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Chebyshevian splines

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CONTENTS

Introduction

A generalization of the system of power functions $\{t^i\}_{i=0}^n$ is an *extended complete Chebyshev system* (ECT-system) $U_n = \{u_i\}_{i=0}^n$ of functions of class C^n in the interval $I = [a, b]$. We can write every such system in the canonical form (see [34, 35]):

$$u_0(t) = w_0(t)\,,$$

$$u_i(t) = w_0(t) \int_a^t w_1(\tau_1) \int_a^{\tau_1} w_2(\tau_2) \ldots \int_a^{\tau_{i-1}} w_i(\tau_i)\, d\tau_i \ldots d\tau_1\,, \tag{1}$$

$i = 1, \ldots, n$, where $w_j \in C^{n-j}(I)$, $w_j(t) > 0$ for $t \in I$, $j = 0, \ldots, n$.

If we assume that the functions w_j are positive and integrable in I, we obtain a *canonical complete Chebyshev system* (CCT-system) (see [43, 50, 56, 57]). The Markov and de Boor inequalities are the main tools in the theory of approximation by splines. In this paper we generalize these inequalities from polynomials and polynomial splines to polynomials and splines w.r.t. a CCT-system (Theorems 1.9 and 2.10) assuming that the weight functions w_j are right continuous and satisfy

$$w_0 = 1\,, \qquad 0 < a_j \le w_j(t) \le b_j < \infty \quad \text{for } t \in I\,, \tag{2}$$

where a_j and b_j are given constants, $j = 1, \ldots, n$. Using these inequalities, we prove a number of important facts in the constructive theory of splines associated with CCT-systems.

Part I is devoted to basic properties of CCT-systems. We give the Lagrange and Newton interpolation formulas for polynomials w.r.t. a CCT-system (cf. [44, 45, 47, 74]).

Further, we prove the Markov inequality. It was proved by H. Johnen and K. Scherer [33] (see also [57]) for null spaces of linear differential operators; their result covers the case of polynomials w.r.t. ECT-systems. Using the Lagrange formula in the proof, we omit the assumption of differ-

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entiability of the weight functions w_j in (1). At the end of this part, using the Newton interpolation formula, we give a new proof of the Mühlbach recurrence formula for generalized divided differences. It was proved by G. Mühlbach [42] (see also [44, 45]) for distinct points. We prove it without this assumption.

In Part II we define Chebyshevian splines w.r.t. the system (1) with assumption (2) (cf. [56, 57]) and we prove their basic properties, including the first and second integral relations. Then we prove de Boor's inequalities. In the algebraic case they were proved by Z. Ciesielski and J. Domsta [20] for dyadic partitions (see also [16]) and by C. de Boor [10] for any partitions (see also [13, 18]). For L-splines they were proved by K. Scherer and L. L. Schumaker [52] (see also [57]). We prove them for general splines, assuming that the weight functions w_j satisfy (2). We also give a new simple proof of the Lyche recurrence relation for B-splines (cf. [39]). For completeness we formulate theorems on zeros of splines and the Marsden identity.

In Part III we are concerned with spline systems and operators associated with them. As in the algebraic case (see [16, 20, 67]) we define biorthogonal systems of Chebyshevian splines w.r.t. CCT-systems. Using de Boor's inequalities we prove that orthogonal spline projections are L_∞-bounded (see [12, 13, 16, 18, 20, 52, 57, 73]). Then we prove that the biorthogonal spline systems form Schauder bases in $L_p(I)$, $1 \le p \le \infty$, and that the orthogonal spline systems are equivalent. This is a generalization of a result of Z. Ciesielski (see [17, 76]). The last section of this part is devoted to a generalization to Chebyshevian splines of a result of P. Sablonnière about positive spline operators and orthogonal splines (see [51]).

In Part IV we generalize moduli of smoothness. We define the modulus of smoothness $\omega_U(f,h)$ w.r.t. the system (1) in such a way that $\omega_U(f,h) = 0$ for any f which is a linear combination of the functions of that system (see [73]). Then we prove the Whitney theorem for this modulus of smoothness. As a consequence we obtain an estimate of the best approximation of a function f by Chebyshevian splines by means of the modulus $\omega_U(f,h)$ in $L_\infty(I)$. We also prove that this modulus is equivalent to a K-functional $K(t,f;\infty,U)$ associated with the system U in $L_\infty(I)$. In the last section of this part we prove a Bernstein type inequality in $L_\infty(I)$. This is a generalization of the author's result given in [71].

Part V is devoted to applications of splines to approximation of analytic functions. We construct Schauder biorthogonal bases in the real (over $\mathbb{R}$) and complex (over $\mathbb{C}$) spaces of functions analytic in the unit disc D and continuous in $\bar{D}$ by means of Chebyshevian splines and the Schwarz formula.This is a generalization of Bochkarev's and the author's results (see [6, 7, 66–70]). We also prove that the conjugate (trigonometric) system to some biorthogonal periodic spline system is a basis in the space $C(T)$ of continuous periodic functions in $T = [0, 2\pi]$. This is a generalization of Bochkarev's result [8] for the periodic Franklin system, and of the author's result given in [77].

Most of the results of this paper were given by the author in [65–77] for polynomial and Chebyshevian splines w.r.t. ECT-systems. We generalize those results to the case of Chebyshevian splines w.r.t. CCT-systems satisfying (2). The most important ideas of the paper are taken from [18] and [67].

Let

$$L = D^{n+1} + \sum_{i=0}^{n} a_i(t) D^i, \tag{3}$$

where D is the differentiation operator and $a_i \in C^i(I)$, $i = 0, \ldots, n$, be a linear differential operator with null space N_L. The space N_L has the following important property (see [57], p. 423):

For every operator L of the form (3) there exists $\delta > 0$ such that, for every subinterval $J \subset I$ with length $|J| < \delta$, the space N_L has a basis $\{u_i^J\}_{i=0}^n$ which is an ECT-system in J.

Hence we may apply our results to the theory of L-splines.

The author would like to express his gratitude to Professor Zbigniew Ciesielski for his kind interest in this paper.

I. Canonical complete Chebyshev systems

1. Canonical complete Chebyshev systems. Let $U = U_n = \{u_i\}_{i=0}^n$ be a CCT-system in I defined by (1) and let the weight functions w_j satisfy (2). Define the following differential operators:

$$D_j f(x) = \lim_{h \to 0} [f(x+h) - f(x)] \Big(\int_x^{x+h} w_j(t)\, dt \Big)^{-1},$$

$$D_j^* f(x) = \frac{d}{dx} \left(\frac{f(x)}{w_j(x)} \right), \qquad j = 0, \ldots, n.$$

For w_j continuous and f differentiable at x_0 we obtain $D_j f(x_0) = f'(x_0) \times (w_j(x_0))^{-1}$. Put

$$(1.1)\qquad \begin{aligned} &L_j f = D_j \dots D_1 f\,, \quad L_j^* f = D_1^* \dots D_j^* f\,, \quad \tilde{L}_j f = D_1 \dots D_j f\,, \qquad j = 1, \dots, n,\\ &L f = D_0 L_n f\,, \quad L^* f = L_n^* D_0^* f\,. \end{aligned}$$

The operator $(-1)^{n+1} L^*$ is the formal adjoint to L. We have $w_j D_j = D_0$ and $L^* f = D_0 \tilde{L}_n f$ almost everywhere.

The *adjoint system* $V = V_n = \{v_i\}_{i=0}^n$ is defined as follows:

$$(1.2)\qquad \begin{aligned} &v_0(t) = 1\,,\\ &v_i(t) = \int_a^t w_n(\tau_1) \int_a^{\tau_1} w_{n-1}(\tau_2) \dots \int_a^{\tau_{i-1}} w_{n-i+1}(\tau_i)\, d\tau_i \dots d\tau_1\,, \end{aligned}$$

$i = 1, \dots, n$, and we put $\tau_0 = t$.

The systems U and V span the null spaces of the operators L and L^* respectively.

An important property of ECT-systems is the fact that certain determinants formed from them are always nonnegative. A similar situation persists for CCT-systems. Define

$$(1.3)\qquad D_U(t_0, \dots, t_k) = D_U \begin{pmatrix} u_0, & \dots, & u_k \\ t_0, & \dots, & t_k \end{pmatrix} = \det[L_{d_j} u_i(t_j)]_{i,j=0}^k\,,$$

$k = 0, \dots, n$, where $d_j = \max\{l : \ t_j = \dots = t_{j-l}\}$, $j = 1, \dots, k$, and we assume that if $t_i = t_{i+m}$, then $t_{i+j} = t_i$ for $j = 1, \dots, m-1$, and $L_0 u = u$.

Remark 1.1. Let $u_j(t) = (t-a)^j$, $j = 0, \dots, n$. In this case we have $w_0 = 1$, $w_i = i$, $i = 1, \dots, n$, $L_k f = L_k^* f = D^k f / k!$, $k = 1, \dots, n$, and $Lf = L^* f = D^{n+1} f / n!$.

THEOREM 1.1 ([57]). *Under the above assumptions we have* $D_U(t_0, \dots, t_k) > 0$ *for all* $t_0 \le t_1 \le \dots \le t_k$ *in* I, $k = 0, \dots, n$.

For the proof in the general case we refer to [57]. In our case the theorem follows from the following lemma, which we prove in detail on account of its importance in the paper.

LEMMA 1.1 ([52]). *Let* $a \le t_0 \le t_1 \le \dots \le t_k \le b$, $t_0 < t_k$. *Then the determinant* $D_U(t_0, \dots, t_k)$ *can be written as a multiple integral over positively oriented subintervals of* I *whose integrand involves only products of the functions* $w_0, \dots, w_k$.

Proof. For $t_0 = \ldots = t_k$, $D_U(t_0,\ldots,t_k) = 1$, $k = 0,\ldots,n$. We proceed by induction on k. Suppose that this lemma holds true for any CCT-system of k functions defined by (1). Let $\{t_0 \le t_1 \le \ldots \le t_k\} = \{\overbrace{x_0,\ldots,x_0}^{\alpha_0} < \ldots < \overbrace{x_d,\ldots,x_d}^{\alpha_d}\}$. Subtracting each column with a 1 in the first row from its successor with a 1 in the first row we obtain (after expanding about the first row)

$$D_U(t_0,\ldots,t_k) = \int_{x_0}^{x_1} w_1(\tau_1)\ldots\int_{x_{d-1}}^{x_d} w_1(\tau_d)$$

$$\times D_{U^1}(\overbrace{x_0,\ldots,x_0}^{\alpha_0-1},\tau_1,\overbrace{x_1,\ldots,x_1}^{\alpha_1-1},\tau_2,\ldots,\tau_d,\overbrace{x_d,\ldots,x_d}^{\alpha_d-1})\,d\tau_1\ldots d\tau_d\,,$$

where $U^1 = \{u_0^1,\ldots,u_{n-1}^1\}$, $u_j^1 = D_1 u_{j+1}$, $j = 0,\ldots,n-1$. Now by the inductive hypothesis the integrand is a multiple integral of products of the w's, and our assertion has been established.

In the sequel $\alpha_U, \beta_U, C_{U,n}, \ldots$ are constants, different in general, depending only on the indicated parameters.

As a corollary of Lemma 1.1 we obtain

LEMMA 1.2 (cf. [52] and [73]). *Let $\tilde{U} = \{(t-a)^i\}_{i=0}^n$. There exist positive constants α_U and β_U such that*

$$(1.4)\qquad \alpha_U D_{\tilde{U}}(t_0,\ldots,t_k) \le D_U(t_0,\ldots,t_k) \le \beta_U D_{\tilde{U}}(t_0,\ldots,t_k)$$

for $a \le t_0 \le t_1 \le \ldots \le t_k \le b$, $k = 1,\ldots,n$.

Proof. In view of Lemma 1.1 we get an upper bound on $D_U(t_0,\ldots,t_k)$ if we replace each weight function w_i in the integral representation by b_i, $i = 1,\ldots,k$. To get the lower bound we substitute a_i for each w_i, $i = 1,\ldots,k$.

Define the following functions in $I \times I$:

$$h_k(t,x) = \int_x^t w_1(\tau_1)\int_x^{\tau_1} w_2(\tau_2)\ldots\int_x^{\tau_{k-1}} w_k(\tau_k)\,d\tau_k\ldots d\tau_1\,,\qquad k = 1,\ldots,n\quad(\tau_0 = t)\,,$$

$$(1.5)\qquad h_k^*(t,x) = \int_x^t w_k(\tau_1)\int_x^{\tau_1} w_{k-1}(\tau_2)\ldots\int_x^{\tau_{k-1}} w_1(\tau_k)\,d\tau_k\ldots d\tau_1\,,\qquad k = 1,\ldots,n\quad(\tau_0 = t)\,,$$

$$h_0(t,x) = h_0^*(t,x) = 1\,,$$

$$\varphi_k(t,x) = \begin{cases} h_k(t,x) & \text{for } a \le x \le t \le b, \\ 0 & \text{for } t < x, \end{cases} \qquad k = 0, \ldots n,$$

(1.6)

$$\varphi_k^*(t,x) = \begin{cases} h_k^*(t,x) & \text{for } a \le x \le t \le b, \\ 0 & \text{for } t < x, \end{cases} \qquad k = 0, \ldots, n.$$

LEMMA 1.3 (see [34, 35]). *The functions φ_n and φ_n^* satisfy*

$$D_U \begin{pmatrix} u_0, \ldots, u_n, \varphi_n \\ t_0, \ldots, t_n, t_{n+1} \end{pmatrix} \ge 0 \quad \text{and} \quad D_V \begin{pmatrix} v_0, \ldots, v_n, \varphi_n^* \\ t_0, \ldots, t_n, t_{n+1} \end{pmatrix} \ge 0$$

for any system $\{t_i\}_{i=0}^{n+1}$ such that $a \le t_0 \le \ldots \le t_{n+1} \le b$, where in the second determinant the operators L_k in (1.3) are replaced by $\hat{L}_k = D_{n-k+1} \ldots D_n$. Further, for $a \le x \le t_0 \le \ldots \le t_{n+1} \le b$

$$D_U \begin{pmatrix} u_0, \ldots, u_n, \varphi_n \\ t_0, \ldots, t_n, t_{n+1} \end{pmatrix} = D_V \begin{pmatrix} v_0, \ldots, v_n, \varphi_n^* \\ t_0, \ldots, t_n, t_{n+1} \end{pmatrix} = 0$$

and for $t_0 < x < t_{n+1}$

$$D_U \begin{pmatrix} u_0, \ldots, u_n, \varphi_n \\ t_0, \ldots, t_n, t_{n+1} \end{pmatrix} > 0 \quad \text{and} \quad D_V \begin{pmatrix} v_0, \ldots, v_n, \varphi_n^* \\ t_0, \ldots, t_n, t_{n+1} \end{pmatrix} > 0 .$$

The functions φ_n and φ_n^* are Green functions associated with the operators L and L^* respectively and the functions $w(t) = \int_a^b \varphi_n(t,x)\,dx$ and $w^*(t) = \int_a^b \varphi_n^*(t,x)\,dx$ are solutions of the equations $Lw(t) = 1$ and $L^*w^*(t) = 1$ respectively.

The proof follows from Lemma 1.1 for the systems U and V and the definitions of φ_n and φ_n^* respectively.

The following theorem shows that $\varphi_n(t,x)$ has the characteristic properties of a Green function:

THEOREM 1.2 ([57]). *For all t and x*

$$h_n(t,x) = \sum_{i=0}^{n} (-1)^{n-i} u_i(t) v_{n-i}(x) . \tag{1.7}$$

Proof. For $n = 1$, $h_1(t,x) = \int_x^t w_1(\tau)\,d\tau = \int_a^t w_1(\tau)\,d\tau - \int_a^x w_1(\tau)\,d\tau = u_1(t)v_0(x) - u_0(t)v_1(x)$. Now we assume that (1.7) holds for $n = k - 1$ and any weight functions w_j and we prove it for $n = k$. We have

$$\int_x^t w_1(\tau_1) \int_x^{\tau_1} w_2(\tau_2) \ldots \int_x^{\tau_{k-1}} w_k(\tau_k)\,d\tau_k \ldots d\tau_1 \tag{1.8}$$

$$= \int_a^t w_1(\tau_1)\psi(\tau_1)\, d\tau_1 - \int_a^x w_1(\tau_1)\psi(\tau_1)\, d\tau_1 ,$$

where (using the induction hypothesis)

$$\begin{aligned}\psi(\tau_1) &= \int_x^{\tau_1} w_2(\tau_2)\dots \int_x^{\tau_{k-1}} w_k(\tau_k)\, d\tau_k \dots d\tau_2 \\ &= \sum_{i=0}^{k-1}(-1)^{k-i-1} D_1 u_{i+1}(\tau_1) v_{k-1-i}(x) .\end{aligned}$$

Substituting this in the first term on the right-hand side of (1.8), we see that it reduces to $\sum_{i=1}^{k}(-1)^{k-i}u_i(t)v_{k-i}(x)$. Now a simple induction argument shows that

$$\begin{aligned}&\int_a^x w_1(\tau_1) \int_x^{\tau_1} w_2(\tau_2) \dots \int_x^{\tau_{k-1}} w_k(\tau_k)\, d\tau_k \dots d\tau_1 \\ &\qquad = (-1)^{k-1} \int_a^x w_k(\tau_k) \int_a^{\tau_k} w_{k-1}(\tau_{k-1}) \dots \int_a^{\tau_2} w_1(\tau_1)\, d\tau_1 \dots d\tau_k .\end{aligned}$$

Since $u_0(t) = 1$, the second term on the right-hand side of (1.8) is equal to $(-1)^k u_0 v_k$, and the theorem is proved.

COROLLARY 1.1.

$$h_n^*(t,x) = \sum_{i=0}^{n}(-1)^{n-i} v_i(t) u_{n-i}(x) = (-1)^n h_n(x,t) . \tag{1.9}$$

EXAMPLE 1.1. Let $U = \{1, t, \dots, t^n\}$ on $I = [0,1]$. In this case $w_0 = v_0 = 1$, $w_j = j$, $v_j(x) = \binom{n}{j} x^j$, $j = 1, \dots, n$, and $h_n(t,x) = (t-x)^n = \sum_{j=0}^{n}(-1)^{n-j} t^j \binom{n}{j} x^{n-j} = (-1)^n h_n^*(t,x) = (-1)^n (x-t)^n$.

THEOREM 1.3 (Generalized Taylor expansion). *Let a function f satisfy the following conditions: f and $L_k f$ are continuous, $k = 0, \dots, n-1$, and $L_n f$ is absolutely continuous in I. Then for any $a \le t \le b$*

$$f(t) = f(a) + \sum_{i=1}^{n} L_i f(a+) u_i(t) + \int_a^t h_n(t,x) L f(x)\, dx . \tag{1.10}$$

Proof (cf. [35]). It follows from (1.5) that

$$\lim_{h\to 0}\frac{1}{h}[h_n(t,x+h)-h_n(t,x)]$$

$$=-\lim_{h\to 0}\frac{1}{h}\int_{x+h}^{t} w_1(\tau_1)\int_{x+h}^{\tau_1} w_2(\tau_2)\dots\int_{x+h}^{\tau_{n-2}} w_{n-1}(\tau_{n-1})\int_{x}^{x+h} w_n(\tau_n)\,d\tau_n\dots d\tau_1$$

$$=-w_n(x)h_{n-1}(x,t)\quad \text{a.e.}$$

Now integrating by parts we obtain (1.10).

Remark 1.2. In the algebraic case we have $h_n(t,x)=(t-x)^n$ and we obtain the Taylor expansion with integral remainder:

$$f(t)=f(a)+\sum_{i=1}^{n}\frac{f^{(i)}(a)}{i!}(t-a)^i+\frac{1}{n!}\int_a^t (t-x)^n f^{(n+1)}(x)\,dx\,.$$

Remark 1.3. We may also write (1.10) as follows:

$$f(t)=f(a)+\sum_{i=1}^{n}L_if(a+)u_i(t)+\int_a^b \varphi_n(t,x)Lf(x)\,dx\,.$$

As a corollary of Theorem 1.3 we obtain

THEOREM 1.4. *Let a function f satisfy the following conditions: f, D_nf, $D_{n-1}D_nf,\dots,D_2\dots D_nf$ are continuous and $D_1\dots D_nf$ is absolutely continuous in I. Then for any $a\le t\le b$*

$$f(t)=f(a)+\sum_{i=1}^{n}D_{n-i+1}f(a+)v_i(t)+\int_a^b \varphi_n(t,x)L^*f(x)\,dx\,. \tag{1.11}$$

2. Interpolation by generalized polynomials and divided differences

DEFINITION 1.1. A function p of the form $p=\sum_{j=0}^{n}a_ju_j$, where the u_j are defined by (1), is said to be a *polynomial w.r.t. the system U*.

The set of all polynomials w.r.t. the system U is denoted by P_U.

Using Theorem 1.1 we obtain the well known generalization of the classical Lagrange interpolation formula.

THEOREM 1.5. *Let f be defined in I and $a\le t_0<t_1<\ldots<t_n\le b$. Then there exists a unique polynomial $p\in P_U$ interpolating the function f*

at the points t_j, $j = 0, \dots, n$. *We may write it as follows:*

$$(1.12) \qquad p(t) = -\frac{D_U\begin{pmatrix} g, u_0, \dots, u_n \\ t, t_0, \dots, t_n \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_n \end{pmatrix}} = \sum_{j=0}^{n} f(t_j)W_j(t),$$

where g *is any function such that* $g(t) = 0$, $g(t_j) = f(t_j)$, $j = 0, \dots, n$, *and*

$$W_j(t) = \frac{D_U(t_0, \dots, t_{j-1}, t, t_{j+1}, \dots, t_n)}{D_U(t_0, \dots, t_n)}, \quad j = 0, \dots, n.$$

Let $t_0 < t_1 < \dots < t_n$ and let $l_i(t)$ be the algebraic Lagrangian function of degree n defined for the points t_j, i.e. $l_i(t_j) = \delta_{i,j}$, $i, j = 0, \dots, n$. Reasoning as in the proof of Lemma 1.1 we get

LEMMA 1.4 (see [73]). *There exist positive constants* α_U *and* β_U *such that for* $t \in I$

$$(1.13) \qquad \alpha_U |l_i(t)| \le |W_i(t)| \le \beta_U |l_i(t)|, \quad i = 0, \dots, n.$$

DEFINITION 1.2. We define the *divided difference* of a function f at the points $t_0 \le \dots \le t_{n+1}$, $t_0 < t_{n+1}$, w.r.t. the system U by

$$(1.14) \qquad [t_0, \dots, t_{n+1}; f]_U = \frac{D_U\begin{pmatrix} u_0, \dots, u_n, f \\ t_0, \dots, t_n, t_{n+1} \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_{n+1} \\ t_0, \dots, t_{n+1} \end{pmatrix}},$$

where

$$u_{n+1}(t) = \int_a^t w_1(\tau_1) \int_a^{\tau_1} w_2(\tau_2) \dots \int_a^{\tau_{n-1}} w_n(\tau_n) \int_a^{\tau_n} w_{n+1}(\tau_{n+1})\, d\tau_{n+1} \dots d\tau_1$$

and $w_{n+1} = 1$.

Generalized divided differences were introduced by T. Popoviciu in [47].

It follows from the definition that the divided difference depends neither on the basis of the space of polynomials P_U nor on the order of the points t_j, $j = 0, \dots, n+1$.

In the algebraic case we have $U_n = \{t^i\}_{i=0}^n$ and $[t_j, \dots, t_{j+n+1}; f]_U = n[t_j, \dots, t_{j+n+1}; f]$, where the last expression is the divided difference in the algebraic case (see [18, 29]).

THEOREM 1.6 (cf. [57]). *The generalized divided difference defined in* (1.14) *is a linear functional such that* $[t_0, \dots, t_{n+1}; u]_U = 0$ *for all* $u \in P_U$ *and* $[t_0, \dots, t_{n+1}; u_{n+1}]_U = 1$.

The proof follows from (1.14) and the properties of determinants.
Further,

$$(1.15)\quad f(t)-p(t) = \frac{D_U\begin{pmatrix} f, u_0, \dots, u_n \\ t, t_0, \dots, t_n \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_n \end{pmatrix}}$$

$$= \frac{D_U\begin{pmatrix} f, u_0, \dots, u_n \\ t, t_0, \dots, t_n \end{pmatrix}}{D_U\begin{pmatrix} u_{n+1}, u_0, \dots, u_n \\ t, \quad t_0, \dots, t_n \end{pmatrix}} \cdot \frac{D_U\begin{pmatrix} u_{n+1}, u_0, \dots, u_n \\ t, \quad t_0, \dots, t_n \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_n \end{pmatrix}}$$

$$= [t, t_0, \dots, t_n; f]_U \cdot W(t),$$

where W is a polynomial from $P_{U_{n+1}}$ $(U_{n+1} = \{u_0, \dots, u_{n+1}\})$, equal to 0 at the points t_j, $j = 0, \dots, n$, and such that $LW = 1$.

Reasoning as in the proof of Lemma 1.1 we obtain

LEMMA 1.5 (see [73]). *There exist positive constants α_U and β_U such that for $t \in I$*

$$(1.16)\qquad \alpha_U|\tilde{W}(t)| \le |W(t)| \le \beta_U|\tilde{W}(t)|,$$

where $\tilde{W}(t) = (t-t_0)\dots(t-t_n)$.

Define

$$\left[\begin{matrix} u_0, \dots, u_j \\ t_0, \dots, t_j \end{matrix}\middle| f\right] = \frac{D_U\begin{pmatrix} u_0, \dots, u_{j-1}, f \\ t_0, \dots, t_{j-1}, t_j \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_j \\ t_0, \dots, t_j \end{pmatrix}}, \quad j = 1, \dots, n.$$

We have the following generalization of the Newton interpolation formula (see [44, 45, 57]).

THEOREM 1.7. *Let $a \le t_0 \le \dots \le t_n \le b$ and let f be such that $[t_0, \dots, t_n; f]_U$ exists. Then there exists a unique polynomial $p \in P_U$ satisfying*

$$L_{d_j}p(t_j) = L_{d_j}f(t_j), \quad j = 0, \dots, n,$$

where $d_j = \max\{l : t_j = \dots = t_{j-l}\}$, $j = 0, \dots, n$ and $L_0 f(t) = f(t)$ and

$$(1.17)\qquad p(t) = f(t_0) + \sum_{j=1}^{n} \left[\begin{matrix} u_0, \dots, u_j \\ t_0, \dots, t_j \end{matrix}\middle| f\right] \frac{D_U\begin{pmatrix} u_0, \dots, u_{j-1}, u_j \\ t_0, \dots, t_{j-1}, t \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_{j-1} \\ t_0, \dots, t_{j-1} \end{pmatrix}}.$$

Proof. The uniqueness of the polynomial p follows from Theorem 1.1. We may write this polynomial in the following form:

$$p(t) = a_0 + \sum_{j=1}^{n} a_j \frac{D_U\begin{pmatrix} u_0, \ldots, u_{j-1}, u_j \\ t_0, \ldots, t_{j-1}, t \end{pmatrix}}{D_U\begin{pmatrix} u_0, \ldots, u_{j-1} \\ t_0, \ldots, t_{j-1} \end{pmatrix}}.$$

Hence

$$\left[\begin{matrix} u_0, \ldots, u_j \\ t_0, \ldots, t_j \end{matrix}\middle| f\right] = \left[\begin{matrix} u_0, \ldots, u_j \\ t_0, \ldots, t_j \end{matrix}\middle| p\right] = a_j \left[\begin{matrix} u_0, \ldots, u_j \\ t_0, \ldots, t_j \end{matrix}\middle| u_j\right] = a_j,$$

$j = 1, \ldots, n$, and we have proved (1.17).

Remark 1.4. In the algebraic case, we obtain the usual Newton interpolation formula and therefore we shall call (1.17) the *Newton interpolation formula*.

Remark 1.5. Theorem 1.7 remains true for any order of the points $t_0, \ldots, t_n$. The proof is the same.

One of the most important properties of ordinary divided differences is the recursion relation. The following theorem gives an analogous relation for generalized divided differences:

THEOREM 1.8 (cf. [42, 44]). *Suppose $t_0 \neq t_{n+1}$. Then*

$$\left[\begin{matrix} u_0, \ldots, u_{n+1} \\ t_0, \ldots, t_{n+1} \end{matrix}\middle| f\right] = \frac{\left[\begin{matrix} u_0, \ldots, u_n \\ t_1, \ldots, t_{n+1} \end{matrix}\middle| f\right] - \left[\begin{matrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{matrix}\middle| f\right]}{\left[\begin{matrix} u_0, \ldots, u_n \\ t_1, \ldots, t_{n+1} \end{matrix}\middle| u_{n+1}\right] - \left[\begin{matrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{matrix}\middle| u_{n+1}\right]}. \tag{1.18}$$

Proof. Let $P_f(t_0, \ldots, t_n; t)$ denote the polynomial interpolating f at t_i, $i = 0, \ldots, n$. Since

$$\begin{aligned} r_f(t_0, \ldots, t_n; t) &= f(t) - P_f(t_0, \ldots, t_n; t) \\ &= \left[\begin{matrix} u_0, \ldots, u_n, u_{n+1} \\ t_0, \ldots, t_n, t \end{matrix}\middle| f\right] \\ &\quad \times \left[D_U\begin{pmatrix} u_0, \ldots, u_n, u_{n+1} \\ t_0, \ldots, t_n, t \end{pmatrix} \Big/ D_U\begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix}\right], \end{aligned}$$

applying the Newton interpolation formula we obtain

$$0 = P_f(t_1, \dots, t_{n+1}; t_0) + r_f(t_1, \dots, t_{n+1}; t_0) - P_f(t_1, \dots, t_n, t_0; t_0)$$

$$= \left[\begin{matrix} u_0, \dots, u_n, & u_{n+1} \\ t_0, \dots, t_{n+1}, & t_0 \end{matrix} \middle| f \right] \cdot \frac{D_U\begin{pmatrix} u_{n+1}, u_0, \dots, u_n \\ t_0, \quad t_1, \dots, t_{n+1} \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_n \\ t_1, \dots, t_{n+1} \end{pmatrix}}$$

$$+ \left(\left[\begin{matrix} u_0, \dots, u_n \\ t_1, \dots, t_{n+1} \end{matrix} \middle| f \right] - \left[\begin{matrix} u_0, \dots, u_{n-1}, u_n \\ t_1, \dots, t_n, \quad t_0 \end{matrix} \middle| f \right] \right)$$

$$\times \frac{D_U\begin{pmatrix} u_0, \dots, u_{n-1}, u_n \\ t_1, \dots, t_n, \quad t_0 \end{pmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_{n-1} \\ t_1, \dots, t_n \end{pmatrix}}.$$

Now applying this formula to the function u_{n+1} and using the fact that

$$\left[\begin{matrix} u_0, \dots, u_{n+1} \\ t_0, \dots, t_{n+1} \end{matrix} \middle| u_{n+1} \right] = 1$$

we obtain (1.18).

Theorem 1.8 was proved by G. Mühlbach [42] (see also [44, 45, 57]) for distinct points. Using the Newton interpolation formula, we have proved it without this assumption.

3. The Markov inequality for generalized polynomials

THEOREM 1.9 (Markov's inequality). *Let u be a polynomial from P_U and $\|u\|_I = \sup_{a \le t \le b} |u(t)|$. There exists a constant C_U such that*

$$\|D_1 u\|_I \le \frac{C_U \|u\|_I}{b-a}. \tag{1.19}$$

Proof. Let $t_j = a + (b-a)j/n$ and $y_j = u(t_j)$, $j = 0, \dots, n$. We may write u as follows:

$$u(t) = -\frac{\begin{vmatrix} 0 & 1 & u_1(t) & \dots & u_n(t) \\ y_0 & 1 & u_1(t_0) & \dots & u_n(t_0) \\ \dots & \dots & \dots & \dots & \dots \\ y_n & 1 & u_1(t_n) & \dots & u_n(t_n) \end{vmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_n \end{pmatrix}}$$

and

$$D_1u(t) = -\frac{\begin{vmatrix} 0 & 0 & 1 & D_1u_2(t) & \dots & D_1u_n(t) \\ y_0 & 1 & u_1(t_0) & u_2(t_0) & \dots & u_n(t_0) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_n & 1 & u_1(t_n) & u_2(t_n) & \dots & u_n(t_n) \end{vmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_n \end{pmatrix}}.$$

Subtracting the jth row from its successor in the determinant from the numerator, $j = n+1, \dots, 2$, and then expanding this determinant w.r.t. the second column we obtain

$$D_1u(t) = -\frac{\begin{vmatrix} 0 & 1 & D_1u_2(t) & \dots & D_1u_n(t) \\ y_1 - y_0 & w_1(t_0, t_1) & w_2(t_0, t_1) & \dots & w_n(t_0, t_1) \\ \dots & \dots & \dots & \dots & \dots \\ y_n - y_{n-1} & w_1(t_{n-1}, t_n) & w_2(t_{n-1}, t_n) & \dots & w_n(t_{n-1}, t_n) \end{vmatrix}}{D_U\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_n \end{pmatrix}},$$

where

$$\begin{aligned} w_k(t_j, t_{j+1}) &= u_k(t_{j+1}) - u_k(t_j) \\ &= \int_{t_j}^{t_{j+1}} w_1(\tau_1) \int_a^{\tau_1} w_2(\tau_2) \dots \int_a^{\tau_{k-1}} w_k(\tau_k)\, d\tau_k \dots d\tau_1, \\ & \qquad\qquad\qquad k = 1, \dots, n. \end{aligned}$$

Expanding now the numerator w.r.t. the first column we may write D_1u in the form

$$D_1u(t) = \sum_{j=1}^{n} (y_j - y_{j-1})\eta_j(t),$$

where $\eta_j(t)$ is the ratio of the algebraic complement to $y_j - y_{j-1}$ in the numerator to the denominator. As in Lemma 1.1 we write η_j in the form

$$\eta_j(t) = \frac{1}{D_U(t_0, \dots, t_n)} \int_{t_0}^{t_1} w_1(x_1) \dots \int_{t_{j-2}}^{t_{j-1}} w_1(x_{j-1}) \int_{t_j}^{t_{j+1}} w_1(x_{n-1}) \times D_{U^1}(t, x_1, \dots, x_{n-1})\, dx_1 \dots dx_{n-1},$$

where $U^1 = \{u_0^1, \dots, u_{n-1}^1\}$, $u_j^1 = D_1u_{j+1}$, $j = 0, \dots, n-1$.

Now applying Lemma 1.2 and the properties of the classical Vandermonde determinant we obtain (1.19).

R e m a r k 1.6. Theorem 1.9 was first proved by H. Johnen and K. Scherer [33] for null spaces of linear differential operators; their theorem covers the case of polynomials w.r.t. ECT-systems. On the other hand, Theorem 1.9 for the null spaces of linear differential operators follows from this theorem for polynomials w.r.t. ECT-systems (see the end of Introduction). Hence Theorem 1.9 is a generalization of the inequality proved by H. Johnen and K. Scherer.

II. Chebyshevian splines

1. Basic properties. Let $U = U_n$ be a CCT-system in I defined by (1) and let

$$\Delta = \{a = t_0 \le t_1 \le \ldots \le t_N = b\} = \{a = x_0 < x_1 < \ldots < x_M = b\}, \tag{2.1}$$

where $t_0, \ldots, t_N = \overbrace{x_0, \ldots, x_0}^{\alpha_0}, \ldots, \overbrace{x_M, \ldots, x_M}^{\alpha_M}$, $\alpha_j \le n+1$ and α_j is the multiplicity of the point x_j, $j = 0, \ldots, M$, $\sum_{j=0}^{M} \alpha_j = N+1$.

DEFINITION 2.1 (cf. [35, 52]). A function s is called a *Chebyshevian spline* w.r.t. the partition Δ and the system U if:

(a) s is a polynomial from P_U in each subinterval $(x_{j-1}, x_j), j = 1, \ldots, M$,
(b) there is $\varepsilon > 0$ such that s, $L_k s$ are continuous in $(x_j - \varepsilon, x_j + \varepsilon)$, $k = 1, \ldots, n - \alpha_j$, $j = 1, \ldots, M-1$.

We denote the set of these functions by $S_\Delta^U(I)$.

A Chebyshevian spline s as above is also said to have knots $x_0, \ldots, x_M$ with associated multiplicities $\alpha_0, \ldots, \alpha_M$.

DEFINITION 2.2. A spline $s \in S_\Delta^U(I)$ is said to be *periodic* of period $b-a$ if $\alpha_0 = \alpha_M$ and $L_k s(a+) = L_k s(b-)$ for $k = 0, \ldots, n - \alpha_0 - 1$ $(L_0 s = s)$.

We denote the space of periodic splines by $\mathring{S}_\Delta^U(I)$.

It follows from the definition that $\varphi_{n-i}(t, x_j) \in S_\Delta^U(I)$ for $j = 1, \ldots, M-1$, $i = 0, \ldots, \alpha_j - 1$. The following lemma exhibits a canonical representation of Chebyshevian splines.

LEMMA 2.1 (see [34, 35]). *A Chebyshevian spline $s \in S_\Delta^U(I)$ admits a unique representation*

$$s(t) = \sum_{i=0}^{n} b_i u_i(t) + \sum_{i=1}^{M-1} \sum_{j=0}^{\alpha_i - 1} a_{i,j} \varphi_{n-j}(t, x_i). \tag{2.2}$$

P r o o f. A function s of the form (2.2) has properties (a) and (b) of Definition 2.1. Conversely, let $s \in S_\Delta^U(I)$. Then s coincides with a polynomial from P_U on each of the intervals (t_{j-1}, t_j), $j = 1, \ldots, M$. The functions

$$g_{i,\alpha_i}(t) = D_{n-\alpha_i+1} \ldots D_1 s(t)\,, \quad i = 1, \ldots, M-1,$$

exist in each of these intervals. Then $s(t) - \sum_{i=1}^{M-1} a_{i,\alpha_i-1}\varphi_{n-\alpha_i+1}(t, x_i)$, where $a_{i,\alpha_i-1} = g_{i,\alpha_i}(x_i+) - g_{i,\alpha_i}(x_i-)$, will display the same knots, with each of the multiplicities lowered by 1. Continuing in this fashion, we reduce the resulting expression

$$s(t) - \sum_{i=1}^{M-1} \sum_{j=0}^{\alpha_{i-1}} a_{i,j}\varphi_{n-j}(t, x_i)$$

to a polynomial from P_U, and the proof is complete.

DEFINITION 2.3. We say that $f \in H_{U,p}^r(I)$ if f and $L_k f$ are continuous for $k = 1, \ldots, r-2$, $L_{r-1} f$ is absolutely continuous and $L_r f$ is p-integrable in I; $f \in C_U^r(I)$ if f and $L_k f$, $k = 1, \ldots, r$, are continuous in I; $f \in \mathring{H}_{U,p}^r(I)$ if $f \in H_{U,p}^r(I)$ and it is $(b-a)$-periodic; $f \in \mathring{C}_U^r(I)$ if $f \in C_U^r(I)$ and it is $(b-a)$-periodic.

Let now $V_{2n+1} = \{u_i\}_{i=0}^{2n+1}$ be a system of functions defined by (1) w.r.t. the system of weight functions $\{w_0, w_n, \ldots, w_1, w_0, w_1 \ldots, w_n\}$, $w_0 = 1$. Assume that $f \in H_{V_{2n+1},2}^{n+1}(I)$, $s \in S_\Delta^{V_{2n+1}}(I)$, Δ is defined by (2.1) and $1 \le \alpha_j \le n+1$, $j = 1, \ldots, M-1$, $\alpha_0 = \alpha_M = n+1$. Integrating by parts and using the equality $w_j D_j = D_0$ a.e., $j = 1, \ldots, n$, we obtain

$$\int_a^b [D_0\tilde{L}_n f(t) - D_0\tilde{L}_n s(t)] D_0\tilde{L}_n s(t)\, dt$$

$$= \sum_{j=1}^{M} \sum_{k=1}^{n} (-1)^{k-1}[D_k \ldots D_n f(t) - D_k \ldots D_n s(t)] D_{k-1} \ldots D_1 D_0 \tilde{L}_n s(t)\big|_{x_{j-1}}^{x_j}$$

$$+ \sum_{j=1}^{M} (-1)^n [f(t) - s(t)] L_n D_0 \tilde{L}_n s(t)\big|_{x_{j-1}}^{x_j}.$$

Hence we obtain

THEOREM 2.1 (First integral relation, cf. [3, 35]). *Suppose that* $f \in H_{V_{2n+1},2}^{n+1}(I)$ *and* $s \in S_\Delta^{V_{2n+1}}(I)$ *satisfy one of the following conditions*:

(a) $f(x_j) = s(x_j)$, $D_i \ldots D_n f(x_j) = D_i \ldots D_n s(x_j)$, $j = 0, \ldots, M$, $i = n, \ldots, n-\alpha_j+2$ *for* $\alpha_j \ge 2$, $\alpha_0 = \alpha_M = n+1$,

(b) $f(x_j) = s(x_j)$, $D_i \ldots D_n f(x_j) = D_i \ldots D_n s(x_j)$, $j = 0, \ldots, M-1$, $i = n, \ldots, n-\alpha_j+2$ *if* $\alpha_j \geq 2$, $\alpha_0 = \alpha_M \leq n+1$, f *and* s *are* $(b-a)$*-periodic.*

Then

$$\int_a^b [L^* f(t) - L^* s(t)]^2 \, dt = \int_a^b [L^* f(t)]^2 \, dt - \int_a^b [L^* s(t)]^2 \, dt. \tag{2.3}$$

COROLLARY 2.1. *If* $f = 0$ *and* $s \in S_\Delta^{V_{2n+1}}(I)$ *satisfy* (a) (*or* (b)) *of Theorem* 2.1, *then* $s = 0$.

Proof (cf. [3]). It follows from (2.3) that $L^* s = 0$. Hence $s \in P_V$. Since s interpolates f on Δ, $s = 0$.

Let $f \in H_{V_n,2}^{n+1}(I)$ and $s \in S_\Delta^{V_{2n+1}}(I)$. Write the spline s in the form (2.2). If f and s satisfy condition (a) (or (b)) of Theorem 2.1, then the $2n + 2 + \sum_{j=1}^{M-1} \alpha_j$ coefficients b_i, $a_{j,k}$, $i = 0, \ldots, 2n+1$, $j = 1, \ldots, M-1$, $k = 0, \ldots, \alpha_j - 1$, satisfy the same number of linear equations. It follows from Corollary 2.1 that the matrix of this system is nonsingular. Hence we obtain

THEOREM 2.2. *For any function* $f \in H_{V_{2n+1},2}^{n+1}(I)$ *there exists a unique Chebyshevian spline* $s \in S_\Delta^{V_{2n+1}}(I)$ *satisfying condition* (a) (*or* (b)) *of Theorem* 2.1.

Integrating by parts, we obtain

THEOREM 2.3 (Second integral relation, cf. [3, 35]). *If* $f \in H_{V_{2n+1},p}^{2n+2}(I)$ $(w_{2n+2} = 1)$ *and* $s \in S_\Delta^{V_{2n+1}}(I)$ *satisfy condition* (a) (*or* (b)) *of Theorem* 2.1 *then*

$$\int_a^b [L^* f(t) - L^* s(t)]^2 \, dt = (-1)^{n+1} \int_a^b [f(t) - s(t)] L L^* f(t) \, dt.$$

Integrating by parts, we obtain the lemma below which will be used later for construction of biorthogonal spline systems (see [3, 67, 68, 70]).

LEMMA 2.2. *Let* $\Delta_i = \{a = t_{0,i} \leq t_{1,i} \leq \ldots \leq t_{N_i,i} = b\}$, $i = 1, 2$, *be two given partitions with* $\Delta_1 \subset \Delta_2$, *i.e. each point of* Δ_1 *is a point of* Δ_2, *and with multiplicities* $\alpha_{i,j}$ *less than or equal to* $n+1$ *and let* $s_i \in S_{\Delta_i}^{V_{2n+1}}(I)$ $i = 1, 2$, *be splines such that* $s_2(t) = 0$ *for* $t \in \Delta_1$ *and either* $\alpha_{i,k} = n+1$, $i = 1, 2$, $k = 0, N_i$, *or* s_1 *and* s_2 *are periodic. Then*

$$\int_a^b L^* s_1(t) L^* s_2(t) \, dt = 0.$$

2. *B*-**splines.** Let $U = U_n$ and $V = V_n$ be CCT-systems defined by (1) and (1.2) respectively, with assumption (2). Assume further that Δ is defined by (2.1) with $\alpha_0 = \alpha_M = n+1$.

DEFINITION 2.4 (cf. [13, 34, 40, 57]). The *ith B-spline*, $i = 0, \ldots, N-n-1$, w.r.t. the system U (LB-spline, $L = D_0 L_n$) and the partition Δ is defined by

$$M_{i,n}(x) = M_{i,n}(t_i, \ldots, t_{i+n+1}; x) = [t_i, \ldots, t_{i+n+1}; \varphi_n^*(t,x)]_V, \tag{2.4}$$

where φ_n^* is defined by (1.6).

As in the proof of Theorem 1.3 we conclude that

$$\frac{d}{dx}\varphi_n^*(t,x) = -w_1(x)\tilde{\varphi}_{n-1}(t,x) \quad \text{a.e.}, \tag{2.5}$$

where

$$\tilde{\varphi}_{n-1}(t,x) = \begin{cases} \displaystyle\int_x^t w_n(\tau_1) \int_x^{\tau_1} w_{n-1}(\tau_2) \ldots \int_x^{\tau_{n-2}} w_2(\tau_{n-1})\, d\tau_{n-1} \ldots d\tau_1 & \text{for } x \le t \le b, \\ 0 & \text{for } t < x. \end{cases}$$

Repeating this reasoning we prove that $M_{i,n} \in S_\Delta^U(I)$.

Let $M_{j,n-1}$ be the jth B-spline w.r.t. U_{n-1}^1 and Δ', where $U_{n-1}^1 = \{u_j^1\}_{j=0}^{n-1}$, $u_j^1 = D_1 u_{j+1}$, $j = 1, \ldots, n$, and Δ' is obtained from Δ by replacing all points of multiplicity $n+1$ with points of multiplicity n. Put

$$\tilde{M}_{j,n-1}(x) = M_{j,n-1}(x) \Big/ \int_{t_j}^{t_{j+1}} w_1(t) M_{j,n-1}(t)\, dt.$$

DEFINITION 2.5. The jth *normalized B-spline*, $j = 0, \ldots, N-n-1$, w.r.t. the system U and the partition Δ is defined by

$$N_{j,n}(x) = \begin{cases} \displaystyle\psi_{t_j}(x) - \int_a^x w_1(t)\tilde{M}_{j+1,n-1}(t)\, dt \\ \qquad\qquad \text{for } t_j = \ldots = t_{j+n} < t_{j+n+1}, \\ \displaystyle\int_a^x w_1(t)[\tilde{M}_{j,n-1}(t) - \tilde{M}_{j+1,n-1}(t)]\, dt \\ \qquad\qquad \text{for } t_j < t_{j+n} \text{ and } t_{j+1} < t_{j+n+1}, \\ \displaystyle\int_a^x w_1(t)\tilde{M}_{j,n-1}(t)\, dt - \psi_{t_{j+n+1}}(x) \\ \qquad\qquad \text{for } t_j < t_{j+1} = \ldots = t_{j+n+1}, \end{cases} \tag{2.6}$$

where $\psi_t(x) = 1$ for $x > t$ and 0 for $x \le t$, and for $j = 0$ we put $N_{0,n}(a) = 1$.

Again by (2.5) we obtain $N_{j,n} \in S_\Delta^U(I)$.

Basic splines w.r.t. Chebyshev systems were defined by S. Karlin [34] and in the special case of trigonometric splines by I. J. Schoenberg [53]. Normalized B-splines were defined in another way by M. J. Marsden [40].

For polynomial B-splines we have (see [13, 18, 57])

$$[t_0, \ldots, t_{n+1}; f] = \frac{1}{(n+1)!} \int_{t_0}^{t_{n+1}} f^{(n+1)}(t) \hat{M}_{0,n}(t)\, dt\,,$$

where $\hat{M}_{0,n}$ is the B-spline in the algebraic case. Since $\int_{t_0}^{t_{n+1}} \hat{M}_{0,n}(t)\, dt = 1$ (see [13, 18, 57]), we have

$$\lim_{t_{n+1} \to 0} [t_0, \ldots, t_{n+1}; f] = \frac{1}{(n+1)!} f^{(n+1)}(t_0) \qquad \text{for} \quad f \in C^{n+1}(I)\,.$$

It follows from (1.11) that

$$\begin{aligned} v_{n+1}(t) &= \int_a^t w_n(\tau_1) \int_a^{\tau_1} w_{n-1}(\tau_2) \ldots \int_a^{\tau_{n-1}} w_1(\tau_n) \int_a^{\tau_n} w_{n+1}(\tau_{n+1})\, d\tau_{n+1} \ldots d\tau_1 \\ &= \int_a^b \varphi_n^*(t,x)\, dx\,, \qquad \text{where} \quad w_{n+1} = 1\,. \end{aligned}$$

Hence by (1.14) and (2.4) we obtain

$$\int_{t_j}^{t_{j+n+1}} M_{j,n}(t)\, dt = 1\,, \quad j = 0, \ldots, N-n-1. \tag{2.7}$$

Just as for polynomial splines, we apply the generalized Taylor formula (1.11) (see [13, 18, 35, 57]) and obtain

$$[t_j, \ldots, t_{j+n+1}; f]_V = \int_{t_j}^{t_{j+n+1}} L^* f(t) M_{j,n}(t)\, dt \quad \text{for} \quad f \in H_{V,p}^{n+1}(I)\,. \tag{2.8}$$

Hence for $f \in C_V^{n+1}(I)$

$$\lim_{t_{j+n+1} \to t_j} [t_j, \ldots, t_{j+n+1}; f]_V = L^* f(t_j)\,. \tag{2.9}$$

Applying this equality for the systems $\{v_0, \ldots, v_k\}$, $k = 1, \ldots, n$, and using the properties of determinants, we obtain

THEOREM 2.4. *Let $\Delta' = \{a \le t'_j < \ldots < t'_{j+n+1} \le b\}$ be a partition of I with distinct points. Assume that there exists $\varepsilon > 0$ such that f and $D_k \ldots D_n f$, $k = n, \ldots, \alpha_j$ (and for $\alpha_j = n+1$ also Lf) are continuous in $(t_j - \varepsilon, t_j + \varepsilon)$, $j = 1, \ldots, N-n-1$, where α_j is the multiplicity of t_j. Then the following limits exist:*

$$(2.10)\qquad \lim_{\substack{t'_i \to t_i \\ i=0,\ldots,n+1}} [t'_j, \ldots t'_{j+n+1}; f]_V = [t_j, \ldots, t_{j+n+1}; f]_V\,, \qquad j = 0, \ldots, N-n-1.$$

Proof. It follows from the definition of the divided differences and the properties of determinants that for any function g

$$[t'_i, \ldots, t'_{i+k+1}; g]_V = \sum_{l=i}^{i+k+l} a_{l,k}\, g(t'_l)\,,$$

where the coefficients $a_{l,k}$ are different from zero and do not depend on g. Hence

$$[t'_j, \ldots, t'_{j+n+1}; f]_V = \det[b_{i,k}] / \det[c_{i,k}]\,,$$

where $b_{i,k} = c_{i,k} = [t'_{k-d_k}, \ldots, t'_k; u_i]_V$, $i = 0, \ldots, n$, $k = j, \ldots, j+n+1$, $b_{n+1,k} = [t'_{k-d_k}, \ldots, t'_k; f]_V$, $c_{n+1,k} = [t'_{k-d_k}, \ldots, t'_k; u_{n+1}]_V$, $k = j, \ldots, j+n+1$, and $d_k = \max\{l : t_k = \ldots = t_{k-l}\}$. Now letting $t'_i \to t_i$, $i = j, \ldots, j+n+1$, and applying (2.9) for the systems $\{v_0, \ldots, v_k\}$, $k = 1, \ldots, n$, we obtain (2.10).

COROLLARY 2.2. *Let Δ' be as in Theorem 2.4. Then*

$$(2.11)\qquad \lim_{\substack{t'_i \to t_i \\ i=j,\ldots,j+n+1}} M_{j,n}(t'_j, \ldots, t'_{j+n+1}; x) = M_{j,n}(t_j, \ldots, t_{j+n+1}; x)$$

at each x except for any knot of Δ with multiplicity $n+1$.

If $t_j \le t'_j$ (or $t'_{j+n+1} \le t_{j+n+1}$) and t_j (or t_{j+n+1}) has multiplicity $n+1$, then (2.11) holds true at t_j (or t_{j+n+1}).

Hence it follows that it suffices to prove the properties of splines for partitions with distinct points.

LEMMA 2.3 (see [74]). *Every spline $\varphi \in S^U_{\Delta_j}(I)$ with $\Delta_j = \{a \le t_j < \ldots < t_{j+n+1} \le b\}$ satisfying*

$$D_i \ldots D_1 \varphi(t_k) = 0\,, \quad i = 1, \ldots, n-1, \quad k = j, j+n+1,$$

can be represented in the form

$$\varphi(x) = \alpha M_{j,n}(x)\,,$$

where α is a constant depending only on φ.

Proof. The partition Δ_j has only distinct points. Put

$$\tilde{\varphi}_{n,i}(t,x) = \begin{cases} \int_t^x w_n(\tau_1) \int_{\tau_1}^x w_{n-1}(\tau_2) \ldots \int_{\tau_{n-i-1}}^x w_{i+1}(\tau_{n-i})\, d\tau_{n-i} \ldots d\tau_1 & \text{for } t \le x, \\ 0 & \text{for } x < t, \end{cases}$$

$i = 0, \ldots, n-1$, and

$$\tilde{\varphi}_{n,n}(t,x) = \begin{cases} 1 & \text{for } t \le x, \\ 0 & \text{for } t > x. \end{cases}$$

We can write φ in the following form (Lemma 2.1):

$$\varphi(x) = \sum_{k=j}^{j+n} b_k \tilde{\varphi}_{n,0}(t_k, x)\,.$$

Therefore

$$\begin{aligned} \varphi(t_{j+n+1}) &= \sum_{k=j}^{j+n} b_k \tilde{\varphi}_{n,0}(t_k, t_{j+n+1}) = 0\,, \\ (-1)^i D_i \ldots D_1 \varphi(t_{j+n+1}) &= \sum_{k=j}^{j+n} b_k \tilde{\varphi}_{n,i}(t_k, t_{j+n+1}) = 0\,, \quad i = 0, \ldots, n-1. \end{aligned}$$

This system has matrix $A_n = [\tilde{\varphi}_{n,i}(t_k, t_{j+n+1});\ i = 0, \ldots, n-1,\ k = j, \ldots, j+n]$. Let B_n be the matrix obtained from A_n by cancelling the last column. We shall prove that $\det B_n > 0$. For $n = 1$, $B_1 = \tilde{\varphi}_1(t_j, t_{j+1}) = \int_{t_j}^{t_{j+1}} w_1(t)\, dt > 0$, because $w_1 > 0$. Assume that $\det B_{n-1} > 0$ for every system of functions $\{\tilde{w}_i\}_{i=0}^{n-1}$, $\tilde{w}_i > 0$, satisfying (2). Subtracting the ith column of B_n from its predecessor and factoring out the integrals of the function w_n, we obtain

$$\det B_n = \int_{t_j}^{t_{j+1}} w_n(x_1) \ldots \int_{t_{j+n-2}}^{t_{j+n-1}} w_n(x_{n-1}) \int_{t_{j+n-1}}^{t_{j+n+1}} w_n(x_n) \tilde{B}_{n-1}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\,,$$

where $\tilde{B}_{n-1}(x_1, \ldots, x_n) = \det[\tilde{\varphi}_{n-1,i}(x_k, t_{j+n+1}),\ i = 0, \ldots, n-1,\ k = 1, \ldots, n]$. Repeating this reasoning for the determinant $\tilde{B}_{n-1}$ and expanding the resulting determinant w.r.t. the last row, we obtain

$$\det B_n = \int_{t_j}^{t_{j+1}} w_n(x_1)\dots\int_{t_{j+n-2}}^{t_{j+n-1}} w_n(x_{n-1})\int_{t_{j+n-1}}^{t_{j+n+1}} w_n(x_n)\int_{x_1}^{x_2} w_{n-1}(y_1)$$

$$\dots\int_{x_{n-1}}^{x_n} w_{n-1}(y_{n-1})B_{n-1}(y_1,\dots,y_{n-1})\,dx_1\dots dx_n\,dy_1\dots dy_{n-1}\,,$$

where $B_{n-1}(y_1,\dots,y_{n-1}) = \det[\tilde{\varphi}_{n-2,i}(y_k,t_{j+n+1});\ i=0,\dots,n-2,\ k=1,\dots,n-1]$. Since $x_i < x_{i+1}$, $i=1,\dots,n-1$, we have $B_{n-1}(y_1,\dots,y_{n-1}) > 0$ by the inductive hypothesis, whence $\det B_n > 0$ and the lemma is proved.

Applying Corollary 2.2 and Lemma 2.3 we obtain

COROLLARY 2.3. *There exists a constant r_j such that*

$$N_{j,n}(x) = r_j M_{j,n}(x)\,. \tag{2.12}$$

Remark 2.1. For the system $\{(t-a)^i\}_{i=0}^n$, $r_j = (t_{j+n+1}-t_j)/(n+1)$.

THEOREM 2.5. *There exists a constant C_U such that*

$$C_U^{-1}\hat{M}_{j,n}(x) \le M_{j,n}(x) \le C_U\hat{M}_{j,n}(x)\,,$$

where $\hat{M}_{j,n}$ is the j-th B-spline for the system $\{(t-j)^i\}_{i=0}^n$.

Proof. Assume that Δ has only distinct points. Then

$$M_{j,n}(x) = D_V\begin{pmatrix} v_0, & \dots, & v_n, & \varphi_n^*(t,x) \\ t_j, & \dots, & t_{j+n}, & t_{j+n+1}\end{pmatrix}\Big/ D_V\begin{pmatrix} v_0, & \dots, & v_{n+1} \\ t_j, & \dots, & t_{j+n+1}\end{pmatrix} = \frac{L_n(x)}{M_n(x)}$$

and analogously $\hat{M}_{j,n}(x) = \hat{L}_n(x)/\hat{M}_n(x)$. Here $L_n(x) = \det[a_{i,j}]_{i,j=0}^{n+1}$, where $a_{0,j}=1$, $j=0,\dots,n+1$,

$$a_{i,j} = \int_a^{t_j} w_n(\tau_1)\int_a^{\tau_1} w_{n-1}(\tau_2)\dots\int_a^{\tau_{i-1}} w_{n-i+1}(\tau_i)\,d\tau_i\dots d\tau_1\,,$$

$$i=1,\dots,n,\quad j=0,\dots,n+1,$$

$$a_{n+1,j} = \varphi_n^*(t_j,x) = \int_a^{t_j} w_n(\tau_1)\int_a^{\tau_1} w_{n-1}(\tau_2)\dots\int_a^{\tau_{n-1}} w_1(\tau_n)\rho_x(\tau_n)\,d\tau_n\dots d\tau_1\,,$$

$$j=0,\dots,n+1,$$

where $\rho_x(t)=1$ for $t \ge x$ and 0 for $t < x$.

Subtracting the jth column of $L_n(x)$ from its successor, then expanding the determinant w.r.t. the first row and applying the properties of determinants, we obtain

$$L_n(x) = \int_{t_0}^{t_1} w_n(x_1) \ldots \int_{t_n}^{t_{n+1}} w_n(x_{n+1}) \det[b_{i,j}]\, dx_1 \ldots dx_{n+1}\,,$$

where $b_{0,j} = 1$, $j = 1, \ldots, n+1$,

$$b_{i,j} = \int_a^{x_j} w_{n-1}(\tau_1) \int_a^{\tau_1} w_{n-2}(\tau_2) \ldots \int_a^{\tau_{i-1}} w_{n-i}(\tau_i)\, d\tau_i \ldots d\tau_1\,,$$

$$i = 1, \ldots, n-1, \quad j = 1, \ldots, n+1,$$

$$b_{n,j} = \int_a^{x_j} w_{n-1}(\tau_1) \int_a^{\tau_1} w_{n-2}(\tau_2) \ldots \int_a^{\tau_{n-2}} w_1(\tau_{n-1}) \rho_x(\tau_{n-1})\, d\tau_{n-1} \ldots d\tau_1,$$

$$j = 1, \ldots, n+1.$$

Repeating this reasoning, we obtain

$$(2.13) \qquad L_n(x) = \int_{t_0}^{t_1} w_n(x_{1,1}) \ldots \int_{t_n}^{t_{n+1}} w_n(x_{1,n+1}) \int_{x_{1,1}}^{x_{1,2}} w_{n-1}(x_{2,1}) \ldots \int_{x_{1,n}}^{x_{1,n+1}} w_{n-1}(x_{2,n}) \int_{x_{2,1}}^{x_{2,2}} w_{n-2}(x_{3,1}) \ldots \int_{x_{n-1,1}}^{x_{n-1,2}} w_1(x_{n,1}) \int_{x_{n-1,2}}^{x_{n-1,3}} w_1(x_{n,2})[\rho_x(x_{n,2}) - \rho_x(x_{n,1})]\, dx_{1,1} \ldots dx_{n,2}\,.$$

Since $x_{j,k} \le x_{j,k+1}$, the difference $\rho_x(x_{n,2}) - \rho_x(x_{n,1})$ admits only two values, 0 and 1. Remember that for the system $\{(t-a)^i\}_{i=0}^n$ we have $w_0 = 1$, $w_i = i$, $i = 1, \ldots, n$. Writing $\hat{L}_n(x)$ in the form (2.13), we obtain

$$(2.14) \qquad \hat{L}_n(x) = 2^3 3^4 \ldots n^{n+1} \int_{t_0}^{t_1} \ldots \int_{t_n}^{t_{n+1}} \int_{x_{1,1}}^{x_{1,2}} \ldots \int_{x_{n-1,2}}^{x_{n-1,3}} [\rho_x(x_{n,2}) - \rho_x(x_{n,1})]\, dx_{1,1} \ldots dx_{n,2}\,.$$

Since $M_n(x) = \int_{t_0}^{t_{n+1}} L_n(x)\, dx$ and $\hat{M}_n(x) = \int_{t_0}^{t_{n+1}} \hat{L}_n(x)\, dx$, we have by

(2), (2.13) and (2.14)

$$\left(\frac{c_1}{d_1}\right)^2 \left(\frac{c_2}{d_2}\right)^3 \cdots \left(\frac{c_n}{d_n}\right)^{n+1} \hat{M}_{j,n}(x) \le M_{j,n}(x)$$
$$\le \left(\frac{d_1}{c_1}\right)^2 \left(\frac{d_2}{c_2}\right)^3 \cdots \left(\frac{d_n}{c_n}\right)^{n+1} \hat{M}_{j,n}(x) .$$

Since the above constants depend only on the functions w_j, by Theorem 2.4 we obtain the assertion.

THEOREM 2.6. *Basic splines have the following properties:*

(M.1) $\operatorname{supp} M_{j,n} = [t_j, t_{j+n+1}], \quad M_{j,n}(x) > 0$ *for* $t_j < x < t_{j+n+1}$,

(N.1) $\operatorname{supp} N_{j,n} = [t_j, t_{j+n+1}], \quad N_{j,n}(x) > 0$ *for* $t_j < x < t_{j+n+1}$,

(M.2) $\displaystyle\int_{t_j}^{t_{j+n+1}} M_{j,n}(x)\, dx = 1,$

(N.2) $\displaystyle\sum_{j=0}^{N-n-1} N_{j,n}(x) = 1.$

P r o o f. The first three properties follow from the definition of the basic splines and Lemma 1.3. Let $x \in [t_k, t_{k+1}]$, $t_k < t_{k+1}$. Then

$$\begin{aligned}\sum_{j=0}^{N-n-1} N_{j,n}(x) &= \sum_{j=k-n}^{k} N_{j,n}(x) \\ &= \int_a^x w_1(t) \sum_{j=k-n}^{k} [\tilde{M}_{j,n-1}(t) - \tilde{M}_{j+1,n-1}(t)]\, dt \\ &= \int_a^x w_1(t)[\tilde{M}_{k-n,n-1}(t) - \tilde{M}_{k+1,n-1}(t)]\, dt = 1 ,\end{aligned}$$

for $t_j < t_{j+n}$ and $t_{j+1} < t_{j+n+1}$. The remaining cases of (N.2) are proved by applying Corollaries 2.2 and 2.3.

Now, we estimate the constants $r_j = N_{j,n}(x)/M_{j,n}(x)$, $x \in (t_j, t_{j+n+1})$. By (2.12), Theorem 2.5 and Remark 2.1 we obtain

$$\text{(2.15)} \quad \frac{t_{j+n+1} - t_j}{C_U^2(n+1)} \le r_j \le \frac{C_U^2}{n+1}(t_{j+n+1} - t_j), \quad j = 0, \ldots, N-n-1.$$

Let Δ be defined by (2.1) with $\alpha_0 = \alpha_M = n+1$ and suppose that $f = \sum_{j=0}^{N-n-1} a_j M_{j,n} = 0$. It follows from (2.4) and (2.5) that the functions $M_{0,n}, \ldots, M_{n,n}$ are linearly independent in the interval $[x_0, x_1]$ and therefore form in this interval a basis in the space P_U of polynomials (cf. [22]). Hence $a_0 = \ldots = a_n = 0$. Considering consecutively the intervals $[x_{i-1}, x_i]$, $i = 2, \ldots, M$, we conclude that $a_j = 0$ for $j = 0, \ldots, N-n-1$ and we have established

THEOREM 2.7. *The splines $M_{0,n}, \ldots, M_{N-n-1,n}$ form a basis in $S_\Delta^U(I)$.*

From (2.6) we obtain

LEMMA 2.4. *Put*

$$\beta_j = \Bigg(\int_{t_j}^{t_{j+n+1}} w_1(t) M_{j,n-1}(t)\, dt \Bigg)^{-1} .$$

Then

$$D_1 N_{j,n} = \begin{cases} \beta_j M_{j,n-1} & \text{for } t_{j+1} = t_{j+n+1}, \\ \beta_j M_{j,n-1} - \beta_{j+1} M_{j+1,n-1} & \text{for } t_j < t_{j+n},\ t_{j+1} < t_{j+n+1}, \\ -\beta_{j+1} M_{j+1,n-1} & \text{for } t_j = t_{j+n}. \end{cases}$$

Hence follows

THEOREM 2.8 (cf. [18, 20]). *Let $f \in S_\Delta^U(I)$ and $f = \sum_{j=0}^{N-n-1} a_j N_{j,n}$. Then*

$$D_1 f = -a_1 \beta_1 M_{1,n-1} + \sum_{j=2}^{N-n-2} (a_j - a_{j-1}) \beta_j M_{j,n-1}$$

$$+ a_{N-n-1} \beta_{N-n-1} M_{N-n-1,n-1} .$$

3. The Marsden identity. Let Δ be defined by (2.1) with $\alpha_0 = \alpha_M = n+1$ and let $M_{i,n} \in S_\Delta^U(I)$, $i = 0, \ldots, N-n-1$, be defined by (2.4) and $N_{i,n} = r_i M_{i,n}$. We shall prove a version of the Marsden identity for normalized B-splines. Let $V_{n+2} = \{v_i\}_{i=0}^{n+2}$ be defined by (1) for the system of weight functions $\{w_0 = 1, w_n, \ldots, w_1, 1, 1\}$. We have

THEOREM 2.9 (see [40, 57]). *Let*

$$\psi_i(t) = \frac{D_V(t_{i+1}, \ldots, t_{i+n}, t)}{D_V(t_{i+1}, \ldots, t_{i+n})}, \quad i = 0, \ldots, N-n-1.$$

Then the function $h_n^*(t,x)$ defined by (1.5) *satisfies*

$$h_n^*(t,x) = \sum_{i=0}^{N-n-1} \psi_i(t) N_{i,n}(x) .$$

Moreover, for all $t \in I$,

$$u_j(x) = \sum_{i=0}^{N-n-1} \eta_i^{(j)} N_i(x)\,, \quad j = 0, \ldots, n,$$

where $\eta_i^{(j)} = \hat{L}_{j+1}\psi_i(a)$, $j = 0, \ldots, n$, $\hat{L}_k\psi = D_k \ldots D_n\psi$, $k = 1, \ldots, n$, *and* $\hat{L}_{n+1}\psi = \psi$. *In particular,*

$$\eta_i^{(0)} = 1\,, \quad \eta_i^{(1)} = \frac{D_V\begin{pmatrix} v_0, & \ldots, v_{n-2}, & v_n \\ t_{i+1}, & \ldots, t_{i+n-1}, & t_{i+n} \end{pmatrix}}{D_V\begin{pmatrix} v_0, & \ldots, v_{n-1} \\ t_{i+1}, & \ldots, t_{i+n} \end{pmatrix}}\,,$$

$$\eta_i^{(2)} = \frac{D_V\begin{pmatrix} v_0, & \ldots, v_{n-3}, & v_{n-1}, & v_n \\ t_{i+1}, & \ldots, t_{i+n-2}, & t_{i+n-1}, & t_{i+n} \end{pmatrix}}{D_V\begin{pmatrix} v_0, & \ldots, v_{n-1} \\ t_{i+1}, & \ldots, t_{i+n} \end{pmatrix}}\,.$$

Proof. Let $V_{n+1} = \{v_i\}_{i=0}^{n+1}$. For $x \in (a, b]$ there exists k such that $t_k < x \le t_{k+1}$ with $n \le k < N - n - 1$. For each i there is a unique polynomial $p_i(t) = p_i(t_{i+1}, \ldots, t_{i+n+1}, t_i; t)$ from $P_{V_{n+1}}$ which interpolates $\varphi_n^*(t, x)$ at $t_{i+1}, \ldots, t_{i+n+1}, t_i$ and a unique polynomial $\tilde{p}_i(\tau) = \tilde{p}_i(t_{i+1}, \ldots, t_{i+n+1}, t; \tau)$ from $P_{V_{n+1}}$ which interpolates $\varphi_n^*(\tau, x)$ at $t_{i+1}, \ldots, t_{i+n+1}, t$. From the Newton interpolation formula (1.17) together with (1.15) for the system V_{n+1} we have

$$\begin{aligned} 0 &= [\varphi_n^*(t, x) - p_i(t)] + p_i(t) - \tilde{p}_i(t) \\ &= \left[\begin{matrix} v_0, \ldots, v_{n+1}, & v_{n+2} \\ t_i, \ldots, t_{i+n+1}, & t \end{matrix} \,\middle|\, \varphi_n^*(t, x)\right] \frac{D_{V_{n+2}}(t_i, \ldots, t_{i+n+1}, t)}{D_{V_{n+2}}(t_i, \ldots, t_{i+n+1})} \\ &\quad + \left(\left[\begin{matrix} v_0, & \ldots, v_n, & v_{n+1} \\ t_{i+1}, & \ldots, t_{i+n+1}, & t_i \end{matrix} \,\middle|\, \varphi_n^*(t, x)\right]\right. \\ &\qquad\qquad \left. - \left[\begin{matrix} v_0, & \ldots, v_n, & v_{n+1} \\ t_{i+1}, & \ldots, t_{i+n+1}, & t \end{matrix} \,\middle|\, \varphi_n^*(t, x)\right]\right) \\ &\quad \times \frac{D_{V_{n+2}}(t_{i+1}, \ldots, t_{i+n+1}, t)}{D_{V_{n+2}}(t_{i+1}, \ldots, t_{i+n+1})}\,. \end{aligned}$$

Define

$$W_{i,k+1}(t) = \frac{D_{V_{n+2}}(t_i, \ldots, t_{i+k}, t)}{D_{V_{n+2}}(t_i, \ldots, t_{i+k})}\,, \quad k = n, n+1.$$

Hence

$$\left[\begin{matrix} v_0, \ldots, v_{n+1}, & v_{n+2} \\ t_i, \ \ldots, t_{i+n+1}, & t \end{matrix} \middle| \varphi_n^*(t,x)\right] W_{i,n+2}(t)$$

$$= \left(\left[\begin{matrix} v_0, & \ldots, v_n, & v_{n+1} \\ t_{i+1}, & \ldots, t_{i+n+1}, & t \end{matrix} \middle| \varphi_n^*(t,x)\right] \right.$$

$$\left. - \left[\begin{matrix} v_0, \ldots, v_{n+1} \\ t_i, \ \ldots, t_{i+n+1} \end{matrix} \middle| \varphi_n^*(t,x)\right] \right) W_{i+1,n+1}(t)$$

and in the same way

$$\left[\begin{matrix} v_0, & \ldots, v_{n+1}, & v_{n+2} \\ t_{i+1}, & \ldots, t_{i+n+2}, & t \end{matrix} \middle| \varphi_n^*(t,x)\right] W_{i+1,n+2}(t)$$

$$= \left(\left[\begin{matrix} v_0, & \ldots, v_n, & v_{n+1} \\ t_{i+1}, & \ldots, t_{i+n+1}, & t \end{matrix} \middle| \varphi_n^*(t,x)\right] \right.$$

$$\left. - \left[\begin{matrix} v_0, & \ldots, v_{n+1} \\ t_{i+1}, & \ldots, t_{i+n+2} \end{matrix} \middle| \varphi_n^*(t,x)\right] \right) W_{i+1,n+1}(t)\,.$$

Subtracting these two expressions and setting $R_i(t) = \varphi_n^*(t,x) - p_i(t)$ we obtain

$$R_i(t) - R_{i+1}(t) = W_{i+1,n+1}(t)[M_{i+1,n}(x) - M_{i,n}(x)]\,.$$

If we sum these identities for $i = k-n-1, \ldots, k$ and note that $M_{k-n-1}(x) = M_{k+1,n}(x) = 0$, then after rearranging we can write

$$R_{k-n-1}(t) - R_{k+1}(t) = \sum_{j=k-n}^{k} M_{j,n}(x)[W_{j,n+1}(t) - W_{j+1,n+1}(t)]\,.$$

Since $R_{k+1}(t) = \varphi_n^*(t,x) - h_n^*(t,x)$ and $R_{k-n-1}(t) = \varphi_n^*(t,x)$ we have

$$h_n^*(t,x) = \sum_{j=k-n}^{k} M_{j,n}(x)[W_{j,n+1}(t) - W_{j+1,n+1}(t)]\,.$$

Since the leading terms of $W_{j,n+1}$ and $W_{j+1,n+1}$ are v_{n+1}, the difference $W_{j,n+1} - W_{j+1,n+1}$ is a polynomial from P_V with zeros at $t_{j+1}, \ldots, t_{j+n}$. Thus

$$W_{j,n+1}(t) - W_{j+1,n+1}(t) = \alpha_j \psi_j(t)$$

for some constant α_j. Putting $t = t_{j+n+1}$ we obtain

$$\alpha_j = \frac{D_V(t_{j+1},\dots,t_{j+n})D_V(t_j,\dots,t_{j+n+1})}{D_V(t_j,\dots,t_{j+n})D_V(t_{j+1},\dots,t_{j+n+1})}.$$

Hence

$$h_n^*(t,x) = \sum_{j=0}^{N-n-1} \alpha_j M_{j,n}(x)\psi_j(t) \tag{2.16}$$

and by (1.9)

$$h_n^*(t,x) = \sum_{i=0}^{n}(-1)^{n-i}v_i(t)u_{n-i}(x).$$

Now applying $D_i \dots D_n$, $i = 1,\dots,n$, to both sides of (2.16) and evaluating at $t = a$, we obtain

$$u_{j-1}(x) = (-1)^{j-1}\sum_{i=0}^{N-n-1} \alpha_i M_{i,n}(x) D_j \dots D_n \psi_i(a), \quad j = 1,\dots,n,$$

$$1 = \sum_{i=0}^{N-n-1} \alpha_i M_{i,n}(x)$$

and

$$u_n = (-1)^n \sum_{i=0}^{N-n-1} \alpha_i M_{i,n}(x)\psi_i(a).$$

Comparing this with (2.12) and applying Theorem 2.7 together with (N.2) we conclude that $\alpha_j = r_j$ and the theorem is proved for $x > a$; for $x = a$ it follows from the right continuity of the functions $h_n^*(t,x)$ and $M_{j,n}(x)$ at $x = a$.

Remark 2.2. Since $h_n(x,t) = (-1)^n h_n^*(t,x)$, we have

$$h_n(x,t) = \sum_{i=0}^{N-n-1} (-1)^n \psi_i(t) N_{i,n}(x).$$

Remark 2.3. We have also proved that

$$r_j = \frac{D_V(t_{j+1},\dots,t_{j+n})D_V(t_j,\dots,t_{j+n+1})}{D_V(t_j,\dots,t_{j+n})D_V(t_{j+1},\dots,t_{j+n+1})}. \tag{2.17}$$

Remark 2.4. Corollary 2.2 also holds for normalized B-splines. We proceed similarly to the proof of (2.11).

4. De Boor's inequalities. Let Δ be defined by (2.1) and let $M_{i,n} \in S_\Delta^U(I)$, $i =, \ldots, N-n-1$, be the B-splines defined by (2.4). Define

$$N_{i,p}^{(n)} = r_i^{-1/p} N_{i,n}, \qquad 1 \le p \le \infty, \quad i = 0, \ldots, N-n-1.$$

In this section we discuss the relationship $\sum_i a_i N_{i,p}^{(n)} \leftrightarrow \{a_i\}$ between a spline and the sequence of its normalized B-spline coefficients (cf. [10, 13, 18, 57]). We define

$$\|f\|_p = \left(\int_a^b |f|^p \, dt \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty, \qquad \|f\|_\infty = \sup_{t \in I} |f(t)| .$$

The functions $N_{i,n}$ are nonnegative and form a partition of unity corresponding to Δ, whence $[N_{i,n}(t)]^p \le N_{i,n}(t) \le 1$ for $p \le 1$. Now by (2.12) and Property (M.2) we get $\|N_{i,p}^{(n)}\|_p \le 1$. By Hölder's inequality and (2.15) we have

$$\begin{aligned} r_j &= \int_{t_j}^{t_{j+n+1}} N_{j,n}(t)\, dt \le (t_{j+n+1} - t_j)^{1/q} \|N_{j,n}\|_p \\ &= (t_{j+n+1} - t_j)^{1/q} r_j^{1/p} \|N_{j,p}^{(n)}\|_p \\ &\le C_U^{2/q} (n+1)^{1/q} r_j \|N_{j,p}^{(n)}\|_p, \quad 1/p + 1/q = 1 . \end{aligned}$$

Hence there exists a constant $\alpha_U > 0$ such that

$$\alpha_U \le \|N_{j,p}^{(n)}\|_p \le 1 .$$

Now we shall prove the inequality

$$\left\| \sum_{i=0}^{N-n-1} a_i N_{i,p}^{(n)} \right\|_p \le \|a\|_p , \tag{2.18}$$

where $\|a\|_p = (\sum_{i=0}^{N-n-1} |a_i|^p)^{1/p}$ for $1 \le p < \infty$ and $\|a\|_\infty = \max_{0 \le i \le N-n-1} |a_i|$. By Jensen's inequality and the properties of $N_{j,n}$ we get

$$\begin{aligned} \left| \sum_i a_i N_{i,p}^{(n)}(t) \right|^p &= \left| \sum_i a_i r_i^{-1/p} N_{i,n}(t) \right|^p \\ &\le \sum_i |a_i|^p r_i^{-1} N_{i,n}(t) = \sum_i |a_i|^p M_{i,n}(t) , \end{aligned}$$

whence after integration of both sides we obtain (2.18).

To estimate from below the left-hand side of (2.18) we need the following lemmas:

LEMMA 2.5 (cf. [18]). *Define*

$$\psi_{i,n}^{+}(t) = \begin{cases} \psi_i(t) & \text{for } t \geq t_{i+n}, \\ 0 & \text{for } t < t_{i+n}, \end{cases}$$

where ψ_i *is defined in Theorem* 2.9. *Then*

$$[t_j, \ldots, t_{j+n+1}; \psi_{i,n}^{+}]_V = r_i^{-1}\delta_{i,j}, \tag{2.19}$$

where r_i *is defined by* (2.12).

Proof. Let $i > j$. We have $t_{j+n+1} \leq t_{i+n}$ and $\psi_{n,i}^{+}(t_k) = 0$ for $k < i + n + 1$. Therefore the left side of (2.19) is zero. If $i < j$, then $\psi_{i,n}^{+}(t_k) = \psi_i(t_k)$ for $k = j, \ldots, j + n + 1$ and since $L^*\psi_j = 0$, we have $[t_j, \ldots, t_{j+n+1}; \psi_{i,n}^{+}]_V = [t_j, \ldots, t_{j+n+1}; \psi_i]_V = 0$. If $i = j$, then by (1.14) and (2.17) we obtain

$$[t_i, \ldots, t_{i+n+1}; \psi_{i,n}^{+}]_V = \psi_i(t_{i+n+1}) \frac{D_V(t_i, \ldots, t_{i+n})}{D_V(t_i, \ldots, t_{i+n+1})} = r_i^{-1}.$$

LEMMA 2.6 (cf. [18]). *Let* $\lambda_i \in L_2(I)$. *Then for any given* $j = 0, \ldots, N - n - 1$, $\int_a^b \lambda_j(t) N_{i,n}(t)\,dt = \delta_{i,j}$ *if and only if* $\lambda_j = L^* f$ *for some* $f \in H_{V,2}^{n+2}(I)$ *such that* $f(t_k) = \psi_{i,n}^{+}(t_k)$, $k = 0, \ldots, N - n - 1$.

Proof. It follows from (2.8) and (2.9) that

$$[t_i, \ldots, t_{i+n+1}; f]_V = r_i^{-1} \int_{t_i}^{t_{i+n+1}} L^* f(t) N_{i,n}(t)\,dt$$

for any $f \in H_{V,2}^{n+1}(I)$. On the other hand, $[t_i, \ldots, t_{i+n+1}; \psi_{i,n}^{+}]_V = r_i^{-1}$. Hence for $\lambda = L^* f$ such that $f(t_i) = \psi_{i,n}^{+}(t_i)$ we have $\int_a^b \lambda(t) N_{j,n}(t)\,dt = \delta_{i,j}$. Conversely, if $\int_a^b \lambda(t) N_{i,n}(t)\,dt = \delta_{i,j}$, then there exists $\lambda = L^* f$ for some f such that $\int_a^b L^* f(t) N_{i,n}(t)\,dt = \delta_{i,j}$ and by (2.19), $[t_i, \ldots, t_{i+n+1}; f]_V = [t_i, \ldots, t_{i+n+1}; \psi_{i,n}^{+}]_V$ for $i = 0, \ldots, N - n - 1$. Since $L^* v = 0$ for any $v \in P_V$, we may assume that

$$f(t_k) = \psi_{i,n}^{+}(t_k) \quad \text{for} \quad k = i + 1, \ldots, i + n + 1. \tag{2.20}$$

Now by induction we conclude that (2.20) holds for $k = 0, \ldots, N - n - 1$.

LEMMA 2.7 (cf. [18]). *For any* $j = 0,\dots,N-n-1$ *there exists a constant* C_U *and a function* $h \in L_\infty(I)$ *such that* $\operatorname{supp} h \subset [t_j,\dots,t_{j+n+1}]$,

$$\|h\|_\infty \le C_U(t_{j+n+1}-t_j)^{-1} \tag{2.21}$$

and

$$\int_a^b h(t)N_{i,n}(t)\,dt = \delta_{i,j}\,, \quad i = 0,\dots,N-n-1.$$

Proof. Let $t_{k+1}-t_k = \max_{j\le i\le j+n}(t_{i+1}-t_i)$. Set $x_0 = t_k$, $x_1 = t_{k+1}$ and $\hat{L}_j = D_j\dots D_n$, $j = 1,\dots,n$. Let $\tilde{V}_{2n+1} = \{v_i\}_{i=0}^{2n+1}$ be the system of functions defined by (1) with $a = x_0$ for the system of weight functions $\{w_0,w_n,\dots,w_1,w_0,w_1,\dots,w_n\}$, $w_0 = 1$. There exists a unique polynomial $v \in P_{\tilde{V}_{2n+1}}$ satisfying $v(x_0) = \hat{L}_n v(x_0) = \dots = \hat{L}_1 v(x_0) = 0$, $v(x_1) = \psi_j(x_1)$, $D_n v(x_1) = \hat{L}_n\psi_j(x_1),\dots,\hat{L}_1 v(x_1) = \hat{L}_1\psi_j(x_1)$. We may write v as follows:

(2.22)

$$v(t) = -\frac{\begin{vmatrix} 0 & 1 & \tilde{v}_1(t) & \dots & \tilde{v}_n(t) & \tilde{v}_{n+1}(t) & \dots & \tilde{v}_{2n+1}(t)\\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0\\ \psi_j(x_1) & 1 & \tilde{v}_1(x_1) & \dots & \tilde{v}_n(x_1) & \tilde{v}_{n+1}(x_1) & \dots & \tilde{v}_{2n+1}(x_1)\\ \hat{L}_n\psi_j(x_1) & 0 & 1 & \dots & \hat{L}_n\tilde{v}_n(x_1) & \hat{L}_n\tilde{v}_{n+1}(x_1) & \dots & \hat{L}_n\tilde{v}_{2n+1}(x_1)\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \hat{L}_1\psi_j(x_1) & 0 & 0 & \dots & 1 & \hat{L}_1\tilde{v}_{n+1}(x_1) & \dots & \hat{L}_1\tilde{v}_{2n+1}(x_1) \end{vmatrix}}{\begin{vmatrix} \tilde{v}_{n+1}(x_1) & \dots & \tilde{v}_{2n+1}(x_1)\\ \hat{L}_n\tilde{v}_{n+1}(x_1) & \dots & \hat{L}_n\tilde{v}_{2n+1}(x_1)\\ \dots & \dots & \dots\\ \hat{L}_1\tilde{v}_{n+1}(x_1) & \dots & \hat{L}_1\tilde{v}_{2n+1}(x_1) \end{vmatrix}}$$

for $t \in [x_0,x_1]$. Now we put

$$g(t) = \begin{cases} 0 & \text{for } a \le t \le x_0,\\ v(t) & \text{for } x_0 < t \le x_1,\\ \psi_j(t) & \text{for } x_1 < t \le b, \end{cases}$$

and $h = L^*g$. The function h satisfies the conditions of Lemma 2.6 and $\operatorname{supp} h = [x_0,x_1] \subset [t_j,t_{j+n+1}]$. Now we prove (2.21). Applying Lemmas 1.1 and 1.2 we prove that

$$\sup\{|\psi_j(t)| : t_j \le t \le t_{j+n+1}\} \le C_U(t_{j+n+1}-t_j)^n\,.$$

Hence by the Markov inequality we conclude that there exists a constant

C_U such that

$$
(2.23)\qquad
\begin{aligned}
&|\psi_j(x_1)| \le C_U(t_{j+n+1}-t_j)^n\,,\\
&|\hat{L}_k\psi_j(x_1)| \le C_U(t_{j+n+1}-t_j)^{k-1}\,,\quad k=n,\dots,1.
\end{aligned}
$$

Expanding the determinant from the numerator in (2.22) w.r.t. the first column we can write v in the form

$$
v(t)=\sum_{k=1}^{n}\hat{L}_k\psi_j(x_1)\frac{N_k(x_1)}{D(x_1)}+\psi_j(x_1)\frac{N_0(x_1)}{D(x_1)}\,,
$$

where $D(x_1)$ denotes the denominator of (2.22) and $N_k(x_1)$, $k=0,\dots,n$, the algebraic complement of $\psi_j(x_1)$ and $\hat{L}_k\psi_j(x_1)$ in the numerator of (2.22). Now we compare each fraction with the respective fraction from the algebraic case similarly to the proofs of Lemmas 1.1 and 1.2 and by (2.23) we obtain (2.21).

THEOREM 2.10 (De Boor's inequalities, see [10, 13, 18, 57]). *Let Δ be defined by* (2.1) *with $\alpha_0=\alpha_M=n+1$. There exists a constant C_U such that for $a=\{a_i\}\in l_p$, $1\le p\le\infty$,*

$$
C_U^{-1}\|a\|_p \le \Big\|\sum_{i=0}^{N-n-1} a_i N_{i,p}^{(n)}\Big\|_p \le \|a\|_p\,.
$$

Proof. The right inequality was proved at the beginning of this section. Let

$$
f=\sum_i a_i N_{i,p}^{(n)}=\sum_i b_i N_{i,n}\,,
$$

where $b_i=a_i r_i^{-1/p}$. It follows from Lemma 2.7 that for each i there exists $h_i\in L_\infty(I)$ such that

$$
\begin{gathered}
\operatorname{supp} h_i\subset[t_i,t_{i+n+1}]\,,\\
\int_a^b h_i(t)N_{j,n}(t)\,dt=\delta_{i,j}\,,\quad j=0,\dots,N-n-1,\\
\|h_i\|_\infty\le C_U/(t_{i+n+1}-t_i)\,.
\end{gathered}
$$

Hence

$$
\int_a^b f(t)h_i(t)\,dt=\sum_j b_j\int_a^b N_{j,n}(t)h_i(t)\,dt=b_i\,,
$$

whence by Hölder's inequality

$$|b_i| = \left| \int_{t_i}^{t_{i+n+1}} f(t)h_i(t)\,dt \right| \le \left(\int_{t_i}^{t_{i+n+1}} |f(t)|^p\,dt \right)^{1/p} (t_{i+n+1} - t_i)^{1/q} \|h_i\|_\infty$$

$$\le C_U \left(\int_{t_i}^{t_{i+n+1}} |f(t)|^p\,dt \right)^{1/p} (t_{i+n+1} - t_i)^{-1/p}.$$

Further,

$$\sum_i |a_i|^p \le C_U \sum_{i=0}^{N-n-1} \int_{t_i}^{t_{i+n+1}} |f(t)|^p\,dt \cdot r_i(t_{i+n+1} - t_i)^{-1} \le C_U \|f\|_p$$

and we have proved the theorem.

Now we shall consider the periodic case. Let Δ be defined by (2.1) with $\alpha_0 = \alpha_M$. We extend Δ to the whole real line putting $t_{j+N} = t_j + b - a$ for $j = 0, \pm 1, \pm 2, \ldots$ We also extend the functions w_j to $(b-a)$-periodic functions and define $M_{i,n}$ and $N_{i,n}$ by (2.4) and (2.6) respectively for $i = 0, \ldots, N-1$. Then we extend $M_{i,n}$ and $N_{i,n}$ to $(b-a)$-periodic functions $\mathring{M}_{i,n}$ and $\mathring{N}_{i,n}$. It follows from Definition 2.2 and (2.2) that $\dim \mathring{S}^U_\Delta(I) = N$ and by Theorem 2.7 we obtain

THEOREM 2.11. *The splines* $\mathring{M}_{0,n}, \ldots, \mathring{M}_{N-1,n}$ *form a basis in* $\mathring{S}^U_\Delta(I)$.

If we integrate the functions w_j in (1.2) from $a' < a$, then the divided difference taken at the points $a \le t_j \le \ldots \le t_{j+n+1} \le b$ with respect to the new system $U' = \{u'_j\}_{i=0}^n$ defined by (1.2) will not change. Hence

$$M_{i+N,n}(x + b - a) = M_{i,n}(x)$$

and

$$N_{i+N,n}(x + b - a) = N_{i,n}(x) \tag{2.24}$$

for $i = 0, \pm 1, \pm 2, \ldots$ and we see that $\mathring{M}_{j,n}$ and $\mathring{N}_{j,n}$ are well defined. A simple consequence of (2.24) and the linear independence of functions $N_{j,n}$ is

LEMMA 2.8 (see [18, 57]). *The function* $f = \sum_i a_i N_{i,n}$ *belongs to* $\mathring{S}^U_\Delta(I)$ *if and only if* $a_i = a_{i+N}$ *for all* i.

For $(b-a)$-periodic functions we set

$$\|f\|_p = \left(\int_a^b |f|^p\,dt\right)^{1/p}, \quad 1 \le p < \infty, \qquad \|f\|_\infty = \sup_{t\in I} |f(t)|.$$

Applying Theorem 2.10, (2.24) and the equality

$$\left\|\sum_{i=0}^{N-1} a_i \mathring{N}_{i,p}^{(n)}\right\|_p = \left\|\sum_{i=0}^{N-1} a_i N_{i,p}^{(n)}\right\|_p$$

we obtain

THEOREM 2.12. *There exists a constant C_U such that for any $f = \sum_i a_i N_{i,p}^{(n)} \in \mathring{S}_\Delta^U(I)$*

$$C_U^{-1}\|a\|_p \le \|f\|_p \le \|a\|_p,$$

where $\|a\|_p^p = \sum_{i=0}^{N-1} |a_i|^p$.

COROLLARY 2.4 (cf. [20]). *There exists a constant C_U such that for any $f \in S_\Delta^U(I)$ and $1 \le p \le \infty$*

$$\|D_1 f\|_p \le C_U \|\!|\Delta|\!\|_n^{-1} \|f\|_p, \tag{2.25}$$

where $\|\!|\Delta|\!\|_n = \min_{0\le i\le N-n}(t_{i+n} - t_i)$.

Proof. This follows from Theorems 2.8 and 2.10.

5. A recurrence relation for B-splines. In the algebraic case we have the following important relation:

$$M_{j,n}(x) = \frac{n+1}{n}\left[\frac{x-t_j}{t_{j+n+1}-t_j} M_{j,n-1}(x) + \frac{t_{j+n+1}-x}{t_{j+n+1}-t_j} M_{j+1,n-1}(x)\right], \tag{2.26}$$

where $M_{j,n} \in S_\Delta^n(I)$ and $M_{i,n-1} \in S_\Delta^{n-1}(I)$ and $S_\Delta^k(I)$ denotes the space of polynomial splines of degree k w.r.t. the partition Δ. A generalization of this formula was obtained by T. Lyche [39]. He proved it for Chebyshevian B-splines w.r.t. an ECT-system U, though in his case, the functions $M_{j,n}$ are not in $S_\Delta^{U_{n-1}}(I)$. Applying the Mühlbach recurrence formula we shall give a new and simple proof in our more general case. First we need the following theorems:

THEOREM 2.13 (cf. [18, 47]). *Let* Δ *be defined by* (2.1) *and* $t_0 \le t_{k_0} \le t_{k_1} \le \dots \le t_{k_{n+1}} \le t_N$ *with* $t_{k_0} < t_{k_{n+1}}$. *Then there exist numbers* α_j, $0 < \alpha_j < 1$, *such that* $\sum_{j=k_0}^{k_n-n-1} \alpha_j = 1$ *and for any function* f *defined on* I

$$[t_{k_0}, \dots, t_{k_{n+1}}; f]_U = \sum_{j=k_0}^{k_n-n-1} \alpha_j [t_j, \dots, t_{j+n+1}; f]_U \,. \tag{2.27}$$

Proof. (2.27) is obvious for $n+1 = N$. Assume that it holds for a partition Δ' obtained from Δ by omission of one point. Put $x_j = t_{k_j}$, $x \ne x_j$, $j = 0, \dots, n+1$, $x_0 < x < x_{n+1}$. Applying Theorem 1.8 we obtain

$$\left[\begin{matrix} u_0, \dots, u_{n+1} \\ x_0, \dots, x_{n+1} \end{matrix}\middle| f\right] = \frac{\left[\begin{matrix} u_0, \dots, u_n \\ x_1, \dots, x_{n+1} \end{matrix}\middle| f\right] - \left[\begin{matrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{matrix}\middle| f\right]}{\left[\begin{matrix} u_0, \dots, u_n \\ x_1, \dots, x_{n+1} \end{matrix}\middle| u_{n+1}\right] - \left[\begin{matrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{matrix}\middle| u_{n+1}\right]} = \frac{L}{M}\,.$$

Further,

$$\begin{aligned} L = {} & \left(\left[\begin{matrix} u_0, \dots, u_n \\ x_1, \dots, x_{n+1} \end{matrix}\middle| f\right] - \left[\begin{matrix} u_0, \dots, u_{n-1}, u_n \\ x_1, \dots, x_n, \quad x \end{matrix}\middle| f\right]\right) \\ & + \left(\left[\begin{matrix} u_0, \dots, u_{n-1}, u_n \\ x_1, \dots, x_n, \quad x \end{matrix}\middle| f\right] - \left[\begin{matrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{matrix}\middle| f\right]\right) \\ = {} & \left(\left[\begin{matrix} u_0, \dots, u_n \\ x_1, \dots, x_{n+1} \end{matrix}\middle| u_{n+1}\right] - \left[\begin{matrix} u_0, \dots, u_{n-1}, u_n \\ x_1, \dots, x_n, \quad x \end{matrix}\middle| u_{n+1}\right]\right) \\ & \qquad \times \left[\begin{matrix} u_0, \dots, u_n, \quad u_{n+1} \\ x_1, \dots, x_{n+1}, x \end{matrix}\middle| f\right] \\ & + \left(\left[\begin{matrix} u_0, \dots, u_{n-1}, u_n \\ x_1, \dots, x_n, \quad x \end{matrix}\middle| u_{n+1}\right] - \left[\begin{matrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{matrix}\middle| u_{n+1}\right]\right) \\ & \qquad \times \left[\begin{matrix} u_0, \dots, u_n, u_{n+1} \\ x_0, \dots, x_n, x \end{matrix}\middle| f\right] \\ = {} & (\alpha - \beta) \left[\begin{matrix} u_0, \dots, u_n, \quad u_{n+1} \\ x_1, \dots, x_{n+1}, x \end{matrix}\middle| f\right] \\ & + (\beta - \gamma) \left[\begin{matrix} u_0, \dots, u_n, u_{n+1} \\ x_0, \dots, x_n, x \end{matrix}\middle| f\right], \quad M = \alpha - \gamma\,. \end{aligned}$$

Hence

$$\begin{bmatrix} u_0, \ldots, u_{n+1} \\ x_0, \ldots, x_{n+1} \end{bmatrix} f \Bigg] = \frac{\alpha-\beta}{\alpha-\gamma} \begin{bmatrix} u_0, \ldots, u_n, & u_{n+1} \\ x_1, \ldots, x_{n+1}, x \end{bmatrix} f \Bigg]$$

$$+ \frac{\beta-\gamma}{\alpha-\gamma} \begin{bmatrix} u_0, \ldots, u_n, u_{n+1} \\ x_0, \ldots, x_n, x \end{bmatrix} f \Bigg] .$$

This formula holds for any function defined on I. Assume that $f(x_j) = 0$ for $j = 0, \ldots, n$, $f(x) = 0$ and $f(x_{n+1}) = 1$. Then $[x_0, \ldots, x_{n+1}; f]_U > 0$, $[x_1, \ldots, x_{n+1}, x; f]_U > 0$ and $[x_0, \ldots, x_n, x; f]_U = 0$, so $(\alpha-\beta)/(\alpha-\gamma) > 0$. Analogously $(\beta-\gamma)/(\alpha-\gamma) > 0$, whence by induction we obtain (2.27).

Applying Lemma 2.5 and (2.27) to the function $\psi_{i,n}^+$ we obtain

THEOREM 2.14. *Under the above assumptions we have*

$$[t_{k_0}, \ldots, t_{k_{n+1}}; f]_V = \sum_{j=k_0}^{k_n-n-1} \beta_j [t_j, \ldots, t_{j+n+1}; f]_V ,$$

where $\beta_j = r_j [t_{k_0}, \ldots, t_{k_{n+1}}; \psi_{j,n}^+]_V$.

In the algebraic case Theorem 2.13 was proved by T. Popoviciu [46] and the formula for the coefficients β_j was obtained by C. de Boor [13] and in the complex case by P. M. Tamrazov [62]. For ECT-systems Theorem 2.13 was proved by T. Popoviciu [47].

Remark 2.5. It follows from Definition 2.4 that

$$M_{k_0,n}(t_{k_0}, \ldots, t_{k_{n+1}}; t) = \sum_{j=k_0}^{k_n-n-1} \beta_j M_{j,n}(t_j, \ldots, t_{j+n+1}; t) . \tag{2.28}$$

Define

$$M_{i,n}^{(k)}(x) = \begin{cases} M_{i,n}(t_i, \ldots, \overbrace{x, \ldots, x}^{n-k+1}, \ldots, t_{i+k}; x) & \text{for } x \in (t_i, t_{i+n+1}), \\ 0 & \text{otherwise,} \end{cases}$$

$$M_{i,n}^{(k)}(t_i) = M_{i,n}(\overbrace{x, \ldots, x}^{n-k+1}, t_{i+1}, \ldots, t_{i+k}) ,$$

$$M_{i,n}^{(k)}(t_{i+k}) = M_{i,n}(t_i, \ldots, t_{i+k-1}, \overbrace{x, \ldots, x}^{n-k+1}) , \quad k = 0, \ldots, n,$$

$$M_{i,n}^{(n+1)} = M_{i,n} .$$

In the algebraic case we have $w_0 = 1$, $w_i = i$, $i = 1,\dots,n$, $[t_0,\dots,t_{n+1};f]_V = n[t_0,\dots,t_{n+1};f]$ (the divided difference in the usual sense) and

$$M_{i,n}^{(k)}(x) = (n+1)[t_i,\dots,\overbrace{x,\dots,x}^{n-k+1},\dots,t_{i+k};(t-x)_+^n] = \frac{n+1}{n+1-k}M_{i,k}(x)\,.$$

Write

$$\begin{aligned}\psi_{i,n}^{(k)+}(t) &= \psi_{i,n}^{+}(t_i,\dots,\overbrace{x,\dots,x}^{n-k+1},\dots,t_{i+k};t)\\ &= \begin{cases} \dfrac{D_V(t_{i+1},\dots,\overbrace{x,\dots,x}^{n-k+1},\dots,t_{i+k},t)}{D_V(t_{i+1},\dots,\underbrace{x,\dots,x}_{n-k+1},\dots,t_{i+k})} & \text{for } t \ge t_{i+k-1},\\ 0 & \text{for } t < t_{i+k-1}.\end{cases}\end{aligned}$$

THEOREM 2.15. *For any* $k = 0,\dots,n$

$$M_{i,n}^{(k+1)}(x) = \lambda_{i,k}(x)M_{i,n}^{(k)}(x) + \lambda_{i+1,k}(x)M_{i+1,n}^{(k)}(x)\,, \tag{2.29}$$

where $\lambda_{i,k}(x) = [t_i,\dots,\overbrace{x,\dots,x}^{n-k+1},\dots,t_{i+k};\psi_{i,n}^{(k)+}(t)]_V$ *for* $t_{i+1} < x < t_{i+k}$, $\lambda_{i,k}(x) = 1$ *for* $a \le x \le t_{i+1}$, $\lambda_{i,k}(x) = 0$ *for* $t_{i+k} \le x \le b$ *and* $0 < \lambda_{i,k}(x) < 1$ *and* $\lambda_{i,k}(x) + \lambda_{i+1,k}(x) = 1$.

Proof. This follows from Theorem 2.14.

Remark 2.6. In the algebraic case we obtain (2.26).

Define

$$N_{i,n}^{(k)}(x) = r_{i,k}(x)M_{i,n}^{(k)}(x)\,, \tag{2.30}$$

where $r_{i,k}(x) = r_i(t_i,\dots,\overbrace{x,\dots,x}^{n-k+1},\dots,t_{i+k})$ is defined by (2.17). Since $N_{i,n}^{(k)}(x)$ is the value at x of the normalized B-spline $N_{i,n}(t_i,,\dots,\overbrace{x,\dots,x}^{n-k+1},\dots,t_{i+k};x)$, it has the following properties:

$$\operatorname{supp} N_{i,n}^{(k)} = [t_i,t_{i+k}]\,, \tag{N$_1$}$$

$$N_{i,n}^{(k)}(x) > 0 \qquad \text{for} \quad t_i < x < t_{i+k}\,, \tag{N$_2$}$$

$$\text{(N}_3\text{)} \qquad \sum_{i=0}^{N-k} N_{i,n}^{(k)}(x) = 1\,.$$

Applying (2.29) and (2.30) we obtain

THEOREM 2.16. *For any* $k = 0, \ldots, n$

$$N_{i,n}^{(k+1)}(x) = \mu_{i,k}(x) N_{i,n}^{(k)}(x) + \tilde{\mu}_{i+1,k}(x) N_{i+1,n}^{(k)}(x)\,, \tag{2.31}$$

where

$$\begin{aligned} \mu_{i,k}(x) &= \lambda_{i,k}(x) r_{i,k+1}(x)/r_{i,k}(x)\,, \\ \tilde{\mu}_{i+1,k}(x) &= \lambda_{i+1,k}(x) r_{i,k+1}(x)/r_{i+1,k}(x)\,. \end{aligned}$$

Remark 2.7. Since the divided difference does not depend on the basis of the space P_V, we may compute the respective differences w.r.t. the basis of P_V defined by (1.2) with $a = x$. Hence the determinants appearing in (2.28)–(2.31) may be written in a simpler form (cf. [39]).

6. Bounds on zeros. In this section we give several results on the zeros of Chebyshevian splines. All material given below may be found in [55–57]; we present it on account of its applications in the next part and for the sake of the completeness of the theory.

DEFINITION 2.6. Let $a = (a_1, \ldots, a_m)$ be a vector of real numbers. We define the *number of strong sign changes of* a by

$S^-(a)$ = the number of sign changes in the sequence $a_1, \ldots, a_m$, where zeros are ignored.

Similarly, we define the *number of weak sign changes of* a by

$S^+(a)$ = the maximum number of sign changes in the sequence $a_1, \ldots, a_m$, where each zero can be regarded as either $+1$ or -1, whichever makes the count largest.

DEFINITION 2.7. Let f be a bounded real-valued function on I. We call

$$S_I^-(f) = \sup\{S^-[f(t_1), \ldots, f(t_m)] : \ m \in \mathbb{N},\ a \le t_1 < \ldots < t_m \le b\}$$

the *number of strong sign changes of* f on I. Similarly, we define the *number of weak sign changes of* f on I by

$$S_I^+(f) = \sup\{S^+[f(t_1), \ldots, f(t_m)] : \ m \in \mathbb{N},\ a \le t_1 < \ldots < t_m \le b\}\,.$$

It follows from the definition that

$$S_I^-(f) \leq S_I^+(f).$$

We illustrate $S_I^-(f)$ and $S_I^+(f)$ in Fig. 1.

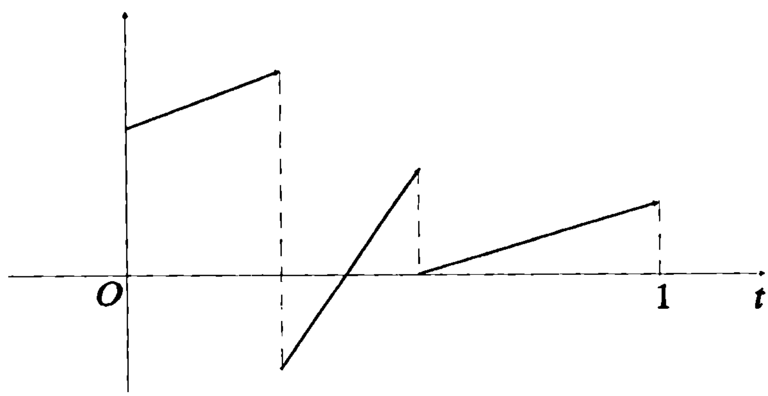

Fig. 1. Strong and weak sign changes: $S_I^-(f) = 2$, $S_I^+(f) = 4$.

DEFINITION 2.8. Let f be defined on I. We say that f *has a zero at* $t \in I$ provided $f(t) = 0$. We shall write

$$Z_I(f) = \text{number of zeros of } f \text{ in } I.$$

If f vanishes identically on a subinterval of I, then $Z_I(f) = \infty$.

DEFINITION 2.9. Let $f \in C_U^k(I)$ and $t \in I$. We say that f *has a zero of multiplicity* k *at* t provided

$$f(t) = L_1 f(t) = \ldots = L_{k-1} f(t) = 0 \neq L_k f(t). \tag{2.32}$$

We define

$$Z_I^*(f) = \text{number of zeros of } f \text{ in } I, \text{ counted with multiplicities.}$$

As usual, if t is an end of I, then D_j is to be understood as either the left or right "derivative". We have the following property: If f has an odd zero at t in (a, b) then f must change sign at t and if f has an even zero at t in (a, b), then f does not change sign at t.

DEFINITION 2.10. If f is an absolutely continuous function on the interval (c, d), then we say that c is a *left Rolle point* of f provided that either $f(c) = 0$ or for every $\varepsilon > 0$, there exists some $c < t < c + \varepsilon$ with $f(t)f'(t) > 0$.

Similarly, we say that d is a *right Rolle point* of f provided either $f(d) = 0$ or for every $\varepsilon > 0$, there exists some $d - \varepsilon < t < d$ with $f(t)f'(t) < 0$.

THEOREM 2.17. *Suppose f is absolutely continuous in (c, d) and that c and d are left and right Rolle points of f, respectively. Then f has at least one sign change in (c, d). If f' is continuous on (c, d), then it has at least one zero in this interval.*

For the proof we refer to [57].

THEOREM 2.18 (Budan–Fourier). *Let U be defined by* (1) *and suppose $u = \sum_{i=0}^{n} c_i u_i$ with $c_n = 0$. Then*

$$Z^*_{(a,b)}(u) \le n - S^+[u(a), -L_1u(a), \ldots, (-1)^n L_n u(a)] - S^+[u(b), L_1u(b), \ldots, L_n u(b)].$$

For the proof we refer to [57].

DEFINITION 2.11 [57] (Isolated zero). Suppose $s \in S^U_\Delta(I)$ does not vanish identically on any interval containing t, and $s(t-) = L_1s(t-) = \ldots = L_{l-1}s(t-) = 0 \neq L_ls(t-)$, while $s(t+) = L_1s(t+) = \ldots = L_{r-1}s(t+) = 0 \neq L_rs(t+)$. Then we say that s has an *isolated zero* at t of multiplicity

$$k = \begin{cases} \alpha + 1 & \text{if } \alpha \text{ is even and } s \text{ changes sign at } t, \\ \alpha + 1 & \text{if } \alpha \text{ is odd and } s \text{ does not change sign at } t, \\ \alpha & \text{otherwise,} \end{cases}$$

where $\alpha = \max(l, r)$.

DEFINITION 2.12 [57] (Interval zeros). Let $s \in S^U_\Delta(I)$, where Δ is defined by (2.1). We define interval zeros of s as follows:

Left-end interval. Suppose that $s(t) = 0$ for $a \le t < x_i$, while $s(t) \neq 0$ for all $x_i < t < x_i + \varepsilon$. Then (a, x_i) is said to be an interval zero of multiplicity

$$k = n + 1 + \sum_{j=1}^{i-1} \alpha_i .$$

Interior interval. Suppose that $s(t) = 0$ for $x_i < t < x_j$ and does not vanish identically on any larger interval containing (x_i, x_j). Then we say that (x_i, x_j) is an interval zero of s of multiplicity

$$k = \begin{cases} \alpha + 1 & \text{if } \alpha \text{ is even and } s \text{ changes sign,} \\ \alpha + 1 & \text{if } \alpha \text{ is odd and } s \text{ does not change sign,} \\ \alpha & \text{otherwise,} \end{cases} \tag{2.33}$$

where $\alpha = n + \sum_{l=i+1}^{j-1} \alpha_l$.

Right-end interval. Suppose that $s(t) = 0$ for $x_j < t < b$, while $s(t) \neq 0$ for all $x_j - \varepsilon < t < x_j$. Then we say that (x_j, b) is a zero of multiplicity

$$k = n + 1 + \sum_{i=j+1}^{M-1} \alpha_i .$$

The meaning of "s changes sign" in (2.33) is that for every $\varepsilon > 0$ there exist $x_i - \varepsilon < \tau_1 < x_i < \tau_2 < x_j + \varepsilon$ with $s(\tau_1)s(\tau_2) < 0$. Just as for isolated zeros, a spline changes sign across an odd interval zero, but it does not change sign across an even one.

This rule counts a jump through 0 at a knot as a zero of multiplicity 1. It can be shown that the multiplicities produced by these rules coincide with the limits of the number of zeros of appropriate sequences of splines converging to s and having simple zeros, i.e. the definitions are natural.

LEMMA 2.9. *Let $s \in S_\Delta^U(I)$. Then $D_1 s$ exists for all t which are not knots of multiplicity $n+1$. Moreover, $D_1 s \in S_{\Delta'}^{U'}(I)$, where $U' = \{1, D_1 u_2, \ldots, D_1 u_n\}$, $\Delta' \subset \Delta$ and the multiplicity of $x_i \in \Delta'$ is $m_i' = \min(n, \alpha_i)$, $i = 1, \ldots, M-1$.*

DEFINITION 2.13. Given a spline $s \in S_\Delta^U(I)$, let $T_1, \ldots, T_d$ be points or intervals where s has zeros of multiplicities $z(T_1), \ldots, z(T_d)$, respectively. We call

$$Z^S(s) = \sum_{i=1}^{d} z(T_i)$$

the *number of zeros* of s in I, *relative to* $S = S_\Delta^U(I)$.

THEOREM 2.19 ([57]). *Suppose $s \in S_\Delta^U(I)$ and that s is continuous. Then*

$$Z^{D_1 S}(D_1 s) \geq Z^S(s) - 1 ,$$

where $D_1 S = S_{\Delta'}^{U'}(I)$ is the space defined in Lemma 2.9.

THEOREM 2.20 ([56, 57]) (Budan–Fourier). *Suppose $s \in S_\Delta^U(I)$ and $L_{\alpha_i - 1} s(x_i) \neq 0$, $L_{\alpha_j - 1} s(x_j) \neq 0$, $i < j - 1$. Then*

$$Z^S_{(x_i, x_j)}(s) \leq n + \sum_{k=i+1}^{j-1} \alpha_k - S^+[s(t_i), -L_1 s(t_i), \ldots, (-1)^{\alpha_i - 1} L_{\alpha_i - 1} s(t_i)]$$
$$- S^+[s(t_j), L_1 s(t_j), \ldots, L_{\alpha_j - 1} s(t_j)] .$$

THEOREM 2.21 ([57]). *For all nontrivial* $s \in S^U_\Delta(I)$

$$Z^S(s) \le n + \sum_{i=0}^{M} \alpha_i .$$

Using this theorem, it can be shown ([55–57]) that the following result holds:

THEOREM 2.22 ([57]). *Let* $n \ge 1$. *Then for any* $0 \le \nu_0 < \ldots < \nu_k \le N - n - 1$ $(0 \le k \le N - n - 1)$

$$D_U \begin{pmatrix} M_{\nu_0,n}, & \ldots, & M_{\nu_k,n} \\ t_0, & & \ldots, t_k \end{pmatrix} \ge 0$$

for all $a \le t_0 \le \ldots \le t_k \le b$ *(with* $t_i < t_{i+n+1}$, $i = 0, \ldots, N - n - 1$*), and moreover, strict inequality holds if and only if*

$$t_i \in \operatorname{int}(\operatorname{supp} M_{\nu_i,n}) = (t_{\nu_i}, t_{\nu_i+n+1})$$

for $i = 1, \ldots, k$, *where* $M_{\nu_i,n}$ *is the B-spline defined by* (2.4).

Now using a generalization of the Cauchy–Binet composition formula (see [9, 11, 34 (pp. 16–17), 56, 57]) we have

$$\int_{\tau_0<\ldots<\tau_k} \cdots \int (\det [M_{\nu_i,n}(\tau_j)]_{i,j=0}^k)^2 \, d\tau_0 \ldots d\tau_k$$

$$= \int_{\tau_0<\ldots<\tau_k} \cdots \int \det \left[\sum_{l=0}^{k} M_{\nu_i,n}(\tau_l) M_{\nu_j,n}(\tau_l) \right]_{i,j=0}^{k} d\tau_0 \ldots d\tau_k$$

$$= \det [(M_{\nu_i,n}, M_{\nu_j,n})]_{i,j=0}^k ,$$

where $(f, g) = \int_a^b fg \, dt$, whence we obtain

THEOREM 2.23. *The Gram matrix* $[(M_{i,n}, M_{j,n})]_{i,j=0}^{N-n-1}$ *is totally positive, i.e. all its minors are nonnegative.*

THEOREM 2.24 (Variation-diminishing property).

$$S_I^-\left(\sum_{i=0}^{N-n-1} a_i N_{i,n} \right) \le S^-(a_0, \ldots, a_{N-n-1})$$

for any $a_0, \ldots, a_{N-n-1}$ *not all zero.*

For the proof we refer to [56] or [57].

III. Spline operators

1. Orthogonal spline projections. Let Δ be defined by (2.1) with $\alpha_0 = \alpha_M = n+1$ and let $S_\Delta^U(I)$ be the space of splines defined in part II. For any $f \in L_2(I)$ there exists a spline $s \in S_\Delta^U(I)$ best approximating f in $L_2(I)$. This spline is defined by the condition: for all $g \in S_\Delta^U(I)$, $\int_a^b (f-s)g\,dt = 0$. Using Theorem 2.1 we can find the spline s of the L_2-approximation to f as follows: for any f there exist a function F and a spline $S \in S_\Delta^{U_{2n+1}}(I)$ such that $D_0 D_1 \dots D_n F = f$ and $F(x_j) = S(x_j)$, $D_i \dots D_n F(x_j) = D_i \dots D_n S(x_j)$, $j = 0, \dots, M$, $i = n, n-1, \dots, n-\alpha_j+2$ for $\alpha_j \geq 2$ and $s = D_0 D_1 \dots D_n S$. If f is periodic then we find periodic functions F and $S \in \overset{\circ}{S}{}_\Delta^{U_{2n+1}}(I)$ such that $D_0 \dots D_n F = f - \int_I f\,dt$, S interpolates F on Δ and $D_0 \dots D_n S = s - \int_I f\,dt$.

Let P_Δ^U be the linear projector on $L_2(I)$ of best L_2-approximation in $S_\Delta^U(I)$. We are interested in estimating the norm of P_Δ^U in $L_p(I)$. First we shall find the estimate of $\|P_\Delta^U\|_\infty$. We need the following notations: For $m \geq 1$, $1 \leq p \leq \infty$, l_p^m denotes the space $\mathbf{R}^m$ with the norm

$$\|a\|_p = \left(\sum_{i=1}^m |a_i|^p\right)^{1/p}, \quad 1 \leq p < \infty, \qquad \|a\|_\infty = \max_i |a_i|.$$

The matrix norm corresponding to a given vector norm is

$$\|A\|_p = \sup\{\|Ax\|_p : \|x\|_p = 1\}.$$

Suppose $\{1, \dots, m\}$ is equipped with the structure of metric space with an integer-valued distance d.

DEFINITION 3.1. We say that a matrix $A = [a_{i,j}]_{i,j=1}^m$ is a *band matrix* with *bandwidth* k if $a_{i,j} = 0$ for $d(i,j) \geq k$.

THEOREM 3.1 (Demko's theorem, see [18, 23, 24]). *Let $A = [a_{i,j}]$ be an $m \times m$ band matrix with bandwidth k and suppose there exist $1 \leq p \leq \infty$ and M such that $\|A\|_p \leq 1$ and $\|A^{-1}\|_p \leq M$. Then there exist constants $C = C_{k,M}$ and $q = q_{k,M}$, $0 < q < 1$, such that for $A^{-1} = [b_{i,j}]$*

$$|b_{i,j}| \leq C q^{d(i,j)}, \quad i,j = 1, \dots, m.$$

For the proof we refer to [18] (see also [23,24]).

A special case of this theorem associated with dyadic partitions of I and B-splines was proved by J. Domsta [25].

Let $s \in S_\Delta^U(I)$. Then $s = \sum_{j=0}^{N-n-1} a_j N_{n,p}^{(n)}$. There exists a system $\{\underline{N}_{j,n},\ j = 0, \dots, N-n-1\}$ in $S_\Delta^U(I)$ biorthogonal to the basis $\{N_{j,n}\}$ w.r.t. the scalar product $(f,g) = \int_I fg\,dt$, i.e. $(N_{i,n}, \underline{N}_{j,n}) = \delta_{i,j}$ for $i,j = 0, \dots, N-n-1$.

Let A be the Gram matrix $A = [a_{i,j}] = [(N_{i,n}, N_{j,n})]_{i,j=0}^{N-n-1}$ and let $B = [b_{i,j}]$ be the inverse matrix to A. Then

$$\underline{N}_{i,n} = \sum_{j=0}^{N-n-1} b_{i,j} N_{j,n} . \tag{3.1}$$

First we estimate the coefficients $b_{i,j}$. Let $A_2 = [a_{i,j}^{(2)}] = [(N_{i,2}^{(n)}, N_{j,2}^{(n)})]$ and $B_2 = [b_{i,j}^{(2)}] = A_2^{-1}$. Write $b = A_2 a$, $a = (a_0, \dots, a_{N-n-1})$, $b = (b_0, \dots, b_{N-n-1})$. Let $f = \sum_{j=0}^{N-n-1} a_j N_{j,2}^{(n)}$. We obtain

$$\begin{aligned} \|b\|_2^2 &= \sum_{j=0}^{N-n-1} |b_j|^2 = \sum_{j=0}^{N-n-1} |(f, N_{j,2}^{(n)})|^2 = \sum_{j=0}^{N-n-1} r_j^{-1} \Big(\int_I f N_j^{1/2} N_j^{1/2}\, dt \Big)^2 \\ &\le \sum_{j=0}^{N-n-1} \int_I f^2(t) N_{j,n}(t)\, dt \le \|f\|_2^2 . \end{aligned}$$

Hence by Theorem 2.10 we obtain

$$\|A_2 a\|_2 \le \|a\|_2 . \tag{3.2}$$

On the other hand, $(A_2 a, a) = \|f\|_2^2$, whence $\|f\|_2^2 \le \|A_2 a\|_2 \|a\|_2$. It follows from Theorem 2.10 that $C_U^{-2} \|a\|_2^2 \le \|f\|_2^2$. Hence

$$\|B_2 b\|_2 \le C_U^2 \|b\|_2 . \tag{3.3}$$

A_2 is a band matrix with bandwidth n and by (3.1)–(3.3) and Theorem 3.1 (with $d(i,j) = |i-j|$) we conclude that there exist constants $C = C_U > 0$ and $q = q_U$, $0 < q < 1$, such that

$$|b_{i,j}^{(2)}| < C q^{|i-j|} .$$

But $A = E A_2 E$, where $E = [e_{i,j}]$ is a diagonal matrix with $e_{i,i} = r_i^{1/2}$. Hence

$$|b_{i,j}| \le C (r_i r_j)^{-1/2} q^{|i-j|} . \tag{3.4}$$

We can write the operator P_Δ^U in the form

$$P_\Delta^U f(t) = \sum_{i=0}^{N-n-1} (f, \underline{N}_{i,n}) N_{i,n}(t) .$$

Using the properties of normalized B-splines, Theorem 2.6, (3.1) and (3.4) . we obtain

$$|P_\Delta^U f(t)| \le \sum_{i=0}^{N-n-1} \sum_{j=0}^{N-n-1} |b_{i,j}(f, N_{j,n})| N_{i,n}(t) \le \frac{2C}{1-q} \lambda \|f\|_\infty \,,$$

where $\lambda = \max_{i,j} r_i^{1/2} r_j^{-1/2}$.

Hence by (2.15) we obtain

THEOREM 3.2 (see [12, 18]). *There exists a constant C_U such that*

$$\|P_\Delta^U\|_\infty \le C_U R_{\Delta,n}^{1/2} \,,$$

where $R_{\Delta,n} = \max_{i,j}(t_{i+n+1} - t_i)/(t_{i+n+1} - t_j)$.

As in [14] (see also [36]) we estimate the L_p norm of P_Δ^U. Let $\{f_i\}_{i=0}^{N-n-1}$ be an orthonormal system in $S_\Delta^U(I)$. Write

$$K_N(t,x) = \sum_{i=0}^{N-n-1} f_i(t) f_i(x) \,.$$

Then

$$P_\Delta^U f(t) = \int_I K_N(t,x) f(x)\, dx \,.$$

Hence using the Hölder inequality we obtain

$$|P_\Delta^U f(t)| \le \int_I |K_N(t,x)|^{1/q} |K_N(t,x)|^{1/p} |f(x)|\, dx$$

$$\le \left(\int_I |K_N(t,x)|\, dx \right)^{1/q} \left(\int_I |K_N(t,x)| |f(x)|^p\, dx \right)^{1/p} , \quad 1/p + 1/q = 1 \,.$$

But by Theorem 3.2

$$\int_I |K_N(t,x)|\, dx = \int_I K_N(t,x) \operatorname{sgn} K_N(t,x)\, dx \le C_U R_{\Delta,n}^{1/2} \,.$$

Hence

$$\int_I |P^U f(t)|^p\, dt \le C_U^{p/q} R_{\Delta,n}^{p/q} \int_I \int_I |K_N(t,x)| |f(t)|^p\, dt\, dx$$

$$\le C_U^{p/q+1} R_{\Delta,n}^{p/(2q)+1/2} \int_I |f(t)|^p\, dt$$

and we have

THEOREM 3.3. *There exists a constant C_U such that*

$$\|P_\Delta^U\|_p \le C_U R_{\Delta,n}^{1/2} \qquad \text{for} \quad 1 \le p \le \infty . \tag{3.5}$$

Remark 3.1. Theorems 3.2 and 3.3 also hold true in the periodic case. This time we use the distance $d(i,j) = \min(|i-j|, N-n-1-|i-j|)$ in Theorem 3.1 and periodic B-splines $N_{i,n}$.

2. Biorthogonal systems. Using Lemma 2.2 we shall construct biorthogonal systems in $L_2(I)$ for both nonperiodic and periodic cases. Let $[c,d] \subset I$. Applying the orthonormal Schmidt procedure to the system U we obtain an orthonormal system $\{f_j\}_{j=1-n}^{1}$ of polynomials from P_U. This system can be obtained as follows: We define $X_0 = 1$. There exists a unique generalized polynomial $X_j(t)$, $j \ge 1$, from $P_{\hat{U}_{2j}}$, where $\hat{U}_{2j} = \{\hat{u}_{0,j}, \hat{u}_{1,j}, \ldots, \hat{u}_{2j,j}\}$ is the system of functions defined by (1) for the sequence of weight functions $\{w_0, w_{j-1}, \ldots, w_1, w_0, w_1, \ldots, w_j\}$, satisfying the following conditions: $X_j(c) = X_j(d) = 0$, $\hat{L}_{kj}X_j(c) = \hat{L}_{kj}X_j(d) = 0$ for $k = j-1, \ldots, 1$, $\|L_{0j}X_j\|_2 = 1$, $j = 1, \ldots, n$, where $L_{kj} = D_k \ldots D_{j-1}$. As in the proof of Theorem 2.1 we can show that $P_0 = 1/(d-c)^{1/2}$, $P_j = L_{0j}X_j$, $j = 1, \ldots, n$, is an orthonormal system in $[c,d]$ and $D_0 L_j P_j = 0$. In the algebraic case we obtain the Legendre orthonormal system of polynomials in $[c,d]$. For the interval $[a,b]$ we have $P_j = \pm f_{j+1-n}$. Comparing the functions X_j with the respective algebraic polynomials, as in Lemmas 1.1 and 1.2, and applying the Markov inequality (Theorem 1.9) we can prove that there exists a constant C_U such that

$$\max_{c \le x,t \le d} \left| \sum_{j=0}^{n} P_j(t)P_j(x) \right| \le C_U (d-c)^{-2} . \tag{3.6}$$

Let $\{\Delta_N\}_{N=2}^{\infty}$ be a given sequence of partitions of I, $\Delta_N = \{t_0 = a,\ t_1 = b,\ t_2, \ldots, t_N\} = \{a = x_{N,0} < x_{N,1} < \ldots < x_{N,N} = b\}$ satisfying the conditions $\Delta_N \subset \Delta_{N+1}$ with $t_N \in \Delta_N \setminus \Delta_{N-1}$. Define the following system of functions $\{\varphi_N\}_{N=2}^{\infty}$: φ_N is a spline from $S_{\Delta_N}^{V_{2n+1}}(I)$ satisfying $D_j \ldots D_n \varphi_N(a) = D_j \ldots D_n \varphi_N(b) = 0$ for $j = n, \ldots, 1$ and $\varphi_N(t_N) = 1$ and $\varphi_N(t) = 0$ for $t \in \Delta_{N-1}$. It follows from Theorem 2.2 that the functions φ_N exist and are unique. Then we conclude from Lemma 2.2 that the system $\{f_N\}_{N=1-n}^{\infty}$, where $f_N = L^*\varphi_N / \|L^*\varphi_N\|_2$, $N \ge 1$, is orthonormal in $L_2(I)$. We assume in the sequel that $R_{N,n} = R_{\Delta_N,n} \le R < \infty$. We have the following estimate for the functions f_N:

THEOREM 3.4 (cf. [21]). *There exist constants $C = C_{U,R}$ and $q = q_{U,R}$ such that*

$$|f_N(t)| \le CN^{1/2} q^{N|t-t_N|} \qquad \textit{for} \quad t \in I\,. \tag{3.7}$$

Proof. $f_N \in S^U_{\Delta_N}(I) \setminus S^U_{\Delta_{N-1}}(I)$. Write f_N in the form $f_N = \sum_{i=-n}^{N} a_{N,i} N_{i,n}$, where $x_{N,0}$ and $x_{N,N}$ are knots of multiplicity $n+1$. Let $t_N = x_{N,k}$. Hence by (3.1) and (3.4) we have

$$\begin{aligned} |a_{N,i}| &= |(f_N, \underline{N}_{i,n})| = \left| \sum_{j=k-n-1}^{k} b_{i,j}(f_N, N_{j,n}) \right| \\ &\le C_U r_i^{-1/2} \sum_{j=k-n-1}^{k} q^{i-j} \le C_U \frac{1}{\sqrt{r_i} q^{n+1}} \frac{q^{|i-k|}}{1-q}\,. \end{aligned}$$

Let $x_{N,i} < t < x_{N,i+1} < x_{N,k}$. Hence by Theorem 2.10 we obtain

$$|f_N(t)| \le \sum_{j=i-n-1}^{i} |a_{N,j}| \le \frac{(n+1)C_U\lambda_N}{(1-q)q^{n+1}} \cdot \frac{q^{|i-k|}}{\sqrt{r_i}}\,,$$

where λ_N is equal to the λ from the proof of Theorem 3.2 for the partition Δ_N. Since $\lambda_N \le R$, there exists a constant α_R such that $\alpha_R|t-t_N| \le |i-k| \times (b-a)(n+1)/N$. Hence

$$q^{|i-k|}/\sqrt{r_i} \le C_{U,R}\sqrt{N} q^{\alpha_R(b-a)^{-1}(n+1)^{-1}N|t-t_N|} \le C_{U,R}\sqrt{N} q'^{N|t-t_N|}$$

and we have proved (3.7) for this case. The case $t > t_N$ is similar.

Let $K_N(t,x) = \sum_{i=1-n}^{N} f_i(t) f_i(x)$ be the Dirichlet kernel for the system $\{f_N\}$. The integral operator with kernel $K_N(t,x)$ is an orthogonal projection of $C(I)$ onto $S^U_{\Delta_N}(I)$ and therefore

$$\int_I K_N(t,x) N_{j,n}(x)\,dx = N_{j,n}(t) \qquad \text{for} \quad t \in I\,,\ j = 1-n, \ldots, N-1,$$

where $N_{j,n}$ is defined for the partition Δ_N. Hence

$$K_N(t,x) = \sum_{i=1-n}^{N-1} N_{i,n}(t) \underline{N}_{i,n}(x) \qquad \text{for} \quad t, x \in I\,.$$

As in the proof of Theorem 3.4, using (3.1), (3.4) and the properties of normalized B-splines we prove

THEOREM 3.5 (see [16, 19, 20]). *There exist constants* $C = C_{U,R}$ *and* $q = q_{U,R}$, $0 < q < 1$, *such that*

$$|K_N(t,x)| \le CNq^{N|t-x|} \quad \text{for} \quad t,x \in I. \tag{3.8}$$

Remark 3.2. Theorems 3.4 and 3.5 also hold in the periodic case. Now the distance $d(t,x) = |t-x|$ in (3.7) and (3.8) is replaced by $d(t,x) = \min(|t-x|, b-a-|t-x|)$.

Estimates (3.7) and (3.8) were first proved by Z. Ciesielski and J. Domsta in [19–21] in the algebraic case for dyadic sequences of partitions. The proofs given above are similar.

Define

$$(H_j f)(t) = \int_t^b w_j(x) f(x)\, dx$$

and in the periodic case

$$(H_j f)(t) = \int_t^b w_j(x) f(x)\, dx - \frac{1}{b-a} \int_a^b w_{j+1}(x) \int_x^b w_j(u) f(u)\, du\, dx. \tag{3.9}$$

Define the following system of functions (cf. [15, 16, 20, 67]):

$$f_j^{(n,k)} = \begin{cases} D_k \dots D_1 f_j & \text{for } k = 1, \dots, n, \\ H_{-k+1} \dots H_0 f_j & \text{for } k = -1, \dots, -n, \end{cases} \tag{3.10}$$

and $f_j^{(n,0)} = f_j$, $j \ge |k| - n + 1$.

In the periodic case we start from $f_1^{(n,k)} = (\int_I w_k\, dt)^{1/2}$ and for $j > 1$, $f_j^{(n,k)}$ is defined by (3.9) and (3.10).

We check easily that the system $\{f_j^{(n,k)}, f_i^{(n,-k)}\}_{i,j=|k|-n+1}^{\infty}$ is biorthonormal w.r.t. the scalar product

$$(f,g)_k = \int_I w_k(t) f(t) g(t)\, dt, \quad k = 1, \dots, n.$$

Applying the interpolation property of splines, we may also obtain these systems from the functions φ_N defined above.

It follows from the definition of $f_j^{(n,k)}$, the interpolation property of the splines φ_N, (3.7) and the Markov inequality that there exist constants $C = C_{U,R_k}$ and $q = q_{U,R_k}$, $0 < q < 1$, where $R_k = \sup_N R_{N,n-k}$ for $k > 0$ and $R_k = R$ for $k < 0$, such that

$$|f_N^{(n,k)}(t)| \le CN^{k+1/2} q^{N|t-t_N|} \quad \text{for} \quad t \in I. \tag{3.11}$$

LEMMA 3.1 (cf. [16]). *There exist constants* $\alpha = \alpha_{U,R_k}$ *and* $\beta = \beta_{U,R_k}$ *such that for* $1 \le p \le \infty$

$$\alpha N^{k+1/2-1/p} \| f_N^{(n,k)} \|_p \le \beta N^{k+1/2-1/p}, \quad -n \le k \le n. \tag{3.12}$$

Proof. It follows from (3.11) that

$$\| f_N^{(n,k)} \|_p \le C N^{k+1/2-1/p}. \tag{3.13}$$

On the other hand, $(f_N^{(n,k)}, f_N^{(n,-k)})_k = 1$ and therefore, by the Hölder inequality and (2),

$$1 \le b_k \| f_N^{(n,k)} \|_p \| f_N^{(n,-k)} \|_q$$

with $q = p/(p-1)$. Hence by (3.13) we obtain (3.12).

COROLLARY 3.1. *There exist constants* $\alpha = \alpha_{U,R_k}$ *and* $\beta = \beta_{U,R_k}$ *such that for* $-n \le k \le n$ *and* $1 \le p \le \infty$

$$\alpha N^{1-2/p} \le \| f_N^{(n,k)} \|_p \| f_N^{(n,-k)} \|_p < \beta N^{1-2/p}.$$

Define

$$K_N^{(n,k)}(t,x) = \sum_{j=|k|-n+1}^{N} f_j^{(n,k)}(t) f_j^{(n,-k)}(x), \quad k = 0, \pm 1, \ldots, \pm n, \tag{3.14}$$

$$P_N^{(n,k)} f(t) = \int_I f(x) K_N^{(n,k)}(t,x) w_k(x)\, dx, \quad k = 0, \pm 1, \ldots, \pm n. \tag{3.15}$$

We shall prove

THEOREM 3.6 (cf. [16, 18, 33, 52, 57]). *There exists a constant* $C = C_{U,R_{N,n-2k}}$ *such that for* $f \in L_p(I)$, $1 \le p < \infty$, $k = 0, \pm 1, \ldots, \pm n$ *and* $p = \infty$ *for* $k = 0, \ldots, n$

$$\| f - P_N^{(n,k)} f \|_p \le C \omega_1^{(p)}(f, 1/N), \tag{3.16}$$

where

$$\omega_1^{(p)}(f,\delta) = \sup_{0<h\le\delta} \left(\int_a^{b-h} |f(t+h) - f(t)|^p\, dt \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty, \tag{3.17}$$

$$\omega_1^{(\infty)}(f,\delta) = \omega_1(f,\delta) = \sup\{ |f(t+h) - f(t)| : \ t, t+h \in [a,b],\ 0 < h \le \delta \}.$$

In the periodic case we integrate in (3.16) over the whole interval $[a,b]$. First we need the following

LEMMA 3.2 (cf. [1]). *Let $f \in L_p(I)$, $1 \le p \le \infty$. Then the Steklov function*

$$f_h(x) = h^{-1} \int_0^h f(x+t)\,dt$$

satisfies

$$\|f - f_h\|_p(h) \le \omega_1^{(p)}(f,h)\,, \tag{3.18}$$

$$\|f_h'\|_p(h) \le h^{-1}\omega_1^{(p)}(f,h)\,, \tag{3.19}$$

where $\|f\|_p(h) = (\int_a^{b-h} |f|^p\,dt)^{1/p}$.

The easy proof will be omitted here.

Let

$$\tilde{f}(t) = \begin{cases} f(t) & \text{for } t \in [a,b], \\ f(2b-t) & \text{for } t \in (b, 2b-a). \end{cases} \tag{3.20}$$

Applying the generalized Minkowski inequality and Lemma 3.2 we obtain

$$\omega_1^{(p)}(\tilde{f},h) \le 5\omega_1^{(p)}(f,h)\,. \tag{3.21}$$

In the periodic case and for $p = \infty$ the constant 5 on the right may be replaced by 1. For the kth modulus of smoothness ($k \ge 1$) this inequality was proved by H. Johnen [32] (see also [18]) but the proof is not so simple.

Proof of Theorem 3.6. First we prove that there exists a constant $C = C_{U,R_{N,n-2k}}$ ($R_{N,m} = R_{N,1}$ for $m \le 0$) such that

$$\|P_N^{(n,k)} f\|_\infty \le C\|f\|_\infty\,. \tag{3.22}$$

We prove this in the periodic case. The proof for non-periodic splines is similar.

Write $P_N^{(n,k)} f = s_{N,f}$. The inequality (3.22) holds true for $k = 0$. We now prove it for $k = -1$. With no loss of generality we may assume that $(f,1)_1 = (s_{N,f},1)_1 = 0$.

Let $S_{N,F}$ be a periodic spline from $S_{\Delta_N}^{U_{2n+1}}(I)$ interpolating the periodic function F at the points of the partition Δ_N, where $D_1 \ldots D_N F = f$. It follows from Theorem 2.1 that $D_1 \ldots D_n S_{N,F} = s_{N,f}$. Hence

$$\|s_{N,f} - f\|_\infty \le \|\Delta_N\|_{n+1} \|D_0 s_{N,f} - D_0 f\|_\infty\,, \tag{3.23}$$

where $\|\Delta_N\|_{n+1} = \max_{1\le j\le N}(x_{N,j+n+1} - x_{N,j})$. Further, $D_0 s_{N,f} = P_N^{(n,0)} D_0 f$. Hence by Theorem 3.3 ($k = 0$)

$$\|D_0 s_{N,f}\|_\infty \le C_{U,R} \|D_0 f\|_\infty$$

for some constant $C_{U,R}$.

We can write $s_{N,f}$ in the form

$$s_{N,f} = \sum_{j=1}^{N} a_j N_{j,n+1} ,$$

where $N_{j,n+1}$ is the normalized B-spline from $S_{\Delta_N}^{U_{n+1}}(I)$. Hence by (3.14) and (3.15) we obtain

$$\sum_{j=1}^{N} a_j (N_{j,n+1}, N_{i,1}^{(n-1)})_1 = (f, N_{i,1}^{(n-1)})_1 , \quad i = 1, \ldots, N, \tag{3.24}$$

where $N_{i,1}^{(n-1)}$ is the ith L_1-normalized B-spline from $S_{\Delta_N}^{D_1 U}(I)$, where $D_1 U = \{D_1 u_j\}_{j=1}^{n}$. Let

$$g_N = \sum_{j=1}^{N} b_j N_{j,n-1} = P_N^{(n-1,0)}(w_1^{-1} f) ,$$

where $P_N^{(n-1,0)}$ is the operator associated with the system $D_1 U$. Then

$$\sum_{j=1}^{N} b_j (N_{j,n-1}, N_{i,1}^{(n-1)}) = (f, N_{i,1}^{(n-1)})_1 , \quad i = 1, \ldots, N. \tag{3.25}$$

Hence by (3.22) for $k = 0$, the system $D_1 U$ and (2), we have

$$\|g_N\|_\infty \le C_U R_{N,n-1}^{1/2} \|f\|_\infty , \tag{3.26}$$

where $R_{N,n-1} = R_{\Delta_N, n-1}$.

It follows from (2) that for each j there exists $b_1^{-1} \le \eta_j \le a_1^{-1}$ such that

$$\sum_{j=1}^{N} b_j \eta_j (N_{j,n-1}, N_{i,1}^{(n-1)})_1 = (f, N_{i,1}^{(n-1)})_1 , \quad i = 1, \ldots, N. \tag{3.27}$$

Let

$$\tilde{g}_N = \sum_{j=1}^{N} b_j \eta_j N_{j,n-1} .$$

Hence by (3.26) and de Boor's inequalities

$$\|\tilde{g}_N\|_\infty \le C_1 \|f\|_\infty , \tag{3.28}$$

for some constant $C_1 = C_{U, R_{N,n-1}}$.

It follows from (3.24) and (3.27) that

$$D_0 s_{n,f} = P_N^{(n,0)}(D_0 \tilde{g}_N), \tag{3.29}$$

whence

$$\|D_0 s_{n,f} - D_0 \tilde{g}_N\|_\infty \le C_2 \|D_0 \tilde{g}_N\|_\infty$$

for some constant $C_2 = C_{U,R_{N,n-1}}$.

Hence by (3.23), (3.29), (3.22) for $k = 0$, the Markov inequality for $\tilde{g}_N$, Theorem 2.8, de Boor's inequalities and (3.28) we obtain

$$\begin{aligned} \|s_{N,f} - \tilde{g}_N\|_\infty &\le \|\Delta_N\|_{n+1} \|D_0 s_{N,f} - D_0 \tilde{g}_N\|_\infty \\ &\le C_3 \|\Delta_N\|_{n+1} \|D_0 \tilde{g}_N\|_\infty \le C_4 R_{N,n-2} \|\tilde{g}_N\|_\infty . \end{aligned} \tag{3.30}$$

We have thus proved (3.22) for $k = -1$.

Let $A = [(N_{i,n+1}, N_{j,1}^{(n-1)})_1]_{i,j=1}^N$, $B = [(N_{i,n-1}, N_{j,1}^{(n-1)})]_{i,j=1}^N$, $a = [a_1, \ldots, a_N]^T$, $b = [b_1, \ldots, b_N]^T$ and $d = [d_1, \ldots, d_N]^T$, where $d_i = (f, N_{i,1}^{(n-1)})_1$, $\|a\| = \max_{1 \le i \le N} |a_i|$. We may write the systems (3.24) and (3.25) as $Aa = d$ and $Bb = d$. Applying de Boor's inequalities, (3.30) and (3.8) to the system $D_0 U$ we obtain

$$\|a\| \le C_1 \|\tilde{g}_N\|_\infty \le C_2 \|g_N\|_\infty \le C_3 \|b\| \le C_4 \|d\|,$$

whence

$$\|a\| \le C_{U,R_{N,n-2}} \|d\| .$$

Now, applying the Demko theorem (Theorem 3.1) to the matrix A, we prove that the elements $\alpha_{i,j}$ of the matrix $A^{-1} = [\alpha_{i,j}]$ satisfy

$$\|\alpha_{i,j}\| \le C q^{Nd(i,j)},$$

where $0 < q < 1$, $C = C_{U,R_{N,n-2}}$ and $q = q_{U,R_{N,n-2}}$.

Reasoning as in Theorem 3.5 we obtain the inequality

$$|K_N^{(n,k)}(t,x)| \le C N q^{Nd(t,x)} \quad \text{for} \quad t, x \in I \quad \text{and} \quad k = \pm 1, \tag{3.31}$$

where $C = C_{U,R_{N,n-2}}$ and $q = q_{U,R_{N,n-2}}$ with $0 < q < 1$. It follows from (3.31) that

$$\|P_N^{(n,k)} f\|_\infty \le C_{U,R_{N,n-2}} \|f\|_\infty .$$

For the remaining k the proof is analogous. The inequality (3.22) for $1 \le p < \infty$ is proved as in Theorem 3.3.

Applying Lemma 3.2, (3.18)–(3.21), Theorem 2.1 (interpolation property of splines) and (3.22) we prove that for given k ($k = 0, \pm 1, \ldots, \pm n$) there exists a function g such that

$$\|f - g\|_p \le C_1 \omega_1^{(p)}(f, 1/N),$$

$$\|D_j g\|_p \le C_2 N \omega_1^{(p)}(f, 1/N), \quad j = 0, \ldots, n,$$

where the constants C_1 and C_2 depend only on U. Further,

$$\begin{aligned}\|P_N^{(n,k)} g - g\|_p &\le C_3 \|\Delta_N\|_{n+1} \|D_k s_{N,g} - D_k g\|_p \\ &\le C_4 \|\Delta_N\|_{n+1} \|D_k g\|_p \le C_5 \omega_1^{(p)}(f, 1/N),\end{aligned}$$

where the constants C_3–C_5 depend only on U and the mesh ratio $R_{N,n-2k}$. Now

$$\begin{aligned}\|f - P_N^{(n,k)} f\|_p &\le \|f - g\|_p + \|g - P_N^{(n,k)} g\|_p + \|P_N^{(n,k)}(g - f)\|_p \\ &\le C_{U,R_{N,n-2}} \omega_1^{(p)}(f, 1/N).\end{aligned}$$

We have also proved the following

THEOREM 3.7. *There exist constants* $C = C_{U,R_{N,n-2},k}$ *and* $q = q_{U,R_{N,n-2},k}$, $0 < q < 1$, $k = 0, \pm 1, \ldots, \pm n$, *such that*

$$|K_N^{(n,k)}(t, x)| \le C N q^{N d(t,x)} \qquad \text{for} \quad t, x \in I, \tag{3.32}$$

where $d(t, x) = |t - x|$ *and in the periodic case* $d(t, x) = \min(b - a - |t - x|, |t - x|)$.

Remark 3.3. In the algebraic case, Theorem 3.7 was proved by Z. Ciesielski and J. Domsta [20] and in the periodic case by Z. Ciesielski [19] for the sequence of dyadic partitions of I. The general algebraic case, with bounded mesh ratio, was proved by the author in [77].

DEFINITION 3.2. A sequence $\{f_N\}_{N=1}^{\infty}$ of elements of a Banach space X is called a *basis* whenever each $f \in X$ has a unique expansion

$$f = \sum_{N=1}^{\infty} a_N f_N$$

convergent in norm.

DEFINITION 3.3. A basis $\{f_N\}_{N=1}^{\infty}$ in $C(I)$ (resp. $\mathring{C}(I)$) is called an *interpolating basis* with nodes $\{t_N\}_{N=1}^{\infty}$ if for each $f = \sum_{i=1}^{\infty} a_i f_i \in C(I)$

(resp. $\mathring{C}(I)$) and each N

$$\sum_{i=1}^{N} a_i f_i(t_k) = f(t_k), \quad k = 1, \ldots, N.$$

DEFINITION 3.4. A basis $\{f_N\}_{N=1}^{\infty}$ in $C_U^k(I)$ (resp. $\mathring{C}_U^k(I)$) is called *simultaneous* if $\{f_N\}_{N=1}^{\infty}$ is a basis in $C_U^r(I)$ (resp. $\mathring{C}_U^r(I)$) for $r = 0, \ldots, k$.

COROLLARY 3.2. *For each k, $|k| \leq n$, the system $\{\mathring{f}_j^{(n,k)}\}_{j=1}^{\infty}$ is a simultaneous basis in $C_{U_{n,k}}^{n-k-1}(I)$, where $U_{n,k} = \{u_{j,k}\}_{j=0}^{n-k}$ and $u_{j,k}$ are defined by means of* (1) *w.r.t. the system of weight functions $\{w_0, w_{k+1}, \ldots, w_n\}$ for $k \geq 0$ and $\{w_0, w_{k+1}, \ldots, w_1, w_0, w_1, \ldots, w_n\}$ for $k < 0$.*

COROLLARY 3.3. *For each $0 \leq k \leq n$, the system $\{f_j^{(n,k)}\}_{j=1-n+k}^{\infty}$ is a simultaneous basis in $C_{U_{n,k}}^{n-k-1}(I)$.*

COROLLARY 3.4. *The system $\{1, t-a, \varphi_N;\ N = 2, 3, \ldots\}$ is an interpolating basis with nodes $\{t_0 = a, t_1 = b, t_N,\ N = 2, 3, \ldots\}$ in $C(I)$ and the system $\{1, \mathring{\varphi}_N,\ N = 2, 3, \ldots\}$ is an interpolating basis in $\mathring{C}(I)$, where $\mathring{\varphi}_N \in \mathring{S}_{\Delta_N}^{U_{2n+1}}(I)$ is zero on Δ_{N-1} and one on $\Delta_N \setminus \Delta_{N-1}$.*

Remark 3.4. In the algebraic case the first simultaneous and interpolating bases in $\mathring{C}^k(I)$ were constructed by S. Schonefeld [54] and Yu. N. Subbotin [61]. Theorem 3.6 was proved by Z. Ciesielski [16] (see also [20]) for dyadic systems of partitions in the algebraic case and the kth modulus of smoothness. For L-splines this theorem follows from the results of K. Scherer and L. Schumaker [52].

3. Equivalence of spline bases. In this section we prove that the bases $\{f_j^{(n,0)}\}$ defined for two different sequences of partitions are equivalent. This was proved by the author [76] for ECT-systems. The proof for CCT-systems is the same but we present it for completeness.

Let $\{\Delta_{N,1}\}_{N=2}^{\infty}$ and $\{\Delta_{N,2}\}_{N=2}^{\infty}$ be given sequences of partitions of the interval I, $\Delta_{N,k} = \{a = t_{N,0}^k < t_{N,1}^k < \ldots < t_{N,N}^k = b\}$, $k = 1, 2$, with $\Delta_{N,k} \subset \Delta_{N+1,k}$ and

$$\sup_{i,j} (t_{N,i+1}^k - t_{N,i}^k)/(t_{N,j+1}^k - t_{N,j}^k) \leq R < \infty, \tag{3.33}$$

and let $t_N^k \in \Delta_{N,k} \setminus \Delta_{N-1,k}$. We assume further that

$$N|t_N^1 - t_N^2| \leq K < \infty, \tag{3.34}$$

where the constant K does not depend on N.

Now, as at the beginning of section 2, we construct the orthonormal systems $\{f_{N,i}\}_{N=1-n}^{\infty}$, $f_{N,i} \in S_{\Delta_{N,i}}^{U^i}(I)$, $i = 1,2$, where the U^i are the systems defined by means of (1) w.r.t. two given systems of weight functions satisfying (2).

We need the following lemmas:

LEMMA 3.3. *For any open set $Q \subset I$ and any sequence $\{\Delta_N\}$ of partitions of I there exists a Whitney decomposition $\{Q_i\}_{i=1}^{\infty}$ of Q such that for each i, $Q_i = [\alpha_i, \beta_i]$ with $\alpha_i, \beta_i \in \Delta_N$ for some N and*

$$Q = \bigcup_{i=1}^{\infty} Q_i\,, \quad \operatorname{int} Q_i \cap Q_j = \emptyset\,, \quad i \neq j\,,$$

$$|Q_i| \leq \operatorname{dist}(Q_i, I \setminus Q) \leq 4R^2 |Q_i|\,.$$

This is a slight modification of the Whitney decomposition given in [60] (pp. 167–168). The proof is similar.

LEMMA 3.4 (cf. [17, 76]). *There exists a constant C depending only on the system U^i and the constants R and K, such that for $t, x \in I$*

$$\sum_{|t-x|>2^{-\mu}} 2^{\mu} \sum_{2^{\mu}<N\leq 2^{\mu+1}} |f_{N,1}(t) f_{N,2}(x)| \leq C/|t-x|^2\,.$$

Proof. In the sequel, we denote by the same letter C different constants depending only on the systems U^i and the constants R and K. Let q_1 and q_2 be the constants from the inequality (3.11) for the systems $\{f_{j,1}\}$ and $\{f_{j,2}\}$ respectively, and let $q = \max(q_1, q_2)$. Applying (3.33), (3.34) and (3.11) we obtain

$$J = \sum_{2^{\mu}<N\leq 2^{\mu+1}} |f_{N,1}(t) f_{N,2}(x)| \leq C \sum_{2^{\mu}<N\leq 2^{\mu+1}} N q^{N(|t-t_N^1|+|x-t_N^2|)}$$

$$< C 2^{\mu} \sum_{2^{\mu}<N\leq 2^{\mu+1}} q^{N(|t-t_N^1|+|x-t_N^1|-|t_N^1-t_N^2|)}$$

$$< C 2^{\mu} \left(\sum\nolimits' q^{2^{\mu}|t-x|} + \sum\nolimits'' q^{2^{\mu}(|t-t_N^1|+|x-t_N^1|)} \right),$$

where the first sum is over t_N^1 between t and x, and the second over the remaining t_N^1. Hence

$$J < C 2^{\mu} q^{2^{\mu}|t-x|} \left(2^{\mu}|t-x| + \sum_{j=1}^{\infty} q^{R^{-1} j} \right) < C 2^{2\mu} q^{2^{\mu}|t-x|} |t-x|\,.$$

On the other hand,

$$\sum_{|t-x|>2^{-\mu}} |t-x| 2^{3\mu} q^{2^\mu |t-x|} \le \frac{C}{|t-x|^2}\,.$$

Combining these two inequalities we complete the proof.

Define the following operator in $L_2(I)$:

$$Tf = \sum_{j=1-n+k}^{\infty} (f, f_{j,1}) f_{j,2}\,.$$

Applying Lemmas 3.3 and 3.4, the orthonormal systems $\{P_j\}$ defined in Section 2 and (3.6) as in [17], we can prove the following

THEOREM 3.8. *There exists a constant C depending only on the systems U^i and the constants R and K such that*

$$|\{t:\ |Tf| > y\}| \le C\|f\|_1/y\,, \quad y > 0\,,$$

i.e. T is of weak type $(1,1)$.

Applying this theorem as in [17], we obtain

THEOREM 3.9. *There exists a constant C depending only on the systems U^i and the constants R, K and p, $1 < p < \infty$, such that*

$$\|Tf\|_p \le C\|f\|_p\,.$$

COROLLARY 3.5. *The systems $\{f_{N,1}\}$ and $\{f_{N,2}\}$ are equivalent in $L_p(I)$ for $1 < p < \infty$, i.e. there exists a constant $C > 0$ depending only on the systems U^i and the constants R, K and p, such that for any N and any numbers $a_{1-n}, \ldots, a_N$*

$$C^{-1}\Big\|\sum_{j=1-n}^{N} a_j f_{j,1}\Big\|_p \le \Big\|\sum_{j=1-n}^{N} a_j f_{j,2}\Big\|_p \le C\Big\|\sum_{j=1-n}^{N} a_j f_{j,1}\Big\|_p\,.$$

COROLLARY 3.6. *If $\{\Delta_{N,1}\}$ is a sequence of dyadic partitions of the interval I, then the system $\{f_{j,2}\}_{j=1-n}^{\infty}$ is an unconditional basis in $L_p(I)$, $1 < p < \infty$, equivalent to the basis given by Z. Ciesielski and J. Domsta* [20]. *What is more, this system is equivalent to the Haar system* (see [17]).

Remark 3.5. Since $Tf_{j,1} = f_{j,2}$ for $j \ge 1-n$, under the assumptions of Theorem 3.9 the orders of convergence of the Nth Fourier sums of f and Tf w.r.t. the systems $\{f_{j,1}\}$ and $\{f_{j,2}\}$ in $L_p(I)$, $1 < p < \infty$, are equal.

Remark 3.6. Theorem 3.9 also holds in the periodic case. The proof is similar. In Lemma 3.4 the distance $|t-x|$ is to be replaced by $d(t,x) = \min(|t-x|, b-a-|t-x|)$.

Remark 3.7. It seems that Theorem 3.9 holds true for the biorthogonal systems $\{h_i^{(n,k)}, h_j^{(n,-k)}\}$, where $h_j^{(n,k)} = f_j^{(n,k)}/\|f_j^{(n,k)}\|_2$. It would suffice to prove Theorem 3.9 for these systems for $p=2$.

4. Positive spline operators and orthogonal splines. In this section we extend the results of P. Sablonnière [51] to Chebyshevian splines. The proofs are similar.

Let $\Delta_N = \{t_{-n} = \ldots = t_{-1} = t_0 = a < t_1 \leq \ldots \leq t_{N-1} < t_N = \ldots = t_{N+n} = b\}$ with $t_j < t_{j+n+1}$. We define the following operator in $L_p(I)$:

$$T_{N,U} f(t) = \sum_{j=-n}^{N-1} (f, M_j) N_j(t), \tag{3.35}$$

where M_j and N_j are the B-splines from $S^U_{\Delta_N}(I)$ defined by (2.4) and (2.6) for the partition Δ_N.

THEOREM 3.10 ([51]).

(i) *$T_{N,U}$ is a positive self-adjoint operator of norm one in $L_p(I)$.*
(ii) *If f is integrable, then*

$$\int_a^b T_{N,U} f(t)\,dt = \int_a^b f(t)\,dt.$$

Proof. We can write $T_{N,U}$ as follows:

$$T_{N,U} f(t) = \int_a^b f(x) K_N(x,t)\,dx,$$

where

$$K_N(x,t) = \sum_{j=-n}^{N-1} M_j(x) N_j(t) = \sum_{j=-n}^{N-1} r_j M_j(x) M_j(t) = K_N(t,x) \geq 0.$$

Since the kernel K_N is positive and symmetric, $T_{N,U}$ is a positive self-adjoint operator. For $f \in L_p(I)$ and $1/p + 1/q = 1$, we get by Hölder's inequality

$$|T_{N,U}f(t)| \le \left(\int_a^b K_N(x,t)\,dx\right)^{1/q}\left(\int_a^b |f(x)|^p K_N(x,t)\,dx\right)^{1/p}$$

Since

$$\int_a^b K_N(x,t)\,dx = \sum_{j=-n}^{N-1}\left(\int_a^b M_i(x)\,dx\right)N_i(t)$$

$$= \sum_{j=-n}^{N-1} N_i(t) = 1 = \int_a^b K_N(t,x)\,dx\,,$$

we get

$$\int_a^b |T_{N,U}f(t)|^p\,dt \le \int_a^b\int_a^b |f(x)|^p K_N(x,t)\,dx\,dt$$

$$= \int_a^b\left(\int_a^b K_N(x,t)\,dt\right)|f(x)|^p\,dx = \int_a^b |f(x)|^p\,dx$$

and we have proved that $\|T_{N,U}f\|_p \le \|f\|_p$ for $1 \le p < \infty$; for $p = \infty$ this follows immediately from (3.35).

For $f = 1$, we have $T_{N,U}f = f$ and

$$\int_a^b T_{N,U}f(t)\,dt = (T_{N,U}f, 1) = (f, T_{N,U}1) = \int_a^b f(t)\,dt\,.$$

We need the following

LEMMA 3.5 (cf. [51]). *For f absolutely continuous in I and* $1 \le p \le \infty$

$$\|T_{N,U}f - f\|_p \le (n+1)\|\Delta_N\|_n\|f'\|_p\,. \tag{3.36}$$

Proof. For $t \in (t_j, t_{j+1})$, $t_j < t_{j+1}$, $0 \le j \le N-1$, we have

$$|T_{N,U}f(t) - f(t)| \le \sum_{i=j-n}^{j} N_i(t)\int_{t_i}^{t_{i+n+1}} M_i(x)\left(\int_t^x |f'(u)|\,du\right)dx\,.$$

Applying the Hölder inequality we obtain successively

$$\int_t^x |f'(u)|^p\,du \le |x-t|^{1/q}\Big(\int_t^x |f'(u)^p\,du\Big)^{1/p},$$

$$\int_{t_i}^{t_{i+n+1}} M_i(x)\Big(\int_t^x |f'(u)|^p\,du\Big)\,dx$$

$$\le \Big(\int_{t_i}^{t_{i+n+1}} M_i(x)|x-t|\,dx\Big)^{1/q}\Big(\int_{t_i}^{t_{i+n+1}} M_i(x)\int_t^x |f'(u)|^p\,du\,dx\Big)^{1/p}.$$

Since $0 \le N_i(x) \le 1$, $|x-t| \le \|\Delta_N\|_{n+1}$ and $\int_I M_i(x)\,dx = 1$, we obtain

$$|T_{N,U}f(t) - f(t)| \le \|\Delta_N\|_{n+1}^{1/q}\Big(\sum_{i=j-n}^{j} N_i(t)\int_{t_i}^{t_{i+n+1}} |f'(u)|^p\,du\Big)^{1/p}.$$

Hence

$$\begin{aligned}\|T_{N,U}f - f\|_p^p &\le \|\Delta_N\|_{n+1}^{p/q}\int_I \sum_{j=0}^{N-1}\sum_{i=j-n}^{j} N_i(t)\int_{t_i}^{t_{i+n+1}} |f'(u)|^p\,du\,dt\\ &\le \|\Delta_N\|_{n+1}^{p/q}\sum_{i=-n}^{N-1}(t_{i+n+1}-t_i)\int_{t_i}^{t_{i+n+1}} |f'(u)|^p\,du\\ &\le (n+1)\|\Delta_N\|_{n+1}^{1+p/q}\int_I |f'(u)|^p\,du\end{aligned}$$

and we have established (3.36) for $1 \le p < \infty$; for $p = \infty$ the proof is similar.

THEOREM 3.11. *For all $f \in L_p(I)$, $1 \le p \le \infty$,*

$$\|T_{N,U}f - f\|_p \le 5(n+3)\omega_1^{(p)}(f, \|\Delta_N\|_{n+1}).$$

Proof. We conclude from Theorem 3.10, Lemma 3.2, (3.21), (3.22) and (3.36) that for $h = \|\Delta_N\|_{n+1}$

$$\|T_{N,U}f - f\|_p \le \|T_{N,U}(f - f_h) - (f - f_h)\|_p + \|T_{N,U}f_h - f_h\|_p$$
$$\le 2\|f - f_h\|_p + (n+1)\|\Delta_N\|_{n+1}\|f_h'\|_p$$
$$\le 5(n+3)\omega_1^{(p)}(f,h)\,.$$

In the periodic case the factor $5(n+3)$ may be replaced by $n+3$.

THEOREM 3.12 (cf. [51]).

(i) *$T_{N,U}$ is a self-adjoint operator in $L_2(I)$ having $N+n$ real positive simple eigenvalues λ_j $(0 \le j \le n+N)$, $0 < \lambda_{N+n} < \ldots < \lambda_1 < \lambda_0 = 1$.*

(ii) *The associated eingenfunctions $V_j(t)$ are orthogonal and they are defined by $V_j(t) = \sum_{j=-n}^{N-1} \omega_{i,j} N_i(t)$, where $\tilde{V}_j = (\omega_{i,j},\ -n \le i \le N-1)$ is the eigenvector of the oscillatory matrix $A_{N,n} = [(M_i, N_j)]_{i,j=-n}^{N-1}$. Moreover, $V_0(t) = 1$ and $S^-(V_j) \le j$ for $1 \le j \le N+n$ (the number of sign changes of V_j on I).*

(iii) *The best least square approximation S of $f \in L_2(I)$ in $S_{\Delta_N}^U(I)$ is*

$$S(t) = \sum_{j=0}^{N+n} \gamma_j (f, V_j) V_j(t)\,,$$

where

$$\gamma_j^{-1} = \lambda_j \sum_{i=-n}^{N-1} r_i \omega_{i,j}^2\,, \quad (f, V_j) = \sum_{i=-n}^{N-1} \omega_{i,j}(f, N_i)\,.$$

Proof (see [51]). $T_{N,U}$ is an operator of finite rank $N+n$ and its restriction to $S_{\Delta_N}^U(I)$ has matrix $A_{N,n}$. It is stochastic, for $\sum_j (M_i, N_j) = (M_i, 1) = 1$, hence $\lambda_0 = 1$ and $\tilde{V}_0 = (1, \ldots, 1) \in \mathrm{R}^{N+n}$ are the associated eigenvalue and eigenvector; moreover, $|\lambda_j| \le 1$ for $j \ge 1$. It follows from Theorem 2.23 that $A_{N,n}$ is totally positive. Since (M_i, N_{i-1}) and (M_i, N_{i+1}) are strictly positive, the theorem of Gantmacher and Krein (see [28, 29]) implies that $A_{N,n}$ is an oscillatory matrix, i.e. some power of it is strictly totally positive: all its minors are positive. Therefore, its eigenvalues are real, positive, simple and the jth eigenvector $\tilde{V}_j = (\omega_{i,j},\ -n \le i \le N-1)$ has exactly j strict sign changes,

$$S^-(V_j) = S^+(V_j) = j\,, \quad 1 \le j \le N+n\,.$$

The jth eigenfunction of $T_{N,U}$ is of course $V_j(t) = \sum_{i=-n}^{N-1} \omega_{i,j} N_i(t)$ (in particular $V_0(t) = \sum_i N_i(t) = 1$) and in view of the variation-diminishing property of B-splines (Theorem 2.24) we have $S^-(V_j) \le j$ for $j \ge 1$.

Since the operator $T_{N,U}$ is self-adjoint in $L_2(I)$, its eigenfunctions V_j form an orthogonal basis in $S^U_{\Delta_N}(I)$. The orthogonal projection of $f \in L_2(I)$ on this space is

$$S(t) = \sum_{j=-n}^{N-1} \gamma_j(f, V_j)V_j(t),$$

where

$$\begin{aligned}\gamma_j^{-1} &= (V_j, V_j) = \sum_{k,l} \omega_{k,j}\omega_{l,j}(N_k, N_l) = \sum_k \omega_{k,j} r_k \Big(\sum_l (M_k, N_l)\omega_{l,j}\Big) \\ &= \sum_k \omega_{k,j} r_k (A_{N,n}\tilde{V}_j)_k = \lambda_j \sum_k r_k \omega_{k,j}^2,\end{aligned}$$

and we have proved the theorem.

Remark 3.8. Theorem 3.12 also holds in the periodic case.

IV. Generalized moduli of smoothness and approximation by splines

1. Generalized moduli of smoothness. Let f be defined in I and let U be defined by (1) with assumption (2). Define $\Delta^U_h f(t) = n!h^{n+1}[t, t+h, \ldots, t+(n+1)h; f]_U$. Let q be a polynomial from P_U interpolating f at $t+jh$, $j = 1, \ldots, n+1$. Then by (1.15) and (1.16) we obtain

$$\alpha|\Delta^U_h f(t)| \le |f(t) - q(t)| \le \beta|\Delta^U_h(t)|,$$

where the constants α and β depend only on the system U.

DEFINITION 4.1 ([73]). We define the *modulus of smoothness* of the function f w.r.t. the system U by the formula

$$\omega_U(f, \delta) = \sup\{|\Delta^U_h f(t)| : 0 < h \le \delta,\ t, t+(n+1)h \in I\}.$$

If $f \in L_p(I)$ for $1 \le p < \infty$, we put

$$\omega^{(p)}_U(f, \delta) = \sup_{0<h\le\delta} \|\Delta^U_h f\|_p((n+1)h),$$

and in the periodic case we integrate over the whole interval I. For $p = \infty$ we put $\omega^{(\infty)}_U(f, \delta) = \omega_U(f, \delta)$.

In the algebraic case we obtain the modulus of smoothness of order $n+1$.

We shall prove the following properties of the moduli of smoothness:

(P.1) $0 \le \omega_U^{(p)}(f,\delta) \le \omega_U^{(p)}(f,\delta')$ for $\delta \le \delta'$.

(P.2) $\omega_U^{(p)}(f,\delta) \le C\|f\|_p$, where the constant C depends only on U.

(P.3) $\omega_U^{(p)}(f+g,\delta) \le \omega_U^{(p)}(f,\delta) + \omega_U^{(p)}(g,\delta)$.

(P.4) $\omega_U^{(p)}(f,m\delta) \le m^{n+1}\omega_U^{(p)}(f,\delta)$, m a positive integer.

(P.5) $\omega_U^{(p)}(f,\lambda\delta) \le (1+\lambda)^{n+1}\omega_U^{(p)}(f,\delta)$, λ a positive number.

(P.6) $\omega_U^{(p)}(f,\delta_1)/\delta_1^{n+1} \le 2^{n+1}\omega_U^{(p)}(f,\delta)/\delta^{n+1}$ for $0 < \delta \le \delta_1$.

(P.7) If $f \in L_p(I)$ and $\omega_U^{(p)}(f,\delta) = o(\delta^{n+1})$ as $\delta \to 0+$, then f is a polynomial from P_U a.e.

(P.8) There exists a constant $C_{m,k}$ depending only on the system U, m and k $(0 < k < m+k \le n)$ such that

$$\omega_{U_{m+k}}^{(p)}(f,\delta) \le C_{m,k}\omega_{U_m}^{(p)}(f,\delta),$$

where U_j is the system of the first j functions from U.

(P.9) $\lim_{\delta\to 0}\omega_U^{(p)}(f,\delta) = 0$ for $f \in L_p(I)$.

Proof (cf. [18, 63, 73]). We shall prove the properties for $1 \le p < \infty$. The proof for $p = \infty$ is analogous. (P.1) follows from the definition of $\omega_U^{(p)}(f,\delta)$.

It follows from the definition of the divided difference of f that $|\Delta_h^U f(t)| = \sum_{j=0}^{n+1} a_j(t,h)f(t+jh)$. Now applying Lemma 1.2 and the Minkowski inequality we obtain (P.2).

(P.3) follows from the Minkowski inequality.

Applying (2.27) we obtain

$$\|\Delta_{mh}^U f(t)\|_p((n+1)mh) = m^{n+1}\left\|\sum_{j=0}^{(m-1)(n+1)} \alpha_j\Delta_h^U f(t+jh)\right\|_p((n+1)mh),$$

where $\alpha_j > 0$ and $\alpha_0 + \ldots + \alpha_{(m-1)(n+1)} = 1$. Hence by the Minkowski inequality $\|\Delta_{mh}^U f\|_p((n+1)mh) \le m^{n+1}\max\{\|\Delta_h^U f(t+jh)\|_p((n+1)mh) : j = 0,\ldots,(n+1)(m-1)\} \le m^{n+1}\|\Delta_h^U f\|_p((n+1)h)$ and we have proved (P.4).

Since $\lambda\delta \le (1+\lambda)\delta$, (P.5) follows from (P.1) and (P.4).

To obtain (P.6) we put $\lambda = \delta_1/\delta$ in (P.5):

$$\omega_U^{(p)}(f,\delta_1) \le (1+\delta_1/\delta)^{n+1}\omega_U^{(p)}(f,\delta) \le (2\delta_1/\delta)^{n+1}\omega_U^{(p)}(f,\delta).$$

(P.7) is proved as follows. Applying (P.6) for $\delta_1 = (b-a)/(n+1)$ we conclude that $\omega_U^{(p)}(f,(b-a)/(n+1)) \doteq 0$. Hence for $0 < h < (b-a)/(n+1)$

$$\|\Delta_h^U f\|_p((n+1)h) = 0$$

and by the definition of $\Delta_h^U f(t)$

$$\|[t,t+h,\ldots,t+(n+1)h;f]_U\|_p((n+1)h) = 0.$$

Applying Theorem 2.13 we obtain

$$\|[t,t+h_1,\ldots,t+h_{n+1};f]_U\|_p(h_{n+1}) = 0 \tag{4.1}$$

for any $0 < h_1 < \ldots < h_{n+1} < b-a$. Let $F(t) = \int_a^t f(x)\,dx$ and let P be a polynomial w.r.t. the system $\{1, t, \int_a^t u_1(x)\,dx, \ldots, \int_a^t u_n(x)\,dx\}$ of best approximation of F in $C(I)$. Since this system is a Haar system, there exist $t_0,\ldots,t_{n+2} \in I$ such that $\eta\|F-P\|_\infty = \Delta(t_0) = -\Delta(t_1) = \ldots = (-1)^{n+2}\Delta(t_{n+2})$, where $\Delta(t) = F(t) - P(t)$, $\eta = \pm 1$ (see e.g. [1, 41]). Hence there exist $t_0 \in I$, $h > 0$, $0 = h_0 < h_1 < \ldots < h_{n+1} < b-a$, $t_0+h+h_{n+1} \in I$ such that $(-1)^k\eta\Delta'(t) > 0$ a.e. for $t \in [t_0+h_k, t_0+h_k+h]$, $k = 0,\ldots,n+1$, and $\int_{t_0}^{t_0+h} |[t,t+h_1,\ldots,t+h_{n+1};f]_U|^p\,dt > 0$. Hence we conclude from (4.1) that $F = P$ and $F' = P' = f$ a.e. is a polynomial from P_U.

(P.8) follows from (1.18) and the following

LEMMA 4.1 (see [75]). *There exist positive constants A_U and B_U such that*

$$A_U(t_{n+1}-t_0) \le \left[\begin{matrix} u_0,\ldots,u_n \\ t_1,\ldots,t_{n+1}\end{matrix}\middle| u_{n+1}\right] - \left[\begin{matrix} u_0,\ldots,u_n \\ t_0,\ldots,t_n\end{matrix}\middle| u_{n+1}\right] \le B_U(t_{n+1}-t_0) \tag{4.2}$$

for $a \le t_0 \le t_1 \le \ldots \le t_{n+1} \le b$, $t_0 < t_{n+1}$.

Proof. Applying (1.18) we obtain

$$\left[\begin{matrix} u_0,\ldots,u_n \\ t_1,\ldots,t_{n+1}\end{matrix}\middle| u_{n+1}\right] - \left[\begin{matrix} u_0,\ldots,u_n \\ t_0,\ldots,t_n\end{matrix}\middle| u_{n+1}\right]$$

$$= \left(\left[\begin{matrix} u_0,\ldots,u_n \\ t_1,\ldots,t_{n+1}\end{matrix}\middle| f\right] - \left[\begin{matrix} u_0,\ldots,u_n \\ t_0,\ldots,t_n\end{matrix}\middle| f\right]\right) \Big/ \left[\begin{matrix} u_0,\ldots,u_{n+1} \\ t_0,\ldots,t_{n+1}\end{matrix}\middle| f\right]$$

for any function f such that the denominator is different from zero. Assume that $f(t_j) = 0$ for $j = 0, \dots, n$ and $f(t_{n+1}) = 1$. Then

$$\left[\begin{matrix} u_0, \dots, u_n \\ t_1, \dots, t_{n+1} \end{matrix} \middle| u_{n+1}\right] - \left[\begin{matrix} u_0, \dots, u_n \\ t_0, \dots, t_n \end{matrix} \middle| u_{n+1}\right] = \frac{D_U(t_0, \dots, t_{n+1}) D_U(t_1, \dots, t_n)}{D_U(t_0, \dots, t_n) D_U(t_1, \dots, t_{n+1})}.$$

In the algebraic case ($u_i = t^i$) the last expression is equal to $t_{n+1} - t_0$. Now applying Lemma 1.2 we get (4.2).

(P.9) follows from (P.8) for $m = 0$ and $k = n$, and the fact that (P.9) holds for $\omega_1^{(p)}(f, h)$.

THEOREM 4.1 ([75]). *There exists a constant* $C = C_{U,r,k}$ $(0 < k < r + k \le n)$ *such that for* $f \in L_p(I)$, $1 \le p \le \infty$,

$$\omega_{U_r}^{(p)}(f, \delta) \le C\delta^{r+1}\left[\omega_{U_r}^{(p)}\left(f, \frac{b-1}{r+1}\right) + \int_\delta^{(b-a)/(r+1)} \frac{\omega_{U_{r+k}}^{(p)}(f, s)}{s^{r+2}}\, ds\right], \tag{4.3}$$

where $0 < \delta(r + k + 1) \le b - a$.

P r o o f. The idea of the proof is the same as in the algebraic case (see [18, 63]). We shall prove (4.3) for $1 \le p < \infty$ by induction on k. The proof for $p = \infty$ is analogous. It suffices to check (4.3) for $k = 1$. Introduce the following notations: $A = (b - a)/(r + 1)$, $\omega(h) = \omega_{U_r}^{(p)}(f, h)$ and $\omega_1(h) = \omega_{U_{r+1}}^{(p)}(f, h)$. For $A/2 < \delta \le A$ the proof of (4.3) is straightforward:

$$\omega(\delta) \le \omega(A) \le \left(\frac{2}{A}\right)^{r+1} \delta^{r+1}\omega(A) \le \left(\frac{2}{A}\right)^{r+1} \delta^{r+1}\left[\omega(A) + \int_\delta^A \frac{\omega_1(s)}{s^{r+2}}\, ds\right].$$

Let now $0 < \delta \le A/2$. There exists a positive integer m such that $A2^{-(m+1)} < \delta \le A2^{-m}$. Let $0 < h \le \delta$. Applying Theorem 1.8 and Lemma 4.1 we prove that there exists a function $c(h, t)$ satisfying $0 < a \le c(h, t) \le b$ such that

$$\Delta_h^{U_{r+1}} f(t) = c(h, t)[\Delta_h^{U_r} f(t + h) - \Delta_h^{U_r} f(t)], \tag{4.4}$$

where the constants a and b depend only on the system U. Applying (2.27) we obtain

$$\begin{aligned}\Delta_{2h}^{U_r} f(t) &= r!(2h)^{r+1}[t, t+2h, \ldots, t+2(r+1)h; f]_{U_r} \\ &= r! 2^{r+1} h^{r+1} \sum_{j=0}^{r+1} \alpha_j [t+jh, \ldots, t+(j+r+1)h; f]_{U_r} \\ &= 2^{r+1} \sum_{j=0}^{r+1} \alpha_j \Delta_h^{U_r} f(t+jh),\end{aligned}$$

where $\sum_{j=0}^{r+1} \alpha_j = 1$ and $0 < \alpha_j < 1$.

Put $c_j = c(h, t+jh)$, $f_j = \Delta_h^{U_r} f(t+jh)$. Then, by (4.4), $f_{j+1} - f_j = (1/c_j)\Delta_h^{U_{r+1}} f(t+jh)$. Further,

$$\begin{aligned}\Delta_{2h}^{U_r} f(t) - 2^{r+1}\Delta_h^{U_r} f(t) &= 2^{r+1} \sum_{j=0}^{r+1} \alpha_j (f_j - f_0) \\ &= 2^{r+1} \sum_{j=1}^{r+1} \alpha_j \sum_{i=1}^{j} (f_i - f_{i-1}) \\ &= 2^{r+1} \sum_{j=1}^{r+1} \alpha_j \sum_{i=1}^{j} \frac{1}{c_{i-1}} \Delta_h^{U_{r+1}} f(t+(i-1)h),\end{aligned}$$

whence

$$|\Delta_{2h}^{U_r} f(t) - 2^{r+1}\Delta_h^{U_r} f(t)| \le \frac{2^{r+1}}{a} \sum_{j=1}^{r+1} \alpha_j \sum_{i=1}^{j} |\Delta_h^{U_{r+1}} f(t+(i-1)h)|,$$

where a is any positive constant which bounds the function $c(h,t)$ from below. Applying the Minkowski inequality we obtain

$$\begin{aligned}&\|\Delta_{2h}^{U_r} f - 2^r \Delta_h^{U_r} f\|_p(2(r+1)h) \\ &\qquad \le \frac{2^{r+1}}{a} \sum_{j=1}^{r+1} \alpha_j \sum_{i=1}^{j} \|\Delta_h^{U_{r+1}} f(t+(i-1)h)\|_p(2(r+1)h) \\ &\qquad \le \frac{(r+1)2^{r+1}}{a} \|\Delta_h^{U_{r+1}} f\|_p((r+2)h),\end{aligned}$$

whence

$$\|2^{-r-1}\Delta_{2h}^{U_r} f - \Delta_h^{U_r} f\|_p(2(r+1)h) \le C\|\Delta_h^{U_{r+1}} f\|_p((r+2)h), \tag{4.5}$$

where $C = 2^{r+1}/a$. Applying (4.5) and the Minkowski inequality to the identity

$$2^{-m(r+1)}\Delta_{2^m h}^{U_r} f - \Delta_h^{U_r} f = \sum_{j=1}^{m} 2^{-(r+1)(j-1)}[2^{-(r+1)}\Delta_{2(2^{j-1}h)}^{U_r} f - \Delta_{2^{j-1}h}^{U_r} f],$$

where m is the integer defined above, we obtain

$$\begin{aligned}\|2^{-m(r+1)}&\Delta_{2^m h}^{U_r} f - \Delta_h^{U_r} f\|_p(2^m(r+1)h)\\ &\le \sum_{j=1}^{m} 2^{-(r+1)(j-1)}\|2^{-(r+1)}\Delta_{2(2^{j-1}h)}^{U_r} f - \Delta_{2^{j-1}h}^{U_r} f\|_p(2^j(r+1)h)\\ &\le C\sum_{j=0}^{m-1} 2^{-(r+1)j}\|\Delta_{2^j h}^{U_{r+1}} f\|_p(2^j(r+2)h).\end{aligned}$$

Hence

$$\begin{aligned}(4.6)\quad &\|\Delta_h^{U_r} f\|_p(2^m(r+1)h)\\ &\le \|2^{-m(r+1)}\Delta_{2^m h}^{U_r} f\|_p(2^m(r+1)h) + C\sum_{j=0}^{m-1} 2^{-(r+1)j}\|\Delta_{2^j h}^{U_{r+1}} f\|_p(2^j(r+2)h).\end{aligned}$$

Let now $g(t) = f(a+b-t)$, $t \in I$. We have $\Delta_h^{U_r} g(t) = (-1)^{r+1}\Delta_h^{U_r} f(a+b-t-(r+1)h)$ and $2^m(r+1)h < \frac{1}{2}$. Hence

$$\begin{aligned}(4.7)\quad &\|\Delta_h^{U_r} f\|_p((r+1)h)\\ &\qquad\le \|\Delta_h^{U_r} f\|_p(2^m(r+1)h) + \|\Delta_h^{U_r} g\|_p(2^m(r+1)h).\end{aligned}$$

Further,

$$(4.8)\quad \|2^{-m(r+1)}\Delta_{2^m h}^{U_r} f\|_p(2^m(r+1)h) = \|2^{-m(r+1)}\Delta_{2^m h}^{U_r} g\|_p(2^m(r+1)h),$$

$$(4.9)\quad \|\Delta_{2^j h}^{U_{r+1}} f\|_p(2^j(r+2)h) = \|\Delta_{2^j h}^{U_{r+1}} g\|_p(2^j(r+2)h).$$

Applying (4.6)–(4.9) we obtain

$$\begin{aligned}(4.10)\quad \|\Delta_h^{U_r} f\|_p(rh) &\le 2\|2^{-m(r+1)}\Delta_{2^m h}^{U_r} f\|_p(2^m(r+1)h)\\ &\quad + 2C\sum_{j=0}^{m-1} 2^{-(r+1)j}\|\Delta_{2^j h}^{U_{r+1}} f\|_p(2^j(r+2)h) = I_1 + I_2.\end{aligned}$$

We estimate the right side by the modulus of smoothness:

$$I_1 \le 2\cdot 2^{-m(r+1)}\omega(2^m\delta) \le 2^{r+1}\left(\frac{2}{A}\right)^{r+1}\delta^{r+1}\omega(A),$$

$$I_2 \le 2C\sum_{j=0}^{m-1} 2^{-(r+1)j}\omega_1(2^j\delta)$$

$$\le \frac{C2^{r+2}}{2^{r+1}-1}\delta^{r+1}\sum_{j=0}^{m-1}\int_{2^j\delta}^{2^{j+1}\delta}\frac{\omega_1(s)}{s^{r+2}}\,ds \le \frac{C2^{r+2}}{2^{r+1}-1}\delta^{r+1}\int_{\delta}^{A}\frac{\omega_1(s)}{s^{r+2}}\,ds\,.$$

Hence by (4.10)

$$\|\Delta_h^{U_r}f\|_p((r+1)h) \le B\delta^{r+1}\left[\omega(A)+\int_{\delta}^{A}\frac{\omega_1(s)}{s^{r+2}}\,ds\right],$$

where $B = \max(C2^{r+2}/(2^{r+1}-1), [4(r+1)]^{r+1})$, whence by the inductive assumption and integration by parts we obtain the theorem.

COROLLARY 4.1. *Let r and k $(0 < k < r+k \le n)$ be given integers and $0 < \alpha < r$. Then for $f \in L_p(I)$, $1 \le p \le \infty$, the following conditions are equivalent*:

(a) $\omega_{U_r}^{(p)}(f,h) = O(h^\alpha)$,

(b) $\omega_{U_{r+k}}^{(p)}(f,h) = O(h^\alpha)$.

2. Generalization of the Whitney Theorem. We shall prove the following

THEOREM 4.2 (see [73]). *Let $f \in C(I)$ and let P_f be a polynomial from P_U interpolating f at $t_i = a+(b-a)i/n$, $i = 0,\dots,n$. Then*

$$(4.11)\qquad |f(t)-P_f(t)| \le C_U\omega_U(f,1/n)\,,$$

where C_U is a constant depending only on the system U.

We need the following lemmas:

LEMMA 4.2. *Let $a = m_0 < m_1 < \dots < m_n$ $(m_n > n+1)$ be given integers. Then for any integer $s \in (m_0, m_n)$, $y \in I$ and h with $y, y+m_nh \in I$ there exist constants a_i and c_j, $i = 0,\dots,m_n-n-1 = l$, $j = 0,\dots,n$, such that for any $f \in C(I)$*

$$(4.12)\qquad f(y+sh) = \sum_{i=0}^{l} a_i\Delta_h^U f(y+ih) + \sum_{j=0}^{n} c_j f(y+m_jh)\,.$$

Moreover, if Q is a polynomial from P_U such that $Q(y+m_jh) = f(y+m_jh)$, $j = 0,\dots,n$, then

$$f(y+sh) = \sum_{i=0}^{l} a_i \Delta_h^U f(y+ih) + Q(y+sh) \tag{4.13}$$

and $\sum_{i=0}^{l} |a_i| \le a$, $\sum_{j=0}^{n} |c_j| \le c$, where the constants a and c depend only on the system U and the integer s.

Proof. Applying (1.12), (1.15) and (2.27) we obtain

$$\begin{aligned} f(y+sh) &= Q(y+sh) + [y+m_jh, j=0,\dots,n, y+sh; f]_U W(y+sh) \\ &= Q(y+sh) + \sum_{j=0}^{l} \alpha_j \Delta_h^U f(y+jh) \frac{W(y+sh)}{n!h^{n+1}}, \end{aligned}$$

where $\sum_{j=0}^{l} \alpha_j = 1$ and $\alpha_j > 0$.

Putting $a_j = \alpha_j W(y+sh)/(n!h^{n+1})$ we obtain (4.13). Hence by (1.16)

$$\sum_{j=0}^{l} |a_j| \le \beta_U |s-m_0| \dots |s-m_n| = a .$$

Writting Q in the form $Q(y+sh) = \sum_{j=0}^{n} f(y+m_jh)W_j(y+sh)$ and putting $c_j = W_j(y+sh)$ we obtain (4.12). Further, by (1.13) we obtain

$$\sum_{j=0}^{n} |c_j| \le \beta_U \sum_{j=0}^{n} |l_j(y+sh)| = c .$$

This completes the proof.

Let now $m_k = k\nu$, $k = 0,\dots,n$, $\nu \ge 2$, $s = 1$. Applying (4.12) we get

$$f(y+h) = \sum_{i=0}^{n\nu-n-1} a_i \Delta_h^U f(y+ih) + \sum_{j=0}^{n} \gamma_j f(y+j\nu h) . \tag{4.14}$$

Since $\gamma_0 = W_0(y+h)$, we have $0 < \gamma_0 < 1$.

LEMMA 4.3. *For every $\varepsilon > 0$ there exists ν such that*

$$\sigma = |\gamma_1| + \dots + |\gamma_n| \le \varepsilon .$$

Proof. Let P be a polynomial from P_U satisfying $P(y) = 0$, $P(y + j\nu h) = 1$ for $\gamma_j \geq 0$ and $P(y + j\nu h) = -1$ for $\gamma_j < 0$. Applying (4.14) we obtain

$$P(y+h) = \sum_{j=1}^{n} |\gamma_j| \,.$$

Writting P in the form (1.12) and applying (1.13) we obtain

$$\begin{aligned} |P(y+h)| &\leq \sum_{j=1}^{n} |W_j(y+h)| \leq \beta_U \sum_{j=1}^{n} |l_j(y+h)| \\ &= \frac{\beta_U}{\nu}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) < \frac{\beta_U}{\nu}(1 + \ln n)\,. \end{aligned}$$

Putting $\nu > (\beta/\varepsilon)(1 + \ln n)$ we obtain the assertion.

Proof of Theorem 4.2. We may assume that $\omega_U(f, 1/n) = 1$ and $f(t_i) = 0, i = 0, \ldots, n$. Put $A_k = \{t : t = a+(b-a)i/(2^k n),\ i = 0, \ldots, 2^k n\}$. Choose ν from Lemma 4.3 such that $\sigma < 1$ and μ and m such that $m = 2^\mu n$. Putting $y = a$ and $m_j h = (b - a)j/n$ we conclude from (4.12) that there exists a constant M such that $|f(t)| \leq M$ for $t \in A_\mu$. Let $|f(t_0)| = \|f\|_\infty$. There exists $t \in A_\mu$ such that $t + (n + 1)(t_0 - t) \in I$. Applying (4.14) for $y = t$ and $y + h = t_0$ we obtain

$$\|f\|_\infty \leq a + M + \sigma \|f\|_\infty \,.$$

Hence

$$\|f\|_\infty \leq \frac{a + M}{1 - \sigma}$$

and putting $C_U = (a + M)/(1 - \sigma)$ we obtain (4.11).

Remark 4.1. In the algebraic case Theorem 4.2 was proved by H. Whitney [64] and its new proofs were given by B. Sendov [58, 59].

3. Best approximation by splines. Let now $\Delta = \{a = t_{-n} = \ldots = t_0 < t_1 < \ldots < t_N = \ldots = t_{N+n} = b\}$, $t_j = a + (b - a)j/N$, $j = 0, \ldots, N$, $U = U_n$ and let $U_{2n+1} = \{u_i\}_{i=0}^{2n+1}$ be a system of functions defined by (1) w.r.t. the system of weight functions $\{w_0, w_1, \ldots, w_n, w_0, w_n, \ldots, w_1\}$.

LEMMA 4.4. *For any $f \in C(I)$ there exists a unique spline $s_f \in S_\Delta^{U_{2n+1}}(I)$ such that $s_f(t_j) = f(t_j)$ for $j = 0, \ldots, N$, $[t_i, \ldots, t_{i+n+1}; s_f]_U = [t_0, \ldots, t_{N+1}; f]_U$ for $i = -n, \ldots, -1$ and $[t_i, \ldots, t_{i+n+1}; s_f]_U = [t_{N-n-1}, \ldots, t_N; f]_U$ for $i = N - n, \ldots, N - 1$ and*

$$\|Ls_f\|_\infty \leq C_U \max\{|[t_j, \ldots, t_{j+n+1}; f]_U| : \ j = 0, \ldots, N - n - 1\}\,,$$

where the operator L is defined by (1.1) *and the constant C_U depends only on the system U.*

Proof. The spline s_f interpolates f at t_j, $j = 0, \ldots, N$, with end conditions on $L_j s_f(t_k)$, $k = 0, N$, $j = 1, \ldots, n$, such that the respective divided differences of f and s_f are equal. It follows from (2.8) for the system U that

$$[t_j, \ldots, t_{j+n+1}; f]_U = \int_{t_j}^{t_{j+n+1}} Lf(t) M_{j,V}(t)\, dt\,,$$

where $M_{j,V}$ is the jth B-spline from $S_\Delta^V(I)$. Write $Ls_f = \sum_{j=-n}^{N-1} a_j N_{j,V}$, where $N_{j,V}$ is the jth normalized B-spline from $S_\Delta^V(I)$. Then we conclude that Ls_f satisfies the following system of equations:

$$\sum_{k=-n}^{N-1} a_j (N_{k,V}, M_{j,V}) = d_j\,, \quad j = -n, \ldots, N-1, \tag{4.15}$$

where $d_j = (f, M_{j,V})$ for $j = 0, \ldots, N-n-1$, $d_i = (f, M_{0,V})$ for $i = -n, \ldots, -1$ and $d_i = (f, M_{N-n-1,V})$ for $i = N-n, \ldots, N-1$. The matrix A of the system (4.15) is similar to the matrix A_2 from the proof of (3.4) for the system V. Hence its inverse matrix $B = [b_{i,j}]$ satisfies (3.4) and by de Boor's inequalities we conclude that

$$\|Ls_f\|_\infty \le C_U \max\{|d_j| : \; j = -n, \ldots, N-n\}\,,$$

where C_U depends only on U, and we have proved the lemma.

Hence for $h = 1/N$

$$\|Ls_f\|_\infty \le C_U (n!h)^{-n-1} \omega_U(f, h)\,. \tag{4.16}$$

Let $t \in (t_j, t_{j+n})$ and let P_f be a polynomial from P_U interpolating f at t_i, $i = j, \ldots, j+n$. We have

$$|f(t) - s_f(t)| \le |f(t) - P_f(t)| + |P_f(t) - s_f(t)|\,.$$

Applying Theorem 4.2 we obtain

$$|f(t) - P_f(t)| \le C_U \omega_U(f, h)\,.$$

To estimate the second term we remark that the polynomial P_f interpolates the spline s_f at t_i, $i = j, \ldots, j+n$. Hence by (1.15), (1.16), (2.8), (2.7) for the system U and (4.16) we obtain

$$|P_f(t) - s_f(t)| \le C_U \omega_U(f, h)\,.$$

Putting these inequalities together and taking N such that $1/(N+1) < \varepsilon \le 1/N$ we obtain

THEOREM 4.3 ([73]). *For any* $\varepsilon > 0$ *and* $f \in C(I)$ *there exists* $f_\varepsilon \in C_U^{n+1}(I)$ *such that*

$$\|f - f_\varepsilon\|_\infty \le C_1 \omega_U(f, \varepsilon), \qquad \|L f_\varepsilon\|_\infty \le C_2 \varepsilon^{-n-1} \omega_U(f, \varepsilon),$$

where the constants C_1 *and* C_2 *depend only on the system* U.

In the algebraic case this theorem was proved for $L_p(I)$, $1 \le p \le \infty$, by G. Freud and V. A. Popov [26, 27].

To each partition $\Delta = \{t_{-n} = \ldots = t_0 = a < t_1 < \ldots < t_N = \ldots = t_{N+n} = b\}$ of I there exists a partition $\Delta' \subset \Delta$ such that $\|\Delta'\|_{n+1} \le 2\|\Delta\|_{n+1}$ and $R_{\Delta,1} \le 4$. We have the following

THEOREM 4.4 (cf. [13, 16, 18, 52, 57]). *For any partition* Δ, *a system* U *satisfying* (2) *and* $f \in H_{U,p}^{n+1}(I)$ *there exists a spline* $s_f \in S_\Delta^U(I)$ *such that*

$$\|f - s_f\|_p \le C_U \|\Delta\|_1^{n+1} \|Lf\|_p,$$

where C_U *is a constant depending only on* U.

Proof. Let $P_{\Delta'}^n$ be the linear projector on $L_2(I)$ of best L_2-approximation in $S_{\Delta'}^U(I)$ defined in Part III and let $s_f = P_{\Delta'}^n f$. It follows from Theorem 2.1 (first integral relation) that there exist a function F and a spline $S \in S_\Delta^{V_{2n+1}}(I)$ satisfying the conditions (a) of that theorem such that $L^*F = f$ and $L^*S = s_f$. Hence by Rolle's theorem, for every $t \in I$ there exist $\tau_0 \in I$ with $|t - \tau_0| \le \|\Delta'\|_{n+1}$ such that $f(\tau_0) = s_f(\tau_0)$ and points τ_k with $|t - \tau_k| \le \|\Delta'\|_{n+k}$ such that $L_k f(\tau_k) = L_k s_f(\tau_k)$, $k = 1, \ldots, n$. Applying the Hölder inequality we obtain (with $L_{n+1} = D_0 L_n$)

$$\begin{aligned} |L_k f(t) - L_k s_f(t)| &\le \int_{\tau_k}^{t} |L_{k+1} f(\tau) - L_{k+1} s_f(\tau)|\, d\tau \\ &\le |t - \tau_k|^{1/q} \left(\int_{\tau_k}^{t} |L_{k+1} f(\tau) - L_{k+1} s_f(\tau)|^p\, d\tau \right)^{1/p} \end{aligned}$$

and there exists a $t_{j_k} \in \Delta'$ such that $[\tau_k, t] \subset [t_{j_k}, t_{j_k+n+k}]$. Hence

$$\int_{t_{j_k}}^{t_{j_k+n+k}} |L_k f(t) - L_k s_f(t)|^p\, dt$$

$$\leq \int_{t_{j_k}}^{t_{j_k+n+k}} |L_{k+1}f(t) - L_{k+1}s_f(t)|^p \, dt \int_{t_{j_k}}^{t_{j_k+n+k}} |t - \tau_k|^{p/q} \, dt$$

and

$$\|L_k f - L_k s_f\|_p \leq C_U \|\Delta\|_{n+k} \|D_{k+1} f - D_{k+1} s_f\|_p$$

for $k = 1, \ldots, n$. But by (3.16) we have

$$\|L_n f - L_n s_f\|_p \leq C_U \omega_1^{(p)}(Lf, \|\Delta\|_1) .$$

Hence

$$\|f - s_f\|_p \leq C_U \|\Delta\|_1^{n+1} \|Lf\|_p , \quad 1 \leq p \leq \infty ,$$

where the constant C_U depends only on U.

DEFINITION 4.2 (cf. [33, 57]). To every system U defined by (1) we associate a one-parameter family of continuous seminorms on $L_p(I)$ by

$$K(t, f; p, U) = \inf\{\|f - g\|_p + t^{n+1}\|Lg\|_p : \ g \in H_{U,p}^{n+1}(I)\}$$

for $0 < t < \infty$, $f \in L_p(I)$.

The *K-functional* $K(t, f; p, U)$ associated with U has the following properties (which follow directly from the definition):

THEOREM 4.5 (see [57]). *Let $1 \leq p \leq \infty$ and let U be defined by* (1). *Then*

(K.1) $K(t_1 + t_2, f; p, U) \leq 2^n [K(t_1, f; p, U) + K(t_2, f; p, U)]$.

(K.2) $K(t, f; p, U) \leq K(t_1, f; p, U)$ *for* $t \leq t_1$.

(K.3) $K(t, f + g; p, U) \leq K(t, f; p, U) + K(t, g; p, U)$.

(K.4) $K(t, f; p, U) \leq \|f\|_p$.

(K.5) $K(t, f; p, U) \leq t^{n+1} \|Lf\|_p$.

(K.6) $\lim_{t \to 0} K(t, f; p, U) = 0$ *for* $f \in L_p(I)$.

THEOREM 4.6 (see [57]). *Suppose that $f \in L_p(I)$, $1 \leq p \leq \infty$, is such that*

$$\liminf_{t_0 \to 0} t^{-n-1} K(t, f; p, U) = 0 .$$

Then there exists $g \in P_U$ such that $f = g$ a.e.

Proof. For any $\varepsilon > 0$ and any $0 < t \leq 1$ there exists $g \in H^{n+1}_{U,p}(I)$ such that

$$K(1,f;p,U) \leq t^{-n-1}(\|f-g\|_p + t^{n+1}\|Lg\|_p) \leq t^{-n-1}[K(t,f;p,U)+\varepsilon]\,.$$

We conclude that $K(1,f;p,U) = 0$ and thus there exists a sequence $\{g_m\}$ in $H^{n+1}_{U,p}(I)$ with

$$\lim_{m\to\infty}\|f-g_m\|_p = \lim_{m\to\infty}\|Lg_m\|_p = 0\,. \tag{4.17}$$

But $H^{n+1}_{U,p}(I)$ is a Banach space with the norm $\|g\|_U = \|g\|_p + \|Lg\|_p$, and thus (4.17) implies that $\{g_m\}$ is a Cauchy sequence in $H^{n+1}_{U,p}(I)$. Hence it converges to some $g \in H^{n+1}_{U,p}(I)$, which by (4.17) satisfies $\|Lg\|_p = 0$ and $\|f-g\|_p = 0$. Thus $f = g$ a.e. and $g \in P_U$.

THEOREM 4.7 (cf. [57]). *For any* $f \in L_p(I)$, $1 \leq p \leq \infty$,

$$\|f - P^n_\Delta f\|_p \leq C_U K(\|\Delta\|_1, f; p, U)\,,$$

where the constant C_U *depends only on* U.

Proof. For any $\varepsilon > 0$ there exists $g \in H^{n+1}_{U,p}(I)$ such that $\|f-g\|_p + \|\Delta\|_1^{n+1}\|Lg\|_p \leq K(\|\Delta\|_1, f; p, U) + \varepsilon$. Since $P^U_\Delta s = s$ for any $s \in S^U_\Delta(I)$, by Theorem 4.4 and (3.5) we obtain

$$\begin{aligned}\|f - P^U_\Delta f\|_p &\leq \|f-g\|_p + \|g - P^U_\Delta g\|_p + \|P^U_\Delta(g-f)\|_p \\ &\leq C_1\|f-g\|_p + C_2\|\Delta\|_1^{n+1}\|Lg\|_p \\ &\leq C_U[K(\|\Delta\|_1, f; p, U) + \varepsilon]\,,\end{aligned}$$

where the constants C_1, C_2 and C_U depend only on U.

As a corollary of Theorems 4.3 and 4.4 we obtain

THEOREM 4.8. *For any* $f \in C(I)$

$$\|f - P^U_\Delta f\|_\infty \leq C_U \omega_U(f, \|\Delta\|_1)\,,$$

where the constant C_U *depends only on* U.

COROLLARY 4.2. *There exists a constant* C *depending only on the system* U *and the mesh ratio* $R_{\Delta,n-2k}$ *such that for* $f \in C^k_U(I)$, $k = 0,\ldots,n$,

$$\|f - P^{(n,k)}_N f\|_\infty \leq C\omega_{U_{n-k}}(f, 1/N)\,,$$

where $P^{(n,k)}_N$ *is defined by* (3.15) *and* $U_{n-k} = \{u_j\}_{j=0}^{n-k}$ *is defined by* (1) *with assumption* (2).

All results given in this part also hold in the periodic case; the proofs are similar or even simpler.

Let $g \in H^{n+1}_{U,p}(I)$. Hence by (2.8) for the system U and $1 \le p \le \infty$

$$\begin{aligned}\|\Delta_h^U f\|_p((n+1)h) &\le \|\Delta_h^U (f-g)\|_p((n+1)h) + \|\Delta_h^U g\|_p((n+1)h)\\ &\le C_1\|f-g\|_p + n!h^{n+1}\|Lg\|_p \le C_2 K(h,f;p,U)\,,\end{aligned}$$

where the constants C_1 and C_2 depend only on U.

On the other hand, it follows from Theorem 4.3 that

$$K(h,f;\infty,U) \le C_U \omega_U(f,h)\,,$$

where C_U depends only on U; we have thus proved

THEOREM 4.9. *There exist constants α_U and β_U such that*

$$\alpha_U \omega_U(f,h) \le K(h,f;\infty,U) \le \beta_U \omega_U(f,h)\,. \tag{4.18}$$

PROBLEM. Prove the right inequality of (4.18) for $1 \le p < \infty$.

4. The Bernstein type inequality for splines. Let $\{\Delta_N\}_{N=2}^{\infty}$ be a sequence of partitions of I defined in Section 2 of Part III. In the algebraic case, the author proved the following inequality (see [71]):

$$\|s^{(k)}\|_p \le CN^k \omega_k^{(p)}(s,1/N)\,, \quad k=1,\dots,n, \tag{4.19}$$

for $1 \le p \le \infty$ and splines of degree n w.r.t. Δ_N. Now we prove this inequality for Chebyshevian splines for $p=\infty$; for $1 \le p < \infty$ the problem is still open.

THEOREM 4.10. *For any $f \in S^U_{\Delta_N}(I)$ and $k=1,\dots,n$*

$$\|L_k f\|_\infty \le CN^k \omega_{U_{k-1}}(f,1/N)\,,$$

where the constant C depends only on the system U and the mesh ratio $R_{\Delta_N,n-k-1}$.

Proof. Let $h = \|\Delta_N\|_{n-k-1}$ and $f \in S^U_{\Delta_N}(I)$. It follows from Theorem 4.3 that there exists $f_h \in C^k_{U_{k-1}}(I)$ such that

$$\|f-f_h\|_\infty \le C_1 \omega_{U_{k-1}}(f,h)\,, \tag{4.20}$$

$$\|L_k f_h\|_\infty \le C_2 h^{-k} \omega_{U_{k-1}}(f,h)\,. \tag{4.21}$$

Further, for $s_f = P^U_{\Delta_N} f_h$ by Theorem 4.8, (3.22) and the property (P.8) of $\omega_{U_{k-1}}(f,\delta)$ we obtain

$$\|f_h - s_f\|_\infty \le C_3 \omega_{U_{k-1}}(f_h,h)\,, \tag{4.22}$$

(4.23) $$\|L_j s_f\|_\infty \le C_4 \|L_j f_h\|_\infty\,, \quad j = 1,\dots,k\,.$$

By (4.20)–(4.23) we have

$$\|s_f - f\|_\infty \le \|s_f - f_h\|_\infty + \|f_h - f\|_\infty \le C_3 \omega_{U_{k-1}}(f_h, h) + C_1 \omega_{U_{k-1}}(f, h)\,.$$

It follows from (4.20) and the properties (P.2) and (P.3) of $\omega_{U_{k-1}}(f,h)$ that there exists a constant C_5 depending only on U and $R_{\Delta_N,k}$ such that $\omega_{U_{k-1}}(f_h,h) \le C_5 \omega_{U_{k-1}}(f,h)$. Hence

$$\|s_f - f\|_\infty \le C_6 \omega_{U_{k-1}}(f,h)\,.$$

Applying k times the Markov inequality to the function $s_f - f \in S^U_{\Delta_N}(I)$ (Corollary 2.4) and the inequalities (4.20)–(4.23) we obtain

$$\|L_k f\|_\infty \le \|L_k f - L_k s_f\|_\infty + \|L_k s_f\|_\infty \le C_U h^{-k} \omega_{U_{k-1}}(f,h)\,.$$

V. Applications to approximation of analytic functions

1. Approximation by analytic splines. Let U be the system of functions defined by (1) with assumption (2) and let $\{\Delta_N\}_{N=2}^\infty$, $\Delta_N = \{t_0 = 0, t_1 = 2\pi, t_2, \dots, t_N\} = \{0 = x_{N,0} < x_{N,1} < \dots < x_{N,N} = 2\pi\}$ be a given sequence of partitions of $T = [0, 2\pi]$ with $\Delta_N \subset \Delta_{N+1}$ and $R_{\Delta_N,1} \le R < \infty$. Let $A_k(D)$ be the Banach space of analytic functions in the unit disc $D = \{z : |z| < 1\}$ and of class C^k in $\bar{D}$ with the norm

$$\|f\|^{(k)} = \max_{|z|=1} \sum_{j=0}^{k} \|f^{(j)}\|\,, \qquad \text{where} \quad \|f^{(j)}\| = \max_{|z|=1} |f^{(j)}(z)|\,.$$

Analytic splines were first defined by means of the Cauchy integral and polynomial splines in the complex variable z (see [2, 4]) and their properties were given in [2, 4, 69, 72]. Analytic splines in the unit disc defined by means of polynomial splines and the Schwarz formula were considered by the author [66, 67]. Now, we define analytic splines by means of Chebyshevian splines.

DEFINITION 5.1. Let $s_\Delta \in \mathring{S}^U_\Delta(T)$, $\Delta = \Delta_N$. The function S_Δ defined by means of the Schwarz formula

$$S_\Delta(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_\Delta(t) \frac{e^{it} + z}{e^{it} - z}\, dt + iA\,, \quad |z| < 1\,,$$

where A is a constant, is said to be an *analytic spline associated with the spline* s_Δ.

Since s_Δ satisfies the Lipschitz condition we can define S_Δ on the unit circle $\Gamma = \{z : |z| = 1\}$ setting $S_\Delta(e^{i\varphi}) = \lim_{r\to 1-} S_\Delta(re^{i\varphi})$. The function S_Δ defined in this way belongs to the space $A(D) = A_0(D)$ and

$$(5.1) \qquad S_\Delta(e^{i\varphi}) = s_\Delta(\varphi) + \frac{i}{2\pi}\int_{-\pi}^{\pi} s_\Delta(\varphi - t)\cot\frac{t}{2}\,dt + iA\,,$$

where the integral is interpreted as the Cauchy principal value (see [31, 66, 67, 78]).

The question of the existence of a basis in $A(D)$ was raised by S. Banach in [5]. S. V. Bochkarev [6, 7] constructed an orthonormal basis $\{G_N\}_{N=0}^{\infty}$ in $A(D)$. He also proved that there exists a constant $B > 0$ such that for $f \in A(D)$, $f(e^{it}) = u(t) + iv(t)$,

$$\|f - S_{N,f}\| \le B[\omega(u, 1/N) + \omega(v, 1/N)]\,,$$

where $S_{N,f}$ is the Nth Fourier sum of f w.r.t. the system $\{G_N\}$.

We may consider $A(D)$ as a real space (over $\mathbb{R}$) or as a complex space (over $\mathbb{C}$). The Bochkarev system is a basis in the complex space $A(D)$. Later Z. Ciesielski [15] constructed a system $\{F_i^{(m,k)}, F_j^{(m,-k)}\}_{i,j=1}^{\infty}$, $|k| \le m$, of biorthogonal spline periodic functions. He proved that $\{F_j^{(m,0)}\}_{j=1}^{\infty}$ is a simultaneous basis in $\mathring{C}^m(T)$ and then he constructed a simultaneous basis $\{G_j^{(k)}\}_{j=1}^{\infty}$ by means of the system $\{F_i^{(m,k)}\}_{i=1}^{\infty}$, $0 \le k \le m$, in the complex space $A_m(D)$. The author constructed simultaneous orthogonal bases in the real and complex spaces $A_m(D)$ (see [66–68, 70]).

Let $F = \{f_i^{(n,k)}, f_j^{(n,-k)}\}_{i,j=1}^{\infty}$ with $f_1^{(n,k)} = (\int_T w_k(t)\,dt)^{-1/2}$, $-n \le k \le n-1$, be the periodic biorthogonal spline system defined by (3.10).

THEOREM 5.1 (cf. [67]). *Let $-n \le k \le n-1$, $f \in A(D)$, $f(e^{it}) = u(t) + iv(t)$ and $s_{N,U} = P_{\Delta_N}^{U} u$. Suppose w_k is absolutely continuous with $\|w_k'(t)\|_\infty \le \beta < \infty$, and let $S_{N,f}$ be the analytic spline associated with $s_{N,u}$ such that* $\operatorname{Im} S_{N,f}(0) = \operatorname{Im} f(0)$. *Then*

$$\|f - S_{N,f}\| \le C_{U,R}[\omega(u, \|\Delta_N\|) + \omega(v, \|\Delta_N\|)]\,,$$

where $\|\Delta_N\| = \|\Delta_N\|_1$ and $C_{U,R}$ is a constant depending only on U and R.

Proof. Let $z = e^{i\varphi}$ and $h = \|\Delta_N\|$. Since addition of a constant to f changes neither the modulus of continuity of f nor the difference $f(z) - S_{N,f}(z)$, we can assume that $f = 0$. From (5.1) we obtain

$$f(e^{i\varphi}) - S_{N,f}(e^{i\varphi})$$
$$= [u(\varphi) - s_{N,u}(\varphi)] + \frac{i}{2\pi} \int\limits_{h\le|t|\le\pi} [u(\varphi - t) - s_{N,u}(\varphi - t)] \cot\frac{t}{2}\, dt$$
$$+ \frac{i}{2\pi} \int\limits_{|t|\le h} u(\varphi - t) \cot\frac{t}{2}\, dt - \frac{i}{2\pi} \int\limits_{|t|\le h} s_{N,u}(\varphi - t) \cot\frac{t}{2}\, dt$$
$$= I_1 + I_2 + I_3 + I_4\,.$$

It follows from Theorem 3.6 that

$$|I_1| \le C_{U,R}\,\omega(u,h)\,. \tag{5.2}$$

Let $r_u(t) = u(t) - s_{N,u}(t)$. It follows from the definition of the operator $P^U_{\Delta_N}$ and the interpolation property of splines (Theorem 2.1) that there exist periodic functions R_u, $\hat{u}$ and $\hat{s}$ such that $R_u = \hat{u} - \hat{s}$, $D_k\hat{u} = u$, $D_k\hat{s} = s_{N,u}$ and

$$\|R_u\|_\infty \le C_{U,R}\, h\omega(u,h)\,. \tag{5.3}$$

Hence

$$\left|\int\limits_h^\pi r_u(\varphi - t)\cot\frac{t}{2}\,dt\right| = \left|R_u(\varphi - t)[w_k(\varphi - t)]^{-1}\cot\frac{t}{2}\right|_h^\pi$$
$$+ \int\limits_h^\pi R_u(\varphi - t)\,\frac{d}{dt}\,[w_k(\varphi - t)]^{-1}\cot\frac{t}{2}\,dt$$
$$\le C_{U,R}\|R_u\|_\infty h \le C_{U,R}\,\omega(u,h)\,.$$

Analogously

$$\left|\int\limits_{-\pi}^{-h} r_u(\varphi - t)\cot\frac{t}{2}\,dt\right| \le C_{U,R}\,\omega(u,h)\,.$$

Then

$$|I_2| \le C_{U,R}\,\omega(u,h)\,. \tag{5.4}$$

Write the integral I_3 as follows:

$$I_3 = \frac{1}{2\pi}\int\limits_{-h}^{h} u(\varphi - t)\cot\frac{t}{2}\,dt = -\frac{i}{2\pi}\int\limits_{-h}^{h} u(\varphi - t)\frac{e^{it}+1}{e^{it}-1}\,dt$$

$$= \frac{i}{2\pi}\int_{\varphi-h}^{\varphi+h} u(\tau)\frac{e^{i\tau}+e^{i\varphi}}{e^{i\tau}-e^{i\varphi}}\,d\tau = \frac{1}{2\pi}\operatorname{Re}\cdot\int_{\varphi-h}^{\varphi+h} if(e^{i\tau})\frac{e^{i\tau}+e^{i\varphi}}{e^{i\tau}-e^{i\varphi}}\,d\tau = \operatorname{Re} J\,.$$

We need the following notations: $\Gamma_1 = \{\varsigma :\ \varsigma = e^{i(\varphi+t)}, -h \le t \le h\}$, $\Gamma_2 = \{\varsigma :\ |\varsigma - z| = |e^{ih}-1| = h_N\} \cap \bar{D}$, $0 < \varepsilon < h_N$, $\eta = (1-\varepsilon)z$. Hence

$$J = \frac{1}{2\pi}\int_{\Gamma_1} \frac{f(\varsigma)}{\varsigma}\frac{\varsigma+z}{\varsigma-z}\,d\varsigma = \frac{1}{\pi}\int_{\Gamma_1}\frac{f(\varsigma)}{\varsigma-z}\,d\varsigma - \frac{1}{2\pi}\int_{\Gamma_2}\frac{f(\varsigma)}{\varsigma}\,d\varsigma = J_1 + J_2\,.$$

From the Sokhotskii and Cauchy theorems (see [38, 48]) we obtain

$$J_1 = \frac{1}{\pi}\int_{\Gamma_1}\frac{f(\varsigma)}{\varsigma-z}\,d\varsigma = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\Gamma_1}\frac{f(\varsigma)}{\varsigma-\eta}\,d\varsigma = \lim_{\varepsilon\to 0} J_{1,\varepsilon}\,,$$

$$\frac{1}{\pi}\int_{\Gamma_1}\frac{f(\varsigma)}{\varsigma-\eta}\,d\varsigma = 2if(\eta) - \frac{1}{\pi}\int_{\Gamma_2}\frac{f(\varsigma)}{\varsigma-\eta}\,d\varsigma\,.$$

Hence

$$|J_{1,\varepsilon}| \le \frac{1}{\pi}\int_{\Gamma_2}\frac{|f(\varsigma)-f(z)|}{|\varsigma-z|}\cdot\left|\frac{\varsigma-z}{\varsigma-\eta}\right|\,|d\varsigma| + 2|f(\eta)|$$

$$\le \omega(f,h_N)\frac{h_N}{h_N-\varepsilon} + 2|f(\eta)| \to \omega(f,h_N) \quad \text{as} \quad \varepsilon \to 0\,.$$

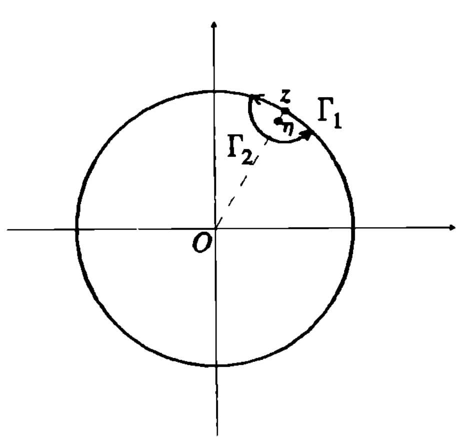

Fig. 2

Since

$$|J_2| \le \frac{1}{2\pi}\int\limits_{\varphi-h}^{\varphi+h} |f(e^{it}) - f(z)|\,dt \le \frac{h}{\pi}\omega(f,h)$$

and $\omega(f,h) \le 3[\omega(u,h)+\omega(v,h)]$ (see [49] or [67]), therefore

$$|I_3| \le C_{U,R}[\omega(u,h)+\omega(v,h)]\,. \tag{5.5}$$

Since $f(z) = 0$, we have for $t \in [\varphi - h, \varphi + h]$

$$|s_{N,u}(t)| \le |s_{N,u}(t) - u(t)| + |u(t) - u(\varphi)| \le C_{U,R}\omega(u,h)\,.$$

Hence from the Markov inequality we conclude that for $t \in [\varphi - h, \varphi + h]$

$$|s'_{N,u}(t)| \le C_{U,R}h^{-1}\omega(u,h) \quad \text{a.e.}$$

Then an application of the Lagrange theorem yields

$$|s_{u,N}(\varphi - t) - s_{u,N}(\varphi)| \le \|s'_{u,N}\|_\infty |t|\,.$$

Hence

$$|I_4| \le \frac{1}{2\pi}\int\limits_{-h}^{h} \|s'_{u,N}\|_\infty \left|t\cot\frac{t}{2}\right| dt \le \frac{2}{\pi}\|s'_{u,N}\|_\infty h$$

and, finally,

$$|I_4| \le C_{U,R}\omega(u,h)\,. \tag{5.6}$$

Adding (5.2)–(5.6), together with an application of the maximum principle, we obtain the theorem.

COROLLARY 5.1. *The system*

$$g_0^{(n,k)} = i\,, \qquad g_N^{(n,k)}(z) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f_N^{(n,k)}(t)\frac{e^{it}+z}{e^{it}-z}\,dt\,,$$

$N = 1,2,\ldots,$ $-n+1 \le k \le n$, *where* $f_N^{(n,k)}$ *are defined by* (3.10) *for the interval* T, *is a basis in the real space* $A(D)$.

Proof. Define a scalar product in the real $A(D)$ as follows: $f,g \in A$, $f = u + iv$, $g = u_1 + iv_1$, $(f,g)_k = \int_T w_k(t)u(e^{it})u_1(e^{it})\,dt + v(0)v_1(0)$. Now the system $\{g_i^{(n,k)}, g_j^{(n,-k)}\}_{i,j=0}^{\infty}$ is biorthogonal and the corollary follows from Theorem 5.1.

2. Biorthogonal systems in the complex space $A(D)$. Let $U = \{u_i\}_{i=0}^n$ be the system of functions defined in the interval $T = [-\pi, \pi]$ by (1) with even weight functions w_j, $j = 0, \dots, n$, satisfying (2) and let $\{\Delta_N\}_{N=1}^\infty$, $\Delta_N = \{t_0 = 0, t_{-1} = -\pi, t_1 = \pi, t_{-2} = -t_2, \dots, t_{-N} = -t_N\} = \{-\pi = x_{N,-N} < x_{N,-N+1} < \dots < x_{N,N-1} < x_{N,N} = \pi\}$ be a given sequence of even partitions of T ($\Delta_N = -\Delta_N$) with $\Delta_N \subset \Delta_{N+1}$, $t_{-N}, t_N \in \Delta_N \setminus \Delta_{N-1}$ and $R_{\Delta_N,1} \le R < \infty$. For given k, $-n \le k \le n$, define the following system of functions $\{\varphi_{N,k}\}_{N=2}^\infty$: $\varphi_{N,k}$ is a 2π-periodic spline from $S_{\Delta_N}^{V_{2n+1}}(T)$ satisfying $\varphi_{N,k}(t_N) = (-1)^{n+k+1}\varphi_{N,k}(-t_N) = 1$ and $\varphi_{N,k}(t) = 0$ for $t \in \Delta_{N-1}$, where V_{2n+1} is defined in Section 1 of Part II. It follows from Theorem 2.2 that the functions $\varphi_{N,k}$ exist and are unique. Then we conclude from Lemma 2.2 that the system $\{F_N\}_{N=1}^\infty$, where $F_1 = 1/\sqrt{2\pi}$, $F_N = L^*\varphi_N / \|L^*\varphi_N\|_2$, $N > 1$, is orthonormal in $L_2(T)$. The periodic system $\{F_j^{(n,k)}\}$ is now defined by (3.10) for the interval T.

Let N and k be given and let

$$\psi_{N,k}(t) = \frac{1}{2}[\varphi_{N,k}(t) + (-1)^{n+k+1}\varphi_{N,k}(-t)] .$$

Since w_j is even $L_n L^* \psi_{N,k}$ is piecewise constant and even for $n+k$ even, and odd for $n+k$ odd. Hence from the uniqueness of interpolating splines we conclude that $\psi_{N,k} = \varphi_{N,k}$ and that $\{F_i^{(n,k)}, F_j^{(n,-k)}\}_{i,j=1}^\infty$ is a biorthogonal system of even 2π-periodic splines. In the algebraic case and $k = 0$ this system was constructed in another way by the author in [68] and [70]. It follows from Theorem 3.6 that it is a basis in the space of continuous, 2π-periodic even functions in T.

Assume that $w_k = 1$ and define the system of functions $\{G_N^{(n,k)}, G_N^{(n,-k)}\}$ by

$$(5.7) \qquad \begin{aligned} G_1^{(n,k)}(z) &= \frac{1}{\sqrt{2\pi}}, \\ G_N^{(n,k)}(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N^{(n,k)}(t)\frac{e^{it}+z}{e^{it}-z}\,dt, \quad N = 2, 3, \dots \end{aligned}$$

In the algebraic case, $n = 1$ and $k = 0$ this system was constructed by S. V. Bochkarev [6, 7] and for $n > 1$ by the author [68, 70].

Let

$$(5.8) \qquad \tilde{F}_N^{(n,k)}(t) = -\frac{1}{2\pi}\int_0^{\pi} [F_n^{(n,k)}(t+\tau) - F_N^{(n,k)}(t-\tau)]\cot\frac{\tau}{2}\,d\tau$$

be the function conjugate to $F_N^{(n,k)}$. Then $\tilde{F}_N^{(n,k)}$ is odd and continuous (see e.g. [31, 78]). Since $F_N^{(n,k)}$ is orthogonal to 1 for $N > 1$, it follows from the above formulas that $G_N^{(n,k)}(0) = 0$. We shall show that the system $\{\tilde{F}_i^{(n,k)}, \tilde{F}_j^{(n,-k)}\}_{i,j=1}^{\infty}$ is also biorthonormal. The scalar product of $G_i^{(n,k)}$ and $G_j^{(n,-k)}$ is

$$(G_i^{(n,k)}, G_j^{(n,-k)}) = \int_{-\pi}^{\pi} G_i^{(n,k)}(e^{it})\overline{G_j^{(n,-k)}(e^{it})}\, dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [F_i^{(n,k)} F_j^{(n,-k)} + \tilde{F}_i^{(n,k)} \tilde{F}_j^{(n,-k)}]\, dt$$

$$+ \frac{i}{2}\int_{-\pi}^{\pi} [\tilde{F}_i^{(n,k)} F_j^{(n,-k)} - F_i^{(n,k)} \tilde{F}_j^{(n,-k)}]\, dt\,.$$

The integral in the imaginary part vanishes because the products $\tilde{F}_i^{(n,k)} \times F_j^{(n,-k)}$ and $F_i^{(n,k)} \tilde{F}_j^{(n,-k)}$ are odd functions. Moreover, if $i \neq j$ then by the Cauchy formula

$$0 = G_i^{(n,k)}(0) G_j^{(n,-k)}(0) = \frac{1}{2\pi i}\int_{|z|=1} \frac{G_i^{(n,k)}(z) G_j^{(n,-k)}(z)}{z}\, dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} G_i^{(n,k)}(e^{it}) G_j^{(n,-k)}(e^{it})\, dt$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} [F_i^{(n,k)} F_j^{(n,-k)} - \tilde{F}_i^{(n,k)} \tilde{F}_j^{(n,-k)}]\, dt$$

$$+ \frac{i}{4\pi}\int_{-\pi}^{\pi} [F_i^{(n,k)} \tilde{F}_j^{(n,-k)} + \tilde{F}_i^{(n,k)} F_j^{(n,-k)}]\, dt\,.$$

Hence $\int_{-\pi}^{\pi} \tilde{F}_i^{(n,k)} \tilde{F}_j^{(n,-k)}\, dt = 0$ and we have proved that the system (5.8) is biorthonormal.

THEOREM 5.2 (cf. [68, 70]). *Let $-n \leq k \leq n-1$. Then the system $\{G_N^{(n,k)}\}_{N=1}^{\infty}$ is a basis in the complex space $A(D)$. Moreover, there exists a constant $C_{U,R}$ such that for $f \in A(D)$, $f(e^{it}) = u(t) + iv(t)$,*

$$\|f - S_{N,f}\| \leq C_{U,R}[\omega(u, 1/N) + \omega(v, 1/N)]\,,$$

where $S_{N,f} = \sum_{j=1}^{N} a_j G_j^{(n,k)}$ and $a_j = (f, G_j^{(n,-k)})$.

Proof. Let $s_{N,f} = \sum_{j=1}^{N} a_j F_j^{(n,k)}$. Since $F_j^{(n,-k)}$ is orthogonal to 1, reasoning as above we conclude by (5.7) that

$$S_{N,f}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_{N,f}(t) \frac{e^{it} + z}{e^{it} - z} \, dt .$$

On the circle Γ

$$S_{N,f}(e^{i\varphi}) = s_{N,f}(\varphi) + \frac{i}{2\pi} \int_{-\pi}^{\pi} s_{N,f}(\varphi - t) \cot \frac{t}{2} \, dt .$$

Decompose u and v into the even and odd parts: $u = u_1 + u_2$, $v = v_1 + v_2$, where $u_1(t) = \frac{1}{2}[u(t) + u(-t)]$, $u_2(t) = \frac{1}{2}[u(t) - u(-t)]$ and v_1 and v_2 are defined analogously. Since $F_N^{(n,k)}$ is even, $\tilde{u}_1 = v_2$ and $\tilde{v}_1 = -u_2 + \operatorname{Re} f(0)$, we have

$$\begin{aligned} f(e^{i\varphi}) - S_{N,f}(e^{i\varphi}) &= [u_1(\varphi) - s_{N,u_1}(\varphi)] + i[v_1(\varphi) - s_{N,v_1}(\varphi)] \\ &\quad + \frac{i}{2\pi} \int_{-\pi}^{\pi} [u_1(\varphi - t) - s_{N,u_1}(\varphi - t)] \cot \frac{t}{2} \, dt \\ &\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} [v_1(\varphi - t) - s_{N,v_1}(\varphi - t)] \cot \frac{t}{2} \, dt . \end{aligned}$$

Applying Theorem 3.6 and reasoning as in the proof of Theorem 5.1 we obtain the assertion.

Let now $\hat{\varphi}_{N,k}$ be a 2π-periodic spline from $S_{\Delta}^{V_{2n+1}}(T)$ satisfying the conditions $\hat{\varphi}_{N,k}(t_N) = (-1)^{n+k}\hat{\varphi}_{N,k}(-t_N) = 1$ and $\hat{\varphi}_{N,k}(t) = 0$ for $t \in \Delta_{N-1}$. Define $\{\hat{F}_N^{(n,k)}\}_{N=1}^{\infty}$ in the same way as $\{F_N^{(n,k)}\}$. The system $\{\hat{F}_i^{(n,k)}, \hat{F}_j^{(n,-k)}\}_{i,j=2}^{\infty}$ is biorthonormal and consists of odd functions. As above we prove that the system $\{\hat{F}_N^{(n,k)}\}$ is a basis in the space of continuous, odd and 2π-periodic functions in T. Let

$$\hat{G}_1^{(n,k)}(z) = \frac{1}{\sqrt{2\pi}} ,$$

$$\hat{G}_N^{(n,k)}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{F}_N^{(n,k)}(t) \frac{e^{it} + z}{e^{it} - z} \, dt , \quad N \geq 2 .$$

THEOREM 5.3. *Let $-n \le k \le n-1$. Then the system $\{\hat{G}_N^{(n,k)}\}_{N=1}^{\infty}$ is a basis in the complex space $A(D)$. Moreover, there exists a constant $C_{U,R}$ such that for $f \in A(D)$, $f(e^{it}) = u(t) + iv(t)$,*

$$\|f - \hat{S}_{N,f}\| \le C_{U,R}[\omega(u,1/N) + \omega(v,1/N)],$$

where $\hat{S}_{N,f} = \sum_{j=1}^{N} a_j \hat{G}_j^{(n,k)}$ and $a_j = (f, \hat{G}_j^{(n,-k)})$.

The proof is analogous.

3. Systems conjugate to biorthogonal spline systems. Let $F = \{f_N\}_{N=1}^{\infty}$ be the periodic Franklin system in $T = [0, 2\pi]$ and $\tilde{F} = \{1/\sqrt{2\pi}, \tilde{f}_2, \tilde{f}_3, \ldots\}$ the conjugate system. In 1985 S. V. Bochkarev proved that $\tilde{F}$ is a Schauder basis in the space $C(T)$ of continuous 2π-periodic functions in T. He thus solved a problem raised by P. Turán in 1961 (see [8]). The author [77] generalized this result to biorthogonal polynomial spline systems and to orthogonal Chebyshevian spline systems. Now we generalize it to biorthogonal systems defined by (3.10).

Let $\Delta_N = \{t_0 = 0, t_1 = 2\pi, t_2, \ldots, t_N\} = \{0 = x_{N,0} < x_{N,1} < \ldots < x_{N,N} = 2\pi\}$, $N = 1, 2, \ldots$, be a given sequence of partitions of T with $\Delta_N \subset \Delta_{N+1}$ and let $F = \{f_i^{(n,k)}, f_j^{(n,-k)}\}_{i,j=1}^{\infty}$ with $f_1^{(n,k)} = (\int_T w_k\, dt)^{-1/2}$ be the periodic biorthogonal spline system defined by (3.10). Assume that $R_{\Delta_N, n-|k|} \le R < \infty$. We shall prove the following

THEOREM 5.4 (cf. [77]). *Let $w_k = 1$. Then $\tilde{F}_k = \{1/\sqrt{2\pi}, \tilde{f}_2^{(n,k)}, \ldots\}$ is a basis in $C(T)$.*

Let

$$K_N^{(n,k)}(x,y) = \sum_{j=1}^{N} f_j^{(n,k)}(x) f_j^{(n,-k)}(y),$$

$$P_N^{(n,k)}(x) = \int_T f(y) K_N^{(n,k)}(x,y)\, dy, \quad -n+2 \le k \le n-2,$$

and let

$$\tilde{K}_N^{(n,k)}(x,y) = \sum_{j=1}^{N} \tilde{f}_j^{(n,k)}(x) \tilde{f}_j^{(n,-k)}(y)$$

be the Dirichlet kernel for the system $\tilde{F}_k$.

Theorem 5.4 is a corollary of the following

THEOREM 5.5 (cf. [8, 77]). *There exists a constant $C_{U,R}$ such that*

$$|\tilde{K}_N^{(n,k)}(x,y)| \le C_{U,R} \min\left(N, \frac{1}{N(x-y)^2}\right) \qquad \text{for} \quad |x-y| \le \pi\,, \quad N \ge 2\,.$$

Proof. Define

$$\Lambda_{N,y}(t) = \begin{cases} \cot \frac{1}{2}(y-t) & \text{for } 1/N \le |y-t| \le \pi, \\ 0 & \text{for } |y-t| < 1/N. \end{cases}$$

Write the kernel $\tilde{K}_N^{(n,k)}(x,y)$ in the following form:

$$\begin{aligned} &\tilde{K}_N^{(n,k)}(x,y) \\ &= \frac{1}{2\pi} + \frac{1}{4\pi^2} \int_0^{1/N}\int_0^{1/N} [K_N^{(n,k)}(x+t,y+\tau) - K_N^{(n,k)}(x-t,y+\tau) \\ &\quad - K_N^{(n,k)}(x+t,y-\tau) + K_N^{(n,k)}(x-t,y-\tau)] \cot\frac{t}{2}\cot\frac{\tau}{2}\,d\tau\,dt \\ &\quad + \frac{1}{4\pi^2} \int\limits_{1/N\le|x-t|\le\pi} \Big[\int\limits_{1/N\le|y-\tau|\le\pi} K_N^{(n,k)}(t,\tau)\cot\frac{y-\tau}{2}\,d\tau - \Lambda_{N,y}(t)\Big] \\ &\qquad\qquad \times \cot\frac{x-t}{2}\,dt \\ &\quad + \frac{1}{4\pi^2} \int\limits_{1/N\le|x-t|\le\pi} \Lambda_{N,y}(t)\cot\frac{x-t}{2}\,dt \\ &\quad + \frac{1}{4\pi^2} \int\limits_{|x-t|\le 1/N} \Big[\int\limits_{1/N\le|x-\tau|\le\pi} K_N^{(n,k)}(t,\tau)\cot\frac{y-\tau}{2}\,d\tau\Big]\cot\frac{x-t}{2}\,dt \\ &\quad + \frac{1}{4\pi^2} \int\limits_{|y-\tau|\le 1/N} \Big[\int\limits_{1/N\le|x-t|\le\pi} K_N^{(n,k)}(t,\tau)\cot\frac{x-t}{2}\,dt\Big]\cot\frac{y-\tau}{2}\,d\tau \\ &= \frac{1}{2\pi} + I_1 + \ldots + I_5\,. \end{aligned}$$

Write the integral I_1 as follows:

$$I_1 = \frac{1}{4\pi^2}\int_0^{1/N}\int_0^{1/N}\Big[\int_{-t}^{t}\int_{-\tau}^{\tau}\frac{\partial^2}{\partial t'\partial\tau'}K_N^{(n,k)}(x+t',y+\tau')\,dt'\,d\tau'\Big]\cot\frac{t}{2}\cot\frac{\tau}{2}\,dt\,d\tau\,.$$

Applying (3.32), the Markov inequality and (2) we obtain

$$|I_1| \le CNq^{N\,d(x,y)}\,,$$

where $d(x,y) = \min(|x-y|, 2\pi - |x-y|)$, $0 < q < 1$ and the constants C and q depend only on U and R. We denote, here and later on, by the same letters C and q different constants depending on U and R only. Since

$$Nq^{N|x-y|} \le \frac{4}{(e\ln q)^2}\frac{1}{N|x-y|^2}\,,$$

we have

$$(5.9) \qquad |I_1| \le C\min\left(N, \frac{1}{N(x-y)^2}\right) \qquad \text{for} \quad |x-y| \le \pi\,.$$

Let $m(n-k) \le N < (m+1)(n-k)$ and let P_N be a spline from $S^{U_{n,k}}_{\Delta_{m(n-k)}}(T)$, where $U_{n,k}$ is the system of functions defined by (1) w.r.t. the system of weight functions $\{w_0, w_{k+1}, \ldots, w_n\}$ for $0 \le k < n$ and $\{w_0, w_{k-1}, \ldots, w_1, w_0, w_1, \ldots, w_n\}$ for $-n < k < 0$, interpolating the function $\Lambda_{N,y}(t)$ at $x_{N,i(n-k)}$, $i = 0, \ldots, m$, such that for $i = 0, \ldots, m$

$$D_{k+j}\ldots D_{k+1}P_N(x_{N,i(n-k)}) = 0\,, \qquad j = 1, \ldots, n-k-1\,, \qquad \text{for} \quad 0 < k < n\,,$$

$$D_{-k+1}\ldots D_{-k+1+j}P_N(x_{N,i(n-k)}) = 0\,, \qquad j = 0, \ldots, n+k-1\,, \qquad \text{for} \quad -n < k < 0\,.$$

Let us define $\varphi_{i(n-k)}(t) = \int_{x_{N,i(n-k)}}^{t} w_{\eta(k+1)}(\tau) M_{i(n-k),n-k-1}(\tau)\,d\tau$, where $M_{i(n-k),n-k-1}$ is the $i(n-k)$th B-spline w.r.t. $\Delta_{m(n-k)}$ and the system $D_{k+1}U_{n,k} = \{D_{k+1}u_j\}$ for $u_j \in U_{n,k}$ with $\eta = 1$ for $k \ge 0$ and $D_{-k-1}U_{n,k}$ with $\eta = -1$ for $k < 0$. Then for $t \in [x_{N,i(n-k)}, x_{N,(i+1)(n-k)}]$

$$P_N(t) = \Lambda_{N,y}(x_{N,i(n-k)}) + [\Lambda_{N,y}(x_{N,(i+1)(n-k)}) - \Lambda_{N,y}(x_{N,i(n-k)})]\frac{\varphi_{i(n-k)}(t)}{\varphi_{i(n-k)}(x_{N,(i+1)(n-k)})}\,.$$

Hence

$$|P_N(t) - \Lambda_{N,y}(t)| \le \omega(\Lambda_{N,y}, 1/N) \quad \text{for} \quad t \in [x_{N,(i-1)(n-k)}, x_{N,i(n-k)}]\,,$$

where $\omega(\Lambda_{N,y}, 1/N)$ is the modulus of continuity of $\Lambda_{N,y}$ restricted to the interval $[x_{N,(i-1)(n-k)}, x_{N,i(n-k)}]$ (cf. [65]). Hence for $1/N \le |y-t| \le \pi$, we have

$$\Lambda'_{N,y}(t) = \frac{1}{2\sin^2\frac{y-t}{2}} \le \frac{\pi^2}{8(y-t)^2}$$

and

$$|P_N(t) - \Lambda_{N,y}(t)| \le \begin{cases} C/(N(y-t)^2) & \text{for } |y-t| \ge 1/N, \\ CN & \text{for } |y-t| < 1/N. \end{cases}$$

Further,

$$\int_T K_N^{(n,k)}(t,\tau) P_N(\tau)\, d\tau = P_N(t)\,.$$

Write

$$S_N(\Lambda_{N,y}, t) = \int_T K_N^{(n,k)}(t,\tau)\Lambda_{N,y}(\tau)\, d\tau\,.$$

Let $t > y$ (the proof for $t < y$ is identical). We have

$$\begin{aligned} |S_N(\Lambda_{N,y}, t) - P_N(t)| &= \left| \int_T K_N^{(n,k)}(t,\tau)[\Lambda_{N,y}(\tau) - P_N(\tau)]\, d\tau \right| \\ &\le C \max_{|t-\tau|\le 1/N} |\Lambda_{N,y}(\tau) - P_N(\tau)| \\ &\qquad + \int_{1/N \le |t-\tau| \le \pi} |K_N^{(n,k)}(t,\tau)| \cdot |\Lambda_{N,y}(\tau) - P_N(\tau)|\, d\tau \\ &\le C \min\left(N, \frac{1}{N(t-y)^2}\right) + \sum_{i=1}^{4} \int_{t_i'}^{t_i''} |K_N^{(n,k)}(t,\tau)| \cdot |\Lambda_{N,y}(\tau) - P_N(\tau)|\, d\tau \\ &= C \min\left(N, \frac{1}{N(t-y)^2}\right) + \sum_{i=1}^{4} J_i\,, \end{aligned}$$

where $t_1' = -\pi$, $t_1'' = t_2' = y - 1/N$, $t_2'' = t_3' = y + 1/N$, $t_3'' = t - 1/N$, $t_4' = t + 1/N$, $t_4'' = \pi$.

Applying (3.32) we estimate the integrals J_i as follows:

$$J_i \le C \int_{t_i'}^{t_i''} N q^{N|t-\tau|} \min\left(N, \frac{1}{N(y-\tau)^2}\right) d\tau \le C \min\left(N, \frac{1}{N(y-t)^2}\right).$$

Therefore

$$|S_N(\Lambda_{N,y}, t) - P_N(t)| \le C \min\left(N, \frac{1}{N(y-t)^2}\right)$$

whence

(5.10) $$|S_N(\Lambda_{N,y},t) - \Lambda_{N,y}(t)| \le C \min\left(N, \frac{1}{N(y-t)^2}\right).$$

Let

$$F_{N,y}(t) = \int_{y-\pi}^{t} [S_N(\Lambda_{N,y},\tau) - \Lambda_{N,y}(\tau)]\, d\tau.$$

The function $F_{N,y}$ is 2π-periodic. Applying Theorem 2.1 (interpolating property of splines) we obtain

(5.11) $$|F_{N,y}(t)| \le C \min\left(1, \frac{1}{N^2(y-t)^2}\right).$$

Therefore

$$I_2 = \frac{1}{4\pi^2} \int_{1/N \le |x-t| \le \pi} [S_N(\Lambda_{N,y},t) - \Lambda_{N,y}(t)] \cot\frac{x-t}{2}\, dt$$

$$= \frac{1}{4\pi^2}[F_{N,y}(x-1/N) + F_{N,y}(x+1/N)] \cot\frac{1}{2n}$$

$$- \frac{1}{8\pi^2} \int_{y-\pi}^{x-1/N} F_{N,y}(t) \sin^{-2}\frac{x-t}{2}\, dt - \frac{1}{8\pi^2} \int_{x+1/N}^{y+\pi} F_{N,y}(t) \sin^{-2}\frac{x-t}{2}\, dt.$$

Hence by (5.11)

(5.12) $$|I_2| \le C \min\left(N, \frac{1}{N(x-y)^2}\right).$$

Next,

$$I_3 = \frac{1}{4\pi^2} \int_J \cot\frac{t-x}{2} \cot\frac{t-y}{2}\, dt,$$

where $J = \{t \in [-\pi,\pi] : \ |t-x| \ge 1/N, |t-y| \ge 1/N\}$.

Let $x > y$ (the proof for $x < y$ is identical). Set $x-y = 2h$, $(x+y)/2 = t_*$, $t = \tau + t_*$, $z = e^{i\tau}$, $w = e^{ih}$,

$$J_N = [-\pi,\pi] \setminus \{(h-1/N, h+1/N) \cup (-h-1/N, -h+1/N)\}.$$

Then

$$I_3 = \frac{1}{4\pi^2}\int_{J_N} \cot\frac{\tau+h}{2}\cot\frac{\tau-h}{2}\,d\tau = \frac{1}{4\pi^2}\int_{J_N} \frac{\cos h + \cos\tau}{\cos h - \cos\tau}\,d\tau$$

$$= \frac{1}{4\pi^2 i}\int_{\Gamma} \frac{(z+w)(\bar w + z)}{(z-w)(\bar w - z)z}\,dz - \frac{1}{4\pi^2 i}\int_{\Gamma_1\cup\Gamma_2} \frac{(z+w)(\bar w + z)}{(z-w)(\bar w - z)z}\,dz = J - J_1,$$

where Γ, Γ_1 and Γ_2 are the curves shown in Fig. 3.

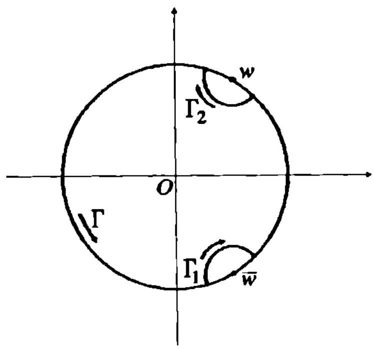

Fig. 3. $-\Gamma_1 = \{z : z = \bar w + \rho e^{i\tau},\ \alpha \le \tau \le \beta\}$, $\Gamma_2 = \{z : z = w + \rho e^{-i\tau},\ \alpha \le \tau \le \beta\}$, $\rho = |e^{i/N} - 1|$

It follows that

$$\frac{1}{2\pi} + I_3 = \frac{1}{2\pi} + \frac{1}{2\pi}\operatorname{res}_0 \frac{(z+w)(\bar w + z)}{(z-w)(\bar w - z)z} - \frac{1}{4\pi^2 i}\int_{\Gamma_1\cup\Gamma_2} \frac{(z+w)(\bar w + z)}{(z-w)(\bar w - z)z}\,dz$$

$$= \frac{1}{4\pi^2}\int_\alpha^\beta \left[\frac{(2w+\rho e^{i\tau})(w+\bar w+\rho e^{i\tau})}{(w-\bar w+\rho e^{i\tau})(w+\rho e^{i\tau})} + \frac{(2\bar w+\rho e^{-i\tau})(w+\bar w+\rho e^{-i\tau})}{(\bar w - w+\rho e^{-i\tau})(\bar w+\rho e^{-i\tau})}\right] d\tau$$

$$= \frac{1}{4\pi^2}\int_\alpha^\beta \frac{\rho g(\tau,h)\,d\tau}{|w-\bar w+\rho e^{i\tau}|^2 |w+\rho e^{i\tau}|^2},$$

where

$$g(\tau,h) = [2\cos(3\tau - h) - 4\cos(\tau - 3h) + 4\cos(\tau - h) + \cos(\tau + h)] + \rho[6\cos 3(\tau - h) - 2\cos 2h + 2\cos 2\tau + 10] + 10\rho^2\cos(\tau - h) + 2\rho^3 .$$

Hence

$$\left|\frac{1}{2\pi} + I_3\right| \le \frac{43}{(|w-\bar{w}|-\rho)^2(1-\rho)^2} \le \frac{C}{N(x-y)^2}\,. \tag{5.13}$$

Now

$$I_4 = -\frac{1}{4\pi^2}\int_0^{1/N} [S_N(\Lambda_{N,y}, x+t) - S_N(\Lambda_{N,y}, x-t)]\cot\frac{t}{2}\,dt\,.$$

Applying (5.10) and the Markov inequality to $S_N(\Lambda_{N,y},t)$ in $[x-1/N, x+1/N]$, we obtain

$$|I_4| \le C\int_0^{1/N} \sup_{x-1/N\le t\le x+1/N} |S_N'(\Lambda_{N,y},t)|t\cot\frac{t}{2}\,dt \le \frac{C}{N(x-y)^2}\,. \tag{5.14}$$

The estimate of I_5 is similar. Hence by (5.9) and (5.12)–(5.14) we obtain the theorem.

Remark 5.1. In the algebraic case the author [77] proved that the system $\tilde{F}_k$ is a simultaneous basis in $\mathring{C}^{n-k-1}(T)$, $k=-n,\ldots,n-1$. In our case this is no more true.

Proof of Theorem 5.4. Let

$$\tilde{P}_N^{(n,k)}f(t) = \int_T f(x)\tilde{K}_N^{(n,k)}(t,x)\,dx\,.$$

Since $(f_j^{(n,k)},1)=0$ for $j>1$ the system $\{\tilde{F}_k,\ \tilde{F}_{-k}\}$ is biorthogonal. Hence it suffices to prove that $\lim_{N\to\infty}\|f - \tilde{P}_N^{(n,k)}f\|_\infty = 0$. It follows from Theorem 5.5 that the operator $\tilde{P}_N^{(n,k)}$ is bounded in $\mathring{C}(T)$, i.e. $|\tilde{P}_N^{(n,k)}f(t)| \le C_{U,R}\|f\|_\infty$. For any $\varepsilon>0$ there exists a trigonometric polynomial p_ε such that $\|f-p_\varepsilon\|_\infty < \varepsilon/3$. Let $\tilde{p}_\varepsilon$ be the trigonometric polynomial conjugate to p_ε. Then $\tilde{p}_\varepsilon(t) - ip_\varepsilon(t) = f_\varepsilon(e^{it})$ with f_ε in $A(D)$. It follows from Theorem 5.1 that there exists N_ε such that $\|p_\varepsilon - \tilde{P}_N^{(n,k)}p_\varepsilon\|_\infty < \varepsilon/(3C_{U,R})$ for $N > N_\varepsilon$. Hence

$$\|f - \tilde{P}_N^{(n,k)}f\|_\infty \le \|f-p_\varepsilon\|_\infty + \|p_\varepsilon - \tilde{P}_N^{(n,k)}p_\varepsilon\|_\infty + \|\tilde{P}_N^{(n,k)}(p_\varepsilon - f)\|_\infty < \varepsilon$$

and we have established the theorem.

Let $\{\Delta_N\}_{N=1}^{\infty}$ be the sequence of partitions of the interval $T = [-\pi, \pi]$ defined in Section 2 and let $w_k = 1$. Reasoning as in the proof of Theorems 5.4 and 5.5 we prove that the systems $\{1/\sqrt{2\pi}, \tilde{\hat{F}}_1^{(n,k)}, \tilde{F}_1^{(n,k)}, \tilde{\hat{F}}_2^{(n,k)}, \tilde{F}_2^{(n,k)}, \ldots\}$ and $\{1/\sqrt{2\pi}, \tilde{\hat{F}}_1^{(n,-k)}, \tilde{F}_1^{(n,-k)}, \tilde{\hat{F}}_2^{(n,-k)}, \tilde{F}_2^{(n,-k)}, \ldots\}$ are biorthonormal and each of them is a basis in $\mathring{C}(T)$. For $k = 0$ we obtain an orthonormal system of even and odd 2π-periodic functions.

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