

POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

5. 4133
[251]

**DISSERTATIONES
MATHEMATICAE**
(ROZPRAWY MATEMATYCZNE)

KOMITET REDAKCYJNY

BOGDAN BOJARSKI redaktor
ANDRZEJ BIAŁYNICKI-BIRULA, ZBIGNIEW CIESIELSKI
JERZY ŁOŚ, ZBIGNIEW SEMADENI

CCLI

PIOTR BORÓWKO and WITOLD RZYMOWSKI

Existence of the value of fixed duration dynamical games

WARSZAWA 1986

PAŃSTWOWE WYDAWNICTWO NAUKOWE

6.7133



PRINTED IN POLAND

© Copyright by PWN – Polish Scientific Publishers, Warszawa 1986

ISBN 83-01-06719-5

ISSN 0012-3862

W R O C I A W S K A D R U K A R N I A N A U K O W A

BUW-EO-87/41/30

1987

CONTENTS

1. Introduction .	5
2. Game	6
3. Approximation condition	8
4. Differential games of Friedman's type	13
4.1. Game without delay	13
4.2. Game with delay	19
5. Varaiya's differential game with dynamical system .	21
5.1. Game with constraints of controls	22
5.2. Game with control by trajectories	24
5.2.1. Approximation condition	25
5.2.2. Alternative way	26
6. Krasovskii's differential game with dynamical system	27
6.1. Correlations between the games $G(X, Y, \mu)$ and $G_K(X, Y, \varphi)$	29
6.2. Sufficient condition .	31
References .	36

1. Introduction

In this paper a general model of two-person zero-sum dynamical games of fixed duration is described. We study the problem of the existence of the value of these games and discuss the estimation of precision of reaching the value in a real game (e.g., when the players take decisions at a finite number of moments only). This model is illustrated by many examples of its application to more definite games (in the main, we assume that the players are not „separated”).

In particular, it appears possible to give a unified treatment of three fundamental concepts of such differential and dynamical games. As a special case of our Example 1 we obtain the well-known theorem on the existence of the value of a differential game of fixed duration presented in Friedman's book [3]. Section 5 contains, among other things, the principal ideas of Varaiya [7], [8] (these concepts together with the concept of „a small game” due to Krasovskii have in great measure inspired us to construct our model). Obviously in this section players have to be „separated”. Moreover, we have to eliminate the incompleteness in the theory of Varaiya that has been noted in [6]. Two ways of such completion are considered. The first (Theorem 2) is based on the assumption of a certain weak condition of approximation and has already been signalled in [1]. The second rests on the assumption of the right to „stop” the player's trajectories for a certain time (see Theorem 3, and Example 3, in which the strategies of the player P are defined on controls and not on trajectories, which means a considerable departure from Varaiya's game, but Lemma 3 used in this example is in the Varaiya spirit). In Section 5 the asymmetry of the model and of the method is particularly apparent – after the separation of variables not only a regularity of the dynamical system but also any regularity of the payoff functional is assumed for one player only (the duality of the theory allows us to choose any player). However, in Section 6 Krasovskii's „differential game with a retention of information” [4], Chapter XVI, is generalized to the game with dynamical systems. The relationship between the Krasovskii's strategy and our strategies is described (Lemma 6). Afterwards it is proved that if there exists the value (in the sense of Krasovskii) of this game then the value of our corresponding game also exists. The sufficient conditions of the existence of the Krasovskii value are also given (Theorem 4).

The approximation method used in this paper had previously been applied to study orientor fields in [1]. Now, it is used in a much more general case. Moreover, in order to obtain an explicit form of the estimations for the games this method is applied by starting, in a sense, from the "other end" (see Lemma 2 and Theorem 1 for example).

2. Game

In the whole paper for $\Delta \subset [0, T]$ and a function x defined on $[0, T]$ we shall denote the restriction of x to the set Δ by $x|_{\Delta}$. Moreover, we fix a number $T \in (0, \infty)$ and two sets A and B . Let X be a nonempty set of functions (controls or trajectories of player E) $x: [0, T] \rightarrow A$, and let Y be a nonempty set of functions (controls or trajectories of player P) $y: [0, T] \rightarrow B$.

Let $\pi = \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$ be the partition of the interval $[0, T]$.

The function $e: Y \rightarrow X$ is called a *lower strategy associated with the partition π* for player E if for each pair of functions $y, \bar{y} \in Y$ the following conditions are fulfilled

$$1^\circ e(y)|_{[0, t_1]} = e(\bar{y})|_{[0, t_1]},$$

2° for each $i = 0, 1, \dots, n-1$ the equality $y|_{[0, t_i]} = \bar{y}|_{[0, t_i]}$ implies that $e(y)|_{[0, t_{i+1}]} = e(\bar{y})|_{[0, t_{i+1}]}$.

The set of all such strategies will be denoted by $E^-(\pi)$.

Analogously, the function $p: X \rightarrow Y$ is called a *lower strategy associated with the partition π* for player P if for all $x, \bar{x} \in X$

$$1^\circ p(x)|_{[0, t_1]} = p(\bar{x})|_{[0, t_1]},$$

2° for each $i = 0, 1, \dots, n-1$, if $x|_{[0, t_i]} = \bar{x}|_{[0, t_i]}$ then $p(x)|_{[0, t_{i+1}]} = p(\bar{x})|_{[0, t_{i+1}]}$.

The set of all such strategies will be denoted by $P^-(\pi)$.

The function $e: Y \rightarrow X$ is called an *upper strategy associated with the partition π* for player E if for all $i = 0, 1, \dots, n$ and $y, \bar{y} \in Y$ the equality

$$y|_{[0, t_i]} = \bar{y}|_{[0, t_i]}$$

implies that

$$e(y)|_{[0, t_i]} = e(\bar{y})|_{[0, t_i]}.$$

Similarly, we define an upper strategy associated with the partition π for player P .

Sets of upper strategies will be denoted by $E^+(\pi)$ and $P^+(\pi)$.

Any real function μ defined on $X \times Y$ is called the *payoff functional*. In

order to simplify our considerations we assume that μ is bounded, which is not an important assumption for the methods used below.

Let us note that for any partitions $\pi, \tilde{\pi}$ and $e \in E^-(\pi), p \in P^-(\tilde{\pi})$:

1° there exists a unique pair $(x, y) \in X \times Y$, such that

$$x = e(y) \quad \text{and} \quad y = p(x),$$

$$2^\circ \inf_{y \in Y} \mu(e(y), y) \leq \sup_{x \in X} \mu(x, p(x)).$$

Moreover, if $\tilde{\pi} \subset \pi$ then

$$E^-(\tilde{\pi}) \subset E^-(\pi) \subset E^+(\pi) \subset E^+(\tilde{\pi})$$

and

$$P^-(\tilde{\pi}) \subset P^-(\pi) \subset P^+(\pi) \subset P^+(\tilde{\pi}).$$

Now, let us consider a game $G(X, Y, \mu)$ in which the aim of the player E will be to maximize the payoff functional and the aim of the player P will be to functional minimize it.

The number V is called the *value of the game* $G(X, Y, \mu)$ when the following condition is satisfied:

for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\pi: \partial(\pi) \leq \delta, \tilde{\pi}: \partial(\tilde{\pi}) \leq \delta$ there exist $e \in E^-(\pi)$ and $p \in P^-(\tilde{\pi})$ for which

$$V - \varepsilon \leq \inf_{y \in Y} \mu(e(y), y), \quad \sup_{x \in X} \mu(x, p(x)) \leq V + \varepsilon.$$

(The symbol $\partial(\pi)$ denotes the diameter of the partition π , i.e., $\max \{t_{i+1} - t_i: i = 0, 1, \dots, n-1\}$). Obviously, there exists only one number V .

Our definition of the value of the game is slightly different from the most frequently used definitions (see, e.g. Friedman [3], Varaiya and Lin [8]). We are particularly emphasize that players E and P may both take decisions (in a „practical” game) at different moments of time.

In most of our paper, assumptions referring to the players are asymmetric and they are given in some sense from the point of view of player E . The dual results may be obtained in an identical way when all the considerations are repeated from the point of view of player P .

LEMMA 1. *If there exists a function $\alpha_0: (0, \delta_0) \rightarrow [0, \infty)$, $\alpha_0(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, having the following property:*

for all $\pi: \partial(\pi) \leq \delta_0$, and $e^+ \in E^+(\pi)$ there exists an $e^- \in E^-(\pi)$ such that

$$\inf_{y \in Y} \mu(e^+(y), y) - \inf_{y \in Y} \mu(e^-(y), y) \leq \alpha_0(\partial(\pi)),$$

then there exists the value V of the game $G(X, Y, \mu)$, and for all $\pi: \partial(\pi) \leq \delta_0$,

and $\tilde{\pi}$: $\partial(\tilde{\pi}) \leq \delta_0$ we have

$$\begin{aligned} V - \kappa_0(\partial(\pi)) &\leq \sup_{e \in E^-(\pi)} \inf_{y \in Y} \mu(e(y), y) \leq V \\ &\leq \inf_{p \in P^-(\tilde{\pi})} \sup_{x \in X} \mu(x, p(x)) \leq V + \kappa_0(\partial(\tilde{\pi})). \end{aligned}$$

Proof. Obviously, the number V satisfying the above inequalities for any partitions $\pi, \tilde{\pi}$: $\partial(\pi), \partial(\tilde{\pi}) \leq \delta_0$ has to be the value of the game. Thus, we shall show the existence of that number. For any increasing sequence of partitions $\pi_j, j \in N$, such that $\partial(\pi_j) \rightarrow 0$ as $j \rightarrow \infty$, the sequences

$$\sup_{e^+ \in E^+(\pi_j)} \inf_{y \in Y} \mu(e^+(y), y)$$

and

$$\sup_{e^- \in E^-(\pi_j)} \inf_{y \in Y} \mu(e^-(y), y)$$

tend to the common limit V . The limit does not depend upon the choice of the sequence $\pi_j, j \in N$. Let us fix any partitions $\pi, \tilde{\pi}$ and assume that $\partial(\pi) \leq \delta_0, \partial(\tilde{\pi}) \leq \delta_0$. Using any increasing sequence $\pi_j, j \in N$ such that $\pi_1 = \pi$ and $\partial(\pi_j) \rightarrow 0$ if $j \rightarrow \infty$, we obtain

$$\begin{aligned} V - \kappa_0(\partial(\pi)) &\leq \sup_{e^- \in E^-(\pi)} \inf_{y \in Y} \mu(e^-(y), y) \leq V \\ &\leq \sup_{e^+ \in E^+(\tilde{\pi})} \inf_{y \in Y} \mu(e^+(y), y) \leq V + \kappa_0(\partial(\tilde{\pi})). \end{aligned}$$

By employing well-known methods [8], [3] it can be proved that

$$\sup_{e^+ \in E^+(\pi)} \inf_{y \in Y} \mu(e^+(y), y) = \inf_{p^- \in P^-(\tilde{\pi})} \sup_{x \in X} \mu(x, p^-(x)),$$

which ends the proof of Lemma 1.

3. Approximation condition

A continuous function $\omega: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is called a *comparative function* if $\omega(t, 0) = 0, t \in [0, \infty)$, ω is nondecreasing with respect to the second variable and the only solution of the equation $v' = \omega(t, v), v(0) = 0$ is the zero solution.

However, a function $\varrho: (0, \infty) \times (0, \infty) \rightarrow [0, \infty)$ is called a *complementary function* if for arbitrary $\varepsilon, \eta > 0$ there exists a $\delta > 0$ such that for any $\xi \in (0, \delta]$ and $\bar{\eta} \in [\eta, \infty)$ we have $\varrho(\xi, \bar{\eta}) \leq \varepsilon$.

LEMMA 2. For any complementary function ϱ there exist a number $T^* \in (0, T]$ and a nondecreasing function $\zeta: (0, T^*] \rightarrow (0, \infty), \zeta(\delta) \rightarrow 0$ for

$\delta \rightarrow 0$ such that

$$\varrho(\delta, \hat{u}) \leq \zeta(\delta)$$

for all $\delta \in (0, T^*]$, $\delta \in (0, \delta]$ and $\hat{u} \in [\zeta(\delta), \infty)$.

Proof. For each $\varepsilon \in (0, \infty)$, let $\chi(\varepsilon)$ be the least upper bound of the set of all $\delta \in (0, T]$ such that

$$\varrho(\delta, \hat{u}) \leq \varepsilon \quad \text{for} \quad \delta \in (0, \delta], \hat{u} \in [\varepsilon, \infty).$$

The function $\chi: (0, \infty) \rightarrow (0, T]$ is nondecreasing, and so there exists an increasing and continuous function $\chi_1: [0, \infty) \rightarrow [0, T]$, $\chi_1(0) = 0$ such that

$$\chi_1(\varepsilon) < \chi(\varepsilon) \quad \text{for} \quad \varepsilon > 0.$$

We choose any $T^* \in (0, \sup \chi_1([0, \infty)))$ and define a function $\zeta: (0, T^*] \rightarrow (0, \infty)$ by the following formula

$$\zeta(\delta) = \chi_1^{-1}(\delta), \quad \delta \in (0, T^*].$$

By definition it is a nonnegative and increasing function. Moreover, $\zeta(\delta) \rightarrow 0$ for $\delta \rightarrow 0$.

Now, let us fix any $\delta \in (0, T^*]$, $\delta \in (0, \delta]$ and $\hat{u} \in [\zeta(\delta), \infty)$. Then

$$\delta = \chi_1(\zeta(\delta)) < \chi(\zeta(\delta)),$$

thus from the definition of the function χ we have

$$\varrho(\delta, \hat{u}) \leq \zeta(\delta).$$

This completes the proof of Lemma 2.

Now we shall present a certain general theorem on the existence of the function κ_0 from the assumptions of Lemma 1, and thus on the existence of the value of the game $G(X, Y, \mu)$. We shall often refer to this theorem and its proof in a further part of the paper.

Let M be any set. It will perform the role of the phase space of the game. Assume that a function $d: M \times M \rightarrow [0, \infty)$ is a pseudometric. Let Z be a nonempty set of equicontinuous functions $z: [0, T] \rightarrow M$ such that $d(z(0), \bar{z}(0)) = 0$ for all $z, \bar{z} \in Z$. It is the set of trajectories of the game.

Let us consider functions: $F: X \times Y \rightarrow Z$, $\varphi: Z \rightarrow \mathbf{R}$ and a nondecreasing function $w: [0, \infty) \rightarrow [0, \infty)$ such that

1° for any $t \in [0, T]$, $(x, y), (x, \bar{y}) \in X \times Y$ if

$$x|_{[0,t]} = \bar{x}|_{[0,t]} \quad \text{and} \quad y|_{[0,t]} = \bar{y}|_{[0,t]}$$

then

$$F(x, y)|_{[0,t]} = F(\bar{x}, \bar{y})|_{[0,t]},$$

2° $w(\delta) \rightarrow 0$ if $\delta \rightarrow 0$, and for $z, \bar{z} \in Z$ we have

$$|\varphi(z) - \varphi(\bar{z})| \leq w\left(\max_{t \in [0, T]} d(z(t), \bar{z}(t))\right).$$

Let us assume in addition that

$$\mu(x, y) = \varphi(F(x, y)), \quad \text{for } (x, y) \in X \times Y.$$

THEOREM 1. *If there exist a comparative function ω and a complementary function ϱ for which the following is true:*

for any $t \in [0, T]$ and $(x_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2) \in X \times Y$, $d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t)) \neq 0$ there exist $x_1^ \in X$, $x_1^*|_{[0, t]} = \bar{x}_1|_{[0, t]}$ and $y_2^* \in Y$, $y_2^*|_{[0, t]} = \bar{y}_2|_{[0, t]}$ such that for all $y_1 \in Y$, $y_1|_{[0, t]} = \bar{y}_1|_{[0, t]}$, $x_2 \in X$, $x_2|_{[0, t]} = \bar{x}_2|_{[0, t]}$, and $s \in (t, T]$ the inequality*

$$\begin{aligned} & d(F(x_1^*, y_1)(s), F(x_2, y_2^*)(s)) - d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t)) \\ & \leq \left[\omega(t, d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t))) + \right. \\ & \quad \left. + \varrho(s-t, d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t))) \right] (s-t) \end{aligned}$$

is satisfied, then there exists a function κ_0 by the assumptions of Lemma 1.

Proof. Let us fix r_0 and $h_0 > 0$ such that for any $r \in [0, r_0]$ and $h \in [0, h_0]$ there exists the minimal solution v of the equation

$$v'(s) = \omega(s, v(s)) + h, \quad v(0) = r$$

defined on the whole interval $[0, T]$. Let us denote this solution by $v(r, h)$. There exists a nondecreasing function $\eta: (0, \infty) \rightarrow [0, \infty)$, $\eta(\delta) \rightarrow 0$ if $\delta \rightarrow 0$, such that

$$\omega(s, v(r, h)(s)) \geq \omega(t, v(r, h)(t)) - \eta(s-t)$$

for all $r \in [0, r_0]$, $h \in [0, h_0]$, $t \in [0, T]$ and $s \in (t, T]$. According to Lemma 2, there exist numbers $T_i^* \in (0, T]$ and nondecreasing functions $\zeta_i: (0, T_i^*] \rightarrow (0, \infty)$, $\zeta_i(\delta) \rightarrow 0$ if $\delta \rightarrow 0$, $i = 1, 2$, such that

$$\varrho(\delta, \hat{u}) \leq \zeta_2(\delta)$$

for all $0 < \delta \leq \min\{T_1^*, T_2^*\}$, $\hat{\delta} \in (0, \delta]$ and $\hat{u} \in [\zeta_1(\delta), \infty)$.

Let us choose a positive number δ_0 for which the following inequalities are fulfilled:

$$\begin{aligned} \delta_0 & \leq T_i^*, \quad i = 1, 2, \\ \zeta_1(\delta_0) + 2w_Z(\delta_0) & \leq r_0, \quad \zeta_2(\delta_0) + \eta(\delta_0) \leq h_0, \end{aligned}$$

where w_Z denotes a nondecreasing common modulus of continuity of trajectories from the set Z .

Let us assume that for $\delta \in (0, \delta_0]$ and $t \in [0, T]$ we have

$$z_1(\delta, t) = t(\zeta_1(\delta) + 2w_z(\delta), \zeta_2(\delta) + \eta(\delta))(t)$$

and

$$z_0(\delta) = w(z_1(\delta, T)).$$

Obviously, $z_0(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Let us fix any partition $\pi = \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$ of the interval $[0, T]$ with the diameter $\partial(\pi) \leq \delta_0$ and an upper strategy $e^+ \in E^+(\pi)$.

It is enough to show that there exists a lower strategy $e^- \in E^-(\pi)$ such that for each $y \in Y$ there exists a $\bar{y} \in Y$ for which

$$d(F(e^-(y), y)(t), F(e^-(\bar{y}), \bar{y})(t)) \leq z_1(\partial(\pi), T), t \in [0, T].$$

We shall construct the strategy $e^- \in E^-(\pi)$.

For any $i = 1, 2, \dots, n-1$ there exists a function

$$Q_i: (X \times Y) \times (X \times Y) \rightarrow X \times Y$$

such that:

1° for any $(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2) \in X \times Y$ if

$$(x, y) = Q_i((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)),$$

then

$$x|_{[0, t_i]} = \bar{x}_1|_{[0, t_i]} \quad \text{and} \quad y|_{[0, t_i]} = \bar{y}_2|_{[0, t_i]},$$

2° if $d(F(\bar{x}_1, \bar{y}_1)(t_i), F(\bar{x}_2, \bar{y}_2)(t_i)) \geq \zeta_1(\partial(\pi))$ then $Q_i((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2))$ is the pair (x_1^*, y_2^*) which is associated with $t = t_i$, (\bar{x}_1, \bar{y}_1) and (\bar{x}_2, \bar{y}_2) according to the assumption of Theorem 1,

3° for $(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2), (\hat{x}_1, \hat{y}_1)$ and $(\hat{x}_2, \hat{y}_2) \in X \times Y$, if

$$\bar{x}_j|_{[0, t_i]} = \hat{x}_j|_{[0, t_i]}, \quad \bar{y}_j|_{[0, t_i]} = \hat{y}_j|_{[0, t_i]}, \quad j = 1, 2$$

then

$$Q_i((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)) = Q_i((\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2)).$$

Let us fix any $x^{(1)} \in X$, $\bar{y}^{(1)} \in Y$ and $y \in Y$. We shall define a value of $e^-(y)$. At the same time we shall construct $\bar{y} \in Y$ such that

$$d(F(e^-(y), y)(t), F(e^-(\bar{y}), \bar{y})(t)) \leq z_1(\partial(\pi), T), \text{ for } t \in [0, T].$$

Let be

$$(x^{(i+1)}, \bar{y}^{(i+1)}) = Q_i((x^{(i)}, y), (e^+(\bar{y}^{(i)}), \bar{y}^{(i)})), \quad \text{for } i = 1, 2, \dots, n-1.$$

Next, let us assume that

$$e^-(y) = x^{(n)} \quad \text{and} \quad \bar{y} = \bar{y}^{(n)}.$$

By induction it is easy to check that $e^- \in E^-(\pi)$.

Finally, let us assume that for any $i = 0, 1, \dots, n-1$ the following inequality is fulfilled

$$d(F(e^-(y), y)(t_i), F(e^+(\bar{y}), \bar{y})(t_i)) \leq \kappa_1(\partial(\pi), t_i).$$

We shall prove that in this case we have

$$d(F(e^-(y), y)(t), F(e^+(\bar{y}), \bar{y})(t)) \leq \kappa_1(\partial(\pi), t) \quad \text{for } t \in [t_i, t_{i+1}].$$

If $d(F(e^-(y), y)(t_i), F(e^+(\bar{y}), \bar{y})(t_i)) < \zeta_1(\partial(\pi))$ then

$$\begin{aligned} d(F(e^-(y), y)(t), F(e^+(\bar{y}), \bar{y})(t)) &\leq \zeta_1(\partial(\pi)) + 2\omega_z(\partial(\pi)) \\ &\leq \kappa_1(\partial(\pi), t), \quad t \in [t_i, t_{i+1}]. \end{aligned}$$

Thus, let us assume that

$$d(F(e^-(y), y)(t_i), F(e^+(\bar{y}), \bar{y})(t_i)) \geq \zeta_1(\partial(\pi)).$$

Then

$$\zeta_1(\partial(\pi)) \leq d(F(x^{(i)}, y)(t_i), F(e^+(\bar{y}^{(i)}), \bar{y}^{(i)})(t_i)) \leq \kappa_1(\partial(\pi), t_i),$$

and according to the property 2° of the function Q_i we have

$$\begin{aligned} &d(F(e^-(y), y)(t), F(e^+(\bar{y}), \bar{y})(t)) \\ &= d(F(x^{(i+1)}, y)(t), F(e^+(\bar{y}^{(i+1)}), \bar{y}^{(i+1)})(t)) \\ &\leq d(F(x^{(i)}, y)(t_i), F(e^+(\bar{y}^{(i)}), \bar{y}^{(i)})(t_i)) + \\ &\quad + [\omega(t_i, d(F(x^{(i)}, y)(t_i), F(e^+(\bar{y}^{(i)}), \bar{y}^{(i)})(t_i))) + \\ &\quad + \varrho(t - t_i, d(F(x^{(i)}, y)(t_i), F(e^+(\bar{y}^{(i)}), \bar{y}^{(i)})(t_i)))](t - t_i) \\ &\leq \kappa_1(\partial(\pi), t_i) + [\omega(t_i, \kappa_1(\partial(\pi), t_i)) + \zeta_2(\partial(\pi))](t - t_i) \\ &\leq \kappa_1(\partial(\pi), t_i) + \int_{t_i}^t [\omega(s, \kappa_1(\partial(\pi), s)) + \eta(\partial(\pi)) + \zeta_2(\partial(\pi))] ds \\ &= \kappa_1(\partial(\pi), t), \quad \text{for } t \in [t_i, t_{i+1}]. \end{aligned}$$

According to the principle of induction, we indeed have

$$d(F(e^-(y), y)(t), F(e^+(\bar{y}), \bar{y})(t)) \leq \kappa_1(\partial(\pi), t) \leq \kappa_1(\partial(\pi), T), \quad t \in [0, T],$$

which completes the proof of Theorem 1.

Remark 1. The inequality from the assumptions of Theorem 1 may be

replaced by

$$\begin{aligned} & d(F(x_1^*, y_1)(s), F(x_2, y_2^*)(s)) - d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t)) \\ & \leq [\omega(t, \max_{\tau \in [0, t]} d(F(\bar{x}_1, \bar{y}_1)(\tau), F(\bar{x}_2, \bar{y}_2)(\tau))) + \\ & \quad + \varrho(s-t, d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t)))](s-t), \end{aligned}$$

and the theorem is also true.

4. Differential games of Friedman's type

In this section we shall check that the assumptions of Theorem 1 are satisfied for two games in which the trajectories of the game are described by a differential equation. Thus, we shall check that there exist the values of those games. Moreover, referring to the method of the proof of Theorem 1, we shall obtain formulas estimating the precision of reaching the value in the games.

4.1. Game without delay

EXAMPLE 1. For $a, b \in \mathbf{R}^n$, $n \in \mathbf{N}$, let $\langle a, b \rangle$ denote the Euclidean scalar product of vectors a and b , $\|a\| = \sqrt{\langle a, a \rangle}$. Next, by $\text{comp}(\mathbf{R}^n)$ we denote the family of all nonempty and compact subsets of the space \mathbf{R}^n .

Let us fix any $n, k, l \in \mathbf{N}$ and $a \in \mathbf{R}^n$, $U \in \text{comp}(\mathbf{R}^k)$, $V \in \text{comp}(\mathbf{R}^l)$. Next, let us fix continuous functions

$$f: [0, T] \times \mathbf{R}^n \times U \times V \rightarrow \mathbf{R}^n,$$

$$h: [0, T] \times \mathbf{R}^n \times U \times V \rightarrow \mathbf{R}$$

such that the following formulas are valid

1° there exists a number $\lambda > 0$ satisfying the inequality

$$\langle b-a, f(t, b, u, v) \rangle \leq \lambda(1 + \|b-a\|^2)$$

for any $t \in [0, T]$, $b \in \mathbf{R}^n$, $u \in U$ and $v \in V$;

2° for each bounded set $\hat{M} \subset \mathbf{R}^n$ there exists a comparative function ω satisfying the inequality

$$\|f(t, b, u, v) - f(t, c, u, v)\|^2 + (h(t, b, u, v) - h(t, c, u, v))^2 \leq \omega^2(t, \|b-c\|)$$

for all $t \in [0, T]$, $b, c \in \hat{M}$, $u \in U$ and $v \in V$;

3° for any $t \in [0, T]$, $b, c \in \mathbf{R}^n$ and $r \in \mathbf{R}$

$$\begin{aligned} \max_{u \in U} \min_{v \in V} [\langle c, f(t, b, u, v) \rangle + rh(t, b, u, v)] \\ = \min_{v \in V} \max_{u \in U} [\langle c, f(t, b, u, v) \rangle + rh(t, b, u, v)]. \end{aligned}$$

We assume that for $b \in \mathbf{R}^n$ and $r \in \mathbf{R}$

$$K(b, r) = \{c \in \mathbf{R}^n: \|b - c\| \leq r\}$$

and take

$$R = (e^{2\lambda T} - 1)^{1/2},$$

$$L_1 = \max \{ \|f(t, b, u, v)\|: t \in [0, T], b \in K(a, R), u \in U \text{ and } v \in V \},$$

$$L_2 = \max \{ \|h(t, b, u, v)\|: t \in [0, T], b \in K(a, R), u \in U \text{ and } v \in V \},$$

and

$$L = (L_1^2 + L_2^2)^{1/2}.$$

Let $M = \mathbf{R}^{n+1}$, $d(b, c) = \|b - c\|$, $b, c \in \mathbf{R}^{n+1}$, and let us denote by Z the set of all functions of the following form: $z = (z_1, z_2)$, where $z_1: [0, T] \rightarrow \mathbf{R}^n$, $z_2: [0, T] \rightarrow \mathbf{R}$, $z_1(0) = a$, $z_2(0) = 0$ and

$$\|z_1(t) - z_1(s)\|^2 + (z_2(t) - z_2(s))^2 \leq L^2(s - t)^2 \quad \text{for } s, t \in [0, T].$$

It is easy to check that the function w_Z given by

$$w_Z(\delta) = L\delta, \quad \delta \in [0, \infty)$$

is the common nondecreasing modulus of continuity of all trajectories from the set Z .

Next, let X denote the set of all measurable functions $x: [0, T] \rightarrow U$ and let Y denote the set of all measurable functions $y: [0, T] \rightarrow V$.

For $(x, y) \in X \times Y$, let $F(x, y)$ denote the pair of trajectories (z_1, z_2) given by the formulas

$$z_1(t) = a + \int_0^t f(s, z_1(s), x(s), y(s)) ds,$$

$$z_2(t) = \int_0^t h(s, z_1(s), x(s), y(s)) ds, \quad t \in [0, T].$$

Obviously, $F(x, y) \in Z$ for $(x, y) \in X \times Y$. One can easily check that F satisfies condition 1° previous to Theorem 1.

We denote by Z_1 the set of all functions $z_1: [0, T] \rightarrow \mathbf{R}^n$, $z_1(0) = a$, $\|z_1(t) - z_1(s)\| \leq L|t - s|$, $t, s \in [0, T]$. Let $\sigma: Z_1 \rightarrow \mathbf{R}$ and $w_1: [0, \infty) \rightarrow [0, \infty)$ be functions such that w_1 is nondecreasing, $w_1(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, and

furthermore

$$|\sigma(z_1) - \sigma(\bar{z}_1)| \leq w_1 \left(\max_{t \in [0, T]} \|z_1(t) - \bar{z}_1(t)\| \right), \quad z_1, \bar{z}_1 \in Z_1.$$

Now, if

$$\varphi((z_1, z_2)) = \sigma(z_1) + z_2(T), \quad (z_1, z_2) \in Z$$

and

$$w(\delta) = w_1(\delta) + \delta, \quad \delta \in [0, \infty),$$

the condition 2° preceding Theorem 1 is also fulfilled.

Let us assume that $\mu(x, y) = \varphi(F(x, y))$, $(x, y) \in X \times Y$. We shall prove that the approximation condition from the assumptions of Theorem 1 is satisfied.

Let us associate with $\hat{M} = K(a, R)$ the comparative function ω according to the condition 2° of this example. Let

$$\Omega(r) = \max_{t \in [0, T]} \omega(t, r).$$

The functions f and h are uniformly continuous on $[0, T] \times K(a, R) \times U \times V$, and so there exists a nondecreasing function $\eta^*: [0, \infty) \rightarrow [0, \infty)$, $\eta^*(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, such that

$$\|f(t, b, u, v) - f(s, b, u, v)\|^2 + \{h(t, b, u, v) - h(s, b, u, v)\}^2 \leq (\eta^*)^2(|s - t|)$$

for $s, t \in [0, T]$, $b \in K(a, R)$, $u \in U$ and $v \in V$.

Let us suppose that for $\tau, r \in (0, \infty)$

$$\varrho(\tau, r) = \frac{6L^2}{r} \tau + 4[\eta^*(\tau) + \Omega(L_1 \tau)].$$

Next, let us fix $t \in [0, T)$ and $(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2) \in X \times Y$ such that

$$d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t)) = \|F(\bar{x}_1, \bar{y}_1)(t) - F(\bar{x}_2, \bar{y}_2)(t)\| > 0.$$

From assumption 3° follows the existence of $u^* \in U$ and $v^* \in V$ which, for any $u \in U$ and $v \in V$, satisfy the inequality

$$\begin{aligned} \langle F(\bar{x}_2, \bar{y}_2)(t) - F(\bar{x}_1, \bar{y}_1)(t), g(t, z_{21}(t), u, v^*) \rangle \\ \leq \langle F(\bar{x}_2, \bar{y}_2)(t) - F(\bar{x}_1, \bar{y}_1)(t), g(t, z_{21}(t), u^*, v) \rangle, \end{aligned}$$

where $g(s, b, u, v) = (f(s, b, u, v), h(s, b, u, v)) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, for $s \in [0, T]$, $b \in \mathbb{R}^n$, $u \in U$ and $v \in V$, where

$$z_{21}(s) = a + \int_0^s f(\tau, z_{21}(\tau), \bar{x}_2(\tau), \bar{y}_2(\tau)) d\tau, \quad s \in [0, T].$$

Let us assume

$$x_1^*(s) = \begin{cases} \bar{x}_1(s) & \text{for } s \in [0, t], \\ u^* & \text{for } s \in (t, T], \end{cases}$$

$$y_2^*(s) = \begin{cases} \bar{y}_2(s) & \text{for } s \in [0, t], \\ v^* & \text{for } s \in (t, T], \end{cases}$$

and fix any $x_2 \in X$, $x_2|_{[0,t]} = \bar{x}_2|_{[0,t]}$ and $y_1 \in Y$, $y_1|_{[0,t]} = \bar{y}_1|_{[0,t]}$.
Let

$$z_1^* = F(x_1^*, y_1) \quad \text{and} \quad z_2^* = F(x_2, y_2^*).$$

Obviously, $z_1^*(t) = F(\bar{x}_1, \bar{y}_1)(t)$ and $z_2^*(t) = F(\bar{x}_2, \bar{y}_2)(t)$.

Hence, for $s \in (t, T]$

$$d(F(x_1^*, y_1)(s), F(x_2, y_2^*)(s)) - d(F(\bar{x}_1, \bar{y}_1)(t), F(\bar{x}_2, \bar{y}_2)(t)) \\ = \|z_2^*(s) - z_1^*(s)\| - \|z_2^*(t) - z_1^*(t)\|.$$

Let $z_{21}^*: [0, T] \rightarrow \mathbf{R}^n$ and $\hat{z}_{21}: [0, T] \rightarrow \mathbf{R}^n$ denote solutions of the equations

$$z_{21}^*(s) = a + \int_0^s f(\tau, z_{21}^*(\tau), x_2(\tau), y_2^*(\tau)) d\tau,$$

$$\hat{z}_{21}(s) = a + \int_0^s f(\tau, \hat{z}_{21}(\tau), \hat{x}_1(\tau), \hat{y}_1(\tau)) d\tau, \quad s \in [0, T],$$

where

$$\hat{x}_1(\tau) = x_2(\tau), \quad \hat{y}_1(\tau) = y_2^*(\tau) \quad \text{for } \tau \in [0, t]$$

and

$$\hat{x}_1(\tau) = u^*, \quad \hat{y}_1(\tau) = y_1(\tau) \quad \text{for } \tau \in [t, T].$$

Of course, $z_{21}^*|_{[0,t]} = \hat{z}_{21}|_{[0,t]} = z_{21}|_{[0,t]}$.

For almost all $s \in (t, T]$ such that $\|z_2^*(s) - z_1^*(s)\| \neq 0$ we have:

$$\frac{d}{ds} (\|z_2^*(s) - z_1^*(s)\|) = \langle z_2^*(s) - z_1^*(s), (z_2^*)'(s) - (z_1^*)'(s) \rangle \times \\ \times (\|z_2^*(s) - z_1^*(s)\|)^{-1} = [\langle z_2^*(t) - z_1^*(t), (z_2^*)'(s) - \\ - g(t, \hat{z}_{21}(t), u^*, y_1(s)) \rangle + \langle z_2^*(s) - z_1^*(s) - (z_2^*(t) - z_1^*(t)), \\ (z_2^*)'(s) - g(t, \hat{z}_{21}(t), u^*, y_1(s)) \rangle + \langle z_2^*(s) - z_1^*(s), \\ g(t, \hat{z}_{21}(t), u^*, y_1(s)) - (z_1^*)'(s) \rangle] (\|z_2^*(s) - z_1^*(s)\|)^{-1}.$$

We shall consider separately each term of the sum in the square brackets.

For almost all $s \in (t, T]$ we have:

$$\begin{aligned} (z_2^*)'(s) - g(t, \hat{z}_{21}(t), u^*, y_1(s)) &= g(s, z_{21}^*(s), x_2(s), v^*) - g(t, \hat{z}_{21}(t), u^*, y_1(s)) \\ &= g(s, z_{21}^*(s), x_2(s), v^*) - g(t, z_{21}^*(s), x_2(s), v^*) \\ &\quad + g(t, z_{21}^*(s), x_2(s), v^*) - g(t, z_{21}^*(t), x_2(s), v^*) \\ &\quad + g(t, z_{21}^*(t), x_2(s), v^*) - g(t, \hat{z}_{21}(t), u^*, y_1(s)). \end{aligned}$$

Using the equality $z_{21}^*(t) = \hat{z}_{21}(t) = z_{21}(t)$ we obtain

$$\begin{aligned} \langle z_2^*(t) - z_1^*(t), (z_2^*)'(s) - g(t, \hat{z}_{21}(t), u^*, y_1(s)) \rangle \\ \leq \|z_2^*(t) - z_1^*(t)\| [\eta^*(s-t) + \omega(t, \|z_{21}^*(s) - z_{21}^*(t)\|)] + \\ + \langle z_2^*(t) - z_1^*(t), g(t, z_{21}(t), x_2(s), v^*) - g(t, z_{21}(t), u^*, y_1(s)) \rangle \\ \leq \|z_2^*(t) - z_1^*(t)\| [\eta^*(s-t) + \Omega(L_1(s-t))] + 0. \end{aligned}$$

The second term of this sum will be estimated in the following way:

$$\begin{aligned} \langle z_2^*(s) - z_1^*(s) - (z_2^*(t) - z_1^*(t)), (z_2^*)'(s) - g(t, \hat{z}_{21}(t), u^*, y_1(s)) \rangle \\ \leq (\|z_2^*(s) - z_2^*(t)\| + \|z_1^*(s) - z_1^*(t)\|) (\|(z_2^*)'(s)\| + \\ + \|g(t, \hat{z}_{21}(t), u^*, y_1(s))\|) \leq 4L^2(s-t), \quad \text{for almost all } s \in (t, T]. \end{aligned}$$

Similarly, for

$$z_{11}^*(s) = a + \int_0^s f(\tau, z_{11}^*(\tau), x_1^*(\tau), y_1(\tau)) d\tau, \quad s \in [0, T]$$

the third term satisfies the inequalities

$$\begin{aligned} \langle z_2^*(s) - z_1^*(s), g(t, \hat{z}_{21}(t), u^*, y_1(s)) - (z_1^*)'(s) \rangle \\ \leq \|z_2^*(s) - z_1^*(s)\| \|g(s, z_{11}^*(s), u^*, y_1(s)) - \\ - g(t, \hat{z}_{21}(t), u^*, y_1(s))\| \leq \|z_2^*(s) - z_1^*(s)\| [\eta^*(s-t) + \\ + \|g(t, z_{11}^*(s), u^*, y_1(s)) - g(t, z_{11}^*(t), u^*, y_1(s))\| + \\ + \|g(t, z_{11}^*(t), u^*, y_1(s)) - g(t, \hat{z}_{21}(t), u^*, y_1(s))\|] \\ \leq \|z_2^*(s) - z_1^*(s)\| [\eta^*(s-t) + \omega(t, \|z_{11}^*(s) - z_{11}^*(t)\|) + \\ + \|g(t, z_{11}^*(t), u^*, y_1(s)) - g(t, z_{21}^*(t), u^*, y_1(s))\|] \\ \leq \|z_2^*(s) - z_1^*(s)\| [\eta^*(s-t) + \Omega(L_1(s-t)) + \\ + \omega(t, \|z_{11}^*(t) - z_{21}^*(t)\|)] \leq \|z_2^*(s) - z_1^*(s)\| [\eta^*(s-t) + \\ + \Omega(L_1(s-t)) + \omega(t, \|z_2^*(t) - z_1^*(t)\|)]. \end{aligned}$$

BU
W

Now, if

$$\Delta = \left(t, t + \frac{\|z_2^*(t) - z_1^*(t)\|}{3L} \right] \cap [t, T]$$

then of course $z_2^*(s) \neq z_1^*(s)$ for $s \in \Delta$. Therefore, employing the above estimations, we obtain

$$\begin{aligned} \frac{d}{ds} (\|z_2^*(s) - z_1^*(s)\|) &\leq \omega(t, \|z_2^*(t) - z_1^*(t)\|) + \eta^*(s-t) + \\ &+ \Omega(L_1(s-t)) + \frac{\|z_2^*(t) - z_1^*(t)\|}{\|z_2^*(s) - z_1^*(s)\|} [\eta^*(s-t) + \Omega(L_1(s-t))] + \frac{4L^2(s-t)}{\|z_2^*(s) - z_1^*(s)\|}, \end{aligned}$$

for almost all $s \in \Delta$.

We have

$$2L(s-t) \leq 2\|z_2^*(t) - z_1^*(t)\|/3, \quad s \in \Delta.$$

Consequently, in view of the inequality

$$1/\|z_2^*(s) - z_1^*(s)\| \leq 1/(\|z_2^*(t) - z_1^*(t)\| - 2L(s-t)), \quad s \in \Delta$$

we have

$$\begin{aligned} \frac{d}{ds} (\|z_2^*(s) - z_1^*(s)\|) &\leq \omega(t, \|z_2^*(t) - z_1^*(t)\|) + \\ &+ 4[\eta^*(s-t) + \Omega(L_1(s-t))] + \frac{12L^2(s-t)}{\|z_2^*(t) - z_1^*(t)\|} \end{aligned}$$

for almost all $s \in \Delta$. The functions η^* and Ω are nondecreasing, and so

$$\begin{aligned} &\|z_2^*(s) - z_1^*(s)\| - \|z_2^*(t) - z_1^*(t)\| \\ &\leq \int_t^s \left\{ \omega(t, \|z_2^*(t) - z_1^*(t)\|) + 4[\eta^*(\tau-t) + \Omega(L_1(\tau-t))] + \right. \\ &\quad \left. + \frac{12L^2(\tau-t)}{\|z_2^*(t) - z_1^*(t)\|} \right\} d\tau \leq [\omega(t, \|z_2^*(t) - z_1^*(t)\|) + \\ &\quad + \varrho(s-t, \|z_2^*(t) - z_1^*(t)\|)](s-t), \quad s \in \Delta. \end{aligned}$$

However, if

$$s \in \left(t + \frac{\|z_2^*(t) - z_1^*(t)\|}{3L}, \infty \right) \cap [t, T]$$

then $\|z_2^*(t) - z_1^*(t)\| < 3L(s-t)$. Hence, taking into account the definition of L ,

we obtain

$$\begin{aligned} \|z_2^*(s) - z_1^*(s)\| - \|z_2^*(t) - z_1^*(t)\| &\leq \|z_2^*(s) - z_2^*(t)\| + \\ &+ \|z_1^*(s) - z_1^*(t)\| \leq 2L(s-t) < \frac{6L^2(s-t)^2}{\|z_2^*(t) - z_1^*(t)\|} \\ &\leq [\omega(t, \|z_2^*(t) - z_1^*(t)\|) + \varrho(s-t, \|z_2^*(t) - z_1^*(t)\|)](s-t). \end{aligned}$$

Thus, we have checked all the assumptions of Theorem 1.

Furthermore, let us observe that, in this example, as T_i^* and ζ_i , $i = 1, 2$, from the proof of Theorem 1 one can assume

$$T_i^* = T, \quad i = 1, 2,$$

$$\zeta_1(\delta) = \sqrt{\delta},$$

$$\zeta_2(\delta) = 6L^2\sqrt{\delta} + 4[\eta^*(\delta) + \Omega(L_1\delta)], \quad \delta \in (0, \infty).$$

Remark 2. Theorem 2.3.1 from Friedman's book [3], p. 38, is a special case of the result described in Example 1.

4.2. Game with delay

EXAMPLE 2. Let us fix $n, k, l \in \mathbf{N}$, $a \in \mathbf{R}^n$, $U \in \text{comp}(\mathbf{R}^k)$, $V \in \text{comp}(\mathbf{R}^l)$, a continuous function $z_0: (-\infty, 0] \rightarrow \mathbf{R}^n$, $z_0(0) = a$ and a continuous function $v: [0, T] \rightarrow \mathbf{R}$, $v(t) \leq t$ for $t \in [0, T]$.

Next, let us fix any continuous functions

$$f: [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \times U \times V \rightarrow \mathbf{R}^n,$$

$$h: [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \times U \times V \rightarrow \mathbf{R},$$

such that

1° there exists a constant $K \geq 0$ for which

$$\begin{aligned} \|f(t, b, c, u, v) - f(t, \bar{b}, \bar{c}, u, v)\|^2 + (h(t, b, c, u, v) - \\ - h(t, \bar{b}, \bar{c}, u, v))^2 \leq (K \max\{\|b - \bar{b}\|, \|c - \bar{c}\|\})^2, \end{aligned}$$

for all $t \in [0, T]$; $b, \bar{b}, c, \bar{c} \in \mathbf{R}^n$, $u \in U$ and $v \in V$;

2° for any $t \in [0, T]$; $b, c, \bar{c} \in \mathbf{R}^n$ and $r \in \mathbf{R}$:

$$\max_{u \in U} \min_{v \in V} [\langle \bar{c}, f(t, b, c, u, v) \rangle + rh(t, b, c, u, v)]$$

$$= \min_{v \in V} \max_{u \in U} [\langle \bar{c}, f(t, b, c, u, v) \rangle + rh(t, b, c, u, v)].$$

Let $v_0 = \min\{v(t): t \in [0, T]\}$, and $R_0 = \max\{\|z_0(t) - a\|: t \in [v_0, 0]\}$.

From assumption 1° follows the existence of $R \geq R_0$ such that: if $x: [0, T] \rightarrow U$, $y: [0, T] \rightarrow V$ are measurable and

$$z(t) = a + \int_0^t f(s, z(s), z(v(s)), x(s), y(s)) ds, \quad t \in [0, T],$$

$$z(v(s)) = z_0(v(s)) \quad \text{when} \quad v(s) < 0,$$

then

$$\|z(t) - a\| \leq R, \quad t \in [0, T].$$

Let us take R fulfilling the above conditions and denote

$$L_1 = \max \{ \|f(t, b, c, u, v)\| : t \in [0, T]; b, c \in K(a, R), u \in U, v \in V \},$$

$$L_2 = \max \{ \|h(t, b, c, u, v)\| : t \in [0, T]; b, c \in K(a, R), u \in U, v \in V \}.$$

As in the previous example, we define M , d , Z , X and Y . For $(x, y) \in X \times Y$, let $F(x, y)$ be the pair of trajectories (z_1, z_2) , given by

$$z_1(t) = a + \int_0^t f(s, z_1(s), z_1(v(s)), x(s), y(s)) ds,$$

$$z_2(t) = \int_0^t h(s, z_1(s), z_1(v(s)), x(s), y(s)) ds, \quad t \in [0, T]$$

$$(z_1(v(s)) = z_0(v(s)) \quad \text{for} \quad v(s) < 0).$$

In the same way as in Example 1, we define φ , w , η^* and check all the assumptions of Theorem 1 modified according to Remark 1. Now, the comparative and the complementary functions assume the following form:

$$\omega(\tau, r) = Kr, \quad \tau, r \in [0, \infty),$$

$$\varrho(\tau, r) = \frac{6L^2\tau}{r} + 4[\eta^*(\tau) + KL_1\tau + KL_1w_v(\tau)], \quad \tau, r \in (0, \infty),$$

where w_v denotes the nondecreasing modulus of continuity of the function v .

Remark 3. If the functions f and h do not depend on t and $v(t) = t - v_0$, $t \in [0, T]$ then

$$\varrho(\tau, r) = \frac{6L^2\tau}{r} + 8KL_1\tau, \quad \tau, r \in (0, \infty)$$

and

$$\kappa_1(\delta, t) = (\sqrt{\delta} + 2L\delta + 8L_1\delta + 6L^2\sqrt{\delta/K})e^{Kt} - 8L_1\delta - 6L^2\sqrt{\delta/K},$$

$$\delta \in (0, \infty).$$

5. Varaiya's differential game with a dynamical system

Now, let us take any set Z and two functions:

$$d_Z: Z \times Z \rightarrow [0, \infty] (= [0, \infty) \cup \{\infty\})$$

and

$$F: X \times Y \rightarrow Z.$$

LEMMA 3. Let us assume that there exist: a function $\kappa_2: (0, \delta_2] \rightarrow [0, \infty)$, a bounded function $\varphi: Z \rightarrow \mathbb{R}$ and a nondecreasing function $w: [0, \infty) \rightarrow [0, \infty)$ such that $\kappa_2(\delta) \rightarrow 0$, $w(\delta) \rightarrow 0$ if $\delta \rightarrow 0$, and:

1° for each partition $\pi = \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$ of the interval $[0, T]$ with the diameter $\partial(\pi) \leq \delta_2$, there exists a function $f: X \rightarrow X$ having the following properties:

a) $\sup_{x \in X} \sup_{y \in Y} d_Z(F(f(x), y), F(x, y)) \leq \kappa_2(\partial(\pi))$,

b) for each pair of functions $x, \bar{x} \in X$,

$$f(\bar{x})|_{[0, t_1]} \cong f(x)|_{[0, t_1]},$$

c) for each pair of functions $x, \bar{x} \in X$ and any $i = 0, 1, \dots, n-1$, if $x|_{[0, t_i]} = \bar{x}|_{[0, t_i]}$ then

$$f(x)|_{[0, t_{i+1}]} = f(\bar{x})|_{[0, t_{i+1}]},$$

2° $|\varphi(F(x, y)) - \varphi(F(\bar{x}, y))| \leq w(d_Z(F(x, y), F(\bar{x}, y)))$, for $x, \bar{x} \in X$ and $y \in Y$ for which

$$d_Z(F(x, y), F(\bar{x}, y)) < \infty.$$

If we assume that $\mu(x, y) = \varphi(F(x, y))$, $(x, y) \in X \times Y$, then there exists a function κ_0 from the assumptions of Lemma 1.

Proof. Let $\delta_0 = \delta_2$ and $\kappa_0(\delta) = w(\kappa_2(\delta))$, $\delta \in (0, \delta_0]$. Let us take any partition π such that $\partial(\pi) \leq \delta_0$, and fix any upper strategy $e^+ \in E^+(\pi)$. It is enough to assume that $e^-(y) = f(e^+(y))$, where f is the function associated with π according to the assumptions. Then

$$\begin{aligned} \inf_{y \in Y} \mu(e^+(y), y) &\leq \inf_{y \in Y} \left[\mu(f(e^+(y)), y) + w(d_Z(F(e^+(y), y), F(f(e^+(y)), y))) \right] \\ &\leq \inf_{y \in Y} \mu(e^-(y), y) + \kappa_0(\partial(\pi)). \end{aligned}$$

This completes the proof.

Remark 4. Let us observe that in the above lemma the payoff functional μ can depend on $y \in Y$ in a quite irregular way.

5.1. Game with constraints of controls

Now we shall present an example of an immediate application of Lemma 3 to the study of a game in which the motion of player E is described by a control system with constraints of controls.

Let us observe that the set Y may be completely unrestricted. For example, it can consist of controls or trajectories of the player P . This remark refers also to all examples from Section 5.2.

EXAMPLE 3. Let us fix any $k, n \in \mathbb{N}$ and $a \in \mathbb{R}^n$. Let $g: [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, where $U \in \text{comp}(\mathbb{R}^k)$, be a continuous function such that:

1° there exists a $\lambda \geq 0$ for which

$$\langle b - a, g(t, b, u) \rangle \leq \lambda(1 + \|b - a\|^2)$$

for any $t \in [0, T]$, $b \in \mathbb{R}^n$ and $u \in U$;

2° for each bounded set $\hat{M} \subset \mathbb{R}^n$ there exists a comparative function ω satisfying the inequality

$$\|g(t, b, u) - g(t, c, u)\| \leq \omega(t, \|b - c\|)$$

for all $t \in [0, T]$, $b, c \in \hat{M}$ and $u \in U$.

Next, let

$$\alpha = \min \{\|u\| : u \in U\},$$

and let β be any fixed number such that $\beta \geq \alpha^2 T$.

We assume that $A = U$ and X is the set of all measurable functions $x: [0, T] \rightarrow U$ satisfying the inequality

$$\int_0^T \|x(s)\|^2 ds \leq \beta.$$

Let

$$R = (e^{2\lambda T} - 1)^{1/2},$$

$$L = \max \{\|g(t, b, u)\| : t \in [0, T], b \in K(a, R), u \in U\},$$

and let Ξ be the set of all functions $\xi: [0, T] \rightarrow \mathbb{R}^n$, $\xi(0) = a$, such that

$$\|\xi(t) - \xi(s)\| \leq L|t - s|, \quad t, s \in [0, T].$$

Next, let us assume that $Z = \Xi \times Y$ (we do not impose any additional restrictions on Y) and

$$d_Z((\xi, y), (\bar{\xi}, \bar{y})) = \max_{t \in [0, T]} \|\xi(t) - \bar{\xi}(t)\|,$$

for $(\xi, y), (\bar{\xi}, \bar{y}) \in Z$.

The function $F: X \times Y \rightarrow Z$ is defined in the following way: for any $x \in X$, let ξ denote a solution of the equation

$$\xi'(t) = g(t, \xi(t), x(t)), \quad \xi(0) = a, \quad t \in [0, T],$$

and for any $y \in Y$ we assume

$$F(x, y) = (\xi, y).$$

As in Example 1 let us observe that

$$\|\xi(t) - a\| \leq R \quad \text{for } t \in [0, T],$$

and associate with $\hat{M} = K(a, R)$ the comparative function ω according to condition 2°. Let

$$\Omega(r) = \max_{t \in [0, T]} \omega(t, r).$$

There exists a nondecreasing function $\eta^*: (0, \infty) \rightarrow [0, \infty)$ such that $\eta^*(\delta) \rightarrow 0$, $\delta \rightarrow 0$, and

$$\|g(t, b, u) - g(s, b, u)\| \leq \eta^*(|t - s|), \quad t, s \in [0, T], \quad b \in K(a, R), \quad u \in U.$$

Let us take $r_0 > 0$ such that for any $r \in [0, r_0]$ there exists a maximal solution v of the equation

$$v'(s) = \omega(s, v(s)), \quad v(0) = r,$$

defined on the interval $[0, T]$. Let us denote this solution by $v(r)$. Now, let us choose any $\delta_2 \in (0, T]$ such that

$$4L\delta_2 + T[\eta^*(\delta_2) + \Omega(L\delta_2)] \leq r_0,$$

and for $\delta \in (0, \delta_2]$ assume

$$x_2(\delta) = v(4L\delta + T[\eta^*(\delta) + \Omega(L\delta)])(T).$$

Let us fix any partition π of the interval $[0, T]$ and assume that $\partial(\pi) \leq \delta_2$.

Let u_0 be any element of the set U such that $\|u_0\| = x$ and let

$$f(x)(t) = \begin{cases} u_0, & t \in [0, \partial(\pi)], \\ x(t - \partial(\pi)), & t \in (\partial(\pi), T], \end{cases}$$

for $x \in X$.

We shall check condition a) from the assumptions of Lemma 3. In order to do it, for any fixed $x \in X$, we shall estimate the difference between

$$\xi(t) = a + \int_0^t g(s, \xi(s), x(s)) ds$$

and

$$\xi^*(t) = a + \int_0^t g(s, \xi^*(s), f(x)(s)) ds.$$

Let us fix any $t \in [0, T]$. If $t \leq \partial(\pi)$ then

$$\|\xi(t) - \xi^*(t)\| \leq 2L\partial(\pi) \leq \kappa_2(\partial(\pi)).$$

However, if $t \in (\partial(\pi), T]$ then after standard calculations we obtain

$$\|\xi(t) - \xi^*(t)\| \leq 4L\partial(\pi) + T[\eta^*(\partial(\pi)) + \Omega(L\partial(\pi))] + \int_0^t \omega(s, \|\xi(s) - \xi^*(s)\|) ds.$$

Hence

$$\|\xi(t) - \xi^*(t)\| \leq \kappa_2(\partial(\pi)) \quad \text{for } t \in [0, T].$$

One can readily prove that κ_2 satisfies the remaining assumptions of Lemma 3.

5.2. Game with control by trajectories

In this section games nearest to those of Varaiya ones [7, 8] will be considered.

Let us assume that $M = A \times B$, $Z = X \times Y$, and d_A is any function $d_A: A \times A \rightarrow [0, \infty)$. Assuming for $(x, y), (\bar{x}, \bar{y}) \in Z$,

$$d_Z((x, y), (\bar{x}, \bar{y})) = \sup_{t \in [0, T]} d_A(x(t), \bar{x}(t)),$$

$$F(x, y) = (x, y), \quad \text{and} \quad \mu(x, y) = \varphi(x, y),$$

we obtain:

LEMMA 4. *If there exist: a function $\kappa_2: (0, \delta_2] \rightarrow [0, \infty)$ and a nondecreasing function $w: [0, \infty) \rightarrow [0, \infty)$ for which the following is true: $\kappa_2(\delta) \rightarrow 0$ and $w(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, and*

1° *for each partition $\pi = \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$ of the interval $[0, T]$ with the diameter $\partial(\pi) \leq \delta_2$ there exists a function $f: X \rightarrow X$ such that*

$$\text{a) } \sup_{x \in X} \sup_{t \in [0, T]} d_A(x(t), f(x)(t)) \leq \kappa_2(\partial(\pi)),$$

b) *for each pair $x, \bar{x} \in X$,*

$$f(x)|_{[0, t_1]} = f(\bar{x})|_{[0, t_1]},$$

c) *for each pair $x, \bar{x} \in X$ and $i = 0, 1, \dots, n-1$, if $x|_{[0, t_i]} = \bar{x}|_{[0, t_i]}$ then*

$$f(x)|_{[0, t_{i+1}]} = f(\bar{x})|_{[0, t_{i+1}]},$$

2° $|\mu(x, y) - \mu(\bar{x}, y)| \leq w\left(\sup_{t \in [0, T]} d_A(x(t), \bar{x}(t))\right)$, for $x, \bar{x} \in X$ and $y \in Y$ such that

$$\sup_{t \in [0, T]} d_A(x(t), \bar{x}(t)) < \infty,$$

then there exists a function x_0 from the assumptions of Lemma 1.

5.2.1. Approximation condition

Let (A, d_A) be a metric space. Furthermore, let us assume that trajectories from the set X are equicontinuous and $x(0) = \bar{x}(0)$ for $x, \bar{x} \in X$.

THEOREM 2. *If there exist a comparative function ω and a complementary function ϱ having the following property:*

for any $t \in [0, T]$ and $\bar{x}_1, \bar{x}_2 \in X$, $\bar{x}_1(t) \neq \bar{x}_2(t)$ there exists $x_1^ \in X$, $x_1^*|_{[0, t]} = \bar{x}_1|_{[0, t]}$ such that for all $x_2 \in X$, $x_2|_{[0, t]} = \bar{x}_2|_{[0, t]}$ and $s \in (t, T]$ the inequality*

$$d_A(x_1^*(s), x_2(s)) - d_A(\bar{x}_1(t), \bar{x}_2(t)) \leq [\omega(t, d_A(\bar{x}_1(t), \bar{x}_2(t))) + \varrho(s-t, d_A(\bar{x}_1(t), \bar{x}_2(t)))](s-t)$$

is true, and the payoff functional μ is the same as in Lemma 4, then there exists a function x_0 from the assumptions of Lemma 1, which results in the existence of the value of the game $G(X, Y, \mu)$.

Proof. Let us assume $d((a, b), (\bar{a}, \bar{b})) = d_A(a, \bar{a})$, $a, \bar{a} \in A$, $b, \bar{b} \in B$. Then, of course, $d(z(0), \bar{z}(0)) = 0$ for all $z, \bar{z} \in Z$, and the approximation condition from Theorem 1 is satisfied. Let us define functions $x_1(\delta, t)$ and $x_0(\delta)$ identically as in the proof of Theorem 1. Next, let us fix any partition π of the interval $[0, T]$ with the diameter $\partial(\pi) \leq \delta_0$ and upper strategy $e^+ \in E^+(\pi)$. By employing the same methods as in the proof of Theorem 1 one can prove an existence of a strategy $e^- \in E^-(\pi)$ which satisfies the condition

$$d(F(e^-(y), y)(t), F(e^+(y), y)(t)) \leq x_1(\partial(\pi), T), \quad t \in [0, T]$$

(in this case $\bar{y} = y$).

In order to construct such a strategy e^- , we should assume, in the definition of the function Q_i , that $Q_i((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2))$ is (for $d(F(\bar{x}_1, \bar{y}_1)(t_i), F(\bar{x}_2, \bar{y}_2)(t_i)) \geq \zeta_1(\partial(\pi))$) the pair (x_1^*, \bar{y}_2) , where x_1^* is associated with t_i and \bar{x}_1, \bar{x}_2 according to the assumption of Theorem 2.

EXAMPLE 4. Let us fix any $a \in \mathbb{R}^n$. Let $G: [0, T] \times \mathbb{R}^n \rightarrow \text{comp}(\mathbb{R}^n)$ be an orientor field which is continuous (in the Hausdorff metric) and such that

1° there exists a $\lambda \geq 0$ satisfying the following inequality

$$\langle b - a, u \rangle \leq \lambda(1 + \|b - a\|^2)$$

for all $b \in \mathbf{R}^n$, $t \in [0, T]$ and $u \in G(t, b)$;

2° for each bounded set $\hat{M} \subset \mathbf{R}^n$ there exists a comparative function ω fulfilling the inequality

$$D(G(t, b), G(t, c)) \leq \omega(t, \|b - c\|)$$

for all $t \in [0, T]$, $b, c \in \hat{M}$ (by D we denote the distance in Hausdorff metric).

Let us assume that $(A, d_A) = (\mathbf{R}^n, \|\cdot\|)$. Let X be the set of all trajectories x of the field G defined on $[0, T]$ and satisfying the condition $x(0) = a$.

If $x \in X$ then $\|x(t) - a\| \leq (e^{2\lambda T} - 1)^{1/2} = R$, $t \in [0, T]$. Let

$$L = \max \{ \|u\| : u \in G(t, b), t \in [0, T], b \in K(a, R) \}.$$

Now, let us associate with $\hat{M} = K(a, R)$ a comparative function ω according to the condition 2°. Let

$$\Omega(r) = \max_{t \in [0, T]} \omega(t, r).$$

There exists a nondecreasing function $\eta^*: [0, \infty) \rightarrow [0, \infty)$, $\eta^*(\delta) \rightarrow 0$ for $\delta \rightarrow 0$, such that

$$D(G(t, b), G(s, b)) \leq \eta^*(|s - t|), \quad t, s \in [0, T], b \in K(a, R).$$

Using the appropriate constructions carried out in the proof of the Filippov theorem [2], one can prove that there exists a nondecreasing function $\xi: (0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

a) $\lim_{r \rightarrow 0} \xi(r) = 0$;

b) for any $t \in [0, T]$, $b \in K(a, R)$ and $u \in G(t, b)$ there exists a trajectory x of the orientor field G , $x(t) = b$, such that $\|x'(s) - u\| \leq \xi(s - t)$ almost everywhere on $(t, T]$.

Identically as in Example 1 one can check that it is enough to take as the complementary function

$$\varrho(\tau, r) = 6L^2 \tau / r + \xi(\tau) + 3[\eta^*(\tau) + \Omega(L\tau)], \quad \tau, r \in (0, \infty).$$

5.2.2. Alternative way

In a further part of this section (A, d_A) denotes the metric space and functions from the set X are equicontinuous. Let us fix any $a \in A$ and assume that $x(0) = a$ for $x \in X$.

For $h \in [0, T)$ and $x \in X$ let

$$x_h(t) = \begin{cases} a, & t \in [0, h], \\ x(t-h), & t \in (h, T]. \end{cases}$$

THEOREM 3. *If $x_h \in X$, for any $h \in [0, \delta^*] \subset [0, T)$ and $x \in X$, then it may be assumed that*

$$\kappa_2(\delta) = w_X(\delta), \quad \delta \in (0, T),$$

where κ_2 is the function from Lemma 4 and w_X denotes the nondecreasing modulus of continuity of the trajectories from the set X .

Proof. For any $\delta_2 \in (0, \delta^*]$ and the partition π of the interval $[0, T]$ with the diameter $\delta(\pi) \leq \delta_2$ it is enough to assume

$$f(x) = x_{\pi(x)}, \quad x \in X.$$

EXAMPLE 5. (See also [5]). Let us assume that (A, d_A) is any metric space. $a \in A$ is any fixed point of it and $w^*: [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function such that $w^*(\delta) \rightarrow 0$ when $\delta \rightarrow 0$. Let X be the set of all functions $x: [0, T] \rightarrow A$, $x(0) = a$, for which

$$d_A(x(s), x(t)) \leq w^*(|s-t|), \quad s, t \in [0, T].$$

Obviously the assumptions of Theorem 3 are fulfilled.

EXAMPLE 6. Let a natural number n be given. Let us fix any element $a \in \mathbb{R}^n$ and an orientor field $G: \mathbb{R}^n \rightarrow \text{comp}(\mathbb{R}^n)$ such that $0 \in G(a)$. Let us assume that $(A, d_A) = (\mathbb{R}^n, \|\cdot\|)$ and X is the set of all trajectories x of the field G defined on $[0, T]$ and satisfying the condition $x(0) = a$. Next, let us assume that X is nonempty and the trajectories from X are equicontinuous.

To use Theorem 3 it is enough to check that $x_h \in X$ for all $h \in (0, \delta^*] \subset [0, T]$, which is obviously true.

EXAMPLE 7. Let us assume in the previous example that $n = 1$, $a = 0$ and

$$G(b) = \begin{cases} \{-1, 0, 1\}, & b = 0, \\ \{1\}, & b > 0, \\ \{-1\}, & b < 0. \end{cases}$$

Then according to Theorem 3 the assumptions of Lemma 4 are fulfilled. However, in this example the approximation condition from Theorem 2 cannot be satisfied.

6. Krasovskii's differential game with a dynamical system

Let M be a fixed set and let a function $d: M \times M \rightarrow [0, \infty)$ be the pseudometric. Let us fix any $m \in M$. Let \hat{Z} be a nonempty set of functions

$z: [0, T] \rightarrow M$ and let us assume that for any $R > 0$ functions $z \in \hat{Z}$ such that $d(z(0), m) \leq R$ are equicontinuous. Next, for any $t \in [0, T]$ let a function $F_t: \hat{Z} \times X \times Y \rightarrow \hat{Z}$ be given. We assume that for $z, \bar{z} \in \hat{Z}$ and $(x, y), (\bar{x}, \bar{y}) \in X \times Y$,

$$1^\circ F_t(z, x, y)|_{[0,t]} = z|_{[0,t]};$$

2° if $x|_{[0,t]} = \bar{x}|_{[0,t]}$ and $y|_{[0,t]} = \bar{y}|_{[0,t]}$ then

$$F_0(z, x, y)|_{[0,t]} = F_0(z, \bar{x}, \bar{y})|_{[0,t]};$$

3° if $d(z(0), \bar{z}(0)) = 0$ then $F_0(z, x, y) = F_0(\bar{z}, x, y)$.

For $t \in [0, T]$, $z \in \hat{Z}$; $x, \bar{x} \in X$, $y, \bar{y} \in Y$ we introduce the following notation:

$$(x \langle t \rangle \bar{x})(s) = \begin{cases} x(s), & s \in [0, t), \\ \bar{x}(s), & s \in [t, T], \end{cases}$$

$$(y \langle t \rangle \bar{y})(s) = \begin{cases} y(s), & s \in [0, t), \\ \bar{y}(s), & s \in [t, T], \end{cases}$$

and assume that

$$x \langle t \rangle \bar{x} \in X, \quad y \langle t \rangle \bar{y} \in Y.$$

$$F_t(F_0(z, x, y), \bar{x}, y \langle t \rangle \bar{y}) = F_0(z, x \langle t \rangle \bar{x}, y \langle t \rangle \bar{y})$$

and

$$F_t(F_0(z, x, y), x \langle t \rangle \bar{x}, \bar{y}) = F_0(z, x \langle t \rangle \bar{x}, y \langle t \rangle \bar{y}).$$

Moreover, let us fix a function $\varphi: \hat{Z} \rightarrow \mathbb{R}$ and assume that for each $R \geq 0$ there exists a nondecreasing function $w: [0, \infty) \rightarrow [0, \infty)$, $w(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, such that for $z, \bar{z} \in \hat{Z}$ if

$$d(z(0), m) \leq R \quad \text{and} \quad d(\bar{z}(0), m) \leq R$$

then

$$|\varphi(z) - \varphi(\bar{z})| \leq w\left(\max_{t \in [0, T]} d(z(t), \bar{z}(t))\right).$$

Let

$$Z = \{z \in \hat{Z}: d(z(0), m) = 0\}.$$

Let us assume that Z is compact in the sense that for each sequence $z_j \in Z$, $j \in \mathbb{N}$ there exist a subsequence z_{j_k} , $k \in \mathbb{N}$, and $z \in Z$ for which

$$\max_{t \in [0, T]} d(z_{j_k}(t), z(t)) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

(In this order it is enough to assume that the pseudometric space (M, d) is locally compact in an analogous sense.)

Let $q: [0, T] \times \hat{Z} \rightarrow X$ be any function such that $q(t, z) = q(t, \bar{z})$, for

any $t \in [0, T]$ and $z; z \in \hat{Z}$, $z|_{[0,t]} = z|_{[0,t]}$. Every such a function is said to be the *Krasovskii strategy* for the player E .

For any partition $\pi = \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$ of the interval $[0, T]$ we say that the trajectory $z \in \hat{Z}$ is consistent with the function q on the partition π (which we denote by the symbol $z \approx q$) if

$$z|_{[t_i, t_{i+1}]} \in F_{t_i}(z, q(t_i, z), Y)|_{[t_i, t_{i+1}]}, \quad i = 0, 1, \dots, n-1.$$

For $R \geq 0$ we denote by $[q, \pi, R]$ the set of all trajectories $z \in \hat{Z}$ consistent with the function q on the partition π which satisfy the inequality

$$d(z(0), m) \leq R.$$

Of course, the set $[q, \pi, R]$ is nonempty.

Next, let us denote by $[q]$ the set of all trajectories $z \in \hat{Z}$ for which there exist: a sequence π_j , $j \in N$ of partitions of the interval $[0, T]$ with diameters $\partial(\pi_j)$ tending to zero, a sequence R_j , $j \in N$ of positive numbers tending to zero, and a sequence $z_j \in [q, \pi_j, R_j]$, $j \in N$ such that

$$\max_{t \in [0, T]} d(z_j(t), z(t)) \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.$$

Obviously, we again have $[q] \neq \emptyset$, and $d(z(0), m) = 0$ for $z \in [q]$.

We shall say that the strategy q secures the player E the result $\lambda \in R$ if, for all $z \in [q]$,

$$\varphi(z) \geq \lambda.$$

The least upper bound of a $\lambda \in R$ for which there exists a strategy q securing player E the result λ we call the lower value V_K^- (in Krasovskii's sense) of the game $G_K(X, Y, \varphi)$. Analogously we define the Krasovskii strategies for the player P , the security of the result $\lambda \in R$ by these strategy, and the upper value V_K^+ (in Krasovskii's sense) of the game $G_K(X, Y, \varphi)$.

If $V_K^- = V_K^+$, we say that the game $G_K(X, Y, \varphi)$ has the value V_K (in Krasovskii's sense). Moreover, if there exist strategies securing both players the value V_K then we say that $G_K(X, Y, \varphi)$ has a saddle point.

Finally, let us fix any $z_0 \in Z$ and assume that for $(x, y) \in X \times Y$

$$F(x, y) = F_0(z_0, x, y) \quad \text{and} \quad \mu(x, y) = \varphi(F(x, y)).$$

6.1. Correlations between the games $G(X, Y, \mu)$ and $G_K(X, Y, \varphi)$

LEMMA 5. For any $\varepsilon > 0$ there exist $\delta > 0$ and $R^* > 0$ such that for each partition π , $\partial(\pi) \leq \delta$, of the interval $[0, T]$, any $R \in [0, R^*]$ and each $z \in [q, \pi, R]$ there exists a $\bar{z} \in [q]$ for which

$$\max_{t \in [0, T]} d(z(t), \bar{z}(t)) \leq \varepsilon.$$

LEMMA 6. For any Krasovskii strategy q of player E and the partition $\pi = \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$, of the interval $[0, T]$ there exists a strategy $e^- \in E^-(\pi)$ such that, for any $R \geq 0$ and any $y \in Y$,

$$F(e^-(y), y) \in [q, \pi, R].$$

Proof. It is enough to show that $F(e^-(y), y) \in [q, \pi, 0]$. Let us fix any Krasovskii strategy q and a partition π . Let $x^{(0)} \in X$ denote any fixed element of the set X . For any $y \in Y$ and $i = 0, 1, \dots, n-1$ let

$$x^{(i+1)} = x^{(i)} \langle t_i \rangle q(t_i, F(x^{(i)}, y)),$$

next, let us assume that

$$e^-(y) = x^{(n)}.$$

By induction one can easily check that $e^- \in E^-(\pi)$. Moreover, it should be proved that $F(e^-(y), y) \in [q, \pi, 0]$, $y \in Y$. Let us fix $y \in Y$ and assume that $z = F(e^-(y), y)$. Obviously $d(z(0), m) = 0$. Next, let us take any $i = 0, 1, \dots, n-1$. From the properties of the functions F and q follows

$$\begin{aligned} z|_{[t_i, t_{i+1}]} &= F(x^{(i+1)}, y)|_{[t_i, t_{i+1}]} \\ &= F_0(z_0, x^{(i)} \langle t_i \rangle q(t_i, F(x^{(i)}, y)), y \langle t_i \rangle y)|_{[t_i, t_{i+1}]} \\ &= F_{t_i}(F_0(z_0, x^{(i)}, y), q(t_i, F(x^{(i)}, y)), y)|_{[t_i, t_{i+1}]} \\ &= F_{t_i}(z, q(t_i, z), y)|_{[t_i, t_{i+1}]} \in F_{t_i}(z, q(t_i, z), Y)|_{[t_i, t_{i+1}]}. \end{aligned}$$

Thus the proof of Lemma 6 has been finished.

CONCLUSION. If there exists a value V_K of the game $G_K(X, Y, \varphi)$ then there exists a value V of the game $G(X, Y, \mu)$ and $V = V_K$.

Proof. Let us take any $V_1, V_2 \in \mathbf{R}$ such that $V_1 < V_2 < V_K^-$. Let q denote a Krasovskii strategy which secures player E the result V_2 . Let $\varepsilon > 0$ be such that $w(\varepsilon) \leq V_2 - V_1$. From Lemma 5 follows the existence of $\delta, R^* > 0$ such that for the partition π of the interval $[0, T]$, $\beta(\pi) \leq \delta$, and $R \in [0, R^*]$ if $z \in [q, \pi, R]$ then there exists a $\hat{z} \in [q]$ for which

$$\max_{t \in [0, T]} d(z(t), \hat{z}(t)) \leq \varepsilon.$$

However, from Lemma 6 follows the existence of a strategy $e^- \in E^-(\pi)$ satisfying the condition $F(e^-(y), y) \in [q, \pi, R]$, $y \in Y$. This being so, for each $y \in Y$ there exists a $\bar{z} \in [q]$ such that

$$\max_{t \in [0, T]} d(F(e^-(y), y)(t), \bar{z}(t)) \leq \varepsilon.$$

Hence

$$\varphi(F(e^-(y), y)) = \varphi(\hat{z}) + \varphi(F(e^-(y), y)) - \varphi(\hat{z}) \geq V_2 - w(\varepsilon) \geq V_1.$$

The remaining part of the proof is analogous to the above.

6.2. Sufficient condition

Now we shall present a theorem which gives the sufficient conditions of the existence of the value V_K of the game $G_K(X, Y, \varphi)$. In the same way as before, one can check the fulfilment of these conditions in the concrete examples of games.

THEOREM 4. *Let us assume that for any $t \in [0, T]$, $z \in Z$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $\bar{z} \in Z$, $z \in \hat{Z}$, $x \in X$ and $y \in Y$ if $d(z(s), \bar{z}(s)) \leq \delta$, $s \in [0, t]$ then*

$$d(F_t(z, x, y)(s), F_t(\bar{z}, x, y)(s)) \leq \varepsilon, \quad s \in [t, T].$$

Let us also assume that for any $R > 0$ there exist a comparative function ω and a complementary function ϱ having the following property:

for any $t \in [0, T]$, $z_1, z_2 \in Z$, $d(z_1(t), z_2(t)) > 0$, $d(z_i(0), m) \leq R$, $i = 1, 2$, there exist $x^ \in X$ and $y^* \in Y$ such that for all $x \in X$, $y \in Y$ and $s \in (t, T]$ the following inequality holds*

$$d(F_t(z_1, x^*, y)(s), F_t(z_2, x, y^*)(s)) - d(z_1(t), z_2(t)) \leq [\omega(t, d(z_1(t), z_2(t))) + \varrho(s-t, d(z_1(t), z_2(t)))](s-t)$$

is true.

Then there exists a value V_K of the game $G_K(X, Y, \varphi)$.

Proof. It is easy to show that all the assumptions of Theorem 1 are fulfilled. Thus there exists a value V of the game $G(X, Y, \mu)$.

Let e_j^- , $j \in N$ be a sequence of lower strategies such that

$$\mu(e_j^-(y), y) \geq V - 1/j, \quad y \in Y.$$

Next, let W denote the set of those $z \in Z$ for which there exist sequences $j_k \in N$ and $y_k \in Y$, $k \in N$ such that

$$F(e_{j_k}^-(y_k), y_k) \xrightarrow{[0, T]} z \quad \text{as } k \rightarrow \infty.$$

The set W is closed, and so it is also compact. We shall prove that it is E -stable, i.e., that for each $t \in [0, T]$, $z \in W$ and $y \in Y$ there exists a $\bar{z} \in \overline{F_t(z, X, y)}$ such that $\bar{z} \in W$, where $\overline{F_t(z, X, y)}$ denotes the closure of the set $F_t(z, X, y)$.

Let us fix any $t \in [0, T]$, $z \in W$ and $y \in Y$. There exist sequences $j_k \in N$ and $y_k \in Y$, $k \in N$ such that

$$F(e_{j_k}^-(y_k), y_k) \xrightarrow{[0, T]} z \quad \text{as } k \rightarrow \infty.$$

Let us note that for each $k \in N$ there exists an $x_k \in X$ such that

$$e_{j_k}^-(y_k \langle t \rangle y) = e_{j_k}^-(y_k) \langle t \rangle x_k.$$

Thus, we have

$$\begin{aligned} F(e_{j_k}^-(y_k \langle t \rangle y), y_k \langle t \rangle y) &= F_0(z_0, e_{j_k}^-(y_k) \langle t \rangle x_k, y_k \langle t \rangle y) \\ &= F_t(F_0(z_0, e_{j_k}^-(y_k), y_k), e_{j_k}^-(y_k) \langle t \rangle x_k, y) \\ &= F_t(F(e_{j_k}^-(y_k), y_k), e_{j_k}^-(y_k) \langle t \rangle x_k, y). \end{aligned}$$

From the compactness of $F_t(z, X, y)$ it follows that there exists a sequence $k_i, i \in N$ such that the subsequence $F_t(z, e_{j_{k_i}}^-(y_{k_i}) \langle t \rangle x_{k_i}, y), i \in N$ of the sequence $F_t(z, e_{j_k}^-(y_k) \langle t \rangle x_k, y), k \in N$ is uniformly convergent on $[0, T]$ to a $\bar{z} \in \overline{F_t(z, X, y)}$. Since

$$F(e_{j_{k_i}}^-(y_{k_i}), y_{k_i}) \underset{[0, T]}{\rightrightarrows} z, \quad i \rightarrow \infty,$$

we have

$$\begin{aligned} \max_{s \in [t, T]} d(F_t(F(e_{j_{k_i}}^-(y_{k_i}), y_{k_i}), e_{j_{k_i}}^-(y_{k_i}) \langle t \rangle x_{k_i}, y)(s), \\ F_t(z, e_{j_{k_i}}^-(y_{k_i}) \langle t \rangle x_{k_i}, y)(s)) \rightarrow 0, \quad i \rightarrow \infty. \end{aligned}$$

Furthermore, since

$$F_t(F(e_{j_{k_i}}^-(y_{k_i}), y_{k_i}), e_{j_{k_i}}^-(y_{k_i}) \langle t \rangle x_{k_i}, y)|_{[0, t]} = F(e_{j_{k_i}}^-(y_{k_i}), y_{k_i})|_{[0, t]}$$

and

$$F_t(z, e_{j_{k_i}}^-(y_{k_i}) \langle t \rangle x_{k_i}, y)|_{[0, t]} = z|_{[0, t]},$$

the maximum may be taken for the whole interval $[0, T]$. This being so,

$$F(e_{j_{k_i}}^-(y_{k_i} \langle t \rangle y), y_{k_i} \langle t \rangle y) = F_t(F(e_{j_{k_i}}^-(y_{k_i}), y_{k_i}), e_{j_{k_i}}^-(y_{k_i}) \langle t \rangle x_{k_i}, y) \underset{[0, T]}{\rightrightarrows} \bar{z},$$

and thus $\bar{z} \in W$.

Let us fix any $R > 0$ and associate with R the comparative function ω and the complementary function ϱ according to the approximation condition. Let us take $r_0 \in (0, R]$ and $h_0 \in (0, \infty)$ such that for any $r \in [0, r_0]$ and $h \in [0, h_0]$ there exists a minimal solution v of the equation

$$v'(s) = \omega(s, v(s)) + h, \quad v(0) = r$$

defined on the interval $[0, T]$.

Let us define $v(r, h), \eta, T_i^*$ and $\zeta_i (i = 1, 2)$ identically as in the proof of Theorem 1, and choose a positive number δ_0 which satisfies the following inequalities:

$$\begin{aligned} \delta_0 &\leq T_i^*, \quad i = 1, 2, \\ \zeta_1(\delta_0) + 2w_R(\delta_0) &\leq r_0, \\ \zeta_2(\delta_0) + \eta(\delta_0) &\leq h_0, \end{aligned}$$

where w_R denotes a nondecreasing common modulus of continuity of trajectories $z \in \hat{Z}$ such that $d(z(0), m) \leq R$.

For $\delta \in (0, \delta_0]$ and $t \in [0, T]$ let us assume

$$\kappa_1(\delta, t) = v(\zeta_1(\delta) + 2w_R(\delta), \zeta_2(\delta) + \eta(\delta))(t)$$

and

$$\kappa_0(\delta) = w(\kappa_1(\delta, T)),$$

where w is the modulus of continuity of the function φ associated with R . Of course, $\kappa_0(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Let us define the function $q: [0, T] \times \hat{Z} \rightarrow X$ as described below.

Let us fix any $t \in [0, T]$ and $z \in \hat{Z}$. Let $z_2 \in W$ be such that

$$\max_{s \in [0, t]} d(z_2(s), z(s)) = \min_{z \in W} \max_{s \in [0, t]} d(\bar{z}(s), z(s)).$$

If

$$d(z(0), m) > R \quad \text{or} \quad d(z_2(t), z(t)) = 0,$$

then $q(t, z)$ is any fixed element from the set X . Conversely, if

$$d(z(0), m) \leq R \quad \text{and} \quad d(z_2(t), z(t)) > 0$$

then $q(t, z)$ is any fixed element x^* which is associated with t , $z_1 = z$ and z_2 according to the approximation condition.

One can easily check that the function q may be constructed in such a way that, in addition,

$$q(t, z) = q(t, \bar{z}),$$

for any $t \in [0, T]$ and $z, \bar{z} \in \hat{Z}$ such that

$$z|_{[0, t]} = \bar{z}|_{[0, t]}.$$

Now, let us fix any partition $\pi = \{t_0, t_1, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = T$ of the interval $[0, T]$ with the diameter $\partial(\pi) \leq \delta_0$. We shall check that for each

$$z \in [q, \pi, \zeta_1(\partial(\pi)) + 2w_R(\partial(\pi))]$$

there exists a $\hat{z} \in W$ for which

$$d(z(t), \hat{z}(t)) \leq \kappa_1(\partial(\pi), T), \quad t \in [0, T].$$

Let $z^{(0)}$ be any element from the set W . Then

$$d(z(0), z^{(0)}(0)) = d(z(0), \hat{m}) + 0 \leq \zeta_1(\partial(\pi)) + 2w_R(\partial(\pi)) = \kappa_1(\partial(\pi), 0).$$

Let us assume that for any $i = 0, 1, \dots, n-1$ there exists a $z^{(i)} \in W$ such that

$$d(z(t), z^{(i)}(t)) \leq \kappa_1(\partial(\pi), t_i), \quad t \in [0, t_i].$$

We shall prove that in this case there exists a $z^{(i+1)} \in W$ for which

$$d(z(t), z^{(i+1)}(t)) \leq \kappa_1(\partial(\pi), t_{i+1}), \quad t \in [0, t_{i+1}].$$

Let us take that $z_2 \in W$ which has been associated with t_i and z in the definition of $q(t_i, z)$. From the definition z_2 we have

$$d(z(t), z_2(t)) \leq d(z(t), z^{(i)}(t)) \leq \kappa_1(\partial(\pi), t_i), \quad t \in [0, t_i].$$

If $d(z_2(t_i), z(t_i)) < \zeta_1(\partial(\pi))$ then we assume

$$z^{(i+1)} = z_2.$$

Then, for $t \in [0, t_i]$,

$$d(z(t), z^{(i+1)}(t)) \leq \kappa_1(\partial(\pi), t_i) \leq \kappa_1(\partial(\pi), t_{i+1}),$$

whereas, for $t \in [t_i, t_{i+1}]$,

$$d(z(t), z^{(i+1)}(t)) = d(z(t), z_2(t)) \leq \zeta_1(\partial(\pi)) + 2w_R(t - t_i) \leq \kappa_1(\partial(\pi), t_{i+1}),$$

and so

$$d(z(t), z^{(i+1)}(t)) \leq \kappa_1(\partial(\pi), t_{i+1}), \quad t \in [0, t_{i+1}].$$

Conversely, if $d(z_2(t_i), z(t_i)) \geq \zeta_1(\partial(\pi))$ then also $d(z_2(t_i), z(t_i)) > 0$, and we associate with $t = t_i$, $z_1 = z$ and z_2 , according to the approximation condition, such $x^* \in X$ and $y^* \in Y$ that $x^* = q(t_i, z)$. The set W is E -stable, and thus for $t = t_i$, $z = z_2$ and $y = y^*$ there exists a $\bar{z} \in \overline{F_{t_i}(z_2, X, y^*)}$ such that $\bar{z} \in W$. Let us assume

$$z^{(i+1)} = \bar{z}.$$

Then there exist sequences $x_j \in X$, $z_j^* \in \hat{Z}$, $j \in \mathbb{N}$ such that

$$z_j^* = F_{t_i}(z_2, x_j, y^*) \underset{[0, T]}{\rightrightarrows} z^{(i+1)} \quad \text{as } j \rightarrow \infty.$$

Let us note that

$$F_{t_i}(z_2, x_j, y^*)|_{[0, t_i]} = z_2|_{[0, t_i]}$$

and thus

$$z^{(i+1)}|_{[0, t_i]} = z_2|_{[0, t_i]},$$

hence, as before, for $t \in [0, t_i]$,

$$d(z(t), z^{(i+1)}(t)) \leq \kappa_1(\partial(\pi), t_{i+1}).$$

However, if $t \in (t_i, t_{i+1}]$ then we shall take advantage of the fact that $z \stackrel{\pi}{\sim} q$, owing to which there exists a $y \in Y$ such that

$$z|_{(t_i, t_{i+1}]} = F_{t_i}(z, q(t_i, z), y)|_{(t_i, t_{i+1}]}.$$

As in the proof of Theorem 1, on the basis of the approximation condition, for any $j \in N$ we obtain

$$\begin{aligned} & d(z(t), z_j^*(t)) - d(z(t_i), z_j^*(t_i)) \\ &= d(F_{t_i}(z, q(t_i, z), y)(t), F_{t_i}(z_2, x_j, y^*)(t)) - d(z(t_i), z_2(t_i)) \\ &\leq \kappa_1(\partial(\pi), t) \leq \kappa_1(\partial(\pi), t_{i+1}). \end{aligned}$$

Thus we have proved by induction that for each $i = 0, 1, \dots, n$ there exists a $z^{(i)} \in W$ such that $d(z(t), z^{(i)}(t)) \leq \kappa_1(\partial(\pi), t_i)$, $t \in [0, t_i]$. Now we only have to assume that $\hat{z} = z^{(n)}$.

From the fact proved above it follows that $\varphi(z) \geq V - \kappa_0(\partial(\pi))$, for $z \in [q, \pi, \zeta_1(\partial(\pi)) + 2w_R(\partial(\pi))]$: In an analogous way one can prove the existence of the Krasovskii strategy \hat{q} of player P for which the fact that $z \in [\hat{q}, \pi, \zeta_1(\partial(\pi)) + 2w_R(\partial(\pi))]$ implies $\varphi(z) \leq V + \kappa_0(\partial(\pi))$. Thus, there exists value V_K of the game $G_K(X, Y, \varphi)$. Obviously, the strategies q and \hat{q} point to the saddle point of the game.

References

- [1] P. Borówko, *Application of a certain method of approximation to the theory of orientor fields and differential games*, J. Differential Equations 48 (1983), 17-34.
- [2] A. F. Filippov, *On the existence of solutions of multivalued differential equations* (in Russian), Mat. Zametki 10 (1971), 307-313.
- [3] A. Friedman, *Differential Games*, Wiley-Interscience, New York/London/Sydney/Toronto 1971.
- [4] N. N. Krasovskii and A. I. Subbotin, *Positional Differential Games* (in Russian), Nauka, Moscow 1974.
- [5] C. Ryll-Nardzewski, *A theory of pursuit and evasion*, in: *Advances in Game Theory* (M. Dresher, L. S. Shapley and A. W. Tucker, Eds.), Princeton University Press, Princeton 1964.
- [6] W. Rzymowski, *A counterexample in differential games with dynamical systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (1981), 417-418.
- [7] P. Varaiya, *Differential games with dynamical systems*, in: *Differential Games and Related Topics* (H. W. Kuhn and G. P. Szegö, Eds.), North-Holland, Amsterdam/London 1971.
- [8] P. Varaiya and J. Lin, *Existence of saddle points in differential games*, SIAM J. Control 7 (1969), 141-157.