POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCLXXI

RALPH HAEGER TOLIVER

Bounds for solutions of two additive equations of odd degree

WARSZAWA 1989
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-08508-8 ISSN 0012-3862

WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

1. Introduction

In this paper we examine the integral non-trivial solubility of a system of two additive equations of odd degree k over $\mathbf{Z}$, that is, two equations, each of the form

$$a_1 x_1^k + \ldots + a_N x_N^k = 0, \quad a_i \in \mathbf{Z}. \tag{1.1}$$

Over the past quarter-century an extensive literature has developed on the integral solubility of such equations. For a brief review of some of the major results and for a bibliography of the work done till 1969, see Lewis (1970). The essential tool for dealing with such problems is the Hardy—Littlewood circle method, as described in Davenport (1962) and in papers of Davenport and Lewis now collected in Volume III of Davenport (1977). A good account of the method is also given in Vaughan (1981), which includes an extensive bibliography covering more recent work.

Lewis began the study of additive equations of degree $k \geqslant 3$. In Lewis (1957), he showed that for cubic additive forms, $N \geqslant 7$ variables are sufficient for non-trivial p-adic solutions in any given p-adic field and for non-trivial solutions in algebraic integers to congruences modulo ideals in any given ring of algebraic integers. The early research on the solubility of a single equation of the form (1.1), for both even and odd k, consisted of finding conditions that implied a solution. It was always required that N be sufficiently large compared to k and for k even, conditions on the coefficients were usually required. The next step was to reduce the size of N. For example, Davenport (1962), Theorems 7 and 9, Davenport and Lewis (1963) and (1966) give lower bounds for N in terms of k.

Then the theory developed in two directions. One was to give conditions which imply the solubility of systems of additive equations and systems of general forms. See for example Davenport and Lewis (1969a) and Cook (1971a) and (1971b). For the case of two equations, which is our focus in this paper, the following papers are of special interest: Davenport and Lewis (1969b), Cook (1972b), and Vaughan (1977). Birch (1957) showed that for systems of general forms, each of odd degree, the only obstruction for a solution is the number of variables.

A second direction was to find a bounded region in which (1.1) has an integral solution. Pitman and Ridout (1967) and Pitman (1971) showed that non-trivial solutions to (1.1) exist that are bounded explicitly in terms of the coefficients of the equations. These two directions of research recently culminated in the work of Schmidt (1980) on Diophantine inequalities for general real forms of odd degree in a large number of variables.

In this paper, we merge the two directions mentioned and find a bounded region in which there exists a non-trivial integral solution to two additive equations of odd degree. In doing this, we develop a quantitative, two-dimensional form of the circle method. The paper is based on my doctoral thesis, Toliver (1975), where fuller details of some proofs are given.

Both the main result of this paper and the similar independent result of Lloyd (1975) on pairs of quadratic equations were used by Pitman (1981) in proving the solubility of the corresponding Diophantine inequalities for pairs of real additive forms. Ponnudurai (1979) extended these ideas to systems of R inequalities of odd degree and obtained a result on R equations which was sufficient for that purpose, though weaker than that given here for the case $R = 2$.

The main result of the paper is as follows:

THEOREM. *Consider the homogeneous system of equations*

$$(1.2)\qquad \begin{aligned} f(\mathbf{x}) &= \sum_{i=1}^{N} a_i x_i^k = 0, \\ g(\mathbf{x}) &= \sum_{i=1}^{N} b_i x_i^k = 0 \end{aligned}$$

with rational integer coefficients, where k is and odd number $\geqslant 3$, and $N = 2^{k+1}+1$. Let θ be any positive number and let

$$(1.3)\qquad A(k) = \begin{cases} \dfrac{2k}{N-4k} = \dfrac{6}{5}, & \text{if } k = 3, \\ \dfrac{k^2}{N-2k^2}, & \text{if } k \geqslant 5 \text{ and } k \text{ is not divisible by a power of } 3, \\ \dfrac{k(2k-1)}{N-2k(2k-1)}, & \text{if } k \geqslant 5 \text{ and } k \text{ is divisible by a power of } 3. \end{cases}$$

Then there is a non-trivial solution in integers to (1.2) which satisfies

$$(1.4)\qquad \left.\begin{aligned} &\sum_{i=1}^{N} |a_i x_i^k| \\ &\sum_{i=1}^{N} |b_i x_i^k| \end{aligned}\right\} \ll \Big(\prod_{i=1}^{N} \max(1, |a_i|, |b_i|)^{k/2+\theta}\Big)\cdot \max\big(1, \Delta(f, g)\big)^{kA(k)/(N-1)+\theta},$$

where $\Delta(f, g)$ is defined below.

The implied constant depends only on k and θ.

The $\Delta(f, g)$ of the theorem is a pseudo-discriminant with fewer distinct linear factors than normally used (see, e.g., Davenport and Lewis (1966), Lemma 1) and is defined as follwos. Consider the matrix of coefficients of the pair f, g:

$$\begin{bmatrix} a_1 & \dots & a_N \\ b_1 & \dots & b_N \end{bmatrix}.$$

The ith column represents the ratio a_i/b_i, and we assume without loss of generality that all columns representing the same ratio are grouped consecutively. We extend the numbering cyclically by taking the $(j+N)$-th column to be the jth. Suppose that for each i, we delete the ith column and partition the remaining $N-1$ columns into 2^k disjoint pairs by pairing the $(i+j)$-th column with the $(i+j+2^k)$-th for $j = 1, \dots, 2^k$. For each i, form the product of the absolute values of the determinants of these pairs of columns and call the product $D_i(f, g)$. Let

$$\Delta(f, g) = \prod_{i=1}^{N} D_i(f, g).$$

If no ratio occurs more than 2^k times, then the $(i+j)$-th and $(i+j+2^k)$-th columns cannot represent the same ratio, and it follows that $\Delta(f, g) > 0$. However, if some ratio occurs more than 2^k times then some $D_i(f, g)$ vanishes and hence $\Delta(f, g) = 0$. (Moreover, in this last situation, on deleting the ith column no other partitioning of the remaining columns will produce a product $D_i(f, g)$ which is non-zero.)

Note that in fact, with an obvious notation,

$$\Delta(f, g) = \Bigl(\prod_{i=1}^{N} \Delta(i, i+2^k)\Bigr)^{2^k}.$$

The power 2^k could be deleted from the definition, but this would appear to make little difference, at least to the conclusions of Chapter 4. However, this change in the definition of $\Delta(f, g)$ would considerably simplify practical computation of the pseudo-discriminant.

For a simpler form of the bound, we have

$$|a_i b_j - a_j b_i| \ll \max(1, |a_i|, |b_i|)\cdot \max(1, |a_j|, |b_j|)$$

so that

$$\Delta(f, g) \ll \prod_{i=1}^{N} (1, |a_i|, |b_i|)^{N-1}. \tag{1.5}$$

In (1.4), then, Δ can be deleted and $kA(k)+(N-1)\theta$ added to the exponent of the first product. If the ratios are close together, the left hand side of (1.5) is much smaller than the right, and the more complex form of the bound represents a great saving over the simpler form. If the ratios are bounded apart, both sides of (1.5) are of the same magnitude and there is no saving.

Throughout the paper, the implied constants depend on k and θ. They may also depend on ε, where ε is not the same throughout, and on other parameters, but all the parameters will be thought of as depending on k and θ. In the paper, "solution" means "a non-trivial solution in integers" and "bounded solution" means "a solution which satisfies (1.4)", unless another meaning is clearly intended.

Our method of argument for the theorem determines its form. The use of Hua's inequality in the analytic argument determines the number of variables used. Weyl's inequality determines the size of the first factor in the bound given in (1.4). The second factor in the bound of (1.4) is contributed by the p-adic argument and the exponent can perhaps be improved by more careful use of local information.

For large k, Davenport and Lewis (1969a) (solutions to two additive equations) and Pitman (1971) (bounded solutions to one additive equation) used Vinogradov's mean-value theorem to get a better lower bound for N in terms of k. This theorem could also be applied in our case (bounded solutions for two additive equations).

The argument for the theorem follows the Hardy–Littlewood circle method, and uses standard estimates and lines of argument, particularly those of Davenport and Lewis. I follow the guide of Pitman and Ridout (1967) in producing a bound which explicitly incorporates the coefficients.

The new contributions made by this paper are: (1) In Chapter 2, an initial reduction is made, without weakening the bounds, to the case of "simply normalized" forms in which for each i, the coefficients a_i and b_i of x_i are of the same sign and approximately the same size. This simplifies the p-adic argument and in the analytic argument, the equations often act as two copies of the same equation. (2) A "global reduction" is made in the p-adic argument. At the price of an explicitly determined increase in the bound of (1.4) (i.e., the addition of the second factor), this reduces the problem to the case where f and g are "p-reduced" for all relevant primes p. Because of the need to explicitly determine the size of the bound of (1.4) in terms of the coefficients, it is necessary to use a pseudo-discriminant which in non-zero simultaneously for all primes. This is in contrast to using a discriminant which must be modified for each p-adic field to ensure that it is non-zero in that p-adic field. This is the motivation for defining our pseudo-discriminant, which for simply normalized pairs f, g is non-zero in Q. (3) A careful analytic argument makes it possible to obtain, in the p-reduced case, a bound for the smallest solution of two equations which exactly corresponds to the bounds found by Pitman and Ridout (1967) for a single diagonal equation.

As in the case of a single equation, the case $k = 3$ generally involves much special detail, and for this detail I refer the reader to my thesis, Toliver (1975).

2. Initial reductions

To each index i, associate the ratio a_i/b_i and its height, $\max(|a_i|, |b_i|)$.

LEMMA 2.1. *If*

(1) *the height of some ratio is* 0, *or*

(2) 2^k+1 *or more of the ratios are equal,*

then the Theorem holds.

Proof. Case (1) is trivial. For case (2), suppose that all heights are positive but that the ratios for $1 \leqslant i \leqslant 2^k+1$ are all equal. On putting $x_i = 0$ for $i > 2^k+1$, we can assume that f is a form in 2^k+1 variables and that $g = cf$, $|c| \leqslant 1$. By Theorem 1 of Pitman and Ridout (1967) and Theorem 1 of Pitman (1971), f has a solution $(x_1, \ldots, x_{2^k+1})$ which satisfies

$$\sum_{i=1}^{2^k+1} |a_i x_i^k| \ll \prod_{i=1}^{2^k+1} |a_i|^{k/2+\theta},$$

where the implied constant depends only on k and θ. This solution and bound, together with the inequality $|c| \leqslant 1$, shows that $f = g = 0$ has a solution in N variables which satisfies (1.4). This shows the lemma.

For the remainder of this paper, we assume that all heights are positive and that no ratio appears more than 2^k times. It follows that $\Delta(f, g)$ is non-zero.

LEMMA 2.2. *It suffices for the proof of the Theorem to show that all pairs of additive forms f, g whose coefficients a_i, b_i satisfy*

$$1 \leqslant a_i/b_i \leqslant 5 \tag{2.1}$$

have a solution $\boldsymbol{x}$ to

$$f(\boldsymbol{x}) = g(\boldsymbol{x}) = 0$$

which satisfies

$$\left.\begin{array}{l} \sum_{i=1}^{N} |a_i x_i^k| \\ \sum_{i=1}^{N} |b_i x_i^k| \end{array}\right\} \ll \Big(\prod_{i=1}^{N} |b_i|^{k/2+\theta}\Big) \cdot \Delta(f, g)^{kA(k)/(N-1)+\theta}. \tag{2.2}$$

Proof. Suppose that all pairs of additive forms f, g whose coefficients satisfy (2.1) have a solution $\boldsymbol{x}$ to $f = g = 0$ which satisfies (2.2). Let F, G be a pair of additive forms with coefficients A_i, B_i. Choose an integer c such that $|c| \leqslant N$ and

$$\left|c + \left(\frac{B_i}{A_i}\right)\right| \geqslant \tfrac{1}{2} \quad \text{whenever } A_i \neq 0.$$

Now consider the pair f, g with coefficients a_i, b_i obtained by an integral unimodular transformation as follows:

$$f = (1+3c)F+3G, \qquad g = cF+G.$$

Then

$$\frac{a_i}{b_i} = \begin{cases} 3 & \text{if } A_i = 0, \\ 3+\dfrac{1}{c+(B_i/A_i)} & \text{if } A_i \neq 0, \end{cases}$$

from which it follows that (2.1) holds. By our hypothesis, there is a solution x of $f = g = 0$ such that (2.2) holds.

It follows from the definition of f and g that x also satisfies $F = G = 0$ and that $\Delta(f, g) = \Delta(F, G)$. And, by using $|c| \ll 1$ as well, we obtain

$$\max(|A_i|, |B_i|) \ll |b_i| \ll \max(|A_i|, |B_i|).$$

Hence (2.2) implies that x satisfies conditions corresponding to (1.4) in terms of A_i and B_i. This proves the lemma.

We can now assume that the pair f, g has coefficients a_i, b_i $(i = 1, \ldots, N)$ which satisfy

$$(2.3) \quad \begin{cases} (1) \ a_i, b_i \in \mathbf{Z}, \\ (2) \ \text{no ratio } a_i/b_i \text{ appears more than } 2^k \text{ times, and} \\ (3) \ 1/5 \leqslant a_i/b_i \leqslant 5. \end{cases}$$

Note that property (3) implies that a_i and b_i are non-zero and have the same sign. Pairs with the three properties of (2.3) will be called *simply normalized.*

The normalizations of this chapter hold for all k.

3. Local information

In view of Chapter 2, for the proof of the Theorem it is sufficient to show that each simply normalized pair f, g has a solution to $f = g = 0$ which satisfies (2.2). To use the Hardy–Littlewood method in the proof, we need a real non-singular solution. Its existence is guaranteed by the fact that k is odd and the pair f, g (by property (2) of (2.3)) is linearly independent over $\mathbf{R}$. We also need a fairly dense set of solutions in each local field $\mathbf{Q}_p$. This chapter gives conditions that ensure the needed density of solutions. In Chapter 4, we perform the reduction to a pair which has these properties.

For a pair f, g and a prime p, let

$$S(B, R) = \sum_{x=1}^{N} e\left(\frac{Bx^k}{R}\right), \quad \text{where } B \text{ and } R \text{ are integers,}$$

$$S_0(p^n) = \sum_{\substack{r=1 \\ (r,s,p)=1}}^{p^n} \sum_{s=1}^{p^n} \prod_{i=1}^{N} p^{-n} S(ra_i + sb_i, p^n),$$

and

$$\chi(p) = \chi(p, f, g) = 1 + \sum_{n=1}^{\infty} S_0(p^n).$$

We relate $\chi(p)$ to the density of local solutions by

LEMMA 3.1. *Let $M(p^n)$ be the number of solutions x for the congruences*

$$f(x) \equiv g(x) \equiv 0 \pmod{p^n},$$

with $0 \leqslant x_i < p^n$ $(i = 1, \ldots, N)$. Then

$$1 + \sum_{m=1}^{n} S_0(p^m) = \frac{M(p^n)}{p^{n(N-2)}}$$

and consequently

$$\chi(p) = \lim_{n \to \infty} \frac{M(p^n)}{p^{n(N-2)}}.$$

Proof. This is the two-dimensional analog of Davenport (1962), Lemma 8 of Section 5.

If $\chi(p) > 0$, then the limit of Lemma 3.1 is positive and we get many solutions to the congruence, for each n. Hence we get a p-adic solution to $f = g = 0$, and in fact get a positive density of p-adic solutions depending on the size of $\chi(p)$. Conversely, if $f = g = 0$ has a non-singular p-adic solution, then, arguing as in Davenport (1962), Lemma 10 of Section 5, the limit and hence $\chi(p)$ is positive. So $\chi(p)$ is a measure of the density of p-adic solutions.

To obtain a bound on integral solutions to $f(x) = g(x) = 0$ such as the bound given in (2.2), we need a minimum density of solutions in each local field. The condition we need is that for each $\varepsilon > 0$,

$$\prod_p \chi(p, f, g) \gg \Delta(f, g)^{-\varepsilon}. \tag{3.1}$$

Here the implied constant depends on k and ε.

We now give conditions on f, g under which (3.1) is satisfied. To do this, we first need estimates on the trigonometric sums.

LEMMA 3.2. *Let R be any integer. Suppose $(B, R) = 1$. Then*

(1) $S(B, R) \ll R^{1-1/k}$,

(2) $S(B, R) \ll R^{1/2}$, *if R is a prime,*

and

(3) $p^{-(n+m)} S(Bp^m, p^{n+m}) = p^{-n} S(B, p^n)$, *for any $m, n \geqslant 0$.*

Here the implied constants depend only on k.

Proof. Parts (1) and (2) are Lemmas 15 and 12, respectively, of Davenport (1962), Section 6. The proof of (3) is trivial.

For all but a finite number of primes, $\chi(p)$ is close to 1. In fact,

LEMMA 3.3. *There exists a constant c_1 that depends only on k such that*

$$\prod_{\substack{p \nmid \Delta(f,g) \\ p > c_1}} \chi(p) > \tfrac{1}{2}.$$

Proof. This follows the argument of Pitman and Ridout (1967), Lemma 11. Suppose $p \nmid \Delta(f, g)$. For a pair r, s, put

$$\frac{ra_i+sb_i}{p^n} = \frac{B_i}{R_i}, \quad \text{where } (B_i, R_i) = 1.$$

By Lemma 3.2,

$$S_0(p^n) \ll \sum_{r,s} \prod_{i=1}^{N} R_i^{-1/k}, \tag{3.2}$$

and

$$S_0(p) \ll \sum_{r,s} \prod_{i=1}^{N} R_i^{-1/2}. \tag{3.3}$$

We now show

$$S_0(p^n) \ll \begin{cases} p^{n(2-(2^k/k))}, & n > 1, \\ p^{2-(2^k/2)}, & n = 1. \end{cases} \tag{3.4}$$

Suppose that $n > 1$. Since $p \nmid \Delta(f, g)$, $p \nmid D_1(f, g)$, a product of 2^k non-zero determinants of the form $|a_ib_j - a_jb_i|$, where the pairs (i, j) partition $2, 3, \ldots, N$. By (3.2) and Hölder's inequality, to show (3.4), it is sufficient to show

$$\sum_{\substack{r=1 \\ (r,s,p)=1}}^{p^n} \sum_{s=1}^{p^n} (R_iR_j)^{-2^k/k} \ll p^{n(2-(2^k/k))} \tag{3.5}$$

for each pair (i, j) such that $|a_ib_j - a_jb_i| \neq 0$ is a factor of $D_1(f, g)$. Estimate (3.5), and hence (3.4) for $n > 1$, is shown by the method used in Davenport and Lewis (1969a), Lemma 23, with $Q = p^n$. In this case, the d of the lemma actually equals 1. By a similar argument, using the estimate of (3.3) instead of (3.2), we obtain the estimate in (3.4) for $n = 1$.

Since $k \geqslant 3$,

$$S_0(p^n) \ll \begin{cases} p^{-2n/3}, & n > 1, \\ p^{-2}, & n = 1, \end{cases}$$

and hence

$$\chi(p)-1 = \sum_{n=1}^{\infty} S_0(p^n) \ll p^{-4/3}.$$

The lemma follows easily from this.

We now find conditions which give good estimates for $\chi(p)$ when either $p|\Delta(f, g)$ or when p is bounded by a function of k and ε. To do this, we separate the coefficients of f and g into different sets, depending on their divisibility by powers of p. In particular, we can put the pair f, g in the form

$$f = f_0+pf_1+p^2 f_2+ \dots,$$
$$g = g_0+pg_1+p^2 g_2+ \dots$$

where f_j, g_j are forms with integer coefficients in $m_j = m_j(p, f, g)$ variables and these sets of variables are disjoint. Moreover, each of the m_j variables occurs in at least one of the f_j, g_j with a coefficient not divisible by p. Also, let $q_j = q_j(p, f, g)$ denote the minimum number of variables appearing in any form $\lambda f_j+\mu g_j$, $((\lambda, \mu) \not\equiv (0, 0) \pmod p)$ with a coefficient not divisible by p. We relabel the variables so that $x_1, \dots, x_{m_0}$ appear in f_0, g_0, $x_{m_0+1}, \dots, x_{m_0+m_1}$ appear in f_1, g_1, and so on.

To any simply normalized pair f, g there is an associated pair which we shall the *canonical form* of the pair f, g. Consider any form F with integral coefficients. There corresponds to F a form F^* with coefficients in the finite field of p elements, these coefficients being congruent (mod p) to the corresponding coefficients of F. Variables explicitly present in F may not be explicitly present in F^*. The *rank* of a form will be the number of variables explicitly appearing in it. As in the argument of Davenport and Lewis (1966), Lemma 11, we can use a unimodular transformation to transform a pair f, g into a pair F, G where G_0^* has minimal rank among all forms which are non-trivial combinations of F_0^* and G_0^*, and where either F_1^* or G_1^* has minimal rank among all forms which are non-trivial combinations of F_1^* and G_1^*. The transformation to the new pair does not disturb the values of the m_j or of the q_j. We say that a pair f, g is in *canonical form* if there are exactly q_0 coefficients in g_0 not divisible by p and there exactly q_1 coefficients in either f_1 or g_1 which are not divisible by p.

For any k and prime p, let $k = p^t k_0$, where $(k_0, p) = 1$ and $t \geqslant 0$. Define

$$\gamma = \begin{cases} 1, & \text{if } t = 0, \\ t+1, & \text{if } t > 0 \text{ and } p \neq 2, \\ t+2, & \text{if } t > 0 \text{ and } p = 2. \end{cases}$$

Note that in all cases, $1 \leqslant \gamma \leqslant k+1$.

LEMMA 3.4. *Let $N(p^\gamma)$ be the number of solutions x for the congruence*

$$f \equiv g \equiv 0 \pmod{p^\gamma}$$

with $0 \leqslant x_i < p^\gamma$ $(i = 1, \ldots, N)$ for which the matrix

$$\begin{bmatrix} a_1 x_1 & \ldots & a_N x_N \\ b_1 x_1 & \ldots & b_N x_N \end{bmatrix}$$

has rank 2 (mod p). *Such solutions are said to be non-singular* (mod p^γ). *Then for any $n \geqslant \gamma$,*

$$M(p^n) \geqslant p^{(n-\gamma)(N-2)} N(p^\gamma).$$

Proof. The lemma is the two-dimensional analog of Davenport (1962), Lemma 10 of Section 5.

We now get an estimate on $\chi(p)$ for large p which will be used when $p | \Delta(f, g)$.

LEMMA 3.5. *Given $\varepsilon > 0$, there exists a constant c_2 that depends only on k and ε such that if $p > c_2$ and $q_0 \geqslant 3$ and $m_0 \geqslant 5$, then*

$$\chi(p) \geqslant p^{-\varepsilon}.$$

Proof. Let $c_2 > k$ so $p \nmid k$ and $\gamma = 1$ always. Then by Lemma 3.4, $M(p^n) \geqslant (p^{(n-1)(N-2)}) N(p)$. Let $N_0(p)$ be the number of solutions (mod p) to

$$f_0 \equiv g_0 \equiv 0 \pmod{p} \tag{3.6}$$

which are non-singular (mod p). Because of the free choice of values of x_i (mod p) for $i > m_0$, there is a 1 to p^{N-m_0} correspondence between the non-singular solutions (mod p) of (3.6) and a subset of those of

$$f \equiv g \equiv 0 \pmod{p}.$$

Thus

$$N(p) \geqslant p^{N-m_0} N_0(p).$$

The lemma follows if we show that $N_0(p) \geqslant p^{m_0-2-\varepsilon}$.

To find a lower bound for $N_0(p)$, we find a lower bound for the total number of solutions to (3.6) and an upper bound for Sing(p), the number of those solutions which are singular (mod p). A singular solution (mod p) is one for which the matrix

$$\begin{bmatrix} a_1 x_1 & \ldots & a_{m_0} x_{m_0} \\ b_1 x_1 & \ldots & b_{m_0} x_{m_0} \end{bmatrix}$$

has rank less than 2 (mod p). This is equivalent to

$$(a_i b_j - a_j b_i) x_i x_j \equiv 0 \pmod{p}$$

for all pairs (i, j) for which $i, j = 1, 2, \ldots, m_0$.

Let x be a singular solution of (3.6) such that $x_i \not\equiv 0 \pmod p$, for a fixed value of i. If $x_j \not\equiv 0 \pmod p$, then

$$a_i b_j - a_j b_i \equiv 0 \pmod p,$$

and so the variable x_j does not appear explicitly (mod p) in $a_i g_0 - b_i f_0$. It follows from the definition of f_0, g_0 and q_0 that there are at most $m_0 - q_0$ values of j for which we can have $x_j \not\equiv 0 \pmod p$. Thus the number of singular solutions x such that $x_i \not\equiv 0 \pmod p$ is less than

$$p^{m_0 - q_0}.$$

Applying this with $i = 1, \ldots, m_0$, and noting that the trivial solution is singular, we see that

$$\mathrm{Sing}(p) < 1 + m_0 p^{m_0 - q_0} \ll p^{m_0 - 3},$$

since $q_0 \geqslant 3$ and $m_0 \leqslant N$.

Now the total number of solutions of (3.6) is

$$\begin{aligned} N_0(p) + \mathrm{Sing}(p) &= p^{m_0 - 2} + \sum_{\substack{r,s \\ (r,s) \neq (0,0)}} p^{-2} \prod_{i=1}^{m_0} S(ra_i + sb_i, p) \\ &= p^{m_0 - 2} + \sum, \end{aligned}$$

say. We need an upper bound on $\sum$. To obtain this, we note that, by Lemma 3.2,

$$\begin{aligned} S(ra_j + sb_j, p) &= p \quad \text{if } ra_j + sb_j \equiv 0 \pmod p \\ &\ll p^{1/2} \text{ otherwise.} \end{aligned}$$

The contribution to $\sum$ from all pairs r, s such that

$$ra_j + sb_j \not\equiv 0 \pmod p \quad \text{for } j = 1, 2, \ldots, m_0$$

is thus

$$\ll p^{-2} \cdot p^2 \cdot p^{(1/2)m_0} \ll p^{(1/2)m_0}.$$

For fixed i, there are exactly $p-1$ pairs $(r, s) \neq (0,0)$ such that

$$ra_i + sb_i \equiv 0 \pmod p.$$

For each such pair, by the definition of q_0, there are at most $m_0 - q_0$ values of j (including $j = i$) such that

$$ra_j + sb_j \equiv 0 \pmod p.$$

Thus the contribution to $\sum$ from such pairs is

$$\ll p^{-2} \cdot p \cdot p^{(1/2)q_0} \cdot p^{m_0 - q_0} \ll p^{m_0 - (5/2)},$$

since $q_0 \geqslant 3$ and $m_0 \geqslant 5$.

Using our estimate for Sing(p), we see that

$$|N_0(p)-p^{m_0-2}| \ll p^{m_0-2}(p^{-1}+p^{-1/2}),$$

where the implied constant depends on k. Hence, given ε, there is a c_2 which depends on k and ε such that

$$N_0(p) \geqslant p^{m_0-2-\varepsilon}$$

when $p \geqslant c_2$. The condition on $\chi(p)$ follows, and the lemma is shown.

We finally get estimates which can be used for the remaining primes, that is, the set of primes p such that $p \nmid \Delta(f, g)$ and $p \leqslant c_1$ or such that $p | \Delta(f, g)$ and $p \leqslant c_2$.

We will need the following six conditions, where c_1 is the constant of Lemma 3.3, c_2 is the constant of Lemma 3.5, and $c_3 = \max(c_1, c_2)$:

(3.7) (1) if $k \geqslant 3$, $p > c_2$, and $p|\Delta(f, g)$, we have $q_0 \geqslant 3$ and $m_0 \geqslant 5$,

(2) if $k \geqslant 5$, $p \leqslant c_3$, and the ordered pairs $(k, p) \neq (3^t k_0, 3)$ (where $t \geqslant 1$ and $(k_0, 3) = 1$, we have $q_0 \geqslant k+1$, $m_0 \geqslant 2k+1$, $m_0+q_1 \geqslant 3k+1$, and $m_0 = m_1 \geqslant 4k+1$,

(3) if $k \geqslant 5$, $p \leqslant c_3$, and the ordered pair $(k, p) = (3^t k_0, 3)$(where $t \geqslant 1$ and $(k_0, 3) = 1$), we have $q_0 \geqslant 2k$ and $m_0 \geqslant 3k$,

(4) if $k = 3$ and $3 < p \leqslant c_3$, we have $q_0 \geqslant 3$ and $m_0 \geqslant 6$,

(5) if $k = p = 3$, we have $m_0 + \dots m_{j-1} \geqslant j\frac{16}{3}$ for $j = 1, 2, 3$ and $m_0 + \dots + m_{j-1} + q_j \geqslant (j+\frac{1}{2})\frac{16}{2}$ for $j = 0, 1, 2$,

(6) if $k = p = 3$, the pair f, g, in its canonical form, does not satisfy: $q_0 = 3$, $q_1 = 0$, $0 \leqslant q_2 \leqslant 1$, the $m_0 - q_0$ ratios in f_0, g_0 which are of the same ratio (mod 3) are also of the same ratio (mod 9), and all the coefficients of g_1 are divisible by p.

If a condition of (3.7) contains only $m_0, \dots, m_{j-1}$, it is an *(m, j)-condition* and if it contains $m_0, \dots, m_{j-1}, q_j$ it is a (q, j)-condition. Condition (6) is the C-condition. A pair f, g which satisfies all appropriate conditions of (3.7) for a given prime p is said to be *p-reduced.* We do not require a p-reduced pair to be simply normalized.

LEMMA 3.6. *If p and k satisfy conditions* (2), (3), *or* (4) *of* (3.7) *or if they satisfy both conditions* (5) *and* (6) *of* (3.7), *then*

$$\chi(p) \geqslant p^{-(k+1)(N-2)} \gg 1.$$

Proof. In each of the cases given, we have $N(p^\gamma) \geqslant 1$ for all primes. For condition (2), see Davenport and Lewis (1969b), particularly Sections 6 and 7, and the beginning of Section 8 (for $t = 0$). For condition (3), see Davenport and Lewis (1969b), Sections 1 to 6 and conditions (18) and (19). For condition

(4), see Davenport and Lewis (1966), Lemma 10. For conditions (5) and (6), see Davenport and Lewis (1966), Lemmas 16 and 17.

Since in each case $N(p^\gamma) \geqslant 1$, we have by Lemma 3.5 that

$$M(p^n) \geqslant p^{(n-\gamma)(N-2)} N(p^\gamma) \geqslant p^{(n-\gamma)(N-2)}.$$

Then

$$\chi(p) = \lim_{n\to\infty} \frac{M(p^n)}{p^{n(N-2)}} \geqslant p^{-\gamma(N-2)} \geqslant p^{-(k+1)(N-2)},$$

whence for $p \leqslant c_3(k, \varepsilon)$, we have $\chi(p) \gg 1$. This shows the lemma.

The form of the conditions (3.7) and the resulting Lemma 3.6 are dependent on k odd. If k is even, we can still show that $N(p^\gamma) \geqslant 1$, but we must increase the number of variables to $7k^3$ for even $k \leqslant 12$. See Davenport and Lewis (1969b), Theorem 3.

LEMMA 3.7. *If a simply normalized pair of additive forms f, g satisfies all conditions of* (3.7) *for all $p|\Delta(f, g)$ and all $p \leqslant c_3$, then for each $\varepsilon > 0$,*

$$\prod_p \chi(p, f, g) \gg \Delta(f, g)^{-\varepsilon}. \tag{3.1}$$

Here the implied constant depends on k and ε.

Proof. Suppose that the pair f, g satisfies the hypothesis of the lemma. Then by Lemmas 3.5 and 3.6

$$\chi(p) = \chi(p, f, g) \begin{cases} \geqslant p^{-\varepsilon}, & \text{if } p > c_2 \text{ and } p|\Delta(f, g), \\ \geqslant p^{-(k+1)(N-2)} \gg 1, & \text{if } p \leqslant c_3. \end{cases} \tag{3.8}$$

Hence, by (3.8) and the estimate of Lemma 3.3,

$$\prod_p \chi(p) = \prod_{\substack{p\nmid\Delta(f,g)\\ p>c_1}} \chi(p) \cdot \prod_{\substack{p|\Delta(f,g)\\ p>c_2}} \chi(p) \cdot \Big(\prod_{\substack{p\nmid\Delta(f,g)\\ p\leqslant c_1}} \chi(p) \cdot \prod_{\substack{p|\Delta(f,g)\\ p\leqslant c_2}} \chi(p)\Big)$$
$$\gg \Delta(f, g)^{-\varepsilon}.$$

This shows the lemma.

So (3.7) gives conditions on a pair f, g which guarantee that it has the needed density of local solutions in each local field.

4. Global reduction

In Chapter 3, we showed that if a simply normalized pair f, g is p-reduced with respect to all relevant primes, that is, if f, g satisfies the conditions of (3.7) for all relevant primes, then it satisfies

$$\prod_p \chi(p, f, g) \gg \Delta(f, g)^{-\varepsilon}. \tag{3.1}$$

The line of argument of this chapter is the following: If a pair f, g does not satisfy (3.7) for a given prime, then we reduce to a pair which does. We iterate this process to find a pair f', g' which satisfies (3.7) for all relevant primes. By keeping tabs on the size of the pseudo-discriminant Δ, we show that if the pair f', g' has a solution which satisfies (2.2) then so does the pair f, g. Thus, for the proof of the theorem, it will be sufficient to show that every simply normalized pair f, g which satisfies (3.1) has a solution in integers which satisfies (2.2).

To gain the needed control on the size of Δ, we need to know how it changes under transformations. Let R be a non-singular diagonal matrix with diagonal entries $r_1, \ldots, r_N$ from the field of rational numbers. Let T be a non-singular 2×2 matrix with rational entries. Then we can form the pairs

$$f(R\mathbf{x}),\ g(R\mathbf{x}) \quad \text{and} \quad \alpha f(\mathbf{x})+\beta g(\mathbf{x}),\ \gamma f(\mathbf{x})+\delta g(\mathbf{x}),$$

where $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. For each of these pairs, the same choice of the pairings used to define the $D_i(f, g)$ can also be used to define $D_i(f(R\mathbf{x}), g(R\mathbf{x}))$ and $D_i\left(T\begin{bmatrix} f \\ g \end{bmatrix}\right)$. We then have

LEMMA 4.1. (1) $\Delta(f(R\mathbf{x}), g(R\mathbf{x})) = (\det R)^{k(N-1)}\Delta(f, g)$,

$$(2)\ \Delta\left(T\begin{bmatrix} f \\ g \end{bmatrix}\right) = \left((\det T)^{N(N-1)/2}\right)\Delta(f, g).$$

Proof. This is the analog of Davenport and Lewis (1966), Lemma 1.

We shall often use a diagonal matrix where the entries are integral powers of a prime p, say $p^{\nu_1}, \ldots, p^{\nu_N}$. We denote this matrix by $P^{(\nu)}$. We let $\nu = \sum_{i=1}^{N} \nu_i$. Then by part (1) of the lemma,

$$\Delta(f(P^{(\nu)}\mathbf{x}), g(P^{(\nu)}\mathbf{x})) = p^{\nu k(N-1)}\Delta(f, g).$$

Consider the set of all pairs with integer coefficients which can be obtained from f, g by application of operations of the type T and R described above. Any two such pairs are said to be *equivalent*. Note that a pair f', g' which is equivalent to a simply normalized pair f, g need not itself be simply normalized, for the pair f', g' will satisfy the first two properties of (2.3) but need not satisfy the third. However, by Lemma 4.1, we always have $\Delta(f', g') > 0$.

Suppose that a T-matrix or an R-matrix takes the pair f, g to an equivalent pair f', g'. If the matrix is a T-matrix, suppose that it has integer entries and $\det T$ is a p-adic unit, or if the matrix is an R-matrix, suppose its diagonal entries are p-adic units. In either case, for each n the solutions (mod p^n) to $f \equiv g \equiv 0$ and $f' \equiv g' \equiv 0$ are in one-to-one correspondence.

Then by Lemma 3.1, $\chi(p, f, g) = \chi(p, f', g')$. Finally, if $\det T = 1$ or $\det R = 1$, then $\Delta(f, g) = \Delta(f', g')$.

Now select and fix a prime p. Consider the set of pairs equivalent to a given pair f, g. From this set, pick a pair f', g' such that the power of p in $\Delta(f', g')$ is minimal. Such a pair is said to be *p-normalized.*

LEMMA 4.2. *If the pair f, g is p-normalized, then*
(1)

$$(4.1)\qquad m_0 + \ldots + m_{j-1} \geqslant j\frac{N}{k} \quad \textit{for} \quad j = 1, 2, \ldots, k$$

and

$$(4.2)\qquad m_0 + \ldots + m_{j-1} + q_j \geqslant (2j+1)\frac{N}{2k} \quad \textit{for} \quad j = 0, 1, \ldots, k-1, \textit{ and}$$

(2) *the pair f, g is p-reduced.*

Proof. Let the m_j and the q_j be determined for the pair f, g and assume that the variables are labelled as in Chapter 3. For a fixed j in the range $1 \leqslant j \leqslant k$, transform x_i to px_i for $1 \leqslant i \leqslant m$, where $m = m_0 + \ldots + m_{j-1}$. On multiplying both forms by p^{-j}, we still have integer coefficients. The resulting form f', g' is of the form

$$(4.3)\qquad \begin{bmatrix} f' \\ g' \end{bmatrix} = \begin{bmatrix} p^{-j} & 0 \\ 0 & p^{-j} \end{bmatrix} \begin{bmatrix} f(P^{(\nu)}\chi \\ g(P^{(\nu)}\chi \end{bmatrix}$$

where $\nu = (\nu_1, \ldots, \nu_N)$ with $\nu_i = 1$ for $i = 1, \ldots, m$ and $\nu_i = 0$ otherwise. We shall call this type of operation an *(m, j)-operation.* Now by Lemma 4.1,

$$(4.4)\qquad \Delta(f', g') = (p^{(N-1)(km-jN)})\Delta(f, g).$$

Since the pair f, g is p-normalized, the power of p in $\Delta(f, g)$ must be at least as small as that in $\Delta(f', g')$ so $km - jN \geqslant 0$. Thus we have (4.1).

Any transformation ψ of the type $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, c any integer, produces pairs f'', g'' equivalent to f, g. We can put such pairs into the form

$$(4.5)\qquad \begin{aligned} &f_0'' + pf_1'' + p^2 f_2'' + \ldots, \\ &g_0'' + pg_1'' + p^2 g_2'' + \ldots \end{aligned}$$

The same variables appear in f_j'', g_j'' as in f_j, g_j, for each j. Given j $(0 \leqslant j \leqslant k-1)$, there is such a transformation for which one of f_j'', g_j'', say g_j'', has exactly q_j variables not divisible by p. We can assume that these are the first q_j variables in f_j'', g_j''. Transform x_i to px_i for $1 \leqslant i \leqslant m+q$, where $q = q_j$. After this substitution, multiply the first form in (4.5) by p^{-j} and the second form by p^{-j-1}. The new pair f', g' has integer coefficients and is of the form

(4.6) $$\begin{bmatrix} f' \\ g' \end{bmatrix} = \begin{bmatrix} p^{-j} & 0 \\ 0 & p^{-j-1} \end{bmatrix} \begin{bmatrix} f''(P^{(\nu)}x) \\ g''(P^{(\nu)}x) \end{bmatrix} = \begin{bmatrix} p^{-j} & 0 \\ 0 & p^{-j-1} \end{bmatrix} \psi \begin{bmatrix} f(P^{(\nu)}x) \\ g(P^{(\nu)}x) \end{bmatrix}$$

where ψ unimodular, $\nu_i = 1$ for $1 \leqslant i \leqslant m+q$ and $\nu_i = 0$ otherwise. We shall call this type of operation a *(q, j)-operation*. Again by Lemma 4.1,

(4.7) $$\Delta(f', g') = (p^{\frac{N-1}{2}(2k(m+q)-(2j+1)N)})\Delta(f, g),$$

from which (4.2) follows.

For (2), we must show that the conditions of (3.7) are satisfied. But if the pair f, g is p-normalized, the inequalities (4.1) and (4.2) are satisfied. Since $N = 2^{k+1}+1 > 16$ and since also $k \geqslant 9$ if $k = 3^t k_0 \geqslant 5$ with k odd and $t \geqslant 1$, these imply that conditions (1) to (5) of (3.7) are satisfied.

It remains to show that condition (6) is satisfied. So suppose that $k = 3$ and that the pair f, g is 3-normalized. There is a unimodular matrix U such that

$$\begin{bmatrix} f' \\ g' \end{bmatrix} = U \begin{bmatrix} f \\ g \end{bmatrix}$$

is in canonical form, that is, exactly q_0 of the coefficients in g'_0 are not divisible by $p = 3$ and exactly q_1 of the coefficients in one of g'_1 or f'_1 are not divisible by 3. Since the coefficients of f', g' are integers and since det U is a 3-adic unit, $\Delta(f', g') = \Delta(f, g)$, and so f', g' is also 3-normalized. Now suppose that f', g' does not satisfy condition (6). Then $q_0 = 3$, all the coefficients of g'_1 are divisible by 3, $0 \leqslant q_2 \leqslant 1$, and the $m_0 - q_0 = m_0 - 3$ ratios in f'_0, g'_0 which are of the same ratio (mod 3) are also of the same ratio (mod 9). Then f'_0, g'_0 have the shape

$$f'_0 = a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + c_4 x_4^3 + \dots + c_{m_0} x_{m_0}^3,$$
$$g'_0 = b_1 x_1^3 + b_2 x_2^3 + b_3 x_3^3 + 3(d_4 x_4^3 + \dots + d_{m_0} x_{m_0}^3),$$

where all the b_i and c_i are $\not\equiv 0 \pmod 3$.

Following the argument of Davenport and Lewis (1966), Lemma 17, let e be an integer congruent to the common ratio $\frac{d_i}{c_i} \pmod 3$, transform x_i to $3x_i$ for $i = 1, 2, 3$ and, using the new variables, let

$$f'' = f,$$
$$g'' = g' - 3ef'.$$

Then f'', $3^{-2}g''$ is a pair with integer coefficients, so by Lemma 4.1,

(4.8) $$\Delta(f'', 3^{-2}g'') = (3^{\frac{N-1}{2}(2k\cdot 3 - 2\cdot N)})\Delta(f', g') = 3^{(N-1)(-8)}\Delta(f', g').$$

But f', g' is 3-normalized, so no equivalent pair can have a smaller power of 3 in Δ, contrary to the last equation. Thus it must be that f', g' does satisfy condition (6).

This completes the proof of the Lemma.

Now in Lemma 4.2,

$$\begin{bmatrix} f'' \\ 3^{-2}g'' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{-2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3e & 1 \end{bmatrix} U \begin{bmatrix} f(P^{(\nu)}x) \\ g(P^{(\nu)}x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{-2} \end{bmatrix} \psi \begin{bmatrix} f(P^{(\nu)}x) \\ g(P^{(\nu)}x) \end{bmatrix}, \tag{4.9}$$

where ψ is a unimodular matrix, $\nu = (\nu_1, \ldots, \nu_N)$, $\nu_i = 1$ if $i = 1, 2, 3$, and $\nu_i = 0$ otherwise. An operation of this type will be called a *C-operation.*

By (4.3), (4.6), and (4.9), an application of an (m, j)-, a (q, j)-, or a C-operation to the pair f, g results in a pair f', g' which satisfies

$$\begin{bmatrix} f' \\ g' \end{bmatrix} = p^{-j}\varphi \begin{bmatrix} f(P^{(\nu)}x) \\ g(P^{(\nu)}x) \end{bmatrix}. \tag{4.10}$$

Here $\varphi = \begin{bmatrix} 1 & 0 \\ 0 & p^{-e} \end{bmatrix} U$, where U is unimodular. The following properties hold:

(1) For an (m, j)-operation, $j = 1, 2, \ldots$, or k,

$$\nu = \sum_{i=1}^{N} \nu_i = m_0 + \ldots + m_{j-1}, \quad \text{and} \quad e = 0.$$

(2) For a (q, j)-operation, $j = 0, 1, \ldots$, or $k-1$,

$$\nu = m_0 + \ldots + m_{j-1} + q_j, \quad \text{and} \quad e = 1.$$

(3) For a C-operation, $j = 0$, $\nu = 3$, and $e = 2$.

If a simply normalized pair f, g is not p-reduced (and hence not p-normalized), we can apply the operations of the proof of Lemma 4.2 and reduce the power of p in $\Delta(f, g)$. We iterate these operations to produce a p-reduced pair. On performing any operation, there will also be a change in the size of the bound given in (2.2), which we estimate by a power of p. We first give an upper bound to this power of p in terms of the change in the size of Δ.

LEMMA 4.3. *Let f, g be a simply normalized pair of additive forms and let θ be a positive real number. If f, g is not p-reduced, then some (m, j)-, (q, j)- or C-condition fails. Apply an (m, j)-, (q, j)-, or C-operation, respectively. On applying an operation, we obtain the pair f', g' defined by*

$$\begin{bmatrix} f' \\ g' \end{bmatrix} = p^{-j}\varphi \begin{bmatrix} f(P^{(\nu)}x) \\ g(P^{(\nu)}x) \end{bmatrix} \tag{4.10}$$

and this pair satisfies

$$(4.11)\qquad (p^{j+(k/2+\theta)(kv-jN)}) \leqslant \left(\frac{\Delta(f,g)}{\Delta(f',g')}\right)^{\frac{k}{N-1}(A(k)+\theta)}$$

where $A(k)$ is given by (1.3).

Proof. We first suppose that a (q, j)-condition fails, so that we apply a (q, j)-operation. Then $v = m_0 + \ldots + m_{j-1} + q_j$ and by (4.7),

$$(4.12)\qquad (p^{j+(k/2+\theta)(kv-jN)}) = \left(\frac{\Delta(f,g)}{\Delta(f',g')}\right)^{\left[\frac{1}{N-1}\cdot\frac{j+(k/2+\theta)(kv-jN)}{(j+1/2)N-kv}\right]}.$$

The application of the (q, j)-operation reduces the power of p in Δ so $\dfrac{\Delta(f,g)}{\Delta(f',g')} > 1$. Hence to show that (4.12) implies (4.11) it suffices to get an upper bound on the exponent on the right of (4.12).

If $k \geqslant 3$, $p > c_2$ and $p|\Delta(f, g)$, a (q, j)-condition fails if $j = 0$ and $q_0 \leqslant 2$. Then

$$\frac{(\frac{1}{2}k+\theta)kq_0}{\frac{1}{2}N-kq_0} \leqslant \frac{(k+2\theta)k\cdot 2}{N-4k} \leqslant k\left(\frac{2k}{N-4k}+\theta\right) \leqslant k(A(k)+\theta).$$

Thus (4.11) holds in this case.

Calculations along the same lines show that (4.11) is satisfied if we apply a (q, j)-operation when any other (q, j)-condition of (3.7) fails. Similarly, (4.11) is satisfied if we apply an (m, j)-operation when an (m, j)-condition fails and if we apply the C-operation when the C-condition is satisfied. This shows the lemma.

In all cases the exponent on the right of (4.12) is largest when $j = 0$ and $v = q_0$. It is this exponent that determines the values of $A(k)$ in each case, and the values are the smallest that can hold for all $\theta > 0$.

LEMMA 4.4. *Let f, g be a simply normalized pair of additive forms of degree k. Let p be a prime and let θ and ε be positive real numbers. Let c_2 and c_3, constants which depend only on k and ε, be the constants of Chapter* 3.

Then there exist

(a) *a matrix Φ with integer entries and determinant a power of p,*

(b) *an N-tuple $\mu = (\mu_1, \ldots, \mu_N)$ with non-negative integer entries, and*

(c) *non-negative integers l, s such that the pair f', g' determined by*

$$\begin{bmatrix} f' \\ g' \end{bmatrix} = \Phi^{-1}\begin{bmatrix} f(P^{(\mu)}x) \\ g(P^{(\mu)}x) \end{bmatrix}$$

satisfies the following:

(1) *f', g' is simply normalized,*

(2) $\chi(p, f', g')\begin{cases} \geqslant p^{-\varepsilon}, & \textit{if } p > c_2 \textit{ and } p|\Delta(f,g) \\ \geqslant p^{-(k+1)(N-2)} \gg 1, & \textit{if } p \leqslant c_3, \end{cases}$

(3) $\Delta(f', g')$ divides $\Delta(f, g)$,

(4) $\left(p^{l+(k/2+\theta)(k\sum_{i=1}^{N}\mu_i - lN)}\right) \ll \left(\dfrac{\Delta(f, g)}{\Delta(f', g')}\right)^{\frac{k}{N-1}(A(k)+\theta)}$,

and

(5) $\frac{1}{5} \leqslant (p^{(l+s-\mu_i \cdot k)})\left(\dfrac{b_i'}{b_i}\right) \leqslant 11$ *for* $i = 1, \ldots, N$, *where the implied constants depend only on* k *and* θ *and are independent of* p.

Proof. Consider a simply normalized pair f, g and a prime p such that $p > c_2$ and $p|\Delta(f, g)$, or that $p \leqslant c_3$. If f, g is not p-reduced, then after a finite number of (m, j)-, (q, j)-, and C-operations, we either reach a pair f'', g'' which satisfies (4.1) and (4.2) for all appropriate j and which satisfies condition C, or, at worst, reach a pair which is p-normalized. In either case, f'', g'' is p-reduced, and satisfies the appropriate conclusions of Lemmas 3.5 and 3.6. Hence part (2) of the lemma is satisfied for f'', g''. Each operation drops the power of p in Δ and leaves other prime powers unchanged, hence

$$\Delta(f'', g'')|\Delta(f, g).$$

So f'', g'' satisfies part (3) of the lemma.

If there are a total of n operations, by (4.10) we can express f'', g'' in terms of f, g by

$$\begin{bmatrix} f'' \\ g'' \end{bmatrix} = (p^{-(j_1+\ldots+j_n)})\,\varphi_n \ldots \varphi_1 \begin{bmatrix} f(P^{(\nu_1+\ldots+\nu_n)}\boldsymbol{x}) \\ g(P^{(\nu_1+\ldots+\nu_n)}\boldsymbol{x}) \end{bmatrix}$$

Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N) = v_1 + \ldots + v_n$ and let $\mu = \sum_{i=1}^{N} \mu_i$. The entries of $v_1, \ldots, v_n$ are non-negative integers, hence so are the entries of $\boldsymbol{\mu}$. (So the condition (b) on $\boldsymbol{\mu}$ in the lemma is satisfied.) Since (4.11) is satisfied for each of the n operations, the non-negative integer $l = j_1 + \ldots + j_n$ is such that

$$p^{(l+(k/2+\theta)(k\mu - lN))} \ll \left(\frac{\Delta(f, g)}{\Delta(f'', g'')}\right)^{\frac{k}{N-1}(A(k)+\theta)}.$$

So the pair f'', g'' satisfies part (4) of the lemma.

By interchanging f and g and interchanging the columns of $\varphi_n \ldots \varphi_1$, if necessary, we can ensure that

$$\varphi_n \ldots \varphi_1 = p^{-t}\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C, D are integers such that g.c.d. $(B, D, p) = 1$, and p^t is the least common denominator of the entries of $\varphi_n \ldots \varphi_1$. Let V be the unimodular matrix which corresponds to application of the Euclidean algorithm to the pair B, D. It can be checked that

$$V\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} p^r & 0 \\ d & 1 \end{bmatrix},$$

where $0 \leqslant r \leqslant t$. By pre-multiplying by a further integral unimodular matrix, if necessary, we obtain an integral unimodular matrix U such that

$$U\varphi_n \ldots \varphi_1 = p^{-t}\begin{bmatrix} p^r & 0 \\ c & 1 \end{bmatrix},$$

where $0 \leqslant r \leqslant t$ and $p^r \leqslant c \leqslant 2p^r$. Let

$$\Phi^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} U(p^{-l})\varphi_n \ldots \varphi_1 = p^{(-l-t)}\begin{bmatrix} p^r + c & 1 \\ c & 1 \end{bmatrix}.$$

Then Φ has integer entries and det Φ is a power of p, as required by part (a) of the lemma. Finally, let

$$\begin{bmatrix} f' \\ g' \end{bmatrix} = \Phi^{-1}\begin{bmatrix} f(P^{(\mu)}(x)) \\ g(P^{(\mu)}(x)) \end{bmatrix}.$$

Thus, to obtain f', g', we have multiplied $\begin{bmatrix} f'' \\ g'' \end{bmatrix}$ on the left by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} U$, a unimodular matrix which has integer entries. Hence $\Delta(f', g') = \Delta(f'', g'')$ and $\chi(p, f', g') = \chi(p, f'', g'')$. So the pair f', g' satisfies parts (2), (3), and (4) of the lemma.

For part (1), f', g' is equivalent to f, g so as in the remarks after Lemma 4.1, f', g' satisfies the first two properties of simple normalization. The coefficients of f', g' satisfy

$$\begin{bmatrix} a_i' \\ b_i' \end{bmatrix} = p^{\mu_i k}\Phi^{-1}\begin{bmatrix} a_i \\ b_i \end{bmatrix}.$$

By using the matrix of Φ^{-1}, the condition on c, and the condition $1/5 \leqslant a_i/b_i \leqslant 5$, it is easily checked that j', g' is indeed simply normalized.

Let $s = t - r$, a non-negative integer. With this s, part (5) of the lemma holds. This completes the proof of the lemma.

The requirement of the preceding lemma that f', g' be simply normalized may seem unnecessary and it adds to the complications. However, we shall later perform the transformation of the lemma for a set of primes in succession whose number is not necessarily bounded in terms of k and θ. Part (1) of the lemma is our guarantee that the ratios a_i/b_i do not stray far from 1. We shall need this fact in the analytic argument.

LEMMA 4.5. *Let f, g be a simply normalized pair of additive forms and f', g' be the associated pair of Lemma 4.4. If $f' = g' = 0$ has a solution $\mathbf{x}$ which satisfies*

$$\left.\begin{array}{l}\sum_{i=1}^{N} |a_i' x_i^k| \\ \sum_{i=1}^{N} |b_i' x_i^k|\end{array}\right\} \leqslant d_1(k, \theta) \prod_{i=1}^{N} |b_i'|^{k/2+\theta} \Delta(f', g')^{\frac{k}{N-1}A(k)+\theta}$$

then $f = g = 0$ has a solution $\mathbf{y}$ which satisfies

$$\left.\begin{array}{l}\sum_{i=1}^{N} |a_i y_i^k| \\ \sum_{i=1}^{N} |b_i y_i^k|\end{array}\right\} \leqslant \left(\max(1, d_2(k, \theta)p^{-\theta})d_1(k, \theta)\right)\left(\prod_{i=1}^{N} |b_i|^{k/2+\theta}\right)\Delta(f, g)^{\frac{k}{N-1}A(k)+\theta}$$

where $d_1(k, \theta)$ and $d_2(k, \theta)$ are constants which depend only on k and θ.

Proof. If f, g is p-reduced, put $f' = f$, $g' = g$ and the bound of the conclusion holds with an initial factor of 1. If f, g is not p-reduced, suppose that $f' = g' = 0$ has a solution $\mathbf{x}$ which satisfies the bound of the hypothesis.

Let $y_i = p^{\mu_i} x_i$ and let $\mathbf{y} = (y_1, \ldots, y_N)$. Then $\mathbf{y}$ is a non-trivial solution in integers to $f = g = 0$. It remains to find the bound on the solution. Now, for each i, a_i and b_i have approximately the same size, and by part (5) of Lemma 4.4, we can rewrite b_i in terms of b_i' to get

$$\left.\begin{array}{l}\sum_{i=1}^{N} |a_i y_i^k| \\ \sum_{i=1}^{N} |b_i y_i^k|\end{array}\right\} \ll \left(\max(1, d_2(k, \theta)p^{-\theta})d_1(k, \theta)\right)\left(\prod_{i=1}^{N} |b_i|^{k/2+\theta}\right)\Delta(f, g)^{\frac{k}{N-1}A(k)+\theta}$$

Rewriting b_i' in terms of b_i, and using part (4) of Lemma 4.4 and the fact that $s \geqslant 0$, the left hand side is

$$\leqslant \left(25 \cdot 11^{N(k/2+\theta)} d_1(k, \theta)\right)\left(\prod_{i=1}^{N} |b_i|^{k/2+\theta}\right)\left(\Delta(f, g)^{\frac{k}{N-1}A(k)+\theta}\right)\left(\frac{\Delta(f', g')}{\Delta(f, g)}\right)^{\left(1-\frac{k}{N-1}\right)\theta}.$$

There is a drop in the power of p in going from $\Delta(f, g)$ to $\Delta(f', g')$. In fact, by (4.4), (4.7), and (4.8), the power is a multiple of $(N-1)/2$, so

$$\left(\frac{\Delta(f', g')}{\Delta(f, g)}\right)^{\left(1-\frac{k}{N-1}\right)\theta} \leqslant p^{-\theta}.$$

Let $d_2(k, \theta) = 25 \cdot 11^{N(k/2+\theta)}$. Then the solution $\mathbf{y}$ to $f = g = 0$ has the bound of the lemma with an initial factor of $d_2(k, \theta)p^{-\theta}$. This proves the lemma.

By Lemma 4.5, for any one prime p such that $p > c_2$ and $p|\Delta(f, g)$ or such that $p \leqslant c_3$, we can reduce from the pair f, g to a pair f', g' which has the needed density of local solutions. We wish now to reduce from f, g to a pair which has the need density for all primes. By Lemma 3.3, the density is always sufficiently great for primes which are large (greater than c_1, a constant depending only on k and θ), and do not divide $\Delta(f, g)$. So consider the finite set E of primes which divide $\Delta(f, g)$ or do not exceed $c_3 = \max(c_1, c_2)$. We seek a pair f', g' which has the needed density of local solutions for all primes in E.

LEMMA 4.6. *Let θ and ε be positive real numbers and let f, g be a simply normalized pair of additive forms. Then there is a simply normalized pair f', g' such that*

(1) $\prod_p \chi(p, f', g') \gg \Delta(f', g')^{-\varepsilon}$, *where the implied constant depends on k and ε, and*

(2) *if $f' = g' = 0$ has a solution x which satisfies*

$$\left.\begin{array}{l}\sum_{i=1}^{N} |a_i' x_i^k| \\ \sum_{i=1}^{N} |b_i' x_i^k|\end{array}\right\} \ll \prod_{i=1}^{N} |b_i'|^{k/2+\theta} \Delta(f', g')^{\frac{k}{N-1}A(k)+\theta}$$

then $f = g = 0$ has a solution y which satisfies

$$\left.\begin{array}{l}\sum_{i=1}^{N} |a_i y_i^k| \\ \sum_{i=1}^{N} |b_i y_i^k|\end{array}\right\} \ll \prod_{i=1}^{N} |b_i|^{k/2+\theta} \Delta(f, g)^{\frac{k}{N-1}A(k)+\theta}.$$

In (2), the implied constants depend on k and θ.

Proof. Let $p_1, p_2, p_3, \ldots$ be a listing of the primes in E. Let $f = F_0$, $g = G_0$ and for each n, $1 \leqslant n \leqslant \operatorname{card} E$, let

$$\begin{bmatrix} F_n \\ G_n \end{bmatrix} = \Phi_n^{-1} \begin{bmatrix} F_{n-1}(P^{(\mu_n)}x) \\ G_{n-1}(P^{(\mu_n)}x) \end{bmatrix}.$$

Here Φ_n and μ_n are chosen as in Lemma 4.4 for the prime p_n, relative to the simply normalized pair F_{n-1}, G_{n-1}, and $P^{(\mu_n)}$ is the $N \times N$ diagonal matrix with diagonal entries $p_n^{\mu_{in}}$, $i = 1, 2, \ldots, N$. Finally, let $f' = F_n$, $g' = G_n$ when $n = \operatorname{card} E$.

By part (1) of Lemma 4.4, f', g' is simply normalized. By part (3) of Lemma 4.4, $\Delta(F_n, G_n)|\Delta(F_{n-1}, G_{n-1})$ for each n, so $\Delta(f', g')|\Delta(f, g)$.

For part (1) of this lemma, we have first of each n,

$$\chi(p_n, F_n, G_n) \geqslant \begin{cases} p_n^{-\varepsilon}, & \text{if } p_n > c_2 \text{ and } p_n | \Delta(f, g), \\ p_n^{-(k+1)(N-2)} \gg 1, & \text{if } p_n \leqslant c_3. \end{cases}$$

But Φ_m, $n+1 \leqslant m \leqslant \operatorname{card} E$, has integer entries and $\det \Phi_m$ is a power of p_m, so the determinant is a p_n-adic unit. The matrix $\Phi_{\operatorname{card} E} \ldots \Phi_{n+1}$ has the some properties. Also $\prod_{m=n+1}^{\operatorname{card} E} P^{(\mu m)}$ is a matrix with entries which are p_n-adic units. Thus, by the remarks after Lemma 4.1, $\chi(p_n, f', g') = \chi(p_n, F_n, G_n)$.

Thus

$$\chi(p, f', g') \geqslant \begin{cases} p^{-\varepsilon}, & \text{if } p > c_2 \text{ and } p | \Delta(f, g), \\ p^{-(k+1)(N-2)} \gg 1, & \text{if } p \leqslant c_3. \end{cases} \tag{4.13}$$

for each p in E. Now $\Delta(f', g') | \Delta(f, g)$ so E contains all primes p dividing $\Delta(f', g')$ and all $p \leqslant c_3$. Then (4.13) is just a restatement of (3.10) for the pair f', g', so by Lemma 3.7,

$$\prod_p \chi(p, f', g') \gg \Delta(f', g')^{-\varepsilon}$$

for each $\varepsilon > 0$. The implied constant depends on k and ε. So part (1) holds.

For part (2), by Lemma 4.5, if $f' = g' = 0$ has a bounded solution $\boldsymbol{x}$, then $f = g = 0$ has a solution y which satisfies

$$\left.\begin{array}{l} \sum_{i=1}^{N} |a_i y_i^k| \\ \sum_{i=1}^{N} |b_i y_i^k| \end{array}\right\} \ll \Big[\prod_{p \in E} \max(1, d_2(k, \theta) \cdot p^{-\theta}) d_1(k, \theta)\Big) \Big(\prod_{i=1}^{N} |b_i|^{k/2+\theta}\Big) \Delta(f, g)^{\frac{k}{N-1} A(k) + \theta}$$

with the same implied constant. Since $d_2(k, \theta) \cdot p^{-\theta} < 1$ for all large primes, the product over the primes in E is $\ll 1$, and so the solution y satisfies the bound of the lemma. This completes the proof of the lemma.

Lemmas 2.1 and 2.2 showed that, for the proof of the Theorem, it is sufficient to show that each simply normalized pair f, g has a bounded solution. Lemma 4.6 shows that it is sufficient to show that each simply normalized pair with a sufficient density of local solutions has a bounded solution. In fact, since $\Delta(f, g) \geqslant 1$ for all simply normalized pairs, we have shown

LEMMA 4.7. *Let θ and ε be positive real numbers. It is sufficient for the proof of the Theorem to show that each simply normalized pair f, g which satisfies*

$$\prod_p \chi(p, f, g) \gg \Delta(f, g)^{-\varepsilon}, \tag{3.1}$$

where the implied constant depends on k and ε, has a solution $\mathbf{x}$ *to*

$$f(\mathbf{x}) = g(\mathbf{x}) = 0$$

which satisfies

$$\left.\begin{array}{l}\sum_{i=1}^{N} |a_i x_i^k| \\ \sum_{i=1}^{N} |b_i x_i^k|\end{array}\right\} \ll \prod_{i=1}^{N} |b_i|^{k/2+\theta}, \tag{4.14}$$

where the implied constant depends on k and θ.

The analytic argument shows that if ε is taken to be a small positive number which depends on k and θ, then the sufficient condition of Lemma 4.7 is met.

5. Preparation for the analytic argument

Let ε be a positive real number and let f, g be a simply normalized pair of additive forms which satisfies

$$\prod_p \chi(p, f, g) \gg \Delta(f, g)^{-\varepsilon}. \tag{3.1}$$

The analytic argument requires a non-singular positive real solution to

$$\begin{aligned} f &= \sum_{i=1}^{N} a_i x_i^k = 0, \\ g &= \sum_{i=1}^{N} b_i x_i^k = 0, \end{aligned} \tag{5.1}$$

that is, a non-singular positive real solution ξ to

$$\begin{aligned} \sum_{i=1}^{N} a_i y_i &= 0, \\ \sum_{i=1}^{N} b_i y_i &= 0. \end{aligned} \tag{5.2}$$

To find such a solution, relabel the indices of f, g so that $\left(\frac{a_1}{b_1} - \frac{a_2}{b_2}\right)$ has the largest positive value among all quantities $\left(\frac{a_i}{b_i} - \frac{a_j}{b_j}\right)$. Then

$$\frac{a_1}{b_1} \geqslant \frac{a_i}{b_i} \geqslant \frac{a_2}{b_2} \quad \text{for all } i = 1, 2, \ldots, N.$$

Put

$$y_i = \frac{\mu_i}{|b_i|} \quad (i = 1, \ldots, N)$$

and solve for μ_1 and μ_2 to obtain

(5.3) $$\mu_1 = \sum_{i=3}^{N} \left[\pm \frac{\frac{a_i}{b_i} - \frac{a_2}{b_2}}{\frac{a_1}{b_1} - \frac{a_2}{b_2}} \right] \mu_i, \quad \mu_2 = \sum_{i=3}^{N} \left[\pm \frac{\frac{a_1}{b_1} - \frac{a_i}{b_i}}{\frac{a_1}{b_1} - \frac{a_2}{b_2}} \right] \mu_i,$$

where the sign of the coefficient of μ_i in the top equation is that of $-b_1 b_i$, and in the bottom equation, that of $-b_2 b_i$. Note that both the numerator and the denominator of the fractions are positive. Since k is odd, we can replace appropriate x_i by $-x_i$ in (5.1) to change the signs of coefficients in (5.1) and guarantee that all coefficients of (5.3) are non-negative. This leaves unchanged the integral solubility of (5.1), the size of the bound of the solution, and the size of the ratios a_i/b_i. This normalization of signs can not be guaranteed for k even without further assumptions. See Pitman (1981), Lemmas 2.3 and 5.2, for possible ways of finding an appropriate non-singular solution in this case.

Put $\mu_i = 1$ for $i = 3, \ldots, N$. There is at least one ratio which is distinct from both the first and the second, so μ_1 and μ_2 are positive. Put $\xi_i = \mu_i/|b_i|$ for all i. Then $\xi = (\xi_1, \ldots, \xi_N)$ is a non-singular positive real solution to (5.2), as desired.

Let

(5.4) $$\varkappa_i = (\tfrac{1}{2}\xi_i)^{1/k}, \quad \varkappa_i' = (2\xi_i)^{1/k} \quad (i = 1, \ldots, N).$$

It is clear that ξ lies in the box determined by

$$(\varkappa_i)^k \leqslant y_i \leqslant (\varkappa_i')^k \quad (i = 1, \ldots, N).$$

Note that for each i,

(5.5) $$\varkappa_i,\ \varkappa_i',\ \varkappa_i' - \varkappa_i,\ (\xi_i)^{1/k},\ \left(\frac{\mu_i}{|b_i|}\right)^{1/k}$$

are all of the same magnitude and all are $\ll |b_i|^{-1/k}$.

Let θ be a positive real constant. Let

(5.6) $$P = D\Big(\prod_{i=1}^{N} |b_i|\Big)^{1/2+\theta},$$

where D is a large positive number which depends on k and θ alone and is independent of the coefficients a_i, b_i of the pair f, g. We will think of D as fixed, but sufficiently large to satisfy our arguments.

Let $N(P)$ be the number of solutions $\boldsymbol{x}$ of $f = g = 0$ with

$$\varkappa_i P \leqslant x_i \leqslant \varkappa_i' P \qquad (i = 1, \ldots, N).$$

Putting

$$\Lambda_i = a_i\alpha + b_i\alpha',$$

and

$$T_i(\Lambda_i) = \sum_{x_i} e(\Lambda_i x_i^k), \qquad \text{where } \varkappa_i P \leqslant x_i \leqslant \varkappa_i' P,$$

$N(P)$ can be expressed analytically as

$$N(P) = \int_0^1\int_0^1 \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha'. \tag{5.7}$$

Our analytic argument will show the asymptotic formula

$$N(P) = CP^{N-2k}(1+O(P^{-\sigma})). \tag{5.8}$$

Here the implied constant and the positive number σ depend only on k and θ and are independent of the coefficients. Also, C is a positive number such that

$$C \gg \frac{(\xi_1\xi_2)^{-1}}{|a_1 b_2 - a_2 b_1|}\Big(\prod_{i=1}^{N} \xi_i^{1/k}\Big)\Delta(f, g)^{-\varepsilon}, \tag{5.9}$$

where ε and the implied constant depend only on k and θ.

If (5.8) holds, then the Theorem holds. For if D is sufficiently large, $1+O(P^{-\sigma})$ and hence $N(P)$ is positive. Thus we have a solution $\boldsymbol{x}$ to $f = g = 0$ which satisfies $\varkappa_i P \leqslant x_i \leqslant \varkappa_i' P$ for all i. The definition of $\varkappa_i$ and P then guarantes that this solution satisfies the bound (4.14) of Lemma 4.7 and hence that the Theorem holds.

For the analytic argument, we need auxiliary variables and estimates on their sizes. Define

$$|b_i| = P^{e_i} \qquad \text{for } i = 1, 2, \ldots, N. \tag{5.10}$$

Then

$$\prod_{i=1}^{N} |b_i| \leqslant P^{2/(1+2\theta)} \leqslant P^2 \tag{5.11}$$

so

$$\sum_{i=1}^{N} e_i \leqslant \frac{2}{1+2\theta} \leqslant 2. \tag{5.12}$$

Also define

$$\mu = \prod_{i=1}^{N} \mu_i = \mu_1\mu_2 = P^{-e}. \tag{5.13}$$

With these variables, we can estimate C of the asymptotic formula by

$$C \gg \mu^{-1}\Big(\prod_{i=1}^{N} \varkappa_i\Big)\Delta(f, g)^{-\varepsilon} \tag{5.14}$$

and by

$$C \gg \Big(\prod_{i=1}^{N} \varkappa_i\Big)P^{e-\varepsilon} \tag{5.15}$$

(where we have replaced $2(N-1)\varepsilon$ by ε).

Finally, when we deal with equations involving only exponents of P, we will systematically abuse notation in that we require the equation to be true only up to a bounded constant. That is, we will write $a \leqslant b$ if $P^a \ll P^b$ and hence write $a = b$ if $P^a \ll P^b \ll P^a$.

We also have

LEMMA 5.1. (1) *Both* μ_1 *and* μ_2 *are* $\ll 1$ *and at most one of them can be small* (*not* $\gg 1$).

If $\mu_2 \leqslant \mu_1$, *we have the properties* (2) *through* (7):

(2) $\mu_2 \ll \mu \ll \mu_2 \ll 1$,

(3) $e \geqslant 0$ *with* $e > 0$ *if and only if* μ_2 *is small* (*not* $\gg 1$),

(4) $\left|\frac{a_i}{b_i} - \frac{a_j}{b_j}\right| \mu_2^{-1} \ll 1$ *if* $i \neq 2$ *and* $j \neq 2$ *represent distinct ratios,*

(5) $e \leqslant \min\limits_{i,j}(e_i+e_j)$, *where* i, j *is over all pairs such that* $i \neq 2$ *and* $j \neq 2$ *represent distinct ratios,*

(6) $e \leqslant 4/(N-1)$,

(7) D *can be chosen depending only on* k *and* θ *so that*

$$\varkappa_i P,\ \varkappa_i' P,\ (\varkappa_i' - \varkappa_i) P \text{ are all } \gg 1.$$

If in fact μ_2 *is small* (*not* $\gg 1$) *we also have*:

(8) *no other ratio* a_i/b_i *has the same values as* a_2/b_2.

Analogous results hold if $\mu_1 \leqslant \mu_2$ *or if is small.*

Proof. Let

$$A_i = \frac{\dfrac{a_i}{b_i} - \dfrac{a_2}{b_2}}{\dfrac{a_1}{b_1} - \dfrac{a_2}{b_2}} \quad \text{and} \quad B_i = \frac{\dfrac{a_1}{b_1} - \dfrac{a_i}{b_i}}{\dfrac{a_1}{b_1} - \dfrac{a_2}{b_2}}.$$

For $i = 1, \ldots, N$, A_i and B_i are non-negative and $A_i + B_i = 1$, so $\mu_1 + \mu_2 = N-2$ and for each i, either A_i or B_i is $\gg 1$.

Part (1) follows immediately.

Now assume that $\mu_2 \leqslant \mu_1$. Parts (2) and (3) follow. If in addition, μ_2 is small, then part (8) follows.

If $i \neq 2$ and $j \neq 2$ represent distinct ratios, then

$$(5.16) \qquad |b_i b_j|^{-1} \leqslant \frac{|a_i b_j - a_j b_i|}{|b_i b_j|} = \left|\frac{a_i}{b_i} - \frac{a_j}{b_j}\right| \ll \max_{l \geqslant 3}(B_l) \ll \mu_2.$$

Parts (4) to (7) then follow easily.

6. Estimates for $T(\Lambda)$

In this chapter we give estimates for

$$T(\Lambda) = \sum_{\varkappa P \leqslant x \leqslant \varkappa' P} e(\Lambda x^k).$$

Here $\varkappa$ and $\varkappa'$ are arbitrary, but fixed, positive real numbers with $\varkappa' > \varkappa$ and with $\varkappa, \varkappa'$, and $\varkappa' - \varkappa$ of the same magnitude, and P is any large positive number such that $P\varkappa \geqslant 1$ and $P(\varkappa' - \varkappa) \geqslant 1$. These lemmas are valid for all integers $k \geqslant 2$.

LEMMA 6.1 (Weyl's Inequality). *Let $f(x)$ be a real polynomial of degree k with highest coefficient Λ:*

$$f(x) = \Lambda x^k + \Lambda_1 x^{k-1} + \dots$$

Suppose Λ has the rational approximation B/R satisfying

$$(B, R) = 1, \quad R > 0, \quad \left|\Lambda - \frac{B}{R}\right| < \frac{1}{R^2}.$$

Then for any $\varepsilon > 0$,

$$\Big|\sum_{\varkappa P \leqslant x \leqslant \varkappa' P} e(f(x))\Big| \ll (\varkappa P)^{1+\varepsilon}\left[(\varkappa P)^{-1/2^{k-1}} + R^{-1/2^{k-1}} + \left(\frac{(\varkappa P)^k}{R}\right)^{-1/2^{k-1}}\right]$$

where the implied constant depends only on k and ε.

Proof. See Davenport (1962), Lemma 1. We have shown explicitly the dependence of the estimate on $\varkappa$.

LEMMA 6.2 (Hua's Inequality). *For any $\varepsilon > 0$,*

$$\int_0^1 |T(\Lambda)|^{2^k} d\Lambda \ll (\varkappa P)^{2^k - k + \varepsilon},$$

where the implied constant depends on k and ε.

Proof. See, for example, Davenport (1962), Lemma 2.

Suppose we have indexed objects $T_i(\Lambda_i)$, $\varkappa_i$, and $\varkappa'_i$ that satisfy the definitions and restrictions of this chapter. Then we have

COROLLARY 6.3. *Suppose that a_i, b_i, a_j, b_j are integers such that $a_i b_j - a_j b_i \neq 0$. Let $\Lambda_i = a_i\alpha + b_i\alpha'$ and let $\Lambda_j = a_j\alpha + b_j\alpha'$. Then for any $\varepsilon > 0$,*

$$(6.1)\qquad \int_0^1\int_0^1 |T_i(\Lambda_i)T_j(\Lambda_j)|^{2^k}\,d\alpha\,d\alpha' \ll [(\varkappa_i P)(\varkappa_j P)]^{2^k-k+\varepsilon},$$

where the implied constant depends on k and ε.

Proof. The left hand side of (6.1) equals

$$\int_0^1\int_0^1 [T_i(\Lambda_i)T_i(-\Lambda_i)T_j(\Lambda_j)T_j(-\Lambda_j)]^{2^{k-1}}\,d\alpha\,d\alpha',$$

which is the number of integral solutions in all variables to the simultaneous equations

$$(6.2)\qquad \begin{cases} a_i F(\boldsymbol{r}, \boldsymbol{s}) + a_j F(\boldsymbol{u}, \boldsymbol{v}) = 0, \\ b_i F(\boldsymbol{r}, \boldsymbol{s}) + b_j F(\boldsymbol{u}, \boldsymbol{v}) = 0 \end{cases}$$

subject to

$$(6.3)\qquad \begin{cases} \varkappa_i P \leqslant r_l, s_l \leqslant \varkappa_i' P, & l = 1, \ldots, 2^{k-1}, \\ \varkappa_j P \leqslant u_l, v_l \leqslant \varkappa_j' P, & l = 1, \ldots, 2^{k-1}, \end{cases}$$

where

$$F(\boldsymbol{x}, \boldsymbol{y}) = \sum_{l=1}^{2^{k-1}} x_l^k - \sum_{l=1}^{2^{k-1}} y_l^k,$$
$$\boldsymbol{x} = (x_1, \ldots, x_{2^{k-1}}), \quad \boldsymbol{y} = (y_1, \ldots, y_{2^{k-1}}).$$

But $a_i b_j - a_j b_i \neq 0$, so the number of solutions to (6.2) is the number of solutions to the (independent) equations

$$(6.4)\qquad F(\boldsymbol{r}, \boldsymbol{s}) = 0, \quad F(\boldsymbol{u}, \boldsymbol{v}) = 0$$

subject to (6.3).

The number of solutions to (6.4) can be represented by the product of 2 integrals, each of the type estimated in Lemma 6.2, so the product is majorized by the left hand side of (6.1), as desired. This shows the lemma.

The simplest way to get a two-dimensional analog to Hua's inequality is to use the lemma of Cook (1972a). Straightforward application of the lemma yields the same estimate, but with an additional factor of

$$\frac{|b_i b_j|}{|a_i b_j - a_j b_i|}.$$

The denominator may be substantially smaller than the numerator. If so, the estimate will requires the use of a much larger P in the argument for the minor arcs, and as a consequence, will produce a larger bound on the solution to $f = g = 0$. The argument for Corollary 6.3 allows us to avoid the factor altogether.

When the denominator of the rational approximation to Λ is small, we have additional estimates for $T(\Lambda)$ (valid for $k \geqslant 3$).

LEMMA 6.4. *Let*

$$I(\gamma) = \int_{\varkappa P}^{\varkappa' P} e(\gamma x^k) dx.$$

Then

$$I(\gamma) \ll \min\left(\varkappa P, (\varkappa P)^{-k+1} |\gamma|^{-1}\right).$$

The implied constant depends only on k.

Proof. We have the trivial upper bound $\varkappa P$. For the other estimate, perform a change of variable to $y = x^k$ and integrate by parts.

LEMMA 6.5. *Let* δ *be a positive real constant. Suppose* $\Lambda = B/R + \gamma$, *where* B, R *are integers such that*

$$(B, R) = 1, \quad 1 \leqslant R \ll (\varkappa P)^{1-\delta}, \quad \text{and} \quad |\gamma| \ll R^{-1} (\varkappa P)^{-k+1-\delta}.$$

Then for any $\varepsilon > 0$,

$$T(\Lambda) = R^{-1} S(B, R) I(\gamma) + O(R^{1-1/k+\varepsilon}). \tag{6.5}$$

When ε *is small compared to* δ,

$$T(\Lambda) \ll R^{-1/k} \min\left(\varkappa P, (\varkappa P)^{-k+1} |\gamma|^{-1}\right). \tag{6.6}$$

The implied constants depend on k, δ, *and* ε.

Proof. Equation (6.5) is the estimate of Davenport (1939a), Lemma 7 (for $k = 3$), and of Devenport (1939b), Lemma 8 (for $k \geqslant 4$). The proofs must be modified since the $I(\gamma)$ of these papers is a sum and not an integral. The size of the error is dictated by estimates on a trigonometric sum given in Davenport and Heilbronn (1936), Lemma 3, and Davenport and Heilbronn (1937), Lemma 2. Equation (6.6) is the estimate of Davenport (1939a), Lemma 9 and Davenport (1939b), Lemma 9.

We will call (6.5) and (6.6) the *Davenport estimates.*

7. The minor arcs

In Chapter 5, we considered $N(P)$, the number of solutions to the simultaneous additive equations $f = g = 0$ in the box $\varkappa_i P \leqslant x_i \leqslant \varkappa_i' P$ $(i = 1, \ldots, N)$. We obtained the analytic expression

$$N(P) = \int_0^1 \int_0^1 \prod_{i=1}^{N} T_i(\Lambda_i) d\alpha d\alpha' \tag{5.7}$$

and want to show

$$N(P) = CP^{N-2k} + O(CP^{N-2k-\sigma}). \tag{5.8}$$

We first find a subset $\mathfrak{M}$ of the unit square, called the *set of major arcs*, such that the points in $\mathfrak{M}$ have good Diophantine properties. The complement $\mathfrak{m}$ of $\mathfrak{M}$ in the unit square, called the *set of minor arcs*, will contribute only to the error term of (5.8).

Define the set of major arcs $\mathfrak{M}$ to be the points (α, α') in the unit square such that for each $i = 1, \ldots, N$, there exist rational integers B_i, R_i such that $\Lambda_i = a_i\alpha + b_i\alpha' = (B_i/R_i) + \gamma_i$ where

$$\begin{gathered}(B_i, R_i) = 1, \quad 1 \leqslant R_i \leqslant (\varkappa_i P)^{k/2^{k-1}}, \quad \text{and} \\ |\gamma_i| \leqslant R_i^{-1/k}(\varkappa_i P)^{-k+1/2^{k-1}}.\end{gathered} \tag{7.1}$$

We show

LEMMA 7.1. *If* $(\alpha, \alpha') \in \mathfrak{m}$, then for some i,

$$|T_i(\Lambda_i)| \ll (\varkappa_i P)^{1-\frac{1-\delta}{2^{k-1}}+\varepsilon}. \tag{7.2}$$

It is the margin below the trivial estimate $\varkappa_i P$ for $T_i(\Lambda_i)$ that allows the analytic argument to proceed.

Proof. By Dirichlet's theorem on Diophantine approximations, for any $\delta > 0$ and for $i = 1, \ldots, N$, there exists an approximation B_i/R_i to $\Lambda_i = a_i\alpha + b_i\alpha'$ with $(B_i, R_i) = 1$ and $\gamma_i = \Lambda_i - (B_i/R_i)$ such that

$$1 \leqslant R_i \leqslant (\varkappa_i P)^{k-1+\delta}, \quad |\gamma_i| < R_i^{-1}(\varkappa_i P)^{-k+1-\delta}. \tag{7.3}$$

If for some i, $R_i > (\varkappa_i P)^{1-\delta}$, then by Weyl's Inequality, (7.2) holds. So we can assume that for all i, $1 \leqslant R_i \leqslant (\varkappa_i P)^{1-\delta}$, and in this case the Davenport estimate (6.6) holds.

Since (α, α') is in $\mathfrak{m}$, for some i the approximation (7.3) does not satisfy (7.1). So either

$$(\varkappa_i P)^{k/2^{k-1}} < R_i \leqslant (\varkappa_i P)^{1-\delta} \tag{7.4}$$

or

$$1 \leqslant R_i \leqslant (\varkappa_i P)^{k/2^{k-1}} \quad \text{and} \quad |\gamma_i| > R_i^{-1/k}(\varkappa_i P)^{-k+1/2^{k-1}}. \tag{7.5}$$

But whether (7.4) or (7.5) holds, (6.6) shows that (7.2) holds. This shows the lemma.

LEMMA 7.2. *We have*

$$\iint_{\mathfrak{m}} \prod_{i=1}^{N} T_i(\Lambda_i)\, d\alpha\, d\alpha' \ll CP^{N-2k-\sigma}. \tag{7.6}$$

Proof. Consider the contribution to the integral from those (α, α') for which (7.2) holds for $i = 1$. Estimate $T_1(\Lambda_1)$ by (7.2). Use Hölder's inequality to estimate the integral of the product over $i \neq 1$ by a product of integrals. This product is over 2^k pairs of indices $i \neq 1$, $j \neq 1$ which represent unequal ratios (i.e., $a_i b_j - a_j b_i \neq 0$). Use Corollary 6.3 to estimate the individual integrals. This yields an estimate for the integral of the lemma of

$$\ll \Big(\prod_{i=1}^{N} \varkappa_i\Big) P^{\Omega}$$

where $\Omega = N - 2k + \frac{e}{2^k} + \frac{2}{1+2\theta}\cdot\frac{1}{2^k} - \frac{1-\delta}{2^{k-1}} + 3\varepsilon.$

Recall that $e \geqslant 0$ (Lemma 5.1, part (3)) and take δ, ε, and σ small compared to k and θ. On using the lower bound (5.15) for C, we see that our estimate is indeed $\ll C\cdot P^{N-2k-\sigma}$.

The same estimate holds for $i = 2, \ldots, N$, so the lemma holds.

By Lemma 7.2, we have

$$N(P) = \iint_{\mathfrak{m}} \prod_{i=1}^{N} T_i(\Lambda_i)\, d\alpha\, d\alpha' + O(CP^{N-2k-\sigma}. \tag{7.7}$$

where the implied constant and σ depend on k and θ alone.

8. Estimates for treatment of the major arcs

For points in $\mathfrak{M}$, by (7.1) the auxiliary variables Λ_i have rational approximations that satisfy the hypotheses of Lemma 6.5. Hence

LEMMA 8.1. *The Davenport estimates hold in* $\mathfrak{M}$.

We now give estimates based on the form of the Davenport estimate (6.6). Recall $\Delta(f, g) = \prod_{t=1}^{N} D_t(f, g)$, where $D_t(f, g)$ is a product of 2^k determinants of the form $|a_i b_j - a_j b_i| \neq 0$ with i, j distinct from one another and from t.

LEMMA 8.2. *For* $i = 1, \ldots, N$, *let*

$$\gamma_i = a_i \beta + b_i \beta'. \tag{8.1}$$

If $\mathscr{W}$ *denotes the whole plane, then for* $t = 1, \ldots, N$,

$$\iint_{\mathscr{W}} \prod_{i \neq t} \min\big(\varkappa_i P, (\varkappa_i P)^{-k+1} |\gamma_i|^{-1}\big)\, d\beta\, d\beta' \tag{8.2}$$

$$\ll \Big(\prod_{i \neq t} \varkappa_i\Big) P^{N-1-2k} \left(\frac{\prod_{i \neq t} \varkappa_i^{-k}}{D_t(f, g)}\right)^{1/2^k}$$

and

$$(8.3)\qquad \iint_{\mathscr{W}} \prod_{i=1}^{N} \min\left(\varkappa_i P, (\varkappa_i P)^{-k+1}|\gamma_i|^{-1}\right) d\beta\, d\beta'$$

$$\ll \left(\prod_{i=1}^{N} \varkappa_i\right) P^{N-2k} \left(\frac{\prod_{i=1}^{N} \varkappa_i^{-k}}{\Delta(f, g)^{1/(N-1)}}\right)^{2/N}.$$

Proof. Fix an index t with $1 \leqslant t \leqslant N$. The tth partition partitions the indices excluding t into 2^k unordered pairs i, j such that $a_i b_j - a_j b_i \neq 0$. By Hölder's inequality, to show (8.2) it is sufficient to show that for each unordered pair i, j in the partition,

$$\iint_{\mathscr{W}} \min\left(\varkappa_i P, (\varkappa_i P)^{-k+1}|\gamma_i|^{-1}\right)^{2^k} \min\left(\varkappa_j P, (\varkappa_j P)^{-k+1}|\gamma_j|^{-1}\right)^{2^k} d\beta\, d\beta'$$

$$\ll \frac{(\varkappa_i \varkappa_j)^{2^k - k}}{|a_i b_j - a_j b_i|} P^{N-1-2k}.$$

To obtain this, we make a change of variable from β, β' to γ_i, γ_j using (8.1). The variables are now separated, so we factor the integral and integrate. The product over these estimates is (8.2).

For the estimate (8.3) first note that

$$\prod_{i=1}^{N} |b_i| = \prod_{j=1}^{N} \left(\prod_{i \neq j} |b_i|^{1/(N-1)}\right).$$

A similar expression holds for the integrand of (8.3). Hence by Hölder's inequality, (8.3) holds if for each $j = 1, \ldots, N$, we have

$$\iint_{\mathscr{W}} \prod_{i \neq j} \min\left(\varkappa_i P, (\varkappa_i P)^{-k+1}|\gamma_i|^{-1}\right)^{N/(N-1)} d\beta\, d\beta'$$

$$\ll \left(\prod_{i \neq j} \varkappa_i\right)^{(N-2k)/(N-1)} D_j(f, g)^{-2/(N-1)} P^{N-2k}.$$

This estimate holds by the same method used to show (8.2).

LEMMA 8.3. *If* $\mathscr{T} = \{(\beta, \beta') \in \boldsymbol{R}^2$ *such that* $\max\{|\beta|, |\beta'|\} > P^{-k+\tau}\}$, *then for* $t = 1, \ldots, N$,

$$(8.4)\qquad \iint_{\mathscr{T}} \prod_{i \neq t} \min\left(\varkappa_i P, (\varkappa_i P)^{-k+1}|\gamma_i|^{-1}\right) d\beta\, d\beta'$$

$$\ll \left(\prod_{i \neq t} \varkappa_i\right) \left(P^{N-1-2k}\right) \left(D_t(f, g)^{-1/2^k}\right) \left(\prod_{i \neq t} \varkappa_i^{-k}\right) \left(P^{-\tau(2^k-1)}\right).$$

Proof. Fix an index t with $1 \leqslant t \leqslant N$. The tth partition partitions the indices excluding t into $(N-1)/2 = 2^k$ pairs i, j such that $a_i b_j - a_j b_i \neq 0$. By Hölder's inequality, it suffices to show that for any pair i, j determined by the tth partition, (8.5) $$\iint_{\mathcal{T}} \min\left(\varkappa_i P, (\varkappa_i P)^{-k+1}|\gamma_i|^{-1}\right)^{2^k} \min\left(\varkappa_j P, (\varkappa_j P)^{-k+1}|\gamma_j|^{-1}\right)^{2^k} d\beta\, d\beta' \ll \frac{(\varkappa_i\varkappa_j)^{(1-k)2^k}}{|a_i b_j - a_j b_i|} P^{N-1-2k-\tau(2^k-1)}.$$

Use (8.1) to solve for β, β' in terms of γ_i, γ_j. If

$$|\gamma_i| \leqslant C_3 \varkappa_j^k(P^{-k+\tau}), \qquad |\gamma_j| \leqslant C_3 \varkappa_i^k(P^{-k+\tau}),$$

then we can choose C_3 to be a small positive constant depending on k and θ such that $|\beta|, |\beta'| \leqslant P^{-k+\tau}$. So if $|\gamma_i|$, $|\gamma_j|$ are small, (β, β') is in the complement of $\mathcal{T}$. Hence $\mathcal{T} \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$, where

$$\mathcal{T}_1 = \{(\gamma_i, \gamma_j)\colon |\gamma_i| > C_3 \varkappa_j^k P^{-k+\tau}\}$$

and

$$\mathcal{T}_2 = \{(\gamma_i, \gamma_j)\colon |\gamma_j| > C_3 \varkappa_i^k P^{-k+\tau}\}.$$

Majorize the integral over $\mathcal{T}$ for the pair i, j by the sum of integrals over $\mathcal{T}_1$ and $\mathcal{T}_2$. Estimate the integral over $\mathcal{T}_1$ by changing variables from β, β' to γ_i, γ_j, factoring, and integrating. We get

$$\iint_{\mathcal{T}_1} \ll \frac{(\varkappa_i\varkappa_j)^{(1-k)2^k}}{|a_i b_j - a_j b_i|} P^{N-1-2k-\tau(2^k-1)}.$$

The same estimate holds for $\iint_{\mathcal{T}_2}$. This shows (8.5) and hence (8.4).

LEMMA 8.4. *For any non-negative real number $a \geqslant 0$ and any $Q \geqslant P^a$, let $\mathfrak{A}_Q$ be the set of pairs of integers A, A' such that*

$$0 \leqslant A, A' < Q \quad \text{and} \quad (A, A', Q) = 1.$$

For each pair A, A' in $\mathfrak{A}_Q$ and for $i = 1, \ldots, N$, define

$$\frac{B_i}{R_i} = \frac{B_i(A, A')}{R_i(A, A')} = a_i \frac{A}{Q} + b_i \frac{A'}{Q}, \quad \text{where } (B_i, R_i) = 1.$$

Then for any $\varepsilon > 0$,

(8.6) $$\sum_{Q \geqslant P^a} \sum_{A, A' \in \mathfrak{A}_Q} \prod_{i \neq t} R_i^{-1/k} \ll \prod_{i \neq t} |b_i|^{1/2^k} \min\left(D_t(f, g)^\varepsilon, \left(D_t(f, g)^{\frac{1}{k} - \frac{2}{2^k}}\right)\left(P^{-a\left(\frac{2^k}{k} - 2 - \varepsilon\right)}\right)\right)$$

and

(8.7) $$\sum_{Q \geqslant P^a} \sum_{A, A' \in \mathfrak{A}_Q} \prod_{i=1}^{N} R_i^{-1/k} \ll \prod_{i=1}^{N} |b_i|^{2/N} \cdot \min\left(\Delta(f, g)^\varepsilon, \left(\Delta(f, g)^{1/(N-1)(1/k - 4/N)}\right)\left(P^{-a(N/(2k) - 2 - \varepsilon)}\right)\right).$$

Here the implied constants depend on k and ε.

Proof. To show (8.6), by Hölder's inequality it is sufficient to show that if i, j is a pair of indices such that $a_i b_j - a_j b_i \neq 0$, we have

$$(8.8) \quad \sum_{Q \geqslant P^a} \sum_{A, A \in \mathfrak{A}_Q} (R_i R_j)^{-2^k/k}$$

$$\ll |b_i b_j| \min\left(|a_i b_j - a_j b_i|^{\varepsilon},\ |a_i b_j - a_j b_i|^{\frac{2^k}{k}-2}(P^{-a(\frac{2^k}{k}-2-\varepsilon)})\right).$$

We first estimate the inner sum in (8.8). Let $R_i = Q/u_i$, where $u_i = (Q, a_i A + b_i A')$. Then $a_i A + b_i A' \equiv 0 \bmod u_i$. Similar equations hold for R_j. Let $d = (u_i, u_j)$. The congruences show that $d \| a_i b_j - a_j b_i | (A, A')$. Now $d | Q$ so in fact $d \| a_i b_j - a_j b_i|$. Hence, $d | (Q, |a_i b_j - a_j b_i|)$. With this restriction on d, we can use the argument of Davenport and Lewis (1969a), Lemma 23, to show that the inner sum is

$$\ll |b_i b_j| \left(\frac{Q}{(Q, |a_i b_j - a_j b_i|)}\right)^{-\frac{2^k}{k}+1+\varepsilon}.$$

We now sum over Q. Consider a particular divisor δ of $a_i b_j - a_j b_i$. For this δ, we have

$$\sum_{\substack{Q \geqslant P^a \\ Q \equiv 0 \bmod \delta}} \sum_{A, A' \in \mathfrak{A}_Q} (R_i R_j)^{-2^k/k} \ll |b_i b_j| \sum_{\substack{Q \geqslant P^a \\ \delta | Q}} \left(\frac{Q}{\delta}\right)^{-2^k/k+1+\varepsilon}$$

$$= |b_i b_j| \sum_{n \geqslant \max(1, P^a/\delta)} n^{-2^k/k+1+\varepsilon} \ll |b_i b_j| \max\left(1, \frac{P^a}{\delta}\right)^{-2^k/k+2+\varepsilon}$$

$$= |b_i b_j| \min\left(1, |a_i b_j - a_j b_i|^{\frac{2^k}{k}-2-\varepsilon}(P^{-a(\frac{2^k}{k}-2-\varepsilon)})\right).$$

There are $\ll |a_i b_j - a_j b_i|^{\varepsilon}$ choices for δ, so the sum in (8.8) is

$$\ll |b_i b_j| \min\left(|a_i b_j - a_j b_i|^{\varepsilon}, |a_i b_j - a_j b_i|^{\frac{2^k}{k}-2} P^{-a(\frac{2^k}{k}-2-\varepsilon)}\right).$$

This shows (8.8) and hence (8.6). The estimate (8.7) is shown similarly, and this proves the lemma.

The argument can be simplified somewhat by estimating $(Q, a_i b_j - a_j b_i)$ by $|a_i b_j - a_j b_i|$ and omitting consideration of different divisors δ of $|a_i b_j - a_j b_i|$. This simplification yields slightly larger estimates for the sums of (8.6) and (8.7). In (8.6), the exponent of $D_t(f, g)$ rises from $\frac{1}{k} - \frac{2}{2^k}$ to $\frac{1}{k} - \frac{1}{2^k}$ and in (8.7) the exponent of $\Delta(f, g)^{1/(N-1)}$ rises from $\frac{1}{k} - \frac{4}{N}$ to $\frac{1}{k} - \frac{2}{N}$. In Chapter 10, we define the set $\mathfrak{M}'$. When $k \geqslant 5$, the larger estimates are sufficient to show that the Davenport estimates hold in $\mathfrak{M}'$, but when $k = 3$, we need the more efficient estimates of (8.6) and (8.7).

9. Rational approximations in the major arcs

Rational approximations to (α, α') induce rational approximations to the auxiliary variables $\Lambda_1, \ldots, \Lambda_N$ and conversely. For suppose there exist integers A, A', Q such that

$$\alpha = \frac{A}{Q}+\beta, \quad \alpha' = \frac{A'}{Q}+\beta' \quad \text{with } (A, A', Q) = 1.$$

For $i = 1, \ldots, N$, these rational approximations induce rational approximations to

$$\Lambda_i = a_i\alpha+b_i\alpha' = B_i/R_i+\gamma_i,$$

where

$$\frac{B_i}{R_i} = a_i\frac{A}{Q}+b_i\frac{A'}{Q}, \quad (B_i, R_i) = 1 \quad \text{and} \quad \gamma_i = a_i\beta+b_i\beta'.$$

Note that $R_i|Q$ for all i.

Conversely, consider a pair of indices i, j for which $a_ib_j-a_jb_i \neq 0$. Then rational approximations B_i/R_i to Λ_i and B_j/R_j to Λ_j induce simultaneous rational approximations $R(i, j) = \left(\frac{A_{ij}}{Q_{ij}}, \frac{A'_{ij}}{Q_{ij}}\right)$ to (α, α'), where

$$\text{(9.1)} \quad \begin{aligned} \frac{A_{ij}}{Q_{ij}} &= \frac{1}{a_ib_j-a_jb_i}\left(b_j\frac{B_i}{R_i} - b_i\frac{B_j}{R_j}\right), \\ \frac{A'_{ij}}{Q_{ij}} &= \frac{1}{a_ib_j-a_jb_i}\left(-a_j\frac{B_i}{R_i} + a_i\frac{B_j}{R_j}\right), \end{aligned}$$

$(A_{ij}, A'_{ij}, Q_{ij}) = 1$, and

$$\text{(9.2)} \quad \begin{aligned} \beta_{ij} &= \alpha-\frac{A_{ij}}{Q_{ij}} = \frac{1}{a_ib_j-a_jb_i}(b_j\gamma_i-b_i\gamma_j), \\ \beta'_{ij} &= \alpha'-\frac{A'_{ij}}{A_{ij}} = \frac{1}{a_ib_j-a_jb_i}(-a_j\gamma_i+a_i\gamma_j). \end{aligned}$$

Note that $Q_{ij}\big|\,|a_ib_j-a_jb_i|R_iR_j$.

Suppose that the rational approximations to Λ_i and Λ_j satisfy (7.1). Then we say that the induced rational approximation $R(i, j) = \left(\frac{A_{ij}}{Q_{ij}}, \frac{A'_{ij}}{Q_{ij}}\right)$ to (α, α') given by (9.1), is the *rational approximation to* (α, α') *with respect to the pair* i, j. As we vary the pair i, j, we may get different approximations $R(i, j)$ to (α, α').

In this chapter we show that when $k \geq 5$, approximations to points in $\mathfrak{M}$ with respect to all relevant pairs are in fact the same. Re-index the coefficients of the pair of forms f, g so that $|b_1| \geqslant |b_2| \geqslant \ldots \geqslant |b_N|$. Recall $\Delta(f, g) = \prod_{t=1}^{N} D_t(f, g)$. The partition that determines $D_t(f, g)$ is called the *t-th partition*. In these terms, the *relevant pairs* (for all odd k) are defined to be:

(9.3)
1. any pair in the first partition, if $k \geqslant 5$ or if $k = 3$ and $|b_1| > P^{1/2}$,
2. any pair in any partition, if $k = 3$ and $|b_1| \leqslant P^{1/2}$.

For appropriate pairs with a common index, we will show that for $k \geqslant 5$ the rational approximations are the same. Strings of such pairs will show that all relevant pairs yield the same rational approximations.

LEMMA 9.1. *To show that $R(i, j) = R(i, l)$ for a given point (α, α') in $\mathfrak{M}$, it is sufficient to show that*

$$|b_i|\, |b_i b_j b_l|^{1-\frac{1}{2^{k-1}}} (\mu_i)^{-\left(1-\frac{1}{2^{k-1}}\right)} \ll P^{\left(k-\frac{3k}{2^{k-1}}+\frac{3(k-2)\delta}{2^{k-1}}\right)}. \tag{9.4}$$

Proof. Let (α, α') be a point in $\mathfrak{M}$ and consider the approximation $R(i, j)$ and $R(i, l)$ to (α, α'). We have, for example,

$$\alpha = \frac{A_{ij}}{Q_{ij}} + \beta_{ij},$$

where by (9.1),

$$\frac{A_{ij}}{Q_{ij}} = \frac{C_{ij}}{D_{ij} R_i R_j},$$

where C_{ij} is an integer and $D_{ij} = |a_i b_j - a_j b_i|$. Similar equations hold for α' with respect to (i, j) and for both α and α' with respect to (i, l).

Suppose that $R(i, j) \neq R(i, j)$. Then

$$\frac{A_{ij}}{Q_{ij}} \neq \frac{A_{il}}{Q_{il}} \quad \text{or} \quad \frac{A'_{ij}}{Q_{ij}} \neq \frac{A'_{il}}{Q_{il}}.$$

If the former holds,

$$\frac{1}{D_{ij} D_{il} R_i R_j R_l} \leqslant \left|\frac{A_{ij}}{Q_{ij}} - \frac{A_{il}}{Q_{il}}\right| \leqslant |\beta_{ij} + \beta_{il}|,$$

whence

$$1 \leqslant D_{ij} D_{il} R_i R_j R_l (|\beta_{ij}| + |\beta_{il}|). \tag{9.5}$$

To show that $R(i, j) = R(i, l)$, it suffices to show that each term of (9.5) is less than 1/2. Now by (9.2) and (7.1), (9.6) $D_{ij}D_{il}R_iR_jR_l|\beta_{ij}|$

$$\ll D_{il}R_l\left[\left(|b_j|R_jR_i^{1-1/k}(\varkappa_iP)^{-k+\frac{1}{2^{k-1}}}\right)+\left(|b_l|R_lR_j^{1-1/k}(\varkappa_jP)^{-k+\frac{1}{2^{k-1}}}\right)\right].$$

Using the estimate for R_i given in (7.1),

$$D_{il}R_l|b_j|R_jR_i^{1-1/k}(\varkappa_iP)^{-k+1/2^{k-1}}$$

$$\ll\left|\frac{a_i}{b_i}-\frac{a_l}{b_l}\right|(\mu_l)^{-\left(1-\frac{1-\delta}{2^{k-1}}\right)}|b_i|\,|b_ib_jb_l|^{1-\frac{1-\delta}{2^{k-1}}}\left(P^{-k+\frac{3k}{2^{k-1}}-\frac{(3k-1)\delta}{2^{k-1}}}\right).$$

Since all ratios a_i/b_i are $\ll 1$, we have

$$\left|\frac{a_i}{b_i}-\frac{a_l}{b_l}\right|\ll 1.$$

The second term of (9.6) is estimated by an expression similar to that of the first, except that μ_l is replaced by μ_j. If $\mu_j^{-1}\ll 1$, the product

$$\left|\frac{a_i}{b_i}-\frac{a_l}{b_l}\right|(\mu_j)^{-\left(1-\frac{1}{2^{k-1}}\right)}\ll 1.$$

If not, we have $i\neq j$, $l\neq j$, so by Lemma 5.1, part (4), the product is still $\ll 1$. Finally note that

$$|\mu_i^{-1}b_ib_jb_l|^{\delta}\ll P^{4\delta}.$$

By these results,

$$\text{(9.7)}\quad D_{ij}D_{il}R_iR_jR_l|\beta_{ij}|\ll|b_i|\,|b_ib_jb_l|^{1-\frac{1}{2^{k-1}}}(\mu_i)^{-\left(1-\frac{1}{2^{k-1}}\right)}P^{-k+\frac{3k}{2^{k-1}}-\frac{(3k-5)\delta}{2^{k-1}}}.$$

The same estimate holds when $|\beta_{ij}|$ is replaced by $|\beta_{il}|$.

Now suppose that (9.4) holds. Then the right hand side of (9.7) is $\ll P^{-\delta/2^{k-1}}$. Take D of (5.6) to be large compared to δ and we make the left hand side of (9.7) $<\frac{1}{2}$. A similar argument holds when $|\beta_{il}|$ replaces $|\beta_{ij}|$, so (9.5) fails, and $R(i,j)=R(i,l)$ as desired.

If $\dfrac{A'_{ij}}{Q_{ij}}\neq\dfrac{A'_{il}}{Q_{il}}$, a similar argument holds. This shows the lemma.

LEMMA 9.2. *Suppose $k\geqslant 5$ and (α,α') is in $\mathfrak{M}$. Then the rational approximations to (α,α') with respect to all pairs (i,j) with unequal ratios are identical.*

Proof. Let (α,α') be a point in $\mathfrak{M}$. To show the lemma, it is sufficient to show that for all pairs (i,j) and (i,l) with unequal ratios, (9.4) holds, whence $R(i,j)=R(i,l)$. For then if (a,b) and (c,d) are any two different pairs, a differs in ratios from c, say, so that $R(a, b) = R(a, c) = R(c, d)$ and hence all rational approximations to (α, α') are equal. Inequality (9.4) holds for $k \geqslant 5$ if it holds for the worst case, $k = 5$ (with $N = 65$). By Lemma 5.1, parts (2) and (6),

$$\mu_i^{-1} \ll \mu^{-1} \ll P^{4/(N-1)} = P^{1/16}.$$

By (5.11),

$$|b_i|\,|b_i b_j b_l|^{1-\frac{1}{2^{k-1}}} \leqslant |b_i b_j b_l|^{2-\frac{1}{2^{k-1}}} \ll P^{31/8},$$

since the three coefficients are distinct. With these estimates, (9.4) holds, and hence so does the lemma.

Lemma 9.2 tells us that for $k \geqslant 5$, the rational approximations to (α, α') in $\mathfrak{M}$ with respect to all relevant pairs (as defined by (9.3)) are identical. Hence these approximations all have the same denominator. We need this fact in the following chapter.

10. Modifying the major arcs

Recall

$$N(P) = \iint_{\mathfrak{M}} \prod_{i=1}^{N} T_i(A_i)\,d\alpha\,d\alpha' + O(CP^{n-2k-\sigma}). \tag{7.7}$$

In this chapter we show that the equation holds with $\mathfrak{M}$ replaced by a much smaller set in $\mathfrak{M}'$. We do this by showing that the integral over the difference of the two sets contributes at most to the error term.

If (α, α') is a point $\mathfrak{M}$, let Q_{ij} be the denominator of the rational approximation to (α, α') with respect to the pair (i, j). By Lemma 9.2, for $k \geqslant 5$ the Q_{ij} for all the relevant pairs (i, j) of (9.3) in fact have a common value, which we denote Q. Let $\mathfrak{M}_\omega$ be the set of points (α, α') in $\mathfrak{M}$ for which $Q \leqslant P^\omega$.

To define ω, we define

$$\prod_{i=2}^{N} |b_i| = P^r, \qquad D_1(f, g) = P^s$$

and recall

$$\mu^{-1} = P^e. \tag{5.13}$$

The exponents have the intervals

$$\begin{cases} 0 \leqslant r \leqslant 2, \\ 0 \leqslant s \leqslant r, \\ 0 \leqslant e \leqslant 4/(N-1), \end{cases}$$

where the interval for e is from Lemma 5.1, parts (3) and (6). Define

$$\omega = \begin{cases} \dfrac{\frac{2}{2^k}r+\left(\frac{1}{k}-\frac{3}{2^k}\right)s-\left(1-\frac{1}{2^k}\right)e}{\frac{2^k}{k}-2}+\theta, & \text{if } k \geqslant 5, \\ \frac{3}{8}r-e+\theta, & \text{if } k=3 \text{ and } |b_1| > P^{1/2}, \\ \frac{4}{7}-e+\theta, & \text{if } k=3 \text{ and } |b_1| \leqslant P^{1/2}, \end{cases} \tag{10.1}$$

where θ is the constant of the Theorem. Definition (10.1) gives the smallest convenient choice of ω for which we can prove

LEMMA 10.1. *If k is odd and $\geqslant 3$,*

$$\iint\limits_{\mathfrak{M}\sim\mathfrak{M}_\omega} \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha' \ll CP^{N-2k-\sigma}.$$

Proof. By (5.15) it is sufficient to show that

$$\iint\limits_{\mathfrak{M}\sim\mathfrak{M}_\omega} \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha' \ll \Big(\prod_{i=1}^{N} \varkappa_i\Big) P^{N-2k+e-\sigma-\varepsilon} \tag{10.2}$$

where σ and ε are functions of k and θ alone.

For $k \geqslant 5$, the relevant pairs are those of the first partition. For these pairs, the denominators of the rational approximations are all equal. To estimate the integral of (10.2), we first estimate $T_1(\Lambda_1)$ trivially by $\varkappa_1 P$. By Lemma 8.1, the Davenport estimated holds in, $\mathfrak{M}$, so we apply (6.6) to $T_i(\Lambda_i)$ for $i \neq 1$. We then extend the area of integration to the whole plane and use (8.2) of Lemma 8.2 and finally sum over A, A', and Q using (8.6) of Lemma 8.4. The integral over $\mathfrak{M} - \mathfrak{M}_\omega$ is then

$$\ll \Big(\prod_{i=1}^{N} \varkappa_i\Big) P^{\Omega}$$

where

$$\Omega = N-2k-\omega\left(\frac{2^k}{k}-2-\varepsilon\right)+\frac{1}{2^k}\left(e+2r+\left(\frac{2^k}{k}-3\right)s\right).$$

The definition of ω insures that the estimate satisfies (10.2), provided we take σ and ε small compared to θ.

There are added difficulties for the case of $k = 3$. I indicate the line of argument used in Toliver (1975), Chapter 10, but omit the proof because of the length and special detail it requires.

We use the relevant pairs of (9.3), but for each partition, we pair the indices so as to minimize the size of $|b_i b_j|$ for each pair (i, j). Let $\mathfrak{M}_\Omega$ (W in the thesis) be those (α, α') in $\mathfrak{M}$ such that $Q_{ij} \leqslant P^{\Omega}$ for all relevant pairs (i, j). Here Ω is a power of P considerably larger than ω. In $\mathfrak{M}_\Omega$, the rational approximations to (α, α') with respect to all relevant pairs are shown to be the same. Hence, by the argument used for $k \geqslant 5$, the lemma holds with $\mathfrak{M}$ replaced by $\mathfrak{M}_\Omega$. There remains the integral over $\mathfrak{M}-\mathfrak{M}_\Omega$. For this integral, we show that we can pare $\mathfrak{M}-\mathfrak{M}_\Omega$ into an absolutely bounded number of parings, each of which contributes only to the error term of (5.8). This proves the lemma.

In the proof of Lemma 10.1 for $k \geqslant 5$, it is critical that for all relevant pairs, the denominators of the rational approximations are the same. For if the denominators varied from pair to pair, a rational approximation with a large denominator would exclude the point from $\mathfrak{M}_\omega$, while a rational approximation with a small denominator would exclude the point from the argument which shows (10.2). This argument depends on Lemma 8.4, which requires large denominators to show that the integral over $\mathfrak{M} - \mathfrak{M}_\omega$ contributes only to the error term. The lack of control over the size of the denominators required the lengthy argument for $k = 3$.

In Toliver (1977), it is shown that we can replace the argument for $k = 3$ with a much shorter combinatoric one. Here I used the same relevant pairs and partitions as before, but in (9.3), the two cases of relevant pairs depended on $|b_1|$ being larger or smaller than $P^{5/14}$ instead of $P^{1/2}$. I showed that in all of $\mathfrak{M}$, the rational approximations with respect to all relevant pairs are equal. Lemma 9.1 is the departure point for this argument. Then Lemma 10.1 can be proved directly for $k = 3$, using the same argument as for $k \geqslant 5$.

We now modify the shape of the arcs, but retain the same rational points. In particular, let

$$\mathfrak{M}' = \bigcup \mathfrak{M}'(Q, A, A'),$$

where the union is over all Q, A, and A' satisfying

$$0 \leqslant A,\, A' < Q, \quad (Q, A, A') = 1, \quad 1 \leqslant Q \leqslant P^{\omega}. \tag{10.3}$$

The typical interval $\mathfrak{M}'(Q, A, A')$ is the set of (α, α') satisfying

$$\left|\alpha - \frac{A}{Q}\right| \leqslant P^{-k+\tau}, \qquad \left|\alpha' - \frac{A'}{Q}\right| \leqslant P^{-k+\tau}.$$

Here

$$\tau = \frac{(2^k+1)r}{(2^k-1)2^k} + \theta.$$

Thus the set $\mathfrak{M}'$ is determined by rational approximations to the primary variables α and α', whereas $\mathfrak{M}$ and hence $\mathfrak{M}_\omega$ are defined in terms of rational approximations to the auxiliary variables $\Lambda_i = a_i\alpha + b_i\alpha'$, $i = 1, \ldots, N$.

LEMMA 10.2. *If θ is small, then*

(1) *the intervals in $\mathfrak{M}'$ are disjoint.*

(2) *Furthermore, suppose (α, α') is in $\mathfrak{M}'(Q, A, A')$. If for $i = 1, \ldots, N$, we put*

$$\Lambda_i = a_i\alpha + b_i\alpha', \qquad \frac{B_i}{R_i} = a_i\frac{A}{Q} + b_i\frac{A'}{Q},$$

with $(B_i, R_i) = 1$, and if we put $\gamma_i = \Lambda_i - (B_i/R_i)$ then the Davenport estimates (6.5) *and* (6.6) *hold for $T_i(\Lambda_i)$.*

Proof. For part (1), if (α, α') is in the intersection of two arcs of $\mathfrak{M}'$, then it is within $P^{-k+\tau}$ of two different rational approximations of the type (10.3). But this is impossible if $D \ll 1$ is taken to be large and θ is small.

For part (2), the rational approximation to Λ_i satisfies $R_i \ll P^{\omega}$, $|\gamma_i| \ll |b_i| P^{-k+\tau}$. Comparing this to Lemma 6.5, the Davenport inequalities hold if

$$\omega + (e_1 + e)\frac{1-\delta}{k} \leqslant 1 - \delta \tag{10.4}$$

and

$$\omega + \tau + e_1\frac{1-\delta}{k} \leqslant 1 - \delta. \tag{10.5}$$

Using $e_1 + r \leqslant 2$, the inequalities hold if θ and δ are small.

In proving part (2) of Lemma 10.2 for $k = 3$, we need an estimate for ω which is smaller than $\frac{3}{8}r - e + \theta$ when $|b_1|$ is small (r large). Thus we have two cases for $k = 3$, depending on the size of $|b_1|$. We also took extra care in proving Lemma 8.4 so that smaller values of ω could satisfy both Lemma 10.1 and equations (10.4) and (10.5).

LEMMA 10.3. *The contribution of $\mathfrak{M}_\omega \sim \mathfrak{M}'$ to $\iint \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha'$ is $\ll CP^{N-2k-\sigma}$.*

Proof. Let $\mathfrak{M}_\omega(Q, A, A')$ be those (α, α') in $\mathfrak{M}_\omega$ whose rational approximation with respect to all pairs in the first partition is $\left(\frac{A}{Q}, \frac{A'}{Q}\right)$. Then

$$\iint\limits_{\mathfrak{M}_\omega \sim \mathfrak{M}'} \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha'$$

$$\ll \sum_{1 \leqslant Q \leqslant P^{\omega}} \; \sum_{A, A' \in \mathfrak{A}_Q} \; \iint\limits_{\mathfrak{M}_\omega(Q,A,A') \sim \mathfrak{M}'(Q,A,A')} \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha'.$$

As in Lemma 10.1, estimate $T_i(\Lambda_i)$ trivially for $i = 1$ and by the Davenport estimate (6.6) for $i \neq 1$. Extend the area of integration to $\mathcal{T}$ and use (8.4) of Lemma 8.3, extend the sum over Q to all $Q \geqslant 1$ and sum over A, A', and Q using (8.6) of Lemma 8.4. The integral is then

$$\ll \left(\prod_{i=1}^{N} k_i\right) P^{\Omega},$$

where $\Omega = N-2k+e+\left(1+\dfrac{1}{2^k}\right)r-\tau(2^k-1)$.

We have $\Omega \leqslant N-2k+e-\sigma-\varepsilon$ when σ and ε are small, so by (5.15) the lemma is shown.

We can now replace $\mathfrak{M}$ by $\mathfrak{M}'$ in

$$N(P) = \iint_{\mathfrak{M}} \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha' + O(CP^{N-2k-\sigma}). \tag{7.7}$$

For let I denote the unit square $[0, 1) \times [0, 1)$. By Chapter 7, Lemma 10.1, and Lemma 10.3, $\mathfrak{m} = I \sim \mathfrak{M}$, $\mathfrak{M} \sim \mathfrak{M}_\omega$, and $\mathfrak{M}_\omega^{\cdot} \sim \mathfrak{M}'$, respectively, contribute only a permissable error to the integral. Now $\mathfrak{M}'$ is generally not a subset of $\mathfrak{M}_\omega$, so we can at most replace $\mathfrak{M}$ by $\mathfrak{M}' \cap \mathfrak{M}_\omega$ in (7.7). However, $I \sim \mathfrak{M}_\omega$ and hence its subset $\mathfrak{M}' \sim \mathfrak{M}_\omega$ contribute only an error to the integral so we can now replace $\mathfrak{M}' \cap \mathfrak{M}_\omega$ by $\mathfrak{M}'$ in (7.7). Finally, by Lemma 10.2, the intervals in $\mathfrak{M}'$ are disjoint, so we can write

$$N(P) = \sum_{1 \leqslant Q \leqslant P^{\omega}} \sum_{A,A' \in \mathfrak{A}_Q} \iint_{\mathfrak{M}'(Q,A,A')} \prod_{i=1}^{N} T_i(\Lambda_i)\,d\alpha\,d\alpha' + O(CP^{N-2k-\sigma}). \tag{10.6}$$

11. The asymptotic formula

In this chapter we show the asymptotic formula

$$N(P) = CP^{N-2k}(1+O(P^{-\sigma})). \tag{5.8}$$

Recall the exponential sum $S(B, R)$ from Chapter 3 and the integral $I(\gamma)$ from Lemma 6.4. We have

LEMMA 11.1. *Let θ be a sufficiently small positive number. Let*

$$\mathfrak{S}(P^{\omega}) = \sum_{1 \leqslant Q \leqslant P^{\omega}} \sum_{(A,A') \in \mathfrak{A}_Q} \prod_{i=1}^{N} R_i^{-1} S(B_i, R_i), \tag{11.1}$$

and let

$$I(P^{-k+\varepsilon}) = \iint \prod_{i=1}^{N} I_i(\gamma_i)\,d\beta\,d\beta',$$

where the area of integration is

$$|\beta|, |\beta'| \leqslant P^{-k+\tau}.$$

Then

$$(11.2)\qquad N(P) = \sum_{1 \leqslant Q \leqslant P^{\omega}} \sum_{(A,A') \in \mathfrak{A}_Q}' \iint_{\mathfrak{M}'(Q,A,A')} \prod_{i=1}^{N} T_i(A_i)\, d\alpha\, d\alpha'$$

$$= \mathfrak{S}(P^{\omega}) I(P^{-k+\tau}) + O(CP^{N-2k-\sigma}).$$

Proof. By Lemma 10.2, the Davenport estimate (6.5) holds in $\mathfrak{M}'$. Applying this estimate to $\prod_{i=1}^{N} T_i(A_i)$, we get a main product $\prod_{i=1}^{N} R_i^{-1} S(B_i, R_i) \times I_i(\gamma_i)$, which when integrated and summed yields the main term of (11.2). The remaining error products are each a product of one or more error terms and the complementary set of main terms.

By Lemmas 3.2 and 6.4, a main term has the estimate

$$(11.3)\qquad R_i^{-1} S(B_i, R_i) I_i(\gamma_i) \ll R_i^{-1/k} \min\left(\varkappa_i P, (\varkappa_i P)^{-k+1} |\gamma_i|^{-1}\right).$$

In $\mathfrak{M}'$,

$$1 \leqslant R_i \ll (\varkappa_i P)^{1-\delta} \quad \text{and} \quad |\gamma_i| \ll R_i^{-1} (\varkappa_i P)^{-k+1-\delta},$$

so if ε is small compared to δ, the error term $R_i^{1-1/k+\varepsilon}$ is truly smaller than the right side of (11.3). So the sum of the error products can be majorized by a product with only one error term.

For each $t = 1, \ldots, N$, let $S(t)$ be the set of (α, α') in $\mathfrak{M}'$ such that the product

$$R_t^{1-1/k+\varepsilon} \prod_{i \neq t} R_i^{-1/k} \min\left(\varkappa_i P, (\varkappa_i P)^{-k+1} |\gamma_i|^{-1}\right),$$

majorizes the sum of the error products. By argument as in Lemma 10.1, the integral over $S(t)$, for each t, contributes only to the error term. So the lemma is shown.

We now give estimates for $\mathfrak{S}(P^{\omega})$ and $I(P^{-k+\tau})$.

LEMMA 11.2. *Let* $\mathfrak{S}(P^{\omega})$ *be defined by* (11.1). *Then* $\mathfrak{S}(P^{\omega}) = \mathfrak{S} + O\left(\Delta(f, g)^{-\varepsilon} P^{-\sigma}\right)$, *where*

$$\mathfrak{S} = \sum_{Q=1}^{\infty} \sum_{(A,A') \in \mathfrak{A}_Q} Q^{-N} S_0(A, A', Q)$$

with

$$S_0(A, A', Q) = \sum_{x_1=1}^{Q} \cdots \sum_{x_N=1}^{Q} e\left(\frac{Af(x) + A'g(x)}{Q}\right).$$

Also, $\mathfrak{S}$ is a positive number $\gg \Delta(f, g)^{-\varepsilon}$, where ε is a small number which depends on k and θ.

Proof. Put

$$\mathfrak{S} = \sum_{Q=1}^{\infty} \sum_{(A,A')\in\mathfrak{A}_Q} \prod_{i=1}^{N} R_i^{-1} S(B_i, R_i).$$

This is equivalent to the form given in the lemma. By Lemmas 3.2 and 8.4 the sum $\mathfrak{S}$ is absolutely convergent. Now

$$\mathfrak{S}(P^{\omega}) = \mathfrak{S} - \sum_{Q>P^{\omega}} \sum_{(A,A')\in\mathfrak{A}_Q} \prod_{i=1}^{N} R_i^{-1} S(B_i, R_i).$$

By Lemma 3.2, the sum of the right is

$$(11.5) \qquad \ll \sum_{Q>P^{\omega}} \sum_{A,A'} \prod_{i=1}^{N} R_i^{-1/k} \leqslant \sum_{Q>P^{\omega}} \sum_{A,A'} \prod_{i\neq 1} R_i^{-1/k}.$$

From (5.16) we can show

$$(11.6) \qquad \frac{D_t(f, g)}{\prod_{i\neq t} |b_i|} \ll P^{-e\cdot\frac{N-3}{2}} = P^{-(2^k-1)e}$$

and hence

$$(11.7) \qquad \frac{\Delta(f, g)}{\prod_{i=1}^{N} |b_i|^{N-1}} \ll P^{-N(2^k-1)e}.$$

If $k \geqslant 5$ or if $k = 3$ and $e_1 > \frac{1}{2}$, use the right-hand estimate of (11.5), (8.6) of Lemma 8.4, and (11.6) to show that the sum over $Q > P^{\omega}$ is $\ll \Delta(f, g)^{-\varepsilon} P^{-\sigma}$. The other case is similar, using (11.7). So

$$\mathfrak{S}(P^{\omega}) = \mathfrak{S} + O(\Delta(f, g)^{-\varepsilon} P^{-\sigma}),$$

where ε and the implied constant depend on k and θ.

Finally, by standard arguments (for example, Davenport (1962), Lemma 7),

$$\mathfrak{S} = \prod_{p} \chi(p).$$

By Lemma 3.1, $\mathfrak{S}$ is a non-negative real number. By (3.1) it is $\gg \Delta(f, g)^{-\varepsilon}$, so is non-zero. Hence $\mathfrak{S}$ is a positive quantity $\gg \Delta(f, g)^{-\varepsilon}$. This shows the lemma.

LEMMA 11.3 *$I(P^{-k+\tau}) = C_0 P^{N-2k}(1+O(P^{-\sigma}))$, where C_0 is a positive number such that*

$$(11.8) \qquad C_0 \gg \frac{(\xi_1 \xi_2)^{-1}}{|a_1 b_2 - a_2 b_1|} \prod_{i=1}^{N} \xi_i^{1/k}.$$

Proof. The integral

$$I(P^{-k+\tau}) = \iint \prod_{i=1}^{N} I_i(\gamma_i)d\beta d\beta'$$

is over $|\beta|, |\beta'| \leqslant P^{-k+\tau}$. We introduce at most an error by extending the integral to the entire plane, by Lemmas 6.4 and 8.3, so we can replace the integral by

$$\lim_{\varphi\to\infty} \int_{-\varphi}^{\varphi} \int_{-\varphi}^{\varphi} \prod_{i=1}^{N} I_i(\gamma_i)d\beta d\beta'.$$

To simplify notation, let f, g again have the indexing chosen in Chapter 5. In particular, $a_1 b_2 - a_2 b_1 \neq 0$ and $\mu = \mu_1 \mu_2$. The Fourier integral theorem can be used, as in the argument of Davenport and Lewis (1969a), Lemma 30, to show that the limit can be written as $P^{N-2k}V$, where

$$V = \frac{1}{|a_1 b_2 - a_2 b_1|} \int (y_1 \ldots y_N)^{-1+1/k} dy_3 \ldots dy_N.$$

The integral is over $(y_1, \ldots, y_N)$ for which

(11.9)
$$(\varkappa_i)^k \leqslant y_i \leqslant (\varkappa_i')^k \quad \text{for } i = 3, \ldots, N, \quad \text{and}$$
$$\sum_{i=1}^{N} a_i y_i = 0, \qquad \sum_{i=1}^{N} b_i y_i = 0.$$

To estimate V, note that any choice of y_i, $i = 3, \ldots, N$, in the range (11.9) determines a positive value for y_1, y_2 which also satisfies (11.9). So $y_i \ll \xi_i$ for $i = 1, 2$. Use this estimate to factor y_1, y_2 out of the integral. Integrate the remaining integral to show that V is a positive number satisfying

$$V \gg \frac{(\xi_1 \xi_2)^{-1}}{|a_1 b_2 - a_2 b_1|} \prod_{i=1}^{N} \xi_i^{1/k}.$$

Put $C_0 = V$ and the lemma is shown.

We can now prove the asymptotic formula and hence the Theorem. Recall that by Lemma 4.7, it is sufficient for the proof of the Theorem to show the following:

Each simply normalized pair f, g which satisfies

(3.1)
$$\prod_p \chi(p) \gg \Delta(f, g)^{-\varepsilon}$$

has a solution $\boldsymbol{x}$ to $f = g = 0$ which satisfies the bound

(4.14)
$$\left.\begin{array}{l} \sum_{i=1}^{N} |a_i x_i^k| \\ \sum_{i=1}^{N} |b_i x_i^k| \end{array}\right\} \ll \prod_{i=1}^{N} |b_i|^{k/2+\theta}.$$

So let f, g be a simply normalized pair which satisfies (3.1). Recall that $N(P)$ is the number of solutions $\boldsymbol{x}$ to $f = g = 0$ with

$$\varkappa_i P \leqslant x_i \leqslant \varkappa_i' P \quad (i = 1, \ldots, N). \tag{11.10}$$

Here $P = D \prod_{i=1}^{N} |b_i|^{1/2+\theta}$, where θ is a small positive constant and D is a constant which depends only on k and θ. We need to show the asymptotic formula

$$N(P) = CP^{N-2k}(1+O(P^{-\sigma})), \tag{5.8}$$

where the implied constant and σ depend only on k and θ, and where C is a positive number such that

$$C \gg \frac{(\xi_1\xi_2)^{-1}}{|a_1b_2-a_2b_1|} \prod_{i=1}^{N} \xi_i^{1/k} \Delta(f, g)^{-\varepsilon}. \tag{5.9}$$

By Lemma 11.1 $N(P) = \mathfrak{S}(P^{\omega}) I(P^{-k+\tau}) + O(CP^{N-2k-\sigma})$. By Lemma 11.2, $\mathfrak{S}(P^{\omega}) = \mathfrak{S} + O(\Delta(f, g)^{-\varepsilon} P^{-\sigma})$, where $\mathfrak{S} = \prod_p \chi(p)$. By (3.1), $\mathfrak{S}$ is a positive number $\gg \Delta(f, g)^{-\varepsilon}$. By Lemma 11.3, $I(P^{-k+\tau}) = C_0 P^{N-2k}(1+O(P^{-\sigma}))$, where C_0 is a positive number which satisfies (11.8). Put $C = C_0\mathfrak{S}$. Then C is a positive number which satisfies (5.9) and hence also (5.14) and (5.15). The analytic argument then holds for this C. So by Lemmas 11.1, 11.2, and 11.3,

$$\begin{aligned} N(P) &= \mathfrak{S} C_0 P^{N-2k} + O(CP^{N-2k-\sigma}) \\ &= CP^{N-2k}(1+O(P^{-\sigma})), \end{aligned}$$

and the asymptotic formula is shown.

On taking D large, but still dependent on k and θ, we have $N(P) > 0$. Hence every simply normalized pair f, g which satisfies (3.1) has a solution in the range (11.10). The definitions of the k_i and P guarantee that this solution satisfies the bound (4.14). Thus the sufficient condition of Lemma 4.7 is met and the Theorem is shown.

References

Birch, B. J. (1957), *Homogeneous forms of odd degree in a large number of variables*, Mathematika 4, 102–105.

Cook, R. J. (1971a), *Simultaneous solutions of additive Diophantine equations and inequalities*, Ph. D. thesis, London.

Cook, R. J. (1971b), *Simultaneous quadratic equations*, J. London Math. Soc. (2) 4, 319–326.

Cook, R. J. (1972a), *A note on a lemma of Hua*, Quart. J. Math. (Oxford) (2) 23, 287–288.

Cook, R. J. (1972b), *Pairs of additive equations*, Michigan Math. J. 19, 325–331.

Davenport, H. (1939a), *On Waring's Problem for cubes*, Acta Math. 71, 123–143.

Davenport, H. (1939b), *On Waring's Problem for fourth powers*, Ann. Math. 40, 731–747.

Davenport, H. (1962), *Analytic methods for Diophantine equations and inequalities*, Campus Publishers, Ann Arbor.

Davenport, H. (1977), *The Collected Works of Harold Davenport*, Vol. III ed. B. J. Birch, H. Halberstam and C. A. Rogers, Academic Press, London.

Davenport, H. and Heilbornn, H. (1936), *On an exponential sum*, Proc. London Math. Soc. (2) 41, 449–453.

Davenport, H. and Heilbronn, H. (1937), *On Waring's Problem: Two cubes and one square*, Proc. London Math. Soc. (2) 43, 73–104.

Davenport, H. and Lewis, D. J. (1963), *Homogeneous additive equations*, Proc. Roy. Soc. A274, 443–460.

Davenport, H. and Lewis, D. J. (1966), *Cubic equations of additive type*, Phil. Trans. Roy. Soc. A261, 97–136.

Davenport, H. and Lewis, D. J. (1969a), *Simultaneous equations of additive type*, Phil. Trans. Roy. Soc. A264, 557–595.

Davenport, H. and Lewis, D. J. (1969b), *Two additive equations*, Amer. Math. Soc. Proc., Symposia in Pure Mathematics 12, 74–98.

Lewis, D. J. (1957), *Cubic congruences*, Michigan Math. J. 4, 85–95

Lewis, D. J. (1970), *Systems of Diophantine equations*, Instituto Nazionale di Alta Matematica, Symposia Mathematica 4, 33–43.

Lloyd, D. P. (1975), *Bounds for solutions of Diophantine equations*, Ph. D. thesis, University of Adelaide.

Pitman, Jane (1971), *Bounds for solutions of diagonal equations*, Acta Arith. 19, 223–247.

Pitman, Jane (1981), *Pairs of diagonal inequalities*, Recent Progress in analytic number theory, Vol. 2 (Durham, 1979), pp. 183–215, Academic Press, London.

Pitman, Jane and Ridout, D. (1967), *Diophantine cubic equations and inequalities*, Proc. Roy. Soc. A297, 476–502.

Ponnudurai, T. (1979), *Diophantine inequalities in many variables*, Ph. D. thesis, University of Adelaide.

Schmidt, W. M. (1980), *Diophantine inequalities for forms of odd degree*, Adv. in Math. 33, no 2, 128–151.

Toliver, R. H. (1975), *Bounds for solutions of two simultaneous additive equations of odd degree*, Ph. D. thesis, The University of Michigan.

Toliver, R. H. (1977), *A note on two additive cubic equations*, unpublished manuscript.

Vaughan, R. C. (1977), *On pairs of additive cubic equations*, Proc. London Math. Soc. Ser. (3) 34, 354–364.

Vaughan, R. C. (1981), *The Hardy–Littlewood Method*, Cambridge University Press.

DEPARMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
U.S.A