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PACKING INDEPENDENT SETS AND TRANSVERSALS

D. DE WERRA

Département de Mathématiques, École Polytechnique Fédérale de Lausanne
Lausanne, Switzerland

Partial colorings of the nodes of a graph are considered; a criterion of optimality is derived by simple network flow techniques for the line-graphs of bipartite multigraphs. More generally such a derivation is possible for graphs whose clique-node incidence matrix is totally unimodular. A similar problem is mentioned for constructing a collection of k disjoint transversal sets with the smallest total number of nodes.

1. Introduction

In this note we examine variations on coloring problems in graphs. We will try to color as many nodes as possible using k colors in a graph. C. Berge has studied this question and developed some optimality criteria [2]. These do not always hold and several classes of graphs for which they hold have been exhibited.

Here we will show that such optimality criteria can be derived by network flow techniques or by linear programming arguments for some classes of graphs. We will examine the special case of line-perfect graphs which generalize bipartite graphs. Also we will develop an analogous optimality criterion for the minimality of a set of nodes which is the union of k disjoint transversal sets.

The reader is referred to [1], [3] for all concepts related to graphs or to hypergraphs which are not defined here.

2. Partial k-colorings

Let $G = (V, E)$ be a simple graph with chromatic number $\varkappa(G)$ and let k be an integer with $1 \leqslant k \leqslant \varkappa(G)$. A *partial k-coloring* of G is a collection

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$\mathcal{S} = (S_1, \ldots, S_k)$ of k disjoint independent sets. The k-coloring $\mathcal{S}$ is *maximum* if $|S_1 \cup \ldots \cup S_k|$ is maximum. Let $\alpha_k(G) = \max |S_1 \cup \ldots \cup S_k|$.

Such colorings have been studied by Berge [2]. Related questions for line-graphs of bipartite graphs are discussed in [9].

A family $\mathscr{C} = (C_1, \ldots, C_p)$ of cliques is called an *associated family* of $\mathcal{S}$ if the following holds:

(1) $S_i \cap S_j = \emptyset$ $(i \neq j)$, $C_i \cap C_j = \emptyset$ $(i \neq j)$.
(2) $(\bigcup_i S_i) \cup (\bigcup_j C_j) = V$.
(3) $|S_i \cap C_j| = 1$ for all i and j.

PROPERTY 2.1 [2]. *If a partial k-coloring $\mathcal{S}$ has an associated family of cliques, then $\mathcal{S}$ is maximum.*

If K_k denotes the complete graph on k nodes, the cartesian sum $G + K_k$ is a graph constructed from $G = (V, E)$ as follows: Its node set is the cartesian product $V \times \{1, \ldots, k\}$; (x, i) and (y, j) are linked if either $x = y$ and $i \neq j$ or if $[x, y] \in E$ and $i = j$. There is a one-to-one correspondence between the partial k-colorings of G and the independent sets of $G + K_k$. Furthermore, $\alpha_k(G) = \alpha(G + K_k)$ where $\alpha(G) = \alpha_1(G)$ is the maximum cardinality of an independent set in G.

Let $\theta(G)$ be the minimum number of cliques covering the nodes of G. Berge has shown that for a graph G, $\alpha(G + K_k) = \theta(G + K_k)$ if and only if every maximum k-coloring has an associated family of cliques [2].

The converse of Property 2.1 is generally not true. Berge has exhibited some classes of graphs for which a k-coloring is optimum if and only if it has an associated family of cliques. Among those are the comparability graphs, their complements. For line-graphs of trees the associated family of cliques is derived in a straightforward way; this derivation is also valid for the line-graphs of a special class of bipartite graphs.

We shall now establish the existence of an associated family of cliques for a more general class of graphs by a linear programming argument.

A graph G is *unimodular* if its clique-node incidence matrix A is totally unimodular ($a_{ij} = 1$ if the inclusionwise maximal clique C_i contains node j and $a_{ij} = 0$ otherwise).

Clearly the characteristic vector $\boldsymbol{x}$ of a partial k-coloring $\mathcal{S}$ of G ($x_i = 1$ if node i is colored) statisfies

$$Ax \leqslant \boldsymbol{k}, \quad \boldsymbol{0} \leqslant \boldsymbol{x} \leqslant \boldsymbol{1}, \tag{2.1}$$

where $\boldsymbol{k} = (k, \ldots, k)$ and A is the clique-node incidence matrix of the graph G.

For an arbitrary graph, any integral solution of (2.1) does not necessarily correspond to a partial k-coloring (take for instance G equal to the chordless cycle on five nodes, $k = 2$ and $\boldsymbol{x} = (1, 1, 1, 1, 1)$). However, for unimodular graphs G we can formulate

PROPERTY 2.2. *If G is unimodular, there is a one-to-one correspondence between the characteristic vectors of partial k-colorings of G and the integral feasible solutions of* (2.1).

Proof. It suffices to observe that any integral vector x satisfying (2.1) can be decomposed into a sum $x^1+\ldots+x^k$ where each x^i is a (0, 1)-vector satisfying $Ax^i \leqslant \mathbf{1}$ (see [10]). Each x^i is the characteristic vector of an independent set S_i.

Hence for finding a maximum k-coloring we have to find an integral optimum solution of

$$\text{(2.2)} \qquad \text{Max } \mathbf{1}x \quad \text{s.t.} \quad Ax \leqslant \mathbf{k},\ \mathbf{0} \leqslant x \leqslant \mathbf{1}.$$

Since A is totally unimodular and since all vectors are integral, the problem has an optimum solution which is integral.

The dual problem of (2.2) is

$$\text{(2.3)} \qquad \text{Min } \mathbf{k}\boldsymbol{\lambda}+\mathbf{1}\boldsymbol{\mu} \quad \text{s.t.} \quad \boldsymbol{\lambda}A+\boldsymbol{\mu} \geqslant \mathbf{1},\ \boldsymbol{\lambda} \geqslant \mathbf{0},\ \boldsymbol{\mu} \geqslant \mathbf{0}.$$

Since both LP's have feasible solutions, finite optima exist. Observe first that (2.3) has an optimum solution with $\mu_j \in \{0, 1\}$ for each node j and $\lambda_i \in \{0, 1\}$ for each maximal clique C_i.

Furthermore, from complementary slackness it follows that if $\lambda_i = 1$ then the clique C_i satisfies $\mathbf{a}_i x = k$ ($\mathbf{a}_i$ is the ith row of A), i.e. $|C_i \cap S_l| = 1$ for each l ($1 \leqslant l \leqslant k$).

Consider the family $F = (C_i\colon i \in I)$ of cliques C_i with $\lambda_i = 1$. If some node u belongs to more than one clique in F, we remove u from all but one of these cliques. Observe that if $u \in C_i \cap C_j$ ($i \neq j$), in an optimum solution of (2.3) we have $\mu_u = 0$; by complementary slackness we can have $x_u = 1$ only if $\mu_u = 1$. So here $x_u = 0$ and u is not colored.

We have now obtained a family of cliques C_i which are pairwise disjoint and satisfy $|C_i \cap S_l| = 1$ for all i, l. Finally, each node u not covered by any of these cliques has $\mu_u = 1$ from the constraints of (2.3) and hence $x_u = 1$. Such a node u is colored, so that $(\bigcup_j S_j) \cup (\bigcup_i C_i)$ covers all nodes of G and we have an associated family of cliques. We have established:

PROPOSITION 2.3. *Let G be a unimodular graph. Then a k-coloring of G is maximum if and only if it has an associated family of cliques.*

COROLLARY 2.4. *For the line-graph $L(G)$ of a bipartite multigraph G a k-coloring is maximum if and only if it has an associated family of cliques.*

This follows from the fact that the clique-node incidence matrix of $L(G)$ is the node-edge incidence matrix of G.

Notice that a maximum k-coloring can be found in $L(G)$ by applying standard network flow techniques in G. The max flow-min cut theorem gives directly the associated family of cliques. Such a construction will be illustrated in a similar situation: the construction of minimum k-tranversals.

COROLLARY 2.5. *If G is unimodular, then for any k*

$$\alpha(G+K_k) = \theta(G+K_k).$$

It should be observed that $G+K_k$ is not necessarily perfect when G is unimodular.

Remark 2.6. Since constructing a maximum k-coloring in G reduces to finding a maximum independent set in $G+K_k$, there exists a polynomial time algorithm for a maximum k-coloring when $G+K_k$ is perfect. This is the case for any $k \leqslant 3$ if and only if G is a graph where every 2-connected component is a clique; if $k = 2$, $G+K_2$ is perfect if and only if G is a parity graph (every odd cycle has at least two crossing chords) [7].

3. Line-perfect graphs

A graph G is *line-perfect* if its line-graph $L(G)$ is perfect. Trotter has shown that G is line-perfect if and only if G does not contain any elementary odd cycle of length five or more [8].

Notice that if G is line-perfect, $L(G)$ is perfect but generally not unimodular (the graph in Fig. 1 is not unimodular; it is the line-graph of the line-perfect graph in Fig. 2). The example in Fig. 1 shows that for the line-graph G^* of a line-perfect graph G we do not generally have $\alpha(G^*+K_k) = \theta(G^*+K_k)$. However, for G we can state

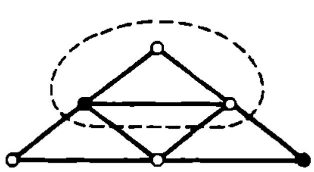

Fig. 1. Black nodes indicate a transversal

PROPOSITION 3.1. *If G is a line-perfect graph with chromatic number $\chi(G)$, then for any $k \leqslant \chi(G)$, we have $\alpha(G+K_k) = \theta(G+K_k)$.*

Proof. It will be sufficient to show that if G is line-perfect, then its clique hypergraph (its edges are the maximal cliques of G) is unimodular. Proposition 3.1 will then follow from Proposition 2.3.

Let $H = (V, \mathscr{E})$ be the clique hypergraph of a line-perfect graph $G = (V, E)$. We claim that H has no odd cycles.

It should be noted that all edges e of H satisfy $2 \leqslant e \leqslant 4$. Furthermore, the following properties hold:

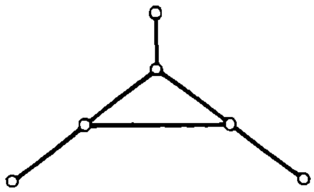

Fig. 2. A line-perfect graph

(1) For any two edges e, f of H we have $|e \cap f| \leqslant 1$ except when $|e| = |f| = 3$; in this case $|e \cap f| \leqslant 2$.

(2) The partial hypergraph spanned by all edges e with $|e| = 2$ is bipartite.

(3) Given any two nodes u, v in an edge e with $|e| \geqslant 3$, there is no elementary chain of length at least three joining u and v in H.

So suppose there is an odd cycle O in H.

From (2), O must use at least one edge e_1 with $|e_1| \geqslant 3$. From (3) we cannot have a cycle O with length at least 5. So O is a triangle. Let a, e_1, b, e_2, c, e_3 be the nodes and the edges of O. Since O is a cycle of H all nodes and all edges are distinct. We cannot have $|e_2| = 2$ or $|e_3| = 2$ (because these would not be maximal cliques of G: they would be included in $\{e_1, e_2, e_3\}$). So we have $|e_2| \geqslant 3$, $|e_3| \geqslant 3$. Notice that we may assume that $c \notin e_1$, $a \notin e_2$ (if two edges e_i, e_j contain the nodes a, b, c we have two edges with 3 common nodes and this contradicts (1)). Then e_1 must contain a third node $d \neq a$. If $d \neq f$, G contains an odd cycle on nodes a, b, c, d, f; this is impossible. So we have $d = f$. Now a, b, c, d form a 4-clique in G and hence an edge $e \neq e_1, e_2$. Then the pair e, e_1 contradicts (1). So we cannot have any odd cycle in H; hence H is unimodular [3]. ∎

4. An associated problem

In $G = (V, E)$ a set $T \subseteq V$ is a transversal if $T \cap K \neq \emptyset$ for every (inclusionwise) maximal clique K. (T is in fact a transversal of the hypergraph of maximal cliques of G). A *k-transversal* is the disjoint union of k transversals. Let $r(G)$ be the maximum value of k for which a k-transversal can be found in G. Clearly $r(G) \leqslant \min\{|K|\colon K \text{ is a maximal clique}\}$. For any k $(1 \leqslant k \leqslant r(G))$ let $\tau_k(G)$ be defined as follows:

$$\tau_k(G) = \min\{|T|\colon T \text{ is a } k\text{-transversal}\}.$$

If T is a k-transversal, we define an *associated family* of T as a family $\mathscr{C}$ of cliques $C_1, \ldots, C_p$ such that

(a) $|C_i \cap T| = k$, $i = 1, \ldots, p$.

(b) $T \supseteq C_j \cap C_i$ $(i \neq j)$.

(c) $\bigcup_i C_i \supseteq T$.

One can easily establish the following:

PROPERTY 4.1. *Let T be a k-transversal of a graph G. If T has an associated family then $|T| = \tau_k(G)$.*

The converse of the property is generally not true (see the graph in Fig. 1 with $k = 1$: we cannot find a family $\mathscr{C}$ satisfying (a)–(c) for the transversal of minimum cardinality).

We can, however, formulate

PROPOSITION 4.2. *Let $G^* = (V, E^*)$ be the line-graph of a bipartite multigraph $G = (X, Y, V)$ and let T be a k-transversal of G^*. Then $|T| = \tau_k(G^*)$ if and only if T has an associated family.*

Proof. We only have to show that for a T with minimum cardinality we can construct an associated family. There is a one-to-one correspondence between the transversals T in G^* and the partial graphs H in G such that for any node z in $X \cup Y$ we have $d_H(z) \geqslant 1$. It is known that T is a k-transversal in G^* if and only if the corresponding partial graph H in G satisfies $d_H(z) \geqslant k$ for any z in $X \cup Y$ [6].

A k-transversal T with minimum cardinality in $G = (X, Y, E)$ can be constructed by network flow techniques. For this we construct a network N from G by introducing a source s and arcs (s, x) for each x in X. Similarly we introduce a sink t and each y in Y is linked to t by an arc (y, t). Capacities $c(x, y)$ and lower bounds $l(x, y)$ are given for each arc (x, y) in Table 1.

Table 1

The network N

Arc (x, y)	Lower bound $l(x, y)$	Capacity $c(x, y)$
(s, x)	k	∞
(x, y)	0	1
(y, t)	k	∞

Clearly a k-transversal T with $|T| = \tau_k(G)$ can be obtained by finding an integral feasible flow from s to t in N with minimum value v. The value of such a flow satisfies

$$v = \tau_k(G) = \max_{\substack{t \in A \\ s \notin A}} \left(l(\bar{A}, A) - c(A, \bar{A})\right)$$

$$= k|A \cap X| + k|\bar{A} \cap Y| - m(A \cap X, \bar{A} \cap Y)$$

for some A with $X \cup Y \supset A$ and $t \in A$, $s \notin A$. Here $m(W, Z)$ is the number of arcs (w, z) with $w \in W$ and $z \in Z$.

For each node $z \in X \cup Y$ let $B(z)$ denote the bundle of z, i.e. the set of edges of G which are adjacent to z. $B(z)$ corresponds to a maximal clique, say C_z, in G^*.

Let H be defined by the edges $[x, y]$ of G for which the flow $f(x, y) = 1$. T consists of the nodes of G^* corresponding to the edges of H as before.

The associated family $\mathscr{C}$ of T is obtained by taking all maximal cliques C_z of G^* corresponding to the bundles $B(z)$ of G with $z \in (A \cap X) \cup (\bar{A} \cap Y)$.

For $x \in A \cap X$ we have $f(s, x) = l(s, x) = k$ (if $f(s, x) > k$, s would be in A and we could decrease the value of the flow). Similarly for $y \in \bar{A} \cap Y$ we have $f(y, t) = k$. Hence the cliques C_z in $\mathscr{C}$ satisfy $|C_z \cap T| = k$. For C_u, C_v in $\mathscr{C}$ we have $C_u \cap C_v \neq \emptyset$ only if $u \in X$, $v \in Y$. Now for any arc (u, v) with $u \in A \cap X$, $v \in \bar{A} \cap Y$ we have $f(u, v) = c(u, v) = 1$. Hence the nodes in $C_u \cap C_v$ are in T.

Finally, consider a node w^* in G^* such that $w^* \notin \bigcup(C_z\colon C_z \in \mathscr{C})$. It corresponds to an edge $[u, v]$ of G and to an arc (u, v) of N with $u \in \bar{A} \cap X$, $v \in A \cap Y$. For such an arc we have $f(u, v) = l(u, v) = 0$ (otherwise we would have $u \in A$). So w^* is not in T and $\bigcup(C_z\colon C_z \in \mathscr{C}) \supseteq T$. We now have a family $\mathscr{C}$ which satisfies conditions (a)–(c); it is an associated family of T.

Remark 4.3. If G is a line-perfect graph, then Proposition 4.2 does not hold for the line-graph of G (see example in Fig. 1).

If G is bipartite, then $r(G) = 2$ and Proposition 4.2 holds for $k = 1$: to any transversal T with minimum cardinality there corresponds a (maximum) matching M with $|M| = |T|$. Each edge $e \in M$ is a maximal clique and it has exactly one endpoint in T. For $k = 2$, it holds trivially.

More generally, for unimodular graphs we can derive the above result by linear programming arguments as in the previous section.

PROPOSITION 4.4. *For a unimodular graph G, a k-transversal is minimum if and only if it has an associated family of cliques.*

As previously, the proposition holds for line-perfect graphs (in such graphs G we have $r(G) = \min\{|K|\colon K$ is a maximal clique$\}$ with $2 \leqslant r(G) \leqslant 4$. It is known that this equality also holds if the clique hypergraph is balanced [4].

Remark 4.5. There is apparently no reduction of the problem of finding a minimum k-transversal in G to determining a subset with a simple characterization in an auxiliary graph like $G + K_k$.

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