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ON THE BUCKLING OF A VISCOELASTIC ROD

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1. Introduction

In [1], the authors consider a problem of buckling of a viscoelastic rod which in the case of linear viscoelastic constitutive relation has the following form.

We look for solutions u of the system:

$$u_{xx}(x,t)-\beta_0\int_0^t\beta(t-s)u_{xx}(x,s)ds+P(t)\sin u(x,t)=0, \tag{1}$$

$$u_x(0,t)=u_x(1,t)=0, \tag{2}$$

$$\int_0^1\sin(u(x,t))dx=0, \tag{3}$$

$$u(x,0)=0, \tag{4}$$

where u is the angle between the tangent to the rod and the x axis, and $P(t)$ is the load.

This problem has been suggested by M. E. Gurtin (see [2]). A numerical approach to this problem is given in [3]. Here we present a rather simple treatment of this problem and an elementary proof of some results given in [1].

2. The existence of a bifurcation point

Concerning the existence of a bifurcation point of the problem (1)-(4), we can formulate the following theorem:

THEOREM 1. *If there is an* $\varepsilon>0$ *such that* $|P(t)|<\lambda_1-\varepsilon$ *for* $t\in[0,T]$, *then every solution of the problem satisfies* $\pi_0u=0$, *so that* $u(t)=k(t)\pi$ *where* $k(t)$ *is an arbitrary function* (π_0 *is a projector to the space* $H:=\{\varphi_n:=\cos n\pi x,\ n=1,2,\ldots,\ x\in[0,1]\}$). *Moreover, if* $\tau\in(0,T)$ *is a bifur-*

cation point for the problem (1)–(4) *then* $P(\tau)$ *is a nonzero eigenvalue of the following Neumann problem*:

$$-\varphi_{xx} = \lambda\varphi, \quad x \in (0,1), \tag{5}$$
$$\varphi_x(0) = \varphi_x(1) = 0;$$

i.e. there is an $n > 1$ *such that* $P(\tau) = \lambda_n = n^2\pi^2$ (see Fig. 1).

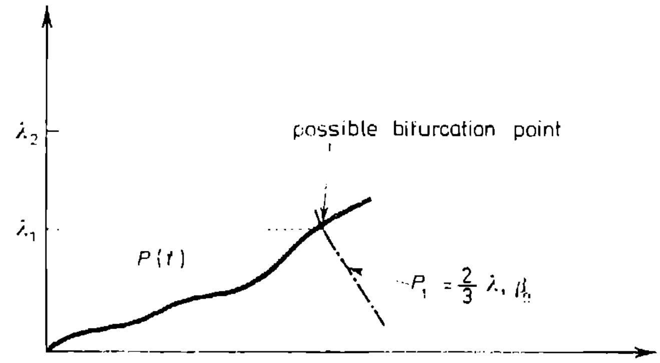

Fig. 1

Proof of Theorem 1. Bifurcation can occur when the linearized problem has at least one nontrivial solution (see, for example, [4]). So we get from (1) the linearized equation

$$u_{xx}(x,t) + P(t)u(x,t) = \beta_0 \int_0^t \beta(t-s)u_{xx}(x,t)ds. \tag{6}$$

Because of the regularity of the right-hand side of (6), we may write for all $t \in [0,T]$ and $x \in (0,1)$

$$\int_0^t \left\{ \frac{\partial}{\partial s}[u_{xx}(x,s) + P(s)u(x,s)] - \beta_0\beta(t-s)u_{xx}(x,s) \right\} ds = 0. \tag{7}$$

It is easy to see that (7) is possible iff $\forall t \in [0,t]$, $\forall x \in [0,1]$ and $\forall s \in [0,t]$

$$\frac{\partial}{\partial s}[u_{xx}(x,s) + P(s)u(x,s)] = \beta_0\beta(t-s)u_{xx}(x,s) \tag{8}$$

holds. But since the right hand side of (8) depends on t, this can be valid iff $\forall x \in [0,1]$ and $\forall s \in [0,t]$

$$\frac{\partial}{\partial s}[u_{xx}(x,s) + P(s)u(x,s)] = c \tag{9}$$

and

$$\beta_0\beta(t-s)u_{xx}(x,s) = c \tag{10}$$

where $c \neq c(t) \neq c(s)$. From (10) it follows that $c = 0$ and $u(x,s) = k(s)$ where $k(s)$ is an arbitrary function such that $\lim_{s\to 0} k(s) = 0$.

From (9) we get

$$u_{xx}(x,s) + P(s)u(x,s) = K. \tag{11}$$

But in view of (4), $K = 0$; and using (2) we can see that if $P(s) \neq \lambda_n$, $n = 1, 2, \ldots$, (11) has only the trivial solution. When $P(s) = \lambda_n$, $n = 1, 2, \ldots$, a nontrivial solution can exist but this question cannot be answered by means of the linearized equation.

Remark. This proof can also be adapted to the case of nonlinear viscoelastic constitutive relation, with $\beta = \beta(u, t-s)$ and with u_{xx} replaced by Lu in (1), L denoting a linear elliptic operator.

3. The problem of bifurcated solutions

The problem of bifurcation of solution of (1)–(4) was solved in [1]. The method of solution is the Lyapunov–Schmidt decomposition (see, for example, [5]). After performing this decomposition, by differentiating the equations with respect to t one gets several nonlinear ordinary differential equation depending on parameters. Here we propose another way of solution to this problem. Having obtained the Lyapunov–Schmidt decomposition, by integrating the problem we can transform it to a nonlinear algebraic equation depending on parameters. The bifurcated solutions are obtained by solving this algebraic equation. The advantage of this approach lies in the simplicity of solution of algebraic equation. Moreover, this approach gives tool for treating problems of this kind in more space variables.

We first prove two lemmas.

LEMMA 1. *If $f(s)$ is an integrable function then*

$$I(t) = \int_0^t \left\{\int_0^s f(\xi)d\xi\right\} ds = -\int_0^t (t-s)f(s)ds.$$

Proof. Let $\varphi(s) = \int_0^s f(\xi)d\xi$. Then

$$I(t) = \int_0^t 1\,\varphi(s)ds = \int_0^t (s)'\varphi(s) = s\varphi(s)|_0^t - \int_0^t s\varphi'(s)ds$$

$$= t\varphi(t) - \int_0^t sf(s)ds = \int_0^t (t-s)f(s)ds.$$

LEMMA 2. *There exist nonzero functions $f(t,s)$ such that $\forall t > 0$*

$$\int_0^t f(t,s)ds = 0. \tag{12}$$

Proof. It is easy to verify that the functions

$$f_{m,m,k}(t,s) = s^m - \frac{1}{\alpha}s^n t^k \neq 0 \tag{13}$$

where $(n+1)\alpha = m+1$, $n+k = m$, n, $m > 0$ satisfy (12) (there also exist other functions satisfying (12)).

In the sequel we assume that

$$P(\quad) \in c^2([0,T)), \qquad t>0,$$
$$(14) \qquad P(t) < \lambda_1, \quad t \in [0,t_0), \quad P(t_0) = \lambda_1, \quad 0<t_0<T,$$
$$P(t) = \lambda_1 + p_1(t-t_0) + Q(t)(t-t_0)^2, \quad t \in [t_0, T],$$

where $Q(\cdot) \in C([t_0,T))$ and $p_1 \in \boldsymbol{R}$.

We may suppose without loss of generality that $t_0 = 0$.

We look for solutions of (1)–(4) of the form

$$(15) \qquad u(t) = a(t) + b(t)\varphi_1 + w(t) = a(t) + v(t), \quad t \in [0,t_1),$$

where $a(t) \in \boldsymbol{R}$, $b(t) \in \boldsymbol{R}$,

$$w \in W := \left\{w; w \in H^0(0,1), \int_0^1 w\,dx = 0 \text{ and } \int_0^1 w\varphi_1\,dx = 0\right\}.$$

With notation and definitions as above, the main part of the bifurcation equation for the problem (1)–(4) is as follows (see [1]):

$$(16) \qquad -p_1 t\cdot b(t) - \lambda_1\beta_0 \int_0^t b(s)\,ds + \lambda_1 \mu b^3(t) = 0,$$

where $\mu = \frac{1}{6}\int_0^1 \varphi_1^4\,dx$.

THEOREM 2. *If* $p_1 > -\frac{2}{3}\lambda_1\beta_0$ (see Fig. 1) *then* $t=0$ *is a bifurcation point and the bifurcated branch* $u(\cdot)$ *in* (15) *given by*

$$b(t) = \left(\frac{3p_1 + 2\lambda_1\beta_0}{3\mu\lambda_1}\right)^{1/2} t^{1/2}(1+O(t^{1/2})).$$

Proof. To find the real solutions $b(t)$ of (16) we integrate (16) over $[0,t]$ and, using Lemma 1, we get

$$(17) \qquad \int_0^t \{-p_1 s b(s) - \lambda_1\beta_0(t-s)b(s) + \lambda_1\mu b^3(s)\}\,ds = 0.$$

Write

$$F(b)(s,t) = -p_1 s b(s) - \lambda_1\beta_0(t-s)b(s) + \lambda_1\mu b^3(s).$$

According to Lemma 2 we have two possibilities: either

$$(18) \qquad \forall t \geqslant s \geqslant 0 \quad F(b)(t,s) = 0$$

or

$$(19) \qquad \forall t \geqslant s \geqslant 0 \quad F(b)(t,s) = f(t,s),$$

where $f(t,s)$ is some nonzero function, e.g. one of functions given in (13).

Assume (18). Separating the variables t and s in $F(b)(s,t)$ and using (15) and (4), we get the equation

$$Ks(\lambda_1\beta_0 - p_1) + \lambda_1 \mu b_k^2(s) = 0, \tag{20}$$

where $b_k(s) = Kb(s)$ and K is a constant. From (20) and (16) we get

$$b_k(s) = \pm\left(\frac{3p_1 + 2\lambda_1\beta_0}{3\lambda_1\mu}\right)^{1/2} s^{1/2}$$

and it is easy to see from (21) that $b_k(s)$ is real if $p_1 > -\frac{2}{3}\lambda_1\beta_0$.

To solve (19) we can use the same technique as in the case of equation (18). After separating the variables s and t and using (13), we get two equations: $\forall t \geqslant s \geqslant 0$

$$-p_1 sb(s) - \lambda_1\beta_0 sb(s) + \lambda_1\mu b^3(s) - Ks^m = 0 \tag{22}$$

and

$$\lambda_1\beta_0 tb(s) + \frac{K}{\alpha}s^n t^k = 0. \tag{23}$$

It is easy to see that (23) has a solution only if $k = 1$ and $n = 1/2$ and that (21) is this solution. Applying the Cardano formula, we can write explicitly the solution of (22). In particular, when $m = 3/2$, we obtain (21).

Since (12) has not only solution of form (13), we cannot claim that (16) has no other solution that (21). So this question remains open.

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