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DENSITY THEOREMS FOR EXCEPTIONAL EIGENVALUES OF THE LAPLACIAN FOR CONGRUENCE GROUPS

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1. Introduction

Let Γ be a congruence subgroup of the full modular group $\mathrm{SL}(2, \mathbf{Z})$ and let $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ denote the non-euclidean Laplacian acting on $L^2(\Gamma \backslash H)$ – the space of Γ-automorphic functions on the upper half-plane $H = \{z: z = x+iy,\ y > 0\}$, square-integrable on the fundamental domain $F = \Gamma \backslash H$ with respect to the invariant measure $dz = y^{-2}dx\,dy$. In the spectral theory of Maass and Selberg it is known that Δ has a point spectrum

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \qquad \lambda_j \to \infty \text{ as } j \to \infty$$

and has continuous spectrum consisting of points in $[1/4, \infty)$ with finite multiplicity equal to the maximal number of Γ-inequivalent cusps. The celebrated conjecture of A. Selberg [10] asserts that positive eigenvalues lie on the continuous spectrum. The conjecture has been proved only for the full modular group and for its few subgroups with a small index, cf. [4], [6] and [7].

Here the situation looks like the one familiar in the theory of Dirichlet's L-series $L(s, \chi)$, for which the non-existence of zeros on the segment $[1/2, 1]$ can be established for any character χ with a small modulus q by means of numerical computations. We should emphasize, however, that the problem concerning eigenvalues $0 < \lambda_j < 1/4$, called exceptional, is much more involved. The approaches of Roelcke, Maass and especially of Huxley are essentially geometrical, to mention just one argument, for example the isoperimetric inequality on the hyperbolic plane.

There have been various unsuccessful attempts to prove Selberg's eigenvalue conjecture by different ideas. The methods of exponential sums seem to be quite promising.

Yet, it was known to Selberg [10] that the problem can be reduced to bounding certain sums of Kloosterman sums associated with cusp of the group Γ. Estimates of A. Weil for individual terms led Selberg to the lower bound

$$\lambda_1 \geqslant 3/16, \tag{1}$$

and any sharper result means that there exists a regular variation of sign of Kloosterman sums. Although that variation is difficult to establish in general, Selberg's observation gives at least convenient objects for numerical computations. Such ideas have already been materialized by N. V. Kuznetsov [5], who found an arithmetic form of Selberg's trace formula in which the traditional weighted sum of norms of primitive hyperbolic classes is replaced by a certain sum of L-series at $s = 1$ with real characters. As an application, V. V. Golovtchanskii and M. N. Smitrov [3] calculated with high precision a few eigenvalues for the modular group.

Apparently, as the level of the group Γ gets large, there may exist a great number of exceptional eigenvalues. J.-M. Deshouillers and H. Iwaniec [2] have studied the case of Hecke congruence group

$$\Gamma = \Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, Z);\ c \equiv 0 \pmod{q} \right\}$$

and established various results of statistical nature which suggest that the exceptional eigenvalues occur very rarely. The results of [2] resemble weighted large sieve inequalities for Dirichlet's characters, the rôle of characters being played by Fourier coefficients of cusp forms.

In this paper we refine the methods of [2] to prove density theorems for exceptional eigenvalues alone, that is, not weighted by Fourier coefficients whose order of magnitude is not known precisely.

2. Statement of results

Before stating the theorems let us recall the Weyl–Selberg formula, cf. [11], which gives us an insight into the topics.

PROPOSITION 1. *Let $N(\lambda, \Gamma)$ denote the number of all eigenvalues $\lambda_j \leqslant \lambda$ counted with their multiplicites. Then, as $\lambda \to \infty$, we have*

$$N(\lambda, \Gamma) \sim \frac{|F|}{4\pi} \lambda$$

where $|F|$ is the volume of a fundamental domain; in case of the Hecke group Γ

$= \Gamma_0(q)$ *we have, say*

$$|F| = \frac{\pi}{3} q \prod_{p|q} \left(1+\frac{1}{p}\right) = \frac{\pi}{3} \nu(q).$$

This formula has the defect of being not uniform in q as it grows with λ. Nevertheless, it indicates that intervals of a fixed length δ might contain $\gg |F|$ eigenvalues, as $q \to \infty$. We believe this is indeed true whenever the interval is contained in the continuous spectrum $[1/4, \infty)$.

Let $\lambda_j = 1/4 - t_j^2$ with $0 < t_j < 1/4$ be exceptional eigenvalues of $\Gamma_0(q)$. For any σ with $0 < \sigma < 1/2$ denote

$$N(\sigma, q) = \# \{\lambda_j;\ 2t_j > \sigma\}.$$

Our first result is

THEOREM 1. *For any $\varepsilon > 0$ we have*

$$N(\sigma, q) \ll q^{1-\sigma+\varepsilon}, \tag{2}$$

the constant implied in $\ll$ depending on ε alone.

This theorem is equivalent to (by partial summation)

THEOREM 1*. *For any $\varepsilon > 0$ we have*

$$\sum_{\lambda_j-\text{except}} q^{2t_j} \ll q^{1+\varepsilon}. \tag{3}$$

Professor M. N. Huxley has informed us (on the telephone) that he has proved the same inequality independently by an appeal to Selberg's trace formula. Our arguments are based on the Kuznetsov trace formula and A. Weil's estimate for Kloosterman sums in the spirit of [2].

THEOREM 2. *For any $\varepsilon > 0$ we have*

$$\sum_{\lambda_j-\text{except}} q^{At_j} \ll q^{1+\varepsilon} \tag{4}$$

with $A = 24/11$, the constant implied in $\ll$ depending on ε alone.

Our proof is a continuation of that of Theorem 1*; the extra arguments are:

A. Weil's estimate for hybrid sums involving a real multiplicative character, an additive character and a quadratic polynomial; precisely, we need the following

LEMMA 1. *Let $d \geq 3$ be an even square-free number, $f(x) = (\alpha x)^2 + \beta x + \gamma$ with $(\alpha, \beta, d) = 1$, $f(x)$ (mod p) not being a square for any $p|d$. We then have*

$$\left|\sum_{x(\bmod d)} \left(\frac{f(x)}{d}\right) e\left(h\frac{x}{d}\right)\right| \leq d^{1/2}\tau(d). \tag{5}$$

The proof will be given in Appendix.

Moreover, we need the famous estimate of D. Burgess [1] for incomplete character sums.

LEMMA 2. *If D is not a perfect square, then*

$$\sum_{1 \leqslant r \leqslant R} \left(\frac{D}{r}\right) \ll R^{1/2} |D|^{3/16+\varepsilon}, \tag{6}$$

the constant implied in $\ll$ depending on ε alone.

It is conjectured that Lemma 2 is true with exponent 0 in place of 3/16. This, indeed, is equivalent to the Lindelöf conjecture for Dirichlet's L-series in D aspect, namely that

$$|L(\tfrac{1}{2}+it, \chi)| \ll D^{\varepsilon}$$

for any character χ (mod D), where the constant implied in $\ll$ may depend on ε and t. From the Lindelöf conjecture Theorem 2 would follow with $A = 3$. Further improvement on the constant A requires sharper estimates for character sums of the type

$$\sum_{1 \leqslant r \leqslant R} \sum_{1 \leqslant a \leqslant A} \left(\frac{a^2-4}{r}\right) \ll (A+R)(AR)^{\varepsilon},$$

which might be true due to variation of sign of a single sum. From an estimate essentially like the one above we could infer the following

DENSITY CONJECTURE. *Theorem 2 holds with $A = 4$.*

It will be seen later that the quality of our density theorems depends on the order of magnitude of the Fourier coefficients of Maass cusp forms. Let $\Gamma = \Gamma_0(q)$ and $u_j(z)$ be a normalized Maass cusp form whose Δ-eigenvalue is $\lambda_j = 1/4 - t_j^2$ and whose Fourier expansion at the cusp ∞ is

$$u_j(z) = \sqrt{y} \sum_{n \neq 0} \varrho_j(n) K_{t_j}(2\pi |n| y) e(nx).$$

It is very likely that most $\varrho_j(n)$ are of order of magnitude $|F|^{-1/2}$. For the purpose of this paper a lower bound for $|\varrho_j(n)|$ is needed. This problem is rather difficult and interesting in itself. By Rankin's method one can prove that

$$\sum_{n \leqslant N} |\varrho_j(n)|^2 \sim c_j \nu^{-1}(q) N$$

as $N \to \infty$ where $c_j = (12/\pi^2) \cos(\pi t_j) > 4/5$, and one can deduce the lower bound

$$\sum_{n \leqslant N} |\varrho_j(n)|^2 \gg \nu^{-1}(q) N \tag{7}$$

if $N \geqslant N_0(q)$, the constant implied in $\ll$ being absolute. The smaller N is admitted the sharper density theorems follow. In this paper we have succeeded to show (7) for $N \geqslant N_0 = O(\nu(q))$, which yields Theorem 2 with $A = 2$. Any result of the kind of $N_0(q) = O(q^\theta)$ with $0 < \theta < 1$ seems to be difficult, and it would yield the constant $A = 2(2-\theta)$. Therefore, the density conjecture is a consequence of another conjecture that (7) holds for $N \geqslant q^\varepsilon$.

For simplicity only, in this paper we deal with q-prime; the general case needs an elementary modification.

3. An application of the Kuznetsov trace formula

The crucial point in what follows is the application of the Kuznetsov trace formula, cf. [2], for the diagonal terms

$$m = n, \qquad \mathfrak{a} = \mathfrak{b}$$

and for a test function $\varphi(X)$ whose graph is

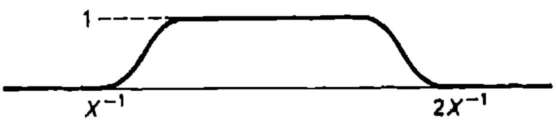

Fig. 1

with $X \geqslant 2$ to be chosen later. We shall appeal to several estimates given in [2], so it is convenient to adopt the same notation. Let us put

$$V_0(\mathfrak{a}, n) = \sum_{0<\lambda_j<1/4} \frac{\hat{\varphi}(\varkappa_j)}{\operatorname{ch} \pi\varkappa_j} |\varrho_{j\mathfrak{a}}(n)|^2,$$

$$V_1(\mathfrak{a}, n) = \sum_{\lambda_j \geqslant 1/4} \frac{\hat{\varphi}(\varkappa_j)}{\operatorname{ch} \pi\varkappa_j} |\varrho_{j\mathfrak{a}}(n)|^2,$$

$$V_2(\mathfrak{a}, n) = \frac{1}{2\pi} \sum_{k \equiv 0 \pmod 2} i^k \tilde{\varphi}(k-1) \frac{(k-1)!}{(4\pi n)^{k-1}} \sum_{1 \leqslant j \leqslant \theta_k} |\psi_{jk}(\mathfrak{a}, n)|^2,$$

$$V_3(\mathfrak{a}, n) = \frac{1}{\pi} \sum_{\mathfrak{c}} \int_{-\infty}^{\infty} \hat{\varphi}(r) |\varphi_{\mathfrak{c}\mathfrak{a}n}(\tfrac{1}{2}+ir)|^2 \, dr,$$

$$S(\mathfrak{a}, n) = \sum_{\gamma} \frac{1}{\gamma} S_{\mathfrak{a}\mathfrak{a}}(n, n; \gamma) \varphi\left(\frac{4\pi n}{\gamma}\right).$$

The trace formula of Kuznetsov says that

$$\sum_{i=0}^{3} V_i(\mathfrak{a}, n) = S(\mathfrak{a}, n). \tag{8}$$

Recall that $S_{\mathfrak{a}\mathfrak{a}}(n, n; \gamma)$ is the Kloosterman sum associated with the cusp $\mathfrak{a} = \mathfrak{b}$ of $\Gamma = \Gamma_0(q)$ and modulus $\gamma > 0$. According to our assumption that q is prime, there are two inequivalent cusps $\mathfrak{a} = \infty$ and $\mathfrak{a} = 0$, and the Kloosterman sums reduce to the classical one,

$$S_{\mathfrak{a}\mathfrak{a}}(n, n; \gamma) = S(n, n; qc), \qquad \gamma = qc,\ c = 1, 2, 3, \ldots$$

In the sequel we shall write $S(n)$ instead of $S(\mathfrak{a}, n)$ because $S(\mathfrak{a}, n)$ happens to be independent of the cusp:

$$S(n) = \sum_{c=1}^{\infty} \frac{1}{qc} S(n, n; qc)\, \varphi\left(\frac{4\pi n}{qc}\right). \tag{9}$$

We have $\tilde{\varphi}(r) \ll (r^2+1)^{-1} \log X$ and $\tilde{\varphi}(k-1) \ll k^{-2}$, therefore the series $V_i(\mathfrak{a}, n)$, $i = 1, 2, 3$ converges, and by Theorem 2 of Deshouillers and Iwaniec [2]

$$V_i(\mathfrak{a}, n) \ll (1+q^{-1} n^{1+\varepsilon}) \log X.$$

Hence, by (8), we get

$$V_0(\mathfrak{a}, n) = S(n) + O\big((1+q^{-1} n^{1+\varepsilon}) \log X\big). \tag{10}$$

Let us remark that, using more elaborate methods, one can reduce the error term in (10) to $O\big((1+q^{-3/2}(n, q)^{1/2} n) \log X\big)$, but this has no significance for our applications.

4. Proof of Theorem 1*

By A. Weil's estimate

$$|S(n, n; qc)| \leqslant (n, qc)^{1/2} (qc)^{1/2} \tau(qc)$$

one easily gets

$$S(n) \ll q^{-1} (n, q)^{1/2} (nX)^{1/2} \tau(n) \log nX. \tag{11}$$

On the other hand, we have $\hat{\varphi}(t_j) \gg X^{2t_j}$ for $0 \leqslant t_j \leqslant 1/2$; thus

$$V_0(\mathfrak{a}, n) \gg \sum_{\lambda_j - \text{except.}} X^{2t_j} |\varrho_{j\mathfrak{a}}(n)|^2. \tag{12}$$

Combining (10), (11) and (12), we conclude that

$$\sum_{\lambda_j - \text{except}} X^{2t_j} |\varrho_{j\mathfrak{a}}(n)|^2 \ll q^{-1} (q + n + \sqrt{nX}) (nX)^{\varepsilon}. \tag{13}$$

Now, in order to get a density theorem involving the exceptional eigenvalues λ_j alone one must get rid of the Fourier coefficients $\varrho_{j\mathfrak{a}}(n)$. To this end we sum up (13) over $n \leqslant N = N_0(q)$ and we appeal to (7), giving (4) with $A = 2(2-\theta)$ after taking $X = q^{2-\theta}$, provided $N_0(q) = O(q^{\theta})$, $0 < \theta \leqslant 1$, is admissible. We are only able to show that $\theta = 1$ is admissible.

Actually we shall give here a proof of the following lemma, which is as useful as (7) for $\theta = 1$.

LEMMA 3. *Let q be a prime and let $\varrho_{j0}(n)$, $\varrho_{j\infty}(n)$ be the n-th Fourier coefficients of the Maass form $u_j(z)$ (normalized) at the cusps $\mathfrak{a} = 0$ and $\mathfrak{a} = \infty$ respectively. Denote*

$$\omega(x) = x^{-1} \exp(-2\pi\sqrt{3}\,x),$$

$$C_{0j} = \sum_{n=1}^{\infty} \omega\left(\frac{n}{q}\right)|\varrho_{j0}(n)|^2, \quad C_{\infty j} = \sum_{n=1}^{\infty} \omega(n)\,|\varrho_{j\infty}(n)|^2.$$

We then have $C_j = C_{0j} + C_{\infty j} \geqslant \sqrt{3}$.

Proof. Let $P(Y)$ stand for the euclidean strip

$$P(Y) = \{z = x+iy,\ -1/2 \leqslant x \leqslant 1/2,\ y > Y\}.$$

For any cusp $\mathfrak{a}$ of Γ take $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2, \mathbf{R})$ such that (cf. [2]) $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$, $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_{\infty}$. One can find $Y(\mathfrak{a}) > 0$ such that

$$\bigcup_{\mathfrak{a}-\text{inequivalent cusps}} \sigma_{\mathfrak{a}} P(Y(\mathfrak{a})) \supseteq F.$$

Hence

$$1 = \int_F |u_j(z)|^2\, dz \leqslant \sum_{\mathfrak{a}} \int_{\sigma_{\mathfrak{a}}(P(Y(\mathfrak{a})))} |u_j(z)|^2\, dz$$

$$= \sum_{\mathfrak{a}} \int_{P(Y(\mathfrak{a}))} |u_j(\sigma_{\mathfrak{a}} z)|^2\, dz = \sum_{\mathfrak{a}} \sum_{n \neq 0} |\varrho_{j\mathfrak{a}}(n)|^2 \int_{2\pi|n|Y(\mathfrak{a})} K_{i\varkappa_j}^2(y)\, \frac{dy}{y}.$$

But

$$K_\nu(y) = \int_0^\infty e^{-y\operatorname{ch}\xi} \operatorname{ch} \nu\xi\, d\xi \leqslant K_{1/2}(y) = \sqrt{\frac{\pi}{2y}}\, e^{-y},$$

$$\int_A^\infty K_\nu^2(y)\, \frac{dy}{y} \leqslant \frac{\pi}{2} \int_A^\infty e^{-2y}\, \frac{dy}{y^2} \leqslant \frac{\pi}{2A}\, e^{-2A} = K_{1/2}^2(A)$$

and $|\varrho_{j\mathfrak{a}}(-n)| = |\varrho_{j\mathfrak{a}}(n)|$; thus

$$\sum_{\mathfrak{a}} \frac{1}{Y(\mathfrak{a})} \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-4\pi n Y(\mathfrak{a})\right)|\varrho_{j\mathfrak{a}}(n)|^2 \geqslant 2. \tag{14}$$

Now, since q is a prime, we find that

$$\mathfrak{a}=\infty,\quad \sigma_\infty=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad Y(\infty)=\sqrt{3}/2,$$

$$\mathfrak{a}=0,\quad \sigma_0=\begin{pmatrix}0&1/\sqrt{q}\\ \sqrt{q}&0\end{pmatrix},\quad Y(0)=\sqrt{3}/2q,$$

see Figures 2 and 3 below.

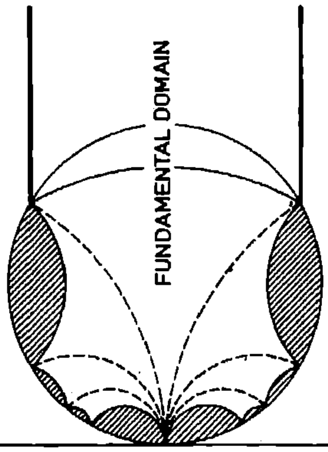

Fig. 2

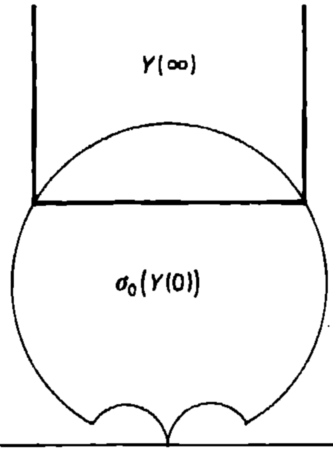

Fig. 3

This completes the proof of Lemma 3.

In order to complete the proof of Theorem 1*, sum up (13) over the cusps $\mathfrak{a}=0$ and $\mathfrak{a}=\infty$, and over $n\geqslant 1$ with weights $\omega(n/q)$ and $\omega(n)$ respectively. Then, it turns out that the optimal value for X is $X=q$, proving Theorem 1* by Lemma 3.

5. Proof of Theorem 2

The crucial idea is to improve (11) on average over n. By (10) and Lemma 3 we deduce that

$$\text{(15)}\qquad \mathscr{E}(X,q):=\sum_{\lambda_j-\text{except.}} C_j\frac{\hat{\varphi}(\varkappa_j)}{\operatorname{ch}\pi\varkappa_j} =\sum_{n=1}^{\infty}\left(\omega\left(\frac{n}{q}\right)+\omega(n)\right)S(n)+O(q^{1+\varepsilon}\log X).$$

The sum of the terms $\omega(n)S(n)$ by (11) is $\ll q^{-1}X^{1/2}\log X$, whence

$$\text{(16)}\quad \mathscr{E}(X,q)=\sum_{c=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{cn}\varphi\left(\frac{4\pi n}{qc}\right)\exp\left(-2\pi\sqrt{3}\,\frac{n}{q}\right)S(n,n;qc)+ +O\big((q^{-1}X^{1/2}+q^{1+\varepsilon})\log X\big).$$

Let $\varrho(\gamma, a)$ stand for the number of solutions of

$$d^2 - ad + 1 \equiv 0 \pmod{\gamma}.$$

We then rearrange the Kloosterman sums as follows:

$$S(n, n; \gamma) = \sum_{a \pmod{\gamma}} \varrho(\gamma, a) e(an/\gamma).$$

For $\gamma = qc$ with q prime, $(q, c) = 1$, we have

$$\varrho(qc, a) = \varrho(q)\varrho(c, a) = \left(1 + \left(\frac{a^2-4}{q}\right)\right)\varrho(c, a).$$

This leads us to the introduction of a new exponential sum

$$T(n, q, c) = \sum_{a \pmod{qc}} \left(\frac{a^2-4}{q}\right)\varrho(c, a)\, e\left(\frac{an}{qc}\right).$$

We find that

$$T(n, q, c) = \begin{cases} S(n, n; qc) - qS(n/q, n/q; c) & \text{if } q|n, \\ S(n, n; qc) & \text{if } q \nmid n. \end{cases}$$

In particular, by A. Weil's estimate, it follows that

$$|T(n, q, c)| \leqslant 4(n, qc)^{1/2} qc^{1/2} \tau(c). \tag{17}$$

Replacing $S(n, n; qc)$ by $T(n, q, c)$ in (16), we get, say,

$$\begin{aligned} \mathcal{E}(X, q) = &\sum_{(c,q)=1} \sum_{n=1}^{\infty} \frac{1}{cn} \varphi\left(\frac{4\pi n}{qc}\right) \exp\left(-2\pi\sqrt{3}\,\frac{n}{q}\right) T(n, q, c) - \\ &- \sum_{(c,q)=1} \sum_{n=1}^{\infty} \frac{1}{cn} \varphi\left(\frac{4\pi n}{c}\right) \exp(-2\pi\sqrt{3}\, n) S(n, n; c) + \\ &+ \sum_{c \equiv 0 \pmod{q}} \sum_{n=1}^{\infty} \frac{1}{cn} \varphi\left(\frac{4\pi n}{qc}\right) \exp\left(-2\pi\sqrt{3}\,\frac{n}{q}\right) S(n, n; qc) + \\ &+ O\left((q^{-1} X^{1/2} + q^{1+\varepsilon}) \log X\right) \\ = &\,\mathcal{E}_1(X, q) - \mathcal{E}_2(X, q) + \mathcal{E}_3(X, q) + O\left((q^{1+\varepsilon} + q^{-1} X^{1/2}) \log X\right). \end{aligned}$$

From A. Weil's estimate for Kloosterman sums we get

$$\mathcal{E}_2(X) \ll X^{1/2} \log X, \quad \mathcal{E}_3(X, q) \ll q^{-1/2} X^{1/2} \log X.$$

Let us remark that it is possible (but not necessary) to show that $\mathcal{E}_2(X) \ll \log X$ by reversing the above arguments for the full modular group, which is known to have no exceptional eigenvalues.

Next split up $\mathcal{E}_1(X, q)$ into its partial sums $\mathcal{E}_0(X, q)$ and $\mathcal{E}_\infty(X, q)$, say, over $c \leqslant q$ and $c > q$ respectively. For $\mathscr{E}_0(X, q)$ we get by (17)

$$\mathscr{E}_0(X, q) \ll q^{1+\varepsilon} \log X.$$

Putting together the results obtained above, we may write

$$\mathscr{E}(X, q) = \mathscr{E}_\infty(X, q) + O\big((X^{1/2} + q^{1+\varepsilon}) \log X\big), \tag{18}$$

where

$$\mathscr{E}_\infty(X, q) = \sideset{}{'}\sum_{\substack{c>q\\(c,q)=1}} \frac{1}{c} \sum_{n=1}^{\infty} \frac{1}{n} \varphi\left(\frac{4\pi n}{qc}\right) \exp\left(-2\pi\sqrt{3}\,\frac{n}{q}\right) T(n, q, c).$$

Now we intend to sum over n by means of Poisson's formula. By the definition of $T(n, q, c)$ the following sum arises:

$$R(a, q, c) = \sum_{n=1}^{\infty} \frac{1}{n} \varphi\left(\frac{4\pi n}{qc}\right) e^{-2\pi\sqrt{3}\,n/q} e\left(\frac{an}{qc}\right)$$

with $-1/2 < a/qc \leqslant 1/2$. Assume that

$$q \leqslant X \leqslant q^{2-\varepsilon}. \tag{19}$$

Then by Poissons's formula

$$R(a, q, c) = \int \varphi(4\pi\xi) e^{-2\pi\sqrt{3}\xi c} e(a\xi) \frac{d\xi}{\xi} + O(X^{-1}).$$

Moreover, if $|a| > q^\varepsilon X$, the integral is $\ll X^{-1}$ by partial integration. From this we conclude that

$$\begin{aligned}(20)\quad & \mathscr{E}_\infty(X, q) \\ &= \int \frac{\varphi(4\pi\xi)}{\xi} \sum_{|a| \leqslant q^\varepsilon X} e(a\xi) \left(\frac{a^2-4}{q}\right) \sum_{\substack{c>q\\(c,q)=1}} \frac{1}{c} e^{-2\pi\sqrt{3}\xi c} \varrho(c, a) + O(q^{1+\varepsilon}) \\ &\ll q^\varepsilon \max_{A, A_1} \max_{C, C_1} C^{-1} |F_q(A, C)| + q^{1+\varepsilon}\end{aligned}$$

by partial summation over a and c, where

$$F_q(A, C) = \sum_{A < a \leqslant A_1} \sum_{\substack{C < c \leqslant C_1\\(c,q)=1}} \left(\frac{a^2-4}{q}\right) \varrho(c, a)$$

and the maximum is taken over A, A_1, C, C_1 with $q < A$, $C \leqslant q^\varepsilon X$, $1 < A/A_1$, $C/C_1 \leqslant 2$, because the terms with $A \leqslant q$ or $C > q^\varepsilon X$ contribute trivially to the error term $q^{1+\varepsilon}$.

6. Estimation of $F_q(A, C)$

Now our nearest aim is to express $\varrho(c, a)$ by means of Jacobi's symbol. To this end write

$$\Delta = a^2-4, \quad c = 2^\alpha \delta_1 \delta_2^2 d, \quad (\Delta, \delta_1 \delta_2^2 d) = \delta_1 \delta_2^2, \quad D = \Delta \delta_2^{-2}$$

where $\delta_1 \delta_2^2 d$ is an odd number and δ_1 is a square-free number. Also denote $\delta = 2^\alpha \delta_1 \delta_2^2$, $\delta_0 = \delta_1 \delta_2^2$.

LEMMA 4. *In the above notation we have*

$$\varrho(c, a) = \varrho(2^\alpha, a)\delta_2 \sum_{r|d} \mu^2(r)\left(\frac{D}{r}\right)$$

provided $(d, D) = 1$ *and* $\varrho(c, a) = 0$ *otherwise.*

Proof. Write $c = 2^\alpha c_1$, c_1 odd, $(\Delta, c_1) = \delta_0 = \delta_1 \delta_2^2$. Since $\varrho(c, a)$ is multiplicative in c, we obtain $\varrho(c, a) = \varrho(2^\alpha, a)\varrho(c_1, a)$, and $\varrho(c_1, a)$ is the number of solutions of

$$x^2 \equiv \Delta \pmod{c_1}, \quad x \pmod{c_1}.$$

The solutions are $x = \delta_1 \delta_2 y$, where y runs over the residue classes $(\mathrm{mod}\ \delta_2 d)$ satisfying

$$\delta_1 y^2 \equiv \delta_0^{-1} \Delta \pmod{d}.$$

Since $(\delta_0^{-1}\Delta, d) = 1$, it follows that $(\delta_1, d) = 1$, $(D, d) = 1$ and the last congruence is equivalent to

$$y^2 \equiv D \pmod{d}.$$

Hence the number of solutions $y \pmod{\delta_2 d}$ is equal to

$$\varrho(c_1, a) = \delta_2 \prod_{p|d}\left(1+\left(\frac{D}{p}\right)\right) = \delta_2 \sum_{r|d} \mu^2(r)\left(\frac{D}{r}\right).$$

This completes the proof of Lemma 4.

By Lemma 4 we deduce that

$$(21)\qquad F_q(A, C) = \sum_{\substack{1 \leqslant \delta s \leqslant C_1 \\ (\delta s, q) = (\delta_1, s) = 1}}\sum \delta_2 \sum_{\substack{R < r \leqslant R_1 \\ (r, 2q) = 1}} \mu^2(r) \sum_{\substack{A < a \leqslant A_1 \\ a^2 \equiv 4 (\mathrm{mod}\ \delta_0) \\ (s, D) = 1}} \varrho(2^\alpha, a)\left(\frac{D}{qr}\right),$$

where for brevity we have denoted $\delta = 2^\alpha \delta_1 \delta_2^2$, $\delta_0 = \delta_1 \delta_2^2 \equiv 1 \pmod 2$, δ_1 — square-free, $R = C/\delta s$, $R_1 = C_1/\delta s$ and $D = \delta_2^{-2}(\alpha^2-4)$.

Notice that the variables r and a of the last two summations are independent. Denote

$$U(r, A) = \sum_{\substack{A < a \leqslant A_1 \\ a^2 \equiv 4 (\mathrm{mod}\, \delta_0) \\ (s,D)=1}} \varrho(2^\alpha, a)\left(\frac{D}{qr}\right) \quad \text{and} \quad U(R, a) = \sum_{\substack{R < r \leqslant R_1 \\ (r,2q)=1}} \mu^2(r)\left(\frac{D}{qr}\right).$$

Then the innermost double sum is equal to

$$U(\delta s, R, A) = \sum_r U(r, A) = \sum_a U(R, a). \tag{22}$$

We are going to estimate $U(\delta s, R, A)$ in two ways.

In the first way we estimate $U(r, A)$ by an appeal to Weil's Lemma 1. The condition that $(s, D) = 1$ can be relaxed by means of the Möbius formula

$$\sum_{\substack{\nu | s, \nu | D \\ (\nu, r)=1}} \mu(\nu) = \begin{cases} 1 & \text{if } (s, D) | r^\infty, \\ 0 & \text{otherwise}. \end{cases}$$

Then we split up the variable of summation a into arithmetic progressions $a = \nu\delta m + a_0$ where $0 \leqslant a_0 < \nu\delta$ and $a_0^2 \equiv 4 \pmod{\nu\delta_0}$. Put

$$f(m) = \delta_2^{-2}[(\nu\delta m + a_0)^2 - 4] = (\alpha' m)^2 + \beta m + \gamma$$

say, where $\alpha' = 2^\alpha \nu\delta_1\delta_2$, $\beta = 2^{\alpha+1}\nu\delta_1 a_0$ and $\gamma = \delta_2^{-2}(a_0^2 - 4)$. We then get

$$U(r, A) = \sum_{\substack{\nu | s \\ (\nu, r)=1}} \mu(\nu) \sum_{\substack{0 \leqslant a_0 < \nu\delta \\ a_0^2 \equiv 4 (\mathrm{mod}\, \nu\delta_0)}} \varrho(2^\alpha, a_0) \sum_{M < m \leqslant M_1} \left(\frac{f(m)}{qr}\right)$$

where for brevity we have denoted $M = (A - a_0)/\nu\delta$ and $M_1 = (A_1 - a_0)/\nu\delta$. It is clear that $f(m)$ satisfies the hypothesis of Lemma 1. Hence, on using standard Fourier technique for completing incomplete exponential sums, it follows that

$$\sum_{M < m \leqslant M_1} \left(\frac{f(m)}{qr}\right) \ll \left(1 + \frac{M_1 - M}{qr}\right)(qr)^{1/2}\tau(qr)\log qr.$$

This yields

$$U(r, A) \ll 2^\alpha\left(1 + \frac{A}{\delta qr}\right)(qr)^{1/2}\tau(\delta qrs)\log q$$

and, on summing over r in $(R, R_1]$, we finally obtain

$$U(\delta s, R, A) \ll 2^\alpha\left(R + \frac{A}{\delta q}\right)(qR)^{1/2} q^\varepsilon \ll \frac{1}{\delta_0}\left(\frac{C}{s} + \frac{A}{q}\right)\left(\frac{qC}{\delta s}\right)^{1/2} q^\varepsilon. \tag{23}$$

Now we proceed to the second way of estimating $U(\delta s, R, A)$ via $U(R, a)$. To this end we appeal to the Burgess Lemma 2. The condition that

r is square-free can be relaxed by the well-known formula

$$\sum_{\nu^2|r} \mu(\nu) = \mu^2(r),$$

and the condition that r is even by introducing a redundant factor $\left(\frac{4}{r}\right)$. We then infer

$$|U(R, a)| \leqslant \sum_{\nu^2 \leqslant R_1} \left| \sum_{R < \nu^2 r \leqslant R_1} \left(\frac{4D}{r}\right)\right| + q^{-1} R_1$$
$$\ll R^{1/2} D^{3/16} q^{\varepsilon} \ll \left(\frac{C}{\delta s}\right)^{1/2} \left(\frac{A}{\delta_2}\right)^{3/8} q^{\varepsilon}.$$

And, summing over a in $(A, A_1]$, $a^2 \equiv 4 \pmod{\delta_0}$ with weights $\varrho(2^{\alpha}, a) \leqslant 4 \cdot 2^{\alpha/2}$, we finally obtain

$$U(\delta s, R, A) \ll \left(1 + \frac{A}{\delta_0}\right)\left(\frac{C}{\delta_0 s}\right)^{1/2} \left(\frac{A}{\delta_2}\right)^{3/8} q^{\varepsilon}. \tag{24}$$

By (21), (23) and (24) we obtain

$$F_q(A, C) \ll \sum_{s} \sum_{\delta} \delta_2 U(\delta s, R, A) \tag{25}$$
$$\ll q^{\varepsilon} \sum_{1 \leqslant s \leqslant 2C} \min\left\{\left(\frac{C}{s} + \frac{A}{q}\right)\left(\frac{qC}{s}\right)^{1/2}, A^{11/8}\left(\frac{C}{s}\right)^{1/2}\right\}$$
$$\ll q^{\varepsilon} \sum_{1 \leqslant s \leqslant 2C} \left[\min\left\{q^{1/2}\left(\frac{C}{s}\right)^{3/2}, A^{11/8}\left(\frac{C}{s}\right)^{1/2}\right\} + A\left(\frac{C}{qs}\right)^{1/2}\right]$$
$$\ll q^{\varepsilon}(q^{1/4} A^{11/16} + q^{-1/2} A) C.$$

7. Completion of the proof of Theorem 2

By (20) and (25) we get

$$\mathscr{E}_{\infty}(X, q) \ll (q^{1/4} X^{11/16} + X q^{-1/2} + q) q^{\varepsilon}.$$

Then by (18)

$$\mathscr{E}(X, q) \ll (q^{1/4} X^{11/16} + X q^{-1/2} + X^{1/2} + q) q^{\varepsilon}.$$

On the other hand, by (15), Lemma 3 and the lower bound $\hat{\varphi}(\varkappa_j) \gg X^{2t_j}$ we have

$$\mathscr{E}(X, q) \gg \sum_{\lambda_j - \text{except.}} X^{2t_j}.$$

On taking $X = q^{12/11}$, we complete the proof of Theorem 2.

8. Appendix

Here we shall give an elementary proof of Lemma 1. The sums

$$S_f(h, d) := \sum_{x (\bmod d)} \left(\frac{f(x)}{d}\right) e\left(\frac{hx}{d}\right)$$

are multiplicative in d in the following sense

$$S_f(h, d_1 d_2) = S_f(h\bar{d}_2, d_1) S_f(h\bar{d}_1, d_2)$$

whenever $(d_1, d_2) = 1$. Therefore it suffices to show that

$$|S_f(h, p)| \leqslant 2p^{1/2}.$$

We infer this inequality from some results of W. Schmidt [9]. To this end notice that $f(x)$ considered as a polynomial over a finite field F_p has $\deg f = m = 1$ or 2 and has exactly m distinct roots. Moreover, $Y^2 - f(X)$ is absolutely irreducible.

If $p \mid h$ then the assertion of Lemma 1 follows from Theorem 2C of [9], and if $p \nmid h$ then the polynomial $Z^p - Z - h(X)$ is absolutely irreducible and the assertion of Lemma 2 follows from Theorem 2G of [9].

Added in proof. The result of Theorem 2 was recently improved by the first named author replacing the exponent $A = 24/11$ by $A = 12/5$, cf. *Character sums and small eigenvalues of* $\Gamma_0(p)$, to appear in the Glasgow Mathematical Journal.

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