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MORSE THEORY ON SINGULAR SPACES AND LEFSCHETZ THEOREMS

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A typical application of usual Morse theory consists in the proof of the classical Lefschetz theorem on hyperplane sections, cf. [A–F], [Mi]. If one tries to generalize this theorem to the case of singular varieties one is led to consider Morse theory on singular spaces. A general theory of this form is the "stratified Morse theory" which has been developed by M. Goresky and R. MacPherson [G–M 1], [G–M 2]. In special cases it is possible to use a simpler technique, such as Morse theory on manifolds with boundary. On the other hand, this kind of Morse theory can serve to illustrate stratified Morse theory in a special case. This will be discussed in the first five sections.

The power of stratified Morse theory becomes clear in more complicated cases. It is important then to get hold of the "normal Morse data" (see [G–M 1], [G–M 2]). In the last two sections it will be shown that Grothendieck's concept of "local homotopical depth" [Gr] is very useful in order to get information about the homotopy type of normal Morse data in certain cases and to prove a general Lefschetz theorem. It is left to the reader to compare this approach with the technique in [Ha 1].

1. The theorems of Lefschetz and Zariski

Let us begin with the following version of the classical Lefschetz theorem on hyperplane sections:

1.1. *Let X be a closed algebraic subset of the complex projective space $\boldsymbol{P}_m$, let H be a hyperplane of $\boldsymbol{P}_m$, $Y = X \cap H$, $X - Y$ non-singular,* $\dim^-(X-Y) = n$. *Then the pair (X, Y) is $(n-1)$-connected.*

Here

$$\dim^-(X-Y) := \min_{x \in X-Y} \dim_x(X-Y).$$

This theorem can be proved in an elegant way by Morse theory, cf. [A–F], [Mi], § 7.

As we will see, there is a relation to the Zariski theorem on the fundamental group of (the complement of) a projective hypersurface. In a slightly more general form it can be stated as follows:

1.2. *Let Z be a hypersurface in $\boldsymbol{P}_m$ and H be a generic hyperplane. Then $(\boldsymbol{P}_m - Z,\ H - Z)$ is $(m-1)$-connected* (cf. [Z], [H–L 1]).

There is the following generalization of 1.2 in the direction of 1.1:

1.3. *Let X and Z be closed algebraic subsets of $\boldsymbol{P}_m$, let H be a generic hyperplane in $\boldsymbol{P}_m$, $Y = X \cap H$, $X - Z$ non-singular,* $\dim^-(X-Z) = n$. *Then $(X-Z,\ Y-Z)$ is $(n-1)$-connected.*

It is essential to choose H generic, so 1.1 is not just a corollary of 1.3. According to an idea of Deligne [D] one may modify the statement in order to avoid the assumption that H is generic:

1.4. *Let X and Z be closed algebraic subsets of $\boldsymbol{P}_m$, let H be a hyperplane in $\boldsymbol{P}_m$ and V a suitable neighbourhood of H in $\boldsymbol{P}_m$, $X-(H \cup Z)$ non-singular,* $\dim^-(X-(H \cup Z)) = n$. *Then the pair $(X-Z,\ V \cap X - Z)$ is $(n-1)$-connected.*

This theorem has been proved in [H–L 2], it is not difficult to see that 1.1 and 1.3 are consequences. There are more general theorems which were conjectured by Deligne [D], cf. [H–L 2], [G–M 1], [G–M 2], but this type of generalization will not be discussed here.

What happens if one weakens the assumption of non-singularity? One possibility is to replace it by the condition that locally we have a (set-theoretic) complete intersection. A particularly easy case is given when $X - Y$ is already a complete intersection. A Lefschetz theorem for this situation will be proved in Section 3 using Morse theory on manifolds with boundary which will be discussed before in Section 2. Another possibility is to replace the non-singularity assumption more generally by an assumption about local homotopy groups, cf. Sect. 7.

2. Morse theory on manifolds with boundary

Just as usual Morse theory gives information about the homotopy type of differentiable manifolds without boundary one can build up an analogous theory for manifolds with boundary. This has been remarked and explained by several authors, see [T], [M–C], [J–R], [H–L 1], [B], [Ha 2]. Let us briefly recall the results.

Let M be a paracompact C^∞ manifold with boundary, $\varphi\colon M \to \boldsymbol{R}$ a C^∞

function. Let us call φ an *m-function* (cf. [J–R]) if the following conditions are satisfied:

(i) φ has no critical points in ∂M;

(ii) the critical points of φ and $\varphi | \partial M$ are non-degenerate.

The m-functions form an open and dense set within the space of all C^∞ functions with respect to the Whitney C^2-topology.

We assume now that φ is an m-function and $\varphi^{-1}(]-\infty, c])$ is compact for all $c \in \boldsymbol{R}$.

Then we have:

2.1 *The manifold M has the homotopy type of a* CW *complex having a cell of dimension i*

(i) *for every critical point of φ of index i;*

(ii) *for every critical point x of $\varphi | \partial M$ of index i provided that the gradient of φ at x (with respect to some Riemannian metric) points into M.*

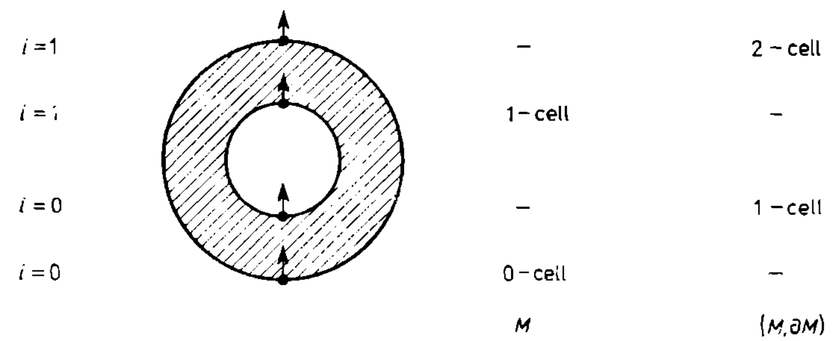
But we will also use another type of Morse theory, namely on manifolds with boundary *modulo the boundary*; here we are interested in the homotopy type of $(M, \partial M)$:

2.2. *The manifold M has the homotopy type of a space which is obtained from ∂M by attaching cells, with a cell of dimension i*

(i) *for every critical point of φ of index i;*

(ii) *for every critical point x of $\varphi | \partial M$ of index $i-1$ provided that the gradient of φ at x points out of M* (cf. [Ha 2], § 1).

A very simple example is given by $M = \{(x, y) \in \boldsymbol{R}^2 | \; 1 \leqslant x^2+y^2 \leqslant 4\}$, $\varphi(x, y) = y$ (cf. also [H–L 1], p. 334). Whereas φ has no critical points the restriction of φ to ∂M has four critical points. In the following drawing let us indicate for every critical point of $\varphi | \partial M$ the index i, the direction of the gradient of φ and the type of cell which has to be attached eventually if we want to study the homotopy type of M and $(M, \partial M)$ in the sense of 2.1 and 2.2, respectively:



See [Ha 2], p. 123, for a more detailed discussion of a different example.

3. A Lefschetz theorem for local complete intersections

In this section the following kind of a Lefschetz theorem will be proved:

THEOREM 3.1. *Let X be a closed algebraic subset of $\boldsymbol{P}_m$, let H be a hyperplane of $\boldsymbol{P}_m$, $Y = X \cap H$,* $\dim^-(X-Y) = n$, *and assume that $X-Y$ is locally a (set-theoretic) complete intersection* (i.e., *any $x \in X-Y$ has an open neighbourhood U in $X-Y$ which can be embedded as a locally closed analytic subset of some $\boldsymbol{C}^{m'}$ such that the image of U is — as a set — the locus of $m'-\dim U$ holomorphic functions*). *Then the pair (X, Y) is $(n-1)$-connected.*

The proof is based on Morse theory on manifolds with boundary. A technical difficulty is the reduction of the global to the local situation. In order to make the central point clear it is useful to prove the following theorem where this reduction is not necessary:

THEOREM 3.2. *Let X be a closed algebraic subset of $\boldsymbol{P}_m$, let H be a hyperplane of $\boldsymbol{P}_m$, $Y = X \cap H$, and let $f_1, \ldots, f_k$ be polynomials such that*

$$X-Y = \{z \in \boldsymbol{C}^m \cong \boldsymbol{P}_m - H \mid f_1(z) = \ldots = f_k(z) = 0\}.$$

Then the pair (X, Y) is $(m-k-1)$-connected.

Note that in the case $m-k = \dim(X-Y)$ we get a special case of Theorem 3.1.

Proof of Theorem 3.2. We may exclude the trivial case $X = \boldsymbol{P}_m$. Let us identify $\boldsymbol{P}_m - H$ with $\boldsymbol{C}^m$ in such a way that we have $0 \notin X-Y$. Let φ_0: $\boldsymbol{P}_m - H \to \boldsymbol{R}$ be defined by $\varphi_0(z) = -(|z_1|^2 + \ldots + |z_m|^2)$, $\varphi = \varphi_0 | X-Y$. Let $\bar{R} \gg 0$, then Y is a deformation retract of $Y \cup \{z \in X-Y \mid -\varphi(z) \geqslant \bar{R}\}$. Let ψ: $\boldsymbol{P}_m - H \to \boldsymbol{R}$ be defined by $\psi(z) = |f_1(z)|^2 + \ldots + |f_k(z)|^2$. Let $\alpha > 0$ be small enough such that the following conditions are satisfied:

(i) $\psi(0) > \alpha$;

(ii) α is a regular value of ψ and of $\psi | \{z \in \boldsymbol{P}_m - H \mid \varphi_0(z) = -\bar{R}\}$;

(iii) $\{z \in X-Y \mid -\varphi(z) \leqslant \bar{R}\}$ is a deformation retract of $\{z \in \boldsymbol{P}_m - H \mid -\varphi_0(z) \leqslant \bar{R},\ \psi(z) \leqslant \alpha\}$;

(iv) $\{z \in X-Y \mid -\varphi(z) = \bar{R}\}$ is a deformation retract of $\{z \in \boldsymbol{P}_m - H \mid -\varphi_0(z) = \bar{R},\ \psi(z) \leqslant \alpha\}$.

Then $M = \{z \in \boldsymbol{P}_m - H \mid \psi(z) \leqslant \alpha\}$ is a C^∞ manifold with boundary. It is sufficient to prove that the pair $(\{z \in M \mid -\varphi_0(z) \leqslant \bar{R}\}, \{z \in M \mid -\varphi_0(z) = \bar{R}\})$ is $(m-k-1)$-connected. Here we use Morse theory as in Section 2, taking an m-function $\tilde{\varphi}_0$ on M which is near enough to $\varphi_0 | M$ such that:

(i) $\tilde{\varphi}_0$ coincides with φ_0 in a neighbourhood of $\{z \in M \mid -\varphi_0(z) = \bar{R}\}$;

(ii) the Levi form of $\tilde{\varphi}_0$ is negative definite everywhere, just as that of φ_0;

(iii) $\tilde{\varphi}_0$ has no critical points, just as $\varphi_0 | M$.

The only critical points to be considered are of the following form: $z \in P_m - H$ with $\psi(z) = \alpha$, $-\tilde{\varphi}_0(z) \leqslant \bar{R}$, $(d\tilde{\varphi}_0)_z = \lambda(d\psi)_z$ with $\lambda < 0$; this means that z is a critical point of $\tilde{\varphi}_0 | \partial M$ where the gradient of $\tilde{\varphi}_0$ at z points into M. Let z be such a point. Now the Hessian H of $\tilde{\varphi}_0 | \partial M$ at z has the following form:

$$H(v) = 2L_{\tilde{\varphi}_0 - \lambda\psi}(v) + 2\operatorname{Re} Q(v)$$

where $L_{\tilde{\varphi}_0 - \lambda\psi}$ is the Levi form of $\tilde{\varphi}_0 - \lambda\psi$ at z, Q a complex quadratic form and $v \in C^m \cong T_z M$ a tangent vector to ∂M at z; cf. [Ha 2], p. 132. Now $L_{\tilde{\varphi}_0}$ is negative definite, and

$$L_\psi(v) = \sum_{j=1}^{k} |(df_j)_z(v)|^2,$$

so $L_{\tilde{\varphi}_0 - \lambda\psi}$ is negative definite on the subspace of C^m defined by

$$(df_1)_z = \ldots = (df_k)_z = 0$$

which is of dimension $\geqslant m-k$. This subspace is contained in the tangent space $\{v \in C^m | \operatorname{Re} \sum_{j=1}^{k} \overline{f_j(z)}(df_j)_z(v) = 0\}$ to ∂M at z. By a well-known argument (cf. [Mi], § 7) the Hessian H has at least $m-k$ negative eigenvalues, so Theorem 3.2 is proved. □

Proof of Theorem 3.1. In contrast to the proof above where $X - Y$ was replaced by a closed neighbourhood M globally we will do this only locally now. We may again assume that $X \neq P_m$ and identify $P_m - H$ with C^m in such a way that $0 \notin X - Y$. Let us fix a complex analytic stratification of X which is Whitney regular [W] and for which $X - Y$ is a union of strata. Let $\bar{R} \gg 0$, then $\{z \in P_m - H | \ \|z\|^2 = \bar{R}\}$ intersects all strata transversally and Y is a deformation retract of $Y \cup \{z \in X - Y | \ \|z\|^2 \geqslant \bar{R}\}$. It is sufficient to prove that the pair $(X - Y, \{z \in X - Y | \ \|z\|^2 \geqslant \bar{R}\})$ is $(n-1)$-connected. We can work now within $P_m - H \cong C^m$. Obviously we may suppose that $X - Y$ is connected. After enlarging m if necessary $X - Y$ is in the neighbourhood of any z with $\|z\|^2 \leqslant \bar{R}$ the locus of $k = m-n$ holomorphic functions.

Let $\varphi_0 \colon P_m - H \to R$ be defined by $\varphi_0(z) = -(|z_1|^2 + \ldots + |z_m|^2)$, $\varphi = \varphi_0 | X - Y$. Now $x \in X - Y$ is called a *critical point* of φ if x is a critical point of the restriction of φ to the stratum which contains x. We approximate φ_0 by a C^∞ function $\tilde{\varphi}_0 \colon P_m - H \to R$ such that:

(i) $\tilde{\varphi}_0$ has no critical point in $X - Y$, just as φ_0;

(ii) the critical points of $\tilde{\varphi} = \tilde{\varphi}_0 | X - Y$ in $\{z \in X - Y | \ -\varphi(z) \leqslant \bar{R}\}$ are isolated;

(iii) $\tilde{\varphi}$ coincides with φ on $\{z \in X - Y | \ -\varphi(z) \geqslant \bar{R}\}$;

(iv) $-\tilde{\varphi}(z) < \bar{R}$ if $-\varphi(z) < \bar{R}$, $z \in X - Y$;

(v) $\tilde{\varphi}_0$ has further properties which will be specified later.

Let $X_r = \{z \in X - Y \mid \tilde{\varphi}(z) \leqslant r\}$. Using controlled vector fields in the sense of Thom and Mather [Ma] we see that X_{r_1} and X_{r_2} have the same homotopy type if $-\bar{R} \leqslant r_1 < r_2$ and $[r_1, r_2]$ contains no critical value of $\tilde{\varphi}$.

Let x be a critical point of $\tilde{\varphi}$ such that $-\tilde{\varphi}(x) \leqslant \bar{R}$, $\gamma = \tilde{\varphi}(x)$. We may choose $\tilde{\varphi}_0$ in such a way that $\tilde{\varphi}^{-1}(\{\gamma\})$ contains no other critical point of $\tilde{\varphi}$. Let ξ: $\boldsymbol{C}^m \to \boldsymbol{R}$ be the distance function from x. Let U be a neighbourhood of x in $\boldsymbol{C}^m$ such that there are holomorphic functions $f_1, \ldots, f_k$ on U with

$$(X-Y) \cap U = \{z \in U \mid f_1(z) = \ldots = f_k(z) = 0\}.$$

Let ψ: $U \to \boldsymbol{R}$ be defined by

$$\psi(z) = |f_1(z)|^2 + \ldots + |f_k(z)|^2.$$

Then we can find a subanalytic stratification of U which satisfies Thom's a_ψ-condition; cf. [T 1], [Hi], § 5. We will consider U always as a stratified space in this sense (this is important for the notion of critical points). Now we may assume that $\tilde{\varphi}_0$ is real analytic in U and that $\tilde{\varphi}_0 | U$ has (with respect to the new stratification) no critical point z different from x (note that we may shrink U if necessary). Let $\varepsilon > 0$ be so small that $\{z \in \boldsymbol{C}^N \mid \xi(z) \leqslant \varepsilon\}$ is contained in U and that $\{z \in \boldsymbol{C}^N \mid \xi(z) = \varepsilon\}$ intersects all strata of U and of $\{z \in U - \{x\} \mid \tilde{\varphi}_0(z) = \gamma\}$ (with respect to the induced stratification) transversally. Now let $\delta > 0$ be so small that $\tilde{\varphi}$ and $\tilde{\varphi}_0 | \{z \in U \mid \xi(z) = \varepsilon\}$ have no critical point except x with value in the interval $[\gamma-\delta, \gamma+\delta]$. Because of the a_ψ-condition we can choose $\alpha > 0$ so small that $\tilde{\varphi}_0 | \{z \in U \mid \xi(z) = \varepsilon, \psi(z) = \alpha\}$ has no critical point with value in $[\gamma-\delta, \gamma+\delta]$, further conditions on α will be specified later on. Because of the choice of δ the space $X_{\gamma-\delta} \cup \{z \in X_{\gamma+\delta} \mid \xi(z) \leqslant \varepsilon\}$ is a deformation retract of $X_{\gamma+\delta}$. It is sufficient to show that $\{z \in X - Y \mid \xi(z) \leqslant \varepsilon, \ \tilde{\varphi}(z) \leqslant \gamma+\delta\}$ has the homotopy type of a space obtained from $\{z \in X - Y \mid \xi(z) \leqslant \varepsilon, \tilde{\varphi}(z) \leqslant \gamma-\delta\}$ by attaching cells of dimension $\geqslant n$. If $\alpha > 0$ is chosen small enough, the space $\{z \in X - Y \mid \xi(z) \leqslant \varepsilon, \ \tilde{\varphi}(z) \leqslant \gamma+\delta\}$ is a deformation retract of $\{z \in U \mid \xi(z) \leqslant \varepsilon, \ \tilde{\varphi}_0(z) \leqslant \gamma+\delta, \ \psi(z) \leqslant \alpha\}$, and the same is true if $\gamma+\delta$ is replaced by $\gamma-\delta$. So we see that it is sufficient to show that $\{z \in U \mid \xi(z) \leqslant \varepsilon, \tilde{\varphi}_0(z) \leqslant \gamma+\delta, \ \psi(z) \leqslant \alpha\}$ has the homotopy type of a space obtained from $\{z \in U \mid \xi(z) \leqslant \varepsilon, \tilde{\varphi}_0(z) \leqslant \gamma-\delta, \psi(z) \leqslant \alpha\}$ by attaching cells of dimension $\geqslant n$.

Now let us forget the stratification which we have chosen on U. The set $\{z \in U \mid \ \xi(z) \leqslant \varepsilon, \psi(z) \leqslant \alpha\}$ is a manifold with corners and therefore has a natural stratification: the strata are defined by $\xi < \varepsilon$ and $\psi < \alpha$, $\xi < \varepsilon$ and $\psi = \alpha$, $\xi = \varepsilon$ and $\psi < \alpha$, $\xi = \varepsilon$ and $\psi = \alpha$ respectively. The restriction of $\tilde{\varphi}_0$ to the first stratum obviously has no critical point. The restrictions of $\tilde{\varphi}_0$ to the third and fourth stratum have no critical points with values in $[\gamma-\delta, \gamma+\delta]$ because of the choices of δ and α. So it remains to consider the restriction to the second stratum, i.e. to the boundary of the manifold $\{z \in U \mid \psi(z) \leqslant \alpha, \xi(z) < \varepsilon\}$.

But now we can apply the same argument as in the proof of Theorem 3.2 in order to see that such critical points give rise to attaching cells of dimension $\geqslant n$ (cf. also [H–L 1] (3.2.6)).

4. Stratified Morse theory

In this section we will recall the fundamental theorem of [G–M 1], [G–M 2] in the "absolute and proper" case and show the connection with Section 2.

Let M be a C^∞ manifold, X a Whitney stratified subset, φ: $M \to \boldsymbol{R}$ a C^∞ function, $a, b \in \boldsymbol{R}$, $a < b$, $\varphi^{-1}([a, b]) \cap X$ compact. Because of Thom's first isotopy lemma (cf. [Ma], § 11) it is easy to see that $\{z \in X \mid \varphi(z) \leqslant a\}$ is homeomorphic to $\{z \in X \mid \varphi(z) \leqslant b\}$ if $\varphi \mid X$ (i.e. the restriction of φ to every stratum of X) has no critical values in $[a, b]$. Now let us consider the case where $\varphi \mid X$ has exactly one critical point x in $\varphi^{-1}([a, b])$ and $\gamma := \varphi(x) \in]a, b[$; let S be the stratum of X which contains x. Furthermore assume that $(d\varphi)_x \mid T \not\equiv 0$ if T is any linear subspace of $T_x M$ such that there exists a stratum $S' \neq S$ of X and a sequence (x_ν) in S' with $x_\nu \to x$ and $T_{x_\nu} S' \to T$. Finally assume that $\varphi \mid S$ has at x a non-degenerate critical point of index i.

Let N' be a submanifold of M transversal to S at x such that $x \in N'$ and

$$\dim_x S + \dim_x N' = \dim_x M.$$

Let d be the distance with respect to some Riemannian metric on N', $\delta > 0$ small, then $N := \{z \in N' \mid d(z, x) \leqslant \delta\}$. Let $\varepsilon > 0$ be small enough, $A_N = \{z \in N \mid \gamma - \varepsilon \leqslant \varphi(z) \leqslant \gamma + \varepsilon\}$, $B_N = \{z \in N \mid \varphi(z) = \gamma - \varepsilon\}$. The pair (A_N, B_N) is called *normal Morse data at* x. Let $s = \dim_x S$.

THEOREM 4.1 (cf. [G–M 1], p. 522, [G–M 2]). *The space* $\{z \in X \mid \varphi(z) \leqslant \gamma + \varepsilon\}$ *is homeomorphic to the space obtained from* $\{z \in X \mid \varphi(z) \leqslant \gamma - \varepsilon\}$ *by attaching* A *along* B, *where*

$$(A, B) = D^{s-i} \times (D^i, \partial D^i) \times (A_N, B_N).$$

Now let us study the connection with Section 2.

Let M be a C^∞ manifold with boundary and φ: $M \to \boldsymbol{R}$ an m-function. We have a natural stratification of M by the strata $\mathring{M} = M - \partial M$ and ∂M. Let x be a critical point of $\varphi \mid \mathring{M}$ or $\varphi \mid \partial M$, $\gamma = \varphi(x)$; assume that there are no other critical points with value γ. Let $\varepsilon > 0$ be small enough and assume that $\{z \in M \mid \gamma - \varepsilon \leqslant \varphi(z) \leqslant \gamma + \varepsilon\}$ is compact. Let us look how the homotopy type changes if we pass from $\{z \in M \mid \varphi(z) \leqslant \gamma - \varepsilon\}$ to $\{z \in M \mid \varphi(z) \leqslant \gamma + \varepsilon\}$.

If $x \in \mathring{M}$ we are essentially in the situation of classical Morse theory, and the theorem above (with $A_N = \{x\}$, $B_N = \emptyset$) as well as Section 2 are compatible with it.

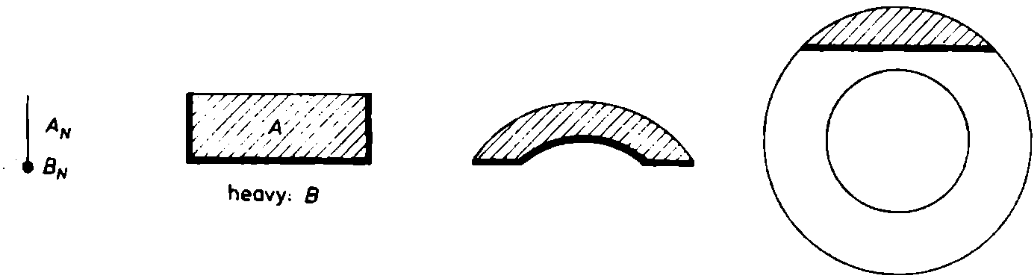
If $x \in \partial M$, A_N is homeomorphic to an interval but B_N depends on the direction of the gradient of φ at x which may point

(i) out of N (B_N consists of one point then) or

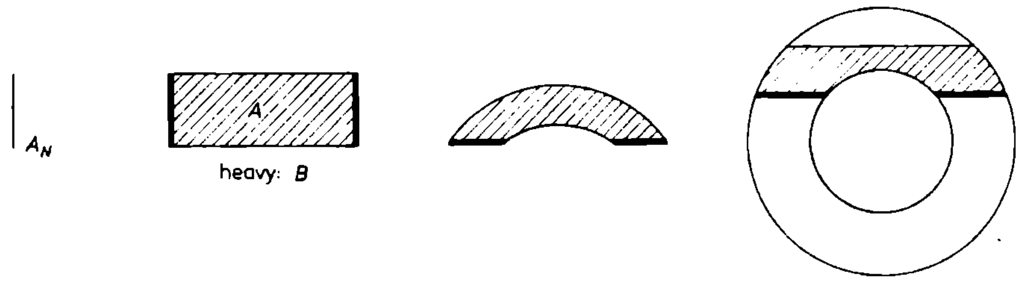
(ii) into N ($B_N = \emptyset$).

So the homotopy type does not change in the first case and we have to attach a cell of dimension i in the second case, i = index of the critical point x of $\varphi | \partial M$. This is compatible with Section 2. Let us sketch (A_N, B_N) and (A, B) in the case of a critical point of index 1 and make 4.1 plausible in this case; the subset along which we attach is drawn as a heavy line:

Case (i)



Case (ii)



In the case of Morse theory on manifolds with boundary modulo the boundary let us replace B_N by $B'_N = B_N \cup (N \cap \partial M)$. From Theorem 4.1 it is plausible that $\{z \in M \mid \varphi(z) \leqslant \gamma + \varepsilon\} \cup \partial M$ should be homeomorphic to the space obtained from $\{z \in M \mid \varphi(z) \leqslant \gamma - \varepsilon\} \cup \partial M$ by attaching A along B', where

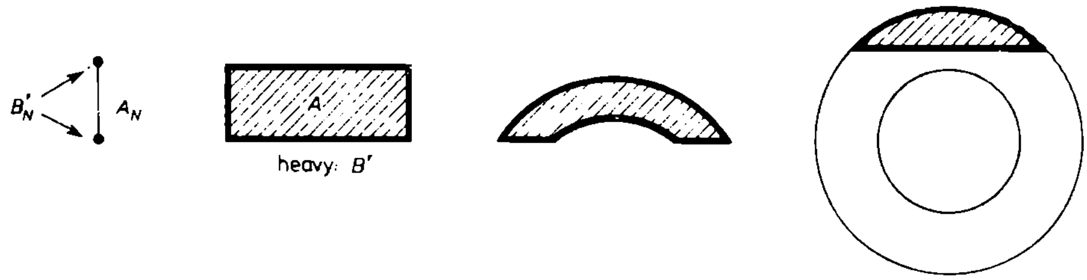
$$(A, B') = D^{s-i} \times (D^i, \partial D^i) \times (A_N, B'_N)$$

and i is the index of the critical point z of $\varphi | \partial M$ or $\varphi | \mathring{M}$.

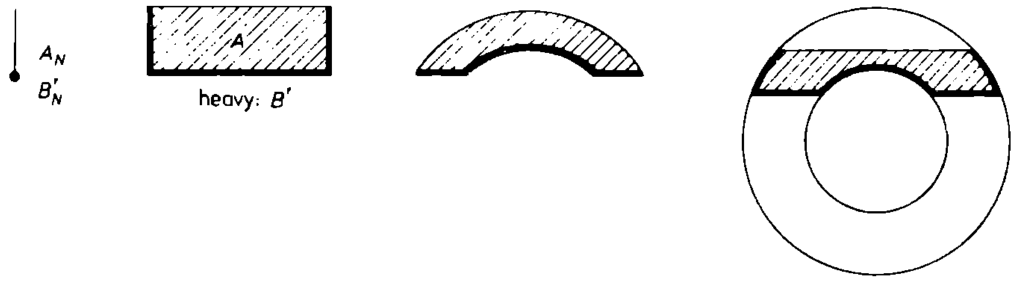
If $x \in \mathring{M}$, the pair (A, B') coincides with (A, B).

If $x \in \partial M$, the space B'_N consists of two points in case (i) and one point in case (ii). Let us sketch (A_N, B'_N) and (A, B') for $i = 1$ and show that the result is as expected:

Case (i)



Case (ii)



So the homotopy type changes as predicted in Section 2.

In order to derive 2.2 literally from the stratified Morse theory of Goresky and MacPherson one uses that the homeomorphism in Theorem 4.1 can be chosen to be compatible with the stratification of X in a certain sense; cf. [G–M 2]. Then one replaces M and ∂M at the same time by spaces obtained by successive attachment according to Theorem 4.1 and looks how the homotopy type changes at each step of attaching.

Much more different cases are possible if one looks at manifolds with corners. We will see an application in the next section.

5. A Zariski–Lefschetz theorem

The results of the preceding section will be illustrated by the proof of the following theorem which is a generalization of Theorem 3.2:

THEOREM 5.1. *Let X and Z be closed algebraic subsets of $\boldsymbol{P}_m$, let H be a hyperplane in $\boldsymbol{P}_m$, $Y = X \cap H$, let V be a suitable neighbourhood of H in $\boldsymbol{P}_m$, and let $f_1, \ldots, f_k$ be polynomials such that identifying $X - H$ with $\boldsymbol{C}^m$ we have*

$$X-(Y \cup Z) = \{z \in \boldsymbol{P}_m-(H \cup Z) \mid f_1(z) = \ldots = f_k(z) = 0\}.$$

Then the pair $(X-Z,\ V \cap X-Z)$ is $(m-k-1)$-connected.

Proof. Let $g_1, \ldots, g_p$ be polynomials such that

$$Z - H = \{z \in C^m \mid g_1(z) = \ldots = g_p(z) = 0\},$$

and let φ_0, χ and ψ be the real-valued functions on C^m defined by

$$\chi(z) = |z_1|^2 + \ldots + |z_m|^2, \qquad \varphi_0(z) = -\sum_{i=1}^{p} |g_i(z)|^2 (1 + \chi(z)),$$

$$\psi(z) = |f_1(z)|^2 + \ldots + |f_k(z)|^2.$$

Let $\bar{R}$ be a non-negative real number and $V = H \cup \{z \in C^m \mid \chi(z) \geqslant \bar{R}\}$ (it is not necessary to take $\bar{R}$ large). Let $r > 0$ be small enough, then $(X \cap V - Z) \cup \{z \in X - V \mid -\varphi_0(z) \geqslant r\}$ is a deformation retract of $X - Z$ and r is a regular value of φ_0 and $\varphi_0 \mid \partial V$. Now it is sufficient to prove that the pair $(\{z \in X - \mathring{V} \mid -\varphi_0(z) \geqslant r\}, \{z \in X \cap \partial V \mid -\varphi_0(z) \geqslant r\}$ is $(m-k-1)$-connected.

Let $\alpha > 0$ be small enough and $M = \{z \in C^m \mid \chi(z) \leqslant \bar{R}, \psi(z) \leqslant \alpha\}$, $M' = \{z \in C^m \mid \chi(z) = \bar{R}, \quad \psi(z) \leqslant \alpha\}$. Then $\{z \in X - \mathring{V} \mid -\varphi_0(z) \geqslant r\}$ and $\{z \in X \cap \partial V \mid -\varphi_0(z) \geqslant r\}$ are deformation retracts of $\{z \in M \mid -\varphi_0(z) \geqslant r\}$ and $\{z \in M' \mid -\varphi_0(z) \geqslant r\}$ respectively, and α is a regular value of the functions ψ, $\psi \mid \partial V$, $\psi \mid \{z \in C^m \mid -\varphi_0(z) = r\}$ and $\psi \mid \{z \in \partial V \mid -\varphi_0(z) = r\}$. We will approximate φ_0 by a C^∞ function $\tilde{\varphi}_0$ which has properties to be precised later on and use $\tilde{\varphi}_0$ in order to apply Morse theory to the manifold with corners M "modulo M'". We have to show that the pair $(\{z \in M \mid -\varphi_0(z) \geqslant r\}, \{z \in M' \mid -\varphi_0(z) \geqslant r\})$ is $(m-k-1)$-connected.

We ask that the Levi form of $\tilde{\varphi}_0$ is – as that of φ_0 – negative definite everywhere and that $-\tilde{\varphi}_0(z) > r$ if $-\varphi_0(z) > r$ and $-\tilde{\varphi}_0(z) < r$ if $-\varphi_0(z) < r$. There is an obvious stratification of the manifold with corners M, we ask that for any stratum S the critical points of $\tilde{\varphi}_0 \mid \bar{S}$ lie in S and are non-degenerate.

In order to prove the theorem we must look at a critical point z of the restriction of $\tilde{\varphi}_0$ to some stratum such that $-\tilde{\varphi}_0(z) \geqslant r$.

If z lies in the interior of M, i.e. if $\chi(z) < \bar{R}$, $\psi(z) < \alpha$, we are essentially in the case of classical Morse theory, so we have to attach a cell of dimension $\geqslant m$ ($\geqslant m-k$) since the Levi form of $\tilde{\varphi}_0$ is negative definite.

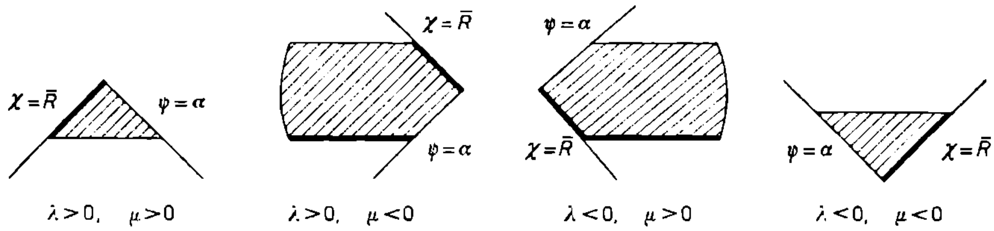
If $\chi(z) < \bar{R}$, $\psi(z) = \alpha$, the argument in the proof of Theorem 3.2 shows that we have to attach a cell of index $\geqslant m-k$ if the homotopy type is affected at all.

If $\chi(z) = \bar{R}$, $\psi(z) < \alpha$, we are essentially in the case of Morse theory on manifolds with boundary modulo the boundary, cf. Section 2. The argument is the same as in [H–L 2], (I.2.6): We have a contribution to the homotopy type only if grad $\tilde{\varphi}_0$ points out of M, i.e. we have only to look at the case where $(d\tilde{\varphi}_0)_z = \lambda (d\chi)_z$ with $\lambda > 0$. But the Levi form of $\tilde{\varphi}_0 - \lambda\chi$ is then negative definite since those of $\tilde{\varphi}_0$ and $-\chi$ are both negative definite. So we have to attach a cell of dimension $\geqslant m$ then.

If $\chi(z) = \bar{R}$, $\psi(z) = \alpha$, i.e. if z lies on the corner, we have an equation of the form

$$(d\tilde{\varphi}_0)_z = \lambda(d\chi)_z + \mu(d\psi)_z,$$

where $\lambda \neq 0$, $\mu \neq 0$. Now we look at the normal Morse data at z (cf Sect. 4): depending on the signs of λ and μ we have one of the following situations (using suitable coordinates for N):



If the restriction of $\tilde{\varphi}_0$ to the corner has index i one has to attach (A, B') along B', where

$$(A, B') = D^{2m-i-2} \times (D^i, \partial D^i) \times (A_N, B'_N),$$

A_N is shaded above and B'_N is drawn as a heavy line. Now B'_N is a deformation retract of A_N unless $\lambda > 0$, $\mu < 0$, so let us consider this special case. Then A_N has the homotopy type of a space obtained from B'_N by attaching a 1-cell, so one has to attach an $(i+1)$-cell. As the Levi form of $\tilde{\varphi}_0 - \lambda\chi - \mu\psi$ is obviously negative definite on the subspace $\{v \in \mathbf{C}^m \mid \sum_{\nu=1}^{N} \bar{z}_\nu v_\nu = (df_1)_z(v) = \ldots = (df_k)_z(v) = 0\}$ of $\mathbf{C}^m \cong T_z \mathbf{C}^m$ which has complex dimension $\geqslant m-k-1$ and is contained in the tangent space to the corner at z, we have $i \geqslant m-k-1$, so one has to attach a cell of dimension $\geqslant m-k$. (Note that the same kind of calculation has been made in [H–L 3].) This proves Theorem 5.1. □

6. Normal Morse data in the complex case

In this section we recall some aspects of the description of the normal Morse data in the complex case (see [G–M 1], Ch. 2, [G–M 2]) and study them using the results of [Ha 1]. An application will be given in the last section.

Let us consider the situation which we had in Section 4, but start with a complex manifold M and a complex analytic subset X of M which is Whitney stratified. In this case the strata and also N' should be complex analytic. If we fix x the normal Morse data at x do not depend on φ, up to homeomorphism; cf. [G–M 1], Ch. 2, [G–M 2]. We may embed N' after shrinking into some affine space $\mathbf{C}^q$. Let l be a linear function on $\mathbf{C}^q$ such that $\ker l$ is a generic hyperplane and let $F = \{z \in N \mid l(z) = \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small. Then:

THEOREM 6.1 (cf. [G–M 1], p. 526, [G–M 2]). *(A_N, B_N) has the same homotopy type as (CF, F), where CF is the cone over F.*

Therefore we get

COROLLARY 6.2. *Let i be the index of the critical point x of $\varphi | S$. If F is $(j-2)$-connected the pair (A, B) is $(j+i-1)$-connected.*

Now F has just been studied in [Ha 1]. Let us use the concept of "rectified homotopical depth" which was introduced by Grothendieck [Gr]. We will take a definition which – as he remarks* – should be equivalent to his.

DEFINITION 6.3. Let X be a complex space, $n \geq -1$.

(a) X is *n-connected* if every continuous map $\varrho\colon \partial D^{k+1} \to X$, $-1 \leq k \leq n$, admits a continuous extension over D^{k+1}.

(b) X has *rectified homotopical depth* $\geq n$ if for any locally closed analytic subset Y of X there is a nowhere dense closed analytic subset Y_0 of Y such that for any $x \in Y - Y_0$ there is a fundamental system of neighbourhoods U of x in X such that $U - Y$ is $(n - \dim_{\mathbf{C}} Y - 2)$-connected.

Let us take a complex analytic stratification of X with connected strata which is Whitney regular, and let X_i be the union of all strata of dimension $\leq i$.

LEMMA 6.4. *The following conditions are equivalent:*

(a) *X has rectified homotopical depth $\geq n$,*

(b) *For any $i \geq 0$ and any $x \in X_i - X_{i-1}$ there is a fundamental system of neighbourhoods U such that $U - X_i$ is $(n-i-2)$-connected.*

Proof. (a) $\Rightarrow$ (b) It is clear that the desired property holds for any $i \geq 0$ and for almost any $x \in X_i - X_{i-1}$, therefore for any $x \in X_i - X_{i-1}$.

(b) $\Rightarrow$ (a) If we take a finer stratification than the given one which is also Whitney regular, property (b) holds also for the finer stratification: for if A is a locally closed complex submanifold of $X_i - X_{i-1}$ of dimension j and U is a suitable neighbourhood of $x \in A$ in X, the pair $(U, U - A)$ has the same homotopy type as $(U, U - X_i) \times (D^{2i-2j}, D^{2i-2j} - \{0\})$, so is $(n+i-2j-1)$-connected, in particular $(n-j-1)$-connected. Now let Y be as in the definition of "rectified homotopical depth"; after shrinking X we may assume that Y is closed in X. Let us take a stratification of X which is finer than the given one and for which Y is a union of (connected) strata. Let Y_0 be the union of all strata of the new stratification which are nowhere dense subsets of Y. Obviously the desired condition is fulfilled. □

Remark 6.5. From Lemma 6.4 we see that our notion of rectified homotopical depth coincides with the one in [H–L 2], (II.1.5).

LEMMA 6.6. *Let X be a closed analytic subset of a complex manifold M, and let us fix a Whitney regular stratification of X. Let N be a complex submanifold of M of codimension p which intersects all strata of X transversally. If X has rectified homotopical depth $\geqslant n$ the space $X \cap N$ has rectified homotopical depth $\geqslant n-p$.*

Proof. This follows from Lemma 6.4 by taking the induced stratification on $X \cap N$. □

Let us now go back to the situation considered at the beginning of this section. Let s be the complex dimension of S at x.

LEMMA 6.7. *If X has rectified homotopical depth $\geqslant n$ the space F is $(n-s-2)$-connected.*

Proof. By Lemma 6.6 N has rectified homotopical depth $\geqslant n-s$, so N is locally strongly $(n-s-2)$-connected in the language of [Ha 1]. Now it follows from the corollary of Proposition 2 in [Ha 1] that F is $(n-s-2)$-connected. □

In total we get

THEOREM 6.8. *If X has rectified homotopical depth $\geqslant n$ the pair (A, B) is $(n-s+i-1)$-connected.*

Note that the theorem can be applied, in particular, if X is locally a set-theoretic complete intersection, $\dim^{-} X \geqslant n$, since X has rectified homotopical depth $\geqslant n$ then.

7. A Lefschetz theorem under assumption about the local homotopical depth

As Theorem (II.1.4) in [H–L 2] shows the condition "nonsingular of dimension n" can be replaced by the condition "rectified homotopical depth $\geqslant n$" in connection with the Lefschetz theorem. Now the results of Section 6 lead to a direct proof of a theorem which is more special than Theorem (II.1.4) in [H–L 2] but more general than Theorem 3.1 above:

THEOREM 7.1. *Let X be a closed algebraic subset of $\boldsymbol{P}_m$, let H be a hyperplane in $\boldsymbol{P}_m$ and $Y = X \cap H$. If $X - Y$ has rectified homotopical depth $\geqslant n$ the pair (X, Y) is $(n-1)$-connected.*

Proof. Let us identify $\boldsymbol{P}_m - H$ with $\boldsymbol{C}^m$, choose a Whitney regular stratification of X and take a suitable Morse function $\tilde{\varphi}\colon X - Y \to \boldsymbol{R}$ which approximates the function $z \mapsto -\|z\|^2$. If S is any stratum of X and $z \in S - Y$

a critical point of $\tilde{\varphi}|S$ of index i we have $i \geqslant \dim_{\mathbf{C}} S$, therefore by Theorem 6.8 the assertion follows. □

COROLLARY 7.2. *Let X be a closed algebraic subset of $\boldsymbol{P}_m$, let L be a linear subspace of $\boldsymbol{P}_m$ of codimension c and $Y = X \cap L$. If $X - Y$ has rectified homotopical depth $\geqslant n$ the pair (X, Y) is $(n-c)$-connected.*

Proof. By induction on $\dim X$. Let us fix a Whitney-regular stratification of X and choose a hyperplane H in $\boldsymbol{P}_m$ such that $L \subset H$ and $H - L$ intersects all strata of X transversally. By Lemma 6.6 the space $(X - Y) \cap H$ has rectified homotopical depth $\geqslant n-1$, by the induction hypothesis the pair $(X \cap H, Y)$ is $((n-1)-(c-1))$-connected. On the other hand, $(X, X \cap H)$ is $(n-1)$-connected by Theorem 7.1. So (X, Y) is $(n-c)$-connected. □

Remark 7.3. (a) Corollary 7.2 can also be proved directly using a suitable function on $X - Y$ and Theorem 6.8.

(b) Theorem 7.1 is even true if $X - Y$ is locally strongly $(n-2)$-connected; cf. [Ha 1], Theorem 3. But this does not lead to a corresponding corollary since the analogue of Lemma 6.6 is not true; cf. the example given in [Ha 1], p. 554.

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