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UNIVERSAL EXACT SEQUENCES FOR TORSION THEORIES

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Dedicated to Professor H. Tachikawa
on the occasion of his 60th birthday

Let A be a finite-dimensional algebra and $(\mathscr{F}, \mathscr{T})$ a torsion theory in A-mod. If $X \in \mathscr{T}$ is an indecomposable torsion module which is not Ext-projective, and $0 \to \tau X \to E \to X \to 0$ is the Auslander–Reiten sequence in A-mod with end X, then it was proved in [AS, Bü, H83] that the induced sequence $0 \to t(\tau X) \to t(E) \to X$ is the relative Auslander–Reiten sequence in $\mathscr{T}$, ending with X (t denotes the torsion radical). The short exact sequence $0 \to t(\tau X) \to \tau X \to Q \to 0$, with $Q \in \mathscr{F}$ the cokernel of the inclusion $t(\tau X) \to \tau X$, has been used successfully in [K, K88] for the description of components of wild tilted algebras. The aim of the first part of this note is to show that this exact sequence is universal: If Z is any module in $\mathscr{F}$ and $0 \to t(\tau X) \to M \to Z \to 0$ is a short exact sequence in $\mathrm{Ext}^1(Z, t(\tau X))$, we can get this sequence by a pull-back construction from the above one. Moreover, for A hereditary Q is Ext-injective in $\mathscr{F}$.

In the second part we will apply this result to the study of wild hereditary algebras with three pairwise nonisomorphic simple modules. We will describe the nonpreprojective component with a projective vertex for a wild tilted algebra with three simple modules rather explicitly.

Preliminaries

By a k-algebra A we understand a finite-dimensional, basic, associative unitary k-algebra, k some commutative field. We write A-mod for the category of finitely generated left A-modules and "module" always means "finitely generated module".

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A tilting module ${}_AT$ defines a torsion theory $(\mathscr{F}(T), \mathscr{G}(T))$, where $\mathscr{F}(T) = \{{}_AX \mid \mathrm{Hom}_A(T, X) = 0\}$, $\mathscr{G}(T) = \{{}_AY \mid \mathrm{Ext}^1_A(T, X) = 0\}$, and a torsion theory $(\mathscr{Y}(T), \mathscr{X}(T))$ in B-mod, $B = \mathrm{End}(T)$, defined by $\mathscr{Y}(T) = \{{}_BN \mid \mathrm{Tor}^B_1(T, N) = 0\}$, $\mathscr{X}(T) = \{{}_BM \mid T \otimes M = 0\}$. Then by the theorem of Brenner and Butler the functor $F = \mathrm{Hom}_A(T, -)$ induces an equivalence between $\mathscr{G}(T)$ and $\mathscr{Y}(T)$, and $F' = \mathrm{Ext}^1_A(T, -)$ an equivalence between $\mathscr{F}(T)$ and $\mathscr{X}(T)$ (see [R84]).

For an algebra A we denote by $\Gamma(A)$ the Auslander–Reiten quiver of A. If A is hereditary and representation-infinite, $\Gamma(A)$ consists of three parts: the preprojective component $\mathscr{P}$, the preinjective component $\mathscr{I}$ and the regular part $\mathscr{R}$. If the context is clear, we shall not distinguish between a component of $\Gamma(A)$, the module class defined by this component, and the full subcategory of A-mod defined by this module class.

The Auslander–Reiten quiver $\Gamma(A)$ of an algebra A is equipped with an additional structure, the Auslander–Reiten translation τ. If A is hereditary, then τ is functorial. If C is a factor algebra of A we have a full exact embedding C-mod $\to A$-mod. Thus for a C-module X we can consider the Auslander–Reiten translation in C-mod and in A-mod; if it is necessary to distinguish, we add a subscript to τ, for example τ_C, τ_A^{-1}.

The term wild algebra always means strictly wild in the sense of [R76]: A k-algebra A is called *strictly wild* if there exists a full exact embedding $k'\{X, Y\} \to A$-mod, where k' is a finite commutative extension of k and $k'\{X, Y\}$ is the free k'-algebra in two generators X and Y.

1. The universal sequence

Definition. Let X be an indecomposable module and $\mathscr{C}$ a class of modules in A-mod. A short exact sequence $0 \to X \to M \to Q \to 0$ with $Q \in \mathrm{add}\,\mathscr{C}$ is called *universal* in $\mathrm{Ext}(\mathscr{C}, X)$ if

(i) $\mathrm{Hom}(Q, M) = \mathrm{rad}(Q, M)$,

(ii) for $D \in \mathrm{add}\,\mathscr{C}$ the induced map δ: $\mathrm{Hom}(D, Q) \to \mathrm{Ext}^1(D, X)$ is surjective.

It is clear that for given X and $\mathscr{C}$ universal short exact sequences in $\mathrm{Ext}(\mathscr{C}, X)$ do not exist in general. But, if existing, they are unique up to isomorphism: Let $0 \to X \to M_1 \to Q_1 \to 0$ and $0 \to X \to M_2 \to Q_2 \to 0$ be universal in $\mathrm{Ext}(\mathscr{C}, X)$. By (ii) we get linear maps r: $Q_1 \to Q_2$ and s: $Q_2 \to Q_1$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & X & \to & M_1 & \to & Q_1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow r & & \\ 0 & \to & X & \to & M_2 & \to & Q_2 & \to & 0 \\ & & \| & & \downarrow & & \downarrow s & & \\ 0 & \to & X & \to & M_1 & \to & Q_1 & \to & 0 \end{array}$$

Let $t = sr \in \mathrm{End}(Q_1)$. By Fitting's lemma we may suppose that $\ker t$ is a direct summand of Q_1. Considering

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \ker t & = & \ker t & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & M_1 & \to & Q_1 & \to & 0 \\ & & \| & & \downarrow & & {}^{t}\downarrow & & \\ 0 & \to & X & \to & M_1 & \to & Q_1 & \to & 0 \end{array}$$

we conclude by condition (i) that the kernel of t is trivial. Doing the same with $rs \in \mathrm{End}(Q_2)$ we see that the sequences are isomorphic.

Let $(\mathscr{F}, \mathscr{T})$ be a torsion theory in A-mod, $\mathscr{F}$ denoting the class of torsion-free modules and $\mathscr{T}$ the class of torsion modules. Let X be an indecomposable torsion module which is not Ext-projective and let $0 \to \tau X \to E \to X \to 0$ be the Auslander–Reiten sequence in A-mod ending with X. If t denotes the torsion radical, then, as mentioned above, the relative Auslander–Reiten sequence in $\mathscr{T}$ with end X is $0 \to t(\tau X) \to t(E) \to X \to 0$.

THEOREM. *Let A be a finite-dimensional algebra and $(\mathscr{F}, \mathscr{T})$ a torsion theory in A-*mod.

(a) *For an indecomposable torsion module X either X is* Ext*-projective, or the short exact sequence*

$$0 \to t(\tau X) \to \tau X \to Q \to 0$$

is universal in $\mathrm{Ext}(\mathscr{F}, t(\tau X))$.

(b) *If A is hereditary, then Q is* Ext*-injective in $\mathscr{F}$.*

Proof. (a) X is Ext-projective if and only if τX is Ext-injective if and only if $t(\tau X) = 0$ (see [AS, H83]). So assume $t(\tau X)$ is not zero. As τX is indecomposable and not torsion-free, we have $\mathrm{Hom}(Q, \tau X) = \mathrm{rad}(Q, \tau X)$. For a torsion-free module Y we apply the covariant functor $\mathrm{Hom}(Y, -)$ to the exact sequence in (a) and get

$$\ldots \to \mathrm{Hom}(Y, Q) \xrightarrow{\delta} \mathrm{Ext}^1(Y, t(\tau X)) \to \mathrm{Ext}^1(Y, \tau X) \to \ldots$$

But by the Auslander–Reiten formula we have $D\mathrm{Ext}(Y, \tau X) \simeq \underline{\mathrm{Hom}}(X, Y) = 0$, since $\mathrm{Hom}(X, Y) = 0$ as X is torsion and Y is torsion-free. Thus δ is surjective.

(b) If X is Ext-projective, $Q = \tau X$ is Ext-injective. So assume $t(\tau X) \neq 0$. Q is torsion-free by construction. Let $Q = Q_1 \oplus Q_2$ with Q_1 injective and Q_2 without injective direct summands. Clearly Q_1 is Ext-injective. Let $p\colon Q \to Q_2$ be the canonical projection. Then the composed map $h\colon \tau X \to Q \to Q_2$ is surjective and, as τ^{-1} is a right-exact functor for A hereditary, also $\tau^{-}(h)\colon X \to \tau^{-1}(Q_2)$ is surjective. But X is a torsion module and so also $\tau^- Q_2$ is torsion. By [H83] a torsion-free module Q is Ext-injective if and only if $\tau^- Q$ is torsion; therefore Q_2 is Ext-injective.

COROLLARY. *Let A be hereditary and $T = \bigoplus_{i=1}^n T_i$ a tilting module. If X is an indecomposable module in $\mathscr{G}(T)$ not* Ext-*projective, then the short exact sequence $0 \to t(\tau X) \to \tau X \to Q \to 0$ is universal in* $\mathrm{Ext}(\mathscr{F}(T), t(\tau X))$ *and Q is in* $\mathrm{add}\{\tau T_i \mid T_i$ *not projective*$\}$.

Remark. Let us mention that in the tilting situation the universal exact sequence in $\mathrm{Ext}(\mathscr{F}, t(\tau X))$ (or, dually in $\mathrm{Ext}(\tau^- X/t(\tau^- X), \mathscr{G}(T))$ if X is torsion-free and not Ext-injective), and thus also τX, can be constructed explicitly: Let $T = T_1 \oplus \ldots. \oplus T_n$ with T_i indecomposable pairwise nonisomorphic and let $\mathscr{Q}$ be the Gabriel quiver of the tilted algebra $B = \mathrm{End}(T)$. The r sources of $\mathscr{Q}$ correspond to r direct summands of T, say $T_1, \ldots, T_r$. If d_i is the dimension of $\mathrm{Ext}(\tau T_i, t(\tau X))$ as a $D_i = \mathrm{End}(T_i)$-module, we consider the universal exact sequence

$$(*) \qquad 0 \to t(\tau X) \to M \to \bigoplus_{i=1}^{r} (\tau T_i)^{d_i} \to 0,$$

which represents all exact sequences in $\mathrm{Ext}(\tau T_i, t(\tau X))$. Let now $T_{r+1}, \ldots, T_s$ be the indecomposable summands of T which correspond to the vertices of $\mathscr{Q}$ next to the sources. For T_m with $r+1 \leqslant m \leqslant s$ we apply the functor $\mathrm{Hom}(\tau T_m, -)$ to the exact sequence $(*)$ and get

$$\ldots \to \mathrm{Hom}(\tau T_m, \bigoplus (\tau T_i)^{d_i}) \xrightarrow{\delta_m} \mathrm{Ext}^1(\tau T_m, t(\tau X)) \to \ldots$$

Let d_m $(r+1 \leqslant m \leqslant s)$ be the dimension of the cokernel of δ_m and consider the exact sequence $0 \to t(\tau X) \to N \to \bigoplus_{i=1}^{s} (\tau T_i)^{d_i} \to 0$ which represents all exact sequences in $\mathrm{Ext}(\tau T_i, t(\tau X))$ for $1 \leqslant i \leqslant s$. Continuing this process, we finally get the universal short exact sequence $0 \to t(\tau X) \to \tau X \to Q \to 0$.

In particular, if the tilting module T is of the form $T = Z \oplus M$ with Z indecomposable regular and M preprojective, then for all regular torsion modules X we have $\tau X/t(\tau X) = (\tau Z)^m$ for some nonnegative m.

Let A be an algebra, not necessarily hereditary, ${}_A T$ a tilting module, and Z an indecomposable torsion module.

If Z is injective, then trivially $0 \to Z = Z \to 0 \to 0$ is the universal exact sequence in $\mathrm{Ext}(\mathscr{F}(T), Z)$. If Z is not injective, by [HR] there exists a relative Auslander–Reiten sequence in $\mathscr{G}(T)$, $0 \to Z \to M \to N \to 0$. Then $0 \to Z \to \tau_A N \to Q \to 0$ is the universal exact sequence in $\mathrm{Ext}(\mathscr{F}(T), Z)$, that is, we have the following result:.

PROPOSITION. *Let A be an algebra and ${}_A T$ a tilting module. If X is in $\mathscr{G}(T)$, then there exists a universal exact sequence in* $\mathrm{Ext}(\mathscr{F}(T), X)$.

2. Hereditary algebras with three simple modules

If A is a finite-dimensional wild hereditary connected algebra, all regular components of $\Gamma(A)$ are of type $\mathbf{Z}A_{\infty}$, by [R78]. If X and Y are indecomposable regular A-modules, there exist integers N_1 and N_2 such that $\mathrm{Hom}(X, \tau^s Y) \neq 0$ for all $s \geqslant N_1$ (see [B]) and $\mathrm{Hom}(X, \tau^{-s} Y) = 0$ for all $s \geqslant N_2$ (see [K]).

It was proved by [H84] that $\mathrm{Ext}(X, X) \neq 0$ for all indecomposable regular modules X with quasilength at least $n-1$, n the number of nonisomorphic simple A-modules. In particular, for $n = 3$ all indecomposable regular modules without self-extensions are quasisimple. Take a regular brick X without self-extensions (such modules always exist, see for example [R88]) and consider the Auslander–Reiten sequences $0 \to \tau^2 X \xrightarrow{e_2} \tau Y \xrightarrow{p_2} \tau X \to 0$ and $0 \to \tau X \xrightarrow{e_1} Y \xrightarrow{p_1} X \to 0$. Then we have:

LEMMA 1. *The map* $\alpha\colon \mathrm{Hom}(X, \tau^2 X) \to \mathrm{Hom}(Y, \tau Y)$ *defined by* $\alpha(f) = p_1 f e_2$ *is an isomorphism.*

Proof. Clearly α is injective. Take $g \in \mathrm{Hom}(Y, \tau Y)$; then $e_1 g p_2 = 0$ since X is a brick and e_1 is irreducible. So there exists $h\colon X \to \tau X$ with $g p_2 = p_1 h$. But, by $\mathrm{Hom}(X, \tau X) \simeq D\mathrm{Ext}(X, X) = 0$ we have $g p_2 = 0$. Therefore $g = r e_2$ for some $r \in \mathrm{Hom}(Y, \tau^2 X)$. But $e_1 r$ is zero, so we see that $g = r e_2 = p_1 f e_2$ for some $f \in \mathrm{Hom}(X, \tau^2 X)$.

If A is wild connected hereditary and X is an indecomposable module without self-extensions, following [GL] we consider the right perpendicular category $X^{\perp}$ of X. It is the full subcategory of A-mod defined by the objects $\{Y \mid \mathrm{Hom}(X, Y) = \mathrm{Ext}^1(X, Y) = 0\}$. By [GL], $X^{\perp}$ is equivalent to C-mod where C is a hereditary algebra. If moreover X is regular, C additionally is wild and connected (see [S]). If T is a tilting module in A-mod, there always exists a decomposition $T = X \oplus T'$ with $T' \in X^{\perp}$.

Let us again consider the case that A has three, pairwise nonisomorphic, simple modules and ${}_A T$ is a (multiplicity-free) tilting module. We are interested in the torsion class $\mathcal{G}(T)$ (or dually in the torsion-free class $\mathcal{F}(T)$). By [K] we may suppose that T has no preinjective direct summands. If T is preprojective, the tilted algebra is concealed. So assume that T has at least one regular direct summand. As mentioned above, we then get a decomposition $T = X \oplus T'$, $T' = T_1 \oplus T_2$ with X regular, $T' \in X^{\perp}$, T_i regular or preprojective for $i = 1, 2$. Let us shortly summarize the known results we will use in the sequel:

(A) T' is a sincere A-module with $\mathrm{Hom}(T_i, X) \neq 0$ for $i = 1, 2$; in the category $X^{\perp}$ the module T' is preprojective (see [U]).

(B) $C = \mathrm{End}(T')$ is a wild hereditary algebra with two projective modules $F(T_1)$ and $F(T_2)$. The tilted algebra $B = \mathrm{End}(T)$ is a one-point extension of C by the indecomposable regular C-module $M = F(R)$, where R is the middle term of the Auslander–Reiten sequence $0 \to \tau X \to R \to X \to 0$ (see [K88, S, U]).

(C) All regular components in $\mathcal{Y}(T)$ are of type $\mathbf{Z}A_\infty$. If $\mathcal{C}$ is a regular component in $\mathcal{Y}(T)$, there exists a quasisimple module $F(S) \in \mathcal{C}$ such that S is a quasisimple regular A-module with $F({}^{>}S) = {}^{>}(F(S))$. The symbol ${}^{>}S$ (${}^{>}(F(S))$) denotes the full subquiver of $\Gamma(A)$ ($\Gamma(B)$) defined by the predecessors of S ($F(S)$) (see [K]).

(D) If $\mathcal{D}$ is a regular component in $\Gamma(C)$, there exists a quasisimple module Z in $\mathcal{D}$ such that $\mathrm{Hom}(M, Y) = 0$ for all $Y \in Z^{<}$ (see [K]). As in (B) we denote by $M = F(R)$ the radical of the projective B-module $F(X)$. So $Z^{<}$ is a full successor-closed part of a component of $\Gamma(B)$ (see [R84, 2.5(6)]).

PROPOSITION 1. *Let A be a connected wild hereditary algebra with three simple modules.*

(a) *If X is a regular brick without self-extensions, then* $\mathrm{Hom}(X, \tau^{-i}X) = 0$ *for all $i > 0$.*

(b) *If $T = X \oplus T_1 \oplus T_2$ is a tilting module with X regular and $T' = T_1 \oplus T_2 \in X^{\perp}$, then ${}^{>}(\tau^2 X)$ is in $\mathcal{G}(T)$, where ${}^{>}(\tau^2 X)$ are the predecessors of $\tau^2 X$ in $\Gamma(A)$.*

Proof. (a) Suppose there exists $i > 0$ such that $\mathrm{Hom}(X, \tau^{-i}X) \neq 0$. Take $f \in \{g \in \mathrm{Hom}(X, \tau^{-i}X) \mid g \neq 0,\ i > 0\}$ such that the image W of f: $X \to \tau^{-n}X$ has minimal composition length. The map f cannot be injective; otherwise we would get a chain of injective maps $\ldots \to \tau^s X \to \ldots \to \tau^2 X \to \tau X \to X$.

If $\mathrm{Ext}^1(W, W) \neq 0$, there exists a nonzero map $g \in \mathrm{Hom}(W, \tau W)$. Notice that the inclusion $W \to \tau^{-n}X$ induces monomorphisms i_r: $\tau^r W \to \tau^{-n+r}X$ for all natural r, since the cokernel $\tau^{-n}X/W$ has no preprojective direct summand. If $n = 1$, then fg is an endomorphism of X which is not injective, but X is a brick. If $n > 1$, the map g has to be injective, otherwise fgi_1: $X \to \tau^{1-n}X$ has smaller image than f. Now $fg(\tau g)\ldots(\tau^{n-1}g)i_n$ is again a noninvertible endomorphism of X.

If $\mathrm{Ext}^1(W, W) = 0$ then W is a quasisimple brick and Lemma 1 with $X = W$ together with the discussion preceding Lemma 1 yield $\mathrm{Hom}(W, \tau^2 W) \neq 0$. In the same way as in the previous case we then construct either a noninjective endomorphism of X or a nonzero map from X to τX.

(b) We consider the Auslander–Reiten sequences

$$(\mathrm{AR1})\ 0 \to \tau X \to R \to X \to 0, \qquad (\mathrm{AR2})\ 0 \to \tau R \to \tau X \oplus R' \to R \to 0.$$

By the cited result (B) we know R is torsion and moreover that R is regular in the category C-mod, with $C = \mathrm{End}(T')$. Consider the universal exact sequence

$$0 \to t(\tau R) \to \tau R \to Q \to 0.$$

We have $Q \in \mathrm{add}(\tau T)$ by the first section. If τT_i is a direct summand of Q, we get $\mathrm{Hom}_A(R, T_i) \neq 0$. Then also $0 \neq \mathrm{Hom}_B(F(R), F(T_i)) = \mathrm{Hom}_C(F(R), F(T_i))$, but $F(T_i)$ is projective in C-mod and $F(R)$ is regular.

Therefore $Q = (\tau X)^s$ for some $s \geqslant 1$. But $\mathrm{Hom}(\tau R, \tau X)$ is one-dimensional, thus we have $s = 1$. From this it can be deduced immediately that $t(\tau R) = \tau^2 X \in \mathscr{G}(T)$; from (AR2) we then get the relative Auslander–Reiten sequence $0 \to \tau^2 X \to R' \to R \to 0$; in particular, R' is a torsion module too.

Assume by induction that $\tau^i X$ is a torsion module for some $i \geqslant 2$. Again we look at the universal exact sequence

$$0 \to t(\tau^{i+1} X) \to \tau^{i+1} X \to Q \to 0.$$

Since $\mathrm{Hom}(\tau^{i+1} X, \tau X) = 0$ by (a), we have $Q \in \mathrm{add}(\tau T')$. For $Q \neq 0$ we get an epimorphism $\tau^{i+1} X \to (\tau T_1)^r \oplus (\tau T_2)^s$ for some r and s and thus an epimorphism $\tau^i X \to T_1^r \oplus T_2^s$. Therefore, by the result (B) we have $\tau^i X \in \mathrm{add}(T')$. But then Unger's result, as mentioned in (A), implies $\mathrm{Hom}(\tau^i X, X) \neq 0$, a contradiction to (a). Therefore $Q = 0$ and $t(\tau^{i+1} X) = \tau^{i+1} X \in \mathscr{G}(T)$. This finishes the proof of (b).

In the last few pages we will apply the results obtained to describe in detail the component of $\Gamma(B)$ which contains the projective module $F(X)$. Again A is a hereditary wild connected algebra with three simple modules and $T = X \oplus T'$ is a tilting module with X regular and $T' \in X^{\perp}$.

We have to consider the Auslander–Reiten sequence $0 \to \tau X \to R \to X \to 0$. Since τX and X are orthogonal bricks, R also is a brick with $R \in X^{\perp} \simeq C$-mod.

LEMMA 2. *Let C be a wild hereditary algebra with two simple modules. If R is a regular brick, we have*:

(i) $\mathrm{Hom}(R, \tau^{-i} R) = 0$ *for all* $i > 0$,

(ii) $\mathrm{Hom}(R, \tau^i R) \neq 0$ *for all* $i > 1$.

Proof. (i) is done in the same way as (a) of Proposition 1.

(ii) By $\mathrm{Ext}^1(R, R) \neq 0$ we get via Auslander–Reiten formulas $\mathrm{Hom}(R, \tau R) \neq 0$. Let $i \geqslant 1$ with $\mathrm{Hom}(R, \tau^i R) \neq 0$ and consider $0 \neq V \neq (R)f$ for $f \in \mathrm{Hom}(R, \tau^i R)$. V is regular, so we have $\mathrm{Hom}(V, \tau V) \neq 0$. For $g\colon V \to \tau V$ nonzero and $e\colon \tau V \to \tau^{i+1} R$ the inclusion we get a nonzero map $fge \in \mathrm{Hom}(R, \tau^{i+1} R)$.

PROPOSITION 2. *Let A be a finite-dimensional connected wild hereditary algebra with three simple modules. If $T = X \oplus T'$ is a tilting module with X indecomposable regular and $T' \in X^{\perp}$ and if $R \to X$ is the irreducible surjective map, then we get for the component $\mathscr{C}$ of $\Gamma(B)$, $B = \mathrm{End}_A(T)$, containing the projective B-module $F(X)$ the following*:

(a) $F(\tau_A^2 X) = \tau_B(F(R))$ *and* ${}^{>}F(\tau_A^2 X) = F({}^{>}(\tau_A^2 X))$.

(b) $\tau_B^-(F(X)) = \tau_C^- F(R)$ *and* $(\tau_C^- F(R))^{<}$ *is a full successor-closed subquiver of $\mathscr{C}$. C denotes the wild hereditary algebra* $\mathrm{End}(T')$.

The component may be visualized as in Fig. 1.

Proof. Let $\mathscr{D}$ be the regular component of $\Gamma(A)$ containing the tilting

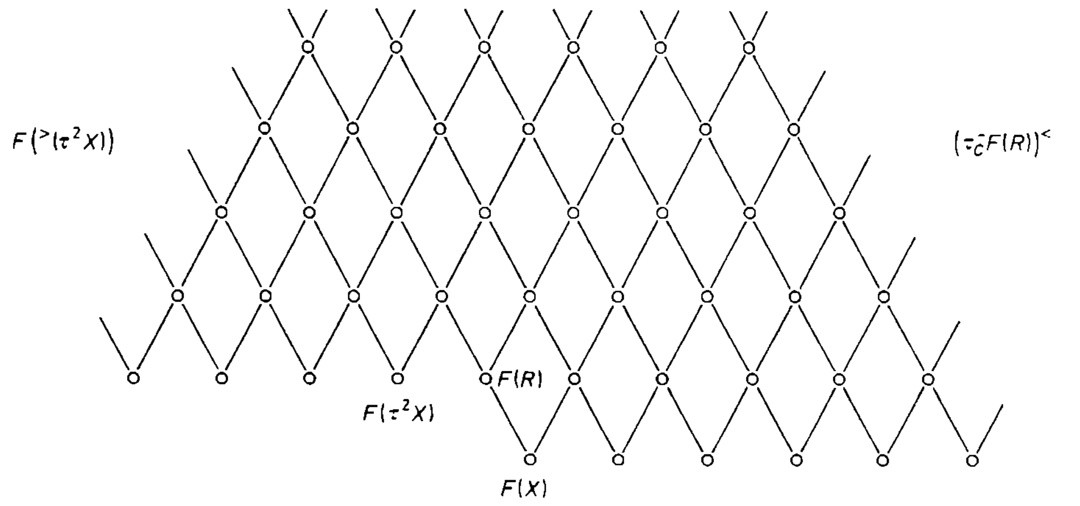

Fig. 1

summand X. Denote the indecomposable module Z in $\mathscr{D}$ with regular length i and regular top $\tau^j X$ by (j, i).

In the proof of Proposition 1 we have shown that ${}^{>}(2, 1) \subset \mathscr{G}(T)$ and that $(0, 2)$ is a torsion module with relative Auslander–Reiten sequence $0 \to (2, 1) \to (0, 3) \to (0, 2) \to 0$; moreover, we have $t(1, 2) = (2, 1)$.

Let us prove by induction that for all $i \geqslant 1$

$$0 \to (2, i) \to (0, i+2) \oplus (2, i-1) \to (0, i+1) \to 0$$

is a relative Auslander–Reiten sequence in $\mathscr{G}(T)$ and that

$$t(1, i+1) = (2, i)$$

is valid.

If the assertion is true for i, then $(0, i+2)$ is a torsion module. We consider the Auslander–Reiten sequence

$$(*) \qquad 0 \to (1, i+2) \to (1, i+1) \oplus (0, i+3) \to (0, i+2) \to 0$$

and the universal exact sequence

$$0 \to t(1, i+2) \to (1, i+2) \to Q \to 0.$$

The module Q is Ext-injective, and the same argument as in the proof of Proposition 1 shows $Q = (1, 1)$, which immediately implies $t(1, i+2) = (2, i+1)$. Using $t(1, i+1) = (2, i)$ we then get from the Auslander–Reiten sequence $(*)$ the relative Auslander–Reiten sequence

$$0 \to (2, i+1) \to (2, i) \oplus (0, i+3) \to (0, i+2) \to 0.$$

We know from [K] that the stable part of $\mathscr{C}$ has the form $\mathbf{Z}A_\infty$; the above considerations then show that the component $\mathscr{C}$ has the form we get from $\mathbf{Z}A_\infty$ by a ray insertion with a branch consisting of one point (see [R84, 4.6]). So it only remains to show that $(\tau_C^- F(R))^<$ is the right part of the component $\mathscr{C}$. Since B is a one-point extension of C by $M = F(R)$ (see (B)), we deduce from [R84, 2.5(6)] that $\tau_B^- F(X) = \tau_C^- M_\epsilon$. Since R is a brick, M is a quasisimple brick in C-mod. By Lemma 2 we have $\mathrm{Hom}_C(M, \tau_C^{-n} M) = 0$ for all $n > 0$. Again from [R84, 2.5(6)] it then follows that $(\tau_C^- M)^<$ is a part of the component with the asserted conditions.

Remark. If Y and Z are indecomposable modules in $\mathscr{C}$ with Z not in the τ^--orbit of $F(X)$, then we have (cf. [K88])

$$\varrho(C) = \lim_{n\to\infty} (\dim_k \tau^{-n} Y)^{1/n} > \lim_{n\to\infty} (\dim_k \tau^n Z)^{1/n} = \varrho(A),$$

ϱ denoting the growth number of the hereditary algebras A and C (see [DR]). By the results (C) and (D) at the beginning of this section we have a lot of regular components in $\mathscr{Y}(T)$ with this property. So there naturally arise the following questions:

Is it true for all regular components $\mathscr{D}$ in $\mathscr{Y}(T)$ that for $X \in \mathscr{D}$, $\varrho(C) = \lim_{n\to\infty}(\dim_k \tau^{-n} X)^{1/n}$? Even stronger:

Do there exist regular components in $\mathscr{Y}(T)$ which are not of the type described in (D)?

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