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SOME REMARKS ON REPRESENTABLE EQUIVALENCES

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We investigate the finite-dimensional modules over finite-dimensional algebras which induce an equivalence satisfying the hypotheses of Menini–Orsatti's Representation Theorem. These modules seem to be a small extension of tilting modules and a large extension of quasiprogenerators.

In a recent paper, Menini and Orsatti obtained a theorem ([4], Representation Theorem 3.1) between categories of modules which extends Fuller's theorem ([1], Theorem 1.1) to a more general situation. In fact, they proved ([4], Section 4) that any tilting module satisfies the hypotheses of their representation theorem, and that a tilting module satisfies the hypotheses of Fuller's theorem if and only if it is projective.

In this paper, we shall see that the equivalences considered in [4] may be induced by finitely generated modules which are neither tilting modules nor quasiprogenerators (the modules that induce the equivalences characterized by Fuller in [1]).

Before we do this, we recall some definitions and results, and we fix the notation used throughout the paper.

Let K be an algebraically closed field, and let A be a finite-dimensional K-algebra. Then a finite-dimensional left module ${}_AT$ is called a *tilting module* [5] if ${}_AT$ satisfies the following conditions:

(i) proj.dim ${}_AT \leqslant 1$.

(ii) $\mathrm{Ext}^1_A({}_AT, {}_AT) = 0$.

(iii) There is an exact sequence $0 \to {}_AA \to T' \to T'' \to 0$, where T' and T'' are direct sums of summands of ${}_AT$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Let $_AT$ be a tilting module with endomorphism ring B, and let $D(T_B)$ denote the module $\mathrm{Hom}_K(T_B, K)$ equipped with its usual structure ([3], Proposition 3.5) of left B-module. Then, by tilting theory ([5], Theorem of Brenner–Butler), the functor $\mathrm{Hom}_A({}_AM, -)$ defines an equivalence between the category of finite-dimensional A-modules generated by $_AM$ and the category of finite-dimensional B-modules cogenerated by $D(M_B)$.

Next, let A and B be two rings, let $\mathscr{G}$ and $\mathscr{D}$ be two full subcategories of A-Mod and B-Mod respectively, and assume that there is an equivalence

$$(*) \qquad \mathscr{G} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

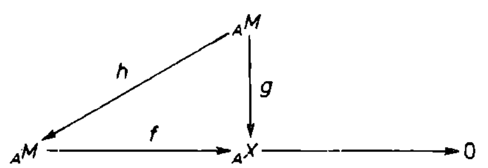
with F and G additive functors. Then, according to [4], we say that a bimodule $_AM_B$ *induces the equivalence* $(*)$ if F is naturally equivalent to $\mathrm{Hom}_A({}_AM, -)|_{\mathscr{G}}$ and G is naturally equivalent to $(M_B \otimes_B -)|_{\mathscr{D}}$.

We say that a module $_AM$ is a *quasiprogenerator* [1] if $_AM$ satisfies the following conditions:

(a) $_AM$ is finitely generated.

(b) $_AM$ generates all its submodules.

(c) $_AM$ is quasiprojective, i.e. if $_AX$ is a module and $f\colon {}_AM \to {}_AX$ is an epimorphism, then, for any morphism $g\colon {}_AM \to {}_AX$, there is an endomorphism h of $_AM$ making the following diagram commutative.



Suppose first that the equivalence $(*)$ has the property that $\mathscr{G}$ is closed under submodules, epimorphic images and direct sums, and that $\mathscr{D} = B$-Mod. Then Fuller's theorem [1] proves that $(*)$ is induced by a module $_AM$ with endomorphism ring B, and that $_AM$ is a quasiprogenerator.

Assume finally that the equivalence $(*)$ has the following properties:

(1) $\mathscr{G}$ is closed under epimorphic images and direct sums.

(2) $\mathscr{D}$ contains $_BB$ and is closed under submodules.

Then Menini–Orsatti's theorem [4] guarantees that $(*)$ is induced by a module $_AM$ with endomorphism ring B, and that the following facts hold:

(1′) $\mathscr{G}$ is the category of A-modules generated by $_AM$.

(2′) $\mathscr{D}$ is the category of B-modules cogenerated by $\mathrm{Hom}_A({}_AM, {}_AQ)$, where $_AQ$ is a fixed, but arbitrary, injective cogenerator of A-Mod.

In the following, for brevity we say that a module $_AM$ is a $*$*-module* if $_AM$ induces an equivalence $(*)$ satisfying the hypotheses of the Representation

Theorem of [4], that is, conditions (1) and (2). We also say that a module ${}_AM$ is a *∗-decomposable module* (resp. *∗-indecomposable module*) if ${}_AM$ is a ∗-module and ${}_AM$ can (resp. cannot) be written as the direct sum of two ∗-modules, say ${}_AM'$ and ${}_AM''$, with the following properties: ${}_AM'$ and ${}_AM''$ are different from zero and, if ${}_AX$ is an indecomposable module generated by ${}_AM$, then either ${}_AX$ is generated by ${}_AM'$ and $\mathrm{Hom}_A({}_AM'', {}_AX) = 0$, or ${}_AX$ is generated by ${}_AM''$ and $\mathrm{Hom}_A({}_AM', {}_AX) = 0$.

In this paper we point out some properties of finite-dimensional ∗-modules over finite-dimensional algebras. In this special situation, we find new ∗-modules which are not obtained from old ∗-modules in an obvious way. More precisely, in Section 1 we give an example of a ∗-indecomposable and sincere [5] module which is neither a tilting module nor a quasiprogenerator. We also note that, if we compare the two subclasses of tilting modules and quasiprogenerators inside the whole class of ∗-modules, then the unknown connection between tilting modules and ∗-modules seems to be stronger than the known connection between quasiprogenerators and ∗-modules. In fact, on the one hand, a first relationship between quasiprogenerators and ∗-modules follows from the results of [1] and [4] already mentioned. Even more, by ([4], Theorem 5.4), quasiprogenerators are exactly the ∗-modules satisfying two of the three properties (a), (b), (c) in the definition of quasiprogenerators, or equivalently the ∗-modules satisfying one of the properties (b), (c) in the same definition. On the other hand, ∗-modules do not arise as a natural generalization of the definition of tilting modules. In addition to this, tilting modules are not the ∗-modules satisfying two of the three properties (i), (ii), (iii) in the definition of tilting modules, and there exist ∗-modules which do not satisfy any of these properties.

However, surprisingly enough, there is a quite obvious, but extremely large, extension of the class of tilting modules over a finite-dimensional algebra A, namely the class of all modules ${}_AM$ with the property that ${}_{\bar{A}}M$ is a tilting module, where $\bar{A}$ denotes the algebra $A/\mathrm{ann}\,{}_AM$. In Section 2 we give an example of a finite-dimensional algebra A with enough ∗-modules, i.e. admitting ∗-indecomposable and sincere modules which are neither tilting modules nor quasiprogenerators, such that any multiplicity-free [2] ∗-module ${}_AM$ satisfies the above property. Up to now, we do not know whether or not there exist a finite-dimensional algebra A and a ∗-module ${}_AM$ such that ${}_AM$ is not a "disguised" tilted module, that is, ${}_{\bar{A}}M$ is not a tilting module, where $\bar{A} = A/\mathrm{ann}\,{}_AM$.

Throughout the paper, the word module usually means left module and, if R is a ring, then we denote by R-Mod (resp. Mod-R) the category of all left modules ${}_RM$ (resp. right modules M_R). We always assume that K is an algebraically closed field, and we define the K-algebra given by a quiver according to [5]. We often identify indecomposable modules and their isomorphism classes. In particular, for brevity we say that there exist

n indecomposable modules with a given property if there exist exactly n isomorphism classes of indecomposable modules with that property. Finally, if R is a finite-dimensional algebra such that any indecomposable R-module is completely determined by its dimension vector [5], then in the Auslander–Reiten quiver $\Gamma(R)$ of R we always denote the indecomposables by their dimension vectors.

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Lemma 1. *Let A be a finite-dimensional K-algebra, let ${}_AM$ be a finite-dimensional module and let $\bar{A} = A/\mathrm{ann}\,{}_AM$. If ${}_{\bar{A}}M$ is a tilting module, then ${}_AM$ is a $*$-module.*

Proof. Since ${}_{\bar{A}}M$ is a $*$-module [4], it suffices to note that the category $\{{}_AX \in A\text{-Mod} \mid {}_AX \text{ is generated by } {}_AM\}$ is isomorphic to the category $\{{}_{\bar{A}}X \in \bar{A}\text{-Mod} \mid {}_{\bar{A}}X \text{ is generated by } {}_{\bar{A}}M\}$, and that the endomorphism ring of ${}_AM$ is isomorphic to the endomorphism ring of ${}_{\bar{A}}M$. ∎

Remark 2. *There are a finite-dimensional algebra A and a $*$-indecomposable and sincere module ${}_AM$ which is neither a tilting module nor a quasiprogenerator.*

Proof. Let A be the K-algebra given by the quiver $\bullet \leftarrow \bullet \leftarrow \bullet$, and let ${}_AM = \bigoplus_{i=1}^3 M_i$, where the M_i's are the indecomposables marked in $\Gamma(A)$

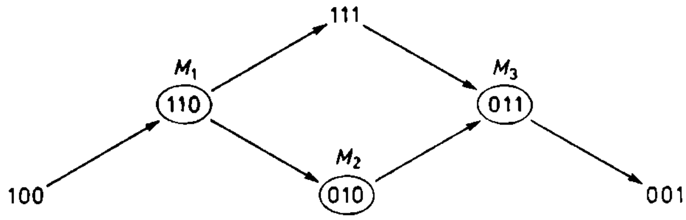

Fig. 1

(see Fig. 1). Then ${}_AM$ is obviously a sincere module. Since ${}_AM$ is not faithful and ${}_AM$ does not generate the socle of M_1, it follows that ${}_AM$ is neither a tilting module nor a quasiprogenerator. We claim that ${}_AM$ is a $*$-module.

To see this, let $\bar{A}$ be the algebra $A/\mathrm{ann}\,{}_AM$. Then $\bar{A}$ is the K-algebra given by the quiver $\bullet \xleftarrow{\beta} \bullet \xleftarrow{\alpha} \bullet$ with relation $\alpha\beta = 0$, and ${}_{\bar{A}}M$ is a tilting module. Hence, by Lemma 1, ${}_AM$ is a $*$-module, as claimed. Since $\mathrm{Hom}_A(M_1, M_2) \neq 0$ and $\mathrm{Hom}_A(M_2, M_3) \neq 0$, the module ${}_AM$ is $*$-indecomposable, and the proof is complete. ∎

The next statement shows that the fact that a tilting module is a $*$-module follows from all the three properties (i)–(iii) of the definition of tilting module, but none of these properties is necessarily satisfied by a $*$-module.

Proposition 3. *Let A be a finite-dimensional algebra and let ${}_AM$ be a finite-dimensional module. Then the following cases are possible:*

(1) *${}_AM$ satisfies two of the three properties* (i)–(iii) *and ${}_AM$ is not a $*$-module.*

(2) *${}_AM$ does not satisfy any (resp. satisfies exactly one or two) of the properties* (i)–(iii) *and ${}_AM$ is a $*$-module.*

Proof. (1) It suffices to consider the following examples, where we always assume that B is the endomorphism ring of ${}_AM$.

EXAMPLE 1. Let A be the K-algebra given by the quiver $\bullet \leftarrow \bullet$, and let ${}_AM$ be the unique indecomposable nonsimple module. Then ${}_AM$ is projective, and so ${}_AM$ satisfies (i) and (ii). Since ${}_AM$ generates two indecomposables and $B \simeq K$, we conclude that ${}_AM$ is not a $*$-module. (The same conclusion follows from ([4], Theorem 5.4) and the remark that ${}_AM$ is a quasiprojective module which does not generate all its submodules.)

EXAMPLE 2. Let A be the K-algebra given by the quiver $\bullet \leftarrow \bullet \leftarrow \bullet$, and let ${}_AM = \bigoplus_{i=1}^4 M_i$, where the M_i's are the indecomposables marked in $\Gamma(A)$

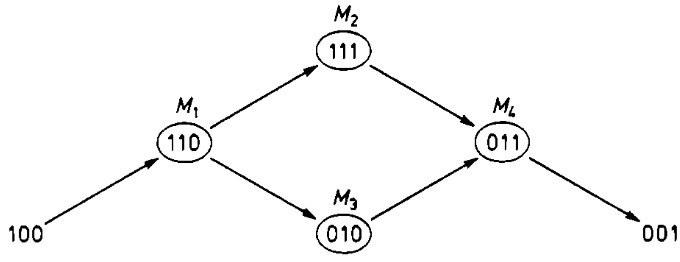

Fig. 2

(see Fig. 2). Then ${}_AM$ satisfies (i) and (iii), and B is isomorphic to the algebra of all 8×8 matrices with entries in K of the form

$$\begin{bmatrix} a & & & e & & f & & g \\ & a & & & e & & & \\ & & b & & & & h & \\ & & & b & & & & h \\ & & & & b & & & \\ & & & & & c & & i \\ & & & & & & d & \\ & & & & & & & d \end{bmatrix}.$$

Consequently B is the K-algebra given by the quiver of Fig. 3 with relation

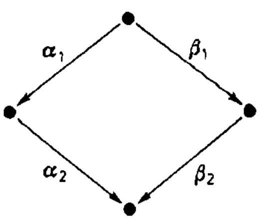

Fig. 3

$\alpha_1\alpha_2 = \beta_1\beta_2$, and—without loss of generality—we may assume that $D(M_B)$ is isomorphic to $\bigoplus_{i=1}^{3} N_i$, where the N_i's are the indecomposables marked in $\Gamma(B)$ (see Fig. 4). Hence $D(M_B)$ cogenerates seven indecomposable modules.

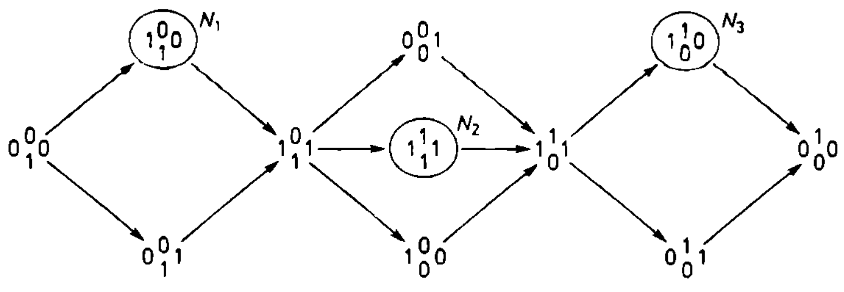

Fig. 4

Since ${}_AM$ generates only five indecomposable modules, it follows that ${}_AM$ is not a $*$-module.

EXAMPLE 3. Let A be the K-algebra given by the quiver of Fig. 5 with

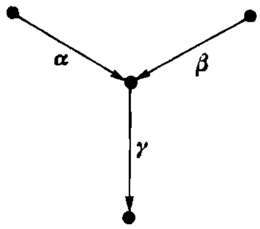

Fig. 5

relations $\alpha\gamma = 0$ and $\beta\gamma = 0$ and let ${}_AM = \bigoplus_{i=1}^{5} M_i$, where the M_i's are the indecomposables marked in $\Gamma(A)$ (see Fig. 6). Then ${}_AM$ clearly satisfies (iii). Moreover, the shape of $\Gamma(A)$ and ([5], Assertion 5, p. 75) guarantee that $\mathrm{Ext}_A^1({}_AM_i, {}_AM_j) = 0$ for $i = 4, 5$ and $j = 1, 3$. Since M_1, M_2, M_3 are projective

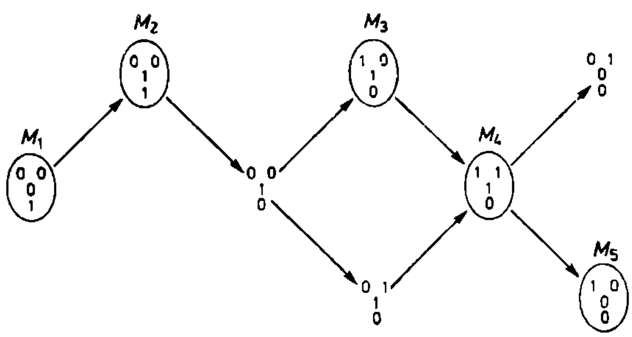

Fig. 6

and M_2, M_4, M_5 are injective, we have $\mathrm{Ext}_A^1({}_AM, {}_AM) = 0$, and so ${}_AM$ satisfies (ii). On the other hand, B is isomorphic to the algebra of all 9×9 matrices with entries in K of the form

$$\begin{bmatrix} a & & f & & & & & & \\ & b & & & g & & & h & \\ & & b & & & & & & \\ & & & c & & i & & & l \\ & & & & c & & & i & \\ & & & & & d & & & m \\ & & & & & & d & & \\ & & & & & & & d & \\ & & & & & & & & e \end{bmatrix}.$$

This implies that B is the K-algebra given by the quiver $\bullet \xleftarrow{\alpha_4} \bullet \xleftarrow{\alpha_3} \bullet \xleftarrow{\alpha_2} \bullet \xleftarrow{\alpha_1} \bullet$ with relations $\alpha_1\alpha_2\alpha_3 = 0$ and $\alpha_3\alpha_4 = 0$, and that $D(M_B)$ is isomorphic to $\bigoplus_{i=1}^4 N_i$, where the N_i's are the indecomposables marked in $\Gamma(B)$ (see Fig. 7). Therefore $D(M_B)$ cogenerates nine indecomposable

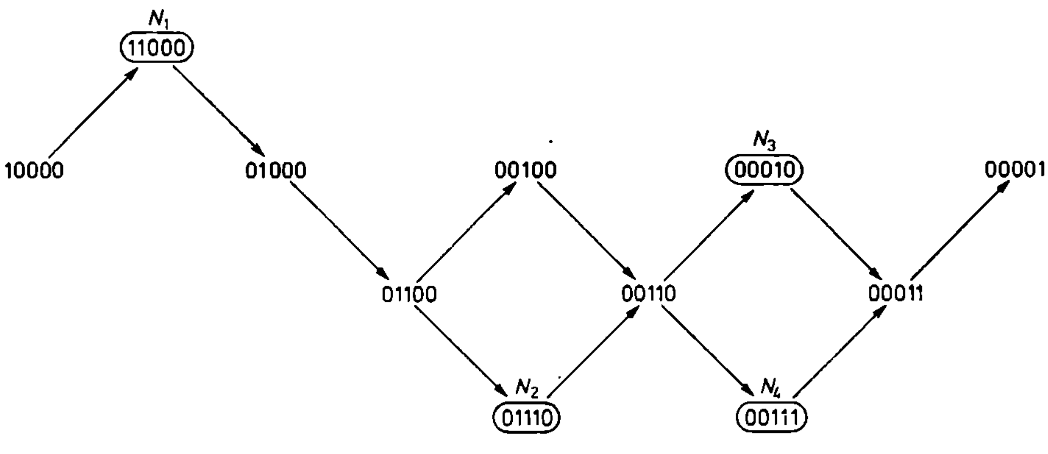

Fig. 7

modules. Since ${}_AM$ generates only seven indecomposable modules, we conclude that ${}_AM$ is not a $*$-module.

(2) Also in this case, we give three examples.

EXAMPLE 1′. Let A be the algebra $K[x]/(x^2)$ and let ${}_AM$ denote the simple socle of ${}_AA$. Then ${}_AM$ is a quasiprogenerator, and so ${}_AM$ is a $*$-module. However, we clearly have $\mathrm{proj.dim}\, {}_AM = \infty$ and $\mathrm{Ext}_A^1({}_AM, {}_AM) \neq 0$. Since ${}_AM$ is not faithful, it follows that ${}_AM$ does not satisfy any of the properties (i)–(iii).

EXAMPLE 2′. Let A and ${}_AM$ be as in the proof of Remark 2. Then ${}_AM$ is a $*$-module satisfying (i), but neither (ii) nor (iii).

EXAMPLE 3′. Let A, M_1 and M_2 be as in the proof of Remark 2, let ${}_AM = M_1 \oplus M_2$ and let $\bar{A} = A/\text{ann}_A M$. Since ${}_{\bar{A}}M$ is a tilting module, we deduce from Lemma 1 that ${}_AM$ is a $*$-module. On the other hand, ${}_AM$ satisfies (i) and (ii), but not (iii).

The proof is finished. ∎

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The next result shows that, dealing with algebras of finite representation type, we cannot conclude that a module is a $*$-module by simply counting the number of some indecomposable modules.

REMARK 4. *Let A be a finite-dimensional K-algebra, let ${}_AM$ be a multiplicity-free module with endomorphism ring B and assume that the following conditions hold: A and B are of finite representation type, and the number of indecomposable A-modules generated by ${}_AM$ is equal to the number of indecomposable B-modules cogenerated by $D(M_B)$. Then ${}_AM$ is not necessarily a $*$-module.*

Proof. Let A be the K-algebra given by the quiver $\underset{1}{\bullet} \xleftarrow{\beta} \underset{2}{\bullet} \xleftarrow{\alpha} \underset{3}{\bullet}$ with relation $\alpha\beta = 0$, and let ${}_AM = \bigoplus_{i=1}^{3} M_i$, where the M_i's are the indecomposables marked in $\Gamma(A)$ (see Fig. 8). Then ${}_AM$ generates four indecomposable modules,

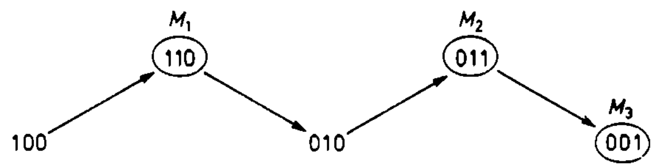

Fig. 8

and B is isomorphic to the algebra of all 5×5 matrices with entries in K of the form

$$\begin{bmatrix} a & & & d & \\ & a & & & \\ & & b & & e \\ & & & b & \\ & & & & c \end{bmatrix}.$$

Consequently B is isomorphic to A and $D(M_B)$ is isomorphic to ${}_BB$. Therefore $D(M_B)$ cogenerates four indecomposables. However, the B-modules

$\mathrm{Hom}_A({}_AM_B, M_1)$ and $\mathrm{Hom}_A({}_AM_B, M_1/\mathrm{soc}M_1)$ are clearly isomorphic, and so ${}_AM$ cannot be a $*$-module. ∎

The situation described in Remark 4 seems to occur only rarely, and the following remark shows that the preceding example is, in a sense, "minimal".

REMARK 5. The algebra given by the quiver $\underset{1}{\bullet} \overset{\beta}{\leftarrow} \underset{2}{\bullet} \overset{\alpha}{\leftarrow} \underset{3}{\bullet}$ with relation $\alpha\beta = 0$ is a minimal algebra satisfying the hypotheses of Remark 4. Moreover, for this choice of A, the module considered in the proof is the unique A-module satisfying the requirements of the remark. In fact, for any $i = 1, 2, 3$, let $S(i)$ denote the simple module corresponding to the vertex i, and let $P(i)$ denote the projective cover of $S(i)$. Next, let ${}_AM$ be a multiplicity-free A-module with endomorphism ring B. Then one of the following conditions hold:

(1) ${}_AM$ is either a tilting module or a quasiprogenerator.

(2) ${}_AM$ is a $*$-indecomposable module, but neither a tilting module nor a quasiprogenerator. In this case, we have either ${}_AM \simeq P(2) \oplus S(2)$ or ${}_AM \simeq P(3) \oplus S(3)$.

(3) ${}_AM$ is a $*$-decomposable module, but not a quasiprogenerator. In this case, we have either ${}_AM \simeq P(2) \oplus S(2) \oplus S(3)$ or ${}_AM \simeq P(3) \oplus S(3) \oplus S(1)$.

(4) ${}_AM$ is not a $*$-module. In this case, either ${}_AM$ is isomorphic to $P(2) \oplus P(3) \oplus S(3)$, i.e. ${}_AM$ is the module considered in the proof of Remark 4, or the number of indecomposable A-modules generated by ${}_AM$ is different from the number of indecomposable B-modules cogenerated by $D(M_B)$.

To see this, we may proceed as follows. Suppose first that ${}_AM$ is faithful. Then ${}_AM$ is the direct sum of n indecomposable summands with $n = 2, 3, 4, 5$. If $n = 3$, then either ${}_AM$ is a tilting module, or ${}_AM$ is isomorphic to $P(2) \oplus P(3) \oplus S(3)$. Hence either (1) or (4) holds. If $n = 2, 4, 5$, then it is easy to check that the number of indecomposable modules generated by ${}_AM$ is different from the number of indecomposable modules cogenerated by $D(M_B)$. Thus in all these cases ${}_AM$ satisfies (4).

Assume now that ${}_AM$ is not faithful, and that ${}_AM$ is not a quasiprogenerator. Then $\bar{A} = A/\mathrm{ann}_A M$ is the K-algebra given by one of the quivers

$$\bullet \leftarrow \bullet \quad \text{or} \quad \bullet \leftarrow \bullet \quad \bullet$$

On the other hand, a multiplicity-free module over the algebra A' given by the quiver $\bullet \leftarrow \bullet$ is a $*$-module (resp. is not a $*$-module) if and only if it is either semisimple, or a projective tilting A'-module, or an injective tilting A'-module (resp. either an indecomposable nonsimple module, or the direct sum of three indecomposable modules). Consequently, if ${}_AM$ is a $*$-module, then ${}_{\bar{A}}M$ is an injective tilting module, and so either (2) or (3) holds. Finally, if ${}_AM$ is not a $*$-module, then one immediately verifies that (4) holds.

It is now easy to see that also the example considered in the proof of Remark 2 is "minimal".

COROLLARY 6. *If A is the algebra given by the quiver* $\bullet \leftarrow \bullet \leftarrow \bullet$, *then*:

(i) *A is a minimal algebra admitting a multiplicity-free $*$-indecomposable and sincere module which is neither a tilting module nor a quasiprogenerator.*
(ii) *A admits exactly one module as in* (i).

Proof. We first note that A is a local algebra and that the algebra considered in Remark 5 is isomorphic to $A/\text{rad}\,A$, where $\text{rad}\,A$ is the Jacobson radical of A. Hence conditions (1)–(4) in Remark 5 guarantee that any multiplicity-free $*$-indecomposable and sincere module over a proper factor algebra of A is either a tilting module or a quasiprogenerator. Thus (i) follows from the proof of Remark 2.

Next, let ${}_AM$ be a faithful and multiplicity-free module with endomorphism ring B, and assume that ${}_AM$ is not a tilting module. If ${}_AA$ is a direct summand – hence a proper direct summand – of ${}_AM$, then ${}_AM$ generates all its submodules, but ${}_AM$ is not quasiprojective. Therefore the conclusion that ${}_AM$ is not a $*$-module follows from ([4], Theorem 5.4). On the other hand, if ${}_AA$ is not a direct summand of ${}_AM$, then we may directly obtain the same conclusion in one of the following ways: either we find two nonisomorphic modules, say ${}_AX$ and ${}_AY$, generated by ${}_AM$ such that $\text{Hom}_A({}_AM_B, {}_AX)$ is isomorphic to $\text{Hom}_A({}_AM_B, {}_AY)$, or we verify that the number of indecomposable A-modules generated by ${}_AM$ is different from the number of indecomposable B-modules cogenerated by $D(M_B)$. Hence (ii) holds, and the corollary is proved. ■

The proof of Corollary 6 also shows that the algebra A given by the quiver $\bullet \leftarrow \bullet \leftarrow \bullet$ has the following property: if ${}_AM$ is a multiplicity-free $*$-module and $\bar{A} = A/\text{ann}_A M$, then ${}_{\bar{A}}M$ is a tilting module. As already observed in the introduction, we do not know any finite-dimensional algebra without this property.

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