TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

ON THE EXISTENCE OF ALMOST SPLIT SEQUENCES IN SUBCATEGORIES

RAYMUNDO BAUTISTA

Instituto de Matemáticas, Universidad Nacional Autónoma de México
México, D.F., México

MARK KLEINER

Department of Mathematics, Syracuse University
New York, U.S.A.

Consider a full subcategory of the category of finitely generated modules over an Artin algebra or of the category of lattices over an order. Suppose the subcategory is closed under isomorphisms, direct sums, and nonzero direct summands. If the direct sum of any two Ext-projectives is Ext-projective, and the direct sum of any two Ext-injectives is Ext-injective, we give a criterion for the existence of almost split sequences in the subcategory. If the subcategory is closed under extensions, the above-mentioned conditions on the Ext-projective and Ext-injective modules are certainly satisfied. Hence in the case of Artin algebras our result is applicable to a wider class of subcategories than the basic existence theorem of Auslander and Smalø. We indicate how to use the criterion to prove the existence of almost split sequences in certain categories of relatively projective modules. These categories contain the categories of representations of BOCS's.

Introduction

Let Λ be an Artin algebra or an order over a commutative noetherian equidimensional Gorenstein ring, and Λ-mod the category of finitely generated left Λ-modules or of Λ-lattices [3]. Throughout the paper we fix $\mathscr{C}$ — a full

This paper is in final form and no version of it will be submitted for publication elsewhere.

subcategory of Δ-mod closed under direct sums, nonzero direct summands, and such that if a module $X \in \mathscr{C}$ is isomorphic to a module $Y \in \Delta$-mod, then $Y \in \mathscr{C}$. When Δ is an Artin algebra, the general theory of almost split sequences in $\mathscr{C}$ was developed by Auslander and Smalø in [2]. We need to recall some notions they have introduced.

An *exact sequence in* $\mathscr{C}$ is an exact sequence $\ldots X_{i-1} \to X_i \to X_{i+1} \ldots$ of modules in Δ-mod in which the nonzero X_i's are all in $\mathscr{C}$. A module $N \in \mathscr{C}$ is called Ext-*projective* if every exact sequence $0 \to X \to Y \to N \to 0$ in $\mathscr{C}$ splits. A module $L \in \mathscr{C}$ is called Ext-*injective* if every exact sequence $0 \to L \to Y \to Z \to 0$ in $\mathscr{C}$ splits.

A morphism g: $M \to N$ in $\mathscr{C}$ is said to be *right almost split* in $\mathscr{C}$ if (i) g is not a splittable epimorphism and (ii) for every morphism h: $W \to N$, where $W \in \mathscr{C}$ and h is not a splittable epimorphism, there exists a morphism j: $W \to M$ satisfying $h = gj$. A morphism f: $L \to M$ in $\mathscr{C}$ is said to be *left almost split* in $\mathscr{C}$ if (i) f is not a splittable monomorphism and (ii) for every morphism h: $L \to W$, where $W \in \mathscr{C}$ and h is not a splittable monomorphism, there exists a morphism j: $M \to W$ satisfying $h = jf$. $\mathscr{C}$ is said to *have right almost split morphisms* if for each indecomposable $N \in \mathscr{C}$ there is an $M \in \mathscr{C}$ and a morphism g: $M \to N$ which is right almost split in $\mathscr{C}$. Dually, $\mathscr{C}$ is said to *have left almost split morphisms* if for each indecomposable $L \in \mathscr{C}$ there is an $M \in \mathscr{C}$ and a morphism f: $L \to M$ which is left almost split in $\mathscr{C}$. Finally, $\mathscr{C}$ *has almost split morphisms* if it has both left and right almost split morphisms.

An exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in $\mathscr{C}$ is called *almost split* if f is a left almost split morphism in $\mathscr{C}$, and g is a right almost split morphism in $\mathscr{C}$. $\mathscr{C}$ is said to *have almost split sequences* if it satisfies the following conditions:

(a) $\mathscr{C}$ has almost split morphisms.

(b) If N is indecomposable non-Ext-projective in $\mathscr{C}$, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

(c) If L is indecomposable non-Ext-injective in $\mathscr{C}$, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

1. The main result

DEFINITION. We say that the Ext-projective (Ext-injective) modules in $\mathscr{C}$ are *closed under direct sums* if whenever X and Y are Ext-projective (Ext-injective) in $\mathscr{C}$, $X \oplus Y$ is Ext-projective (Ext-injective) in $\mathscr{C}$.

THEOREM. *Suppose that both the* Ext-*projective and the* Ext-*injective modules in* $\mathscr{C}$ *are closed under direct sums. Then* $\mathscr{C}$ *has almost split sequences if and only if it satisfies the following conditions*:

(i) $\mathscr{C}$ *has almost split morphisms.*

(ii) *If* N *is indecomposable non*-Ext-*projective in* $\mathscr{C}$, *then there is an exact sequence* $0 \to L \to M \xrightarrow{g} N \to 0$ *in* $\mathscr{C}$ *with* g *right almost split in* $\mathscr{C}$.

(iii) *If L is indecomposable non-*Ext*-injective in $\mathscr{C}$, then there is an exact sequence $0 \to L \xrightarrow{f} M \to N \to 0$ in $\mathscr{C}$ with f left almost split in $\mathscr{C}$.*

Proof. The necessity is obvious. We prove the sufficiency.

Let N be indecomposable non-Ext-projective in $\mathscr{C}$. Consider an exact sequence $0 \to X \xrightarrow{s} Y \xrightarrow{t} N \to 0$, where t is a minimal morphism corresponding to the right almost split morphism g given by condition (ii). Then X is a direct summand of L, so that $X \in \mathscr{C}$, and t is minimal right almost split in $\mathscr{C}$. We only have to show that s is left almost split in $\mathscr{C}$. Let $X_1, \ldots, X_r$ be the indecomposable direct summands of X. They are not all Ext-injective because t is not a splittable epimorphism, and the Ext-injective modules in $\mathscr{C}$ are closed under direct sums. For $j = 1, \ldots, r$ consider an exact sequence

$$0 \to X_j \xrightarrow{h_j} V_j \xrightarrow{k_j} W_j \to 0 \tag{1.1}$$

in $\mathscr{C}$, where if X_j is not Ext-injective, h_j is minimal left almost split in $\mathscr{C}$ (use condition (iii)), and if X_j is Ext-injective, then $V_j = X_j$, $W_j = 0$, $h_j = 1$. Note that h_j is a minimal morphism in both cases. Let $0 \to X \xrightarrow{h} V \xrightarrow{k} W \to 0$ be the direct sum of the exact sequences (1.1) for all j; it does not split because some of the sequences (1.1) do not split (remember that not all X_j's are Ext-injective). For each $j = 1, \ldots, r$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X_j & \xrightarrow{h_j} & V_j & \xrightarrow{k_j} & W_j & \to & 0 \\ & & \downarrow{\scriptstyle u_j} & & \downarrow{\scriptstyle v_j} & & \downarrow{\scriptstyle w_j} & & \\ 0 & \to & X & \xrightarrow{s} & Y & \xrightarrow{t} & N & \to & 0 \end{array}$$

where u_j: $X_j \to X$ is the natural inclusion. Really, this is obvious if X_j is Ext-injective, because $ts = 0$. If X_j is not Ext-injective, then note that su_j is not a splittable monomorphism because of the minimality of t, and use the fact that h_j is left almost split in $\mathscr{C}$. Hence we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \xrightarrow{h} & V & \xrightarrow{k} & W & \to & 0 \\ & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \to & X & \xrightarrow{s} & Y & \xrightarrow{t} & N & \to & 0 \end{array}$$

where v is induced by the v_j's, and w by the w_j's. If w is not a splittable epimorphism, there exists a morphism f: $W \to Y$ satisfying $w = tf$, whence the top sequence splits by [7, Ch. III, Lemma 3.3, p. 74], a contradiction. Hence w is a splittable epimorphism, and $wq = 1$ for some q: $N \to W$. We now arrive at the following commutative diagram of Λ-modules:

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Z & \to & N & \to & 0 \\ & & \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ 0 & \to & X & \xrightarrow{h} & V & \xrightarrow{k} & W & \to & 0 \\ & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \to & X & \xrightarrow{s} & Y & \xrightarrow{t} & N & \to & 0 \end{array}$$

Since $wq = 1$, vp is an isomorphism, whence $Z \in \mathscr{C}$, $V = \operatorname{Im} p \oplus \operatorname{Ker} v$, and

$\operatorname{Im} h \subseteq \operatorname{Im} p$. If v is not an isomorphism, then h is not a minimal morphism, which contradicts the minimality of all h_j's because a direct sum of minimal morphisms is a minimal morphism. Thus v is an isomorphism, w is an isomorphism, and W is indecomposable. Since t is a minimal morphism, so is k, whence none of the X_j's is Ext-injective. Therefore $r = 1$, X is indecomposable, and s is left almost split.

If L is indecomposable non-Ext-injective in $\mathscr{C}$, an exact sequence $0 \to L \xrightarrow{h} V \xrightarrow{k} W \to 0$, where h is a minimal morphism corresponding to the left almost split morphism f given by condition (iii), is almost split in $\mathscr{C}$. The proof is similar to the preceding arguments. ∎

Recall that when Λ is an Artin algebra, the basic existence theorem of [2] states that if $\mathscr{C}$ is a dualizing R-variety closed under extensions, then $\mathscr{C}$ has almost split sequences. Since Ext is an additive bifunctor, if a subcategory is closed under extensions, then both the Ext-projective and the Ext-injective modules in it are closed under direct sums. Thus our theorem may be viewed as an extension of the above-mentioned result of Auslander and Smalø.

2. Applications

Let R be a field or a Dedekind domain, Λ a finite-dimensional R-algebra or an R-order [1, p. 85], respectively, and $i\colon \Lambda \to A$ a Λ-ring. Denote by induc$A$ the full subcategory of A-mod determined by the induced modules, i.e. by the modules isomorphic to $A \otimes_\Lambda M$ with $M \in \Lambda$-mod. Let $\mathbf{p}(A, \Lambda)$ be the full subcategory of A-mod consisting of the direct summands of all modules in induc$A$. The induced modules and their direct summands are called *relatively projective*, or *(A, Λ)-projective*, modules. We use the theorem of Section 1 to prove in [4] the existence of almost split sequences in $\mathbf{p}(A, \Lambda)$, assuming that the unit $i\colon \Lambda \to A$ of the Λ-ring A is injective, and Coker i as a Λ-bimodule is isomorphic to $\bigoplus_{s=1}^{n} I_s \otimes_R P_s$, where I_s is injective in Λ-mod, and P_s is projective in mod-Λ, the category of right Λ-modules, for all s. That both the Ext-projective and the Ext-injective modules in $\mathbf{p}(A, \Lambda)$ are closed under direct sums follows from the following descriptions given in [4]. A module N is projective in A-mod if and only if it is Ext-projective in $\mathbf{p}(A, \Lambda)$. A module L in $\mathbf{p}(A, \Lambda)$ is Ext-injective if and only if it is injective in Λ-mod.

As shown in [6], the set of morphisms $C = \operatorname{Hom}_{-\Lambda}(A, \Lambda)$ in mod-Λ is a Λ^{op}-coring which as a right Λ^{op}-module is finitely generated projective, and there is a duality between $\mathbf{p}(A, \Lambda)$ and $\mathbf{i}(C, \Lambda^{\mathrm{op}})$, the category of relatively injective C-comodules. In fact, as we prove in [4], the above-mentioned conditions on the Λ-ring A are satisfied if and only if the counit $\varepsilon\colon C \to \Lambda^{\mathrm{op}}$ of the Λ^{op}-coring C is surjective, and Kerε is isomorphic as a Λ^{op}-bimodule to $\bigoplus_{s=1}^{n} Q_s \otimes P_s$, where Q_s (P_s) is projective in Λ^{op}-mod (mod-Λ^{op}) for all s. Thus our restriction on Kerε means exactly that C is a free BOCS in the language of

[5]. It is shown in [6] that the category of induced C-modules is equivalent to the category of representations of the corresponding BOCS. If the BOCS is triangular [8], we show in [4] that idempotents split in induc$C$, whence every direct summand of an induced comodule is induced, i.e. every relatively injective comodule is induced. Therefore almost split sequences exist for representations of free triangular BOCS's.

Another application is the existence of almost split sequences in the category of relatively projective modules over a Frobenius group G with respect to its nontrivial proper subgroup H whose intersection with gHg^{-1} is trivial whenever $g \in G-H$.

References

[1] M. Auslander, *Functors and morphisms determined by objects*, in: Representation Theory of Algebras, R. Gordon (ed.), Lecture Notes in Pure and Appl. Math. 37, Dekker, New York 1978, 1–244.

[2] M. Auslander and S. O. Smalø, *Almost split sequences in subcategories*, J. Algebra 69 (1981), 426–454.

[3] —, —, *Lattices over orders: finitely presented functors and preprojective partitions*, Trans. Amer. Math. Soc. 273 (1982), 433–446.

[4] R. Bautista and M. Kleiner, *Almost split sequences for relatively projective modules*, J. Algebra, to appear.

[5] W. W. Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. (3) 56 (1988), 451–483.

[6] M. Kleiner, *Induced modules and comodules and representations of BOCS's and DGC's*, in: Representations of Algebras, Lecture Notes in Math. 903, Springer, Berlin 1981, 168–185.

[7] S. MacLane, *Homology*, Springer, New York 1967.

[8] A. V. Roiter, *Matrix problems and representations of BOCS's*, in: Representation Theory I, Lecture Notes in Math. 831, Springer, Berlin 1980, 288–324.