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Weighted H^p spaces

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Introduction

The purpose of this paper is to study two kinds of weighted H^p spaces and to investigate the relations between them.

Weights arise naturally in many problems in analysis. For example, this occurs in the Sturm–Liouville problem. Given a second order differential operator, there is a weight $w(x)$, with respect to which, the operator is formally selfadjoint. Thus, one is led to consider eigenfunction expansions in $L^2(w(x)\,dx)$. A wide range of classical expansions fall into this category.

From the Harmonic Analysis point of view, the main role of the classical H^p spaces is that of providing a natural extension, for $0 < p \leqslant 1$, of the family of Lebesgue spaces $\{\mathrm{Re}\,L^p\colon 1 < p\}$. It seems reasonable to expect that the development of a theory for the corresponding weighted spaces will help to clarify some questions concerning these classical expansions and many other problems which involve a weight. After the work of Fefferman–Stein ([8]) and Coifman ([1]) our knowledge of the classical H^p spaces is sufficient to allow us to construct such a theory.

We will consider a weight $w(x)$ satisfying the condition A_∞ introduced by Muckenhoupt ([13]). These weights arise in a variety of different situations, which include the one described above, and, also, their properties make it possible to use in the study of the weighted spaces, most of the techniques developed for studying the classical H^p spaces. Even though the family of conditions A_p, $p \geqslant 1$ and A_∞ were introduced by Muckenhoupt, we should point out that very similar conditions were first studied by Rosenblum ([16]). In fact it is in [16] that the idea to develop a theory of weighted H^p spaces appears for the first time.

Following the classical definition of H^p, we are led to consider the space $H^p(w(x)\,dx)$ of all analytic functions $F(z)$ on the upper half plane such that

$$\|F\|^p_{H^p(w(x)dx)} = \sup_{t>0} \int_{-\infty}^{\infty} |F(x+it)|^p w(x)\,dx < \infty.$$

These spaces are studied in II. The main results are: a maximal function characterization of $H^p(w(x)\,dx)$ and the identification of the "building blocks" into which every function in $H^p(w(x)\,dx)$ can be decomposed. We call them atoms. This atomic decomposition is similar to the one

obtained for the classical case by Coifman ([1]). This decomposition allows us to identify the dual of $H^p(w(x)\,dx)$ (Theorem II. 4.4) by means of several equivalent integral conditions. The equivalence of these conditions is a by-product of the theory which extends the theorem of John and Nirenberg ([11]) giving the equivalence of different BMO conditions.

A parallel theory is possible for the space $H^p(w(\theta)\,d\theta)$ of all analytic functions $F(z)$ on the unit disk $|z| < 1$, for which

$$\|F\|^p_{H^p(w(\theta)d\theta)} = \sup \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p w(\theta)\,d\theta < \infty.$$

If we let

$$W(z) = \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log w(\theta)\,d\theta\right\},$$

then the mapping $F \mapsto FW^{1/p}$ is an equivalence between $H^p(w(\theta)d\theta)$ and $H^p(d\theta)$ which is the classical space. Of course, this allows us to identify the dual of $H^p(w(\theta)\,d\theta)$. This way of setting up the duality, however, is not very practical. For example, for $p = 1$, it identifies the dual of $H^1(w(\theta)\,d\theta)$ with the space of functions $L = P(w \cdot e^{-i(\log w)^{\sim}} \cdot B)$, where B is a holomorphic BMO function, $\sim$ represents the conjugate operator and P is the projection on holomorphic functions.

Our way of setting up the duality by real variable techniques leads us to more meaningful descriptions of the dual spaces.

In general, it is not true that for $F \in H^p(w(x)\,dx)$, the functions $\{F(x+it)\colon t > 0\}$ will have a limit in the sense of distributions as $t \to 0$. For $w(x) \equiv 1$, we always have such a limit. Thus, for a general weight $w(x)$, the spaces $H^p(w(x)\,dx)$, $0 < p \leqslant 1$, are not spaces of distributions. This makes them more interesting in a sense but, on the other hand, it urges us to seek a more natural kind of weighted H^p.

We study the spaces $H^p(w(x)\,dx)$ for $0 < p \leqslant 1$. The atomic approach cannot be applied to study $H^p(w(x)\,dx)$ for $p > 1$. If $w(x)$ has critical exponent $q_0 > 1$ (see I) we know that $H^p(w(x)\,dx)$ is equivalent to $\mathrm{Re}\,L^p(w(x)\,dx)$ for $p > q_0$. However, the spaces $H^p(w(x)\,dx)$ for $1 < p \leqslant q_0$ still remain a mystery.

Probably the most natural kind of weighted H^p spaces are the ones we have called $\mathfrak{H}^p(w(x)\,dx)$ which are defined directly by means of atoms in the homogeneous space given by the measure $w(x)\,dx$ (see also [4] and [12]). These spaces are studied in III. The atoms are of a different kind that those considered in II. However, it is proved (Theorem III.2.9) that there is an equivalence between $\mathfrak{H}^p(w(x)\,dx)$ and $H^p(w(x)^{1-p}\,dx)$ for a range of p's close to 1 and this equivalence is given, for atoms, by essentially multiplying times the weight.

IV is devoted to examples and applications. The space of radial functions in $H^1(\boldsymbol{R}^n)$ is found to be equivalent to the space of even functions in $\mathfrak{H}^1(|r|^{n-1}dr)$, the equivalence being the obvious one which assigns to $f(|x|)$ $(x \in \boldsymbol{R}^n)$, the function $f(|r|)$ $(r \in \boldsymbol{R})$. Two proofs of this equivalence are given. One starts from the atomic decomposition in $H^1(\boldsymbol{R}^n)$ and depends on the geometry of the euclidean space. The other is based upon the study of the operators arising from the restriction to radial functions, of different systems of Riesz transforms. The latter proof can be extended to provide a relation between the space of even functions in $\mathfrak{H}^1(|r|^{2\lambda}dr)$ (where $2\lambda \geqslant 1$ does not have to be an integer any more) and the space $\mathfrak{h}^1(|r|^{2\lambda}dr)$ introduced by Muckenhoupt and Stein ([14]).

Finally the relation between the space of radial H^1 functions on an euclidean space and the ordinary $H^1(\boldsymbol{R})$ is used to find a radial function F in $L^1(\boldsymbol{R}^2)$ but not in $H^1(\boldsymbol{R}^2)$ for which the complex-valued singular integral

$$\tilde{\tilde{F}}(w) = \text{p.v.} \int_{\boldsymbol{R}^2} \frac{1}{z^2} F(w-z)\,dz$$

is in $L^1(\boldsymbol{R}^2)$. The two components of this vector-valued singular integral form a system of second-order Riesz transforms. C. Fefferman has conjectured that "nice" conjugate systems, such as the second order Riesz transforms, would give a characterization of $H^1(\boldsymbol{R}^2)$ (see [7]). Our example disproves this conjecture. A similar phenomenon occurs for $H^1(K)$, where K is a local field (see [9]).

Although we have restricted our attention to the real line $\boldsymbol{R}$, a similar theory can be developed for the euclidean space $\boldsymbol{R}^n$, $n \geqslant 2$. The most significant modification should be the use of the Whitney decomposition to prove an analog of Lemma II.3.5 which is the key to obtain the atomic decomposition.(*)

(*) For the case $w(x) \equiv \boldsymbol{R}^n$, the atomic decomposition has been obtained by Latter. See his article in Studia Mathematica.

Chapter I

Some preliminary lemmas

Let $w(x)$ be a weight on the real line $\boldsymbol{R}$ satisfying the condition A_q of Muckenhoupt ([13]) for some $q \geqslant 1$. We will say briefly that w is an A_q weight or that $w \in A_q$. The following lemma is a simple consequence of Hölder's inequality:

LEMMA I.1. *If $f \in L^q(w(x)\,dx)$, it is locally integrable. In particular, for every finite interval I:*

$$\frac{1}{|I|}\int_I |f(x)|\,dx \leqslant (\text{const})\left(\frac{1}{w(I)}\int_I |f(x)|^q w(x)\,dx\right)^{1/q},$$

where (const) *depends only on the constant in the A_q condition for w. $w(I)$ stands for $\int_I w(x)\,dx$.*

Given $a > 1$ and a finite interval I, consider its a-dilation $\delta_a(I)$, that is: $\delta_a(I) = C_I + a(I - C_I)$, where C_I is the center of I. Substituting χ_I for f and $\delta_a(I)$ for I in the previous result, we obtain:

LEMMA I.2. $w(\delta_a(I)) \leqslant (\text{const})\, a^q w(I)$, *where* (const) *depends only on the constant of the A_q condition for w.*

For the rest of the chapter we will assume that w is an A_q weight for some $q > 1$. We will denote by q_0 the critical exponent for w, that is, the infimum of all the q's such that w satisfies the condition A_q. If w satisfies A_q for some $q > 1$, then it also satisfies $A_{q-\varepsilon}$ for some $\varepsilon > 0$ (see [2]). Thus, unless $q_0 = 1$, w is never an A_{q_0} weight.

LEMMA I.3. *There exists a, $0 < a < 1$, depending on the weight and on q, such that if $f \in L^q(w(x)\,dx)$ and φ satisfies the condition $|\varphi(x)| \leqslant (\text{const})/(1+|x|)^a$; the convolution $(f*\varphi_t)(x)$ is well defined ($\varphi_t(x)$ being $\frac{1}{t}\varphi\left(\frac{x}{t}\right)$, $t > 0$) and $|(f*\varphi_t)(x)| \leqslant (\text{const})\,\|f\|_{L^q(w(x)dx)}\, w(I(x;t))^{-1/q}$, where $I(x;t) = \{y\colon |x-y| < t\}$.*

Proof. $w^{-1/(q-1)}$ satisfies $A_{q'}$, where $q' = q/(q-1)$, exponent conjugate to q. Taking $a < 1$ sufficiently close to 1, $w^{-1/(q-1)}$ still satisfies $A_{aq'}$. If

this is so, we will see that for every x, $\varphi_t(x-y)/w(y)$ belongs to $L^{q'}(w(y)\,dy)$ and

$$\left\| \frac{\varphi_t(x-y)}{w(y)} \right\|_{L^{q'}(w(y)dy)} \leqslant (\text{const})\, w\big(I(x;\ t)\big)^{-1/q}$$

from which the lemma follows immediately.

It is clear, just because φ is bounded, that:

$$\text{(I.4)} \qquad \int\limits_{|x-y|<t} \left| \frac{\varphi_t(x-y)}{w(y)} \right|^{q'} w(y)\,dy \leqslant (\text{const})\, t^{-q/(q-1)} \int\limits_{I(x;t)} w(y)^{-1/(q-1)}\,dy\,.$$

But

$$\int\limits_{t<|x-y|} \left| \frac{\varphi_t(x-y)}{w(y)} \right|^{q'} w(y)\,dy \leqslant t^{-q/(q-1)} t^{aq'} \int\limits_{y\notin I(x;t)} \frac{(\text{const})}{|x-y|^{aq'}}\, w(y)^{-1/(q-1)}\,dy$$

so that applying Lemma 1 of [10] to the weight $w(y)^{-1/(q-1)}$ we obtain that also:

$$\text{(I.5)} \qquad \int\limits_{t<|x-y|} \left| \frac{\varphi_t(x-y)}{w(y)} \right|^{q'} w(y)\,dy \leqslant (\text{const})\, t^{-q/(q-1)} \int\limits_{I(x;t)} w(y)^{-1/(q-1)}\,dy\,.$$

Putting together (I.4) and (I.5) we get:

$$\int\limits_{-\infty}^{\infty} \left| \frac{\varphi_t(x-y)}{w(y)} \right|^{q'} w(y)\,dy \leqslant (\text{const})\, w\big(I(x;\ t)\big)^{-1/(q-1)}$$

which ends the proof of (I.3).

With a function φ as above we can associate the non-tangential maximal operator φ_∇^* defined on functions $f \in L^q(w(x)\,dx)$ by:

$$\varphi_\nabla^*(f)(x) = \sup_{|x-y|<t} |(f*\varphi_t)(y)|\,.$$

If $|\varphi(x)| \leqslant (\text{const})/(1+|x|)^a$ for some $a > 1$, then $\varphi_\nabla^*(f)(x) \leqslant (\text{const})\, f^*(x)$, where f^* is the Hardy–Littlewood maximal function of f. Thus, it follows from Theorem 1 in [2] that φ_∇^* is bounded in $L^q(w(x)\,dx)$. In particular, the function defined on the upper half plane $\boldsymbol{R}_+^2 = \{(x, t)\colon t > 0\}$ by $f(x, t) = (f*\varphi_t)(x)$, is in $L^q(w(x)\,dx)$ uniformly in t.

LEMMA I.6. *Let $s(x, t)$ be a non-negative subharmonic function on the upper half plane, which is in $L^q(w(x)\,dx)$ uniformly in $t > 0$. Then there exists a with $0 < a < 1$ such that $s(x, t) \leqslant (\text{const})(1+|x|)^a$, where* (const) *depends on t but can be taken to be the same for a strip $0 < t_0 \leqslant t \leqslant t_1 < \infty$.*

Proof. It follows from Lemma I.3 that, taking $a < 1$ close enough to 1

$$\int\limits_{-\infty}^{\infty} s(x, t)(1+|x|)^{-a}\,dx \leqslant (\text{const})\, w\big(I(0;\ t)\big)^{-1/q}.$$

Of course, the estimate $s(x, t) \leqslant (\text{const})\ (1+|x|)^a$ needs to be proved only for x large, since s is continuous. Take $(\bar{x}, \bar{t})$ in the strip and such that, say $|\bar{x}| > t_0$. By the subharmonicity:

$$s(\bar{x}, \bar{t}) \leqslant (\text{const}) \frac{1}{t_0^2} \iint\limits_{B((\bar{x},\bar{t});\, t_0/2)} s(x, t)\, dx\, dt$$

$$\leqslant (\text{const}) \frac{1}{t_0^2} \left(1 + \left|\bar{x} \pm \frac{t_0}{2}\right|\right)^a t_1 \leqslant (\text{const})(1+|\bar{x}|)^a.$$

The same method leads, via Hölder's inequality to:

LEMMA I.7. *Under the same hypotheses of* I.6:

$$s(x, t) \leqslant (\text{const}) \sup_{u>0} \Big(\int_{-\infty}^{\infty} s(y, u)^q w(y)\, dy \Big)^{1/q} w\big(I(x;\ t)\big)^{-1/q}.$$

In particular $s(x, t) = o(t)$ *as* $t \to \infty$.

I.6 and I.7 provide information about the growth of s at ∞. They will be used in connection with:

LEMMA I.8. *Let* $s(x, t)$ *be a continuous function on* $\overline{\boldsymbol{R}_+^2} = \{(x, t) \in \boldsymbol{R}^2\colon t \geqslant 0\}$ *which is subharmonic on* $\boldsymbol{R}_+^2$ *and has the following growth restrictions*:

(i) $s(x, t) = o(t)$ *as* $t \to \infty$.

(ii) $s(x, t) = o(e^{a|x|})$ *as* $|x| \to \infty$ *for all* $a > 0$, *in each strip* $\{(x, t) \in \boldsymbol{R}^2\colon 0 < t \leqslant t_0\}$.

Then if $s(x, 0) \leqslant A < \infty$ *for all* $x \in \boldsymbol{R}$, *it follows that* $s(x, t) \leqslant A$ *for all* $(x, t) \in \boldsymbol{R}_+^2$.

This is an easy extension of (5.2)(a) in Chapter II of [18].

LEMMA I.9. *Let* $f \in L^q\big(w(x)\,dx\big)$. *The Poisson integral of* f, $f(x, t) = (f * P_t)(x)$ *is a harmonic function on* $\boldsymbol{R}_+^2$ *which is in* $L^q\big(w(x)\,dx\big)$ *uniformly in* t. *Suppose also that* f *is continuous in* $\boldsymbol{R}$ *and* $|f(x)| \leqslant (\text{const})(1+|x|)^a$ *for some* $a < 1$, *then the function defined by* $f(x, t) = (f * P_t)(x)$ *if* $t > 0$ *and* $f(x, 0) = f(x)$ *is continuous in* $\overline{\boldsymbol{R}_+^2}$ *and satisfies* (i) *and* (ii) *in Lemma* I.8.

The proof is routine.

LEMMA I.10. *Let* $s(x, t)$ *be a non-negative subharmonic function on* $\boldsymbol{R}_+^2$ *which is in* $L^q\big(w(x)\,dx\big)$ *uniformly in* $t > 0$. *Then* s *has a least harmonic majorant which is the Poisson integral of some* $s_0 \in L^q\big(w(x)\,dx\big)$.

Proof. For every $t > 0$ let $s_t(x) = s(x, t)$. $s_t \in L^q\big(w(x)\,dx\big)$ uniformly in $t > 0$. Thus, the set of functions $\{s_t\colon t > 0\}$ is contained in a closed ball of $L^q\big(w(x)\,dx\big) = \big(L^{q'}\big(w(x)\,dx\big)\big)^*$, dual space of $L^{q'}\big(w(x)\,dx\big)$, where $q' = q/(q-1)$. This ball, with the weak-* topology given on $L^q\big(w(x)\,dx\big)$ by $L^{q'}\big(w(x)\,dx\big)$, is a compact space. Therefore, there will be a sequence $t_n \to 0$ such that $s_{t_n} \to s_0 \in L^q\big(w(x)\,dx\big)$, the convergence being in the weak-*

topology; that is: for every $\varphi \in L^{q'}(w(x)\,dx)$, $\int_{-\infty}^{\infty} s_{t_n}(x)\varphi(x)w(x)\,dx \to \int_{-\infty}^{\infty} s_0(x)\varphi(x)w(x)\,dx$. Let us consider, for each n, the harmonic function $(x, t)\mapsto(s_{t_n}*P_t)(x)$. For each n, x, t, we have:

$$\text{(I.11)}: \qquad s(x, t+t_n) \leqslant (s_{t_n}*P_t)(x).$$

This can be obtained by applying (I.8) to the function $(x, t)\mapsto s(x, t+t^n) - (s_{t_n}*P_t)(x)$ if $t > 0$

$$(x, 0)\mapsto s(x, t_n) - s_{t_n}(x) = 0.$$

That (I.8) can actually be applied to this function follows from (I.6), (I.7) and (I.9). Now, letting $n\to\infty$ in (I.11) and using the fact that $P_t(x-y)/w(y)$ is in $L^{q'}(w(y)\,dy)$, we get: $s(x, t) \leqslant (s_0*P_t)(x)$.

In the next two lemmas we investigate the relations between different powers of a given weight.

LEMMA I.11. *Let w be a weight with critical exponent q_0. Let $\delta > 0$ be such that w satisfies a reverse Hölder's inequality with exponent $1+\delta$. Then for each p such that $1-\delta/q_0 < p \leqslant 1$, there exists $q > q_0(1-p)+p$ and a constant such that for every I:*

(I.12)

$$\left(\frac{1}{\int_I w(x)^{1-p}\,dx}\int_I w(x)^q w(x)^{1-p}\,dx\right)^{1/q} \leqslant (\text{const})\left(\frac{1}{\int_I w(x)^{1-p}\,dx}\int_I w(x)\,dx\right)^{1/p}.$$

Proof. For $p = 1$ (I.12) is just a reverse Hölder's inequality and there is nothing to be proved (see [2]). Let us assume, therefore, that $p < 1$. $w(x)^{1-p}$ is as good a weight as $w(x)$; its critical exponent is $\leqslant q_0(1-p)+p \leqslant q_0$. Let $q_1 > q_0$. Then $w(x)^{1-p}$ is an A_{q_1} weight. It follows that there are two constants c_1 and c_2 such that:

$$\text{(I.13)} \qquad c_1 \leqslant \left(\frac{1}{|I|}\int_I w(x)^{1-p}\,dx\right)^{1/(1-p)}\left(\frac{1}{|I|}\int_I w(x)^{-s}\,dx\right)^{1/s} \leqslant c_2,$$

where $s = (1-p)/(q_1-1)$. If $p > 1-\delta/q_0$, then $\delta+p > q_0(1-p)+p$. Let $q = \delta+p$. Then the left-hand side of (I.12) is

$$\left(\frac{1}{\int_I w(x)^{1-p}\,dx}\int_I w(x)^{1+\delta}\,dx\right)^{1/q} \leqslant \frac{(\text{const})}{\left[\frac{1}{|I|}\int_I w(x)^{1-p}\,dx\right]^{1/q}}\left(\frac{1}{|I|}\int_I w(x)\,dx\right)^{(1+\delta)/q}$$

$$= (\text{const})\left(\frac{\int_I w(x)\,dx}{\int_I w(x)^{1-p}\,dx}\right)^{1/p}\left\{\left(\frac{1}{|I|}\int_I w(x)^{1-p}\,dx\right)^{1/p-1/q}\left(\frac{1}{|I|}\int_I w(x)\,dx\right)^{(1+\delta)/q-1/p}\right\}.$$

We will have proved (I.12) as soon as we show that the expression between brackets is bounded. But, by (I.13) this is

$$\leqslant (\text{const}) \left\{ \left(\frac{1}{|I|} \int_I w(x)^{-s} dx \right)^{1/s} \left(\frac{1}{|I|} \int_I w(x) dx \right) \right\}^{-(1-p)(1/p-1/q)}$$

and $\left(\frac{1}{|I|} \int_I w(x)^{-s} dx \right)^{1/s} \left(\frac{1}{|I|} \int_I w(x) dx \right)$ is bounded away from 0 and ∞ since $s = 1/(N-1)$ for $N = 1+(q_1-1)/(1-p) > q_1$ so that w is an A_N weight.

LEMMA I.14. *Let w be a weight with critical exponent q_0. Let $0 < p < 1$. Then for every r such that $1 < r < 1+(1-p)/(q_0-1)$, there exists a constant such that for every bounded interval I:*

$$\left(\frac{\int_I w(x)^{1-r} dx}{\int_I w(x) dx} \right)^{1/r} \leqslant (\text{const}) \left(\frac{\int_I w(x)^{1-p} dx}{\int_I w(x) dx} \right)^{1/p}.$$

Proof. Let $q_1 = (r-p)/(r-1) > q_0$. Then $r = 1+(1-p)/(q_1-1)$. Let $s = (1-p)/(q_1-1) > 0$. The exponent conjugate to r will be $r' = 1+1/s = 1+(q_1-1)/(1-p) > q_1$. Therefore w satisfies the condition $A_{r'}$ so that we have:

$$1 \leqslant \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-s} dx \right)^{1/s} \leqslant (\text{const}).$$

The fact that w is in $A_{r'}$ is equivalent to the fact that $w(x)^{-1}$ satisfies a reverse Hölder's inequality with respect to $w(x)$ with exponent r (see [2]).

Also $w(x)^{1-p}$ satisfies A_{q_1} so that we have:

$$1 \leqslant \left(\frac{1}{|I|} \int_I w(x)^{1-p} dx \right)^{1/(1-p)} \left(\frac{1}{|I|} \int_I w(x)^{-s} \right)^{1/s} \leqslant (\text{const}).$$

Putting all these facts together we have:

$$\left(\frac{\int_I w(x)^{1-r} dx}{\int_I w(x) dx} \right)^{1/r} \leqslant (\text{const}) \frac{|I|}{w(I)} \leqslant \frac{(\text{const})}{\left(\frac{1}{|I|} \int_I w(x) dx \right)^{1/p} \left(\frac{1}{|I|} \int_I w(x)^{-s} dx \right)^{\left(\frac{1}{p}-1\right)\frac{1}{s}}}$$

$$\leqslant (\text{const}) \left(\frac{\int_I w(x)^{1-p} dx}{\int_I w(x) dx} \right)^{1/p}.$$

Chapter II

Weighted H^p spaces of analytic functions

1. Behaviour at the boundary

For a weight w with critical exponent q_0 and $p > 0$ let $H^p(w(x)dx)$ be the space of all analytic functions $F(x+it)$ on the upper half plane such that

$$\|F\|_{H^p(w(x)dx)} = \sup_{t>0} \Big\{ \int_{-\infty}^{\infty} |F(x+it)|^p w(x)\,dx \Big\}^{1/p} < \infty.$$

The classical results about the behaviour at the boundary of an H^p function extend to this situation in the following way:

THEOREM II.1.1. *Let $F \in H^p(w(x)dx)$. Then:*

(i) $\lim_{t\to 0} F(x+it)$ *exists for a.e. $x \in \boldsymbol{R}$ and the function $F(x) = \lim_{t\to 0} F(x+it)$ is in $L^p(w(x)dx)$. Actually F has non-tangential limits at a.e. $x \in \boldsymbol{R}$.*

(ii) $\lim_{t\to 0} \int_{-\infty}^{\infty} |F(x+it) - F(x)|^p w(x)\,dx = 0$ *and consequently*

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |F(x+it)|^p w(x)\,dx = \int_{-\infty}^{\infty} |F(x)|^p w(x)\,dx.$$

(iii) $\|F\|_{H^p(w(x)dx)} \leqslant (\text{const}) \Big\{ \int_{-\infty}^{\infty} |F(x)|^p w(x)\,dx \Big\}^{1/p}$.

Proof. We will show that the non-tangential maximal function $m_F(x) = \sup_{|x-y|<t} |F(y+it)|$ is in $L^p(w(x)dx)$. By a result of Calderón (Theorem 3.19 in [18]) (i) follows immediately and (ii) is a consequence of Lebesgue's dominated convergence theorem. If q_0 is the critical exponent of w, let $s(z) = |F(z)|^\varepsilon$ with $0 < \varepsilon < p/q_0$. Then $s(x+it)$ is a non-negative subharmonic function on $\boldsymbol{R}^2_+$ which is uniformly in $L^q(w(x)dx)$ with $q = p/\varepsilon > q_0$. By Lemma I.10: $s(x+it) \leqslant (s_0 * P_t)(x)$ for some $s_0 \in L^q(w(x)dx)$. Thus $m_F(x) \leqslant \big(P_F^*(s_0)(x)\big)^{1/\varepsilon} \leqslant (\text{const}) s_0^*(x)^{1/\varepsilon}$ and $\int_{-\infty}^{\infty} m_F(x)^p w(x)\,dx \leqslant (\text{const}) \times \int_{-\infty}^{\infty} |s_0(x)|^q w(x)\,dx < \infty$. In order to prove (iii) it is only necessary to

observe that $s_0(x) = |F(x)|^s$. Actually, since it follows from (ii) that $\int_{-\infty}^{\infty} |F(x)|^p w(x)\,dx \leqslant \|F\|_{H^p(w(x)dx)}$; $F \mapsto \{\int_{-\infty}^{\infty} |F(x)|^p w(x)\,dx\}^{1/p}$ is a "norm" equivalent to $\|F\|_{H^p(w(x)dx)}$.

LEMMA II.1.2. *For $F \in H^p(w(x)\,dx)$:*

$$|F(x,t)| \leqslant (\text{const})\,\|F\|_{H^p(w(x)dx)}\, w\big(I(x;t)\big)^{-1/p}.$$

Proof. Apply (I.7) to $s(x,t) = |F(x,t)|^s$. An important consequence of (II.1.2) is that the imbedding of $H^p(w(x)\,dx)$ into the space $H(\boldsymbol{R}^2_+)$ of all the functions holomorphic in the upper half plane with the topology of uniform convergence on compact subsets, is continuous. The following result also follows from (II.1.2).

LEMMA II.1.3. *For any $p > 0$, $H^p(w(x)\,dx)$ is complete.*

The next result is very useful in determining whether or not a given function belongs to $H^p(w(x)\,dx)$.

LEMMA II.1.4. *Let w_1 and w_2 be A_∞ weights. Suppose that $F \in H^{p_1}(w_1(x)\,dx)$ and its boundary function belongs to $L^{p_2}(w_2(x)\,dx)$ with $p_2 > p_1$. Then $F \in H^{p_2}(w_2(x)\,dx)$.*

Proof. Let $q > 1$ be such that $w_1, w_2 \in A_q$. Then $|F|^{p_1/q}$ is a subharmonic function which is uniformly in $L^q(w_1(x)\,dx)$. It follows from (II.1.1) and (I.10) that:

$$|F(x+it)|^{p_1/q} \leqslant P_t * (|F|^{p_1/q})(x),$$

where we use the notation $F(x)$ for the boundary function of F. But $|F|^{p_1/q}$ belongs to $L^{qp_2/p_1}(w_2(x)\,dx)$ and, since $\frac{q}{p_1}\,p_2 > q$ we have:

$$\sup_{|x-y|<t} |F(y+it)|^{p_1/q} \leqslant P_t^*(|F|^{p_1/q})(x) \leqslant (\text{const})\,|F(x)|^{\frac{p_1}{q}*}$$

which belongs to $L^{qp_2/p_1}(w_2(x)\,dx)$. Then $\sup_{|x-y|<t} |F(y+it)| \leqslant (\text{const})\,|F(x)|^{\frac{p_1}{q}*\frac{q}{p_1}}$ which belongs to $L^{p_2}(w_2(x)\,dx)$. It follows that $F \in H^{p_2}(w_2(x)\,dx)$.

If $F \in H^p(w(x)\,dx)$ with $p > q_0$, then $F(x+it) = (F * P_t)(x)$ and $F(x) = f(x) + i\tilde{f}(x)$, where f is real and $\tilde{f}$ is its Hilbert transform. The correspondence $F \leftrightarrow f$ is an equivalence of metric linear spaces between $H^p(w(x)\,dx)$ and $\operatorname{Re} L^p(w(x)\,dx)$. The situation is quite different for $p \leqslant q_0$. Simple examples show that the correspondence $F(x+it) \leftrightarrow \operatorname{Re} F(x)$ is no longer $1:1$. We will concentrate our attention on these spaces, in particular on those for which $p \leqslant 1$. The first thing we do is to avail ourselves of a nice dense subclass.

THEOREM II.1.5. *For any positive integer N, let $\mathscr{I}^p_N$ be the subspace formed by the functions $F \in H^p(w(x)\,dx)$, such that:*

(i) *F is continuous in* $\overline{\boldsymbol{R}^2_+} = \{(x, t) \mid t \geqslant 0\}$.

(ii) $F(x+it) = O(|x|^{-N})$ *as* $|x| \to \infty$ *in each strip* $0 \leqslant t \leqslant t_0$.

Then $\mathscr{I}^p_N$ *is dense in* $H^p(w(x)\,dx)$.

Proof. Since every $F(z)$ in $H^p(w(x)\,dx)$ is the limit in $H^p(w(x)\,dx)$ of the functions $F(z+it)$ as $t \to 0$, we can assume that F is continuous in $\overline{\boldsymbol{R}^2_+}$ and also that $F(x+it) = O(|x|^M)$ as $|x| \to \infty$ in each strip $0 \leqslant t \leqslant t_0$. For every n, positive integer, consider the analytic function in $\boldsymbol{R}^2_+$

$$G_n(z) = \left(i \Big/ \left(\frac{z}{n} + i\right)\right)^{M+N}; \qquad |G_n(z)| \leqslant 1.$$

Clearly $G_n(z)F(z)$ is in $H^p(w(x)\,dx)$, is continuous in $\overline{\boldsymbol{R}^2_+}$ and, besides, on any strip $0 \leqslant t \leqslant t_0$: $|G_n(x+it)F(x+it)| \leqslant (\text{const})/(1+|x|)^N$. Thus, for every n, $G_n(z)F(z)$ is in $\mathscr{I}^p_N$; $G_n(z)F(z) \to F(z)$ in $H^p(w(x)\,dx)$ as $n \to \infty$. Indeed

$$\|G_nF - F\|^p_{H^p(w(x)dx)} \leqslant (\text{const}) \int_{-\infty}^{\infty} \left|\left(i \Big/ \left(\frac{x}{n} + i\right)\right)^{M+N} - 1\right|^p |F(x)|^p w(x)\,dx \to 0$$

as $n \to \infty$

because $\left|\left(i \Big/ \left(\frac{x}{n} + i\right)\right)^{M+N} - 1\right| \to 0$ boundedly.

Corollary II.1.6. *For any q, $H^p(w(x)\,dx) \cap H^q(w(x)\,dx)$ is dense in $H^p(w(x)\,dx)$.*

Proof. This follows from the fact that for every $N > q_0/q$, $\mathscr{I}^p_N \subset H^p(w(x)\,dx) \cap H^q(w(x)\,dx)$. Indeed, if $F \in \mathscr{I}^p_N$, the boundary function will be in $L^q(w(x)\,dx)$. Then it follows from (II.1.4) that $F \in H^q(w(x)\,dx)$.

From (II.1.5) and the equivalence between $H^p(w(x)dx)$ and the space of boundary functions, it follows that we can view $H^p(w(x)\,dx)$ for $p \leqslant 1$ as the completion with respect to the quasi-norm (see [19]) $\|f\|^p_{L^p(w(x)dx)} + \|\tilde{f}\|^p_{L^p(w(x)dx)}$ (or the equivalent one $\|P^*_\gamma(f+i\tilde{f})\|^p_{L^p(w(x)dx)}$) of a space of "nice" real functions.

2. Maximal function characterization

On $\operatorname{Re} L^q(w(x)\,dx)$, $q > q_0$, we consider the gauge

$$f \mapsto \|P^*_\gamma(f+i\tilde{f})\|^p_{L^p(w(x)dx)} \tag{II.2.1}$$

The space of functions for which it is finite becomes, once completed, an equivalent copy of $H^p(w(x)\,dx)$. In this section we introduce several gauges which will turn out to be equivalent to (II.2.1) (in the obvious

sense of dominating each other, i.e.: bounding each other up to multiplicative constants) thus providing new ways to look at $H^p(w(x)\,dx)$. The starting point will be the gauge

$$f \mapsto \|P_\gamma^*(f)\|^p_{L^p(w(x)dx)} \tag{II.2.2}$$

which is, of course, dominated by (II.2.1). (II.2.2) is given by the non-tangential maximal operator associated to the Poisson kernel. Now we will consider other approximations to the identity. In general if φ is such that $|\varphi(x)| \leqslant (\text{const})/(1+|x|)^a$ with $a > 1$, we can associate with each $f \in \operatorname{Re} L^q(w(x)\,dx)$ a function defined on $\boldsymbol{R}^2_+$ as $f(x,t) = (f*\varphi_t)(x)$ and derive from it:

(i) the non-tangential maximal function

$$\varphi_\gamma^*(f)(x) = \sup_{|x-y|<t} |f(y,t)|;$$

(ii) In general, for $N \geqslant 1$, the non-tangential maximal function of amplitude N

$$\varphi_{\gamma,N}^*(f)(x) = \sup_{|x-y|<Nt} |f(y,t)|;$$

(iii) For $M \geqslant 1$, the tangential maximal function with exponent M:

$$\varphi_M^{**}(f)(x) = \sup_{(y,t)\in \boldsymbol{R}^2_+} |f(y,t)| \left(\frac{t}{|x-y|+t}\right)^M.$$

Next we investigate the relations among these different maximal operators.

LEMMA II.2.3. $\|\varphi_{\gamma,N}^*(f)\|^p_{L^p(w(x)dx)} \leqslant (\text{const}) N^q \|\varphi_\gamma^*(f)\|^p_{L^p(w(x)dx)}$, *where* (const) *depends only on w, p and q.*

Proof. For any $\lambda > 0$ consider the open set $E_\lambda = \{x \in \boldsymbol{R}\colon \varphi_\gamma^*(f)(x) > \lambda\}$ and let $E_\lambda = \bigcup_i I_i$ be its decomposition in connected components. Then, clearly $\{x \in \boldsymbol{R} \mid \varphi_{\gamma,N}^*(f)(x) > \lambda\} \subset \bigcup_i \delta_N(I_i)$. Applying (I.2) the lemma follows.

LEMMA II.2.4. *If $M > q/p$, then*

$$\|\varphi_M^{**}(f)\|^p_{L^p(w(x)dx)} \leqslant (\text{const}) \|\varphi_\gamma^*(f)\|^p_{L^p(w(x)dx)},$$

where (const) *depends only on M, p, q and w.*

Proof. Fix $x \in \boldsymbol{R}$. Then, breaking up $\boldsymbol{R}^2_+$ as the union of the sets $\{(y,t)\colon |x-y| < t\}$, $\{(y,t)\colon 2^k t \leqslant |x-y| < 2^{k+1}t\}$, $k = 0, 1, \ldots$; we obtain:

$$(\varphi_M^{**}(f)(x))^p \leqslant (\varphi_\gamma^*(f)(x))^p + \sum_0^\infty 2^{-kMp} (\varphi_{\gamma,2^{k+1}}^*(f)(x))^p.$$

Thus, by (II.2.3):

$$\|\varphi_M^{**}(f)\|^p_{L^p(w(x)dx)} \leqslant \|\varphi_V^*(f)\|^p_{L^p(w(x)dx)}\Big(1+(\text{const})\sum_0^\infty (2^{q-Mp})^k\Big)$$
$$= (\text{const})\,\|\varphi_V^*(f)\|^p_{L^p(w(x)dx)},$$

since $q-Mp<0$. As a consequence of (II.2.4) we see that the gauge $f\mapsto \|P_M^{**}(f)\|^p_{L^p(w(x)dx)}$ is dominated by (II.2.2) provided $M>q/p$.

Now following Fefferman and Stein ([8]) we pass from the Poisson kernel to a kernel σ in the class $\mathscr{S}$ of Schwartz. We start with a continuous function ψ defined on $[1,\infty[$ which is rapidly decreasing at ∞ and such that $\int_1^\infty \psi(\lambda)\,d\lambda = 1$ and $\int_1^\infty \lambda^k\psi(\lambda)\,d\lambda = 0$ for $k=1,2,\ldots$ (see [17], p. 182). Then define $\sigma(x) = \int_1^\infty \psi(s)P_s(x)\,ds$. It is easy to see that $\sigma\in\mathscr{S}$. Other interesting properties of σ are: (i) σ is even; (ii) σ has all moments of order $\geqslant 1$ equal to zero; (iii) $\int_{-\infty}^\infty \sigma(x)\,dx = 1$.

σ gives rise to the approximate identity $\{\sigma_t\}$, where $\sigma_t(x) = \dfrac{1}{t}\sigma\Big(\dfrac{x}{t}\Big)$ $= \int_1^\infty \psi(s)P_{st}(x)\,ds$. It is clear that:

LEMMA II.2.5. $\sigma_V^*(f)(x)\leqslant(\text{const})\,P_M^{**}(f)(x)$.

Thus, we have the following chain of gauges, where $<$ signifies "is dominated by":

$$\|\sigma_V^*(f)\|^p_{L^p(w(x)dx)} < \|P_M^{**}(f)\|^p_{L^p(w(x)dx)} < \|P_V^*(f)\|^p_{L^p(w(x)dx)}.$$

Now we pass from $\{\sigma_t\}$ to a very general kind of approximations to the identity. We do it in several steps.

LEMMA II.2.6. *Let φ be in $\mathscr{S}$ and suppose that $\eta = \xi*\varphi_s$ with $\xi\in\mathscr{S}$ and $0<s\leqslant 1$. Then*

$$\eta_V^*(f)(x)\leqslant(\text{const})s^{-M}\Big(\int_{-\infty}^\infty |\xi(\lambda)|(1+|\lambda|)^M\,d\lambda\Big)\,\varphi_M^{**}(f)(x).$$

The proof is in [8].

LEMMA II.2.7. *Let σ be as in* (II.2.5). *Let η be $\mathscr{C}^\infty$ with support contained in $[-A,A]$. Then*

$$\eta_V^*(f)(x)\leqslant(\text{const})\Big(\int_{-\infty}^\infty |\eta(u)|\,du + \int_{-\infty}^\infty \Big|\frac{d^{M+1}\eta(u)}{du^{M+1}}\Big|\,du\Big)\,\sigma_M^{**}(f)(x),$$

where (const) *depends on A and M but is otherwise independent of either η or f.*

Proof.

$$\eta = \eta*\sigma*\sigma + \sum_{k=0}^{\infty} \eta*\sigma_{2^{-k-1}}*\sigma_{2^{-k-1}} - \eta*\sigma_{2^{-k}}*\sigma_{2^{-k}}$$

$$= \eta*\sigma*\sigma + \sum_{k=0}^{\infty} \eta*(\sigma_{2^{-k-1}} - \sigma_{2^{-k}})*(\sigma_{2^{-k-1}} + \sigma_{2^{-k}})$$

$$= \eta*\sigma*\sigma + \sum_{k=0}^{\infty} \eta*(\sigma_-)_{2^{-k}}*(\sigma_+)_{2^{-k}},$$

where $\sigma_- = \sigma_{2^{-1}} - \sigma$ and $\sigma_+ = \sigma_{2^{-1}} + \sigma$.

Now we apply Lemma (II.2.6) with σ and σ_+ in place of φ and, since $(\sigma_+)_M^{**}(f)(x) \leqslant (2^M+1)\sigma_M^{**}(f)(x)$:

$$\text{(II.2.8)} \qquad \eta_p^*(f)(x) \leqslant (\text{const})\,\sigma_M^{**}(f)(x)\Big\{\int_{-\infty}^{\infty} |(\eta*\sigma)(\lambda)|(1+|\lambda|)^M d\lambda + \sum_{k=0}^{\infty} 2^{kM} \int_{-\infty}^{\infty} |(\eta*(\sigma_-)_{2^{-k}})(\lambda)|(1+|\lambda|)^M d\lambda\Big\},$$

where (const) depends only on M

$$\int_{-\infty}^{\infty} |(\eta*\sigma)(\lambda)|(1+|\lambda|)^M d\lambda \leqslant (\text{const}) \int_{-\infty}^{\infty} |\eta(u)|\,du,$$

where (const) depends on A and M.

Now, using the fact that σ_- is even and has all moments equal to zero, we can write:

$$(\eta*(\sigma_-)_s)(\lambda) = \int_{-\infty}^{\infty} \Big\{\eta(\lambda+u) - \eta(\lambda) - u\eta'(\lambda) - \ldots - \frac{u^M}{M!}\eta^{(M)}(\lambda)\Big\}(\sigma_-)_s(u)\,du$$

$$= \int_{-\infty}^{\infty} \Big\{\frac{u^{M+1}}{M!}\int_0^1 (1-r)^M \eta^{(M+1)}(\lambda+ru)\,dr\Big\}(\sigma_-)_s(u)\,du.$$

Therefore:

$$\int_{-\infty}^{\infty} |(\eta*(\sigma_-)_s)(\lambda)|(1+|\lambda|)^M d\lambda$$

$$\leqslant \frac{1}{M!}\int_0^1 (1-r)^M \int_{-\infty}^{\infty} |u|^{M+1}|(\sigma_-)_s(u)| \int_{-\infty}^{\infty} (1+|\lambda|)^M |\eta^{(M+1)}(\lambda+ru)|\,d\lambda\,du\,dr.$$

But

$$\int_{-\infty}^{\infty} (1+|\lambda|)^M |\eta^{(M+1)}(\lambda+ru)|\,d\lambda \leqslant (\text{const})(1+|u|)^M \int_{-\infty}^{\infty} |\eta^{(M+1)}(\lambda)|\,d\lambda$$

for every r with (const) depending only on A and M, so that:

$$\int_{-\infty}^{\infty} |(\eta*(\sigma_-)_s)(\lambda)|(1+|\lambda|)^M d\lambda$$

$$\leqslant (\text{const})\Big(\int_{-\infty}^{\infty} |u|^{M+1}|(\sigma_-)_s(u)|(1+|u|)^M du\Big)\Big(\int_{-\infty}^{\infty} |\eta^{(M+1)}(\lambda)|\,d\lambda\Big)$$

$$\leqslant (\text{const})\, s^{M+1} \int_{-\infty}^{\infty} |\eta^{(M+1)}(\lambda)|\,d\lambda.$$

Going back to (II.2.8), we get:

$$\eta_r^*(f)(x) \leqslant (\text{const})\,\sigma_M^{**}(f)(x)\Big(\int_{-\infty}^{\infty} |\eta(u)|\,du + \sum_{k=0}^{\infty} 2^{kM}2^{-k(M+1)} \int_{-\infty}^{\infty} |\eta^{(M+1)}(u)|\,du\Big)$$

$$= (\text{const})\Big(\int_{-\infty}^{\infty} |\eta(u)|\,du + \int_{-\infty}^{\infty} \Big|\frac{d^{M+1}\eta(u)}{du^{M+1}}\Big|\,du\Big)\,\sigma_M^{**}(f)(x)$$

as we wanted to show.

Suppose now that φ is a $\mathscr{C}^\infty$ function with support contained in $[-\delta, \delta]$ and $\mathscr{C}$ a fixed constant > 0. Take $(y, t) \in \boldsymbol{R}_+^2$ and let x be such that $|x-y| < \mathscr{C}\delta t$. $\varphi_{1/\mathscr{C}\delta}$ has support contained in $[-1/\mathscr{C}, 1/\mathscr{C}]$. Then:

$$|(\varphi_t*f)(y)| \leqslant (\varphi_{1/\mathscr{C}\delta})_r^*(f)(x)$$

$$\leqslant (\text{const})\Big(\int_{-\infty}^{\infty} |\varphi_{1/\mathscr{C}\delta}(u)|\,du + \int_{-\infty}^{\infty} |(\varphi_{1/\mathscr{C}\delta})^{(M+1)}(u)|\,du\Big)\,\sigma_M^{**}(f)(x)$$

$$\leqslant (\text{const})\Big(\int_{-\infty}^{\infty} |\varphi(u)|\,du + \delta^{M+1} \int_{-\infty}^{\infty} |\varphi^{(M+1)}(u)|\,du\Big)\,\sigma_M^{**}(f)(x)$$

with (const) depending only on $\mathscr{C}$. In particular, we have proved the following:

LEMMA II.2.9. *Let φ be $\mathscr{C}^\infty$, supported in $[-\delta, \delta]$ and x and y be such that $|x-y| < \mathscr{C}\delta$. Then:*

$$|(\varphi*f)(y)| \leqslant (\text{const})\Big(\int_{-\infty}^{\infty} |\varphi(u)|\,du + \delta^{M+1} \int_{-\infty}^{\infty} |\varphi^{(M+1)}(u)|\,du\Big)\,\sigma_M^{**}(f)(x),$$

where (const) *depends only on C.*

Now let φ be any $\mathscr{C}^\infty$ function with compact support. Let I_φ be the smallest closed interval containing the support of φ and C_φ the center

of I_φ. Let x be such that dist $(x, I_\varphi) < C|I_\varphi|$ for some fixed C. Then

$$\Big|\int_{-\infty}^{\infty} f(t)\varphi(t)\,dt\Big| = \big|(f*\varphi(C_\varphi-\cdot))(C_\varphi)\big|$$
$$\leqslant (\text{const})\Big(\int_{-\infty}^{\infty}|\varphi(u)|\,du + |I_\varphi|^{M+1}\int_{-\infty}^{\infty}|\varphi^{(M+1)}(u)|\,du\Big)\sigma_M^{**}(f)(x),$$

where (const) depends only on C.

Let us define the maximal operator:

$$S_M^*(f)(x) = \sup\left\{\frac{\Big|\int_{-\infty}^{\infty} f(t)\varphi(t)\,dt\Big|}{\int_{-\infty}^{\infty}|\varphi(u)|\,du + |I_\varphi|^{M+1}\int_{-\infty}^{\infty}|\varphi^{(M+1)}(u)|\,du}\right\},$$

where the supremum is taken over all the $\mathscr{C}^\infty$ functions φ with compact support and such that dist $(x, I_\varphi) < |I_\varphi|$. Then, we have seen that:

THEOREM II.2.10. $S_M^*(f)(x) \leqslant (\text{const})\,\sigma_M^{**}(f)(x)$.

Taking $M > q/p$ and $f \in \operatorname{Re} L^q(w(x)\,dx)$, this result adds a new link to our chain of gauges which now looks like this:

$$\|S_M^*(f)\|^p_{L^p(w(x)dx)} < \|\sigma_M^{**}(f)\|^p_{L^p(w(x)dx)} < \|\sigma_r^*(f)\|^p_{L^p(w(x)dx)}$$
$$< \|P_M^{**}(f)\|^p_{L^p(w(x)dx)} < \|P_r^*(f)\|^p_{L^p(w(x)dx)}.$$

3. Atomic decomposition

Let w be a weight with critical exponent q_0. For $0 < p \leqslant 1$ and r such that $r > p$ and $w \in A_r$, a (p, r)-atom with respect to w will be a real-valued function a, supported in an interval I and satisfying:

(i) $\left(\dfrac{1}{w(I)}\displaystyle\int_I |a(x)|^r w(x)\,dx\right)^{1/r} \leqslant w(I)^{-1/p}$ if $r < \infty$ or $\|a\|_\infty \leqslant w(I)^{-1/p}$ if $r = \infty$.

(ii) $\displaystyle\int_{-\infty}^{\infty} a(x)x^k\,dx = 0$ for $k = 0, 1, \ldots, [q_0/p]-1$, where $[q_0/p]$ stands for the biggest integer $\leqslant q_0/p$.

For the remaining of the section we will write $N_0 = [q_0/p]$. A (p, ∞)-atom is always a (p, r)-atom for any other r. Most of the results and proofs that follow will be written for $r < \infty$. In the next four propositions we will use the following notation: a will be a (p, r)-atom with respect to w, I the smallest closed interval containing the support of w, C_I the center of I and $I^* = \delta_2(I) = C_I + 2(I - C_I)$.

LEMMA II.3.1. $\int_{I^*} |\tilde{a}(x)|^p w(x)\,dx \leqslant (\text{const})$.

Proof.

$$\left(\frac{1}{w(I^*)}\int_{I^*} |\tilde{a}(x)|^p w(x)\,dx\right)^{1/p} \leqslant (\text{const})\left(\frac{1}{w(I)}\int_I |a(x)|^r w(x)\,dx\right)^{1/r}$$
$$\leqslant (\text{const})\, w(I)^{-1/p}.$$

Thus $\int_{I^*} |\tilde{a}(x)|^p w(x)\,dx \leqslant (\text{const})\, w(I^*)\, w(I)^{-1} \leqslant (\text{const})$.

LEMMA II.3.2. *For* $x \notin I^*$:

$$|\tilde{a}(x)| \leqslant (\text{const})\, w(I)^{-1/p}(|I|/|x-C_I|)^{N_0+1}.$$

Proof.

$$|\tilde{a}(x)| \leqslant (\text{const})\left|\text{p.v.}\int_{-\infty}^{\infty} \frac{a(y)}{x-y}\,dy\right|$$
$$= (\text{const})\left|\int_I a(y)\left(\frac{1}{x-y} - \sum_{k=1}^{N_0} \frac{(y-C_I)^{k-1}}{(x-C_I)^k}\right)dy\right|$$
$$\leqslant (\text{const})\left(\int_I |a(y)|^r w(y)\,dy\right)^{1/r} \left\|\frac{(y-C_I)^{N_0}}{(x-C_I-\theta_y(y-C_I))^{N_0+1}}\, w(y)^{-1/r}\chi_I(y)\right\|_{r'}$$

with $0 < \theta_y < 1$. But $|y-C_I| < |x-C_I|/2$ and consequently $|x-C_I-\theta_y(y-C_I)| \geqslant |x-C_I|/2$ so that:

$$|\tilde{a}(x)| \leqslant (\text{const})\, w(I)^{1/r-1/p}\frac{|I|^{N_0}}{|x-C_I|^{N_0+1}}\, \|w(y)^{-1/r}\chi_I(y)\|_{r'}$$
$$\leqslant (\text{const})\, w(I)^{-1/p}(|I|/|x-C_I|)^{N_0+1}.$$

THEOREM II.3.3. $\|\tilde{a}\|^p_{L^p(w(x)dx)} \leqslant (\text{const})$.

Proof.

$$\int_{x\notin I^*} |\tilde{a}(x)|^p w(x)\,dx \leqslant (\text{const})\frac{1}{w(I)}|I|^{p(N_0+1)}\int_{x\notin I^*}\frac{w(x)\,dx}{|x-C_I|^{p(N_0+1)}} \leqslant (\text{const})$$

because $p(N_0+1) > q_0$ and we can apply Lemma 1 in [10]. This inequality together with (II.3.1) gives (II.3.3).

THEOREM II.3.4. $\|P_\nu^*(a+i\tilde{a})\|^p_{L^p(w(x)dx)} \leqslant (\text{const})$.

Proof. The proof parallels that of (II.3.3). First we can see that $\int_{I^*}(P_\nu^*(a+i\tilde{a})(x))^p w(x)\,dx \leqslant (\text{const})$. Then, for $x \notin I^*$ we estimate $P_\nu^*(a)(x)$ and $P_\nu^*(\tilde{a})(x)$.

For $|x-y|<t$ we have:

$$\begin{aligned}
|(P_t*a)(y)| &= \left|\int_{-\infty}^{\infty} P_t(y-u)a(u)\,du\right| \\
&= \left|\int_{-\infty}^{\infty}\left\{P_t(y-u)-\sum_{k=0}^{N_0-1}\frac{(-1)^k}{k!}P_t^{(k)}(y-C_I)\cdot(u-C_I)^k\right\}a(u)\,du\right| \\
&\leqslant (\text{const})\int_{-\infty}^{\infty}|P_t^{(N_0)}(y-C_I-\theta_u(u-C_I))||u-C_I|^{N_0}|a(u)|\,du \\
&\leqslant (\text{const})w(I)^{1/r-1/p}t_0^{-N_0-1}\left\|P^{(N_0)}\left(\frac{y-C_I-\theta_u(u-C_I)}{t}\right)\times\right. \\
&\qquad\qquad \left.\times(u-C_I)^{N_0}w(u)^{-1/r}\chi_I(u)\right\|_{r'}.
\end{aligned}$$

Using the estimate $|P^{(N)}(s)|\leqslant\dfrac{C_N}{(1+|s|)^{N+1}}$ we get:

$$\left|P^{(N_0)}\left(\frac{y-C_I-\theta_u(u-C_I)}{t}\right)\right|\leqslant\frac{(\text{const})t^{N_0+1}}{(t+|y-C_I-\theta_u(u-C_I)|)^{N_0+1}}$$

and since $t+|y-C_I-\theta_u(u-C_I)|>|x-C_I|/2$ we have finally:

$$\begin{aligned}
|(P_t*a)(y)| &\leqslant (\text{const})w(I)^{1/r-1/p}\frac{|I|^{N_0}}{|x-C_I|^{N_0+1}}\|w(u)^{-1/r}\chi_I(u)\|_{r'} \\
&\leqslant (\text{const})w(I)^{-1/p}(|I|/|x-C_I|)^{N_0+1}.
\end{aligned}$$

Thus, if $x\notin I^*$: $P_r^*(a)(x)\leqslant(\text{const})w(I)^{-1/p}(|I|/|x-C_I|)^{N_0+1}$ which is the same estimate obtained in (II.3.2). $P_r^*(\tilde{a})(x)=\sup\limits_{|x-y|<t}|((\tilde{P})_t*a)(y)|$. Since $\tilde{P}(s)=s/(1+s^2)$ satisfies, like P, the condition: $|\tilde{P}^{(N)}(s)|\leqslant C_N/(1+|s|)^{N+1}$ the same proof renders that if $x\notin I^*$: $P_r^*(\tilde{a})(x)\leqslant(\text{const})w(I)^{-1/p}(|I|/|x-C_I|)^{N_0+1}$. The rest is as in (II.3.3).

We have seen that atoms provide very simple examples of functions in $H^p(w(x)\,dx)$. The fundamental result will be that every function in $H^p(w(x)\,dx)$ can be decomposed into atoms. In order to achieve such a decomposition, we will use the following lemma which is a refinement of the classical Calderón–Zygmund decomposition.

LEMMA II.3.5. *Let f be a function in* $\operatorname{Re}L^q(w(x)\,dx)$ *for some $q>q_0$, λ a real number >0 and N an integer $\geqslant 0$. Then $f(x)=g_\lambda(x)+\sum_i b_\lambda^i(x)$ a.e. and in $L^q(w(x)\,dx)$ with $|g_\lambda(x)|\leqslant(\text{const})\,\lambda$, b_λ^i supported in an interval I_λ^i and $\int_{-\infty}^{\infty}b_\lambda^i(x)x^l\,dx=0$ for $l=0,\ldots,N$. The intervals $\{I_\lambda^i\}$ are going to be the connected components of the open set $\{x\colon S_M^*(f)(x)>\lambda\}$.*

Proof. For every I^i, a Whitney type decomposition $\{I^{i,j}\}_{j=-\infty}^{\infty}$ is obtained by first dividing I^i into three intervals of equal length, of which, the one in the middle will be $I^{i,0}$, and the ones in the extremities will be split in half. Of the two intervals next to $I^{i,0}$, the one to the right will be $I^{i,1}$ and the one to the left will be $I^{i,-1}$. The remaining intervals are split again and the process continues indefinitely. With this decomposition we associate a partition of unity in the following way: Let η be an even $\mathscr{C}^\infty$ function with support in $[-1-\delta, 1+\delta]$ for some $\delta < \frac{1}{2}$, identically equal to 1 on $[-1, 1]$, decreasing to 0 between 1 and $1+\delta$. Then if C_j^i is the center of $I^{i,j}$, let

$$\eta_j^i(t) = \eta\left(2\frac{t-C_j^i}{|I^{i,j}|}\right) = \eta\big(2^{|j|+1}|I^{i,0}|(t-C_j^i)\big).$$

Our choice of δ guarantees that every point has a neighborhood that meets at most two of the supports of the η_j^i's. Thus, $\sum_{j=-\infty}^{\infty} \eta_j^i(t)$ is a $\mathscr{C}^\infty$ function vanishing outside I^i and such that for any $t \in I^i$, $1 \leqslant \sum_{j=-\infty}^{\infty} \eta_j^i(t) \leqslant 2$. Let $\varphi_j^i(t) = \eta_j^i(t) / \sum_{k=-\infty}^{\infty} \eta_k^i(t)$. φ_j^i is a $\mathscr{C}^\infty$ function which has the same support as η_j^i. Clearly $\sum_{j=-\infty}^{\infty} \varphi_j^i(t) = \chi_{I^i}(t)$. Let $P_j^i(x)$ be the unique polynomial of degree N such that

$$\int_{-\infty}^{\infty} \big(f(x) - P_j^i(x)\big) x^l \varphi_j^i(x)\, dx = 0 \quad \text{for every } l = 0, \ldots, N.$$

Then we can write:

$$f(x) = f(x)\chi_{R-\bigcup_i I^i}(x) + \sum_i \sum_j P_j^i(x)\varphi_j^i(x) + \sum_i \sum_j \big(f(x) - P_j^i(x)\big)\varphi_j^i(x).$$

Define

$$g_\lambda(x) = f(x)\chi_{R-\bigcup_i I^i}(x) + \sum_i \sum_j P_j^i(x)\varphi_j^i(x)$$

and for every i,

$$b_\lambda^i(x) = \sum_j \big(f(x) - P_j^i(x)\big)\varphi_j^i(x).$$

Then $f(x) = g_\lambda(x) + \sum_i b_\lambda^i(x)$ a.e. The important fact is that $|P_j^i(x)\varphi_j^i(x)| \leqslant (\text{const})\,\lambda$, where (const) does not depend on i, j or λ. Let us prove this. If $N = 0$, $P_j^i(x)$ is just the constant

$$m_j^i = \frac{\int_{-\infty}^{\infty} f(x)\varphi_j^i(x)\, dx}{\int_{-\infty}^{\infty} \varphi_j^i(x)\, dx}.$$

By the definition of $S_M^*(f)$ if we take as x the extremity of I^i closest to $I^{i,j}$ we get:

$$\Big|\int_{-\infty}^{\infty} f(x)\varphi_j^i(x)\,dx\Big| \leqslant S_M^*(f)(x)\Big(\int_{-\infty}^{\infty}\varphi_j^i(u)\,du + (\text{const})\,|I^{i,j}|^{M+1}\int_{-\infty}^{\infty}|(\varphi_j^i)^{M+1}(u)|\,du\Big).$$

But, by the definition of the I^i's, $S_M^*(f)(x) \leqslant \lambda$ and for the kind of functions that we are using:

$$\int_{-\infty}^{\infty}\varphi_j^i(u)\,du + (\text{const})\,|I^{i,j}|^{M+1}\int_{-\infty}^{\infty}|(\varphi_j^i)^{(M+1)}(u)|\,du \leqslant (\text{const})\int_{-\infty}^{\infty}\varphi_j^i(u)\,du.$$

Consequently $|m_j^i| \leqslant (\text{const})\,\lambda$. For the case $N > 0$, we see that $|P_j^i(x)\varphi_j^i(x)| \leqslant (\text{const})\,|m_j^i|$. Indeed, this inequality is invariant under translations and dilations so that we just need to prove it for a fixed φ, for example φ can be the η considered above. Then

$$P(x) = \sum_{l=0}^{N}\Big(\int_{-\infty}^{\infty} f(x)\,\xi_l(x)\varphi(x)\,dx\Big)\,\xi_l(x),$$

where ξ_l is an orthonormal basis for the span of $\{1, x, \ldots, x^N\}$ in $L^2(\varphi(x)\,dx)$, say, the one obtained by the Gramm–Schmidt process. Clearly

$$|P(x)\varphi(x)| \leqslant (\text{const})\,\frac{\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx}{\int_{-\infty}^{\infty}\varphi(x)\,dx}.$$

From $|P_j^i(x)\varphi_j^i(x)| \leqslant (\text{const})\,\lambda$, it follows that $|g_\lambda(x)| \leqslant (\text{const})\,\lambda$ and also $|b_\lambda^i(x)| \leqslant (\text{const})\,S_M^*(f)(x)$. Then, Lebesgue's dominated convergence theorem implies that $f(x) = g_\lambda(x) + \sum_i b_\lambda^i(x)$ makes sense also in $L^q(w(x)\,dx)$. Finally

$$\int_{-\infty}^{\infty} b_\lambda^i(x)x^l\,dx = \sum_j\int_{-\infty}^{\infty}\big(f(x) - P_j^i(x)\big)x^l\varphi_j^i(x)\,dx = 0 \quad \text{for } l = 0, \ldots, N$$

and the lemma is completely proved. Now, from this lemma we get an atomic decomposition for nice functions in $H^p(w(x)\,dx)$, $0 < p \leqslant 1$.

THEOREM II.3.6. *Let $f \in \operatorname{Re} L^q(w(x)\,dx)$ be such that $S_M^*(f) \in L^p(w(x)\,dx)$, where $M > q/p$. Then there exist:*

(i) *A sequence (a_l) of (p, ∞)-atoms with respect to w, and*

(ii) *A sequence (λ_l) of real numbers satisfying*

$$\sum_l |\lambda_l|^p \leqslant (\text{const})\int_{-\infty}^{\infty}|S_M^*(f)(x)|^p w(x)\,dx$$

such that $f(x) = \sum_l \lambda_l a_l(x)$ a.e. and also in $L^q(w(x)\,dx)$.

Proof. For each integer k, let $f(x) = g_k(x) + \sum_i b_k^i(x)$ a.e. and in $L^q(w(x)dx)$, be the decomposition obtained in (II.3.5) with $\lambda = 2^k$ and $N = [q_0/p]-1$. Now we put all these decompositions together. $g_k(x) \to 0$ a.e. as $k \to -\infty$ since $|g_k(x)| \leqslant (\text{const})\, 2^k \to 0$ as $k \to -\infty$. We also have convergence in $L^q(w(x)dx)$ since $|g_k(x)| \leqslant (\text{const})\, S_M^*(f)(x)$ which is in $L^q(w(x)dx)$. Also $g_k(x) \to f(x)$ a.e. as $k \to \infty$ because $f(x) - g_k(x)$ lives in the set $\{x\colon S_M^*(f)(x) > 2^k\}$ which decreases to a set of measure 0 as $k \to \infty$. As before, there is convergence in $L^q(w(x)dx)$ as well, since $|f(x) - g_k(x)| \leqslant (\text{const})\, S_M^*(f)(x)$. These facts imply that f is the sum of the telescopic series $\sum_{k=-\infty}^{\infty} (g_{k+1}(x) - g_k(x))$ converging a.e. and in $L^q(w(x)dx)$. Thus

$$f(x) = \sum_{k=-\infty}^{\infty} (g_{k+1}(x) - g_k(x)) = \sum_{k=-\infty}^{\infty} \Big(\sum_i b_k^i(x) - \sum_j b_{k+1}^j(x)\Big)$$
$$= \sum_{k=-\infty}^{\infty} \sum_i \Big(b_k^i(x) - \sum_{\{j: I_{k+1}^j \subset I_k^i\}} b_{k+1}^j(x)\Big) = \sum_{k=-\infty}^{\infty} \sum_i \beta_k^i(x),$$

where

$$\beta_k^i(x) = b_k^i(x) - \sum_{\{j: I_{k+1}^j \subset I_k^i\}} b_{k+1}^j(x).$$

Of course this can be done because each I_{k+1}^j is contained in a unique I_k^i. $\beta_k^i(x) = g_{k+1}(x) - g_k(x)$ if $x \in I_k^i$ and $\beta_k^i(x) = 0$ elsewhere. Therefore $|\beta_k^i(x)| \leqslant (\text{const})\, 2^k$. Let $a_k^i(x) = \big(w(I_k^i)^{1/p} (\text{const})\, 2^k\big)^{-1} \beta_k^i(x)$. Then a_k^i is a (p, ∞)-atom with respect to w. $f(x) = \sum_k \sum_i \lambda_k^i a_k^i$ with $\lambda_k^i = (\text{const})\, 2^k w(I_k^i)^{1/p}$.

$$\sum_k \sum_i |\lambda_k^i|^p$$
$$= \sum_k \sum_i (\text{const})^p 2^{kp} w(I_k^i) = (\text{const}) \sum_{k=-\infty}^{\infty} 2^{kp} w\big(\{x\colon S_M^*(f)(x) > 2^k\}\big)$$
$$\leqslant (\text{const}) \int_0^{\infty} \lambda^{p-1} w\big(\{x\colon S_M^*(f)(x) > \lambda\}\big) d\lambda = (\text{const}) \int_{-\infty}^{\infty} \big(S_M^*(f)(x)\big)^p w(x)\, dx.$$

This finishes the proof of (II.3.6).

For $f \in \operatorname{Re} L^q(w(x)dx)$ let $N_{p,r}(f) = \inf\big\{\big(\sum_i |\lambda_i|^p\big)^{1/p}\colon f(x) = \sum_i \lambda_i a_i(x)$ a.e. and in $L^q(w(x)dx)$ the a_i's being (p, r)-atoms with respect to $w\big\}$. Since for $r_1 < r_2$, a (p, r_2)-atom is always a (p, r_1)-atom, we clearly have: $N_{p,r_1}(f) \leqslant N_{p,r_2}(f)$. (II.3.6) shows that $\big(N_{p,\infty}(f)\big)^p \leqslant (\text{const})\, \|S_M^*(f)\|_{L^p(w(x)dx)}^p$. Thus, we can extend our chain of gauges over $\operatorname{Re} L^q(w(x)dx)$:

$$\big(N_{p,r}(f)\big)^p < \big(N_{p,\infty}(f)\big)^p < \|S_M^*(f)\|_{L^p(w(x)dx)}^p < \|\sigma_M^{**}(f)\|_{L^p(w(x)dx)}^p$$
$$< \|\sigma_\nabla^*(f)\|_{L^p(w(x)dx)}^p < \|P_M^{**}(f)\|_{L^p(w(x)dx)}^p < \|P_\nabla^*(f)\|_{L^p(w(x)dx)}^p$$
$$< \|P_\nabla^*(f + i\tilde{f})\|_{L^p(w(x)dx)}^p, \qquad w \in A_r,\ r < \infty.$$

Actually all these gauges are equivalent as we will see now: Let $f(x) = \sum_j \lambda_j a_j(x)$ a.e. and in $L^q(w(x)\,dx)$ with $\sum_j |\lambda_j|^p < \infty$ and the a_j's being (p, r)-atoms with respect to w. Then

$$P_\nu^*(f+i\tilde{f})(x) \leqslant \sum_j |\lambda_j| P_\nu^*(a_j+i\tilde{a}_j)(x)$$

so that

$$\|P_\nu^*(f+i\tilde{f})\|^p_{L^p(w(x)dx)} \leqslant \sum_j |\lambda_j|^p \int_{-\infty}^{\infty} \big(P_\nu^*(a_j+i\tilde{a}_j)(x)\big)^p w(x)\,dx \leqslant (\text{const}) \sum_j |\lambda_j|^p.$$

Since this is true for every decomposition, we get finally:

$$\|P_\nu^*(f+i\tilde{f})\|^p_{L^p(w(x)dx)} \leqslant (\text{const})\big(N_{p,r}(f)\big)^p$$

which closes the chain of gauges and shows that all of them are equivalent. The conclusion is that the completion of the space determined on $\operatorname{Re} L^q(w(x)\,dx)$ by any of the equivalent gauges above, is an equivalent copy of $H^p(w(x)\,dx)$.

Next we give an atomic characterization of $H^p(w(x)\,dx)$ in the orginal setting of analytic functions.

THEOREM II.3.7. *For $F \in H(\boldsymbol{R}^2_+)$, $0 < p \leqslant 1$ and r such that $w \in A_r$, let $N_{p,r}(F) = \inf\big\{\sum_{j=1}^{\infty} |\lambda_j|^p\big\}^{1/p}$ over all decompositions $F(z) = \sum_{j=1}^{\infty} \lambda_j A_j(z)$ converging uniformly on compact sets, where $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ and, for every j, $A_j(x+it) = \big(P_t*(a_j+i\tilde{a}_j)\big)(x)$, a_j being a (p, r)-atom with respect to w. Then $F \mapsto \big(N_{p,r}(F)\big)^p$ and $F \mapsto \|F\|^p_{H^p(w(x)dx)}$ are equivalent gauges over $H(\boldsymbol{R}^2_+)$, so that $H^p(w(x)\,dx)$ can be viewed as the space determined on $H(\boldsymbol{R}^2_+)$ by $F \mapsto \big(N_{p,r}(F)\big)^p$.*

Proof. Clearly if $F(z) = \sum_{j=1}^{\infty} \lambda_j A_j(z)$ in $H(\boldsymbol{R}^2_+)$ with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ and $A_j(x+it) = \big(P_t*(a_j+i\tilde{a}_j)\big)(x)$ with a_j (p, r)-atoms; then for every $t > 0$:

$$\int_{-\infty}^{\infty} |F(x+it)|^p w(x)\,dx$$
$$\leqslant \sum_{j=1}^{\infty} |\lambda_j|^p \int_{-\infty}^{\infty} \Big(\big(P_\nu^*(a_j+i\tilde{a}_j)\big)(x)\Big)^p w(x)\,dx \leqslant (\text{const}) \sum_{j=1}^{\infty} |\lambda_j|^p.$$

Thus $F \in H^p(w(x)\,dx)$ and $\|F\|^p_{H^p(w(x)dx)} \leqslant (\text{const})\big(N_{p,r}(F)\big)^p$.

Conversely, if $F \in H^p(w(x)\,dx)$, then by (II.1.5) F is the limit in $H^p(w(x)\,dx)$ of a sequence of functions $F_k \in \mathscr{I}^p_N$ (some fixed N) with $\|F_k\|_{H^p(w(x)dx)} \leqslant \|F\|_{H^p(w(x)dx)}$. The F_k's can be taken in such a way that $\|F_{k+1} - F_k\|^p_{H^p(w(x)dx)} < 2^{-k}$. By (II.1.2) also $F = \lim F_k$ uniformly on com-

pact subsets of $\boldsymbol{R}^2_+$. For every k, $F = F_k + \sum_{j=k}^{\infty}(F_{j+1}-F_j)$. Let f_j be the real part of the boundary function of F_j. Then $f_j \in \operatorname{Re} L^q\big(w(x)\,dx\big)$ for every $q > q_0$ and $F_j(x+it) = \big(P_t*(f_j+i\tilde{f}_j)\big)(x)$. Now $f_{j+1}(x)-f_j(x) = \sum_{m=1}^{\infty}\lambda_{jm}a_{jm}(x)$ a.e. and in $L^q\big(w(x)\,dx\big)$

$$\sum_{m=1}^{\infty}|\lambda_{jm}|^p \leqslant (\text{const})\,\|F_{j+1}-F_j\|^p_{H^p(w(x)dx)} \leqslant (\text{const})\,2^{-j}$$

and a_{jm} are (p,∞)-atoms with respect to w. Likewise $f_k(x) = \sum_{m=1}^{\infty}\mu_{km}b_{km}(x)$ a.e. and in $L^q\big(w(x)\,dx\big)$ with

$$\sum_{m=1}^{\infty}|\mu_{km}|^p \leqslant (\text{const})\,\|F_k\|^p_{H^p(w(x)dx)} \leqslant (\text{const})\,\|F\|^p_{H^p(w(x)dx)}.$$

Thus

$$F_k(x+it) = \sum_{m=1}^{\infty}\mu_{km}\big(P_t*(b_{km}+i\tilde{b}_{km})\big)(x) = \sum_{m=1}^{\infty}\mu_{km}B_{km}(x+it)$$

and

$$F_{j+1}(x+it)-F_j(x+it) = \sum_{m=1}^{\infty}\lambda_{jm}\big(P_t*(a_{jm}+i\tilde{a}_{jm})\big)(x) = \sum_{m=1}^{\infty}\lambda_{jm}A_{jm}(x+it)$$

uniformly on compact subsets of $\boldsymbol{R}^2_+$. Therefore

$$F(z) = F_k(z)+\sum_{j=k}^{\infty}\big(F_{j+1}(z)-F_j(z)\big) = \sum_{m=1}^{\infty}\mu_{km}B_{km}(z)+\sum_{j=k}^{\infty}\sum_{m=1}^{\infty}\lambda_{jm}A_{jm}(z)$$

uniformly on compact subsets with

$$\sum_{m=1}^{\infty}|\mu_{km}|^p+\sum_{j=k}^{\infty}\sum_{m=1}^{\infty}|\lambda_{jm}|^p$$
$$\leqslant (\text{const})\Big(\|F\|^p_{H^p(w(x)dx)}+\sum_{j=k}^{\infty}2^{-j}\Big) \leqslant (\text{const})\big(\|F\|^p_{H^p(w(x)dx)}+2^{-k}\big).$$

This implies that $\big(N_{p,r}(F)\big)^p \leqslant (\text{const})\,\|F\|^p_{H^p(w(x)dx)}$.

4. Dual spaces

$\big(H^p\big(w(x)\,dx\big)\big)^*$, dual of $H^p\big(w(x)\,dx\big)$, is the Banach space formed by all continuous linear functionals Λ: $H^p\big(w(x)\,dx\big)\to\boldsymbol{R}$ with the norm given by:

$$\|\Lambda\| = \sup\{|\Lambda(F)|:\ \|F\|_{H^p(w(x)dx)} = 1\}.$$

Given a bounded interval I and r such that $r > p$, $r < \infty$, and $w \in A_r$ let $[\mathrm{Re}\, L^r(I, w(x)\,dx)]_0$ be the space of all real functions in $L^r(w(x)\,dx)$ living in I and such that $\int_{-\infty}^{\infty} f(x)x^l\,dx = 0$ for $l = 0, \ldots, [q_0/p]-1$. For $f \in [\mathrm{Re}\, L^r(I;\ w(x)\,dx)]_0$, $F(x+it) = (P_t * (f+i\tilde{f}))(x)$ is in $H^p(w(x)\,dx)$ with

$$\|F\|_{H^p(w(x)dx)} \leqslant (\mathrm{const})\, N_{p,r}(f) \leqslant (\mathrm{const})\, w(I)^{1/p-1/r}\, \|f(x)\chi_I(x)\|_{L^r(w(x)dx)}.$$

The following result is a simple consequence of Riesz's representation theorem.

THEOREM II.4.1. *Let $\Lambda \in (H^p(w(x)\,dx))^*$. Then there exists a function b locally in $\mathrm{Re}\, L^{r'}(w(x)\,dx)$ such that for any $f \in [\mathrm{Re}\, L^r(I;\ w(x)\,dx)]_0$ for some I, if we let $F(x+it) = (P_t * (f+i\tilde{f}))(x)$, we have:*

$$\Lambda(F) = \int_{-\infty}^{\infty} f(x)\, b(x)\, w(x)\,dx.$$

Any two such functions b differ by $P(x)/w(x)$, where $P(x)$ is a polynomial of degree $\leqslant [q_0/p]-1$.

Thus, we have a 1 : 1 correspondence $\Lambda \mapsto [bw]$ between $(H^p(w(x)\,dx))^*$ and some subspace of $w \cdot \mathrm{Re}\, L^{r'}_{\mathrm{loc}}(w(x)\,dx)/\{$polynomials of degree $\leqslant [q_0/p]-1\}$. Next we identify this subspace. We will make use of the following lemma and corollary:

LEMMA II.4.2. *If g is locally integrable and I is a bounded interval, let $P_I(g)$ be the unique polynomial of degree $[q_0/p]-1$ such that $\int_I (g(x) - P_I(g)(x))x^k\,dx = 0$ for $k = 0, \ldots, [q_0/p]-1$. Then for $x \in I$:*

$$|P_I(g)(x)| \leqslant \frac{(\mathrm{const})}{|I|} \int_I |g(x)|\,dx.$$

Proof. It is the same proof that the one of $|P_j^i(x)\varphi_j^i(x)| \leqslant (\mathrm{const})\, |m_j^i|$ in (II.3.5).

COROLLARY II.4.3. *If $g \in L^r(I, w(x)\,dx)$, then for $x \in I$:*

$$|P_I(g)(x)| \leqslant (\mathrm{const}) \left(\frac{1}{w(I)} \int_I |g(x)|^r w(x)\,dx \right)^{1/r}.$$

Proof. It follows from (I.1).

THEOREM II.4.4. *Let $\Lambda \in (H^p(w(x)\,dx))^*$. If l is a function such that $l(x)/w(x)$ is locally in $\mathrm{Re}\, L^{r'}(w(x)\,dx)$ and for every I and every $f \in [\mathrm{Re}\, L^r(I;\ w(x)\,dx)]_0$, $\Lambda(F) = \int_I f(x)\, l(x)\,dx$; then there is a constant $C \leqslant (\mathrm{const})\, \|\Lambda\|$ such that for every I:*

$$\left(\int_I \left| \frac{l(x) - P_I(l)(x)}{w(x)} \right|^{r'} \frac{w(x)\,dx}{w(I)} \right)^{1/r'} \leqslant C w(I)^{1/p-1}.$$

Conversely, if such a C exists, $F \mapsto \int_I f(x) l(x)\,dx$ extends in a unique way to a functional $\Lambda \in \big(H^p(w(x)\,dx)\big)^*$ and $\|\Lambda\| \leqslant (\text{const})\, C$.

Proof. For any (p, r)-atom a we will have:

$$\Big| \int_{-\infty}^{\infty} a(x) l(x)\,dx \Big| \leqslant (\text{const})\, \|\Lambda\|.$$

Let I be the smallest interval containing the support of a. Then, the last inequality can be rewritten as

$$\left| \int_I a(x) \frac{l(x) - P_I(l)(x)}{w(x)} \frac{w(x)\,dx}{w(I)} \right| \leqslant (\text{const})\, \|\Lambda\| w(I)^{-1}.$$

Now, given any function f living in I such that

$$\left(\int_I |f(x)|^r \frac{w(x)\,dx}{w(I)} \right)^{1/r} \leqslant 1 \quad \text{let} \quad b(x) = w(I)^{-1/p} \big(f(x) - P_I(f)(x)\big).$$

(II.4.3) implies that $b(x) = (\text{const})\, a(x)$, where $a(x)$ is a (p, r)-atom with respect to w. Thus:

$$\left| \int_I f(x) \frac{l(x) - P_I(l)(x)}{w(x)} \frac{w(x)\,dx}{w(I)} \right|$$

$$= (\text{const}) \left| \int_I w(I)^{1/p} a(x) \frac{l(x) - P_I(l)(x)}{w(x)} \frac{w(x)\,dx}{w(I)} \right| \leqslant (\text{const})\, \|\Lambda\| w(I)^{1/p-1}.$$

Therefore:

$$\text{(II.4.5)} \qquad \left(\int_I \left| \frac{l(x) - P_I(l)(x)}{w(x)} \right|^{r'} \frac{w(x)\,dx}{w(I)} \right)^{1/r'} \leqslant (\text{const})\, \|\Lambda\| w(I)^{1/p-1}.$$

The converse is clear. Even though we had to exclude $r = \infty$ in (II.4.1), once the existence of $l(x)$ has been established we can use (p, ∞)-atoms, getting (II.4.5) also for $r' = 1$.

We obtain as a byproduct the equivalence of conditions (II.4.5) for different values of r'. In particular for $1 \leqslant r' < q_0'$. If $q_0 = 1$ and $p < 1$ the range will include also $q_0' = \infty$, i.e., it will be: $1 \leqslant r' \leqslant \infty$. For $w(x) \equiv 1$ and $p = 1$, this equivalence is the famous theorem of John and Niremberg ([11]) about the characterizations of BMO, which is $(H^1)^*$.

In general, for $p = 1$ and $r' = 1$ (II.4.5) reads:

$$\frac{1}{w(I)} \int_I |l(x) - P_I(l)(x)|\,dx \leqslant (\text{const})$$

which coincides with the definition given by Muckenhoupt and Wheeden ([15]) of the space of functions of bounded mean oscillation with respect to w. This space can be defined equivalently by any of the conditions

$$\left(\int_I \left|\frac{l(x)-P_I(l)(x)}{w(x)}\right|^{r'} \frac{w(x)\,dx}{w(I)}\right)^{1/r'} \leqslant (\text{const})$$

for $1 \leqslant r' < q_0'$. This result is proved independently by Muckenhoupt and Wheeden in [15]. Finally let us point out that in the characterization of the dual we do not need to have necessarily the polynomial $P_I(l)$. The following condition is seen to be equivalent by applying (II.4.3): There is a constant C such that for any bounded interval I there is a polynomial Q_I of degree $\leqslant [q_0/p]-1$ such that

$$\left(\int_I \left|\frac{l(x)-Q_I(x)}{w(x)}\right|^{r'} \frac{w(x)\,dx}{w(I)}\right)^{1/r'} \leqslant Cw(I)^{1/p-1}.$$

The infimum of all the C's for which the above condition holds is a norm equivalent to the one considered previously.

Chapter III

H^p Spaces associated with the space of homogeneous type $(\boldsymbol{R}, w(x)\,dx)$

1. The space $\mathfrak{H}^1(w(x)dx)$

For $r > 1$ a $(1, r)$-atom of homogeneous type (h.t.) with respect to the weight w will be a real valued function a supported in an interval I and satisfying:

(i) $\left(\frac{1}{w(I)}\int_I |a(x)|^r w(x)\,dx\right)^{1/r} \leqslant w(I)^{-1}$ if $r < \infty$ or $\|a\|_\infty \leqslant w(I)^{-1}$ for $r = \infty$.

(ii) $\int_{-\infty}^{\infty} a(x)w(x)\,dx = 0$.

These are the natural kind of atoms for the space of homogeneous type (see [3], [4], and [12]) $(\boldsymbol{R}, w(x)\,dx)$ for which the metric is $d(x, y) = w(I_{x,y})$, $I_{x,y}$ being the smallest interval containing both x and y.

We define $\mathfrak{H}^1_r(w(x)\,dx)$ as the space of all real-valued functions $f(x)$ for which there is a decomposition $f(x) = \sum_i \lambda_i a_i(x)$ a.e. with $\sum_i |\lambda_i| < \infty$ and each a_i a $(1, r)$-atom of h.t. with respect to w.

It is clear that $\mathfrak{H}^1_r(w(x)\,dx)$ is a linear subspace of $L^1(w(x)\,dx)$. On $\mathfrak{H}^1_r(w(x)\,dx)$ we consider the norm $\mathfrak{N}_{1,r}(f) = \inf\{\sum_i |\lambda_i|\}$, where the infimum is taken over all decompositions of f into $(1, r)$-atoms of h.t. It can be proved that all the spaces $\mathfrak{H}^1_r(w(x)\,dx)$ coincide as sets and all the norms $\mathfrak{N}_{1,r}$ are equivalent, since this is the case in the general setting of a space of homogeneous type (see [4] and [12]). In our case we do not need to appeal to the general theory because we have a direct equivalence between each $\mathfrak{H}^1_r(w(x)\,dx)$ and $\mathrm{Re}\,H^1(\boldsymbol{R})$, the ordinary real H^1 of the line. Actually, in order to define $\mathfrak{H}^1_r(w(x)\,dx)$ we do not need $w(x)$ to be a Muckenhoupt weight. It is enough for it to be locally integrable and > 0 a.e. If this is the case, we have the following result:

THEOREM III.1.1. *Let $W(x)$ be a primitive of w and $B(x)$ the inverse*

function of $W(x)$. Then the mapping $f(x) \mapsto f(B(y))$ is an equivalence between $\mathfrak{H}_r^1(w(x)\,dx)$ and $\operatorname{Re} H^1(\boldsymbol{R})$.

Proof. The reason is that $a(x)$ is a $(1, r)$-atom of h.t. with respect to w if and only if $a(B(y))$ is a $(1, r)$-atom with respect to Lebesgue measure since $\int_I |a(x)|^r w(x)\,dx = \int_{W(I)} |a(B(y))|^r dy$; $w(I) = |W(I)|$ and

$$\int_{-\infty}^{\infty} a(x) w(x)\,dx = \int_{-\infty}^{\infty} a(B(y))\,dy.$$

In view of the equivalence among the different $\mathfrak{H}_r^1$'s, we can simply write $\mathfrak{H}^1(w(x)\,dx)$. If w is a Muckenhoupt weight, we have a much less trivial equivalence between $\mathfrak{H}^1(w(x)\,dx)$ and $\operatorname{Re} H^1(\boldsymbol{R})$.

THEOREM III.1.2. *If w is an A_∞ weight, then the mapping $f(x) \mapsto f(x) w(x)$ is an equivalence between $\mathfrak{H}^1(w(x)\,dx)$ and* $\operatorname{Re} H^1(\boldsymbol{R})$.

Proof. Let a be a $(1, \infty)$-atom of h.t. with respect to w. We will see that there is some $r > 1$ and a constant, both depending only on the weight, such that (const) $a(x) w(x)$ is a $(1, r)$-atom with respect to Lebesgue measure. The reason is that the weight satisfies a reverse Hölder's inequality, that is: there is a $\delta > 0$ and a constant such that for every bounded interval J:

$$\left(\frac{1}{|J|} \int_J w(x)^{1+\delta}\,dx\right)^{1/(1+\delta)} \leqslant (\text{const}) \frac{1}{|J|} \int_J w(x)\,dx$$

(see [2]). Let $r = 1 + \delta > 1$. Then, for our atom a:

$$\left(\frac{1}{|I|} \int_I |a(x) w(x)|^r\,dx\right)^{1/r} \leqslant w(I)^{-1} (\text{const}) \frac{w(I)}{|I|} \leqslant (\text{const})\, |I|^{-1}.$$

This, together with the fact that $\int_{-\infty}^{\infty} a(x) w(x)\,dx = 0$ implies that $a(x) w(x)/(\text{const})$ is a $(1, r)$-atom in $H^1(\boldsymbol{R})$. This proves that the operator $f(x) \mapsto f(x) w(x)$ is bounded.

Next we prove that its inverse $g(x) \mapsto g(x)/w(x)$ is also bounded. Let $b(x)$ be a $(1, \infty)$-atom in $H^1(\boldsymbol{R})$. We will see that there exists $q > 1$ and a constant both depending only on the weight, such that (const) $b(x)/w(x)$ is a $(1, q)$-atom of h.t. with respect to the weight. It is enough to take a q such that $1 < q < \infty$ and $w \in A_{q'}$. Then

$$\left(\frac{1}{w(I)} \int_I \left|\frac{b(x)}{w(x)}\right|^q w(x)\,dx\right)^{1/q} \leqslant \frac{1}{|I|} w(I)^{-1/q} \left(\int_I w(x)^{1-q}\,dx\right)^{1/q}$$

$$\leqslant \frac{1}{|I|} w(I)^{-1/q} \left(\int_I w(x)^{-1/(q'-1)}\,dx\right)^{(q'-1)/q'} \leqslant (\text{const})\, w(I)^{-1}.$$

This, together with the fact that $\int_{-\infty}^{\infty} \frac{b(x)}{w(x)} w(x)dx = 0$ implies that $\frac{1}{(\text{const})} \frac{b(x)}{w(x)}$ is a $(1, q)$-atom of h.t. with respect to w. And (III.1.2) is proved.

2. The spaces $\mathfrak{H}^p(w(x)dx)$ for $p < 1$

For $a > 0$, let $\mathfrak{L}_a(w(x)dx)$ be the space of equivalent classes $[l]$ modulo constants of functions l for which there is a constant C such that for any x, y and any interval I containing x and y: $|l(x)-l(y)| \leqslant Cw(I)^a$. The infimum of all these constants C will be denoted by $\|[l]\|_{\mathfrak{L}_a(w(x)dx)}$. $[l] \mapsto \|[l]\|_{\mathfrak{L}_a(w(x)dx)}$ is a norm on $\mathfrak{L}_a(w(x)dx)$ which, endowed with it, becomes a Banach space.

These spaces are well defined not only for weights but for any locally integrable function w. If $w(x) > 0$ a.e. the following result holds.

THEOREM III.2.1. *Let $W(x)$ be a primitive of w and $B(x)$ the inverse of $W(x)$. Then the mapping $[l] \mapsto [l \circ B]$ is an equivalence between $\mathfrak{L}_a(w(x)dx)$ and $\mathfrak{L}_a(dx)$ which is the ordinary Lipschitz space.*

Proof. Indeed $|l(x)-l(y)| \leqslant Cw(I)^a$ for $x, y \in I$ is equivalent to: $|l(B(s))-l(B(t))| \leqslant C|W(I)|^a$ for $s, t \in W(I)$.

For $a > 1$, the space $\mathfrak{L}_a(dx)$ reduces to 0, so that, by the above result, also $\mathfrak{L}_a(w(x)dx) = 0$. We will only consider $0 < a < 1$.

For $\frac{1}{2} < p < 1$ and $r \geqslant 1$, a (p, r)-atom of h.t. with respect to w will be a real-valued function a supported in an interval I and such that:

(i) $\left(\frac{1}{w(I)} \int_I |a(x)|^r w(x)dx\right)^{1/r} \leqslant w(I)^{-1/p}$ if $r < \infty$ or $\|a\|_\infty \leqslant w(I)^{-1/p}$ if $r = \infty$.

(ii) $\int_{-\infty}^{\infty} a(x)w(x)dx = 0$.

LEMMA III.2.2. *If $a = 1/p - 1$ and a is a (p, r)-atom of h.t. with respect to w, then the mapping $[l] \mapsto \int_{-\infty}^{\infty} l(x)a(x)w(x)dx$ is a bounded linear functional L_a on $\mathfrak{L}_a(w(x)dx)$ with norm dominated by a constant independent of a.*

Proof. Indeed

$$\left|\int_{-\infty}^{\infty} l(x)a(x)w(x)dx\right| = \left|\int_I (l(x)-l(C_I))a(x)w(x)dx\right|$$
$$\leqslant w(I)^a \|[l]\|_{\mathfrak{L}_a(w(x)dx)} w(I)w(I)^{-1/p} = \|[l]\|_{\mathfrak{L}_a(w(x)dx)},$$

i.e., $L_a \in (\mathfrak{L}_a(w(x)dx))^*$ and $\|L_a\|_* = \|L_a\|_{(\mathfrak{L}_a(w(x)dx))^*} \leqslant 1$.

LEMMA III.2.3. *a is a (p, r)-atom of h.t. with respect to w if and only if $a(B(y))$ is a (p, r)-atom with respect to Lebesgue measure.*

Proof. $\int_I |a(x)|^r w(x)\,dx = \int_{W(I)} |a(B(y))|^r dy$ and $w(I) = |W(I)|$.

The characterization of $\operatorname{Re} H^p(\boldsymbol{R})$ for $\frac{1}{2} < p < 1$ as the space of functionals $M \in (\mathfrak{L}_{1/p-1}(dx))^*$ which can be decomposed in the topology of $(\mathfrak{L}_{1/p-1}(dx))^*$ as $M = \sum_i \lambda_i M_{a_i}$ with $\sum_i |\lambda_i|^p < \infty$, the a_i's being (p, r)-atoms with respect to dx and $M_{a_i}([m]) = \int_{-\infty}^{\infty} m(x) a_i(x)\,dx$; with the quasi-norm given by the infimum of $\sum_i |\lambda_i|^p$ taken over all such decompositions (see [1] and [4]) suggests the following definitions: $\mathfrak{H}_r^p(w(x)\,dx)$ will be the subspace of $(\mathfrak{L}_{1/p-1}(w(x)\,dx))^*$ formed by those functionals $L\colon \mathfrak{L}_{1/p-1}(w(x)\,dx) \to \boldsymbol{R}$ for which there is a decomposition $L = \sum_i \lambda_i L_{a_i}$ converging in the topology of $(\mathfrak{L}_{1/p-1}(w(x)\,dx))^*$ such that $\sum_i |\lambda_i|^p < \infty$ and a_i are (p, r)-atoms of h.t. with respect to w. Also for $L \in \mathfrak{H}_r^p(w(x)\,dx)$ let $\mathfrak{N}_{p,r}(L) = \inf\{\sum_i |\lambda_i|^p\}^{1/p}$, where the inf is taken over all decompositions as above. $L \mapsto (\mathfrak{N}_{p,r}(L))^p$ is a quasi-norm on $\mathfrak{H}_r^p(w(x)\,dx)$. Given a decomposition $L = \sum_i \lambda_i L_{a_i}$ with $\sum_i |\lambda_i|^p < \infty$ we have $\|L\|_* \leqslant \sum_i |\lambda_i|\, \|L_{a_i}\|_* \leqslant \sum_i |\lambda_i| \leqslant (\sum_i |\lambda_i|^p)^{1/p}$. Thus, for $L \in \mathfrak{H}_r^p(w(x)\,dx)$: $\|L\|_* \leqslant \mathfrak{N}_{p,r}(L)$. In other words the inclusion map $\mathfrak{H}_r^p(w(x)\,dx) \mapsto (\mathfrak{L}_{1/p-1}(w(x)\,dx))^*$ is continuous.

We already know that the mapping $\Phi\colon \mathfrak{L}_{1/p-1}(dx) \to \mathfrak{L}_{1/p-1}(w(x)\,dx)$ given by $\Phi([m]) = [m \circ W]$ is an equivalence of Banach spaces. It provides an equivalence between the duals: $\Phi^*\colon (\mathfrak{L}_{1/p-1}(w(x)\,dx))^* \to (\mathfrak{L}_{1/p-1}(dx))^*$ given by $\Phi^*(L) = L \circ \Phi$.

THEOREM III.2.4. *$\Phi^*(\mathfrak{H}_r^p(w(x)\,dx)) = \operatorname{Re} H^p$ and the restriction of Φ^* to $\mathfrak{H}_r^p(w(x)\,dx)$ is an equivalence of quasi-normed spaces. As a consequence we see that all the $\mathfrak{H}_r^p$'s coincide as sets and the quasi-norms $\mathfrak{N}_{p,r}$ are equivalent, so that we have a single space $\mathfrak{H}^p(w(x)\,dx)$. This space is equivalent to $\operatorname{Re} H^p(\boldsymbol{R})$ and, therefore, complete.*

Proof. Let $L \in \mathfrak{H}_r^p(w(x)\,dx)$ and $L = \sum_i \lambda_i L_{a_i}$ with $\sum_i |\lambda_i|^p < \infty$ and a_i (p, r)-atoms of h.t. with respect to w. Then $\Phi^*(L) = \sum_i \lambda_i \Phi^*(L_{a_i})$ in $(\mathfrak{L}_{1/p-1}(dx))^*$. But

$$(\Phi^*(L_{a_i}))([m]) = (L_{a_i}\Phi)([m]) = L_{a_i}([m \circ W])$$
$$= \int_{-\infty}^{\infty} m(W(x))\, a_i(x) w(x)\,dx = \int_{-\infty}^{\infty} m(y)\, a_i(B(y))\,dy = M_{a_i \circ B}([m]).$$

Thus $\Phi^*(L) = \sum_i \lambda_i M_{a_i \circ B}$. Since $a_i \circ B$ are (p, r)-atoms with respect to Lebesgue measure, we have: $\Phi^*(L) \in \operatorname{Re} H^p(\boldsymbol{R})$ and $\|\Phi^*(L)\|_{H^p} \leqslant$

(const) $\mathfrak{N}_{p,r}(L)$. The inverse change of variables would give us the other half of the theorem. The equality of the spaces $\mathfrak{H}_r^p(w(x)\,dx)$ for different r's and a weight w follows also from the general H^p theory on spaces of homogeneous type (see [4] and [12]).

Next we undertake the extension to $p < 1$ of the equivalence between $\mathfrak{H}^1(w(x)\,dx)$ and $\mathrm{Re}\,H^1(\boldsymbol{R})$ obtained by multiplying times the weight. We start by examining the image of an atom.

LEMMA III.2.5. *Let w be a weight with critical exponent q_0. Let $\delta > 0$ be such that w satisfies a reverse Hölder's inequality with exponent $1+\delta$. Let p be such that $1-\delta/q_0 < p < 1$ and also $p > q_0/(1+q_0)$. Then if $a(x)$ is a (p, ∞)-atom of h.t. with respect to w, there is a $q > q_0(1-p)+p$ and a constant (independent of a) such that* (const) $a(x)w(x)$ *is a (p, q)-atom with respect to $w(x)^{1-p}$.*

Proof. a will be a function with support contained in an interval I; and such that:

(i) $\|a\|_\infty \leqslant w(I)^{-1/p}$.

(ii) $\int_{-\infty}^{\infty} a(x)w(x)\,dx = 0$.

Let $b(x) = a(x)w(x)$. Since $w(x)^{1-p}$ has critical exponent $\leqslant q_0(1-p)+$ $+p$ and $p > q_0/(1+q_0)$ implies that $(q_0(1-p)+p)/p-1 < 1$, the only vanishing moment required for a (p, q)-atom with respect to $w(x)^{1-p}$ is the mean, and this does vanish for $b(x)$.

Now Lemma I.11 guarantees the existence of $q > q_0(1-p)+p$ and a constant such that

$$\left(\frac{1}{\int_I w(x)^{1-p}\,dx}\int_I w(x)^q w(x)^{1-p}\,dx\right)^{1/q} \leqslant (\text{const})\left(\frac{1}{\int_I w(x)^{1-p}\,dx}\int_I w(x)\,dx\right)^{1/p}.$$

Then

$$\left(\frac{1}{\int_I w(x)^{1-p}\,dx}\int_I |b(x)|^q w(x)^{1-p}\,dx\right)^{1/q} \leqslant (\text{const})\left(\int_I w(x)^{1-p}\,dx\right)^{-1/p}.$$

In order to go in the opposite direction we need less limitations in the range of p.

LEMMA III.2.6. *Let $w(x)$ be a weight with critical exponent q_0. Let $\frac{1}{2} < p < 1$. If $b(x)$ is a (p, ∞)-atom with respect to $w(x)^{1-p}$, then for every r such that $1 < r < 1+(1-p)/(q_0-1)$ there is a constant (independent of b) such that* (const) $b(x)/w(x)$ *is a (p, r)-atom of h.t. with respect to w.*

Proof. Let b be supported in I and such that

(i) $\|b\|_\infty \leqslant \left(\int_I w(x)^{1-p}\,dx\right)^{-1/p}$,

(ii) at least $\int_I b(x)\,dx = 0$.

Let $a(x) = b(x)/w(x)$. Then $\int_{-\infty}^{\infty} a(x)w(x)\,dx = 0$. Let r be such that $1 < r < 1+(1-p)/(q_0-1)$. Then:

$$\left(\frac{1}{w(I)}\int_I |a(x)|^r w(x)\,dx\right)^{1/r}$$
$$\leqslant \left(\int_I w(x)^{1-p}\,dx\right)^{-1/p}\left(\frac{\int_I w(x)^{1-r}\,dx}{\int_I w(x)\,dx}\right)^{1/r} \leqslant (\text{const})\,w(I)^{-1/p},$$

the last inequality being a consequence of (I.14).

In Lemma III.2.5, the restriction $1-\delta/p_0 < p$ is rather unfortunate, since there is no relation between δ and q_0. It will prove more fruitful to look at the Hilbert transform or the Poisson non-tangential maximal function of an atom. The results are as follows:

LEMMA III.2.7. *Let w be a weight with critical exponent q_0. Let $p > q_0/(1+q_0)$ and $r > 1$. Suppose $a(x)$ is a (p, r)-atom of h.t. with respect to w. Then $\int_{-\infty}^{\infty} |(a\cdot w)^{\sim}(x)|^p w(x)^{1-p}\,dx \leqslant$ (const), where* (const) *does not depend on a.*

Proof. Let I be the smallest closed interval containing the support of a. Take $q_1 > q_0$ so large that $q_1' = q_1/(q_1-1) < r$. Let $q = q_1'$. Clearly, $a(x)$ is a (p, q)-atom. Applying Hölder's inequality and realizing that $1-q = -1/(q_1-1)$, so that $w(x)^{1-q}$ satisfies the condition A_q, we get:

$$\int_{I^*} |(aw)^{\sim}(x)|^p w(x)^{1-p}\,dx \leqslant \left(\int_{I^*} |(aw)^{\sim}(x)|^q w(x)^{1-q}\,dx\right)^{p/q} w(I^*)^{1-p/q}$$
$$\leqslant (\text{const})\left(\int_{-\infty}^{\infty} |a(x)|^q w(x)\,dx\right)^{p/q} w(I^*)^{1-p/q} \leqslant (\text{const}).$$

Let, now, $x \notin I^*$. Then, for $y \in I$: $|x-y| \geqslant |x-C_I|/2$. Thus:

$$|(aw)^{\sim}(x)| = (\text{const})\left|\int_I \left(\frac{1}{x-y} - \frac{1}{x-C_I}\right) a(y)w(y)\,dy\right|$$
$$\leqslant (\text{const})\,\frac{|I|}{|x-C_I|^2}\,w(I)^{1-1/p}.$$

Since the critical exponent for $w(x)^{1-p}$ is $\leqslant q_0(1-p)+p < 2p$, we have:

$$\int_{x\notin I^*} \frac{w(x)^{1-p}\,dx}{|x-C_I|^{2p}} \leqslant (\text{const})\,\frac{w(I)^{1-p}}{|I|^p}.$$

Thus

$$\int_{x\notin I^*} |(a\cdot w)^{\sim}(x)|^p w(x)^{1-p}\,dx \leqslant (\text{const}).$$

LEMMA III.2.8. *Under the same hypotheses:*

$$\int_{-\infty}^{\infty} \left|P_\nu\left(a\cdot w+i(a\cdot w)^{\sim}\right)(x)\right|^p w(x)^{1-p}\,dx \leqslant (\text{const}).$$

Proof. Select q as above. Then

$$\int_{I^*} \left|P_\nu^*\left(aw+i(aw)^{\sim}\right)(x)\right|^p w(x)^{1-p}\,dx \leqslant (\text{const})\Big(\int_{-\infty}^{\infty} |a(x)|^q w(x)\,dx\Big)^{p/q} w(I^*)^{1-p/q}$$
$$\leqslant (\text{const}).$$

Let $x\notin I^*$. Let $(y,t)\in \boldsymbol{R}_+^2$ such that $|x-y|<t$

$$\left|\left(P_t*(a\cdot w)\right)(y)\right| = \Big|\int_{-\infty}^{\infty}\left(P_t(y-u)-P_t(y-C_I)\right)a(u)w(u)\,du\Big|$$

$$\leqslant \int_{-\infty}^{\infty}\left|P_t'\left(y-C_I-\theta_u(u-C_I)\right)\right||u-C_I|\,|a(u)|w(u)\,du, \qquad \text{where } 0<\theta_u<1.$$

But $|P'(s)|\leqslant (\text{const})/(1+|s|)^2$ leads to:

$$\left|\left(P_t*(a\cdot w)\right)(y)\right| \leqslant \int_{-\infty}^{\infty} \frac{(\text{const})}{\left(t+|y-C_I-\theta_u(u-C_I)|\right)^2}|u-C_I|\,|a(u)|w(u)\,du$$

and since $t+|y-C_I-\theta_u(u-C_I)|\geqslant \dfrac{|x-C_I|}{2}$, we get: $P_\nu^*(a\cdot w)(x)$ $\leqslant (\text{const})\dfrac{|I|}{|x-C_I|^2}\,w(I)^{1-1/p}$ which is the same estimate obtained in (III.2.7) for $|(a\cdot w)^{\sim}(x)|$ with $x\notin I^*$. Then exactly as in (III.2.7) we obtain:

$$\int_{x\notin I^*} |P_\nu^*(a\cdot w)(x)|^p w(x)^{1-p}\,dx \leqslant (\text{const}).$$

Now since $P_\nu^*\left((a\cdot w)^{\sim}\right)(x) = (\tilde{P})_\nu^*(a\cdot w)(x)$ and $\tilde{P}(s) = s/(1+s^2)$ also satisfies $|\tilde{P}'(s)|\leqslant (\text{const})/(1+|s|)^2$ the same proof yields:

$$\int_{x\notin I^*} \left|P_\nu^*\left((a\cdot w)^{\sim}\right)(x)\right|^p w(x)^{1-p}\,dx \leqslant (\text{const})$$

which finishes the proof of (III.2.8).

Next we establish the main result which relates the "atomic" or "homogeneous type" $\mathfrak{H}^p$ spaces and the "analytic type" H^p spaces.

THEOREM III.2.9. *Let w be a weight with critical exponent q_0. Let $q_0/(1+q_0)<p\leqslant 1$. Then there is an equivalence between $\mathfrak{H}^p(w(x)dx)$ and $H^p(w(x)^{1-p}dx)$ which for (p,r)-atoms of h.t. with respect to w, is given by:*

$$a(x)\mapsto P_t*\left(a\cdot w+i(a\cdot w)^{\sim}\right)(x).$$

Proof. From (III.1.2) we know that the result holds for $p = 1$. Here we are concerned with $p < 1$. Fix r such that $1 < r < 1+(1-p)/(q_0-1)$. Let $\mathfrak{A}$ be the subspace of $\left(\mathfrak{L}_{1/p-1}(w(x)\,dx)\right)^*$ formed by those functionals L which can be written as finite linear combinations of functionals L_{a_j} corresponding to (p, r)-atoms of h.t. with respect to w, a_j. Clearly, $\mathfrak{A}$ is a dense subspace of $\mathfrak{H}^p(w(x)\,dx)$. For $L \in \mathfrak{A}$, let

$$\overline{\mathfrak{N}}_{p,r}(L) = \inf\left\{\left(\sum_{j=1}^{n} |\lambda_j|^p\right)^{1/p}:\ L = \sum_{j=1}^{n} \lambda_j L_{a_j},\ a_j(p, r)\text{-atoms}\right\}.$$

Obviously $\mathfrak{N}_{p,r}(L) \leqslant \overline{\mathfrak{N}}_{p,r}(L)$. It turns out that $L \mapsto (\mathfrak{N}_{p,r}(L))^p$ and $L \mapsto (\overline{\mathfrak{N}}_{p,r}(L))^p$ are equivalent quasi-norms on $\mathfrak{A}$. This is not totally obvious. It can be proved by considering the completion of $\mathfrak{A}$ with respect to $L \mapsto (\overline{\mathfrak{N}}_{p,r}(L))^p$ which will be a subspace of $\mathfrak{H}^p(w(x)\,dx)$, and realizing that it has to coincide with $\mathfrak{H}^p(w(x)\,dx)$. Then the identity mapping is continuous and onto and the open mapping theorem guarantees that its inverse is also continuous and, therefore, the two quasi-norms are equivalent.

Let $\mathfrak{B}$ be the space of all the holomorphic functions F in the upper half plane, which can be written as a finite sum:

$$F(x+it) = \sum_{j=1}^{n} \lambda_j\left(P_t * \left(a_j \cdot w + i(a_j \cdot w)^{\sim}\right)\right)(x),$$

where λ_j are real numbers and a_j are (p, r)-atoms of h.t. with respect to w. For $F \in \mathfrak{B}$ define $\overline{N}_{p,r}(F) = \inf\left\{\left(\sum_{j=1}^{n} |\lambda_j|^p\right)^{1/p}\right\}$ taken over all the decompositions of the above type. It follows from (III.2.8) that $\mathfrak{B} \subset H^p(w(x)^{1-p}\,dx)$ and for $F \in \mathfrak{B}$: $\|F\|_{H^p(w(x)^{1-p}dx)} \leqslant \overline{N}_{p,r}(F)$. Actually, on $\mathfrak{B}$, the quasi-norms $F \mapsto (\overline{N}_{p,r}(F))^p$ and $F \mapsto \|F\|^p_{H^p(w(x)^{1-p}dx)}$ are equivalent. The proof is as follows: First of all, $\mathfrak{B}$ is a dense subspace of $H^p(w(x)^{1-p}\,dx)$ since, according to (III.2.6) it contains all the finite linear combinations of the form $F(x+it) = \sum_{j=1}^{n} \lambda_j\left(P_t * (b_j + i\tilde{b}_j)\right)(x)$, where b_j are (p, ∞)-atoms with respect to $w(x)^{1-p}$. Consider the completion of $\mathfrak{B}$ with respect to the quasi-norm $F \mapsto (\overline{N}_{p,r}(F))^p$. It coincides with $H^p(w(x)^{1-p})\,dx$ because of (II.3.7). Then the equivalence of the two quasi-norms follows from the open mapping theorem.

To $L = \sum_{j=1}^{n} \lambda_j L_{a_j} \in \mathfrak{A}$ we associate the analytic function in the upper half plane:

$$F(x+it) = \sum_{j=1}^{n} \lambda_j\left(P_t * \left(a_j \cdot w + i(a_j \cdot w)^{\sim}\right)\right)(x),$$

$F \in \mathfrak{B}$. Let us prove that we obtain this way, a well defined mapping from $\mathfrak{A}$ to $\mathfrak{B}$. It amounts to showing that if $\sum_{j=1}^{n} \lambda_j L_{a_j} = 0$ in $\left(\mathfrak{L}_{1/p-1}(w(x)\,dx)\right)^*$, then:

$$\left(P_t * \left(\sum_{j=1}^{n} \lambda_j a_j w + i\left(\sum_{j=1}^{n} \lambda_j a_j w\right)^{\sim}\right)\right)(x) \equiv 0.$$

Suppose that $\sum_{j=1}^{n} \lambda_j L_{a_j} = 0$ in $\left(\mathfrak{L}_{1/p-1}(w(x)\,dx)\right)^*$. This means that for every $[l] \in \mathfrak{L}_{1/p-1}(w(x)\,dx)$, $\int_{-\infty}^{\infty} l(x) \left(\sum_{j=1}^{n} \lambda_j a_j(x)\right) w(x)\,dx = 0$. It follows that $\sum_{j=1}^{n} \lambda_j a_j(x) w(x) = 0$ for a.e. x, because $\mathfrak{L}_{1/p-1}(w(x)\,dx)$ contains $[l]$ for each function which is $\mathscr{C}^\infty$ and has compact support, provided $p > q_0/(1+q_0)$. (Indeed, let l be $\mathscr{C}^\infty$ with support contained in the interval K. $p > q_0/(1+q_0)$ is equivalent to $1/q_0 > 1/p - 1$. Take $r > q_0$ so that $1/q_0 > 1/r > 1/p - 1$. Then for $x, y \in I \subset K$ we have:

$$|l(x) - l(y)| \leqslant (\text{const})\,|x-y| \leqslant (\text{const})\,|K| \left(\frac{w(I)}{w(K)}\right)^{1/r}$$

$$\leqslant (\text{const})\,|K| \left(\frac{w(I)}{w(K)}\right)^{1/p-1} \leqslant (\text{const})\,w(I)^{1/p-1}.)$$

It follows that

$$\left(P_t * \left(\sum_{j=1}^{n} \lambda_j a_j w + i\left(\sum_{j=1}^{n} \lambda_j a_j w\right)^{\sim}\right)\right)(x) \equiv 0.$$

It is clear that $L \leftrightarrow F$ is an equivalence between $\mathfrak{A}$ and $\mathfrak{B}$. The density of these subspaces allows to extend it to a unique equivalence between $\mathfrak{H}^p(w(x)\,dx)$ and $H^p(w(x)^{1-p}\,dx)$.

Chapter IV

Applications and examples

1. A weighted Hilbert transform

Let w be a weight, W a primitive of w and B the inverse of W. In (III.1) we established the following equivalences: $f(x)$ is in $\mathfrak{H}^1(w(x)dx) \Leftrightarrow$ $\Leftrightarrow f(B(y))$ is in $H^1(dy) \Leftrightarrow f(x)w(x)$ is in $H^1(dx)$.

For $f \in L^1(w(x)dx)$, $(f \circ B)^{\sim}(W(r)) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(s)w(s)}{W(r)-W(s)}\, ds$ so that $f \in \mathfrak{H}^1(w(x)dx)$ if and only if p.v. $\int_{-\infty}^{\infty} \frac{f(s)w(s)w(r)}{W(r)-W(s)}\, ds$ is in $L^1(dr)$. Also $f \in \mathfrak{H}^1(w(x)dx)$ if and only if p.v. $\int_{-\infty}^{\infty} \frac{f(s)w(s)}{r-s}\, ds$ is in $L^1(dr)$. Thus, for a function $g \in L^1$, p.v. $\int_{-\infty}^{\infty} g(s) \frac{w(r)}{W(r)-W(s)}\, ds$ is in $L^1(dr)$ if and only if p.v. $\int_{-\infty}^{\infty} g(s) \frac{1}{r-s}\, ds$ is in $L^1(dr)$. That is, for a weight w, the singular integral with kernel $\frac{w(r)}{W(r)-W(s)}$ can be used in place of the Hilbert transform, with kernel $\frac{1}{\pi}\frac{1}{r-s}$ in order to characterize H^1.

2. Equivalence between the space of radial functions in $H^1(R^n)$ and the space of even functions in $\mathfrak{H}^1(|r|^{n-1}dr)$

LEMMA IV.2.1. *Let $w(r)$ be an even weight. Then, for a function $f(r)$ living in the right half line $[0, \infty[$, the following properties are equivalent:*

(i) $f \in \mathfrak{H}^1(w(r)dr)$;

(ii) *The even extension g of f is in $\mathfrak{H}^1(w(r)dr)$;*

(iii) $f(r) = \sum_{i=1}^{\infty} \lambda_i a_i(r)$ *with* $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ *and a_i $(1, \infty)$-atoms of h.t. with respect to w living all of them in $[0, \infty[$.*

Also $\|f\|_{\mathfrak{H}^1(w(r)dr)}$, $\|g\|_{\mathfrak{H}^1(w(r)dr)}$ *and* $\inf\{\sum_{i=1}^{\infty}|\lambda_i|\}$, *where the* inf *is taken over all decompositions allowed in* (iii) *are equivalent norms.*

Proof. (i) $\Rightarrow$ (ii). If $f \in \mathfrak{H}^1(w(r)dr)$, then $f(r)\chi_{[0,\infty[}(r) = f(r) = \sum_{i=1}^{\infty} \lambda_i a_i(r)$ for a.e. $r \in \boldsymbol{R}$ with $\sum_{i=1}^{\infty} |\lambda_i| \leqslant (\text{const})\|f\|_{\mathfrak{H}^1(w(r)dr)}$ and the a_i's being $(1, \infty)$-atoms of h.t. with respect to w. Then $f(-r)\chi_{]-\infty,0]}(r) = f(-r) = \sum_{i=1}^{\infty} \lambda_i a_i(-r)$ for a.e. $r \in \boldsymbol{R}$. For $j = 1, 2, \ldots$ let $b_j = a_j$. For $j = -1, -2, \ldots$ let $b_j = a_{-j}(-r)$. From the fact that $w(r)$ is even it follows immediately that for $j = -1, -2, \ldots$, b_j is a $(1, \infty)$-atom of h.t. with respect to w. Let $b_0 = 0$ and $\lambda_0 = 0$. Then $g(r) = f(r)\chi_{[\ ,\infty[}(r) + f(-r)\chi_{]-\infty,0]}(r) = \sum_{j=-\infty}^{\infty} \lambda_{|j|} b_j(r)$.

Thus $g(r)$ is in $\mathfrak{H}^1(w(r)dr)$ and $\|g\|_{\mathfrak{H}^1(w(r)dr)} \leqslant (\text{const})\|f\|_{\mathfrak{H}^1(w(r)dr)}$.

(ii) $\Rightarrow$ (iii). If $g(r)$ is in $\mathfrak{H}^1(w(r)dr)$ we will have: $g(r) = \sum_{i=1}^{\infty} \lambda_i a_i(r)$, where $\sum_{i=1}^{\infty}|\lambda_i| \leqslant (\text{const})\|g\|_{\mathfrak{H}^1(w(r)dr)}$ and the a_i's are $(1, \infty)$-atoms of h.t. with respect to w. Since g is even: $g(r) = \frac{1}{2}(g(r) + g(-r)) = \sum_{i=1}^{\infty} \lambda_i \dfrac{a_i(r) + a_i(-r)}{2}$.

Then $f(r) = g(r)\chi_{[0,\infty[}(r) = \sum_{i=1}^{\infty} \lambda_i \dfrac{a_i(r) + a_i(-r)}{2} \chi_{[0,\infty[}(r)$. Let $b_i(r) = \dfrac{a_i(r) + a_i(-r)}{2} \chi_{[0,\infty[}(r)$. If a_i lives in $[0, \infty[$, then $b_i(r) = a_i(r)/2$. If a_i lives in $]-\infty, 0]$, then $b_i(r) = a_i(-r)/2$. If we are not in any of the previous two cases, still b_i is a $(1, \infty)$-atom of h.t. with respect to w living in $[0, \infty[$. Indeed, clearly $\int b_i(r)w(r)dr = 0$ and if I_i is the smallest interval containing the support of a_i; b_i will live in $I_i \cap [0, \infty[$ and $|b_i(r)| \leqslant 1/w(I_i) \leqslant 1/w(I_i \cap [0, \infty[)$. Thus $f(r) = \sum_{i=1}^{\infty} \lambda_i b_i$ proves that (iii) holds and also $\sum_i |\lambda_i| \leqslant \|g\|_{\mathfrak{H}^1(w(r)dr)}$.

That (iii) implies (ii) is trivial.

(IV.2.1) is true, in particular, for $w(r) \equiv 1$, for which $\mathfrak{H}^1(w(r)dr) = \mathfrak{H}^1(dr) = H^1(\boldsymbol{R})$.

THEOREM IV.2.2. *A radial function* $F(x) = f(|x|)$ *defined on* $\boldsymbol{R}^n$, *is in* $H^1(\boldsymbol{R}^n)$ *if and only if* $f(r)\chi_{[0,\infty[}(r) = \sum_{i=1}^{\infty} \lambda_i a_i(r)$ *for a.e.* r *with* $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ *and each* a_i *being a* $(1, \infty)$*-atom of h.t. with respect to the weight* $|r|^{n-1}$ *and living in the half line* $[0, \infty[$. *Also* $\inf\{\sum |\lambda_i|\}$, *where the* inf *is taken over all such decompositions, is equivalent to* $\|F\|_{H^1(\boldsymbol{R}^n)}$.

Proof. Let $F(x) = f(|x|)$ be in $H^1(\boldsymbol{R}^n)$. Then F admits an atomic decomposition $F(x) = \sum_{i=1}^{\infty} \lambda_i A_i(x)$, where $\sum_{i=1}^{\infty} |\lambda_i| \leqslant (\text{const}) \|F\|_{H^1(\boldsymbol{R}^n)}$, each A_i is supported in a ball B_i, has $\int_{\boldsymbol{R}^n} A_i(x)\,dx = 0$ and $|A_i(x)| \leqslant 1/|B_i|$. This atomic decomposition for functions in $H^1(\boldsymbol{R}^n)$ is equivalent to the duality result found in [8]. It can also be obtained directly by modifying accordingly (II.3.6). This has been done by Latter. See the note at the end of the Introduction.

Since F is radial, we will have for a.e. $r > 0$: $f(r) = \sum_{i=1}^{\infty} \lambda_i a_i(r)$, where, for each i, $a_i(r) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} A_i(rx')\,dx'$. We will see that (const) a_i is a $(1, \infty)$-atom of h.t. with respect to the weight $|r|^{n-1}$ for every i. Let us consider, in general, a $(1, \infty)$-atom A in $H^1(\boldsymbol{R}^n)$ and set, for $r > 0$, $a(r) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} A(rx')\,dx'$. Let B be the smallest ball containing the support of A. Let $r_0 = \operatorname{dist}(0, B)$ and $\delta = \operatorname{diam}(B)$. Then a is supported in $[r_0, r_0+\delta]$. $\int_0^{\infty} a(r) r^{n-1}\,dr = (\text{const}) \int_{\boldsymbol{R}^n} A(x)\,dx = 0$ and also $|a(r)| \leqslant (\text{const})/\delta^n$. If $r_0 < \delta$ this is enough to realize that $|a(r)| \leqslant (\text{const})/ / \int_{r_0}^{r_0+\delta} r^{n-1}\,dr$ because $\int_{r_0}^{r_0+\delta} r^{n-1}\,dr \leqslant (\text{const})\,\delta^n$. However, the estimate is not good enough for $r_0 > \delta$. We can obtain a much better one. Consider N rotations $\{\varrho_1, \ldots, \varrho_N\}$ such that the balls $\{\varrho_1^{-1}(B), \ldots, \varrho_N^{-1}(B)\}$ are disjoint. The number N will be $\geqslant (\text{const})(r_0/\delta)^{n-1}$ with (const) depending only on the dimension.

For each j: $\frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} A(\varrho_j rx')\,dx' = a(r)$ so that

$$a(r) = \frac{1}{N} \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} \sum_{j=1}^{N} A(\varrho_j rx')\,dx'.$$

For r fixed, each rx' belongs to at most one of the $\varrho_j^{-1}(B)$'s; so that the sum $\sum_{j=1}^{N} A(\varrho_j rx')$ has actually one single term at most for every x'. Thus $|\sum_{j=1}^{N} A(\varrho_j rx')| \leqslant (\text{const})/\delta^n$. Then

$$|a(r)| \leqslant \left(\frac{\delta}{r_0}\right)^{n-1} \frac{(\text{const})}{\delta^n} \leqslant \frac{(\text{const})}{\int_{r_0}^{r_0+\delta} r^{n-1}\,dr}$$

and we have seen that $a(r)/(\text{const})$ is always a $(1, \infty)$-atom of h.t. with respect to the weight $|r|^{n-1}$. This way we obtain our desired decompo-

sition $f(r)\chi_{[0,\infty[}(r) = \sum \lambda_i(\text{const})\, a_i(r)/(\text{const})$ and see that $\sum |\lambda_i|\,(\text{const}) \leqslant (\text{const})\, \|F\|_{H^1(\boldsymbol{R}^n)}$.

Suppose conversely that $f(r)\chi_{[0,\infty[}(r) = \sum_{i=1}^{\infty} \lambda_i a_i(r)$ for a.e. r with $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ and each a_i a $(1, \infty)$-atom of h.t. with respect to $|r|^{n-1}$ living in the half line $[0, \infty[$. Let $a(r)$ be a general $(1, \infty)$-atom of h.t. with respect to the weight $|r|^{n-1}$ and living in $[0, \infty[$. Let $[r_0, r_0+\delta]$ be the smallest interval containing the support of a. That a is an atom means:

$$\int_0^{\infty} a(r) r^{n-1} dr = 0 \qquad \text{and} \qquad |a(r)| \leqslant 1/\int_{r_0}^{r_0+\delta} r^{n-1}\, dr.$$

Associate with a, the radial function $A(x) = a(|x|)$ defined on $\boldsymbol{R}^n$. By this method we will associate with each atom a_i in the decomposition $f(r)\chi_{[0,\infty[}(r) = \sum_{i=1}^{\infty} \lambda_i a_i(r)$, a radial function $A_i(x) = a_i(|x|)$. Then $F(x) = f(|x|) = \sum_{i=1}^{\infty} \lambda_i a_i(|x|) = \sum_{i=1}^{\infty} \lambda_i A_i(x)$. Going back to our atom $a(r)$ supported in $[r_0, r_0+\delta]$, let us study the radial function $A(x) = a(|x|)$ to which it gives rise. First, if $r_0 < \delta$, there is a constant such that $(\text{const})A(x)$ is a $(1, \infty)$-atom in $H^1(\boldsymbol{R}^n)$. In fact: $A(x)$ is supported in the ball of center 0 and radius $r_0+\delta$,

$$\int_{\boldsymbol{R}^n} A(x)\, dx = |\Sigma_{n-1}| \int_0^{\infty} a(r) r^{n-1} dr = 0$$

and

$$|A(x)| \leqslant (\text{const})/(r_0+\delta)^{n-1}\delta \leqslant (\text{const})/(r_0+\delta)^n.$$

If $\delta < r_0$ we can still decompose A into a finite linear combination of atoms in $H^1(\boldsymbol{R}^n)$ with the sum of the absolute values of the coefficients bounded by a constant. In fact we partition the region $r_0 \leqslant |x| \leqslant r_0+\delta$ into N spherical cubes Q^j of measure $\leqslant (\text{const})\,\delta^n$, where $N \leqslant (\text{const})\,(r_0/\delta)^{n-1}$. Then let $A^{(j)}(x) = A(x)\chi_{Q_j}(x)$. Clearly $A(x) = \sum_{j=1}^{N} \frac{1}{N} N A^{(j)}(x)$ and it turns out that $NA^{(j)}(x)$ is an atom in $H^1(\boldsymbol{R}^n)$ for every j. Indeed $A^{(j)}$ is supported in B_j, the smallest ball containing Q_j and $|B_j| \leqslant (\text{const})\delta^n$. Also

$$\int_{\boldsymbol{R}^n} A^{(j)}(x)\, dx = \Big(\int_{r_0}^{r_0+\delta} a(r) r^{n-1} dr\Big) \cdot \Big(\int_{\Sigma_{n-1}} \chi_{Q_j}(r_0 x')\, dx'\Big) = 0$$

and

$$|NA^{(j)}(x)| \leqslant (\text{const}) \left(\frac{r_0}{\delta}\right)^{n-1} \frac{1}{(r_0+\delta)^{n-1}\delta} \leqslant \frac{(\text{const})}{|B_j|}.$$

Thus, no matter whether $r_0 \leqslant \delta$ or $\delta \leqslant r_0$ we always have $A(x) = \sum a_j B^{(j)}(x)$, where $B^{(j)}$ are atoms in $H^1(\boldsymbol{R}^n)$ and $\sum |a_j| \leqslant$ (const). Going back to our particular problem where we had $F(x) = f(|x|) = \sum_{j=1}^{\infty} \lambda_i A_i(x)$, we can decompose every $A_i(x)$ as $A_i(x) = \sum_{j=1}^{\infty} a_{ij} B_i^{(j)}(x)$ with the $B_i^{(j)}$'s being atoms in $H^1(\boldsymbol{R}^n)$ and $\sum_{j=1}^{\infty} |a_{ij}| \leqslant$ (const) independently of i. Thus

$$F(x) = \sum_{i=1}^{\infty} \lambda_i \Big(\sum_{j=1}^{\infty} a_{ij} B_i^{(j)}(x) \Big) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda_i a_{ij} B_i^{(j)}(x)$$

which is an atomic decomposition with

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_i a_{ij}| \leqslant \sum_{i=1}^{\infty} |\lambda_i| \Big(\sum_{j=1}^{\infty} |a_{ij}| \Big) \leqslant (\text{const}) \sum_{i=1}^{\infty} |\lambda_i|$$

which tells us that $F \in H^1(\boldsymbol{R}^n)$ and $\|F\|_{H^1(\boldsymbol{R}^n)} \leqslant \text{const} \sum_{i=1}^{\infty} |\lambda_i|$. This finishes the proof of (IV.2.2).

Let $H^1_{\text{rad}}(\boldsymbol{R}^n)$ be the closed subspace of $H^1(\boldsymbol{R}^n)$ formed by the radial functions. Let $\mathfrak{H}^1_{\text{even}}(|r|^{n-1}dr)$ be the closed subspace of $\mathfrak{H}^1(|r|^{n-1}dr)$ formed by the even functions. Then by Lemma IV.2.1, Theorem IV.2.2 can be restated as follows:

THEOREM IV. 2.3. *The mapping $f(r) \mapsto F(x) = f(|x|)$ is an equivalence between the Banach spaces $\mathfrak{H}^1_{\text{even}}(|r|^{n-1}dr)$ and $H^1_{\text{rad}}(\boldsymbol{R}^n)$.*

Theorem III.1.2 provides another equivalence: $f(r) \to f(r)\,|r|^{n-1}$ between $\mathfrak{H}^1_{\text{even}}(|r|^{n-1}dr)$ and $H^1_{\text{even}}(\boldsymbol{R})$ the space of even functions in $\operatorname{Re} H^1(\boldsymbol{R})$. Putting this together with (IV.2.3) we have:

COROLLARY IV.2.4. *The mapping $F(x) = f(|x|) \to f(|r|)\,|r|^{n-1}$ is an equivalence between $H^1_{\text{rad}}(\boldsymbol{R}^n)$ and $H^1_{\text{even}}(\boldsymbol{R})$.*

Let $H^1_+(\boldsymbol{R})$ be the subspace of $\operatorname{Re} H^1(\boldsymbol{R})$ formed by the functions living in $[0, \infty[$. Then, again by Lemma IV.2.1, we have:

COROLLARY IV.2.5. *The mapping:*

$$F(x) = f(|x|) \mapsto f(r) r^{n-1} \chi_{[0,\infty[}(r)$$

is an equivalence between $H^1_{\text{rad}}(\boldsymbol{R}^n)$ and $H^1_+(\boldsymbol{R})$.

3. Integral operators in the line obtained by restricting to radial functions some systems of Riesz transforms in higher dimensions

(IV.2.5) suggests to investigate the relation between the system of Riesz transforms of a radial function $F(x) = f(|x|)$ in $\boldsymbol{R}^n$, that is

$$\left\{ R_j F(x) = \text{p.v.} \int_{\boldsymbol{R}^n} F(y) \frac{x_j - y_j}{|x-y|^{n+1}} dy \right\}_{j=1,\ldots,n}$$

and the Hilbert transform of $f(r)r^{n-1}\chi_{[0,\infty[}(r)$. The study of this relation will lead to another proof of (IV.2.5) independent of the atomic decomposition in $H^1(\boldsymbol{R}^n)$.

The first observation to make is that for a radial function $F(x) = f(|x|)$, the n Riesz transforms are not essentially different. Indeed, there is a radial function $RF(x)$ such that for $j = 1, \ldots, n$:

$$(R_jF)(x) = -\frac{1}{n-1}\frac{x_j}{|x|}(RF)(x).$$

If $|x| = r$ and $\boldsymbol{I}$ is a fixed point on the unit sphere Σ_{n-1}, we have:

$$(RF)(x) = \frac{d}{dr}\int_{\boldsymbol{R}^n} F(y)\frac{1}{|r\boldsymbol{I}-y|^{n-1}}dy.$$

Since

$$|RF(x)| = (n-1)\Big\{\sum_{j=1}^{n}|R_jF(x)|^2\Big\}^{1/2},$$

it follows that for a radial function $F(x)$ in $\boldsymbol{R}^n$, the fact that all the Riesz transforms $(R_jF)(x)$ are in $L^1(\boldsymbol{R}^n)$ is equivalent to the fact that $(RF)(x)$ is in $L^1(\boldsymbol{R}^n)$. In other words: the radial operator R can be used to characterize $H^1_{\mathrm{rad}}(\boldsymbol{R}^n)$.

We can view R as an operator in the line applied to $f(s)s^{n-1}\chi_{[0,\infty[}(s)$. Indeed

$$(RF)(x) = \text{p.v.}\int_0^\infty f(s)s^{n-1}\mathfrak{K}(r, s)\,ds, \qquad r = |x|,$$

where

$$\mathfrak{K}(r, s) = \frac{\partial}{\partial r}\Big(\int_{\Sigma_{n-1}}\frac{dy'}{|r\boldsymbol{I}-sy'|^{n-1}}\Big).$$

Thus, for $F(x) = f(|x|)$ in $L^1(\boldsymbol{R}^n)$ or, what is the same, $f(r)$ in $L^1([0, \infty[;\ r^{n-1}dr)$ we have: $F(x)$ is in $H^1(\boldsymbol{R}^n) \Leftrightarrow \text{p.v.}\int_0^\infty f(s)s^{n-1}\mathfrak{K}(r, s)ds$ is in $L^1([0, \infty[;\ r^{n-1}dr)$. Next we compute the kernel of this singular integral:

$$\int_{\Sigma_{n-1}}\frac{dy'}{|r\boldsymbol{I}-sy'|^{n-1}} = \int_{\Sigma_{n-1}}\frac{dy'}{(r^2+s^2-2rs\langle\boldsymbol{I}, y'\rangle)^{(n-1)/2}}$$
$$= C_n\int_0^\pi\frac{(\sin\theta)^{n-2}d\theta}{(r^2+s^2-2rs\cos\theta)^{(n-1)/2}},$$

C_n depends only on the dimension n.

Muckenhoupt and Stein have studied by "complex methods" the space of functions $f(r)$ in $L^1([0, \infty[; r^{2\lambda}dr)$ for which p.v. $\int_0^\infty f(s)s^{2\lambda}\mathfrak{K}_\lambda(r,s)\,ds$ is also in $L^1([0, \infty[; r^{2\lambda}dr)$ with

$$\mathfrak{K}_\lambda(r, s) = C_\lambda \frac{\partial}{\partial r}\left(\int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(r^2+s^2-2rs\cos\theta)^\lambda}\,d\theta\right).$$

See [14]. For $\lambda = (n-1)/2$ they obtain the space that we are considering, i.e., the one corresponding to $H^1_{\mathrm{rad}}(\boldsymbol{R}^n)$. In general, we will study the kernel $\mathfrak{K}_\lambda(r, s)$ for $\lambda \geqslant \frac{1}{2}$. As a first step we analyze the situation for $\lambda = 1$ which corresponds to $n = 3$. In this case, the kernel can be computed very easily. In fact

$$\int_0^\pi \frac{\sin\theta}{r^2+s^2-2rs\cos\theta}\,d\theta = \frac{1}{rs}\log\left|\frac{r+s}{r-s}\right|$$

and, therefore:

$$\mathfrak{K}(r, s) = -(\mathrm{const})\left(\frac{2}{r}\frac{1}{r^2-s^2}+\frac{1}{r^2 s}\log\left|\frac{r+s}{r-s}\right|\right).$$

Since $F(x)$ is in $H^1(\boldsymbol{R}^3)$ if and only if p.v. $\int_0^\infty f(s)s^2r^2\mathfrak{K}(r, s)\,ds$ is in $L^1([0, \infty[)$, we are actually interested in the kernel:

$$\begin{aligned} r^2\mathfrak{K}(r, s) &= -(\mathrm{const})\left(\frac{2r}{r^2-s^2}+\frac{1}{s}\log\left|\frac{r+s}{r-s}\right|\right) \\ &= -(\mathrm{const})\left(\frac{1}{r-s}+\frac{1}{r+s}+\frac{1}{s}\log\left|\frac{r+s}{r-s}\right|\right). \end{aligned}$$

From this decomposition of the kernel, we obtain the following relation between our singular integral and the Hilbert transform:

$$\text{(IV.3.1)}\qquad \begin{aligned} \text{p.v.}\int_0^\infty f(s)s^2\big(r^2\mathfrak{K}(r, s)\big)\,ds = -(\mathrm{const})\Big\{&\big(f(s)s^2\chi_{[0,\infty[}(s)\big)^\sim(r) - \\ &-\big(f(s)s^2\chi_{[0,\infty[}(s)\big)^\sim(-r)+\int_0^\infty f(s)s^2\cdot\frac{1}{s}\log\left|\frac{r+s}{r-s}\right|ds\Big\}. \end{aligned}$$

LEMMA IV.3.2. *$g(r) \mapsto \int_0^\infty g(s)\cdot\frac{1}{s}\log\left|\frac{r+s}{r-s}\right|ds$ is a bounded operator from $H^1_+(\boldsymbol{R})$ into $L^1([0, \infty[)$.*

Proof. By using the power series expansion

$$\log\left|\frac{1+z}{1-z}\right| = 2\sum_{k=0}^{\infty}\frac{z^{2k+1}}{2k+1}, \qquad |z|<1,$$

we can obtain a decomposition:

$$\int_0^\infty g(s)\cdot\frac{1}{s}\log\left|\frac{r+s}{r-s}\right|ds$$

$$= 2\left\{\sum_{k=0}^{\infty}\frac{1}{2k+1}\frac{1}{r^{2k+1}}\int_0^r g(s)s^{2k}\,ds + \sum_{k=0}^{\infty}\frac{1}{2k+1}r^{2k+1}\int_0^\infty\frac{g(s)}{s^{2k+2}}\,ds\right\}$$

$$= 2\left\{\sum_{k=0}^{\infty}\frac{1}{2k+1}A_{2k}(g)(r) + \sum_{k=0}^{\infty}\frac{1}{2k+1}B_{2k+2}(g)(r)\right\},$$

where, for $t\geqslant 0$ we let $(A_t g)(r) = r^{-t-1}\int_0^r g(s)s^t\,ds$ and $(B_t g)(r) = r^{t-1}\int_r^\infty g(s)s^{-t}\,ds$. It is an immediate consequence of Fubini's theorem that, for $t>0$, A_t and B_t are bounded operators in $L^1([0,\infty[)$ with operator norms $\|A_t\| = \|B_t\| = 1/t$. Clearly neither A_0 nor B_0 map $L^1([0,\infty[)$ into itself. For example if $g = \chi_{[0,1]}$:

$$A_0(g)(r) = \begin{cases} 1 & \text{if } 0<r\leqslant 1,\\ \dfrac{1}{r} & \text{if } 1<r,\end{cases}$$

and

$$B_0(g)(r) = \begin{cases} \dfrac{1}{r}-1 & \text{if } 0<r\leqslant 1,\\ 0 & \text{if } 1<r,\end{cases}$$

which are not in $L^1([0,\infty[)$. However, A_0 and B_0 still map $H^1_+(\boldsymbol{R})$ boundedly into $L^1([0,\infty[)$. Indeed if a is a $(1,\infty)$-atom and $[r_0, r_0+\delta]$ with $r_0, \delta>0$, is the smallest interval containing its support, $A_0(a)$ is also supported inside I, since $\int_{-\infty}^{\infty} a(s)\,ds = 0$. Also

$$|A_0(a)(r)| \leqslant \frac{1}{r}\int_0^r |a(s)|\,ds \leqslant \frac{1}{|I|}.$$

Thus $\int_0^\infty |A_0(a)(r)|\,dr \leqslant 1$. Also $B_0(a)(r) = -A_0(a)(r)$ since $\int_{-\infty}^{\infty} a(r)\,dr = 0$. We have thus realized our operator as a sum of a series of bounded operators

from $H^1_+(\boldsymbol{R})$ to $L^1([0, \infty[)$ which converges in norm. This finishes the proof. Combining this result with the decomposition (IV.3.1) we obtain:

THEOREM IV.3.3. *If $f(s)s^2$ is in $H^1_+(\boldsymbol{R})$, then the radial function $F(x) = f(|x|)$ is in $H^1(\boldsymbol{R}^3)$ and*

$$\|F\|_{H^1(\boldsymbol{R}^3)} \leqslant (\text{const})\, \|f(s)s^2\|_{H^1_+(\boldsymbol{R})}.$$

Of course this is part of (IV.2.5) but the new proof does not depend on the geometry of $\boldsymbol{R}^3$ and can be extended to the non-euclidean cases considered by Muckenhoupt and Stein. We will consider, for $\lambda \geqslant \frac{1}{2}$, the kernel $r^{2\lambda}\mathfrak{K}_\lambda(r, s)$, where

$$\mathfrak{K}_\lambda(r, s) = C_\lambda \frac{\partial}{\partial r}\left(\int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(r^2+s^2-2rs\cos\theta)^\lambda}\, d\theta\right).$$

In particular, for $\lambda = (n-1)/2$, n being a positive integer, we get the kernel corresponding to the Riesz transforms of a radial function in $\boldsymbol{R}^n$.

$$K_\lambda(r, s) = \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(r^2+s^2-2rs\cos\theta)^\lambda}\, d\theta = K_\lambda(s, r).$$

Thus, we can restrict our attention to $s < r$. Then, changing variables and expanding in Gegenbauer polynomials (see [5] and [6]) we obtain:

$$K_\lambda(r, s) = \frac{1}{r^{2\lambda}} \int_{-1}^{1} \frac{(1-t^2)^{\lambda-1}\, dt}{\left(1+\left(\frac{s}{r}\right)^2 - 2\left(\frac{s}{r}\right)t\right)^\lambda}$$

$$= \frac{1}{r^{2\lambda}}\left((\text{const}) + \frac{r}{s}\log\left|\frac{r+s}{r-s}\right| + \sum_{l=1}^{\infty}\left(\frac{s}{r}\right)^{2l} O\left(\frac{1}{l^2}\right)\right) \quad \text{for } s < r.$$

Consequently

$$\mathfrak{K}_\lambda(r, s) = C_\lambda\left\{\frac{(\text{const})}{r^{2\lambda+1}} - \frac{1}{r^{2\lambda-1}}\,\frac{2}{r^2-s^2} + \sum_{l=1}^{\infty}\frac{s^{2l}}{r^{2l+2\lambda+1}}\, O\left(\frac{1}{l}\right)\right\}.$$

Thus, for $s < r$, our kernel is:

$$r^{2\lambda}\mathfrak{K}_\lambda(r, s) = C_\lambda\left\{\frac{(\text{const})}{r} - \frac{1}{r-s} - \frac{1}{r+s} + \sum_{l=1}^{\infty}\frac{s^{2l}}{r^{2l+1}}\, O\left(\frac{1}{l}\right)\right\}.$$

For $s > r$:

$$K_\lambda(r, s) = K_\lambda(s, r) = \frac{1}{s^{2\lambda}}\left((\text{const}) + \frac{s}{r}\log\left|\frac{r+s}{r-s}\right| + \sum_{l=1}^{\infty}\left(\frac{r}{s}\right)^{2l} O\left(\frac{1}{l^2}\right)\right).$$

Then

$$\mathfrak{K}_\lambda(r, s) = \frac{C_\lambda}{s^{2\lambda}}\left\{-\frac{2s^2}{r(r^2-s^2)} + \sum_{l=0}^{\infty}\frac{r^{2l+1}}{s^{2l+2}}O\left(\frac{1}{l+1}\right)\right\} \quad \text{for } s > r$$

and our kernel

$$r^{2\lambda}\mathfrak{K}_\lambda(r, s) = C_\lambda\left\{-\left(\frac{r}{s}\right)^{2\lambda-2}\left(\frac{1}{r-s}+\frac{1}{r+s}\right)+\sum_{l=0}^{\infty}\frac{r^{2l+2\lambda+1}}{s^{2l+2\lambda+2}}O\left(\frac{1}{l+1}\right)\right\}$$

for $s > r$.

Thus

$$\text{p.v.}\int_0^\infty (f(s)s^{2\lambda})(r^{2\lambda}\mathfrak{K}_\lambda(r, s))\,ds = C_\lambda\left\{-\text{p.v.}\int_0^\infty (f(s)s^{2\lambda})\frac{ds}{r-s}+\right.$$

$$+\int_0^\infty (f(s)s^{2\lambda})\frac{ds}{-r-s}+\int_r^\infty\left(1-\left(\frac{r}{s}\right)^{2\lambda-2}\right)\frac{1}{r-s}(f(s)s^{2\lambda})\,ds+$$

$$+\int_r^\infty\left(1-\left(\frac{r}{s}\right)^{2\lambda-2}\right)\frac{1}{r+s}(f(s)s^{2\lambda})\,ds+\sum_{l=0}^{\infty}A_{2l}(f(s)s^{2\lambda})(r)\,O\left(\frac{1}{l+1}\right)+$$

$$\left.+\sum_{l=0}^{\infty}B_{2l+2\lambda+2}(f(s)s^{2\lambda})(r)\,O\left(\frac{1}{l+1}\right)\right\}.$$

As we see, the situation is not very different from the one obtained for $\lambda = 1$. The only "new" operators are:

$$g(r)\mapsto\int_r^\infty\left(1-\left(\frac{r}{s}\right)^{2\lambda-2}\right)\frac{g(s)}{r-s}\,ds$$

and

$$g(r)\mapsto\int_r^\infty\left(1-\left(\frac{r}{s}\right)^{2\lambda-2}\right)\frac{g(s)}{r+s}\,ds.$$

We will prove that these operators map $H^1_+(\boldsymbol{R})$ into $L^1([0, \infty[)$ boundedly, from what an extension on (IV.3.3) to this situation follows immediately

$$\int_r^\infty\left(1-\left(\frac{r}{s}\right)^{2\lambda-2}\right)\frac{g(s)}{r-s}\,ds = \int_r^\infty\frac{s^a-r^a}{r-s}\,\frac{g(s)}{s^a}\,ds,$$

where $a = 2\lambda-2 \geqslant -1$.

For $a \geqslant 1$

$$\left|\frac{s^a-r^a}{r-s}\right|\frac{1}{s^a} \leqslant as^{-1},$$

that is, the kernel is dominated by a constant times the kernel of B_1. Therefore the operator is bounded in $L^1([0, \infty[)$. If $0 < a < 1$

$$\left|\frac{s^a - r^a}{r-s}\right| \frac{1}{s^a} \leqslant a \frac{r^{a-1}}{s^a},$$

i.e.: the kernel is dominated by a constant times the kernel of B_a. Thus, also in this case we have an operator bounded in $L^1([0, \infty[)$. If $-1 < a < 0$, let $-a = \beta$. Then

$$\left|\frac{s^a - r^a}{r-s}\right| \frac{1}{s^a} = \left|\frac{s^\beta - r^\beta}{r-s}\right| \frac{1}{r^\beta} \leqslant \frac{1}{(s-r)^{1-\beta} r^\beta}$$

which is also the kernel of an operator bounded in $L^1([0, \infty[)$.

It remains to consider the case $a = -1$ which arises for $\lambda = \frac{1}{2}$ which corresponds to the euclidean case $n = 2$. In that case $\dfrac{s^{-1} - r^{-1}}{r-s} \dfrac{1}{s^{-1}} = \dfrac{1}{r}$, so that the operator coincides with B_0 which maps $H^1_+(\boldsymbol{R})$ into $L^1([0, \infty[)$ boundedly,

$$\int_r^\infty \left(1 - \left(\frac{r}{s}\right)^{2\lambda-2}\right) \frac{1}{r+s} g(s)\, ds = \int_r^\infty g(s) \frac{ds}{r+s} - \int_r^\infty \left(\frac{r}{s}\right)^{2\lambda-2} g(s) \frac{ds}{r+s},$$

$1/(r+s) \leqslant 1/s$ so that the kernel of the first part is dominated by the kernel of B_1. Also

$$\left(\frac{r}{s}\right)^{2\lambda-2} \frac{1}{r+s} \leqslant \frac{1}{s} \quad \text{for } s > r$$

if $2\lambda - 2 \geqslant 0$ so that the only case left is $g(r) \mapsto \int_r^\infty \left(\frac{r}{s}\right)^a \frac{1}{r+s} g(s)\, ds$ for $-1 \leqslant a < 0$. If $-1 < a < 0$ and $-a = \beta$,

$$\left(\frac{r}{s}\right)^a \frac{1}{r+s} = \left(\frac{s}{r}\right)^\beta \frac{1}{r+s} \leqslant \frac{1}{s^{1-\beta}} \frac{1}{r^\beta}$$

which corresponds to the kernel of $B_{1-\beta}$.

It still remains the case $a = -1$, which gives rise to

$$g(r) \mapsto \int_r^\infty \frac{s}{r(r+s)} g(s)\, ds.$$

For $g \in H^1_+(\boldsymbol{R})$, integration by parts yields:

$$\int_r^\infty \frac{s}{r(r+s)} g(s)\, ds = \tfrac{1}{2} A_0(g)(r) - \int_r^\infty A_0(g)(s) \frac{s}{(r+s)^2} ds.$$

Since A_0 maps $H^1_+(\boldsymbol{R})$ boundedly into $L^1([0, \infty[)$ and $h(r) \mapsto \int_r^\infty h(s) \frac{s}{(r+s)^2} ds$ is a bounded operator in $L^1([0, \infty[)$, it follows that $g(r) \mapsto \int_r^\infty \frac{s}{r(r+s)} g(s) ds$ is also a bounded operator from $H^1_+(\boldsymbol{R})$ to $L^1([0, \infty[)$. We have, thus, proved the following result:

THEOREM IV.3.4. *Let $\lambda \geqslant \frac{1}{2}$. If $f(s)s^{2\lambda}$ is in $H^1_+(\boldsymbol{R})$, then the singular integral*

$$\mathfrak{K}_\lambda(f)(r) = \text{p.v.} \int_0^\infty f(s) s^{2\lambda} \mathfrak{K}_\lambda(r, s) ds$$

is in $L^1([0, \infty[; r^{2\lambda} dr)$, where

$$\mathfrak{K}_\lambda(r, s) = C_\lambda \frac{\partial}{\partial r} \left(\int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(r^2+s^2-2rs\cos\theta)^\lambda} d\theta \right).$$

In other words: $f(s)$ is in the weighted H^1 space considered by Muckenhoupt and Stein in [14].

By taking in particular $\lambda = (n-1)/2$ for $n = 2, 3, \ldots$ we get the generalization of (IV.3.3) to the euclidean space $\boldsymbol{R}^n$. This is one of the implications contained in (IV.2.5). That is:

THEOREM IV.3.5. *Let $n = 2, 3, \ldots$ If $f(s)s^{n-1}$ is in $H^1_+(\boldsymbol{R})$, then the radial function in $\boldsymbol{R}^n$ given by $F(x) = f(|x|)$ is in $H^1(\boldsymbol{R}^n)$ and $\|F\|_{H^1(\boldsymbol{R}^n)} \leqslant (\text{const}) \|f(s)s^{n-1}\|_{H^1(\boldsymbol{R})}$.*

Notice that in order to prove the converse of (IV.3.4) all we would need is to make sure that for a function $f(s)$ in the H^1 space of Muckenhoupt and Stein corresponding to a certain λ, $A_0(f(s)s^{2\lambda})(r)$ and $B_0(f(s)s^{2\lambda})(r)$ are in $L^1([0, \infty[)$. The reason is that, apart from the Hilbert transform of $f(s)s^{2\lambda}$, the rest of the operators occurring in the decomposition of $\text{p.v.} \int_0^\infty (f(s)s^{2\lambda})(r^{2\lambda} \mathfrak{K}_\lambda(r, s)) ds$ are either bounded in $L^1([0, \infty[)$ (with norms forming a converging sum) or depend upon $A_0(f(s)s^{2\lambda})$ or $B_0(f(s)s^{2\lambda})$. This is particularly the case for $\int_0^\infty f(s)s^{2\lambda} \frac{ds}{r+s}$ although we can also view it as a Hilbert transform. Indeed

$$\int_0^\infty g(s) \frac{1}{r+s} ds = \int_0^r g(s) \frac{1}{r+s} ds + \int_r^\infty g(s) \frac{1}{r+s} ds.$$

The second operator is bounded in $L^1([0, \infty[)$ since its kernel is dominated by that of B_1. As for the first one, integration by parts yields:

$$\int_0^r g(s) \frac{1}{r+s} ds = \tfrac{1}{2} A_0(g)(r) + \int_0^r A_0(g)(s) \frac{s}{(r+s)^2} ds$$

and $h \mapsto \int_0^r h(s) \frac{s}{(r+s)^2} ds$ is bounded in $L^1([0, \infty[)$ since its kernel is dominated by that of A_2.

The fact that $A_0(f(s)s^{2\lambda})$ is in $L^1([0, \infty[)$ will be proved for the euclidean cases $\lambda = (n-1)/2$, by looking at the system of second order Riesz transforms. These are the singular integral operators obtained by convolution with a principal value distribution of the form $P(x)/|x|^{n+2}$, where $P(x)$ is a homogeneous harmonic polynomial of degree 2. Since $\int_0^\infty f(s)s^{2\lambda} ds = 0$ we will not need to worry about $B_0(f(s)s^{2\lambda})$. We assume $n > 2$ for definiteness in the calculations but the conclusion holds also for $n = 2$, as will be seen in the next section. Let

$$(\mathscr{P}F)(x) = \text{p.v.} \int_{R^n} F(y) \frac{P(x-y)}{|x-y|^{n+2}} dy.$$

We will study the action of $\mathscr{P}$ on a radial function $F(x) = f(|x|)$. Since

$$\text{p.v.} \frac{P(y)}{|y|^{n+2}} = (\text{const}) P(D) \left(\frac{1}{|x|^{n-2}} \right)$$

in the sense of distributions and

$$(P(D)F)(y) = \frac{P(y)}{|y|^2} \left\{ \sum_{j=1}^{2} C_j \frac{f^{(j)}(|y|)}{|y|^{2-j}} \right\},$$

we have:

$$(\mathscr{P}F)(x) = (\text{const}) \int_0^\infty \left\{ \sum_{j=1}^{2} C_j \frac{f^{(j)}(s)}{s^{2-j}} \right\} \left(\int_{\Sigma_{n-1}} \frac{P(y')}{|x - sy'|^{n-2}} dy' \right) s^{n-1} ds.$$

Now if $\boldsymbol{I}$ is any point in the sphere Σ_{n-1} and $Z_{\boldsymbol{I}}^{(2)}(y')$ is the zonal spherical harmonic of degree 2 with pole $\boldsymbol{I}$:

$$\int_{\Sigma_{n-1}} \frac{P(y')}{|rx' - sy'|^{n-2}} dy' = P(x') \int_{\Sigma_{n-1}} \frac{Z_{\boldsymbol{I}}^{(2)}(y')}{|r\boldsymbol{I} - sy'|^{n-2}} dy'.$$

We will use the following formula (see [3], p. 47):

$$\frac{1}{|x - t\boldsymbol{I}|^{n-2}} = \frac{n-2}{|x|^{n-2}} \sum_{j=0}^{\infty} \frac{Z_{\boldsymbol{I}}^{(j)}\left(\frac{x}{|x|^2}\right) t^j}{n+2(j-1)}; \qquad -1 < t < 1.$$

Thus, for $s > r$:

$$\frac{1}{|r\boldsymbol{I} - sy'|^{n-2}} = \frac{n-2}{s^{n-2}} \sum_{j=0}^{\infty} \frac{Z_{\boldsymbol{I}}^{(j)}(y') \left(\frac{r}{s}\right)^j}{n+2(j-1)}$$

from which:

$$\int_{\Sigma_{n-1}} \frac{Z_I^{(2)}(y')}{|rI - sy'|^{n-2}}\, dy' = (\text{const})\frac{r^2}{s^n} \quad \text{for } s > r.$$

Since

$$\frac{1}{|rI - sy'|^{n-2}} = \frac{1}{|sI - ry'|^{n-2}}$$

we have, for $s < r$:

$$\int_{\Sigma_{n-1}} \frac{Z_I^{(2)}(y')}{|rI - sy'|^{n-2}}\, dy' = (\text{const})\frac{s^2}{r^n}.$$

Let

$$\mathfrak{B}(r, s) = \int_{\Sigma_{n-1}} \frac{Z_I^{(2)}(y')}{|rI - sy'|^{n-2}}\, dy'.$$

Then, the behavior of $(\mathscr{P}\tilde{\tilde{F}})(x)$ will depend on the operators in the line:

$$f(s) \mapsto \int_0^\infty \frac{f'(s)}{s}\mathfrak{B}(r, s)s^{n-1}ds \quad \text{and} \quad f(s) \mapsto \int_0^\infty f''(s)\mathfrak{B}(r, s)s^{n-1}ds.$$

Apart from the operators due to the singularity, $\int_0^\infty \frac{f'(s)}{s}\mathfrak{B}(r, s)s^{n-1}ds$ splits into $\frac{1}{r^n}\int_0^r f(s)s^{n-1}ds$ and $r^2\int_r^\infty \frac{f(s)s^{n-1}}{s^{n+2}}ds$ and the same is true for $\int_0^\infty f''(s)\mathfrak{B}(r, s)s^{n-1}ds$. The contribution of the singularity reduces to the identity operator times a constant. Thus:

$$(\mathscr{P}F)(rx') = (\text{const})P(x')\left\{\frac{1}{r^{n-1}}\frac{1}{r}\int_0^r f(s)s^{n-1}ds + \right.$$

$$\left. + \frac{1}{r^{n-1}}r^{n+1}\int_r^\infty \frac{f(s)s^{n-1}}{s^{n+2}}ds - (\text{const})f(r)\right\}$$

$$= (\text{const})P(x')\left\{\frac{1}{r^{n-1}}\left(A_0\left(f(s)s^{n-1}\right)(r) + B_{n+2}\left(f(s)s^{n-1}\right)(r)\right) - (\text{const})f(r)\right\}.$$

Then if $F(x) = f(|x|)$ is in $L^1(\boldsymbol{R}^n)$ (that is: $f(s)s^{n-1}$ is in $L^1([0, \infty[)$ and for every P homogeneous harmonic polynomial of degree 2 $(\mathscr{P}F)(x)$ is also in $L^1(\boldsymbol{R}^n)$, since B_{n+2} is bounded in $L^1([0, \infty[)$, we conclude that $A_0\left(f(s)s^{n-1}\right)(r)$ is in $L^1([0, \infty[)$. This is particularly the case for $F \in H^1_{\text{rad}}(\boldsymbol{R}^n)$. We have proved the converse of (IV.3.5) and obtained another proof of (IV.2.5) which does not use the atomic decomposition in $\boldsymbol{R}^n$.

The fact that the second order Riesz transforms of a radial function $F(x) = f(|x|)$ in $\boldsymbol{R}^n$ are essentially given by $\frac{1}{r^{n-1}} A_0\big(f(s)s^{n-1}\big)(r)$ is exploited in the next section to show that the second order Riesz transforms do not characterize $H^1(\boldsymbol{R}^n)$. We will do that for $n = 2$, that way completing our calculations of this section. Similar computations work for the Riesz transforms of any even order.

4. The kernel z^{-2}

For $F \in \mathrm{Re}\,L^1(\boldsymbol{R}^2)$, let

$$\tilde{\tilde{F}}(w) = \text{p.v.} \int_{\boldsymbol{R}^2} F(z-w)\frac{dz}{z^2}, \qquad w \in \boldsymbol{R}^2.$$

This is a complex-valued singular integral whose real and imaginary parts are two independent second order Riesz transforms in the plane. We will show that this singular integral operator does not characterize $H^1(\boldsymbol{R}^2)$. In particular we will find a radial function $F(z) = f(|z|)$ in $\mathrm{Re}\,L^1(\boldsymbol{R}^2)$ but not in $H^1(\boldsymbol{R}^2)$, such that $\tilde{\tilde{F}}$ is in $L^1(\boldsymbol{R}^2)$.

The kernel

$$\frac{e^{-i\theta}}{|z|^2} = \frac{x}{|z|^3} - i\frac{y}{|z|^3} \qquad (z = |z|e^{i\theta} = x+iy)$$

whose real and imaginary parts give two independent first order Riesz transforms, characterizes $H^1(\boldsymbol{R}^2)$. This characterization is actually the original definition of $H^1(\boldsymbol{R}^2)$. It comes as a surprise that the kernel $1/z^2 = e^{-i2\theta}/|z|^2$ does not characterize $H^1(\boldsymbol{R}^2)$, against what C. Fefferman conjectured (see [7]). The situation is similar for $H^1(K)$, where K is a local field (see [9]).

The key result is the following:

LEMMA IV.4.1. *For a radial function $F(z) = f(|z|)$ in $\mathrm{Re}\,L^1(\boldsymbol{R}^2)$:*

$$\tilde{\tilde{F}}(re^{i\varphi}) = \pi e^{-i2\varphi}\left\{\frac{2}{r^2}\int_0^r f(s)s\,ds - f(r)\right\}$$

$$= \pi e^{-i2\varphi}\left\{\frac{2}{r}A_0\big(f(s)s\big)(r) - f(r)\right\}.$$

Proof.

$$\tilde{\tilde{F}}(w) = -2\int_{\boldsymbol{R}^2} (\log|w-z|)\frac{\partial^2}{\partial z^2}F(z)\,dz$$

$$= -2\int_{\boldsymbol{R}^2} (\log|w-z|)\tfrac{1}{4}\left(f''(|z|)\frac{(\bar{z})^2}{|z|^2} - f'(|z|)\frac{(\bar{z})^2}{|z|^3}\right)dz.$$

Using polar coordinates $z = se^{i\theta}$ and $w = re^{i\varphi}$:

$$\tilde{\tilde{F}}(re^{i\varphi}) = -\tfrac{1}{2}e^{-i2\varphi}\int_0^\infty \big(sf''(s)-f'(s)\big)\Big(\int_0^{2\pi} e^{-i2\theta}\log|r-se^{i\theta}|\,d\theta\Big)\,ds.$$

Let

$$K(r,s) = \int_0^{2\pi} e^{-i2\theta}\log|r-se^{i\theta}|\,d\theta = K(s,r).$$

We can assume $s < r$. Then

$$K(r,s) = \int_0^{2\pi} e^{-i2\theta}\log\left|1-\frac{s}{r}e^{i\theta}\right|d\theta = -\frac{\pi}{2}\left(\frac{s}{r}\right)^2.$$

For $s > r$

$$K(r,s) = K(s,r) = -\frac{\pi}{2}\left(\frac{r}{s}\right)^2.$$

Thus

$$\tilde{\tilde{F}}(re^{i\varphi}) = \frac{\pi}{4}e^{-i2\varphi}\left(\int_0^r \big(sf''(s)-f'(s)\big)\left(\frac{s}{r}\right)^2 ds + \int_r^\infty \big(sf''(s)-f'(s)\big)\left(\frac{r}{s}\right)^2 ds\right)$$

$$= \pi e^{-i2\varphi}\left\{\frac{2}{r^2}\int_0^r f(s)s\,ds - f(r)\right\}.$$

The following is an immediate consequence of the previous lemma:

THEOREM IV.4.2. *For a radial function $F(z) = f(|z|)$ in* $\mathrm{Re}L^1(\boldsymbol{R}^2)$, *$\tilde{\tilde{F}}(z)$ is in $L^1(\boldsymbol{R}^2)$ if and only if $A_0\big(f(s)\cdot s\big)(r)$ is in $L^1([0,\infty[)$.*

Now we combine (IV.4.2) and (IV.2.5) to get a radial function $F(z) = f(|z|)$ in $\mathrm{Re}L^1(\boldsymbol{R}^2)$ but not in $H^1(\boldsymbol{R}^2)$, for which $\tilde{\tilde{F}}(z)$ is in $L^1(\boldsymbol{R}^2)$. All we need is to find $f(s)$ living in $[0,\infty[$, such that $f(s)s$ is in $L^1([0,\infty[)$ but not in $H^1(\boldsymbol{R})$ and yet $A_0\big(f(s)s\big)(r)$ is in $L^1([0,\infty[)$. It will be sufficient to take $f(s) = g(s)/s$, where g is as given in the following

THEOREM IV.4.3. *There exists a function $g(s)$ living in $[0,\infty[$, which belongs to $L^1([0,\infty[)$ but not to $H^1(\boldsymbol{R})$ and for which $A_0(g)(r)$ is in $L^1([0,\infty[)$.*

Proof. We saw that A_0 takes $H^1_+(\boldsymbol{R})$ into $L^1([0,\infty[)$ by taking a $(1,\infty)$-atom $a(s)$ supported in an interval $I \subset [0,\infty[$ and realizing that $A_0(a)$ lives in I also (because $\int_{-\infty}^{\infty} a(s)\,ds = 0$), and besides we have

the estimate

$$|A_0(a)(r)| \leqslant \frac{1}{r}\int_0^r |a(s)|\,ds \leqslant \frac{1}{|I|}.$$

This is a very rough estimate. We can find a function l also supported in I, with $\int_I l(x)\,dx = 0$ so that $A_0(l)$ lives also in I and for which $|A_0(l)(r)| \leqslant 1/|I|$ without being $|l(s)| \leqslant 1/|I|$. We will get our function g as a linear combination of functions like l. For $k = 1, 2, \ldots$, let $l_k(r) = k\chi_{[k,k+1/k]}(r) - (1/k)\chi_{[k+1/k,2k+1/k]}(r)$. Since $\int_{-\infty}^{\infty} l_k(s)\,ds = 0$, $A_0(l_k)(r) = (1/r)\int_0^r l_k(s)\,ds$ is going to be supported in the same interval that l_k; that is, in $[k, 2k+1/k]$. Clearly $A_0(l_k)(r)$ increases from k to $k+1/k$, where it attains the value $1/(k+1/k)$ and then decreases to become 0 for $r \geqslant 2k+1/k$. Thus $\int_{-\infty}^{\infty} |A_0(l_k)(r)|\,dr \leqslant 1$. Any $g(s) = \sum \lambda_k l_k$ with $\sum |\lambda_k| < \infty$ will be in $L^1([0, \infty[)$ and will have $A_0(g)$ in $L^1([0, \infty[)$ also. From

$$\tilde{l}_k(r) = k\log\left|\frac{r-k}{r-k-1/k}\right| - \frac{1}{k}\log\left|\frac{r-k-1/k}{r-2k-1/k}\right|$$

it follows that there is a constant > 0 such that for k bigger than some k_0

$$\int_{\frac{k}{2}}^{k} |\tilde{l}_k(r)|\,dr \geqslant (\text{const})\log k.$$

At this point we are able to conclude the existence of g without actually constructing it. We just need to consider the inclusion of $H^1_+(\boldsymbol{R})$ into the space of functions f in $L^1([0, \infty[)$ for which $A_0(f)$ is also in $L^1([0, \infty[)$, with the norm $\|f\|_1 + \|A_0(f)\|_1$. This is a Banach space and the inclusion is continuous. Were it onto, its inverse would be continuous too, and the norms $\|f\|_1 + \|A_0(f)\|_1$ and $\|f\|_1 + \|\tilde{f}\|_1$ would be equivalent by the open mapping theorem. This is in contradiction with the existence of the l_k's and, therefore, the existence of g is proved.

Here is an explicit way to construct g as a linear combination of l_k's. Take a sequence k_n of positive integers such that $\frac{1}{2}k_{n+1} > 2k_n + 1/k_n$ for every n and let k_0 be big enough to have $\int_{\frac{k}{2}}^{k} |\tilde{l}_k(r)|\,dr \geqslant (\text{const})\log k$ for every $k \geqslant k_0$. Also take a sequence (λ_n) of positive real numbers such that $\sum_{n=0}^{\infty} \lambda_n < \infty$ but $\sum_{n=0}^{\infty} \lambda_n \log k_n = \infty$. Then $g(s) = \sum_0^{\infty} \lambda_n l_{k_n}(s)$ will be in $L^1([0, \infty[)$

and $\|A_0(g)\|_{L^1([0,\infty[)} \leqslant \sum_0^\infty \lambda_n < \infty$. However,

$$\|\tilde{g}\|_{L^1(\boldsymbol{R})} = \int_{-\infty}^{\infty} \Big| \sum_{n=0}^{\infty} \lambda_n \tilde{l}_{k_n}(r) \Big| \, dr \geqslant \sum_{m=0}^{\infty} \int_{\frac{k_m}{2}}^{k_m} \Big| \sum_{n=0}^{\infty} \lambda_n \tilde{l}_{k_n}(r) \Big| \, dr$$

$$\geqslant \sum_{m=0}^{\infty} \lambda_m \int_{\frac{k_m}{2}}^{k_m} |\tilde{l}_{k_m}(r)| \, dr \geqslant (\text{const}) \sum_0^\infty \lambda_m \log k_m = \infty.$$

Thus $\tilde{g} \notin L^1(\boldsymbol{R})$, i.e., $g \notin H^1(\boldsymbol{R})$. One such choice of sequences will be, for instance: $k_n = 6^n n_0$ with n_0 large enough and $\lambda_n = 1/n^2$.

References

[1] R. R. Coifman, *A real variable characterization of H^p*, Studia Math. 51 (1974), p. 259–274.

[2] — and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), p. 241–250.

[3] — and G. Weiss, *Analyse harmonique non commutative sur certains espaces homogènes*, Lecture notes in Mathematics 242, Springer-Verlag, Berlin 1971.

[4] — —, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. (to appear).

[5] A. Erdelyi (director), *Higher transcendental functions*, Vol. I–III, Bateman Manuscript Project, McGraw-Hill, New York 1953.

[6] — (director), *Tables of integral transforms*, Vol. I–II, Bateman Manuscript Project, McGraw-Hill, New York 1954.

[7] C. Fefferman, *Harmonic analysis and H^p spaces*, in Studies in Harmonic Analysis (J. M. Ash, Editor), M.A.A.

[8] — and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), p. 137–193.

[9] A. Gandulfo, J. Garcia-Cuerva and M. Taibleson, *Conjugate system characterizations of H^1: counterexamples for the euclidean plane and local fields*, Bull. Amer. Math. Soc. Vol. 82, 1 (1976), p. 83–85.

[10] R. A. Hunt, B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for the conjugate function and the Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), p. 227–251.

[11] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure App. Math. 14 (1961), p. 415–426.

[12] R. Macias, *Interpolation theorems on generalized Hardy spaces*, Ph. D. dissertation, Washington University, St. Louis, Missouri 1974.

[13] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), p. 207–226.

[14] — and E. M. Stein, *Classical expansions and their relations to conjugate harmonic functions*, Trans. Amer. Math. Soc. 118 (1965), p. 17–92.

[15] — and R. L. Wheeden, *Weighted bounded mean oscillation and the Hilbert transform*, Studia Math. 54 (1976), p. 221–237.

[16] M. Rosenblum, *Summability of Fourier series in $L^p(d\mu)$*, Trans. Amer. Math. Soc. 105, 1 (1962), p. 32–42.

[17] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton 1970.

[18] — and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton 1971.

[19] K. Yosida, *Functional analysis*, Springer-Verlag, 1968.

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