

POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

5.7133

[424]

DISSERTATIONES  
MATHEMATICAE  
(ROZPRAWY MATEMATYCZNE)

KOMITET REDAKCYJNY

KAROL BORSUK *redaktor*

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CXXIV

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**The categories of presheaves  
containing any category of algebras**

WARSZAWA 1975

P A Ń S T W O W E W Y D A W N I C T W O N A U K O W E

5.7133



PRINTED IN POLAND

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W R O C Ł A W S K A   D R U K A R N I A   N A U K O W A

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## Introduction

Following [3], a category  $K$  is said to be *binding* if every category of universal algebras can be fully embedded in it. A lot of categories can be fully embedded in any binding category and, under a set-theoretical assumption, even every concrete category can ([1], [4]).

The present paper is a contribution to the problem what functor categories  $\mathcal{S}^k$  are binding (where  $\mathcal{S}$  is the category of all sets). The small categories  $k$ , such that the functor category  $\mathcal{S}^k$  is binding, are called *rich* in [2], [9].

The problem of the characterization or a lucid description of the class of all rich categories is far from being solved. Papers [7], [8] investigate the case where a category  $k$  has exactly one object, i.e., where  $k$  is a monoid, and some theorems about rich monoids are proved there. In the present paper we consider the categories of presheaves, i.e., the functor categories with a thin domain. The class of all thin rich categories is fully described here. We prove that there exist 31 finite and 4 infinite thin categories such that a thin category is rich if and only if it contains some of them as a full subcategory.

The paper has six parts. The first part contains conventions, known definitions and facts about binding categories and functor categories. In the second one the main theorem and its corollaries are formulated. The rest of the paper, i.e., Parts III–VI, contain lemmas and propositions needed for the proof of the theorem and the proof itself.

The authors would like to express their gratitude to L. Kučera for reading the manuscript carefully and to A. Pultr for remarks and references.

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## I. Preliminaries

**1. Conventions.** If  $K$  is a category, then we denote the class of all its objects (or the class of all its morphisms) by  $K^o$  (or  $K^m$ ). If  $a, b \in K^o$ , then  $K(a, b)$  is the class of all morphisms from  $a$  to  $b$ . A category  $K$  is said to be *thin* if  $\text{card} K(a, b) \leq 1$  for all  $a, b \in K^o$ . The symbol  $\simeq$  denotes an isomorphism of categories. If  $A \subset K^o$  is a class, then by  $KA$  we mean the full subcategory of  $K$  such that  $A$  is the class of all its objects. If  $H \simeq KA$  for some  $A \subset K^o$ , i.e., if  $H$  can be fully embedded in  $K$ , we denote it by  $H \dot{\subset} K$ . If  $H$  is isomorphic to an arbitrary subcategory of  $K$ , we write  $H \subset K$ . If  $k$  is a small category, then  $K^k$  is the category of all covariant functors from  $k$  to  $K$  and all their transformations. The category of all (or all non-empty) sets and all their mappings is denoted by  $\mathcal{S}$  (or  $\mathcal{S}_n$ , respectively). Each semi-group with the unit, i.e., each monoid, is also considered as a category with precisely one object.  $c_2$  denotes the group of all integers modulo 2 and also the category with one object  $a$ , and precisely two morphisms  $1_a, \tau \in c_2(a, a)$  such that  $1_a \neq \tau, \tau \circ \tau = 1_a$ . We recall that a category  $K$  is called *connected* if for every  $a, b \in K^o$ , there exist  $a_0 = a, a_1, \dots, a_n = b$  in  $K^o$  such that  $K(a_i, a_{i+1}) \cup K(a_{i+1}, a_i) \neq \emptyset$  for all  $i = 0, \dots, n-1$ . A maximal connected subcategory of  $K$  is called its *component*.

**2.** Following [3], a category  $K$  is said to be *binding* if every category of universal algebras can be fully embedded in it. We recall

**THEOREM ([3]).** *The following properties of a category  $K$  are equivalent:*

- (i)  $K$  is binding;
- (ii) The category  $\mathcal{U}(1, 1)$  of all universal algebras with two unary operations can be fully embedded in  $K$ ;
- (iii) The category  $\mathcal{G}$  of all graphs can be fully embedded in  $K$ .

**3.** Let  $h, k$  be small categories, let  $F: h \rightarrow k$  be a functor, and let  $K$  be a cocomplete category. Define  $\bar{F}: K^k \rightarrow K^h$  by  $\bar{F}(f) = fF$ . Then  $\bar{F}$  has a left adjoint, say  $G: K^h \rightarrow K^k$  (see e.g. [5], p. 74). If  $F$  is a full embed-

ding, so is  $G$ , which follows immediately from the construction of  $G$ . Thus, we have the following proposition, suitable for our purposes.

**PROPOSITION.** *Let  $h, k$  be small categories,  $h \dot{\subset} k$ . Then  $\mathcal{G}^h \dot{\subset} \mathcal{G}^k$ .*

**4. DEFINITION** ([2], [9]). A small category  $k$  is called *rich* if  $\mathcal{G}^k$  is binding. It is called *poor* otherwise.

## II. Main theorem

**THEOREM.** *Let  $k$  be a small thin category. The following assertions are equivalent:*

- (i)  *$k$  is rich;*
- (ii) *The group  $c_2$  can be fully embedded in  $\mathcal{S}^k$ , i.e.,  $c_2 \dot{\subset} \mathcal{S}^k$ ;*
- (iii)  *$k$  contains some of the categories  $k_1, \dots, k_{35}$  as a full subcategory.*

**Note.** The categories  $k_1, \dots, k_{35}$  are drawn on the next page (Fig. 1). The identical morphisms and the composed morphism are not pictured.

**COROLLARY.** 1) *Every thin rich category has a rich component. A thin category is rich iff some of its countable parts is rich. A finite thin category is rich iff some of its at most 7-objects part is rich.*

2) *There are infinite rich categories with all finite full subcategories poor (e.g.  $k_{33}, k_{34}, k_{35}$ ). There are rich categories, the duals of which are poor (e.g.  $k_{13}, k_{30}$ ).*

3) *Every rich category has at least 8 morphisms. If it is connected, then it has either just 8 or at least 10 morphisms (the identical morphisms are included).*

**Note.** With the aim of the theorem it is possible to form a computer program for deciding whether a given finite thin category is rich or not. In fact, such a program was really created by J. Kastl.

The following Assertion a, as well as the unicity in Assertion b, are not immediate corollaries of the theorem, but they follow from the proof of it, presented further.

**ASSERTION a.** *If  $k$  is a poor connected thin category, then always there exists a bigger poor connected thin category  $h$ , in the sense that  $k \dot{\subset} h$ ,  $\text{card } h^0 > \text{card } k^0$ . For every cardinal number  $m$  there exists a connected poor thin category  $k$  with  $\text{card } k^0 = m$ .*

**ASSERTION b.** *There exists a thin connected category  $k$  such that*

- ( $\alpha$ )  *$k$  is poor;*
- ( $\beta$ ) *If  $h$  is a connected thin category such that  $k \dot{\subset} h$ ,  $0 < \text{card}(h^m - k^m) \leq 3$ , then  $h$  is rich.*

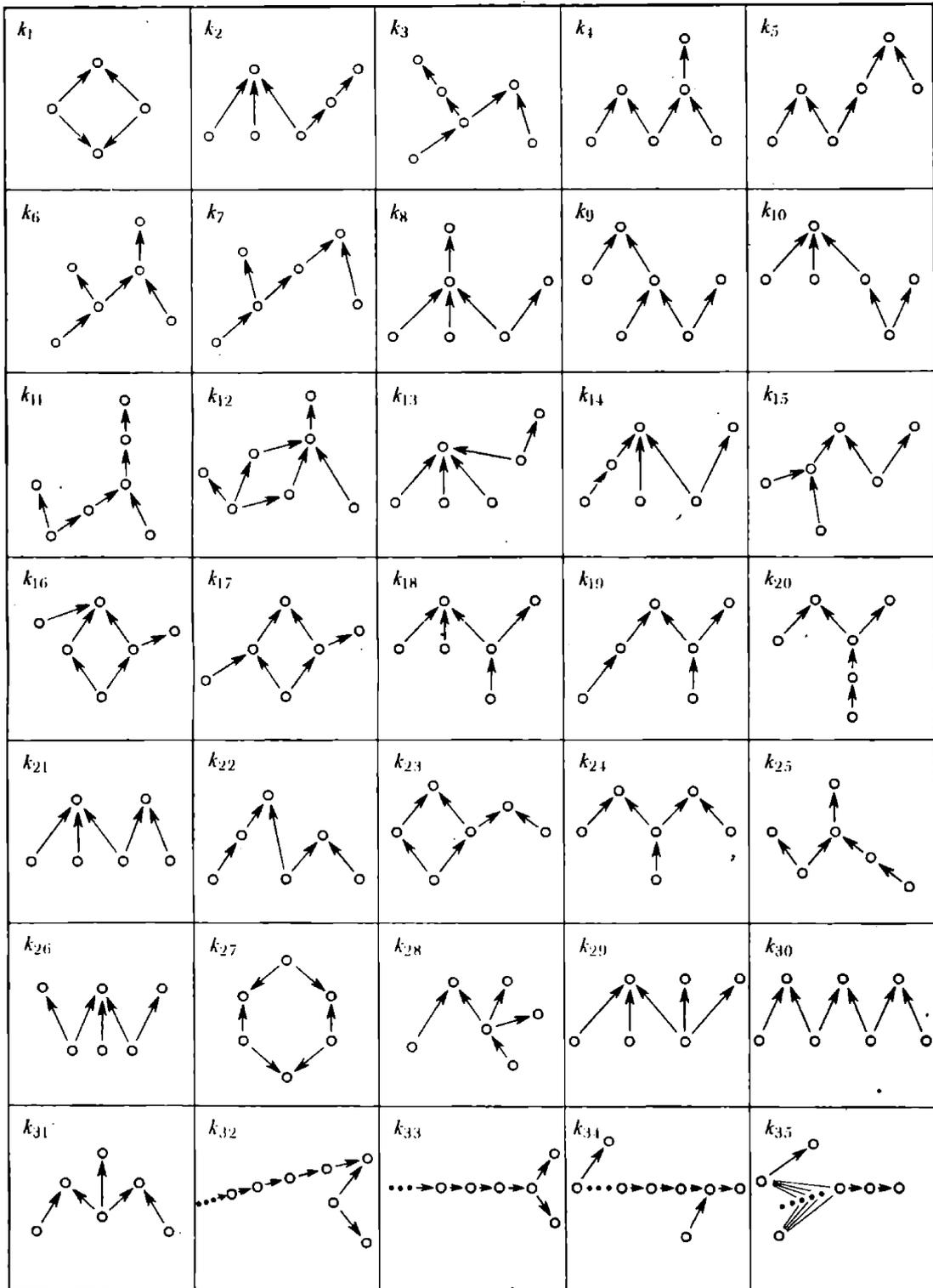
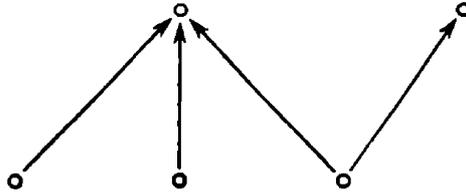


Fig. 1

*The category  $k$  is unique (up to isomorphism), it is*



Now we describe the framework of the proof of the theorem. In Part III we prove that all the categories  $k_1, \dots, k_{35}$  are rich. So, using Proposition of I.3, (iii)  $\Rightarrow$  (i). The implication (i)  $\Rightarrow$  (ii) follows immediately from the known fact ([6]) that every small category can be fully embedded in any binding category. The proof of the implication (ii)  $\Rightarrow$  (iii) has the following parts: In Part IV we define a cone and forms  $a_1, \dots, a_{10}$  of thin categories and prove that, if a thin category  $k$  has a form  $a_i$ ,  $i \in \{1, \dots, 10\}$ , then there exists no connected functor  $f: k \rightarrow \mathcal{S}_n$  such that all endotransformations of  $f$  form  $c_2$ . In Part V we prove that a connected skeletical thin category contains some  $k_i$ ,  $i = 1, \dots, 35$ , or it is a cone, or it has some of the forms  $a_1, \dots, a_{10}$ . In Part VI the proof of (ii)  $\Rightarrow$  (iii) is finished. If a small thin category  $k$  satisfying (ii) is given, then it is reduced by a transfinite process and we receive a category  $h$  such that  $h \subset k$ ,  $h$  is not a cone, it is skeletical, connected and there exists a connected functor  $f: h \rightarrow \mathcal{S}_n$  such that all endotransformations of  $f$  form  $c_2$ . Then it is sufficient to use Parts V and IV.

### III. The categories $k_1, \dots, k_{35}$ are rich

**Conventions and notation.** (a) If  $f: X \rightarrow X'$  is a mapping, denote by  $\bar{f}: X \times X \rightarrow X' \times X'$  the mapping with  $\bar{f}(x, y) = (f(x), f(y))$ .

(b) As usual, a *graph*  $G$  is a couple  $G = (X, R)$ , where  $X$  is a set and  $R \subset X \times X$ . Denote by  $\pi_1: R \rightarrow X$ ,  $\pi_2: R \rightarrow X$ , the projections, i.e.  $\pi_1(x, y) = x$ ,  $\pi_2(x, y) = y$ . This denotation will be kept in the sequel. If  $G, G'$  are two graphs, then the notation for  $G'$  will be the same, only primed, i.e.,  $G' = (X', R')$ , and so on.

(c) Let  $G, G'$  be graphs,  $f: X \rightarrow X'$  be a mapping. As usual,  $f$  is said to be *compatible* iff  $\bar{f}(R) \subset R'$ . Then we write  $f: G \rightarrow G'$ . All graphs as objects and all their compatible mappings as morphisms form the category  $\mathfrak{G}$  of all graphs and all compatible mappings.

(d) Let  $k$  be a thin category,  $Z \subset k^m$ . We shall write  $k^m = \text{gen} Z$  whenever every non-identical  $a \in k^m$  can be received as a composition of some morphisms from  $Z$ . If  $a, b \in k^0$ ,  $a \in k(a, b)$ , we write also  $a: a \rightarrow b$ .

**A. Construction** (see Fig. 2). Let  $\mathfrak{s}$  be a thin category defined as follows:

$$\mathfrak{s}^0 = \{s_0, s_1, s_2, s_3\} \quad (\text{where all } s_i \text{ are distinct}),$$

$$\mathfrak{s}^m = \text{gen}\{\sigma_1: s_0 \rightarrow s_1, \sigma_2: s_0 \rightarrow s_2, \sigma_3: s_2 \rightarrow s_3\}.$$

If  $G = (X, R)$  is a graph, denote by  $\Sigma_G: \mathfrak{s} \rightarrow \mathcal{S}$  the following functor:

$$\Sigma_G(s_0) = (R \times \{1, 2\}) \cup (X \times \{3, 4\});$$

$$\Sigma_G(s_1) = (R \times \{0\}) \cup (X \times \{5, 6\});$$

$$\Sigma_G(s_2) = X \times \{7, 8\};$$

$$\Sigma_G(s_3) = X \times \{9\};$$

$$[\Sigma_G(\sigma_1)](r, 1) = [\Sigma_G(\sigma_1)](r, 2) = (r, 0) \quad \text{whenever } r \in R;$$

$$[\Sigma_G(\sigma_1)](x, i) = (x, i + 2), \quad \text{whenever } x \in X, i \in \{3, 4\};$$

$$[\Sigma_G(\sigma_2)](r, i) = (\pi_i(r), i + 6), \quad \text{whenever } r \in R, i \in \{1, 2\};$$

$$[\Sigma_G(\sigma_2)](x, i) = (x, i + 4), \quad \text{whenever } x \in X, i \in \{3, 4\};$$

$$[\Sigma_G(\sigma_3)](x, 7) = [\Sigma_G(\sigma_3)](x, 8) = (x, 9), \quad \text{whenever } x \in X.$$

If  $G, G'$  are graphs,  $f: G \rightarrow G'$  is a compatible mapping, denote by  $\tau_f$  the transformation  $\tau_f: \Sigma_G \rightarrow \Sigma_{G'}$  such that

$$\begin{aligned} \tau_f(r, i) &= (\bar{f}(r), i) & \text{for all } r \in R, i = 0, 1, 2, \\ \tau_f(x, i) &= (f(x), i), & \text{for all } x \in X, i = 3, \dots, 9. \end{aligned}$$

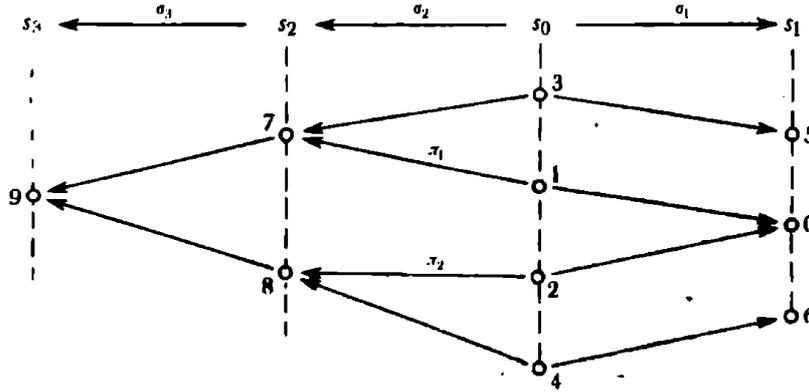


Fig. 2

PROPOSITION 1. Let  $\tau: \Sigma_G \rightarrow \Sigma_{G'}$  be a transformation such that either (α) or (β) is fulfilled:

- (α)  $(\tau(X \times \{5\}) \subset X' \times \{5\}) \ \& \ (\tau(X \times \{6\}) \subset X' \times \{6\});$
- (β)  $(\tau(X \times \{5\}) \subset X' \times \{5\}) \ \& \ (\tau(X \times \{4\}) \subset X' \times \{4\}).$

Then there exists a compatible mapping  $f: G \rightarrow G'$  such that  $\tau = \tau_f$ .

Proof. Let (α) or (β) be fulfilled. Since  $\tau$  commutes with  $\Sigma_G(\sigma_1)$  and  $\Sigma_G(\sigma_2)$ , one can easily see that

$$\tau(X \times \{i\}) \subset X' \times \{i\} \quad \text{for } i = 3, \dots, 9.$$

If  $\tau(r, 0) \in X' \times \{5\}$  for some  $r \in R$ , then necessarily  $\tau(\pi_2(r), 8) \in X' \times \{7\}$ , which is impossible. Analogously,  $\tau(r, 0) \in X' \times \{6\}$  is impossible. Thus,  $\tau(R \times \{0\}) \subset R' \times \{0\}$ . Clearly,  $\tau(R \times \{i\}) \subset R' \times \{i\}$  for  $i = 1, 2$ , now. Define the mappings  $f_i: X \rightarrow X'$ ,  $g_j: R \rightarrow R'$ ,  $j = 0, \dots, 2$ ,  $i = 3, \dots, 9$ , such that

$$\tau(x, i) = (f_i(x), i), \quad \tau(r, j) = (g_j(r), j).$$

Then  $g_0 = g_1 = g_2$ ,  $f_3 = \dots = f_9$ , and  $g_0 = \bar{f}_3$ .

COROLLARY.

- (α) The category  $k_2$  is rich.

Proof. Let  $k_2^0 = s^0 \cup \{a, b\}$ ,  $k_2^m = \text{gen}(s^m \cup \{a: a \rightarrow s_1; \beta: b \rightarrow s_1\})$ . If  $G$  is a graph, define a functor  $\Phi_G: k_2 \rightarrow \mathcal{S}$  such that  $\Sigma_G$  is a domain-restriction of  $\Phi_G$ ,  $\Phi_G(a) = X \times \{5\}$ ,  $\Phi_G(b) = X \times \{6\}$ ,  $\Phi_G(a)$  and  $\Phi_G(b)$  are

inclusions. If  $f: G \rightarrow G'$  is a compatible mapping of the graphs, denote by  $\mu_f: \Phi_G \rightarrow \Phi_{G'}$ , the transformation such that  $\mu_f(r, j) = (\bar{f}(r), j)$ ,  $\mu_f(x, i) = (f(x), i)$ , for all  $r \in R$ ,  $x \in X$ ,  $j = 0, \dots, 2$ ,  $i = 3, \dots, 9$ . Define  $\Psi: \mathfrak{G} \rightarrow \mathcal{S}^{k_2}$  by  $\Psi(G) = \Phi_G$ ,  $\Psi(f) = \mu_f$ . One can easily see that  $\Psi$  is a faithful functor. Proposition 1 ( $\alpha$ ) implies that  $\Psi$  is a functor onto a full subcategory.

( $\beta$ ) *The category  $k_3$  is rich.*

Proof is quite analogous; use Proposition 1 ( $\beta$ ).

Construction: If  $G = (X, R)$  is a graph, denote by  $\bar{\Sigma}_G: \mathfrak{s} \rightarrow \mathcal{S}$  the following functor:

$$\bar{\Sigma}_G(s_0) = \Sigma_G(s_0) \cup (X \times \{10\}), \quad \bar{\Sigma}_G(s_1) = \Sigma_G(s_1),$$

$$\bar{\Sigma}_G(s_2) = \Sigma_G(s_2) \cup (X \times \{11\}), \quad \bar{\Sigma}_G(s_3) = \Sigma_G(s_3) \cup (X \times \{12\});$$

$\Sigma_G$  is a subfunctor of  $\bar{\Sigma}_G$  and  $[\bar{\Sigma}_G(\sigma_1)](x, 10) = (x, 5)$ ,  $[\bar{\Sigma}_G(\sigma_2)](x, 10) = (x, 11)$ ,  $[\bar{\Sigma}_G(\sigma_3)](x, 11) = (x, 12)$ . If  $f: G \rightarrow G'$  is a compatible mapping of the graphs, denote by  $\bar{\tau}_f: \bar{\Sigma}_G \rightarrow \bar{\Sigma}_{G'}$  the transformation such that

$$\bar{\tau}_f(r, j) = (\bar{f}(r), j), \quad \bar{\tau}_f(x, i) = (f(x), i)$$

for all  $r \in R$ ,  $x \in X$ ,  $j = 0, \dots, 2$ ,  $i = 3, \dots, 12$ .

PROPOSITION 2. Let  $\bar{\tau}: \bar{\Sigma}_G \rightarrow \bar{\Sigma}_{G'}$  be a transformation such that one of the following conditions is fulfilled:

- ( $\alpha$ )  $(\bar{\tau}(X \times \{6\}) \subset X' \times \{6\}) \ \& \ (\bar{\tau}(X \times \{11\}) \subset X' \times \{11\});$
- ( $\beta$ )  $(\bar{\tau}(X \times \{6\}) \subset X' \times \{6\}) \ \& \ (\bar{\tau}(X \times \{12\}) \subset X' \times \{12\});$
- ( $\gamma$ )  $(\bar{\tau}(X \times \{4\}) \subset X' \times \{4\}) \ \& \ (\bar{\tau}(X \times \{11\}) \subset X' \times \{11\});$
- ( $\delta$ )  $(\bar{\tau}(X \times \{4\}) \subset X' \times \{4\}) \ \& \ (\bar{\tau}(X \times \{12\}) \subset X' \times \{12\}).$

Then there exists a compatible mapping  $f: G \rightarrow G'$  such that  $\bar{\tau} = \bar{\tau}_f$ .

Proof. If one of the conditions ( $\alpha$ ), ..., ( $\delta$ ) is fulfilled, then, clearly,  $\bar{\tau}(X \times \{i\}) \subset X' \times \{i\}$ , whenever  $i \in \{4, 5, 6, 10, 11, 12\}$ . If  $\bar{\tau}(x, 9) \in X' \times \{12\}$  for some  $x \in X$ , then necessarily  $\bar{\tau}(x, 6) \in X' \times \{5\}$ , which is impossible. Now, prove  $\bar{\tau}(\Sigma_G) \subset \Sigma_{G'}$  and use Proposition 1.

COROLLARY.

- ( $\alpha$ ) *The category  $k_7$  is rich.*
- ( $\beta$ ) *The category  $k_5$  is rich.*
- ( $\gamma$ ) *The category  $k_6$  is rich.*
- ( $\delta$ ) *The category  $k_4$  is rich.*

**Construction:** If  $G = (X, R)$  is a graph, denote by  $\bar{\Sigma}_G: \mathbf{s} \rightarrow \mathcal{S}$  the following functor:

$$\begin{aligned}\bar{\Sigma}_G(s_0) &= \bar{\Sigma}_G(s_0) \cup (X \times \{13\}), & \bar{\Sigma}_G(s_1) &= \bar{\Sigma}_G(s_1), \\ \bar{\Sigma}_G(s_2) &= \bar{\Sigma}_G(s_2) \cup (X \times \{14\}), & \bar{\Sigma}_G(s_3) &= \bar{\Sigma}_G(s_3) \cup (X \times \{15\});\end{aligned}$$

$\bar{\Sigma}_G$  is a subfunctor of  $\bar{\Sigma}_G$  and  $[\bar{\Sigma}_G(\sigma_1)](x, 13) = (x, 6)$ ,  $[\bar{\Sigma}_G(\sigma_2)](x, 13) = (x, 14)$ ,  $[\bar{\Sigma}_G(\sigma_3)](x, 14) = (x, 15)$ . If  $f: G \rightarrow G'$  is a compatible mapping of the graphs, denote by  $\bar{\tau}_f: \bar{\Sigma}_G \rightarrow \bar{\Sigma}_{G'}$  the transformation such that

$$\bar{\tau}_f(r, j) = (\bar{f}(r), j), \quad \bar{\tau}_f(x, i) = (f(x), i),$$

whenever  $r \in R$ ,  $x \in X$ ,  $j = 0, \dots, 2$ ,  $i = 3, \dots, 15$ .

**PROPOSITION 3.** Let  $\bar{\tau}: \bar{\Sigma}_G \rightarrow \bar{\Sigma}_{G'}$  be a transformation such that one of the following conditions is fulfilled:

- ( $\alpha$ )  $(\bar{\tau}(X \times \{14\}) \subset X' \times \{14\})$  &  $(\bar{\tau}(X \times \{11\}) \subset X' \times \{11\})$ ;
- ( $\beta$ )  $(\bar{\tau}(X \times \{14\}) \subset X' \times \{14\})$  &  $(\bar{\tau}(X \times \{12\}) \subset X' \times \{12\})$ ;
- ( $\gamma$ )  $(\bar{\tau}(X \times \{15\}) \subset X' \times \{15\})$  &  $(\bar{\tau}(X \times \{12\}) \subset X' \times \{12\})$ .

Then there exists a compatible mapping  $f: G \rightarrow G'$  such that  $\bar{\tau} = \bar{\tau}_f$ .

**Proof.** Prove  $\bar{\tau}(\bar{\Sigma}_G) \subset \bar{\Sigma}_{G'}$  and use Proposition 2.

**COROLLARY.**

- ( $\alpha$ ) The category  $k_8$  is rich.
- ( $\beta$ ) The category  $k_9$  is rich.
- ( $\gamma$ ) The category  $k_{10}$  is rich.

**PROPOSITION 4.** The category  $k_{11}$  is rich.

**Proof.** Put  $k_{11}^o = \mathbf{s}^o \cup \{c, b, \bar{b}\}$ ,  $k_{11}^m = \text{gen}(\mathbf{s}^m \cup \{\gamma: c \rightarrow s_3, \beta: s_3 \rightarrow b, \bar{\beta}: b \rightarrow \bar{b}\})$ . If  $G = (X, R)$  is a graph, denote by  $\Phi_G: k_{11} \rightarrow \mathcal{S}$  the following functor:

$$\begin{aligned}\Phi_G(s_0) &= \bar{\Sigma}_G(s_0), & \Phi_G(s_1) &= \bar{\Sigma}_G(s_1), \\ \Phi_G(s_2) &= \bar{\Sigma}_G(s_2) \cup (X \times \{16\}), & \Phi_G(s_3) &= \bar{\Sigma}_G(s_3) \cup (X \times \{17, 18\}), \\ \Phi_G(c) &= X \times \{17, 18\}, & \Phi_G(b) &= \bar{\Sigma}_G(s_3) \cup (X \times \{17\}), & \Phi_G(\bar{b}) &= \bar{\Sigma}_G(s_3);\end{aligned}$$

$\bar{\Sigma}_G$  is a subfunctor of the domain-restriction  $\Phi_G/\mathbf{s}$  and  $\Phi_G(\gamma)$  is an inclusion; if  $x \in X$ , then  $[\Phi_G(\sigma_2)](x, 16) = (x, 17)$ ,  $[\Phi_G(\beta)](x, 17) = (x, 17)$ ,  $[\Phi_G(\beta)](x, 18) = (x, 15)$ ,  $[\Phi_G(\bar{\beta})](x, 17) = (x, 12)$ ; if  $z \in \bar{\Sigma}_G(s_3)$ , then  $[\Phi_G(\beta)](z) = [\Phi_G(\bar{\beta})](z) = z$ . If  $f: G \rightarrow G'$  is a compatible mapping of the graphs, denote by  $\mu_f: \Phi_G \rightarrow \Phi_{G'}$  the transformation such that

$\mu_f(r, j) = (\bar{f}(r), j)$ ,  $\mu_f(x, i) = (f(x), i)$  for all  $r \in R$ ,  $x \in X$ ,  $j = 0, \dots, 2$ ,  $i = 3, \dots, 18$ . It is easy to see that  $\Psi: \mathfrak{G} \rightarrow \mathcal{S}^{k_{11}}$  such that  $\Psi(G) = \Phi_G$ ,  $\Psi(f) = \mu_f$  is an embedding. Let  $\mu: \Phi_G \rightarrow \Phi_{G'}$  be a transformation; we prove that  $\mu = \mu_f$  for some compatible mapping  $f: G \rightarrow G'$ . Since  $X \times \{17\} = [\Phi_G(\sigma_3)](\Phi_G(s_2)) \cap [\Phi_G(\gamma)](\Phi_G(o))$ , necessarily

$$\mu^{s_3}(X \times \{17\}) \subset X' \times \{17\}, \quad \mu^b(X \times \{12\}) \subset X' \times \{12\}.$$

The last inclusion and  $X \times \{12\} \subset [\Phi_G(\sigma_3 \circ \sigma_2)](\Phi_G(s_0))$  imply  $\mu^{s_3}(X \times \{12\}) \subset X' \times \{12\}$ . Since  $X \times \{15\} = [\Phi_G(\beta \circ \gamma)](\Phi_G(o)) \cap [\Phi_G(\beta \circ \sigma_3 \circ \sigma_2)](\Phi_G(s_0))$ , necessarily

$$\mu^b(X \times \{15\}) \subset X' \times \{15\}, \quad \mu^{s_3}(X \times \{15\}) \subset X' \times \{15\}.$$

Now, prove that  $\mu(\bar{\Sigma}_G) \subset \bar{\Sigma}_{G'}$  and use Proposition 3.

**PROPOSITION 5.** *The category  $k_{12}$  is rich.*

**Proof.** Put

$$k_{12}^o = s^o \cup \{s'_2, c, b\},$$

$$k_{12}^m = \text{gen}(s^m \cup \{s'_2: s_0 \rightarrow s'_2, s'_3: s'_2 \rightarrow s_3, \gamma: c \rightarrow s_3, \beta: s_3 \rightarrow b\}).$$

We construct a full embedding  $\Psi: \mathfrak{G} \rightarrow \mathcal{S}^{k_{12}}$  such that if  $G$  is a graph we put  $\Psi(G) = \Phi_G$ , where  $\Phi_G$  is defined as follows:

$$\Phi_G(s_0) = \bar{\Sigma}_G(s_0), \quad \Phi_G(s_1) = \bar{\Sigma}_G(s_1), \quad \Phi_G(s_2) = \bar{\Sigma}_G(s_2) \cup (X \times \{16\}),$$

$$\Phi_G(s'_2) = \bar{\Sigma}_G(s_2) \cup (X \times \{17\}), \quad \Phi_G(s_3) = \bar{\Sigma}_G(s_3) \cup (X \times \{16, 17\}),$$

$$\Phi_G(o) = X \times \{16, 17\}, \quad \Phi_G(b) = \bar{\Sigma}_G(s_3);$$

$\bar{\Sigma}_G$  is a subfunctor of  $\Phi_G/s$ ,  $\Phi_G(\gamma)$  is an inclusion and if  $z \in \Phi_G(s_0)$ , then

$$[\Phi_G(\sigma'_2)](z) = [\Phi_G(\sigma_2)](z);$$

if  $z \in \bar{\Sigma}_G(s_2)$ , then

$$[\Phi_G(\sigma'_3)](z) = [\Phi_G(\sigma_3)](z);$$

if  $x \in X$ , then

$$[\Phi_G(\sigma_2)](x, 16) = (x, 16), \quad [\Phi_G(\sigma'_3)](x, 17) = (x, 17),$$

$$[\Phi_G(\beta)](x, 16) = (x, 12), \quad [\Phi_G(\beta)](x, 17) = (x, 15).$$

**B. DEFINITION.** Let  $k$  be a small category,  $\Phi: k \rightarrow \mathcal{S}$  a functor,  $Y = \bigvee_{o \in k^o} \Phi(o)$ ,  $y, y' \in Y$ . Every sequence  $a_0, \dots, a_n$  such that  $a_0 = y$ ,  $a_n = y'$ , and that for every  $i = 1, \dots, n$  there exists  $a_i \in k^m$  such that either  $a_i = [\Phi(a_i)](a_{i-1})$  or  $a_{i-1} = [\Phi(a_i)](a_i)$ , is called a *path (in  $Y$ ) from  $y$  to  $y'$ ,  $n$  is its length.*

**Construction (Fig. 3):** Let  $\mathcal{I}$  be a thin category such that

$$\mathcal{I}^0 = \{l_0, l_1, l_2\} \text{ (where all } l_i \text{ are different),}$$

$$\mathcal{I}^m = \text{gen}\{\lambda_1: l_0 \rightarrow l_1, \lambda_2: l_0 \rightarrow l_2\}.$$

If  $G = (X, R)$  is a graph, denote by  $\Lambda_G: \mathcal{I} \rightarrow \mathcal{S}$  the following functor:

$$\Lambda_G(l_1) = \left(\bigcup_{i=0}^4 X \times \{4i\}\right) \cup \left(\bigcup_{j=5}^6 R \times \{4j\}\right),$$

$$\Lambda_G(l_0) = \left(\bigcup_{i=0}^3 X \times \{4i+1, 4i+3\}\right) \cup \left(\bigcup_{j=4}^6 R \times \{4j+1, 4j+3\}\right),$$

$$\Lambda_G(l_2) = \left(\bigcup_{i=0}^3 X \times \{4i+2\}\right) \cup \left(\bigcup_{j=4}^6 R \times \{4j+2\}\right);$$

if  $x \in X$ ,  $i = 0, \dots, 3$ , then

$$[\Lambda_G(\lambda_1)](x, 4i+1) = (x, 4i), \quad [\Lambda_G(\lambda_1)](x, 4i+3) = (x, 4i+4);$$

if  $r \in R$ , then

$$[\Lambda_G(\lambda_1)](r, 4j+1) = (r, 4j) \quad \text{whenever } j = 5, 6,$$

$$[\Lambda_G(\lambda_1)](r, 4j+3) = (r, 4j+4) \quad \text{whenever } j = 4, 5,$$

$$[\Lambda_G(\lambda_1)](r, 17) = (\pi_1(r), 16), \quad [\Lambda_G(\lambda_1)](r, 27) = (\pi_2(R), 0);$$

$$[\Lambda_G(\lambda_2)](z, 4n+1) = [\Lambda_G(\lambda_2)](z, 4n+3) = (z, 4n+2)$$

whenever either  $z \in X$ ,  $n = 0, \dots, 3$  or  $z \in R$ ,  $n = 4, 5, 6$ .

If  $f: G \rightarrow G'$  is a compatible mapping of the graphs, denote by  $\nu_f: \Lambda_G \rightarrow \Lambda_{G'}$  the transformation such that  $\nu_f(x, i) = (f(x), i)$ ,  $\nu_f(r, j) = (\bar{f}(r), j)$  for all  $x \in X$ ,  $r \in R$ ,  $i = 0, \dots, 16$ ,  $j = 17, \dots, 27$ .

**PROPOSITION 6.** Let  $\nu: \Lambda_G \rightarrow \Lambda_{G'}$  be a transformation such that one of the following conditions is fulfilled:

- ( $\alpha$ )  $(\nu(X \times \{0\}) \subset X' \times \{0\})$  &  $(\nu(X \times \{16\}) \subset X' \times \{16\})$  &  
&  $(\nu(X \times \{8\}) \subset X' \times \{0, 8, 16\})$ ;
- ( $\beta$ )  $(\nu(X \times \{1\}) \subset X' \times \{1\})$  &  $(\nu(X \times \{16\}) \subset X' \times \{16\})$  &  
&  $(\nu(X \times \{8\}) \subset X' \times \{8, 16\})$ ;
- ( $\gamma$ )  $(\nu(X \times \{1\}) \subset X' \times \{1\})$  &  $(\nu(X \times \{16\}) \subset X' \times \{16\})$  &  
&  $(\nu(X \times \{7\}) \subset X' \times \{1, 7\})$ .

Then there exists a compatible mapping  $f: G \rightarrow G'$  such that  $\nu = \nu_f$ .

Proof. If  $(\beta)$  or  $(\gamma)$  is satisfied, then  $(\alpha)$  is also satisfied. Now, suppose  $(\alpha)$  is fulfilled. Put

$$Y = \bigcup_{i=0}^2 \Lambda_G(l_i), \quad Y' = \bigcup_{i=0}^2 \Lambda_{G'}(l_i).$$

If  $a_0, \dots, a_n$  is a path in  $Y$ , then  $\nu(a_0), \dots, \nu(a_n)$  is a path in  $Y'$ .

1) First prove  $\nu(X \times \{8\}) \subset X' \times \{8\}$ : Let  $x \in X$ . Then there exists a path with the length 8 from  $(x, 8)$  to  $(x, 0)$  in  $Y$  but for no  $z \in X'$  there

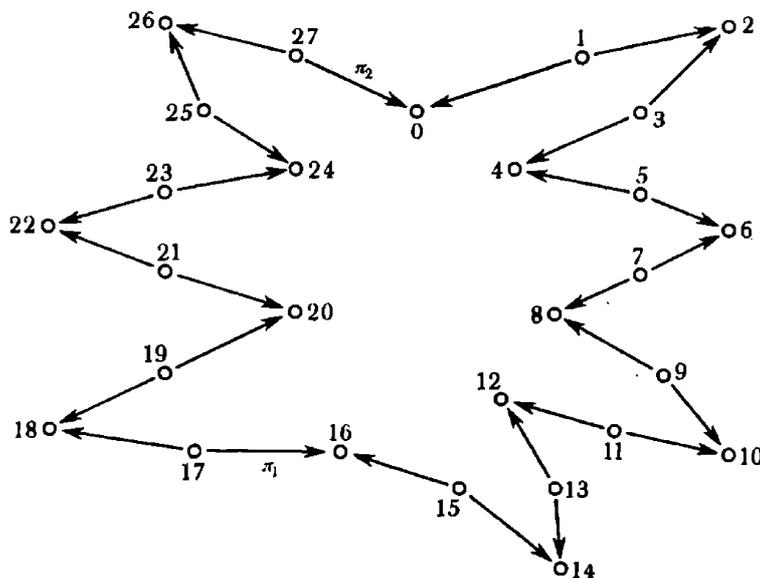


Fig. 3

exists a path with the length 8 from  $(z, 16)$  to a point of  $X' \times \{0\}$  in  $Y'$ . Since  $\nu(X \times \{0\}) \subset X' \times \{0\}$ , necessarily  $\nu(x, 8) \notin X' \times \{16\}$ . One can prove that  $\nu(x, 8) \notin X' \times \{0\}$ , analogously.

2) Now prove  $\nu(X \times \{i\}) \subset X' \times \{i\}$  for  $i = 0, \dots, 16$ : We prove, for example,  $\nu(X \times \{4\}) \subset X' \times \{4\}$ . If  $x \in X$ , then there exists a path with the length 4 from  $(x, 4)$  to a point of  $X \times \{0\}$  and another path with the length 4 from  $(x, 4)$  to a point of  $X \times \{8\}$ . If  $z \in Y'$  and there exists a path with the length 4 from  $z$  to a point of  $X' \times \{0\}$  and another path with the length 4 from  $z$  to a point  $X' \times \{8\}$ , then necessarily  $z \in X' \times \{4\}$ .

3) Now, prove  $\nu(R \times \{j\}) \subset R' \times \{j\}$ ,  $j = 17, \dots, 27$ : The proof is analogous.

4) Define the mappings  $f_i: X \rightarrow X'$ ,  $g_j: R \rightarrow R'$ ,  $i = 0, \dots, 16$ ,  $j = 17, \dots, 27$ , such that  $\nu(x, i) = (f_i(x), i)$ ,  $\nu(r, j) = (g_j(r), j)$ . One can easily see that  $f_0 = f_1 = \dots = f_{16}$ ,  $g_{17} = \dots = g_{27}$  and  $g_{17} = \bar{f}_0$ . Consequently,  $f_0: G \rightarrow G'$  is a compatible mapping and  $\nu = \nu_{f_0}$ .



**Proof.** If  $(\beta)$  is fulfilled, then  $(\alpha)$  is also fulfilled. So, we may suppose  $(\alpha)$ . Then, clearly,  $\bar{v}(X \times \{28\}) \subset X' \times \{28\}$ . Now, prove  $\bar{v}(A_G) \subset A_G$  and use Proposition 6.

**COROLLARY.**

- ( $\alpha$ ) *The categories  $k_{21}, k_{22}, k_{23}$  are rich.*  
 ( $\beta$ ) *The category  $k_{24}$  is rich.*

**PROPOSITION 8.** *The category  $k_{25}$  is rich.*

**Proof.** Let

$$k_{25}^o = \mathcal{I}^o \cup \{a, \tilde{a}, b\},$$

$$k_{25}^m = \text{gen}(\mathcal{I}^m \cup \{\alpha: a \rightarrow \tilde{a}, \tilde{a}: \tilde{a} \rightarrow l_1, \beta: l_1 \rightarrow b\}).$$

We construct a full embedding  $\Psi: \mathfrak{G} \rightarrow \mathcal{S}^{k_{25}}$  such that, if  $G$  is a graph, we put  $\Psi(G) = \Phi_G$ , where  $\Phi_G$  is the following functor:

$$\begin{aligned} \Phi_G(l_0) &= A_G(l_0), & \Phi_G(l_2) &= A_G(l_2), \\ \Phi_G(l_1) &= A_G(l_1) \cup (X \times \{28, 29\}), & \Phi_G(a) &= X \times \{28\}, \\ \Phi_G(\tilde{a}) &= X \times \{28, 29, 30\}, & \Phi_G(b) &= A_G(l_1); \end{aligned}$$

$A_G$  is a subfunctor of the domain-restriction  $\Phi_G/\mathcal{I}$ ,  $\Phi_G(a)$  is an inclusion and if  $x \in X$ , then

$$\begin{aligned} [\Phi_G(\tilde{a})](x, 28) &= (x, 28), & [\Phi_G(\tilde{a})](x, 29) &= (x, 29), \\ [\Phi_G(\tilde{a})](x, 30) &= (x, 0), \\ [\Phi_G(\beta)](x, 28) &= (x, 16), & [\Phi_G(\beta)](x, 29) &= (x, 8), \\ [\Phi_G(\beta)](z) &= z \quad \text{whenever} \quad z \in A_G(l_1). \end{aligned}$$

**PROPOSITION 9.** *The category  $k_{26}$  is rich.*

**Proof**<sup>(1)</sup>. Let

$$k_{26}^o = \mathcal{I}^o \cup \{a, b, c\},$$

$$k_{26}^m = \text{gen}(\mathcal{I}^m \cup \{\alpha: a \rightarrow l_1, \beta: b \rightarrow l_1, \gamma: b \rightarrow c\}).$$

The full embedding  $\Psi: \mathfrak{G} \rightarrow \mathcal{S}^{k_{26}}$  is defined as follows: if  $G$  is a graph, put  $\Psi(G) = \Phi_G$ , where

$$\begin{aligned} \Phi_G(l_0) &= A_G(l_0), & \Phi_G(l_2) &= A_G(l_2), & \Phi_G(l_1) &= A_G(l_1) \cup (X \times \{28\}), \\ \Phi_G(a) &= X \times \{0, 28\}, & \Phi_G(b) &= X \times \{8, 16, 28\}, & \Phi_G(c) &= X \times \{8, 16\}, \end{aligned}$$

<sup>(1)</sup> The proof, shorter than our original one, was given by J. Kastl.

$A_G$  is a subfunctor of the domain-restriction  $\Phi_G/l$ ,  $\Phi_G(a)$  and  $\Phi_G(\beta)$  are inclusions and

$$[\Phi_G(\gamma)](x, 8) = (x, 8), \quad [\Phi_G(\gamma)](x, 16) = [\Phi_G(\gamma)](x, 28) = (x, 16)$$

for all  $x \in X$ .

**PROPOSITION 10.** *The categories  $k_1$ ,  $k_{27}$  are rich.*

**Proof.** Choose an object  $a$  of  $k_1$  (or  $k_{27}$ ) such that  $a$  is a domain of two distinct non-identical morphisms, say  $a_1, a_2$ . If  $G = (X, R)$  is a graph, denote by  $\Phi_G: k_1 \rightarrow \mathcal{S}$  (or  $\Phi_G: k_{27} \rightarrow \mathcal{S}$ ) the functor with  $\Phi_G(a) = R$ ,  $\Phi_G(c) = X$  for  $c \neq a$ ,  $\Phi_G(a_i) = \pi_i$ ,  $i = 1, 2$ ,  $\Phi_G(\gamma) = 1_X$  whenever  $\gamma$  is a non-identical morphism of  $k_1$  (or  $k_{27}$ , respectively),  $\gamma \neq a_i$ ,  $i = 1, 2$ .

**PROPOSITION 11.** *The categories  $k_{28}$ ,  $k_{29}$  are rich.*

**Proof.** Let

$$k_{28}^o = \{a, b_1, b_2, c, d_1, d_2\},$$

$$k_{28}^o = \text{gen}(\{a_1: a \rightarrow b_1, a_2: a \rightarrow b_2, \gamma: a \rightarrow c, \delta_1: d_1 \rightarrow c, \delta_2: d_2 \rightarrow a\})$$

(or  $k_{29}^o = k_{28}^o$ ,  $k_{29}^m = \text{gen}(\{a_1, a_2, \gamma, \delta_1\} \cup \{\delta_2: d_2 \rightarrow c\})$ ). If  $G$  is a graph, denote by  $\Phi_G: k_{28} \rightarrow \mathcal{S}$  (or  $\Phi_G: k_{29} \rightarrow \mathcal{S}$ , respectively) the following functor:  $\Phi_G(a) = (X \times \{1\}) \cup (R \times \{2\}) = \Phi_G(c)$ ,  $\Phi_G(\gamma)$  is the identity,  $\Phi_G(d_1) = X \times \{1\}$ ,  $\Phi_G(d_2) = R \times \{2\}$ ,  $\Phi_G(\delta_1)$  and  $\Phi_G(\delta_2)$  are inclusions,

$$\Phi_G(b_1) = \Phi_G(b_2) = X, \quad [\Phi_G(a_1)](x, 1) = [\Phi_G(a_2)](x, 1) = x,$$

$$[\Phi_G(a_i)](r, 2) = \pi_i(r), \quad \text{for all } x \in X, r \in R, i = 1, 2.$$

**PROPOSITION 12.** *The category  $k_{30}$  is rich.*

**Proof.** Let

$$k_{30}^o = \{a, b, c, d, e, f, g\},$$

$$k_{30}^m = \text{gen}\{a: a \rightarrow b, \beta: c \rightarrow b, \gamma: c \rightarrow d, \delta: e \rightarrow d, \varepsilon: e \rightarrow f, \xi: g \rightarrow f\}.$$

We construct the full embedding  $\Psi: \mathfrak{G} \rightarrow \mathcal{S}^{k_{30}}$  such that, if  $G = (X, R)$  is a graph, we put  $\Psi(G) = \Phi_G$ , where  $\Phi_G$  is the following functor:

$$\Phi_G(a) = X \times \{1\}, \quad \Phi_G(b) = X \times \{1, 2\}, \quad \Phi_G(c) = X \times \{3, 4, 5\},$$

$$\Phi_G(d) = X \times \{6, 7\}, \quad \Phi_G(e) = (X \times \{8\}) \cup (R \times \{10, 11\}),$$

$$\Phi_G(f) = (X \times \{9\}) \cup (R \times \{12\}), \quad \Phi_G(g) = X \times \{9\},$$

$\Phi_G(a)$  and  $\Phi_G(\xi)$  are inclusions,

$$[\Phi_G(\beta)](x, 3) = (x, 1), \quad [\Phi_G(\beta)](x, 4) = [\Phi_G(\beta)](x, 5) = (x, 2),$$

$$[\Phi_G(\gamma)](x, 3) = [\Phi_G(\gamma)](x, 4) = (x, 6), \quad [\Phi_G(\gamma)](x, 5) = (x, 7),$$

$$[\Phi_G(\delta)](x, 8) = (x, 7), \quad [\Phi_G(\delta)](r, 10) = (\pi_1(r), 7),$$

$$[\Phi_G(\delta)](r, 11) = (\pi_2(r), 6),$$

$$[\Phi_G(\varepsilon)](x, 8) = (x, 9), \quad [\Phi_G(\varepsilon)](r, 10) = [\Phi_G(\varepsilon)](r, 11) = (r, 12)$$

for all  $x \in X$ ,  $r \in R$ .

PROPOSITION 13. *The category  $k_{31}$  is rich.*

Proof. Let

$$k_{31}^o = \{a, b, c, \tilde{b}, \tilde{a}, d\},$$

$$k_{31}^m = \text{gen}\{a: a \rightarrow b, \beta: c \rightarrow b, \tilde{\beta}: c \rightarrow \tilde{b}, \tilde{a}: \tilde{a} \rightarrow \tilde{b}, \gamma: c \rightarrow d\}.$$

We construct a full embedding of the category  $\mathcal{U}(1, 1)$  of all universal algebras with two unary operations in  $\mathcal{S}^{k_{31}}$ . If  $A = (X, u_1, u_2)$  is an algebra with two unary operations  $u_1, u_2$ , denote by  $\Phi_A: k_{31} \rightarrow \mathcal{S}$  the following functor:

$$\begin{aligned} \Phi_A(a) = \Phi_A(\tilde{a}) = \Phi_A(d) &= X \times \{1\}, & \Phi_A(b) = \Phi_A(\tilde{b}) &= X \times \{1, 2\}, \\ \Phi_A(c) &= X \times \{3, 4, 5\}, \end{aligned}$$

$\Phi(a), \Phi(\tilde{a})$  are inclusions,

$$\begin{aligned} [\Phi_A(\beta)](x, 3) &= [\Phi_A(\beta)](x, 4) = (x, 2), & [\Phi_A(\beta)](x, 5) &= (x, 1), \\ [\Phi_A(\tilde{\beta})](x, 3) &= (x, 1), & [\Phi_A(\tilde{\beta})](x, 4) &= [\Phi_A(\tilde{\beta})](x, 5) = (x, 2), \\ [\Phi_A(\gamma)](x, 3) &= (x, 1), & [\Phi_A(\gamma)](x, 4) &= (u_1(x), 1), \\ [\Phi_A(\gamma)](x, 5) &= (u_2(x), 1). \end{aligned}$$

If  $f: A \rightarrow A'$  is a homomorphism of algebras, denote by  $\varphi_f: \Phi_A \rightarrow \Phi_{A'}$  the transformation such that  $\varphi_f(x, i) = (f(x), i)$  for  $x \in X, i = 1, \dots, 5$ . One can easily prove that the mapping  $\Psi$ , such that  $\Psi(A) = \Phi_A, \Psi(f) = \varphi_f$ , is a full embedding of  $\mathcal{U}(1, 1)$  in  $\mathcal{S}^{k_{31}}$ .

**C. DEFINITION.** Let  $\mathcal{R}$  be the following category: objects are all triples  $(X, R, e)$ , where  $X$  is a set;  $R$  is a reflexive symmetric relation on  $X$  without isolated points (that is,

- 1)  $R \subset \text{exp } X$ ;
- 2)  $h \in R \Rightarrow \text{card } h \leq 2$ ;
- 3)  $(h \subset X) \ \& \ (\text{card } h = 1) \Rightarrow h \in R$ ;
- 4)  $(\forall x \in X)(\exists h \in R)((\text{card } h = 2) \ \& \ (x \in h))$ ;

$e$  is an evaluation of  $X$ , i.e.,  $e: X \rightarrow N$  is a mapping, where  $N = \{0, 1, 2, 3, \dots\}$ ; morphisms are precisely evaluation-increasing compatible mappings, i.e., if  $g = (X, R, e), g' = (X', R', e')$ , are objects of  $\mathcal{R}$ , then morphisms from  $g$  to  $g'$  are precisely the mappings  $f: X \rightarrow X'$  such that

$$\begin{aligned} h = \{x, y\} \in R &\Rightarrow \{f(x), f(y)\} \in R', \\ (\forall x \in X)(e'(f(x)) &\geq e(x)). \end{aligned}$$

**DEFINITION.** We say that  $f: (X, R, e) \rightarrow (X', R', e')$  is a *strongly compatible mapping* if it is a morphism of  $\mathcal{R}$  such that

$$\{x, y\} \in R, \quad x \neq y \Rightarrow f(x) \neq f(y).$$

Convention: Let  $(X, R, e)$  be an object of  $\mathcal{R}$ . Every finite sequence  $a_0, \dots, a_n$  of elements of  $X$  such that  $\{a_{i-1}, a_i\} \in R$  for every  $i = 1, \dots, n$ , is called a *path in  $(X, R, e)$* . (We do not assume  $a_{i-1} \neq a_i$ .) The number  $n$  is called the *length of the path  $a_0, \dots, a_n$* .

PROPOSITION 14. *There exists a full embedding  $\Gamma: \mathcal{U}(1, 1) \rightarrow \mathcal{R}$  such that for every objects  $A, A'$  of  $\mathcal{U}(1, 1)$  the following statements are equivalent:*

- 1)  $f: \Gamma(A) \rightarrow \Gamma(A')$  is a morphism of  $\mathcal{R}$ ;
- 2)  $f: \Gamma(A) \rightarrow \Gamma(A')$  is a strongly compatible mapping;
- 3)  $f = \Gamma(h)$  for some homomorphism  $h: A \rightarrow A'$ .

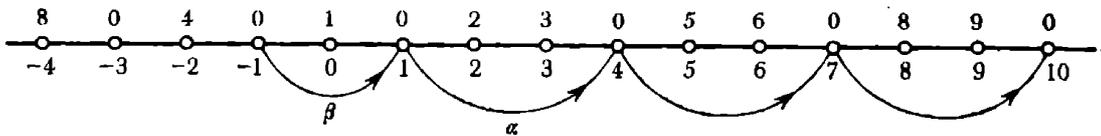


Fig. 4

Proof. If  $A = (X, \alpha, \beta)$  is an object of  $\mathcal{U}(1, 1)$ , put  $\Gamma(A) = g$ , where  $g = (Y, R, e)$  is defined as follows:  $Y = X \times Z$ ,  $Z$  being the set of all integers;

$$\begin{aligned}
 R = & \{ \{(x, n), (x, n+1)\}; x \in X, n \in Z \} \cup \\
 & \cup \{ \{(x, 3n+1), (x, 3n+4)\}; x \in X, n = 1, 2, 3, \dots \} \cup \\
 & \cup \{ \{(x, 1), (\alpha(x), 4)\}; x \in X \} \cup \\
 & \cup \{ \{(x, -1), (\beta(x), 1)\}; x \in X \} \cup \\
 & \cup \{ \{(x, n)\}; x \in X, n \in Z \};
 \end{aligned}$$

$$\begin{aligned}
 e: e(x, n) = 0 \text{ for } (x, n) \in X \times (\{-1, -3, -5, -7, \dots\} \cup \\
 \cup \{1, 4, 7, 10, 13, \dots\}),
 \end{aligned}$$

$$e(x, n) = 2|n| \quad \text{for } (x, n) \in X \times \{-2, -4, -6, -8, \dots\},$$

$$e(x, 0) = 1 \quad \text{for all } x \in X,$$

$$e(x, n) = n \quad \text{for all } (x, n) \in X \times \{2, 3, 5, 6, 8, 9, 11, 12, \dots\}.$$

If  $A = (X, \alpha, \beta)$ ,  $A' = (X', \alpha', \beta')$  are objects of  $\mathcal{U}(1, 1)$ ,  $h: A \rightarrow A'$  is a homomorphism, put  $[\Gamma(h)](x, n) = (h(x), n)$  for all  $(x, n) \in X \times Z$ . It is easy to see that  $\Gamma: \mathcal{U}(1, 1) \rightarrow \mathcal{R}$  is a faithful functor and every  $\Gamma(h)$  is a strongly compatible mapping. Thus it suffices to prove: *if  $f: \Gamma(A) \rightarrow \Gamma(A')$  is a morphism of  $\mathcal{R}$ , then  $f = \Gamma(h)$  for some homomorphism  $h: A \rightarrow A'$ .*

Thus, write

$$\Gamma(A) = g = (X, R, e), \quad \Gamma(A') = g' = (X', R', e').$$

Let  $f: g \rightarrow g'$  be fixed in the sequel.

1) First we prove  $f(X \times \{0\}) \subset X' \times \{0\}$ :

(a) Let us assume that  $f(x, 0) \in X' \times \{-n\}$  for some  $x \in X$  and some positive integer  $n$ . As  $e(x, 0) = 1$ , we have  $e'(f(x, 0)) \geq 1$  and so  $n$  must be even. Let us consider the path

$$b, c_0, c_1, \dots, c_{4n}, d,$$

where  $b = (x, 0)$ ,  $c_0 = (x, 1)$ ,  $c_k = (\alpha(x), 3k+1)$  for  $k = 1, 2, \dots, 4n$ ,  $d = (\alpha(x), 12n+2)$ . The length of the path is  $4n+2$  and  $e(d) = 12n+2$ . Then

$$f(b), f(c_0), f(c_1), \dots, f(c_{4n}), f(d)$$

is also a path with the same length. Since  $f(b) \in X' \times \{-n\}$ , necessarily  $f(d) \in X' \times \{l\}$  such that

$$-n - 4n - 2 \leq l \leq 1 + 3(3n + 2).$$

Consequently,

$$e'(f(d)) \leq \max\{2(n + 4n + 2), 1 + 3(3n + 2)\}.$$

Since  $n$  is even,  $e'(f(d)) < e(d)$ , which is impossible.

(b) Let us assume  $f(x, 0) \in X' \times \{n\}$  for some  $x \in X$  and some positive integer  $n$ . Consider the path

$$a_0, a_1, \dots, a_{2n},$$

where  $a_i = (x, -i)$  for  $i = 0, 1, \dots, 2n$ . Its length is  $2n$  and  $e(a_{2n}) = 4n$ . Since either  $e(a_{i-1}) > 0$  or  $e(a_i) > 0$  for every  $i = 1, 2, \dots, 2n$ , there is no  $i \in \{1, \dots, 2n\}$  such that

$$\{f(a_{i-1}), f(a_i)\} \subset X' \times (\{-1\} \cup \{3k+1; k = 0, 1, 2, \dots\}).$$

Since  $f(a_0) \in X' \times \{n\}$ , necessarily  $f(a_{2n}) \in X' \times \{l\}$ , where

$$n - 2n \leq l \leq n + 2n.$$

Then  $e'(f(a_{2n})) \leq \max(2n, 3n)$ , which is impossible.

2) Now, we prove  $f(X \times \{-n\}) \subset X' \times \{-n\}$  for every  $n \in \mathbb{Z}$  positive: Let  $x \in X$ ,  $n \in \mathbb{Z}$  positive be given. Consider the path

$$a_0, a_1, \dots, a_{2k},$$

where  $2k \geq n$ ,  $a_i = (x, -i)$  for  $i = 0, \dots, 2k$ . Since  $e'(f(a_{2m})) \geq 4m$  for every  $m = 1, \dots, k$  and  $f(a_0) \in X' \times \{0\}$ , necessarily  $f(a_i) \in X' \times \{-i\}$  for all  $i = 0, 1, \dots, 2k$ .

3) Further prove  $f(X \times \{1\}) \subset X' \times \{1\}$ : Let  $x \in X$ . Consider the path

$$(x, 0), (x, 1), (\alpha(x), 4), (\alpha(x), 5).$$

Since  $f(x, 0) \in X' \times \{0\}$ ,  $e'(f(\alpha(x), 5)) \geq 5$ , necessarily  $f(x, 1) \in X' \times \{1\}$ .

4) Now prove  $f(X \times \{i\}) \subset X' \times \{i\}$ ,  $i = 2, 3$ . Let  $x \in X$ . Consider the path  $(x, 1), (x, 2), (x, 3)$ .

5) Now prove  $f(X \times \{4\}) \subset X' \times \{4\}$ : If  $x \in X$ , consider the path  $(x, 3), (x, 4), (x, 5)$ .

6) Further prove  $f(X \times \{l\}) \subset X' \times \{l\}$ ,  $f(x \times \{l+1\}) \subset X' \times \{l+1\}$  whenever  $l = 4 + 3k$ ,  $k = 0, 1, 2, \dots$ : Let  $x \in X$ ,  $l = 4 + 3k$  be given. Consider the path

$$a_0, a_1, \dots, a_k, b,$$

where  $a_i = (x, 4 + 3i)$ ,  $i = 0, 1, \dots, k$ ,  $b = (x, 4 + 3k + 1)$ . Since  $f(a_0), f(a_1), \dots, f(a_k), f(b)$  is a path with the length  $k + 1$  and such that  $f(a_0) \in X' \times \{4\}$ ,  $e'(f(b)) \geq 4 + 3k + 1$ , necessarily  $f(b) \in X' \times \{4 + 3k + 1\}$ ,  $f(a_k) \in X' \times \{4 + 3k\}$ .

7) Finally prove  $f(X \times \{l\}) \subset X' \times \{l\}$  whenever  $l = 4 + 3k + 2$ ,  $k = 0, 1, 2, \dots$ : If  $x \in X$ , consider the path

$$(x, 4 + 3k + 1), (x, 4 + 3k + 2), (x, 4 + 3(k + 1)).$$

We have proved  $f(X \times \{n\}) \subset X' \times \{n\}$  for every  $n \in \mathbb{Z}$ . Define  $h_n: X \rightarrow X'$  such that  $f(x, n) = (h_n(x), n)$ . Put  $h = h_0$ . One can easily see that  $h_n = h$  for all  $n \in \mathbb{Z}$  and  $h: A \rightarrow A'$  is a homomorphism. Thus  $f = \Gamma(h)$ .

Convention: Let  $\mathcal{A}$  be the full subcategory of  $\mathcal{R}$  such that  $\mathcal{A}^\circ$  is the class of all  $\Gamma(A)$ ,  $A \in \mathcal{U}(1, 1)^\circ$ . Thus  $\mathcal{A}$  is isomorphic to  $\mathcal{U}(1, 1)$  and every morphism of  $\mathcal{A}$ , i.e., every evaluation-increasing compatible mapping from an object of  $\mathcal{A}$  to another one, is a strongly compatible mapping.

If  $g = (X, R, e) \in \mathcal{A}^\circ$ , write

$$\bar{R} = \{r \in R; \text{card } r = 2\}.$$

PROPOSITION 15. *The category  $k_{32}$  is rich.*

Proof. Let

$$k_{32}^\circ = \{a; b; o_n, n = 0, 1, \dots\},$$

$$k_{32}^m = \text{gen}\{a: a \rightarrow o_0; \beta: a \rightarrow b; \omega_n: o_n \rightarrow o_{n-1}, n = 1, 2, \dots\}.$$

Define a full embedding  $\Psi: \mathcal{A} \rightarrow \mathcal{S}^{k_{32}}$  as follows: if  $g = (X, R, e) \in \mathcal{A}^\circ$ , put  $\Psi(g) = \Phi$ , where  $\Phi: k_{32} \rightarrow \mathcal{S}$  is the following functor:

$$\Phi(o_n) = \{x \in X; e(x) \geq n\}, \quad n = 0, 1, 2, \dots;$$

$\Phi(\omega_n)$  is the inclusion;  $\Phi(a) = \{(x, r) \in X \times \bar{R}; x \in r\}$ ;  $\Phi(b) = \bar{R}$ ;  $[\Phi(a)](x, r) = x$ ;  $[\Phi(\beta)](x, r) = r$ . The definition of  $\Psi(f)$ , where  $f \in \mathcal{A}^m$ , is evident.

PROPOSITION 16. *The category  $k_{33}$  is rich.*

Proof. Let

$$k_{33}^o = \{a; b; o_n, n = 0, 1, 2, \dots\},$$

$$k_{33}^m = \text{gen}\{\alpha: o_0 \rightarrow a; \beta: o_0 \rightarrow b; \omega_n: o_n \rightarrow o_{n-1}, n = 1, 2, \dots\}.$$

Define a full embedding as follows: if  $g = (X, R, e) \in \mathcal{A}^o$ , put  $\Psi(g) = \Phi$ , where

$$\Phi(a) = X, \quad \Phi(b) = \bar{R}, \quad \Phi(o_n) = \{(x, r) \in X \times \bar{R}; x \in r, e(x) \geq n\}, \\ n = 0, 1, 2, \dots;$$

$\Phi(\omega_n)$  are inclusions,  $[\Phi(\alpha)](x, r) = x$ ,  $[\Phi(\beta)](x, r) = r$ .

PROPOSITION 17. *The category  $k_{34}$  is rich.*

Proof. Let

$$k_{34}^o = \{a, b, c, d\} \cup \{o_n, n = 0, 1, 2, \dots\},$$

$$k_{34}^m = \text{gen}\{\alpha: o_0 \rightarrow a; \beta: b \rightarrow o_0; \omega_n: o_n \rightarrow o_{n-1}, n = 1, 2, \dots; \\ \gamma_m: c \rightarrow o_m, m = 0, 1, 2, \dots; \delta: c \rightarrow d\}.$$

Define a full embedding  $\Psi: \mathcal{A} \rightarrow \mathcal{S}^{k_{34}}$  as follows: if  $g = (X, R, e) \in \mathcal{A}^o$ , put  $\Psi(g) = \Phi$ , where  $\Phi(o_n) = (X \times \{0\}) \cup (\{x \in X; e(x) \geq n\} \times \{1\})$ ,  $\Phi(\omega_n)$  are inclusions;

$$\Phi(b) = X, \quad [\Phi(\beta)](x) = (x, 1);$$

$$\Phi(a) = X, \quad [\Phi(\alpha)](x, 0) = [\Phi(\alpha)](x, 1) = x;$$

$$\Phi(o) = \{(x, r) \in X \times \bar{R}; x \in r\}, \quad [\Phi(\gamma_m)](x, r) = (x, 0);$$

$$\Phi(d) = \bar{R}, \quad [\Phi(\delta)](x, r) = r.$$

PROPOSITION 18. *The category  $k_{35}$  is rich.*

Proof. Let

$$k_{35}^o = \{a, b, c\} \cup \{o_n; n = 0, 1, 2, \dots\},$$

$$k_{35}^m = \text{gen}(\{\gamma: b \rightarrow c\} \cup \{\alpha_n: a \rightarrow o_n, \beta_n: b \rightarrow o_n, n = 0, 1, 2, \dots\} \cup \\ \cup \{\omega_n: o_n \rightarrow o_{n-1}, n = 1, 2, \dots\}).$$

Let  $h$  be a thin category such that

$$h^o = k_{35}^o \cup \{d\},$$

$$h^m = \text{gen}(k_{35}^m \cup \{\alpha: a \rightarrow d, \beta: b \rightarrow d\} \cup \{\delta_n: d \rightarrow o_n, n = 0, 1, 2, \dots\}).$$

1) First we define a full embedding  $\Psi: \mathcal{A} \rightarrow \mathcal{S}^h$ : if  $g = (X, R, e) \in \mathcal{A}^o$ , put  $\Psi(g) = \Phi$ , where  $\Phi: h \rightarrow \mathcal{S}$  is the following functor: choose an element  $\xi$ ,

$\xi \notin X$  and put  $\Phi(d) = \{\xi\} \cup X$ ,  $\Phi(a) = \{\xi\}$ ,  $\Phi(a)$  is the inclusion;  $\Phi(b) = \{(x, r) \in X \times \bar{R}, x \in r\}$ ,  $\Phi(c) = \bar{R}$ ,  $[\Phi(\beta)](x, r) = x$ ,  $[\Phi(\gamma)](x, r) = r$ ;  $\Phi(o_n) = \{\xi\} \cup \{x \in X; e(x) < n\}$ ,  $[\Phi(\delta_n)](\xi) = \xi$ ,  $[\Phi(\delta_n)](x) = x$  whenever  $e(x) < n$ ,  $[\Phi(\delta_n)](x) = \xi$  whenever  $e(x) \geq n$ ;  $\Phi(\omega_n)$  is the mapping such that  $\Phi(\omega_n) \circ \Phi(\delta_n) = \Phi(\delta_{n-1})$ . One can easily define  $\Psi(f)$  for  $f \in \mathcal{A}^m$  and prove that  $\Psi$  is a full embedding.

2) Let  $l$  be a full subcategory of  $k_{35}$ ,  $l^o = \{o_n; n = 0, 1, 2, \dots\}$ . Denote by  $\Phi_i: l \rightarrow \mathcal{S}$  the domain-restriction of  $\Phi$  to  $l$ . One can easily see that

$$\langle \Phi(d); \{\Phi(\delta_n); n = 0, 1, 2, \dots\} \rangle = \lim \Phi_i.$$

Thus, the functors  $\Phi = \Psi(g): h \rightarrow \mathcal{S}$ ,  $g$  running through  $\mathcal{A}^o$ , and their transformations, are uniquely determined by their values on  $k_{35}$ .

#### IV. Forms $a_1, \dots, a_{10}$ and the cone

In the present part the forms  $a_1, \dots, a_{10}$  of small thin categories are defined. We prove here: *If  $k$  has some of the forms  $a_1, \dots, a_{10}$ , then there exists no connected functor from  $k$  to  $\mathcal{S}_n$  such that all its endotransformations form precisely  $c_2$ .*

The proofs are rather similar for the forms  $a_1, \dots, a_6$ . To prove it for  $a_7, \dots, a_{10}$  we need some lemmas about endomorphisms of a certain graph. At the end of the present part we define a cone.

CONVENTION 1. (a) Let  $h: X \rightarrow Y$  be a mapping. We write  $h = \text{const } y$  iff  $h(x) = y$  for all  $x \in X$ .

(b) The set-theoretical symbols  $\subset, \cup, \cap, \vee$  will be used also for functors to  $\mathcal{S}$ . If  $f: k \rightarrow \mathcal{S}$  is a functor, we denote by  $1_f$  the identity endotransformation.  $f$  is said to be *rigid* if  $1_f$  is the only endotransformation of  $f$ .

(c) Let  $k$  be a thin category. We write  $a < b$  whenever  $a, b \in k^0$ ,  $a \neq b$  and there is a morphism from  $a$  to  $b$  in  $k$ . This morphism is sometimes denoted by  $\begin{pmatrix} a \\ b \end{pmatrix}$ . If  $k(a, b) \cup k(b, a) = \emptyset$ , we write  $a \# b$ . As usual, thin small categories are considered also as quasi-ordered sets (by  $<$ ) and the terminology for quasi-ordered sets is sometimes used for them.

(d) Let  $k$  be a category,  $A \subset k^0$  be a class. We denote by  $kA$  the full subcategory of  $k$  such that  $(kA)^0 = A$ .

(e) We recall that a functor  $f: k \rightarrow \mathcal{S}$  is said to be *trivial* iff  $f(o) = \emptyset$  for every  $o \in k^0$ . The trivial functor is denoted by  $C_0$ . The other functors are called *non-trivial*. A functor  $f$  is said to be *connected* if it cannot be expressed as  $f = f_1 \cup f_2$ , where  $f_1$  and  $f_2$  are non-trivial,  $f_1 \cap f_2 = C_0$ .

LEMMA 1. *Let  $f: k \rightarrow \mathcal{S}$  be a non-trivial functor from a small thin category. Then it is connected iff for every  $o, o' \in k^0$ ,  $x \in f(o)$ ,  $x' \in f(o')$ , there exist  $o_0, o_1, \dots, o_n \in k^0$  and  $x_i \in f(o_i)$  such that  $o_0 = o$ ,  $o_n = o'$ ,  $x_0 = x$ ,  $x_n = x'$  and either*

$$\left[ f \begin{pmatrix} o_i \\ o_{i+1} \end{pmatrix} \right] (x_i) = x_{i+1} \quad \text{or} \quad \left[ f \begin{pmatrix} o_{i+1} \\ o_i \end{pmatrix} \right] (x_{i+1}) = x_i, \quad i = 0, \dots, n-1.$$

Proof is easy.

CONVENTION 2. Let  $k$  be a small category,  $f: k \rightarrow \mathcal{S}$  be a functor. Denote by  $\mathbf{Z}(f)$  the following three conditions:

- (a)  $f$  is connected;
- (b)  $f(o) \neq \emptyset$  for every  $o \in k^o$ ;
- (c) the monoid of all endotransformations of  $f$  is isomorphic to  $e_2$ .

DEFINITION 1. We say that a small thin category  $k$  has a form  $\alpha_1$  iff it has exactly one minimal object.

PROPOSITION 1. Let  $k$  have the form  $\alpha_1$ . Then  $\mathbf{Z}(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

Proof. Let  $k$  have the form  $\alpha_1$ , let  $a$  be its minimal object. Let  $f: k \rightarrow \mathcal{S}$  be a functor satisfying (a), (b) from  $\mathbf{Z}(f)$ . Choose  $x_a \in f(a)$  and define a transformation  $\tau: f \rightarrow f$  such that  $\tau^o = \text{const } x_o$  for every  $o \in k^o$ , where  $x_o = \left[ f \begin{pmatrix} a \\ o \end{pmatrix} \right] (x_a)$ .

Then  $\tau$  is idempotent and if  $\tau = 1_f$ , then  $f$  is rigid. Consequently, (c) is not fulfilled,  $\mathbf{Z}(f)$  does not hold.

DEFINITION 2. We say that a small thin category  $k$  has the form  $\alpha_2$  (see Fig. 5) iff

$$(2) k^o = \{a_1, a_2\} \cup R \cup C \cup D \cup M \cup L$$

so that:

- (2a) the sets on the right-hand side of (2) are disjoint;
- (2b)  $a_1, a_2$  are two different minimal elements of  $k$ ;
- (2c)  $R = \{o \in k^o; o > a_i, i = 1, 2\}$  is well ordered and it can be written as  $R = \{r_\delta; \delta \leq \beta + 1\}$  for some ordinal  $\beta > 0$  such that  $\delta < \delta' \leq \beta + 1 \Rightarrow r_\delta < r_{\delta'}$ ;

Denote by  $r$  the last element  $r_{\beta+1}$ .

(2d)  $C$  is non-void and well ordered,  $C = \{c_i; i < \alpha\}$  for some ordinal  $\alpha$ ,  $i \leq i' < \alpha \Rightarrow a_1 < c_i \leq c_{i'} < r_\beta$ ;

(2e)  $L$  is linearly ordered,  $a_2 < l < r$  for every  $l \in L$ ;

(2f)  $a_1 < d \# r$  for every  $d \in D$ ;

(2g)  $r > m > a_1$  for every  $m \in M$ ;

$l \# r_\delta \# m$  for every  $l \in L, m \in M, r_\delta \in R, r_\delta < r$ ;

(2h) if  $A = \{d, m, c_i, r_\delta, l\}$  for some  $d \in D, m \in M, i < \alpha, \delta < \beta, l \in L$ , then  $x \# y$  for every  $x, y \in A, x \neq y$ .

Note 1. (a) The sets  $M, L, D$  can be empty, but necessarily  $R \neq \emptyset, C \neq \emptyset$ .

(b) (2a), ..., (2h) imply  $c \# a_2$  for every  $c \in C$ . Analogously,  $m \# a_2, d \# a_2, l \# a_1$ , for every  $m \in M, d \in D, l \in L$ .

**PROPOSITION 2.** *Let  $f: k \rightarrow \mathcal{S}$  be a functor. Let  $k$  have the form  $\alpha_2$ . Then  $\mathbf{Z}(f)$  does not hold.*

**Proof.** Let (a), (b) from the definition of  $\mathbf{Z}(f)$  be fulfilled. We construct an endotransformation  $\tau: f \rightarrow f$  with  $\tau \circ \tau = \tau$  such that either  $\tau = 1_f$  and  $f$  is rigid or  $\tau \neq 1_f$ . Thus, (c) from  $\mathbf{Z}(f)$  cannot hold.

Put

$$\gamma = \min \left\{ \delta; \left[ f \left( \begin{smallmatrix} a_1 \\ r_\delta \end{smallmatrix} \right) \right] (f(a_1)) \cap \left[ f \left( \begin{smallmatrix} a_2 \\ r_\delta \end{smallmatrix} \right) \right] (f(a_2)) \neq \emptyset \right\}.$$

(Note that the last set is non-void because  $f$  is connected.)

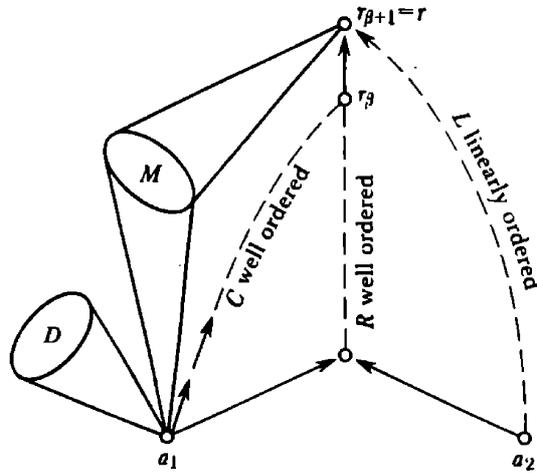


Fig. 5

1) Let  $\gamma \leq \beta$ . Choose  $p_i \in f(a_i)$ ,  $i = 1, 2$ , such that

$$(1) \quad \left[ f \left( \begin{smallmatrix} a_1 \\ r_\gamma \end{smallmatrix} \right) \right] (p_1) = \left[ f \left( \begin{smallmatrix} a_2 \\ r_\gamma \end{smallmatrix} \right) \right] (p_2),$$

and define  $q_o$ ,  $o \in k^o - \{r_\delta; \delta < \gamma\}$ , by

$$(2) \quad \begin{aligned} q_o &= \left[ f \left( \begin{smallmatrix} a_1 \\ o \end{smallmatrix} \right) \right] (p_1) && \text{if } o > a_1, \\ q_o &= \left[ f \left( \begin{smallmatrix} a_2 \\ o \end{smallmatrix} \right) \right] (p_2) && \text{if } o > a_2. \end{aligned}$$

Further define  $\bar{a}_\delta^1, \bar{a}_\delta^2$  for  $\delta < \gamma$  by

$$(3) \quad \bar{a}_\delta^i = \left[ f \left( \begin{smallmatrix} a_i \\ r_\delta \end{smallmatrix} \right) \right] (p_i), \quad i = 1, 2.$$

Define  $\tau: f \rightarrow f$  by

$$(4) \quad \begin{aligned} \tau^o &= \text{const } q_o & \text{for } o \in k^o - \{r_\delta; \delta < \gamma\}; \\ \tau^{r_\delta}/A_\delta &= \text{const } \bar{a}_\delta^1 & \text{for } \delta < \gamma; \\ \tau^{r_\delta}/f(r_\delta) - A_\delta &= \text{const } \bar{a}_\delta^2 & \text{for } \delta < \gamma, \end{aligned}$$

where

$$A_\delta = \bigcup_{\delta \leq \kappa < \gamma} \left[ f \left( \begin{smallmatrix} r_\delta \\ r_\kappa \end{smallmatrix} \right) \right]^{-1} \left( \left[ f \left( \begin{smallmatrix} a_1 \\ r_\kappa \end{smallmatrix} \right) \right] (f(a_1)) \right).$$

Evidently,  $\tau$  is idempotent. If  $\tau = 1_f$ , then  $f(o) = \{q_o\}$  for  $o \in k^o - \{r_\delta; \delta < \gamma\}$ , and  $f(r_\delta) = \{\bar{a}_\delta^1, \bar{a}_\delta^2\}$  for  $\delta < \gamma$ . But then  $f$  is rigid.

2) Let  $\gamma = \beta + 1$  and there be  $\iota < \alpha$  such that

$$\left[ f \left( \begin{smallmatrix} a_2 \\ r_\beta \end{smallmatrix} \right) \right] (f(a_2)) \cap \left[ f \left( \begin{smallmatrix} c_\iota \\ r_\beta \end{smallmatrix} \right) \right] (f(c_\iota)) \neq \emptyset.$$

Let  $\lambda$  be the least ordinal such that the last assertion is fulfilled. Choose  $p_2 \in f(a_2)$ ,  $p \in f(c_\lambda)$  such that

$$\left[ f \left( \begin{smallmatrix} c_\lambda \\ r_\beta \end{smallmatrix} \right) \right] (p) = \left[ f \left( \begin{smallmatrix} a_2 \\ r_\beta \end{smallmatrix} \right) \right] (p_2).$$

As  $f$  is connected, the mapping  $f \left( \begin{smallmatrix} a_1 \\ r_{\beta+1} \end{smallmatrix} \right)$  is onto.

Now choose  $p_1 \in f(a_1)$  with

$$\left[ f \left( \begin{smallmatrix} a_1 \\ r_{\beta+1} \end{smallmatrix} \right) \right] (p_1) = \left[ f \left( \begin{smallmatrix} a_2 \\ r_{\beta+1} \end{smallmatrix} \right) \right] (p_2).$$

Define  $\{q_o; o \in k^o - \{r_\delta; \delta \leq \beta\}\}$  by (2) and  $\{\bar{a}_\delta^i; \delta \leq \beta, i = 1, 2\}$  by (3); put  $h_\iota = \left[ f \left( \begin{smallmatrix} c_\lambda \\ c_\iota \end{smallmatrix} \right) \right] (p)$  for every  $\iota, \lambda \leq \iota < \alpha$ . Finally, define  $\tau: f \rightarrow f$  by (4), changing only the definition of  $\tau^{c_\iota}$ ,  $\iota < \alpha$ , to

$$\begin{aligned} \tau^{c_\iota}(x) &= h_\iota & \text{for every } x \in \left[ f \left( \begin{smallmatrix} c_\iota \\ r_\beta \end{smallmatrix} \right) \right]^{-1} (Y), \iota < \alpha, \\ \tau^{c_\iota}(x) &= q_{c_\iota} & \text{for every } x \in f(c_\iota) - \left( \left[ f \left( \begin{smallmatrix} c_\iota \\ r_\beta \end{smallmatrix} \right) \right]^{-1} (Y) \right), \iota < \alpha, \end{aligned}$$

where

$$Y = \left[ f \left( \begin{smallmatrix} a_2 \\ r_\beta \end{smallmatrix} \right) \right] (f(a_2)).$$

It is easy to see that  $\tau$  is an idempotent and if  $\tau = 1_f$ , then  $f$  is rigid.

3) Let  $\gamma = \beta + 1$  and

$$\left[ f \left( \begin{smallmatrix} a_2 \\ r_\beta \end{smallmatrix} \right) \right] (f(a_2)) \cap \left[ f \left( \begin{smallmatrix} c_\iota \\ r_\beta \end{smallmatrix} \right) \right] (f(c_\iota)) = \emptyset.$$

We proceed formally in the same way as in 1).

**DEFINITION 3.** We say that a small thin category  $k$  has a form  $a_3$  (see Fig. 6) iff (2), (2a), (2b), (2c'), (2d), (2e), (2f), (2g) hold, where (2c') is obtained from (2c) by changing the assumption  $\beta > 0$  to  $\beta = 0$ .

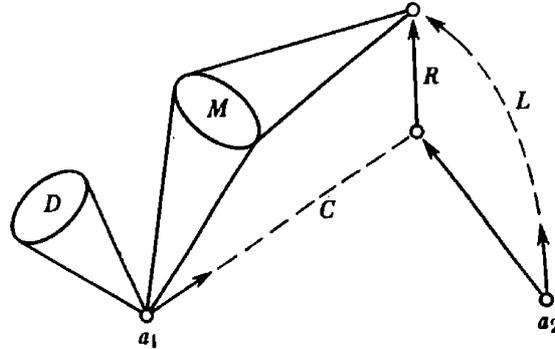


Fig. 6

**PROPOSITION 3.** Let  $k$  have the form  $a_3$ . Then  $\mathbf{Z}(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

Proof is the same as that of Proposition 2 (it was not essential that  $\beta > 0$ ).

**DEFINITION 4.** We say that a small thin category  $k$  has the form  $a_4$  (see Fig. 7) iff (2), (2a), (2b), (2c''), (2d'), (2e), (2f), (2g), (2h') are fulfilled, where

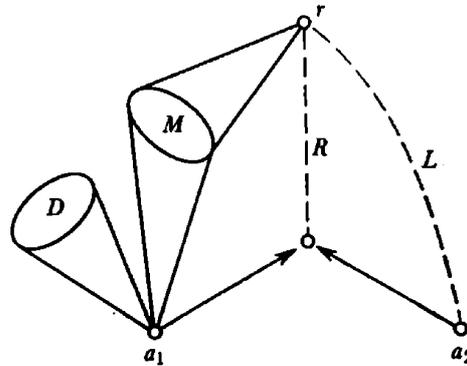


Fig. 7

(2c'')  $R = \{o \in k^o; o > a_i, i = 1, 2\}$  is well ordered,  $|R| \geq 2$ , and it has a last element, say  $r$ ;

(2d')  $C = \emptyset$ ;

(2h') if  $A = \{d, m, l, r_s\}$  for some  $d \in D$ ,  $m \in M$ ,  $l \in L$ ,  $r_s \in R - \{r\}$ , then  $x \# y$  for every  $x, y \in A$ ,  $x \neq y$ .

**PROPOSITION 4.** Let  $k$  have the form  $a_4$ . Let  $f: k \rightarrow \mathcal{S}$  be a functor. Then  $\mathbf{Z}(f)$  does not hold.

**Proof** is similar to that of Proposition 2 and therefore it is omitted.

**DEFINITION 5.** We say that a small thin category  $k$  has the form  $a_5$  (see Fig. 8) iff

$$(5) k^0 = \{a_1, a_2\} \cup R \cup L_1 \cup L_2 \cup D_1 \cup D_2$$

so that

(5a) the sets on the right-hand side of (5) are disjoint;

(5b)  $a_1, a_2$  are two distinct minimal elements of  $k$ ;

(5c)  $R = \{o \in k^0; o > a_1, o > a_2\}$  is well ordered,  $R \neq \emptyset$ ;

(5d)  $L_i$  is linearly ordered and  $a_i < l$  for every  $l \in L_i, i = 1, 2$ ;

(5e) if  $L_1 \cup L_2 \neq \emptyset$ , then  $|R| \geq 2$  and  $R$  has a last element, say  $r$ ;

(5f) if  $L_1 \cup L_2 \neq \emptyset$ , then  $l < r$  for every  $l \in L_1 \cup L_2$ ;

(5g)  $d > a_i$  for every  $d \in D_i, i = 1, 2$ ;

(5h) if  $A = \{d_1, d_2, z\}$ , where  $d_i \in D_i, z \in R$ , then  $x \# y$  whenever  $x, y \in A, x \neq y$ ;

(5i) if  $x, y \in D_1 \cup D_2 \cup L_1 \cup L_2 \cup (R - \{r\})$ ,  $x < y$ , then either  $x, y \in R$  or  $x, y \in D_i$ , or  $x, y \in L_i, i = 1, 2$ .

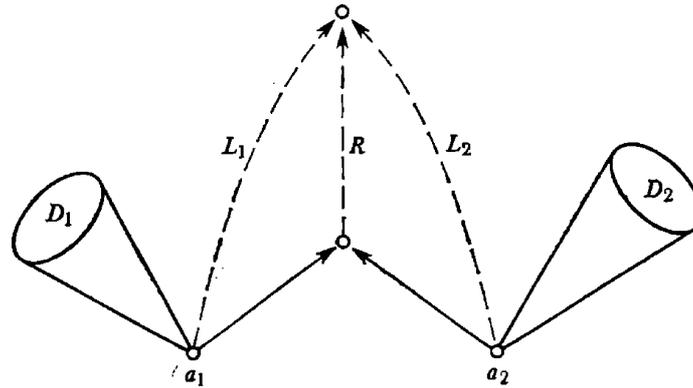


Fig. 8

**PROPOSITION 5.** Let  $k$  have the form  $a_5$ . Then  $Z(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

**Proof** is similar to that of Proposition 2 and therefore it is omitted.

**DEFINITION 6.** We say that a small thin category  $k$  has the form  $a_6$  (see Fig. 9) iff

$$(6) k^0 = \{a_1, a_2, a_3\} \cup D_1 \cup D_2 \cup M_1 \cup M_2$$

so that

(6a) the sets on the right-hand side of (6) are disjoint;

(6b)  $a_1, a_2$  are two distinct minimal elements of  $k$ ;

(6c)  $a_i < m < a_3$  for every  $m \in M_i, i = 1, 2$ ;

(6d)  $a_i < d \# a_j$  for every  $d \in D_i$ ,  $i = 1, 2$ ; if  $d \in D_1$ , then  $d \# a_2$ ; if  $d \in D_2$ , then  $d \# a_1$ ;

(6e) if  $A = \{d_1, d_2, m_1, m_2\}$ , where  $d_i \in D_i$ ,  $m_i \in M_i$ , then  $x \# y$  for every  $x, y \in A$ ,  $x \neq y$ .

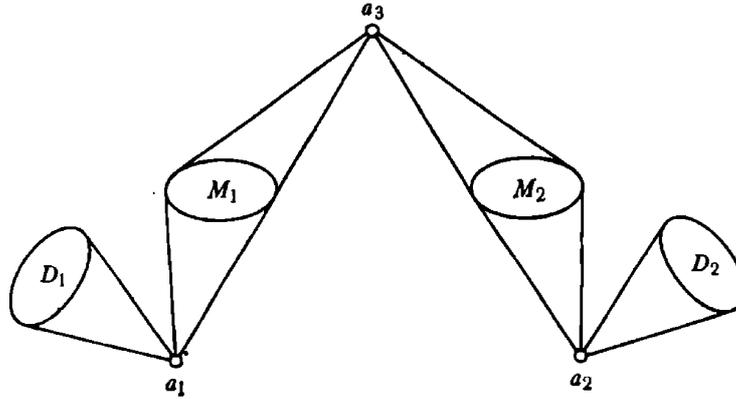


Fig. 9

PROPOSITION 6. Let a small thin category  $k$  have the form  $a_6$ . Then  $Z(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

Proof. If  $f$  is connected, then

$$\left[ f \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right] (f(a_1)) \cap \left[ f \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} \right] (f(a_2)) \neq \emptyset.$$

The rest of the proof is evident.

CONVENTION 3. Let  $\mathbf{l}$  be the category described in III.B, i.e.,

$$\mathbf{l}^0 = \{l_0, l_1, l_2\}, \quad \mathbf{l}^m = \text{gen}\{\lambda_1: l_0 \rightarrow l_1; \lambda_2: l_0 \rightarrow l_2\}.$$

Let  $f: \mathbf{l} \rightarrow \mathcal{S}$  be a functor. Define an oriented graph  $G_f = (X, R)$  such that

$$X = f(l_1) \vee f(l_2),$$

$$(x_1, x_2) \in R \Leftrightarrow (x_i \in f(l_i), i = 1, 2) \ \& \ (\exists z \in f(l_0) ([f(\lambda_i)](z) = x_i, i = 1, 2)).$$

Clearly, there exists an epimorphism  $\varepsilon: f(l_0) \rightarrow R$  such that  $f(\lambda_i) = \pi_i \circ \varepsilon$ .

LEMMA 2. Let  $f: \mathbf{l} \rightarrow \mathcal{S}$  be a functor. Let  $\tau^i: f(l_i) \rightarrow f(l_i)$ ,  $i = 1, 2$ , be mappings. Then there exists a mapping  $\tau^0: f(l_0) \rightarrow f(l_0)$  such that  $\tau = \{\tau^0, \tau^1, \tau^2\}$  is an endotransformation of  $f$  iff the mapping  $\sigma = \tau^1 \vee \tau^2$  is an endomorphism of  $G_f$ . If  $\sigma$  is idempotent, then  $\tau$  can be chosen also idempotent.

Proof is easy.

CONVENTION 4. Let  $G = (X, R)$  be a (directed) graph. We recall that every sequence  $x_0, \dots, x_n$  such that either  $(x_i, x_{i+1}) \in R$  or  $(x_{i+1}, x_i) \in R$  is said to be a path in  $G$ ,  $n$  is called its length. Let  $A_1, A_2 \subset X$ . We write  $\varrho(A_1, A_2) = n$  iff  $n$  is the minimal length of a path  $x_0, \dots, x_n$  with  $x_0 \in A_1$ ,

$x_n \in A_2$ . If there exists no path  $x_0, \dots, x_n$  with  $x_0 \in A_1, x_n \in A_2$ , we write  $\rho(A_1, A_2) = \infty$ . We write  $\rho(x, y)$  instead of  $\rho(\{x\}, \{y\})$ . The identical endomorphism of  $G$  is denoted by  $1_G$ .

**LEMMA 3.** *Let  $G = (X, R)$  be a graph such that*

$$(*) \quad X = X_1 \vee X_2, (x_1, x_2) \in R \Rightarrow x_i \in X_i, \quad i = 1, 2.$$

*Let  $A_1, A_2$  be non-void sets such that  $A_1 \subset X_1$  and either  $A_2 \subset X_2$  or  $A_2 \subset X_1, A_1 \cap A_2 = \emptyset$ .*

*Then there exists an idempotent endomorphism  $\sigma$  of  $G$  such that:*

- ( $\alpha$ )  $\sigma = \tau^1 \vee \tau^2$ , where  $\tau^i: X_i \rightarrow X_i, i = 1, 2$ ;
- ( $\beta$ )  $\sigma(A_i) \subset A_i, \sigma/A_i$  are constant,  $i = 1, 2$ ;
- ( $\gamma$ )  $\sigma = 1_G \Rightarrow |A_1| = |A_2| = 1$  and  $1_G$  is the only endotransformation of  $G$  with  $\sigma(A_i) \subset A_i, i = 1, 2$ .

**Proof.** We may omit the trivial case  $R = \emptyset$ .

1) Let  $\rho(A_1, A_2) = \infty$ : Put

$$Y_i = \{x \in X; \rho(x, A_i) < \infty\}, \quad i = 1, 2.$$

Let  $j \in \{1, 2\}$ ; if there exists  $(x_j, y_j) \in R$  such that either  $x_j \in A_j$  or  $y_j \in A_j$ , put  $\sigma(x) = x_j$  for every  $x \in Y_j \cap X_1, \sigma(x) = y_j$  for every  $x \in Y_j \cap X_2$ . If not (i.e. if  $Y_j = A_j$ ), choose  $a_j \in A_j$  and put  $\sigma(x) = a_j$  for every  $x \in Y_j$ . It remains to define  $\sigma(x)$  for  $x \in X - (Y_1 \cup Y_2) = Z$ . If  $Y_1 - A_1 \neq \emptyset$ , put  $\sigma(x) = x_1$  for  $x \in Z \cap X_1, \sigma(x) = y_1$  for  $x \in Z \cap X_2$ . If  $Y_1 = A_1, Y_2 - A_2 \neq \emptyset$ , put  $\sigma(x) = x_2$  for  $x \in Z \cap X_1, \sigma(x) = y_2$  for  $x \in Z \cap X_2$ . If  $A_1 = Y_1, A_2 = Y_2$ , choose  $\langle z_1, z_2 \rangle \in R$  and put  $\sigma(x) = z_1$  for  $x \in Z \cap X_1, \sigma(x) = z_2$  for  $x \in Z \cap X_2$ . It is easy to see that  $\sigma$  is an idempotent endomorphism of  $G$  satisfying all the required properties.

2) Let  $\rho(A_1, A_2) = n < \infty$ . Clearly,  $n > 0$  and so either  $n = 2d$  or  $n = 2d + 1, d$  is an integer,  $d \geq 0$ . Let  $x_0, \dots, x_n$  be a path in  $G$  such that  $x_0 \in A_1, x_n \in A_2$ . Let  $x \in X$ ; if  $k = \rho(x, A_1) \leq d$ , put  $\sigma(x) = x_k$  (in particular,  $\sigma(x) = x_0 \in A_1$  for every  $x \in A_1$ ); if  $k = \rho(x, A_2) \leq d$ , put  $\sigma(x) = x_{n-k}$  (in particular,  $\sigma(x) = x_n \in A_2$  for every  $x \in A_2$ ); if both inequalities  $\rho(x, A_1) \leq d, \rho(x, A_2) \leq d$  hold, then  $n = 2d, \sigma(x) = x_d = x_{n-d}$  so that the definition is correct; if  $\rho(x, A_1) > d < \rho(x, A_2)$ , put  $\sigma(x) = x_{d+1}$  whenever  $x, x_{d+1} \in X_i$  for some  $i = 1, 2, \sigma(x) = x_d$  whenever  $x, x_d \in X_i$  for some  $i = 1, 2$ . One can prove easily that  $\sigma$  is an endomorphism of  $G$  with all the required properties.

**LEMMA 4.** *Let a graph  $G = (X, R)$  satisfy (\*). Let  $A_1 \subset X_1, B \subset R$  be non-void sets. Then there exists an idempotent endomorphism  $\sigma$  of  $G$  such that:*

- ( $\alpha$ )  $\sigma = \tau^1 \vee \tau^2, \tau^i: X_i \rightarrow X_i$  is a mapping,  $i = 1, 2$ ;
- ( $\beta_1$ )  $\sigma(A_1) \subset A_1, \sigma/A_1$  is constant;

- ( $\beta_2$ ) the mapping  $\delta: R \rightarrow R$ , defined by  $\delta(x, y) = (\tau_1(x), \tau_2(x))$ , satisfies  $\delta(B) \subset B$ ,  $\delta/B$  is a constant;
- ( $\gamma$ ) if  $\sigma = 1_G$ , then  $|B| = |R| = |A_1| = 1$  and  $1_G$  is the only endomorphism of  $G$  such that  $\sigma(A_1) \subset A_1$ ,  $\delta(B) \subset B$ .

Proof. 1) Let  $\pi_1(B) \cap A_1 \neq \emptyset$ . Then Lemma 4 is trivial.

2) Let  $\pi_1(B) \cap A_1 = \emptyset$ ,  $\rho(\pi_1(B), A_1) = \infty$ . Put  $A_2 = \pi_1(B)$  and apply Lemma 3; evidently, the point  $(x_2, y_2)$  from the proof of Lemma 3 can be chosen such that  $(x_2, y_2) \in B$  so that  $\delta/B$  is constant.

3) Let  $\pi_1(B) \cap A_1 = \emptyset$ ,  $\rho(\pi_1(B), A_1) < \infty$ . Then either

- (i)  $\rho(\pi_1(B), A_1) = \rho(\pi_2(B), A_1) + 1$  or  
 (ii)  $\rho(\pi_2(B), A_1) = \rho(\pi_1(B), A_1) + 1$ .

In case (i) put  $A_2 = \pi_1(B)$ , in case (ii) define  $A_2 = \pi_2(B)$ . Choose a path  $x_0, \dots, x_n$  with the minimal length from  $A_1$  to  $A_2$  such that  $x_{n-1} \in \pi_2(B)$  whenever (i) holds,  $x_{n-1} \in \pi_1(B)$  whenever (ii) holds. Then define  $\sigma$  as in the proof of Lemma 3.

LEMMA 5. Let  $f: \mathcal{I} \rightarrow \mathcal{S}$  be a functor. Let  $A_1, A_2$  be non-void sets,  $A_1 \subset f(l_1)$ ,  $A_2 \subset f(l_i)$  for some  $i \in \{0, 1, 2\}$ ; moreover,  $A_1 \cap A_2 = \emptyset$  in the case  $A_2 \subset f(l_1)$ . Then there exists an idempotent transformation  $\tau: f \rightarrow f$  such that:

- $\tau(A_i) \subset A_i$ ,  $\tau/A_i$  is constant,  $i = 1, 2$ ;  
 (+)  $\tau \neq 1_f$  or the following statement is true:  $\tau = 1_f$ ,  $|A_1| = |A_2| = 1$  and  $1_f$  is the only endotransformation of  $f$  such that  $\tau(A_i) \subset A_i$ ,  $i = 1, 2$ .

Proof. If either  $A_2 \subset f(l_1)$  or  $A_2 \subset f(l_2)$ , then Lemma 5 follows from Lemma 2 and Lemma 3. It is only necessary to choose  $\tau^0$  to be constant on every non-void set  $Z_{x_1, x_2}$ , where  $Z_{x_1, x_2} = \{x \in f(l_0); [f(\lambda_i)](x) = x_i, i = 1, 2\}$ .

If  $A_2 \subset f(l_0)$  put  $B = \varepsilon(A_2)$  and apply Lemma 2 and Lemma 4 and again choose  $\tau^0$  carefully.

CONVENTION. Denote by  $\mathbf{c}$  a thin category such that

$$\mathbf{c}^0 = \mathcal{I}^0 \cup \{l_3\}, \quad \mathbf{c}^m = \text{gen } \mathcal{I}^m \cup \{\gamma_1: l_1 \rightarrow l_3, \gamma_2: l_2 \rightarrow l_3\}$$

(where  $\mathcal{I}$  is the category defined in III.B). Since it is a thin category, necessarily  $\gamma_1 \circ \lambda_1 = \gamma_2 \circ \lambda_2$ .

LEMMA 6. Let  $f: \mathbf{c} \rightarrow \mathcal{S}$  be a functor,  $A_1 \subset f(l_1)$ ,  $A_2 \subset f(l_0)$ ,  $[f(\gamma_1)](A_1) \cap [f(\gamma_1 \circ \lambda_1)](A_2) \neq \emptyset$ . Then there exists an idempotent endotransformation  $\tau: f \rightarrow f$  such that (+) holds and  $\tau^3$  is constant.

Proof. Denote by  $\tilde{f}$  the domain-restriction of  $f$  to  $\mathcal{I}$  and consider the graph  $G_{\tilde{f}}$ . Notice that

$$[f(\gamma_1)](x_0) = [f(\gamma_2)](x_1) = [f(\gamma_1)](x_2) = [f(\gamma_2)](x_3) = \dots$$

for every path  $x_0, x_1, x_2, x_3, \dots$  in  $G_f$  with  $x_0 \in f(l_1)$ . Use this fact and recall the construction of the transformation  $\tilde{\tau}: \tilde{f} \rightarrow \tilde{f}$  satisfying (+) (where we write  $\tilde{\tau}, \tilde{f}$  instead of  $\tau, f$ ) in Lemma 5.

DEFINITION 7. We say that a small thin category  $k$  has the form  $a_7$  (see Fig. 10) iff

$$(7) \quad k^o = \{a_1, a_2\} \cup \{l_0, l_1, l_2, l_3\} \cup D \cup M \cup B$$

so that

(7a) the sets on the right-hand side of (7) are disjoint;

(7b)  $a_1, a_2$  are two distinct minimal elements of  $k$ ;

(7c)  $l_0 < l_1 < l_3, l_0 < l_2 < l_3, l_1 \# l_2$ ;

(7d)  $a_1 < l_i, i = 0, \dots, 3; a_2 < l_1, l_3; a_2 \# l_0, l_2$ ;

(7e)  $d > a_1$  for every  $d \in D$ ; if  $d \in D, x \in k^o - (D \cup \{a_1\})$ , then  $x \# d$ ;

(7f)  $a_1 < m < l_3$  for every  $m \in M$ ; if  $m \in M, x \in k^o - (M \cup \{a_1, l_3\})$ , then  $x \# d$ ;

(7g)  $a_2 < b < l_3$  for every  $b \in B$ ; if  $b \in B, x \in k^o - (B \cup \{a_2, l_3\})$ , then  $x \# d$ ;

(7h)  $B$  is linearly ordered.

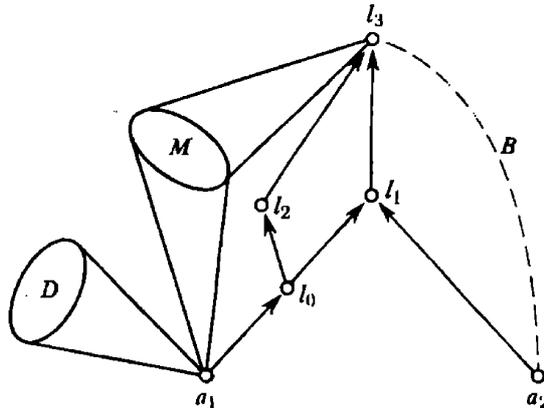


Fig. 10

Note. Some of the sets  $D, M, B$  may be empty. The following proposition and its proof remain true if we do not suppose that  $B$  is linearly ordered.

PROPOSITION 7. Let a small thin category  $k$  have the form  $a_7$ . Then  $Z(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

Proof. Let  $f: k \rightarrow \mathcal{S}$  satisfy (a), (b) from the definition of  $Z(f)$ . Put

$$A_1 = \left[ f \left( \begin{smallmatrix} a_2 \\ l_1 \end{smallmatrix} \right) \right] (f(a_2)), \quad A_2 = \left[ f \left( \begin{smallmatrix} a_1 \\ l_0 \end{smallmatrix} \right) \right] (f(a_1)).$$

Then  $A_1, A_2$  are non-void sets.

Denote by  $\tilde{f}$  the domain-restriction of  $f$  to  $k(l_0, l_1, l_2, l_3) \simeq c$ . Since  $f$  is connected,

$$\left[ f \begin{pmatrix} l_1 \\ l_3 \end{pmatrix} \right] \circ \left[ f \begin{pmatrix} l_0 \\ l_1 \end{pmatrix} \right] (A_2) \cap \left[ f \begin{pmatrix} l_1 \\ l_3 \end{pmatrix} \right] (A_1) \neq \emptyset.$$

Then (Lemma 6) there exists an idempotent transformation  $\tilde{\tau}: \tilde{f} \rightarrow \tilde{f}$  satisfying (+) such that  $\tilde{\tau}^3$  is constant. Clearly,  $\tilde{\tau}$  can be extended to  $\tau: f \rightarrow f$  such that  $\tau^o$  is constant for every  $o \in k^o - \{l_0, l_1, l_2, l_3\}$ . Then either  $\tau \neq 1$ , or  $\tau = 1$ , and  $f$  is rigid. Thus,  $\mathbf{Z}(f)$  does not hold.

DEFINITION 8. We say that a small thin category  $k$  has the form  $\alpha_8$  (see Fig. 11) iff

$$(8) \quad k^o = \{a_1, a_2\} \cup \{l_0, l_1, l_2\} \cup D_1 \cup D_2$$

so that

(8a) the sets on the right-hand side of (8) are disjoint;

(8b)  $a_1, a_2$  are two distinct minimal objects in  $k$ ;

(8c)  $l_0 < l_1, l_2$ ;  $l_1 \# l_2$ ;

(8d)  $a_1 < l_i, i = 0, 1, 2$ ;  $a_2 < l_1, a_2 \# l_0, l_2$ ;

(8e)  $a_i < d$  for every  $d \in D_i$ ; if  $d \in D_i, x \in k^o - (D_i \cup \{a_i\})$ , then  $x \# d$  ( $i = 1, 2$ ).

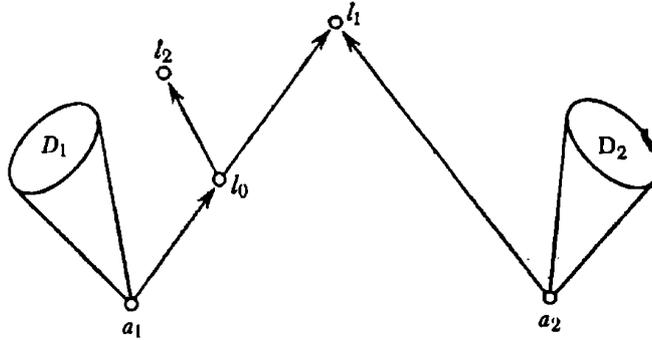


Fig. 11

PROPOSITION 8. Let a small thin category  $k$  have the form  $\alpha_8$ . Then  $\mathbf{Z}(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

Proof. Let  $f: k \rightarrow \mathcal{S}$  satisfy (a), (b). Denote by  $\tilde{f}$  the domain-restriction of  $f$  to  $k(l_0, l_1, l_2) \simeq l$ , put

$$A_1 = \left[ f \begin{pmatrix} a_2 \\ l_1 \end{pmatrix} \right] (f(a_2)), \quad A_2 = \left[ f \begin{pmatrix} a_1 \\ l_0 \end{pmatrix} \right] (f(a_1))$$

and use Lemma 5.

DEFINITION 9. We say that a small thin category  $k$  has the form  $\alpha_9$  (see Fig. 12) iff

$$(9) \quad k^o = \{a_1, a_2\} \cup \{l_0, l_1, l_2\} \cup D_1 \cup D_2,$$

so that

(9a) the sets on the right-hand side of (9) are disjoint;

(9b)  $a_1, a_2, l_0$  are three distinct minimal objects of  $k$ ;

(9c)  $a_1 < l_1 > l_0, l_0 < l_2 > a_2, l_1 \# l_2$ ;

(9d)  $d > a_i$  for every  $d \in D_i$ ; if  $d \in D_i, x \in k^o - (D_i \cup \{a_i\})$ , then  $x \# d, i = 1, 2$ .

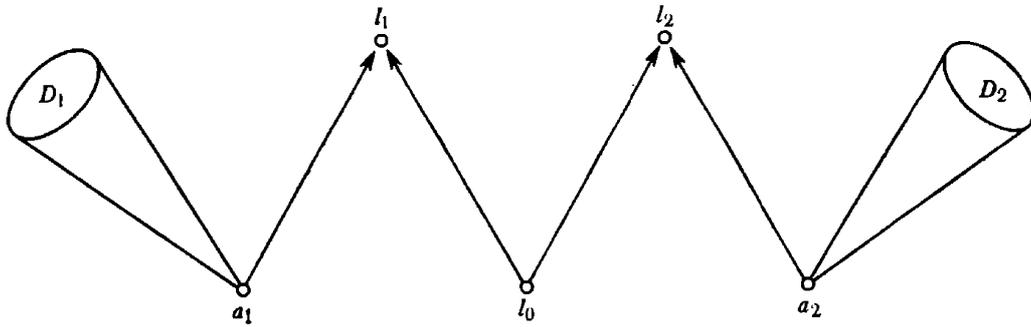


Fig. 12

PROPOSITION 9. Let a small thin category  $k$  have the form  $\alpha_9$ . Then  $\mathbf{Z}(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

Proof. Denote by  $f$  the domain-restriction of  $f$  to  $k(l_0, l_1, l_2) \simeq \mathbf{l}$ , put

$$A_i = \left[ f \begin{pmatrix} a_i \\ l_i \end{pmatrix} \right] (f(a_i)), \quad i = 1, 2$$

and use Lemma 5.

DEFINITION 10. We say that a small thin category  $k$  has the form  $\alpha_{10}$  (see Fig. 13) iff

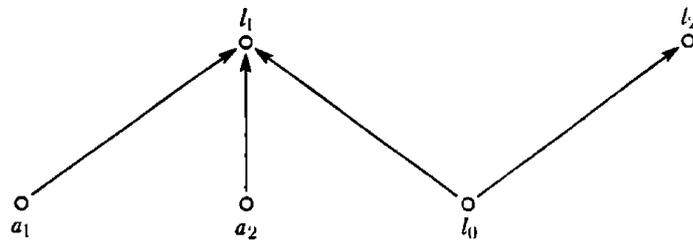


Fig. 13

(10)  $k^o = \{a_1, a_2, l_0, l_1, l_2\}$  and

(10a)  $a_1, a_2, l_0$  are three distinct minimal objects of  $k$ ;

(10b)  $a_1, a_2, l_0 < l_1, l_0 < l_2$  and  $x \# l_2$  whenever  $x \in \{a_1, a_2, l_1\}$ .

PROPOSITION 10. Let a small thin category  $k$  have the form  $a_{10}$ . Then  $\mathbf{Z}(f)$  does not hold for any  $f: k \rightarrow \mathcal{S}$ .

Proof. Let  $k$  have the form  $a_{10}$ , let  $f: k \rightarrow \mathcal{S}$  be a functor satisfying (a), (b) from  $\mathbf{Z}(f)$ . Put

$$A_i = \left[ f \begin{pmatrix} a_i \\ l_1 \end{pmatrix} \right] (f(a_i)), \quad L = \left[ f \begin{pmatrix} l_0 \\ l_1 \end{pmatrix} \right] (f(l_0)).$$

Since  $f$  is connected, necessarily  $L = f(l_1)$ .

1) Let  $A_1 \cap A_2 \neq \emptyset$ : Choose  $x_{l_1} \in A_1 \cap A_2$ ,  $x_{a_i}, x_{l_0}, x_{l_2}$  such that

$$\left[ f \begin{pmatrix} a_i \\ l_1 \end{pmatrix} \right] (x_{a_i}) = \left[ f \begin{pmatrix} l_0 \\ l_1 \end{pmatrix} \right] (x_{l_0}) = x_{l_1}, \quad x_{l_2} = \left[ f \begin{pmatrix} l_0 \\ l_2 \end{pmatrix} \right] (x_{l_0})$$

and define  $\tau: f \rightarrow f$  such that  $\tau^o = x_o$  for every  $o \in k^o$ .

2) Let  $A_1 \cap A_2 = \emptyset$ ; use Lemma 5.

DEFINITION 11. We say that a small thin category  $k$  is a cone (see Fig. 14) iff

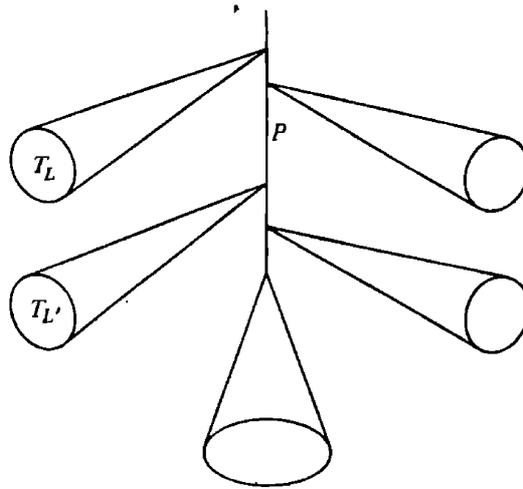


Fig. 14

$$(11) \quad k^o = P \cup \bigcup_{i \in I} T_i,$$

so that

(11a) the sets on the right-hand side of (11) are disjoint;

(11b)  $P$  is linearly ordered non-void;

(11c)  $I$  is the set of all cuts  $\iota = (A_i, B_i)$  in  $P$  such that  $A_i \neq \emptyset$  (the cut  $(P, \emptyset)$  is included);

(11d) if  $t \in T_i$ , then  $t < a$ ,  $t \# b$  for every  $a \in A_i$ ,  $b \in B_i$ ;

(11e) if  $t \in T_i$ ,  $t' \in T_{i'}$ ,  $i \neq i'$ , then  $t \# t'$ .

## V. Classification of thin categories

The aim of the present part is to prove the following proposition.

**PROPOSITION.** *Let  $k$  be a connected skeletal thin category,  $k^\circ \neq \emptyset$ . Then  $k$  is a cone or it has some of the forms  $a_1, \dots, a_{10}$  or  $k_i \subset k$  for some  $i = 1, \dots, 35$ .*

**Remark.** We recall that a category  $k$  is said to be *skeletal* if two objects are identical provided they are isomorphic.

**Proof.**

A) Let  $k$  contain a chain without a lower bound. Denote by  $R$  a maximal chain without a lower bound. Then either  $k$  is a cone or the class  $k^\circ - R$  is non-empty. We shall consider the latter. Since  $k$  is connected, one can find a path from  $R$  to  $o$  for every  $o \in k^\circ - R$ . Fix the shortest one, say  $p_o = \{a_0^\circ, \dots, a_n^\circ\}$ , where  $a_0^\circ \in R$ ,  $a_n^\circ = o$ , put  $l(p_o) = n$ .

1) Let there exist  $o \in k^\circ - R$  such that  $l(p_o) \geq 2$ . One from the following two cases must occur:

I.  $a_1^\circ < a_0^\circ$ : Then necessarily  $a_1^\circ < a_2^\circ$ ,  $a_1^\circ \notin R$ ,  $a_2^\circ \notin R$ ,  $a_0^\circ \neq a_2^\circ$ . Put  $R_0 = \{r \in R; r < a_0^\circ\}$ . Since  $a_1^\circ$  is not a lower bound of  $R$ , either there exists  $r_0 \in R_0$  such that  $r_0 < a_1^\circ$  and then  $k_{33} \subset k$  or  $r \neq a_1^\circ$  for every  $r \in R_0$ . Then  $a_2^\circ < r$  for no  $r \in R_0$ . If  $r_0 < a_2^\circ$  for some  $r_0 \in R_0$ , then  $k(a_0^\circ, a_1^\circ, a_2^\circ, r_0) \simeq k_1$ . If  $r \neq a_2^\circ$  for every  $r \in R_0$ , then  $k_{32} \subset k$ .

II.  $a_1^\circ > a_0^\circ$ : Then necessarily  $a_1^\circ \notin R$ . Since  $R$  is maximal, there exists  $r_0 \in R$  such that  $a_1^\circ \neq r_0$ . Then  $k_{33} \subset k$ .

2)  $l(p_o) = 1$  for every  $o \in k^\circ - R$ :

I. Let there exist  $o \in k^\circ - R$  such that  $r_0 < o$  for some  $r_0 \in R$ . Then  $k_{33} \subset k$  (see the case 1.II).

II. Let I does not hold. Then every  $o \in k^\circ - R$  determines the cut  $\iota_o = (A_o, B_o)$  in  $R$  such that  $A_o = \{r \in R; o < r\}$ ,  $B_o = \{r \in R; r \neq o\}$ . Then either  $k$  is a cone or there exist  $o, o' \in k^\circ - R$  such that  $\iota_o \neq \iota_{o'}$  and  $o, o'$  are not incomparable, say  $o < o'$ . Then  $A_o \cap B_{o'} \neq \emptyset$  and  $o' \neq r$  for every  $r \in A_o \cap B_{o'}$ . Thus  $k_{33} \subset k(\{o, o'\} \cup B_{o'})$ .

B) *Every chain in  $k$  has a lower bound.* Particularly, every object of  $k$  is greater than a minimal one. Then either  $k$  has the form  $a_1$  or it has at least two minimal elements.

I. Let  $k$  have exactly two minimal objects.

Denote them by  $a_1, a_2$ . Put

$$R = \{r \in k^o; (r > a_1) \ \& \ (r > a_2)\}.$$

$R$  is a non-empty class because  $k$  is connected. If there exist  $r_1, r_2 \in R$  such that  $r_1 \# r_2$ , then  $k(a_1, a_2, r_1, r_2) \simeq k_1$ . So we may suppose that  $R$  is linearly ordered.

a) Let  $R$  be not well ordered. Then there exist  $r_i \in R, r_{i+1} < r_i, i = 1, 2, \dots$ . Then either  $k$  is a cone or there exists  $o \in k^o, o \# r_1$ . Then  $o \# r_i$  for all  $i = 1, 2, \dots$  and either  $a_1 < o$  or  $a_2 < o$ . Consequently,  $k_{35} \dot{\subset} k$ .

b) Let  $R$  be well ordered. A lot of cases must be considered now. Denote by  $b$  the smallest element of  $R$  and put ( $i = 1, 2$ )

$$\begin{aligned} A_i &= \{o \in k^o; a_i < o < b\}, \\ T_i &= \{o \in k^o - R; (\exists r \in R)(a_i < o < r)\}, \\ D_i &= \{o \in k^o; a_i < o\} - (R \cup T_i), \\ L_i &= T_i - A_i. \end{aligned}$$

We consider the following cases (the lexicographic enumeration for cases is used):

1.  $|R| \geq 2, D_1 \neq \emptyset, D_2 = \emptyset$

Now we prove: either  $k_i \dot{\subset} k$  for some  $i = 1, \dots, 35$  or  $k$  has the form  $a_2$  or  $a_3$  or  $a_4$  or  $a_7$ .

Let  $d$  denote an arbitrary but fixed element of  $D_1$ .

1.1.  $A_2 \neq \emptyset$ . Then  $k\{d, a_1, b, a_2, x, r\} \simeq k_{25}$ , where  $x \in A_2, r \in R - \{b\}$ .

1.2.  $A_2 = \emptyset$ .

1.2.1. The class  $T_2 = L_2$  is not linearly ordered. Choose  $t, t' \in T_2, t \# t'$ . Denote by  $r_t$  or  $r_{t'}$  the smallest element of  $R$  such that  $t < r_t$  or  $t' < r_{t'}$ , respectively. If  $r_t = r_{t'}$ , then  $k\{d, a_1, b, r_t, t, t'\} \simeq k_{10}$ . If  $r_t \neq r_{t'}$ , say  $r_t < r_{t'}$ , then  $k\{d, a_1, a_2, t, r_t, r_{t'}\} \simeq k_{26}$ .

1.2.2. The class  $T_2 = L_2$  is linearly ordered. For every  $t \in T_2$  denote by  $r_t$  the smallest element of the set  $R$  such that  $t < r_t$ .

1.2.2.1. There exists  $t_0 \in T_2$  such that  $r_{t_0}$  is not the greatest element of  $R$ : Then

$$k\{d, a_1, a_2, t_0, r_{t_0}, x\} \simeq k_{25}, \quad \text{where } x \in R, x > r_{t_0}.$$

1.2.2.2.  $T_2$  has no element  $t_0$  with the property from 1.2.2.1. Then either  $R$  has the greatest element, say  $f$ , and  $t \# r$  for every  $t \in T_2, r \in R, r < f$ , or  $T_2 = \emptyset$ .

1.2.2.2.1. There exist  $d_0 \in D_1$  and  $t_0 \in T_1$  such that  $t_0 < d_0$ . If  $t_0 \in A_1$ , then  $k\{a_1, t_0, d_0, b, a_2, x\} \simeq k_6$ , where  $x \in R$ ,  $x \neq b$ ; if  $t_0 \in L_1$ , then  $t_0 \# b$ , consequently

$$k\{a_1, t_0, d_0, b, a_2, x\} \simeq k_{17}, \quad \text{where } x \in R, x > t_0.$$

1.2.2.2.2. There exist no  $d_0 \in D_1$  and  $t_0 \in T_1$  such that  $t_0 < d_0$ . Consequently,  $(\forall d \in D_1)(\forall t \in T_1)(d \# t)$ . For every  $t \in T_1$  denote by  $r_t$  the smallest element of  $R$  such that  $t < r_t$ . Put

$$F = \{r_t; t \in T_1\}.$$

We consider the following four cases for the power of  $F$ :

1.2.2.2.2.1.  $\text{card } F \geq 3$ : Let  $m, n, p \in F$ ,  $m < n < p$ . Choose  $t \in T_1$  such that  $r_t = m$ . Then

$$k\{d, a_1, t, m, n, p, a_2\} \simeq k_{11}.$$

1.2.2.2.2.2.  $\text{card } F = 2$ : Let  $F = \{p, q\}$ ,  $p < q$ . If there exists  $r \in R$  such that either  $r > q$  or  $p < r < q$ , then

$$k\{d, a_1, t, p, q, a_2, r\} \simeq k_{11}, \quad \text{where } t \in T_1, r_t = p.$$

Now let  $q$  be the greatest element of  $R$  and there is no  $r \in R$  between  $p$  and  $q$ . Put

$$Q = \{t \in T_1; r_t = q\}, \quad P = \{t \in T_1; r_t = p\}.$$

Then  $P \neq \emptyset$ ,  $Q \neq \emptyset$ .

1.2.2.2.2.2.1.  $P$  is not linearly ordered: Let  $p_1, p_2 \in P$ ,  $p_1 \# p_2$ . Then

$$k\{d, a_1, p_1, p_2, p, q, a_2\} \simeq k_{12}.$$

1.2.2.2.2.2.2.  $P$  is linearly ordered but not well ordered: Choose  $p_i \in P$ ,  $p_{i+1} < p_i$ ,  $i = 1, 2, \dots$ . Then

$$k\{d, a_1, a_2, p, q\} \cup \{p_i; i = 1, 2, \dots\} \simeq k_{34}.$$

1.2.2.2.2.2.3.  $P$  is well ordered.

1.2.2.2.2.2.3.1. There exist  $z_1, z_2 \in Q$ ,  $s \in P$  such that  $s < z_1$ ,  $s < z_2$ ,  $z_1 \neq z_2$ : If  $z_1 \# z_2$ , then

$$k(a_1, s, z_1, z_2, p, a_2) \simeq k_{28}.$$

If  $z_1 < z_2$ , then

$$k(a_1, s, z_1, z_2, p, a_2) \simeq k_3.$$

1.2.2.2.2.2.3.2. For every  $s \in P$  there exists at most one  $z_s \in Q$  such that  $s < z_s$ : Then either  $k$  has the form  $\alpha_2$  or the form  $\alpha_3$  or a  $z_{s_0}$  really

exists for some  $s_0 \in P$ . Consider the last case. If  $P = \{s_0\}$ , then either  $p > b$  and  $k(a_1, b, a_2, p, s_0, z_{s_0}) \simeq k_{17}$  or  $k$  has the form  $a_7$ . If there exists  $s \in P - \{s_0\}$ , then

$$k(a_1, s_0, z_{s_0}, s, p, a_2) \simeq k_{20} \quad \text{for } s < s_0$$

and

$$k(a_1, s_0, z_{s_0}, s, p, a_2) \simeq k_7 \quad \text{for } s_0 < s.$$

1.2.2.2.2.3.  $\text{card } F = 1$ : Put  $F = \{p\}$ .

1.2.2.2.2.3.1. There exist  $r_1, r_2 \in R$  such that  $p < r_1 < r_2$ : Then

$$k(d, a_1, t, p, r_1, r_2, a_2) \simeq k_{11}, \quad \text{where } t \in T_1.$$

1.2.2.2.2.3.2. There exists exactly one element of  $R$  greater than  $p$ : Denote it by  $r$ . If there exist  $t_1, t_2 \in T_1$ ,  $t_1 \neq t_2$ , then

$$k(d, a_1, t_1, t_2, p, r, a_2) \simeq k_{12}.$$

Now, let  $T_1$  be linearly ordered. If  $T_1$  is well ordered, then  $k$  has the form  $a_2$ . Now, let  $T_1$  be not well ordered. Choose  $t_i \in T_1$ ,  $t_{i+1} < t_i$ ,  $i = 1, 2, \dots$ . Then

$$k(\{t_i; i = 1, 2, \dots\} \cup \{d, a_1, a_2, r, p\}) \simeq k_{34}.$$

1.2.2.2.2.3.3.  $p$  is the last element of  $R$ : Then  $k$  has the form  $a_4$ .

1.2.2.2.2.4.  $F = \emptyset$ : Then  $T_1 = \emptyset$  and  $k$  has the form  $a_4$ .

2.  $|R| \geq 2$ ,  $D_1 \neq \emptyset$ ,  $D_2 \neq \emptyset$

Put  $l_1 = k(k^\circ - D_1)$ ,  $l_2 = k(k^\circ - D_2)$ . Now use the case 1. Then  $l_i$  ( $i = 1, 2$ ) either contains some  $k_j$  ( $j = 1, 2, \dots, 35$ ) as a full subcategory or it has the form  $a_2$  or  $a_3$  or  $a_4$  or  $a_7$ . If both categories  $l_1, l_2$  have some of the forms  $a_2, a_3, a_4, a_7$ , then they have necessarily the form  $a_4$  and then  $k$  has the form  $a_5$ .

3.  $|R| \geq 2$ ,  $D_1 = \emptyset = D_2$ . For every  $t \in T_1 \cup T_2$  define  $r_t$  as in 1.2.1. Then either  $k$  is a cone or there exist  $t, t' \in T_1 \cup T_2$  such that  $t < t'$ ,  $b < r_t < r_{t'}$ . Then

$$k(a_1, a_2, b, t, r_t, t') \simeq k_{17}.$$

4.  $|R| = 1$ . Then  $R = \{b\}$ . Put

$$V_i = \{o \in D_i; \exists p \in A_i, p < o\}, \quad i = 1, 2.$$

4.1.  $V_1 \neq \emptyset \neq A_2$  (or  $V_2 \neq \emptyset \neq A_1$ ): Choose  $v \in V_1$ ;  $p \in A_1$ ,  $p < v$ ,  $a \in A_2$ . Then

$$k(a_1, a_2, v, p, a, b) \simeq k_{19}.$$

4.2.  $V_1 \neq \emptyset = A_2$  (or  $V_2 \neq \emptyset = A_1$ ):

4.2.1.  $|V_1| > 1$ : Choose  $v, v' \in V_1$ ,  $v \neq v'$ ,  $q, q' \in A_1$ ,  $v > q$ ,  $v' > q'$ . If  $v < v'$ , then

$$k(a_1, a_2, b, q, v, v') \simeq k_3.$$

Let  $v \# v'$ . If  $q \# q'$ , then

$$k(a_1, a_2, b, q, q', v) \simeq k_{16}.$$

If either  $q < q'$  or  $q = q'$ , then

$$k(a_1, a_2, q, v, v', b) \simeq k_{28}.$$

4.2.2.  $|V_1| = 1$ : Put  $V_1 = \{v\}$ .

4.2.2.1.  $|A_1| > 1$ : Choose  $a, q \in A_1$ ,  $q < v$ ,  $a \neq q$ . Consider the category  $h = k(a_1, a_2, b, a, q, v)$ . If  $a < q$ , then  $h \simeq k_{20}$ . If  $q < a$ , then either  $a < v$  and then  $h \simeq k_{20}$  or  $a \# v$  and then  $h \simeq k_7$ . Now, let  $a \# q$ . If  $a \# v$ , then  $h \simeq k_{18}$ , if  $a < v$ , then

$$k(q, a, v, b) \simeq k_1.$$

4.2.2.2.  $|A_1| = 1$ : Put  $A_1 = \{a\}$ . Then either  $k$  has the form  $a_8$  or there exists  $d \in D_1$  such that  $d < v$ . Then  $d \notin V_1$ , consequently  $d \# a$ . Then

$$k(a_1, a_2, b, v, d, a) \simeq k_{23}.$$

4.3.  $V_1 = V_2 = \emptyset$ : Then  $k$  has the form  $a_6$ .

II.  $k$  has at least three minimal objects.

Denote by  $M$  the set of all minimal objects of  $k$ . If  $a, a' \in M$ ,  $a \neq a'$ , put  $M_{a,a'} = \{o \in k^0; (o > a) \ \& \ (o > a')\}$ . If there exist  $a, a' \in M$ ,  $a \neq a'$  such that  $s \# t$  for some  $s, t \in M_{a,a'}$ , then  $k(a, a', s, t) \simeq k_1$ . We shall suppose in the sequel that

(\*) for every  $a, a' \in M$ ,  $a \neq a'$ , the set  $M_{a,a'}$  is linearly ordered.

1. Let there exist different  $a_1, a_2, a_3 \in M$  and  $b \in k^0$  such that  $a_i < b$  for all  $i = 1, 2, 3$ . Since  $k$  is connected, for every  $o \in k^0 - \{b\}$  there exists a path from  $b$  to  $o$ . If  $p_o = \{c_0, \dots, c_{n+1}\}$  is such a path with the minimal length, then  $c_0 = b$ ,  $c_{n+1} = o$ ,  $|i-j| > 1 \Rightarrow c_i \# c_j$ .

1.1. Let there exist  $o \in k^0 - \{b\}$  such that  $c_0 > c_1 < c_2$  for some shortest path  $p_o = \{c_0, c_1, c_2, \dots, c_{n+1}\}$  from  $b$  to  $o$ .

1.1.1. Let  $c_1 = a_i$  for some  $i = 1, 2, 3$ , say  $c_1 = a_1$ . Then  $h = k(a_1, a_2, a_3, b, c_2)$  has the form  $a_{10}$ . If  $k \neq h$ , then  $k^0 - h^0 \neq \emptyset$ . We suppose this is the case throughout the 1.1.1.

1.1.1.1. Let there exist  $t \in k^0 - h^0$  comparable to at least one element of  $h^0$  but not to all of them. Put  $l = k(a_1, a_2, a_3, b, c_2, t)$ . Since  $a_i, i = 1, 2, 3$ , are minimal objects and since (\*) is supposed, the following cases are the only possible:

1.1.1.1.1.  $a_i \# t$ ,  $i = 1, 2, 3$ .

1.1.1.1.1.1.  $t < c_2$ ,  $t \# b$ ; then  $l \simeq k_{21}$ ;

1.1.1.1.1.2.  $t < c_2$ ,  $t < b$ ; then  $k(a_1, t, c_2, b) \simeq k_1$ ;

1.1.1.1.1.3.  $t \# c_2$ ,  $t < b$ ; then  $l \simeq k_{13}$ .

1.1.1.1.2. Put  $\{2, 3\} = \{m, n\}$ . Let  $a_i \# t$ ,  $i = 1, m$ ,  $t > a_n$ .

1.1.1.1.2.1.  $t \# c_2$ ,  $t \# b$ ; then  $l \simeq k_{26}$ ;

1.1.1.1.2.2.  $t \# c_2$ ,  $t < b$ ; then  $l \simeq k_{14}$ ;

1.1.1.1.3.  $t > a_1$ ,  $t \# a_i$ ,  $i = 2, 3$ .

1.1.1.1.3.1.  $t \# c_2$ ,  $t \# b$ ; then  $l \simeq k_{29}$ ;

1.1.1.1.3.2.  $t \# b$  and either  $t < c_2$  or  $c_2 < t$ ; then  $l \simeq k_2$ ;

1.1.1.1.3.3.  $t < b$ ,  $t \# c_2$ ; then  $l \simeq k_{10}$ ;

1.1.1.1.3.4.  $t < b$ ,  $t < c_2$ ; then  $l \simeq k_{18}$ .

1.1.1.1.4.  $t \# a_1$ ,  $a_i < t$ ,  $i = 2, 3$ ; then (\*) implies  $t < b$ . Since necessarily  $t \# c_2$ ,  $l \simeq k_{15}$ .

1.1.1.1.5. Put  $\{2, 3\} = \{m, n\}$ . Let  $t \# a_m$ ,  $t > a_1$ ,  $t > a_n$ . Then  $t < b$ ,  $t \# c_2$  and  $l \simeq k_9$ .

1.1.1.1.6.  $a_i < t$ ,  $i = 1, 2, 3$ . Then  $t$  is not incomparable to  $b$ . Since  $t$  is not comparable to all elements of  $h^o$ , necessarily  $t \# c_2$ . Then  $l \simeq k_8$ .

1.1.1.2. There exists no  $t \in k^o - h^o$  with the property from 1.1.1.1. So, if  $t \in k^o - h^o$ , then either  $t$  is comparable to all objects of  $h$  or it is incomparable to all objects of  $h$ . If  $t \in k^o - h^o$  is comparable to all objects of  $h$ , then  $t > x$  for all  $x \in h^o$ . Put

$$A = \{t \in k^o - h^o; t > x \text{ for all } x \in h^o\}.$$

(\*) implies that  $A$  is linearly ordered. If  $s \in k^o - (h^o \cup A)$ , then  $s \# x$  for all  $x \in h^o$ . Since  $k$  is connected, either  $k$  is a cone or there exist  $d_1, d_2, d_3$  such that  $d_1 \in A$ ,  $d_2, d_3 \in k^o - (h^o \cup A)$ ,  $d_1 > d_2$ ,  $d_2 < d_3$ ,  $d_1 \# d_3$ . Then

$$k(d_1, d_2, d_3, a_1, a_2, a_3) \simeq k_{13}.$$

1.1.2. Let  $c_1 \neq a_i$ ,  $i = 1, 2, 3$ .

1.1.2.1. Let  $c_1 > a_i$  for some  $i = 1, 2, 3$ , say  $c_1 > a_1$ . Since  $b \# c_2 > c_1$ , (\*) implies  $a_2 \# c_1$ ,  $a_3 \# c_1$ . Then

$$k(a_1, a_2, a_3, b, c_1, c_2) \simeq k_{18}.$$

1.1.2.2. Let  $c_1 \# a_i$ ,  $i = 1, 2, 3$ . If  $a_i \# c_2$ , then

$$k(a_1, a_2, a_3, b, c_1, c_2) \simeq k_{13};$$

if  $a_j < c_2$  for some  $j$ , then

$$k(a_j, b, c_1, c_2) \simeq k_1.$$

1.2. Let there exist  $o \in k^o - \{b\}$  such that  $c_0 < c_1 > c_2 < c_3$  for some path with the minimal length  $p_o = \{c_0, c_1, c_2, c_3, \dots, c_{n+1}\}$ . Then nece-

ssarily  $c_2 \# a_i$ ,  $c_3 \# a_i$  for all  $i = 1, 2, 3$ . Then

$$k(a_1, a_2, a_3, c_1, c_2, c_3) \simeq k_{13}.$$

1.3. For no  $o \in k^\circ - \{b\}$  there exists a path with the minimal length from  $b$  to  $o$  with the property from 1.1 or 1.2. So either  $p_o = \{c_0, c_1\}$  or  $p_o = \{c_0, c_1, c_2\}$ ,  $c_0 < c_1 > c_2$  for all  $o \in k^\circ - \{b\}$  and all paths with the minimal length from  $b$  to  $o$ . Put  $A = \{x \in k^\circ; x \geq b\}$ . (\*) implies that  $A$  is linearly ordered. Clearly, for every  $o \in k^\circ$  there exists  $t_o \in A$  such that  $o \leq t_o$ . Then either  $k$  is a cone or there exist  $o, o' \in k^\circ - (A \cup \{a_1, a_2, a_3\})$ ,  $t_o, t_{o'} \in A$  such that  $o < t_o$ ,  $o' < t_{o'}$ ,  $o < o'$ ,  $t_o < t_{o'}$  and  $o' \# t_o$ . If  $o > a_i$  for some  $i = 1, 2, 3$ , then  $\{b, a_i, o\}$  is a path with the minimal length from  $b$  to  $o$  with the property from 1.1, which is impossible. So  $o \# a_i$  for all  $i = 1, 2, 3$ . Analogously  $o' \# a_i$  for all  $i = 1, 2, 3$ . Then

$$k(a_1, a_2, a_3, o, t_o, o') \simeq k_{13}.$$

2. Let there exist] no  $b \in k^\circ$  and distinct  $a_1, a_2, a_3 \in M$  with  $a_i < b$ ,  $i = 1, 2, 3$ . Define a symmetric graph  $R$  on  $M$  as follows:  $(a, a') \in R \Leftrightarrow M_{a, a'} \neq \emptyset$ . Since  $k$  is connected,  $R$  is a connected graph.

2.1. Let  $\text{card } M = 3$ , say  $M = \{a_1, a_2, a_3\}$ . Then  $\text{card } R \in \{2, 3\}$ .

2.1.1. Let  $\text{card } R = 3$ : for every  $i, j = 1, 2, 3$ ,  $i \neq j$  choose  $a_{i, j} \in M_{a_i, a_j}$ . Then  $k(a_1, a_2, a_3, a_{1,2}, a_{2,3}, a_{3,1}) \simeq k_{27}$ .

2.1.2. Let  $\text{card } R = 2$ : say  $R = \{(a_1, a_2), (a_2, a_3)\}$ . Choose  $a_{1,2} \in M_{a_1, a_2}$ ,  $a_{2,3} \in M_{a_2, a_3}$ .

2.1.2.1. Let there exist  $o \in k^\circ$ ,  $o > a_2$ ,  $o \neq a_{1,2}$ ,  $o \neq a_{2,3}$ . Put  $l = k(a_1, a_2, a_3, a_{1,2}, a_{2,3}, o)$ . The following cases are the only possible:

2.1.2.1.1.  $o \# a_{1,2}$ ,  $o \# a_{2,3}$ . (\*) implies  $o \# a_1$ ,  $o \# a_3$  and then  $l \simeq k_{31}$ ;

2.1.2.1.2. Put  $\{m, n\} = \{1, 3\}$ . Let  $o \# a_{m,2}$ ,  $o < a_{n,2}$ . (\*) implies  $o \# a_m$ . If  $o \# a_n$ , then  $l \simeq k_5$ . If  $o > a_n$ , then  $l \simeq k_4$ ;

2.1.2.1.3. Let  $o < a_{1,2}$ ,  $o < a_{2,3}$ . If  $o > a_1$ , then  $a_{2,3} > a_i$  for all  $i = 1, 2, 3$ , which is impossible. So,  $o \# a_1$  and analogously  $o \# a_3$ . Then  $l \simeq k_{24}$ ;

2.1.2.1.4. Put  $\{m, n\} = \{1, 3\}$ , let  $o \# a_{n,2}$ ,  $o > a_{m,2}$ . Then  $o \# a_n$  and  $l \simeq k_4$ .

2.1.2.2. Let there exist  $o \# a_2$  such that either  $a_1 < o < a_{1,2}$  or  $a_3 < o < a_{2,3}$ . Then

$$k(a_1, a_2, a_3, a_{1,2}, a_{2,3}, o) \simeq k_{22}.$$

2.1.2.3. Let neither 2.1.2.1 nor 2.1.2.2 be satisfied. Then  $k$  has the form  $a_9$ .

2.2. Let card  $M > 3$ :

2.2.1. Let there exist  $a_0, a_1, a_2, a_3 \in M$  such that  $(a_0, a_i) \in R$  for all  $i = 1, 2, 3$ . Choose  $a_{0,i} \in M_{a_0, a_i}$ ,  $i = 1, 2, 3$ . Then

$$k(a_0, a_1, a_2, a_{0,1}, a_{0,2}, a_{0,3}) \simeq k_{31}.$$

2.2.2. Let 2.2.1 be not satisfied. Since the graph  $(M, R)$  is connected there exist  $a_0, a_1, a_2, a_3$  such that  $(a_i, a_{i+1}) \in R$ ,  $i = 0, 1, 2$  and such that

$$k(a_0, a_1, a_2, a_3, a_{0,1}, a_{1,2}, a_{2,3}) \simeq k_{30},$$

where  $a_{i,j} \in M_{a_i, a_j}$ .

## VI. Operations $o_1, o_2$ and reducing of a category by these operations

### 1. The operation $o_1$ .

**DEFINITION 1.** We write  $r(k, f, \tau)$  whenever  $k$  is a small thin category,  $f: k \rightarrow \mathcal{S}$  is a functor,  $\tau: f \rightarrow f$  is a transformation such that  $\tau \neq 1_f$ ,  $\tau \circ \tau = 1_f$  and  $\tau, 1_f$  are the only endotransformations of  $f$ .

**CONVENTION.** If  $k$  is a cone, we always suppose that one expression of  $k$  in the form

$$k^o = P \cup \bigcup_{i \in I} T_i$$

is chosen and kept in the following (where (11a), ..., (11e) from Definition 11 of Part IV are satisfied).

**LEMMA 1.** *Let  $r(k, f, \tau)$ , let  $k$  be a cone. Then  $\tau^p(x) = x$  for every  $p \in P$ ,  $x \in f(p)$ .*

**Proof.** Let there exist  $p_0 \in P$ ,  $a_1, a_2 \in f(p_0)$  such that  $a_1 \neq a_2$ ,  $\tau(a_1) = a_2$ ,  $\tau(a_2) = a_1$ . Put  $a_i^p = \left[ f \left( \begin{smallmatrix} p_0 \\ p \end{smallmatrix} \right) \right] (a_i)$ ,  $i = 1, 2$ . Clearly,  $\tau(a_1^p) = a_2^p$ ,  $\tau(a_2^p) = a_1^p$ . Put  $P_0 = \{p \geq p_0; a_1^p \neq a_2^p\}$ . If  $o \in k^o$ , put

$$g(o) = \left\{ x \in f(o); (\exists p \in P_0) \left( \left[ f \left( \begin{smallmatrix} o \\ p \end{smallmatrix} \right) \right] (x) = a_1^p \right) \right\}.$$

Let  $o, o' \in k^o$ ,  $o < o'$ ,  $x \in f(o)$ ,  $y = \left[ f \left( \begin{smallmatrix} o \\ o' \end{smallmatrix} \right) \right] (x)$ .

One can easily see:

if  $x \notin g(o)$ , then  $y \notin g(o')$ ;

if  $x \in g(o)$ , then either  $y \in g(o')$  or  $\tau(y) = y$ .

Define  $\nu: f \rightarrow f$  by  $\nu^o(x) = \tau^o(x)$  whenever  $o \in k^o$ ,  $x \in g(o)$ ,  $\nu^o(x) = x$  otherwise. One can easily see that  $\nu$  is a transformation such that  $1_f \neq \nu \neq \tau$ .

**LEMMA 2.** *Let  $r(k, f, \tau)$ , let  $k$  be a cone. Then there exists exactly one cut  $i_0 = (A_{i_0}, B_{i_0}) \in I$  and exactly one collection  $\{z_p; p \in A_{i_0}\}$  such that*

- 1)  $z_p \in f(p)$ ,  $\left[ f \left( \begin{smallmatrix} p' \\ p \end{smallmatrix} \right) \right] (z_{p'}) = z_p$ ;
- 2) if  $o \in k^o$ ,  $x \in f(o)$ ,  $\tau(x) \neq x$ , then
  - a)  $o \in T_{i_0}$ ;
  - b)  $(\forall p \in A_{i_0}) \left( \left[ f \left( \begin{smallmatrix} o \\ p \end{smallmatrix} \right) \right] (x) = z_p \right)$ .

Proof. Lemma 1 implies  $\tau(x) = x$  for every  $x \in f(p)$ ,  $p \in P$ . Choose  $o \in k^o$ ,  $a \in f(o)$  such that  $\tau(a) \neq a$ . Then necessarily  $o \in T_{i_0}$  for some  $i_0 = (A_{i_0}, B_{i_0}) \in I$ . Put

$$z_p = \left[ f \begin{pmatrix} o \\ p \end{pmatrix} \right] (a).$$

1) If there exist  $i_1 \neq i_0$ ,  $q \in T_{i_1}$ ,  $b \in f(q)$  such that  $\tau(b) \neq b$ , define  $\alpha(x) = \tau(x)$  whenever  $x \in f(t)$ ,  $t \in T_{i_1}$  and  $\alpha(x) = x$  otherwise. Clearly,  $\alpha: f \rightarrow f$  is a transformation such that  $\alpha \neq \tau$ ,  $\alpha \neq 1_f$ , which is impossible.

2) If there exist  $q \in T_{i_0}$ ,  $b \in f(q)$  such that  $\tau(b) \neq b$ ,  $\left[ f \begin{pmatrix} q \\ p \end{pmatrix} \right] (b) \neq z_p$  for some  $p \in A_{i_0}$ , put  $\alpha(x) = \tau(x)$  whenever  $x \in f(t)$ ,  $t \in T_{i_0}$ ,  $\left[ f \begin{pmatrix} t \\ p \end{pmatrix} \right] (x) = \left[ f \begin{pmatrix} q \\ p \end{pmatrix} \right] (b)$  for all  $p \in A_{i_0}$ ,  $\alpha(x) = x$  otherwise. Clearly,  $\alpha: f \rightarrow f$  is a transformation such that  $\alpha \neq \tau$ ,  $\alpha \neq 1_f$ , which is impossible.

DEFINITION 2. Let  $r(k, f, \tau)$ , let  $k$  be a cone. Let  $i_0 = (A_{i_0}, B_{i_0})$ ,  $\{z_p; p \in A_{i_0}\}$  be as in Lemma 2. Define an operation  $o_1$  such that

$$o_1(k, f, \tau) = (k_1, f_1, \tau_1),$$

where

$$1) k_1 = kT_{i_0};$$

2)  $f_1: k_1 \rightarrow \mathcal{S}$  is a subfunctor of the domain-restriction of  $f$ ,

$$f_1(o) = \left\{ x \in f(o); (\forall p \in A_{i_0}) \left( \left[ f \begin{pmatrix} o \\ p \end{pmatrix} \right] (x) = z_p \right) \right\};$$

3)  $\tau_1: f_1 \rightarrow f_1$  is the domain-range-restriction of  $\tau$ .

LEMMA 3. Let  $r(k, f, \tau)$ , let  $k$  be a cone. Let  $o_1(k, f, \tau) = (k_1, f_1, \tau_1)$ . Then  $r(k_1, f_1, \tau_1)$ . If  $o \in k^o$ ,  $x \in f(o)$ ,  $\tau^o(x) \neq x$ , then  $o \in k_1^o$ ,  $x \in f_1(o)$ .

Proof. It follows immediately from the definition of  $o_1$ .

## 2. The operation $o_2$ .

Note. Recall Conventions 1, 2 and Lemma 1 from Part IV.

LEMMA 4. Let  $f: k \rightarrow \mathcal{S}$  be a connected functor. Let  $A = \{o \in k^o; f(o) \neq \emptyset\}$ . Then  $kA$  is a connected category.

Proof. It is evident.

LEMMA 5. Let  $f: k \rightarrow \mathcal{S}$  be a non-trivial functor. It may be uniquely written in a form

$$f = \bigcup_{a \in A} f_a,$$

where all  $f_a$  are connected and pairwise disjoint.

Proof. It is evident.

Note. The subfunctors  $f_a$  will be called *components* of  $f$ .

**LEMMA 6.** *Let  $f: k \rightarrow \mathcal{S}$  be a non-trivial functor,  $\{f_a; a \in A\}$  be the system of all its components,  $\tau: f \rightarrow f$  be a transformation. Then for every  $a \in A$  there exists  $a' \in A$  such that  $\tau$  maps  $f_a$  into  $f_{a'}$ .*

**Proof.** It is evident.

**Note.** We shall write  $\tau(f_a) \subset f_{a'}$  in this case.

**LEMMA 7.** *Let  $r(k, f, \tau)$ , let  $\{f_a; a \in A\}$  be the system of all components of  $f$ . Then  $\tau(f_a) \subset f_a$  for every  $a \in A$ .*

**Proof** is easy.

**Note.** If  $r(k, f, \tau)$ ,  $f_a$  is a component of  $f$ , denote by  $\tau_a: f_a \rightarrow f_a$  the domain-range-restriction of  $f$ .

**LEMMA 8.** *Let  $r(k, f, \tau)$ , let  $\{f_a; a \in A\}$  be the system of all components of  $f$ . Then there exists exactly one  $a_0 \in A$  such that  $r(k, f_{a_0}, \tau_{a_0})$ . The other components are rigid.*

**Proof.** It is evident.

**DEFINITION 3.** Let  $r(k, f, \tau)$ . Let  $f_{a_0}$  be the component of  $f$  such that  $r(k, f_{a_0}, \tau_{a_0})$ . Put

$$o_2(k, f, \tau) = (k_2, f_2, \tau_2),$$

where  $k_2 = kA$ ,  $A = \{o \in k^o; f_{a_0}(o) \neq \emptyset\}$ ,  $f_2$  is a domain-restriction of  $f_{a_0}$  to  $k_2$ ,  $\tau_2(x) = \tau_{a_0}(x)$  for every  $x \in f_2(o)$ ,  $o \in A$ .

**LEMMA 9.** *Let  $r(k, f, \tau)$ . Put  $o_2(k, f, \tau) = (k_2, f_2, \tau_2)$ . Then  $r(k_2, f_2, \tau_2)$ ,  $k_2$  is connected and  $Z(f_2)$  holds. If  $o \in k^o$ ,  $x \in f(o)$ ,  $\tau^o(x) \neq x$ , then  $o \in k_2^o$ ,  $x \in f_2(o)$ .*

**Proof** is easy.

**3. Construction of reducing.** Let  $r(k, f, \tau)$ . We use the definition by transfinite induction.

1)  $h_0 = k$ ,  $f_0 = f$ ,  $\tau_0 = \tau$ ;

2) let  $\alpha = \beta + 1$ ,  $r(h_\beta, f_\beta, \tau_\beta)$ ; if  $h_\beta$  is a cone, put  $(h_\alpha, f_\alpha, \tau_\alpha) = o_1(h_\beta, f_\beta, \tau_\beta)$  (where some expression from IV, Definition 11, is chosen), and put  $(h_\alpha, f_\alpha, \tau_\alpha) = o_2(h_\beta, f_\beta, \tau_\beta)$  otherwise; then  $r(k_\alpha, f_\alpha, \tau_\alpha)$  again;

3) let  $\alpha$  be limit: put  $A = \bigcap_{\beta < \alpha} h_\beta^o$ ,  $h_\alpha = kA$ ,  $f_\alpha$  is a subfunctor of the domain-restriction of  $f$ ,  $f_\alpha(o) = \bigcap_{\beta < \alpha} f_\beta(o)$ ,  $\tau_\alpha^o: f_\alpha(o) \rightarrow f_\alpha(o)$  is the domain-range-restriction of  $\tau^o$  for every  $o \in h_\alpha^o$ .

We must prove  $r(h_\alpha, f_\alpha, \tau_\alpha)$  in this case: If  $o \in k^o$ ,  $x \in f(o)$ ,  $\tau(x) \neq x$ , then necessarily  $o \in h_\alpha^o$ ,  $x \in f_\alpha(o)$ . Consequently,  $\tau_\alpha \neq 1_{f_\alpha}$ . The equality  $\tau_\alpha \circ \tau_\alpha = 1_{f_\alpha}$  is evident. Let  $\nu: f_\alpha \rightarrow f_\alpha$  be a transformation. We want to prove either  $\nu = \tau_\alpha$  or  $\nu = 1_{f_\alpha}$ . We show that it is possible to find a transformation  $\varrho: f \rightarrow f$  which is an extension of  $\nu$ . So, either  $\varrho = \tau$  or  $\varrho = 1_f$ , consequently, either  $\nu = \tau_\alpha$  or  $\nu = 1_{f_\alpha}$ .

Put  $\varrho^o(x) = x$  whenever  $x \in f(o)$  and either  $o \in k^o - h_a^o$  or  $o \in h_a^o, x \notin f_a(o)$ , and put  $\varrho^o(x) = \nu^o(x)$ , otherwise.

We prove that  $\varrho: f \rightarrow f$  is a transformation. Let  $o, o' \in k^o, o < o', x \in f(o), x' = \left[ f \begin{pmatrix} o \\ o' \end{pmatrix} \right](x)$ . We prove that

$$(*) \quad \left[ f \begin{pmatrix} o \\ o' \end{pmatrix} \right](\varrho^o(x)) = \varrho^{o'}(x').$$

a) Let  $x' \in f_a(o')$ : if  $x \in f_a(o)$ , then  $(*)$  holds because  $\varrho^{o'}(x') = \nu^{o'}(x')$ ,  $\varrho^o(x) = \nu^o(x)$ . We show that  $x \notin f_a(o)$  (or  $o \notin h_a^o$ ) is not possible. Suppose  $x \notin f_a(o)$  (or  $o \notin h_a^o$ ). Let  $\beta$  be the smallest ordinal such that  $x \notin f_\beta(o)$  (or  $o \notin h_\beta^o$ ). Then  $\beta$  is isolated, say  $\beta = \gamma + 1$ . Then  $x \in f_\gamma(o), x' \in f_\gamma(o')$ . Then neither  $(h_\beta, f_\beta, \tau_\beta) = o_1(h_\gamma, f_\gamma, \tau_\gamma)$  nor  $(h_\beta, f_\beta, \tau_\beta) = o_2(h_\gamma, f_\gamma, \tau_\gamma)$  as it follows from the definitions of  $o_1, o_2$ .

b) Let  $x' \notin f_a(o')$  (or  $o' \notin h_a^o$ ): if  $x \notin f_a(o)$  (or  $o \notin h_a^o$ ), then  $(*)$  holds because  $\varrho^{o'}(x') = x', \varrho^o(x) = x$ . Let  $x \in f_a(o)$ . Let  $\beta$  be the smallest ordinal such that  $x' \notin f_\beta(o')$  (or  $o' \notin h_\beta^o$ ). Then necessarily  $\beta$  is isolated, say  $\beta = \gamma + 1$ , and  $(h_\beta, f_\beta, \tau_\beta) = o_1(h_\gamma, f_\gamma, \tau_\gamma)$ . But then  $\left[ f_\beta \begin{pmatrix} o \\ o' \end{pmatrix} \right](z) = x'$  for every  $z \in f_\beta(o)$  (see the definition of  $o_1$ ) and thus  $(*)$  holds again.

So, we obtain  $(h_\beta, f_\beta, \tau_\beta)$  for every ordinal  $\beta$ .

One can easily see that there exists an ordinal  $\lambda$  such that

$$(h_\lambda, f_\lambda, \tau_\lambda) = (h_{\lambda+1}, f_{\lambda+1}, \tau_{\lambda+1}).$$

Then

- ( $\alpha$ )  $h_\lambda$  is not a cone;
- ( $\beta$ )  $\mathbf{Z}(f_\lambda)$  holds;
- ( $\gamma$ )  $h_\lambda$  is connected and it is a full subcategory of  $k$ .

Now, it is easy to prove the implication (ii)  $\Rightarrow$  (iii) of the theorem. Let  $k$  be a small thin category,  $o_2$  can be fully embedded in  $\mathcal{S}^k$ . We may suppose that  $k$  is skeletal. Choose  $f: k \rightarrow \mathcal{S}$  and  $\tau: f \rightarrow f$  such that  $r(k, f, \tau)$ . Find an ordinal  $\lambda$  such that  $h_\lambda, f_\lambda, \tau_\lambda$  satisfy ( $\alpha$ ), ( $\beta$ ), ( $\gamma$ ). Then  $h_\lambda$  is also skeletal. Now, use the proposition from V.

Since  $\mathbf{Z}(f_\lambda)$  holds,  $h_\lambda$  has not any of the forms  $a_1, \dots, a_{10}$  (see Lemmas 1, ..., 10 from Part IV). Since  $h_\lambda$  is not a cone, it must contain some  $k_i, i = 1, \dots, 35$ , as a full subcategory.

## Appendix

The main theorem was announced as Theorem 4 in [10].

In [9], we announced a little stronger theorem for finite thin categories. The following properties are equivalent for a finite thin category  $k$ :

- (1)  $k$  is rich.
- (2)  $\mathcal{S}^k$  contains  $\aleph_1$  non-isomorphic rigid objects.
- (3) A non-trivial monoid without a non-trivial (i.e. non-identical) idempotent can be fully embedded into  $\mathcal{S}^k$ .
- (4) Some of the categories  $k_1, \dots, k_{31}$  is a full subcategory of  $k$ .

The proof of the equivalence of (1)–(4) uses analogous methods as the proof of the main theorem. One can see that if a finite category  $k$  has some of the forms  $a_1, \dots, a_{10}$ , then (2) cannot be satisfied by connected functors to  $\mathcal{S}_n$  and, if a non-trivial monoid  $M$  without a non-trivial idempotent is given, then there is no connected functor  $\Phi: k \rightarrow \mathcal{S}_n$  such that all endotransformations of  $\Phi$  form a monoid isomorphic to  $M$ . The cone is very simple for finite categories and the transfinite reduction described in VI may be replaced by usual induction on the objects of  $k$ . We put the following problems:

- a) Let  $k$  be a small thin category such that  $\mathcal{S}^k$  contains a proper class of pairwise non-isomorphic rigid objects. Is  $k$  rich?
- b) Is the condition (3) sufficient for an infinite thin category to be rich?

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