{R}^n \times H^1(-h, 0; \mathbf{R}^n)$ is retained by its topological dual $W' = \mathbf{R}^n \times H^1(-h, 0; \mathbf{R}^n)'$. As a result we can define

$$(4.28) \quad \tilde{Q}: W' \rightarrow W, \quad \tilde{Q}(w_0, w_1) = (Qw_0, 0).$$

The cost function (3.3) now takes the form (3.25). So finally the minimizing control u^* is fully characterized by the optimality system:

$$(4.29a) \quad \begin{cases} \frac{dy}{dt} = (A_V^T)^* y + (B^T)^* u^*(t), \\ y(0) = j^*(\varphi^0, H\varphi^1 + Kw) \in V', \end{cases}$$

$$(4.29b) \quad \frac{dq}{dt} + A^T q + \tilde{Q}y = 0, \quad q(T) = 0,$$

$$(4.29c) \quad u^*(t) = -N^{-1} B^T q(t).$$

By the technique of invariant embedding, a family $\{\Pi(t): 0 \leq t \leq T\}$ of continuous linear operator from V' to V can be constructed with the property that

$$(4.30) \quad q(t) = \Pi(t) j^* \hat{x}(t).$$

They are solution of the following operator Riccati equation: for all y in $\mathcal{V}(0, T; V', V') \cap L^2(0, T; W')$, Πy is the unique solution in $\mathcal{V}(0, T; V, W)$ of

$$(4.31) \quad \begin{cases} \frac{d}{dt} (\Pi y) + A^T (\Pi y) - \Pi \tilde{R} (\Pi y) + \tilde{Q} y + \Pi \left(-\frac{dy}{dt} + (A_V^T)^* y \right) = 0, \\ (\Pi y)(T) = 0, \quad R = (B^T)^* N^{-1} B^T. \end{cases}$$

In particular for all h in M^2 , $t \rightarrow \Pi(t)h$ is the unique solution in $\mathcal{V}(0, T; V, W)$ of

$$(4.32) \quad \begin{cases} \frac{d}{dt} \Pi(t)h + A^T \Pi(t)h + \Pi(A^T)^* \Pi(t)h - \Pi \tilde{R} \Pi(t)h + \tilde{Q}h = 0, \\ \Pi(T) = 0. \end{cases}$$

Further details can be found in M. C. Delfour [26].

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