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Global solutions to initial value problems in nonlinear hyperbolic thermoelasticity

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Abstract

The aim of this paper is to present an elementary, self-contained introduction into some important aspects of the theory of global, small, smooth solutions to initial value problems for nonlinear hyperbolic equations of thermoelasticity theory. This system of equations is a new one describing thermoelastic solids in the three-dimensional space. It describes the propagation of thermal perturbations with finite velocity.

The theory is presented using the classical method of continuation of local solutions with the help of a priori estimates obtained for small data.

The corresponding global existence theorems have been proved using the $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates for the solution of the associated linearized problem. The $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates were obtained by constructing the matrix of fundamental solutions to the linearized system of equations of thermoelasticity theory with the help of the Radon transform or by providing the fundamental solution for the linearized system of the hyperbolic heat equation by the Hörmander method.

This approach to the $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates based on the Radon transform is new; we indicate some possibilities of extending it.

1. Introduction

The investigation of existence of global-in-time solutions to initial value problems under small initial data (in the sense of norms in Sobolev spaces) for nonlinear partial differential equations has been extensively developed during the years 1980–1990. It was begun in 1980 by S. Klainerman [73]. He proved the existence of a global-in-time solution to the following initial value problem for the nonlinear wave equation:

$$\Box y = F(\nabla y, D\nabla y), \tag{1.1}$$

$$y(0,x) = h(x), \qquad (\partial_t y)(0,x) = g(x), \tag{1.2}$$

where

$$y = y(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \partial_t = \frac{\partial}{\partial t},$$

$$\Box y = \partial_t^2 y - \Delta y, \quad \Delta = \sum_{j=1}^{n} \partial_j^2, \quad \partial_j = \frac{\partial}{\partial x_j},$$

$Dy = (\partial_1 y, \ldots, \partial_n y)$ is the spatial gradient,

$\nabla y = (\partial_t y, Dy) = (\partial_t y, \partial_1 y, \ldots, \partial_n y)$ is the time-spatial gradient

under the assumption that the initial data are small enough and that in some neighbourhood of the origin,

$$F(U) = O(|U|^2) \tag{1.3}$$

for the space dimension $n \geq 6$. In 1981 F. John [59] proved a theorem on the blow-up in finite time for the solution to (1.1)–(1.2) in $\mathbb{R}^3$ under the assumption that F is given by (1.3)

S. Klainerman and G. Ponce [79] proved the existence of a global-in-time solution to (1.1)–(1.2) in $\mathbb{R}^n$ (where $n \geq 3$) under the assumption that

$$F(U) = O(|U|^3). \tag{1.4}$$

In 1982 analogous theorems were proved by J. Shatah [122] for the nonlinear heat equation, nonlinear Schrödinger equation and nonlinear Klein–Gordon equation using a more general functional analytic setup.

In 1985 S. Klainerman [75] proved the existence of a global-in-time solution to (1.1)–(1.2) in $\mathbb{R}^3$ under the assumption that

$$F(U) = O(|U|^2) \tag{1.5}$$

using the techniques of Lorentz invariants and a special assumption about the nonlinearity, the so-called null condition, which is sufficient to prove the existence of a global solution for quadratic nonlinearities in $\mathbb{R}^3$. F. John [60] proved that there exists an "almost global solution" to the following initial value problem:

$$\square u = C^{rs}(Du)\, D_r D_s u, \tag{1.6}$$

$$u = \varepsilon\, f(x), \quad D_0 u = \varepsilon\, g(x) \quad \text{for } t = 0, x \in \mathbb{R}^3, \tag{1.7}$$

where

$$D_0 \equiv \partial_t, \qquad D_m \equiv \partial_m,$$

$$\square_{ik} = \delta_{ik} D_0^2 - (c_2^2 \delta_{rs}\delta_{ik} + (c_1^2 - c_2^2)\,\delta_{ri}\delta_{sk})\, D_r D_s, \qquad i,k,r,s = 1,2,3. \tag{1.8}$$

Here C^{rs} are matrices with elements C_{ik}^{rs} depending on the space gradient Du and satisfying

$$C_{ik}^{rs}(0) = 0, \quad C_{ik}^{rs}(Du) = C_{ki}^{sr}(Du),$$

where $C_{ik}^{rs}(Du)$ are of class $\mathbb{C}^\infty$ and have bounded derivatives in $\mathbb{R}^9$ (Du will be restricted here to a small neighbourhood of the origin), ε is a positive constant, the vectors f and g belong to $\mathbb{C}_0^\infty(\mathbb{R}^3)$, $c_1^2 = (\lambda + 2\mu)/\varrho$, $c_2^2 = \mu/\varrho$, λ, μ are Lamé's constants, ϱ is the density and δ_{rs} denotes Kronecker's symbol.

John obtained the following result: *There exist positive constants A, B, ε_0 (depending on f, g, C_{ik}^{rs}, but not on ε) such that a $\mathbb{C}^\infty$ solution $u(t,x)$ of* (1.6), (1.7) *exists for*

$$0 \le t < B \exp(1/(A\varepsilon)), \qquad x \in \mathbb{R}^3, \tag{1.9}$$

provided $\varepsilon < \varepsilon_0$.

The slab described by (1.9) (in which u exists) is exponentially large for small ε. We say that u *exists almost globally*.

R. Racke and G. Ponce [112] proved the existence of a global-in-time solution in $\mathbb{R}^3$ (for small data) to the following initial value problem of classical thermoelasticity (the so-called hyperbolic-parabolic thermoelasticity):

$$\partial_t^2 u - ((2\mu + \lambda)\,\mathrm{grad}\,\mathrm{div} - \mu\,\mathrm{curl}\,\mathrm{curl})u + \mathrm{grad}\,\Theta = f^1(Du, D^2u, \Theta, D\Theta),$$
$$\partial_t \Theta - \Delta\Theta + \mathrm{div}\,\partial_t u = f^2(Du, D\partial_t u, D^2 u, \Theta, D\Theta, D^2\Theta),$$
$$u(0,x) = u^0(x), \quad (\partial_t u)(0,x) = u^1(x), \quad \Theta(0,x) = \Theta^0(x),$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $u = u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))^*$ is the displacement vector of the medium, and $\Theta = \Theta(t,x)$ is the temperature of the medium; we assume that, near the origin,

$$f^1(w) = O(|w|^3), \qquad f^2(w) = O(|w|^3) + \Theta\Delta\Theta.$$

The aim of this paper is to prove the existence of a global-in-time solution to initial value problems for a new nonlinear hyperbolic PDE system describing thermoelastic solids in the three-dimensional space (the so-called temperature-rate-dependent thermoelastic solids).

It is known that the classical thermoelasticity theory leads to a parabolic differential equation for temperature distribution in rigid heat conductors. This implies that thermal perturbations are felt instantaneously (cf. [99], [5]) in every part of the body. Although, at first sight, this outcome of the theory seems to contradict the physical intuition, it can be justified by resorting to the fact that molecular motion, which plays a crucial part in transport phenomena, is very rapid except at extremely low temperatures. Hence a finite velocity of propagation for thermal perturbations is usually nonobservable unless experiments are performed in some neighbourhood of absolute zero as in the case of liquid helium. In fact, thermal waves, commonly known as *second sound*, are detected in some metals cooled approximately down to 20°K. For a short survey the reader is referred to the works of Ackerman and Guyer [1], Taylor *et al.* [140] and Jackson and Walker [50].

Below, we shall describe a theory of thermoelasticity [138] by considering the temperature rate dependence and assigning an appropriate constitutive function to the entropy flux. Such a theory leads to a hyperbolic differential equation for thermal perturbations different from the equation describing propagation of thermal perturbations in classical thermoelasticity (which is a parabolic equation). One approach is to include the temperature rate among the constitutive variables, which results in the presence of the second order time derivative of the temperature field in the energy balance. However, the Clausius–Dühem inequality, in the form employed up to now, eliminates the temperature rate dependence from all the constitutive functions except for the constitutive function of heat flux. Hence, in order to obtain a properly posed theory for temperature-rate-dependent thermoelastic solids we have to resort to an entropy principle in its full generality presented in [19, 98]. Such a theory of thermoelasticity was proposed by Müller in [98], who advocated rather special constitutive relations for the entropy supply in rigid conductors which are simple generalizations of the conventional form. Suhubi [138] extended his results to thermoelasticity theory and obtained a hyperbolic system of equations describing temperature-rate-dependent thermoelastic solids.

The aim of our paper is to investigate the (global-in-time) solvability of initial value problems for the nonlinear hyperbolic PDE system describing thermoelastic solids in three-dimensional space.

We prove the existence of global-in-time solutions for small data in suitably chosen Sobolev spaces, for two variants of this system:

1) The coupled nonlinear hyperbolic PDE system describing the temperature-rate-dependent thermoelastic solids under the assumption that the coefficients in the nonlinear terms are smooth functions of their arguments (which are first order derivatives of the unknown functions u^1, u^2, u^3 and T) and behave near the origin like $O(|\eta|^{k_0})$ for $k_0 \geq 2$.

2) Nonlinear hyperbolic heat equations describing the propagation of thermal perturbations with finite velocity in a rigid heat conductor (i.e., a material whose deformations are negligible in the analysis of temperature) under the assumption that the coefficients in the nonlinear terms are smooth functions of their arguments which are first order derivatives of the unknown function Θ and behave near the origin like $O(|\eta|^{k_0})$ for $k_0 \geq 2$.

First, using a suitable transformation, we reduce the above initial value problems to equivalent one for quasilinear first order hyperbolic PDE systems. Applying the local existence theorems for those systems we get the existence and uniqueness of solutions to the initial value problems in a finite time interval. Basing on the matrix of fundamental solutions of the linearized system (in case 1) and the Radon transform, or on the fundamental solution (Hörmander's theorem) of the linearized equation (in case 2) and using a convolution type representation of the local solutions we prove a priori energy estimates with constants independent of time. This enables us to extend the local solutions to global ones. The energy estimates also yield the asymptotic behaviour of the solutions as $t \to \infty$.

1.1. Main Theorem 1.1. We first consider the following initial value problem of hyperbolic thermoelasticity (cf. [29]):

$$\varrho\partial_t^2 u - \mu\Delta u - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u + \beta\,\mathrm{grad}\,\partial_t T = F(\nabla u, \nabla T, D\nabla u, D\nabla T), \tag{1.10}$$

$$\varrho\tau\partial_t^2 T - k\Delta T + \beta\,\mathrm{div}\,\partial_t u = Q(\nabla u, \nabla T, D\nabla u, D\nabla T), \tag{1.11}$$

with the initial conditions

$$u(+0,x) = u^0(x), \qquad (\partial_t u)(+0,x) = u^1(x), \tag{1.12}$$

$$T(+0,x) = T^0(x), \qquad (\partial_t T)(+0,x) = T^1(x), \tag{1.13}$$

where $u = (u^1(t,x), u^2(t,x), u^3(t,x))^*$ is the displacement vector and $T = T(t,x)$ is the temperature, both depending on $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^3$. Here $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\Delta = \sum_{j=1}^3 \partial_j^2$, where ϱ, μ, λ, β, τ, k are positive physical constants, $u^0(x), u^1(x), T^0(x), T^1(x)$ are given functions, the asterisk denotes transposition and div stands for divergence with respect to x. Moreover, $\nabla u = (\partial_t u, \partial_1 u, \partial_2 u, \partial_3 u)^*$ and $\nabla T = (\partial_t T, \partial_1 T, \partial_2 T, \partial_3 T)^*$ denote the space-time gradients of u and T, while $Du = (\partial_1 u, \partial_2 u, \partial_3 u)$ and $DT = (\partial_1 T, \partial_2 T, \partial_3 T)$ denote their spatial gradients.

We assume that the nonlinear terms in (1.10), (1.11) have the following form:

$$\begin{aligned} F_j(\nabla u, \nabla T, D\nabla u, D\nabla T) = {} & \sum_{\substack{n=1\\k,m=0}}^{3} a_{jkmn}(\nabla u, \nabla T)\partial_k\partial_m u^n \\ & + \sum_{m=1}^{3} \widetilde{c}_{jm}(\nabla u, \nabla T)\partial_m\partial_t T, \qquad j = 1,2,3, \end{aligned} \tag{1.14}$$

$$\begin{aligned} Q(\nabla u, \nabla T, D\nabla u, D\nabla T) = {} & \sum_{\substack{j,k=0\\j+k\neq 0}}^{3} \widetilde{a}_{jk}(\nabla u, \nabla T)\partial_j\partial_k T \\ & + \sum_{\substack{j=1\\m=1}}^{3} \widetilde{c}_{jm}(\nabla u, \nabla T)\partial_t\partial_m u^j, \qquad j = 1,2,3, \end{aligned} \tag{1.15}$$

where a_{jkmn}, $\widetilde{c}_{jm}$, $\widetilde{a}_{jk}$ are $\mathbb{C}^\infty$ functions of $\eta \in \mathbb{R}^{16}$ such that, for some $k_0 \geq 2$,

$$|a_{jkmn}(\eta)| = O(|\eta|^{k_0}), \tag{1.16}$$

$$|\widetilde{a}_{jk}(\eta)| = O(|\eta|^{k_0}), \tag{1.17}$$

$$|\widetilde{c}_{jm}(\eta)| = O(|\eta|^{k_0}), \tag{1.18}$$

for $|\eta| < 1$ and satisfy the following symmetry conditions:

$$\begin{aligned} a_{jkmn}(\eta) &= a_{nmkj}(\eta), \\ \widetilde{a}_{jk}(\eta) &= \widetilde{a}_{kj}(\eta), \\ \widetilde{c}_{jm}(\eta) &= \widetilde{c}_{mj}(\eta) \qquad \text{for } \eta \in \mathbb{R}^{16}. \end{aligned} \tag{1.19}$$

We shall prove the following main theorem about the existence and asymptotic behaviour of global solutions.

THEOREM 1.1. *Let* $s \geq 10$ *be an integer and* $p = (2k_0+2)/(2k_0+1)$ ($k_0 \geq 2$ *is given by* (1.16)–(1.18)). *Suppose that*

$$(u^1, Du^0, T^1, DT^0)^* \in \mathbb{W}^{s,2}(\mathbb{R}^3) \cap \mathbb{W}^{s,p}(\mathbb{R}^3). \tag{1.20}$$

Then for a sufficiently small positive constant δ, *if*

$$\|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} + \|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)} < \delta$$

then there exists a unique smooth solution $(u, T)^*$ *to the Cauchy problem* (1.10)–(1.13) *under the assumptions* (1.19) *with the following properties*:

$$\nabla u \in \mathbb{C}^0([0,\infty),\, \mathbb{W}^{s,2}(\mathbb{R}^3)) \cap \mathbb{C}^1([0,\infty),\, \mathbb{W}^{s-1,2}(\mathbb{R}^3)),$$
$$\nabla T \in \mathbb{C}^0([0,\infty),\, \mathbb{W}^{s,2}(\mathbb{R}^3)) \cap \mathbb{C}^1([0,\infty),\, \mathbb{W}^{s-1,2}(\mathbb{R}^3)).$$

Moreover,

$$\begin{aligned}
\|\nabla u(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} &= O(t^{-k_0/(k_0+1)}),\\
\|\nabla T(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} &= O(t^{-k_0/(k_0+1)}),\\
\|\nabla u(t)\|_{\mathbb{L}^{2k+2}(\mathbb{R}^3)} &= O(t^{-k_0/(k_0+1)}),\\
\|\nabla T(t)\|_{\mathbb{L}^{2k+2}(\mathbb{R}^3)} &= O(t^{-k_0/(k_0+1)}),\\
\|\nabla u(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} &= O(1),\\
\|\nabla T(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} &= O(1).
\end{aligned}$$

1.2. Main Theorem 1.2. We now consider the following initial value problem for a nonlinear hyperbolic heat equation (describing the propagation of thermal perturbations with finite velocity in a rigid heat conductor, i.e., a material whose deformations are negligible when analysing its temperature) (cf. [16]):

$$\partial_t^2 \Theta + 2m\partial_t \Theta - a^2 \Delta\Theta = \sum_{j,k=1}^{3} a_{jk}(\nabla\Theta)\partial_j\partial_k \Theta + 2\sum_{j=1}^{3} a_{j0}(\nabla\Theta)\partial_j\partial_t \Theta, \tag{1.21}$$

$$\Theta(+0, x) = \Theta^0(x), \qquad (\partial_t\Theta)(+0, x) = \Theta^1(x), \tag{1.22}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $\Theta = \Theta(t, x)$ is the temperature of the medium,

$$\nabla\Theta = (\partial_t\Theta,\, D\Theta)^*, \qquad D\Theta = (\partial_1\Theta, \partial_2\Theta, \partial_3\Theta), \tag{1.23}$$

m and a are positive physical constants given by $2m = c/\tau$ and $a^2 = k/(\varrho\tau)$, where ϱ is the material density, c is the specific heat, k is the conductivity coefficient and τ is the relaxation time (cf. [16], [138]). We assume that the coefficients $a_{jk} \in \mathbb{C}^\infty(\mathbb{R}^4)$ are symmetric in $j, k = 0, 1, 2, 3$ with $a_{jk}(0) = 0$ and that there exists $\nu > 0$ such that for all $\xi \in \mathbb{R}^3$ and all $\nabla\Theta$ with $|\nabla\Theta|_{\mathbb{L}^\infty} < R_0$ we have

$$a^2|\xi|^2 + \sum_{j,k=1}^{3} a_{jk}(\nabla\Theta)\xi_j\xi_k \geq \nu|\xi|^2, \tag{1.24}$$

where

$$|a_{jk}(\eta)| = O(|\eta|^{k_0}) \tag{1.25}$$

for $|\eta| < 1$ and some $k_0 \geq 2$.

THEOREM 1.2. *Let s be an integer ≥ 10 and $p = (2k_0+2)/(2k_0+1)$, where $k_0 \geq 2$ is given by* (1.25). *Suppose that*

$$(\Theta^1, D\Theta^0)^* \in \mathbb{H}^s(\mathbb{R}^3) \cap \mathbb{W}^{s,p}(\mathbb{R}^3). \tag{1.26}$$

Then for $\delta > 0$ small enough, if

$$\|(\Theta^1,\, D\Theta^0)^*\|_{\mathbb{H}^s(\mathbb{R}^3)} + \|(\Theta^1,\, D\Theta^0)^*\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)} < \delta \tag{1.27}$$

then there exists a unique smooth solution Θ to the initial value problem (1.21)–(1.23) *under the assumption* (1.24)–(1.25) *with the property*

$$\nabla\Theta \in \mathbb{C}^0([0,\infty),\, \mathbb{H}^s(\mathbb{R}^3)) \cap \mathbb{C}^1([0,\infty),\, \mathbb{H}^{s-1}(\mathbb{R}^3)). \tag{1.28}$$

Moreover,

$$\begin{aligned} &\|\nabla\Theta(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} = O(t^{-3k_0/(2(k_0+1))}),\\ &\|\nabla\Theta(t)\|_{\mathbb{L}^{2k_0+2}(\mathbb{R}^3)} = O(t^{-3k_0/(2(k_0+1))}),\\ &\|\nabla\Theta(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} = O(1). \end{aligned} \tag{1.29}$$

The paper is organized as follows.

In Section 2 we describe the Radon transform and its application to the construction of fundamental solutions of hyperbolic homogeneous differential operators of order m having characteristic polynomials with multiple zeros. Each zero is assumed to be of constant multiplicity. We also describe the construction of a matrix of fundamental solutions of a hyperbolic PDE system with constant coefficients.

In the next section we prove $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates. Section 4 presents the local existence theorems for solutions to the initial value problems. Section 5 is devoted to the proof of energy estimates. Finally, in Section 6 the proofs of the main Theorems 1.1, 1.2 are presented and some applications of the above method and results are indicated.

2. Radon transform

2.1. Definition of the Radon transform

DEFINITION 2.1. The *Radon transform* of a function $f \in \mathbb{S}(\mathbb{R}^n)$ is defined as follows (cf. [107], [64], [24], [42]):

$$\mathbb{S}^{n-1} \times \mathbb{R} \ni (\omega, s) \to (Rf)(\omega, s) = \int\limits_{y\cdot\omega=s} f(y)\, dy, \tag{2.1}$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere with centre at 0 and radius 1, and $\mathbb{S}(\mathbb{R}^n)$ is the Schwartz space of test functions.

Main properties of the Radon transform

$$Rf(\omega,s)=\int_{\mathbb{R}^n} f(y)\delta(s-y\omega)\,dy, \tag{P1}$$

$$Rf(-\omega,-s)=Rf(\omega,s), \tag{P2}$$

$$R(\partial^b f)=\omega^b\partial_s^{|b|}Rf, \tag{P3}$$

$$\mathcal{F}f(\xi)=(2\pi)^{-n/2}\int_{-\infty}^{\infty} e^{-is|\xi|}(Rf)\left(\frac{\xi}{|\xi|},s\right)ds, \tag{P4}$$

where $\partial_s=d/ds$, $b=(b_1,\dots,b_n)$ is a multiindex, $\omega^b=\omega_1^{b_1}\dots\omega_n^{b_n}$, $\mathcal{F}$ is the Fourier transform, and $f\in\mathbb{S}(\mathbb{R}^n)$.

Now, we present the fundamental theorem about the connection of the Radon and Fourier transforms.

Theorem 2.1. *Let $f\in\mathbb{S}(\mathbb{R}^n)$. Then the equality $f=\mathcal{F}^{-1}[\mathcal{F}f]$ can be written in the form*

$$f=R^*(k_q^{(n+q)}*Rf), \tag{2.2}$$

where

$$R^*(k_q^{(n+q)}*Rf)(x)=\int_{|\omega|=1}(k_q^{(n+q)}*Rf)(\omega,x)\,d\omega, \tag{2.3}$$

$$(k_q^{(n+q)}*Rf)(\omega,x)=\int_{-\infty}^{\infty}[(\partial_s^{(n+q)}k_q)(x\cdot\omega-s)Rf(\omega,s)]\,ds, \tag{2.4a}$$

i.e.,

$$\begin{aligned} R^*(k_q^{(n+q)}*Rf)(x)&=\int_{\mathbb{S}^{n-1}}[(\partial_s^{(n+q)}k_q)*Rf)(\omega,s)\,ds]\,d\omega\\ &=\int_{\mathbb{S}^{n-1}}\Big[\int_{-\infty}^{\infty}(\partial_s^{(n+q)}k_q)(x\cdot\omega-s)Rf(\omega,s)\,ds\Big]\,d\omega \end{aligned} \tag{2.4b}$$

or

$$f(\cdot)=\int_{\mathbb{S}^{n-1}}(k_q^{(n+q)}*Rf)(\omega,\cdot\omega)\,d\omega, \tag{2.4c}$$

where

$$\mathbb{R}\ni s\to k_q(s)=\begin{cases}-\dfrac{s^q\ln|s|}{i(2\pi i)^{n-1}q!} & \textit{for } n \textit{ even},\\ \dfrac{s^q\,\mathrm{sgn}(s)}{4(2\pi i)^{n-1}q!} & \textit{for } n \textit{ odd},\end{cases} \tag{2.5}$$

and q is a positive integer of the same parity as n.

Theorem 2.2. *For all $f\in\mathbb{S}(\mathbb{R}^n)$,*

$$f(x)=\Delta_x^{(n+q)/2}\int_{\mathbb{S}^{n-1}}(k_q*Rf)(\omega,x\omega)\,d\omega=\Delta^{(n+q)/2}R(k_q*Rf), \tag{2.6}$$

where $\Delta=\sum_{j=1}^n\partial_j^2$.

2.2. Remark. The following formula holds:

$$\delta(x) = \Delta^{(n+q)/2} \int\limits_{|\omega|=1} k_q(x\omega)\, d\omega. \tag{2.7}$$

Definition 2.2. The linear partial differential operator

$$P(\partial, \partial_t) = \sum_{|b|+l=m} a_b \partial^b \partial_t^l \tag{2.8}$$

with constant coefficients a_b is called *hyperbolic with respect to* t if its characteristic polynomial $P(\xi, \tau)$ has m real zeros in τ for every real ξ.

If the zeros are all different for every real $\xi \neq 0$, then the operator $P(\partial, \partial_t)$ is called *strongly hyperbolic with respect to* t.

Using the Radon transform one can construct a fundamental solution E of the strongly hyperbolic differential operator $P(\partial, \partial_t)$, i.e., a distribution E on $\mathbb{R}^n \times \mathbb{R}^1$ such that

$$P(\partial, \partial_t)E(x,t) = \delta(x,t).$$

We recall this construction. The first step is to construct explicitly in the half-space $t > 0$ a solution u of the Cauchy problem

$$\begin{aligned} P(\partial, \partial_t)u(x,t) &= 0 && \text{for } t > 0, \\ \partial_t^l u(x, 0+) &= 0 && \text{for } l = 0, 1, \ldots, m-2, \\ \partial_t^{m-1} u(x, 0+) &= \delta(x). \end{aligned} \tag{2.9}$$

For this purpose we use the Radon transform and the formulae (P1–P4) to reduce to the auxiliary Cauchy problem

$$\begin{aligned} P(\omega\partial_s, \partial_t)Ru(\omega, s, t) &= 0 && \text{for } t > 0, \\ \partial_t^l Ru(\omega, s, 0+) &= 0 && \text{for } l = 0, 1, \ldots, m-2, \\ \partial_t^{m-1} Ru(\omega, s, 0+) &= \delta(s). \end{aligned} \tag{2.10}$$

Convolution with k_q (cf. (2.5)) gives

$$\begin{aligned} k_q * P(\omega, \partial_s, \partial_t)\, Ru(\omega, s, t) &= 0, && t > 0, \\ k * \partial_t^l\, Ru(\omega, s, 0+) &= 0, && l = 0, 1, \ldots, m-2, \\ k * \partial_t^{m-1}\, Ru(\omega, s, 0+) &= k_q * \delta(s), \end{aligned}$$

because $k_q * \delta(s) = k_q(s)$. Then

$$\begin{aligned} P(\omega\partial_s, \partial_t)\, U(\omega, s, t) &= 0 && \text{for } t > 0, \\ \partial_t^l\, U(\omega, s, 0+) &= 0 && \text{for } l = 0, 1, \ldots, m-2, \\ \partial_t^{m-1}\, U(\omega, s, 0+) &= k_q(s), \end{aligned} \tag{2.11}$$

where

$$U(\omega, s, t) = (k_q * Ru)(\omega, s, t) = \int\limits_{-\infty}^{\infty} k_q(s - \delta)[Ru(\omega, \delta, t)]\, d\delta$$

(compare with (2.4a)).

Since the operator $P(\partial,\partial_t)$ is strongly hyperbolic, we can write

$$P(\omega,\tau)=\prod_{j=1}^{m}(\tau-\tau_j(\omega)),$$

where $\tau_1(\omega)<\ldots<\tau_m(\omega)$ are the zeros of $P(\omega,\tau)$. Then the first equation of (2.11) reads

$$\prod_{j=1}^{m}(\partial_t-\tau_j(\omega)\partial_s)\,U(\omega,s,t)=0 \quad \text{for } t>0$$

and the general solution is

$$U(\omega,s,t)=\sum_{j=1}^{m}f_k(s+\tau_k(\omega)t), \tag{2.12}$$

where $f_k\in C^m(\mathbb{R})$ are arbitrary.

The construction of the solution of the Cauchy problem (2.11) is based on (2.12) and on the following well known

LEMMA 2.1. *If a polynomial $P(\omega,\tau)$ of degree m has m different zeros in τ (i.e. the operator $P(\partial,\partial_t)$ is strongly hyperbolic), then*

$$\sum_{j=1}^{m}\frac{\tau_j^l(\omega)}{P'_\tau(\omega,\tau_j(\omega))}=\begin{cases}0 & \text{for } l=0,1,\ldots,m-2,\\ 1 & \text{for } l=m-1.\end{cases}$$

This follows from a simple evaluation of the Cauchy integral.

LEMMA 2.2. *The function*

$$U(\omega,s,t)=\sum_{j=1}^{m}\frac{k_{q+m-1}[s+\tau_j(\omega)t]}{P'_\tau[\omega,\tau_j(\omega)]}, \tag{2.13}$$

with k_{q+m-1} given by (2.5), *is a solution of the Cauchy problem* (2.11).

Proof. Putting

$$f_j[s+\tau_j(\omega)t]=\frac{k_{q+m-1}[s+\tau_j(\omega)t]}{P'_\tau[\omega,\tau_j(\omega)]},$$

we see that U satisfies the first equation of (2.11). Now, we check the initial condition: for $l=0,\ldots,m-2$,

$$\begin{aligned}\partial_t^l U(\omega,s,0+) &= \sum_{j=1}^{m}\frac{\partial_t^l k_{q+m-1}[s+\tau_j(\omega)t]}{P'_\tau[\omega,\tau_j(\omega)]}\bigg|_{t=0}\\ &= \sum_{j=1}^{m}\frac{\tau_j^l(\omega)k_{q+m-1}^{(l)}[s+\tau_j(\omega)t]}{P'_\tau[\omega,\tau_j(\omega)]}\bigg|_{t=0}\\ &= \sum_{j=1}^{m}\frac{\tau_j^l(\omega)k_{q+m-1}^{(l)}(s)}{P'_\tau[\omega,\tau_j(\omega)]}\\ &= 0\cdot k_{q+m-1}^{(l)}(s)=0.\end{aligned}$$

For $l = m-1$ we have

$$\partial_t^{m-1} U(\omega, s, 0+) = k_{q+m-1}^{(m-1)} \cdot 1 = k_{q+m-1}^{(m-1)}(s),$$

but

$$k_{q+m-1}^{(m-1)}(s) = k_q(s).$$

which follows from the formula for $k_q(s)$. ■

Since $U(\omega, s, t) = (k_q * Ru)(\omega, s, t)$, we have the solution of the problem (2.9), in the form

$$u = \Delta^{(n+q)/2} R(k_q * Ru) = \Delta^{(n+q)/2} \int\limits_{|\omega|=1} (k_q * Ru)(\omega x)\, d\omega = \Delta^{(n+q)/2} \int\limits_{|\omega|=1} U(\omega, (\omega x), t)\, d\omega. \tag{2.14}$$

Now, the last step consists in demonstrating that the distribution

$$E(x,t) = \varepsilon(t)\, u(x,t), \tag{2.15}$$

where u is given by (2.14) and

$$\varepsilon(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0, \end{cases}$$

is a fundamental solution of $P(\partial, \partial_t)$. The proof is very easy and is left to the reader.

Thus we have the following

THEOREM 2.3. *If a homogeneous differential operator $P(\partial, \partial_t)$ of order m is strongly hyperbolic with respect to t, then it has a fundamental solution E given by* (2.15).

Now, we adapt the above construction to a more general case, where the characteristic polynomial has multiple zeros (in τ). Each zero is assumed to be of constant multiplicity.

In this case we have

$$P(\tau, \omega) = \prod_{j=1}^{p} (\tau - \tau_j(\omega))^{rj}, \tag{2.16}$$

where

$$\tau_1(\omega) < \ldots < \tau_p(\omega), \quad p < m,\ r_1 + \ldots + r_p = m. \tag{2.17}$$

The first step is the same as above. By using the Radon transform and then a suitable convolution we reduce the Cauchy problem (2.9) for the operator $P(\partial, \partial_t)$ hyperbolic with respect to t to the auxiliary Cauchy problem (2.10) and then to the problem (2.11).

Next, by (2.16), the first equation of (2.11) can be replaced by

$$\prod_{j=1}^{p} (\partial_t - \tau_j(\omega)\partial_s)^{rj}\, U(\omega, s, t) = 0. \tag{2.18}$$

LEMMA 2.3. *The solution of the equation* (2.18) *is given by*

$$U(\omega, s, t) = \sum_{k=1}^{p} \sum_{d=0}^{r_k - 1} t^d f_{kd}(s + \tau_k(\omega) t), \tag{2.19}$$

where f_{kd} are arbitrary differentiable functions on $\mathbb{R}$ (cf. (2.12)).

P r o o f. The proof goes in the same way as that of Lemma 2.2, with

$$f_k(s+\tau_k(\omega)t)=\sum_{d=0}^{r_k-1} t^d\, f_{kd}(s+\tau_k(\omega)t),\qquad k=1,\ldots,p. \tag{2.20}$$

LEMMA 2.4. *If a polynomial* $P(\omega,\tau)$ *of degree* m *has* p *different zeros with constant multiplicities* (see (2.16)), *then*

$$\sum_{k=1}^{p}\left[\frac{z^l}{\prod_{j=1,j\neq k}^{p}(z-\tau_j(\omega))^{r_j}(r_k-1)!}\right]^{(r_k-1)}\Bigg|_{z=\tau_{k(\omega)}} =\begin{cases}0 & \text{for } l=0,1,\ldots,m-2,\\ 1 & \text{for } l=m-1,\end{cases} \tag{2.21}$$

where $[\ldots]^{(r_k-1)}=\frac{d^{r_k-1}}{dz^{r_k-1}}[\ldots]$.

The proof goes in the same way as the proof of Lemma 2.1.

LEMMA 2.5. *If the characteristic polynomial* $P(\omega,\tau)$ *of the hyperbolic homogeneous operator* $P(\partial,\partial_t)$ *of order* m *has* p *zeros* $\tau_1(\omega),\ldots,\tau_p(\omega)$ *with constant multiplicities* $r_1,\ldots,r_p$ *for* $\omega\neq 0$, *then the formula*

$$U(\omega,s,t)=\sum_{e=1}^{p}\sum_{d=0}^{r_e-1} t^d\,\frac{\binom{r_e-1}{d}}{(r_e-1)!}\,\frac{k_{q+m-d-1}(s+\tau_e(\omega)t)}{[\prod_{j=1,j\neq e}^{p}(z-\tau_j(\omega))^{r_j}]^{(r_e-d-1)}|_{z=\tau_e(\omega)}}, \tag{2.22}$$

where $k_{q+m-d-1}$ *is given by* (2.5), *gives a solution of the auxiliary Cauchy problem* (2.11) *for the hyperbolic operator* $P(\omega\partial_s,\partial_t)$.

S k e t c h o f p r o o f. It follows from (2.22) that the function U satisfies equation (2.18) for $t>0$. Taking the derivatives of U with respect to t and then letting $t\to 0+$ we see by Lemma 2.4 that U satisfies the initial condition of the Cauchy problem (2.11). This completes the proof.

Now it is easy to see that under the assumption of Lemma 2.5 the auxiliary Cauchy problem (2.11) can be solved by the formulae (2.22). As an immediate corollary from (2.14) and (2.15) we have the following

THEOREM 2.4. *If the homogeneous operator* $P(\partial,\partial_t)$ *of order* m *is hyperbolic with respect to* t *and its characteristic polynomial* $P(\omega,t)$ *has* p *zeros with constant multiplicities for* $\omega\neq 0$, *then the formula*

$$E(x,t)=\varepsilon(t)\Delta^{(n+q)/2}\int_{|\omega|=1} U(\omega,\omega x,t)\,d\omega, \tag{2.23}$$

where U *is given by* (2.22), *gives a fundamental solution for* $P(\partial,\partial_t)$.

In mathematical physics, matrices of fundamental solutions for hyperbolic systems are more important than fundamental solutions for hyperbolic operators considered in this section.

Now, we describe the construction of a matrix of fundamental solutions for a hyperbolic PDE system with constant coefficients.

Let $P_{jl}(\partial, \partial_t)_{j,l=1,\ldots,N}$ be a system of linear hyperbolic partial differential operators of the form

$$P_{jl}(\partial, \partial_t) = \sum_{|b|+l=m} a^{jl} \partial^b \partial_t^l, \quad j,l = 1,\ldots,N, \tag{2.24}$$

with constant coefficients a^{jl}. Let

$$P(\xi,\tau) = \det(P_{jl}(\xi,\tau)) \tag{2.25}$$

Assume that $P(\xi,\tau)$ has p real zeros $\tau_1(\xi),\ldots,\tau_p(\xi)$ with multiplicities $r_1,\ldots,r_p$, respectively, and $r_1+\ldots+r_p = mN$, when $p \le Nm$.

Definition 2.2. A matrix $\mathbb{E}(x,t) = (E_{jl}(x,t))_{jl}$ is called a *matrix of fundamental solutions* for the system $(P(\partial,\partial_t))_{j,l=1\ldots,N}$ if

$$P_{jk}(\partial,\partial_t)E_{kl}(x,t) = \delta_{jl}\delta(x,t), \quad j,l = 1,\ldots,N, \tag{2.26}$$

in the sense of distributions, where δ_{jk} denotes the Kronecker symbol and $\delta(x,t)$ is the Dirac function.

The construction of the elements $E_{jl}(x,t)$ is divided into three steps:

I. Using the Radon transform we solve the Cauchy problem

$$P(\partial,\partial_t)W(x,t) = 0 \quad \text{for } t>0, \tag{2.27}$$

$$\partial_t^l W(x,+0) = 0 \quad \text{for } l=1,\ldots,m-2; \quad \partial_t^{m-1}W(x,0+) = \delta(x), \tag{2.28}$$

where $P(\partial,\partial_t)$ is given by (2.15).

II. The solution $W(x,t)$ of (2.27), (2.28) yields a fundamental solution

$$E(x,t) = \varepsilon(t)W(x,t) \tag{2.29}$$

of $P(\partial,\partial_t)$.

III. Acting on $E(x,t)$ by the adjugate matrix $P^d(\partial,\partial_t)$ of $(P_{jl}(\partial,\partial_t))_{j,l=1,\ldots,N}$ we get the desired matrix $\mathbb{E}(x,t)$:

$$E_{jl}(x,t) = P^d_{jl}(\partial,\partial_t)E(x,t), \tag{2.30}$$

where

$$P^d_{jl}(\xi,\tau) = (-1)^{j+l}\det[P_{ik}(\xi,\tau)]_{i\ne j,k\ne l}. \tag{2.31}$$

2.2. Basic notation and formulae. We shall use the following notations:

$$D_x^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2}\ldots D_{x_n}^{\alpha_n} \quad \text{for} \quad \left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_n}^{\alpha_n}}$$

with $\alpha = (\alpha_1,\ldots,\alpha_n) \in (\mathbb{N}\cup\{0\})^3$.

$\mathbb{L}^p(\mathbb{R}^n)$, $1 \le p < \infty$, denotes the space of p-integrable functions with the L_p norm.

$\mathbb{L}^\infty(\mathbb{R}^n)$ is the space of essentially bounded measurable functions on $\mathbb{R}^n$ with the esssup norm.

Here for $0 < s \le \infty$, $1 \le p \le m$ we denote by $\mathbb{W}^{s,p}(\mathbb{R}^n)$ the usual Sobolev space with the norm $\|\cdot\|_{\mathbb{W}^{s,p}(\mathbb{R}^n)}$ (cf. [2], [134]); $\mathbb{W}^{s,p}(\mathbb{R}^n) = \mathbb{L}^p(\mathbb{R}^n)$ with the norm $\|\cdot\|_{\mathbb{L}^p(\mathbb{R}^n)}$; $\mathbb{W}^{s,2}(\mathbb{R}^n) = \mathbb{H}^s(\mathbb{R}^n)$. The norm $\|\cdot\|_{\mathbb{L}^p_{,s}(\mathbb{R}^n)}$ stands sometimes for the Sobolev norm $\|\cdot\|_{\mathbb{W}^{s,p}(\mathbb{R}^n)}$. Instead of $\mathbb{W}^{s,p}(\mathbb{R}^n)$ we write $\mathbb{L}^p_{,s}(\mathbb{R}^n)$.

Let X be a Banach space and I an interval. The $C^l(I,X)$ ($l \geq 0$, an integer) denotes the space of l times continuously differentiable functions f on I with values in X. We recall the definition of mollifiers. Let J be a nonnegative real-valued $\mathbb{C}_0^\infty(\mathbb{R}^n)$ function with $J(x) = 0$ if $|x| \geq 1$ and $\int_{\mathbb{R}^n} J(x)\,dx = 1$. For example, we may take

$$J(x) = \begin{cases} C\exp\left(\frac{-1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

If $\varepsilon > 0$, then the function $J_\varepsilon(x) = \varepsilon^{-n} J(x/\varepsilon)$ is nonnegative, belongs to $\mathbb{C}_0^\infty(\mathbb{R}^n)$ and satisfies $J_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$, $\int_{\mathbb{R}^n} J(x)\,dx = 1$. J_ε is called a *mollifier* and the convolution

$$J_\varepsilon * u(x) = \int_{\mathbb{R}^n} J_\varepsilon(x-y)u(y)dy, \tag{2.32}$$

defined for a function u for which the right side of (2.32) makes sense, is called a *mollifier of u*. Below, we give some properties (cf. [90]) of mollification which we use in our paper. If $u \in \mathbb{H}^s(\mathbb{R}^n)$ then

$$J_\varepsilon * u \in \mathbb{C}^\infty(\mathbb{R}^n) \cap \mathbb{H}^s(\mathbb{R}^n), \quad \|u - J_\varepsilon * u\|_{\mathbb{H}^s} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.33}$$

If $u, v \in \mathbb{H}^s(\mathbb{R}^n)$, then

$$\|J_\varepsilon * (uD_x^1 v) - u(J_\varepsilon * D_x^1 v)\|_{\mathbb{H}^s} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.34}$$

We use the following Sobolev imbedding theorems (cf. [2], [134]):

$$\begin{gathered} \mathbb{W}^{s,p}(\mathbb{R}^n) \hookrightarrow \mathbb{C}^k(\mathbb{R}^n) \quad \text{if } s > k + n/p, \\ \mathbb{W}^{s,p}(\mathbb{R}^n) \hookrightarrow \mathbb{W}^{t,q}(\mathbb{R}^n) \quad \text{if } \begin{cases} 1 < p \leq q < \infty, \\ s > t, \\ 1/q \geq 1/p - (s-t)/n. \end{cases} \end{gathered} \tag{2.35}$$

Lemma 2.6 (Gronwall's inequality). *If $y \in \mathbb{C}^1(\mathbb{R})$ satisfies*

$$\frac{dy}{dt} + p(t)y \leq q(t) \tag{2.36}$$

for $p, q \in \mathbb{C}^0(\mathbb{R})$, then

$$y(t) \leq \Big[y(0) + \int_0^t q(\sigma)\exp\Big(\int_0^\sigma p(\tau)\,d\tau\Big)\,d\sigma\Big] \cdot \exp\Big(-\int_0^t p(\tau)\,d\tau\Big) \quad \textit{for } t > 0.$$

We also use the formula for the partial derivative of order α ($\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = s$) for the composite function $a(V(x))$ (where $V(x) = (V_1(x), \ldots, V_m(x))$ is a vector function with m components, depends only on $x \in \mathbb{R}^3$) of the following form:

$$\begin{aligned} D^{(\alpha_1,\alpha_2,\alpha_3)} a(V(x)) &= \sum D^{(\Lambda_1,\ldots,\Lambda_m)} a \\ &\times \frac{(\alpha_1!\alpha_2!\alpha_3!)(V_1^{(1,0,0)})^{k^1_{(1,0,0)}} \ldots (V_m^{(\lfloor s/2\rfloor,0,0)})^{k^m_{(\lfloor s/2\rfloor,0,0)}} \ldots (V_1^{(\lceil s/2\rceil,0,0)})^{k^1_{(\lceil s/2\rceil,0,0)}} \ldots (V_m^{(0,0,s)})^{k^m_{(0,0,s)}}}{(k^1_{(1,0,0)})! \ldots (k^m_{(0,0,s)})! \ldots (1!0!0!)^{k^1_{(1,0,0)}} \ldots (1!0!0!)^{k^m_{(1,0,0)}} \ldots (0!0!s!)^{k^1_{(0,0,s)}} \ldots (0!0!s!)^{k^m_{(0,0,s)}}}, \end{aligned} \tag{2.37}$$

where summation is over all $\Lambda_1, \dots, \Lambda_m$ satisfying

$$\Lambda_1 + \Lambda_2 + \dots + \Lambda_m \le s,$$

$$\Lambda_1 = k^1_{(1,0,0)} + \dots + k^1_{(1,1,1)} + \dots + k^1_{(s,0,0)} + \dots + k^1_{(0,s,0)} + \dots + k^1_{(0,0,s)},$$

$$\Lambda_2 = k^2_{(1,0,0)} + \dots + k^2_{(1,1,1)} + \dots + k^2_{(s,0,0)} + \dots + k^2_{(0,s,0)} + \dots + k^2_{(0,0,s)},$$

$$\vdots$$

$$\Lambda_m = k^m_{(1,0,0)} + \dots + k^m_{(1,1,1)} + \dots + k^m_{(s,0,0)} + \dots + k^m_{(0,s,0)} + \dots + k^m_{(0,0,s)},$$

$$\begin{aligned}
1(k^1_{(1,0,0)} &+ \dots + k^m_{(1,0,0)} + \dots + k^1_{(1,0,s-1)} + \dots + k^m_{(1,0,s-1)}) \\
&+ 2(k^1_{(2,0,0)} + \dots + k^m_{(2,0,0)} + \dots + k^1_{(2,0,s-2)} + \dots + k^m_{(2,0,s-2)}) \\
&+ \lfloor s/2 \rfloor (k^1_{(\lfloor s/2 \rfloor,0,0)} + \dots + k^m_{(\lfloor s/2 \rfloor,0,s-\lfloor s/2 \rfloor)}) \\
&+ \lceil s/2 \rceil (k^1_{(\lceil s/2 \rceil,0,0)} + \dots + k^m_{(\lceil s/2 \rceil,0,s-\lceil s/2 \rceil)}) + \dots \\
&+ s(k^1_{(s,0,0)} + \dots + k^m_{(s,0,0)}) = \alpha_1,
\end{aligned}$$

$$\begin{aligned}
1(k^1_{(0,1,0)} &+ \dots + k^m_{(0,1,0)} + \dots + k^1_{(0,1,s-1)} + \dots + k^m_{(0,1,s-1)}) \\
&+ 2(k^1_{(0,2,0)} + \dots + k^m_{(0,2,0)} + \dots + k^1_{(0,2,s-2)} + \dots + k^m_{(0,2,s-2)}) \\
&+ \lfloor s/2 \rfloor (k^1_{(0,\lfloor s/2 \rfloor,0)} + \dots + k^m_{(0,\lfloor s/2 \rfloor,s-\lfloor s/2 \rfloor)}) \\
&+ \lceil s/2 \rceil (k^1_{(0,\lceil s/2 \rceil,0)} + \dots + k^m_{(0,\lceil s/2 \rceil,s-\lceil s/2 \rceil)}) + \dots \\
&+ s(k^1_{(0,s,0)} + \dots + k^m_{(0,s,0)}) = \alpha_2,
\end{aligned}$$

$$\begin{aligned}
1(k^1_{(0,0,1)} &+ \dots + k^m_{(0,0,1)} + \dots + k^1_{(0,s-1,1)} + \dots + k^m_{(0,s-1,1)}) \\
&+ 2(k^1_{(0,0,2)} + \dots + k^m_{(0,0,2)} + \dots + k^1_{(0,s-2,2)} + \dots + k^m_{(0,s-2,2)}) \\
&+ \lfloor s/2 \rfloor (k^1_{(0,0,\lfloor s/2 \rfloor)} + \dots + k^m_{(0,s-\lfloor s/2 \rfloor,\lfloor s/2 \rfloor)}) \\
&+ \lceil s/2 \rceil (k^1_{(0,0,\lceil s/2 \rceil)} + \dots k^m_{(0,s-\lceil s/2 \rceil,\lceil s/2 \rceil,0,)}) + \dots \\
&+ s(k^1_{(0,0,s)} + \dots + k^m_{(0,0,s)}) = \alpha_3.
\end{aligned}$$

The proof of (2.37) is based on the Taylor formula (cf. [121]).

3. $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates for the Cauchy problem for hyperbolic thermoelasticity

3.1. Matrix of fundamental solutions for linear hyperbolic thermoelasticity. The system describing an isotropic and homogeneous thermoelastic medium in three-dimensional Euclidean space (cf. [24], [138]) is

$$\varrho \partial_t^2 u - \mu \Delta u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u + \beta \operatorname{grad} \partial_t T = \mathbf{0}, \tag{3.1}$$

$$\beta \operatorname{div} \partial_t u + \varrho \tau \partial_t^2 T - k \Delta T = 0. \tag{3.2}$$

The system (3.1), (3.2) can be written in matrix form

$$L(\partial,\partial_t)U(x,t)=0, \tag{3.3}$$

where $L(\partial,\partial_t)$ is the 4×4 matrix with elements

$$\begin{aligned}L_{jk}(\partial,\partial_t)&=[(\varrho\partial_t^2-\mu\Delta)\delta_{jk}-(\lambda+\mu)\partial_{jk}(1-\delta_{4j})](1-\delta_{k4})\\&\quad+\beta\partial_j\partial_t(1-\delta_{jk})\delta_{k4}+[\varrho\tau\partial_t^2-k\Delta]\delta_{jk}\delta_{k4}+\beta\partial_k\partial_t(1-\delta_{jk})\delta_{4j},\end{aligned}$$

and $U=(u,T)^*$ is a four-dimensional vector of displacement u and temperature T.

A matrix $H(x,t)$ of fundamental solutions of (3.3) satisfies in $\mathbb{R}^4$ the system

$$L(\partial,\partial_t)H(x,t)=\delta(x,t)I,$$

where $\delta(x,t)$ is the Dirac function and I is the 4×4 unit matrix. The operator $L(\partial,\partial_t)$ is hyperbolic because its symbol, i.e., the symbol of the operator $P(\partial,\partial_t)$ given by

$$P(\partial,\partial_t)=\det L(\partial,\partial_t)=\varrho^4\tau(\partial_t^2-a_1^2\Delta)(\partial_t^2-a_2^2\Delta)(\partial_t^2-b^2\Delta)^2, \tag{3.4}$$

has eight roots

$$r_1=-a_1,\quad r_2=a_1,\quad r_3=-a_2,\quad r_4=a_2,\quad r_{5,6}=b,\quad r_{7,8}=-b$$

with multiplicities

$$m_1=1,\quad m_2=1,\quad m_3=1,\quad m_4=1,\quad m_5=2,\quad m_6=2,\quad m_7=2,\quad m_8=2,$$

respectively, where

$$\begin{gathered}a_1^2=\frac{k+\frac{\beta^2}{\varrho}+\varrho\tau a^2-\sqrt{\sigma}}{2\varrho\tau},\qquad a_2^2=\frac{k+\frac{\beta^2}{\varrho}+\varrho\tau a^2+\sqrt{\sigma}}{2\varrho\tau},\\ b^2=\frac{\mu}{\varrho},\qquad a^2=\frac{\lambda+2\mu}{\varrho}\end{gathered} \tag{3.5}$$

with

$$\sigma=(k-\varrho\tau a^2)^2+\left(\frac{\beta^2}{\varrho}\right)^2+\frac{2k\beta^2}{\varrho}+2\beta^2\tau a^2>0.$$

It follows from (3.5) that

$$\sigma=\left(k+\frac{\beta^2}{\varrho}+\varrho\tau a^2\right)^2-4\varrho\tau a^2>0,$$

so $\sqrt{\sigma}<k+\beta^2/\varrho+\varrho\tau a^2$. Hence $a_1^2>0$ and $a_2^2>0$.

In formula (3.5), a is the velocity of propagation of longitudinal waves, b the velocity of propagation of transversal elastic waves and $\vartheta=\sqrt{k/(\varrho\tau)}$ the velocity of propagation of thermal waves.

The construction of elements $H_{jk}(x,t)$ of $H(x,t)$ is divided (cf. (2.27)–(2.31)) into three steps:

I. Using the Radon transform we solve the Cauchy problem

$$P(\partial,\partial_t)W(x,t)=0\quad\text{for } t>0, \tag{3.6}$$

$$\partial_t^lW(x,+0)=0\quad\text{for } l=1,\ldots,6\quad\text{and}\quad\partial_t^7W(x,+0)=\delta(x), \tag{3.7}$$

where the operator $P(\partial,\partial_t)$ is given by (3.4).

II. From the solution $W(x,t)$ of (3.6), (3.7) we get

$$E(x,t) = \begin{cases} W(x,t) & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

This is a fundamental solution of (3.6), i.e.,

$$P(\partial, \partial_t)E(x,t) = \delta(x,t).$$

III. Acting on $E(x,t)$ by the adjugate matrix $L^d(\partial, \partial_t)$ of $L(\partial, \partial_t)$ we get the required matrix $H(x,t)$.

It is easy to see that the elements $L^d_{jk}(\partial, \partial_t)$ of the adjugate matrix $L^d(\partial, \partial_t)$ are

$$\begin{aligned}
(3.8)\quad & L^d_{jk}(\partial, \partial_t) \\
&= [\varrho^3\tau\partial_t^6 - (\varrho^2 k + \varrho^2\tau(\lambda + 3\mu) + \varrho\beta^2)\partial_t^4\Delta \\
&\quad + (\mu\varrho\tau(\lambda+2\mu) + \varrho k(\lambda+2\mu) + \mu\beta^2)\partial_t^2\Delta^2 - k\mu(\lambda+2\mu)\Delta^3]\delta_{jk}(1-\delta_{k4}) \\
&\quad + [(\varrho^2\tau(\lambda+\mu)\varrho\beta^2)\partial^4\partial_{jk} - (k\varrho(\lambda+\mu) + \varrho\tau\mu(\lambda+\mu) + \mu\beta^2)\partial_t^2\Delta\partial_{jk} \\
&\quad + \mu(\lambda+\mu)k\Delta^2\partial_{jk}]\delta_{jj}(1-\delta_{k4}) + [-\varrho^2\beta^2\partial_t^5\partial_j + 2\mu\varrho\beta\partial_t^3\Delta\partial_j \\
&\quad + \mu^2\beta\partial_t\Delta^2\partial_j](1-\delta_{jk})\delta_{k4} + [\varrho^3\partial_t^6 - \varrho^3(a^2+2b^2)\partial_t^4\Delta + \varrho^3(2a^2b^2+b^4)\partial_t^4\Delta^2 \\
&\quad + \varrho^3a^2b^4\Delta^3]\delta_{jk}\delta_{k4} + [-\varrho^2\beta\partial_t^5\partial_k + 2\mu\varrho\beta\partial_t^3\Delta\partial_k - \mu^2\beta\partial_t\Delta^2\partial_k](1-\delta_{kj})\delta_{4j}.
\end{aligned}$$

To use the Radon transform, we reduce the Cauchy problem (3.6), (3.7) to

$$\begin{aligned}
P(\omega\partial_s, \partial_t)RW(\omega,s,t) &= 0 && \text{for } t > 0, \\
\partial_t^l RW(\omega,s,+0) &= 0 && \text{for } l = 0,\dots,6, \\
\partial_t^7 RW(\omega,s,+0) &= \delta(s).
\end{aligned}$$

Then, by convolution of RW with k_q (cf. (2.5)), we obtain

$$(3.9)\qquad \begin{aligned}
P(\omega\partial_s, \partial_t)V(\omega,s,t) &= 0 && \text{for } t > 0, \\
\partial_t^l V(\omega,s,+0) &= 0 && \text{for } l = 0,\dots,6, \\
\partial_t^7 V(\omega,s,+0) &= k_q(s),
\end{aligned}$$

where

$$V(\omega,s,t) = \int_{-\infty}^{\infty} k_q(s-\sigma)[RW(\omega,\sigma,t)]\,d\sigma.$$

Since $P(\partial, \partial_t)$ is hyperbolic (cf. (3.4)), we may seek the solution of the problem (3.9) (cf. (2.22)) in the form of plane waves

$$\begin{aligned}
(3.10)\qquad V(\omega,s,t) &= \varphi_1^0(s+a_1t) + \varphi_2^0(s-a_1t) + \varphi_1(s+a_2t) + \varphi_2(s-a_2t) \\
&\quad + \varphi_3(s+bt) + \varphi_4(s-bt) + t\varphi_5(s+bt) + t\varphi_6(s-bt),
\end{aligned}$$

where

$$(3.11)\qquad \begin{aligned}
\varphi_1^0(s+a_1t) &= \frac{k_{q+7}(s+a_1t)}{\partial_r P(r)|_{r=a_1}}, \qquad \varphi_2^0(s-a_1t) = \frac{k_{q+7}(s-a_1t)}{\partial_r P(r)|_{r=-a_1}}, \\
\varphi_1(s+a_2t) &= \frac{k_{q+7}(s+a_2t)}{\partial_r P(r)|_{r=a_2}}, \qquad \varphi_2(s-a_2t) = \frac{k_{q+7}(s-a_2t)}{\partial_r P(r)|_{r=-a_2}},
\end{aligned}$$

$$\varphi_3(s+bt) = \frac{-k_{q+7}(s+bt)\partial_r P(r)|_{r=b}}{[\partial_r P(r)|_{r=b}]^2}, \quad \varphi_4(s-bt) = \frac{-k_{q+7}(s-bt)\partial_r P(r)|_{r=-b}}{[\partial_r P(r)|_{r=-b}]^2},$$

(3.11)
[cont.]

$$\varphi_5(s+bt) = \frac{k_{q+6}(s+bt)}{P_1(r)|_{r=b}}, \quad \varphi_6(s-bt) = \frac{k_{q+6}(s-bt)}{P_2(r)|_{r=-b}}$$

with

$$P_1(r) = \varrho^4\tau(r^2-a_1^2)(r^2-a_2^2)(r+b)^2, \qquad P_2(r) = \varrho^4\tau(r^2-a_1^2)(r^2-a_2^2)(r-b)^2.$$

Without loss of generality we assume that $q = 1$. Substituting (3.11) into (3.10) and taking into account (2.5) we obtain

$$\begin{aligned}(3.12)\quad & V(\omega,s,t) \\ &= \frac{1}{128\pi^2 7!\varrho^4\tau}\Bigg\{\frac{1}{a_1^2-a_2^2}\Bigg[\frac{1}{(a_1^2-b^2)^2 8a_1} \\ &\times[-(s+a_1t)^8\operatorname{sgn}(s+a_1t)+(s-a_1t)^8\operatorname{sgn}(s-a_1t)] \\ &+\frac{1}{(a_2^2-b^2)^2 8a_2}[-(s+a_2t)^8\operatorname{sgn}(s+a_2t)+(s-a_2t)^8\operatorname{sgn}(s-a_2t)]\Bigg] \\ &+\frac{(5b^4-3b^2(a_1^2+a_2^2)+a_1^2\cdot a_2^2)}{4b^3(b^2-a_1^2)^2(b^2-a_2^2)^2}[(s+bt)^8\operatorname{sgn}(s+bt)-(s-bt)^8\operatorname{sgn}(s-bt)] \\ &-\frac{1}{b^2(a_1^2-b^2)(b^2-a_2^2)}[t(s+bt)^7\operatorname{sgn}(s+bt)+t(s-bt)^7\operatorname{sgn}(s-bt)]\Bigg\}.\end{aligned}$$

Using (2.2) and (2.6), we get the following solution of (3.6), (3.7):

$$W(x,t) = \Delta_x^2 \int_{|\omega|=1} V(\omega,s,t)|_{s=\omega x}\,d\omega. \tag{3.13}$$

We illustrate the calculation of the integral (3.13) using the first term in (3.12). We have

$$\begin{aligned}I(x,t) &= \int_{|\omega|=1}(s+a_1t)^8\operatorname{sgn}(s+a_1t)|_{s=\omega x}\,d\omega \\ &= \int_{|\omega|=1}(\omega x+a_1t)^8\operatorname{sgn}(\omega x+a_1t)d\omega.\end{aligned} \tag{3.14}$$

We change variables in (3.14) by means of $x\omega = |x|p$ where $p\in[-1,1]$ and $d\omega = 2\pi dp$ to get

$$I(x,t) = 2\pi\int_{-1}^{1}(|x|p+a_1t)^8\operatorname{sgn}(|x|p+a_1t)\,dp. \tag{3.15}$$

After integration we have

$$\begin{aligned}I(x,t) = \frac{2}{9|x|}\Bigg\{&[(|x|+a_1t)^9-(|x|-a_1t)^9]\bigg[\varepsilon(t)-\varepsilon\bigg(t-\frac{|x|}{a_1}\bigg)\bigg] \\ &+[(|x|+a_1t)^9+(|x|-a_1t)^9]\varepsilon\bigg(t-\frac{|x|}{a_1}\bigg)\Bigg\},\end{aligned}$$

where $\varepsilon(t)$ denotes Heaviside's function.

Integrating the remaining terms in (3.12) in the same way we obtain

$$\int\limits_{|\omega|=1} V(\omega,s,t)|_{s=\omega x}\,d\omega \tag{3.16}$$
$$= \frac{1}{32\pi 7!\varrho^4\tau|x|}\bigg\{\frac{1}{(a_1^2-a_2^2)}\bigg\{-\frac{1}{9a_1(a_1^2-b^2)^2}$$
$$\times(|x|-a_1t)^9\varepsilon\Big(t-\frac{|x|}{a_1}\Big)-\frac{1}{9a_2(a_2^2-b^2)^2}\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\bigg\}$$
$$+\frac{(5b^4-3b^2(a_1^2+a_2^2)+a_1^2a_2^2)}{18b^3(b^2-a_1^2)^2(b^2-a_2^2)^2}(|x|-bt)^9\varepsilon\Big(t-\frac{|x|}{b}\Big)$$
$$+t\frac{(|x|-bt)^8\varepsilon\Big(t-\frac{|x|}{b}\Big)}{4b^2(a_1^2-b^2)(b^2-a_2^2)}$$
$$+\varepsilon(t)\bigg[\frac{1}{(a_1^2-a_2^2)}\bigg[\frac{(|x|-a_1t)^9-(|x|+a_1t)^9}{18(a_1^2-b^2)^2a_1}$$
$$+\frac{(|x|-a_2t)^9-(|x|+a_2t)^9}{18(a_2^2-b^2)^2a_2}\bigg]+\frac{5b^4-3b^2(a_1^2+a_2^2)+a_1^2a_2^2}{36b^3(b^2-a_1^2)^2(b^2-a_2^2)^2}$$
$$\times((|x|+bt)^9-(|x|-bt)^9)+t\frac{(|x|+bt)^8+(|x|-bt)^8}{8b^2(a_1^2-b^2)(b^2-a_2^2)}\bigg]\bigg\}.$$

Applying the operator Δ_x^2 to the integral (3.16) we have

$$W(x,t)=\frac{1}{32\pi 7!\varrho^4\tau|x|}\bigg\{\frac{1}{a_1^2-a_2^2}\bigg[-\frac{336}{a_1(a_1^2-b^2)^2}(|x|-a_1t)^5\varepsilon\Big(t-\frac{|x|}{a_1}\Big) \tag{3.17}$$
$$+\frac{336}{a_1(a_1^2-b^2)^2}(|x|-a_1t)^5\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\bigg]$$
$$+\frac{168\Big[5b^4-3b^2(a_1^2+a_2^2)+a_1^2a_2^2\Big]}{b^3(b^2-a_1^2)^2(b^2-a_2^2)^2}(|x|-bt)^5\varepsilon\Big(t-\frac{|x|}{b}\Big)$$
$$+\frac{1680}{2b^2(a_1^2-b^2)(a_2^2-b^2)}t(|x|-bt)^4\varepsilon\Big(t-\frac{|x|}{b}\Big)\bigg\}.$$

Now, in agreement with the 3rd step in Section 3, we calculate the elements of the matrix $H(x,t)$. For simplicity, we introduce the following notations:

$$\begin{aligned}\Psi_i(x,t)&=\frac{1}{|x|}(|x|-a_it)^5\varepsilon\Big(t-\frac{|x|}{a_i}\Big),\qquad i=1,2,3,\quad a_3=b,\\ \Psi_4(x,t)&=\frac{t}{|x|}(|x|-bt)^4\varepsilon\Big(t-\frac{|x|}{b}\Big).\end{aligned} \tag{3.18}$$

Acting on the functions (3.18) by the partial differential operators occurring in (3.8) we get

$$
\begin{aligned}
\partial_t^6 \Psi_i &= -\frac{a_i^5 5!}{|x|}\delta\Big(t-\frac{|x|}{a_i}\Big),\\
\Delta\partial_t^4 \Psi_i &= -\frac{a_i^3 5!}{|x|}\delta\Big(t-\frac{|x|}{a_i}\Big),\\
\Delta^2\partial_t^2 \Psi_i &= -\frac{a_i 5!}{|x|}\delta\Big(t-\frac{|x|}{a_i}\Big),\\
\Delta^3 \Psi_i &= -\frac{5!}{a_i|x|}\delta\Big(t-\frac{|x|}{a_i}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\partial_t^4\partial_{jk}\Psi_i &= a_i^5 5!t\Big(\frac{\delta_{kj}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\varepsilon\Big(t-\frac{|x|}{a_i}\Big)-a_i^3 5!\frac{x_kx_j}{|x|^3}\delta\Big(t-\frac{|x|}{a_i}\Big),\\
\partial_t^2\Delta\partial_{jk}\Psi_i &= a_i^3 5!t\Big(\frac{\delta_{kj}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\varepsilon\Big(t-\frac{|x|}{a_i}\Big)-a_i 5!\frac{x_jx_k}{|x|^3}\delta\Big(t-\frac{|x|}{a_i}\Big),\\
\Delta^2\partial_{jk}\Psi_i &= a_i 5!t\Big(\frac{\delta_{kj}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\varepsilon\Big(t-\frac{|x|}{a_i}\Big)-\frac{5!}{a_i}\frac{x_jx_k}{|x|^3}\delta\Big(t-\frac{|x|}{a_i}\Big),\\
\partial_t^5\partial_j\Psi_i &= a_i^5 5!\frac{x_j}{|x|^3}\varepsilon\Big(t-\frac{|x|}{a_i}\Big)+a_i^4 5!\frac{x_j}{|x|^2}\delta\Big(t-\frac{|x|}{a_i}\Big),\\
\partial_t^3\Delta\partial_j\Psi_i &= a_i^2 5!\frac{x_j}{|x|^2}\delta\Big(t-\frac{|x|}{a_i}\Big)+a_i^3 5!\frac{x_j}{|x|^3}\varepsilon\Big(t-\frac{|x|}{a_i}\Big),\\
\partial_t\Delta\partial_j\Psi_i &= 5!\frac{x_j}{|x|^2}\delta\Big(t-\frac{|x|}{a_i}\Big)+a_i 5!\frac{x_j}{|x|^3}\varepsilon\Big(t-\frac{|x|}{a_i}\Big), \quad i=1,2,3.
\end{aligned} \tag{3.19}
$$

In the same way, we have

$$
\begin{aligned}
\partial_t^6\Psi_4 &= 144\frac{b^4}{|x|}\delta\Big(t-\frac{|x|}{b}\Big)+24b^4\frac{t}{|x|}\delta'\Big(t-\frac{|x|}{b}\Big),\\
\Delta\partial_t^4\Psi_4 &= 96\frac{b^2}{|x|}\delta\Big(t-\frac{|x|}{b}\Big)+24b^2\frac{t}{|x|}\delta'\Big(t-\frac{|x|}{b}\Big),\\
\Delta^2\partial_t^4\Psi_4 &= 48b\frac{t}{|x|^2}\delta\Big(t-\frac{|x|}{b}\Big)+24\frac{t}{|x|}\delta'\Big(t-\frac{|x|}{b}\Big),\\
\Delta^3\Psi_4 &= 4!\frac{t}{b^2|x|}\delta'\Big(t-\frac{|x|}{b}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\partial_t^4\partial_{jk}\Psi_4 &= -120b^4t\Big(\frac{\delta_{kj}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\varepsilon\Big(t-\frac{|x|}{b}\Big)-24b^3t\Big(\frac{\delta_{kj}}{|x|^2}-\frac{7x_jx_k}{|x|^4}\Big)\\
&\quad\times\delta\Big(t-\frac{|x|}{b}\Big)+24b^2t\frac{x_jx_k}{|x|^3}\delta'\Big(t-\frac{|x|}{b}\Big),\\
\partial_t^2\Delta\partial_{jk}\Psi_4 &= -72b^2t\Big(\frac{\delta_{kj}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\varepsilon\Big(t-\frac{|x|}{b}\Big)-24bt\Big(\frac{\delta_{kj}}{|x|^2}-\frac{5x_jx_k}{|x|^4}\Big)\\
&\quad\times\delta\Big(t-\frac{|x|}{b}\Big)+24t\frac{x_jx_k}{|x|^3}\delta'\Big(t-\frac{|x|}{b}\Big),\\
\Delta^2\partial_{jk}\Psi_4 &= -4!t\Big(\frac{\delta_{kj}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\varepsilon\Big(t-\frac{|x|}{b}\Big)-4!\frac{t}{b}\Big(\frac{\delta_{kj}}{|x|^2}-\frac{3x_jx_k}{|x|^4}\Big)
\end{aligned} \tag{3.20}
$$

$$\times\,\delta\Big(t-\frac{|x|}{b}\Big)+4!\frac{t}{b^2}\frac{x_jx_k}{|x|^3}\delta'\Big(t-\frac{|x|}{b}\Big), \tag{3.20 [cont.]}$$

$$\partial_j\partial_t^5\Psi_4 = -120b^4\frac{x_j}{|x|^3}\varepsilon\Big(t-\frac{|x|}{b}\Big)-144b^3\frac{x_j}{|x|^2}\delta\Big(t-\frac{|x|}{b}\Big)+24b^3t\frac{x_j}{|x|^2}\delta'\Big(t-\frac{|x|}{b}\Big),$$

$$\partial_t^3\Delta\partial_j\Psi_4 = -72b^2\frac{x_j}{|x|^3}\varepsilon\Big(t-\frac{|x|}{b}\Big)-96b\frac{x_j}{|x|^2}\delta\Big(t-\frac{|x|}{b}\Big)+24bt\frac{x_j}{|x|^2}\delta'\Big(t-\frac{|x|}{b}\Big),$$

$$\partial_j\partial_t\Delta^2\Psi_4 = -4!\frac{x_j}{|x|^3}\varepsilon\Big(t-\frac{|x|}{b}\Big)-48\frac{x_j}{b|x|^2}\delta\Big(t-\frac{|x|}{b}\Big)+24t\frac{x_j}{b|x|^2}\delta'\Big(t-\frac{|x|}{b}\Big).$$

After some calculations we get

$$\begin{aligned}
H_{jk}(x,t) &= (32\pi 7!\varrho^4\tau)^{-1}\\
&\times\Big\{\delta_{jk}(1-\delta_{k4})\Big[\frac{A_1}{a_1^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_1}\Big)-\frac{A_2}{a_2^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_2}\Big)-\frac{B}{b^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{b}\Big)\Big]\\
&+\delta_{jk}\delta_{k4}\Big[\frac{A_3}{a_1^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_1}\Big)-\frac{A_4}{a_2^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_2}\Big)\Big]\\
&+\delta_{jj}(1-\delta_{k4})\Big[\frac{A_5}{a_1^2}\frac{x_jx_k}{x^3}\delta\Big(t-\frac{|x|}{a_1}\Big)-\frac{A_6}{a_2^2}\frac{x_jx_k}{|x|^3}\delta\Big(t-\frac{|x|}{a_2}\Big)+\frac{B}{b^2}\frac{x_jx_k}{|x|^3}\delta\Big(t-\frac{|x|}{b}\Big)\Big]\\
&-\delta_{jj}(1-\delta_{k4})\Big[A_7t\Big(\frac{\delta_{jk}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\Big]\Big[\varepsilon\Big(t-\frac{|x|}{a_1}\Big)-\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\Big]\\
&+A_8t\Big(\frac{\delta_{jk}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\times\Big[\varepsilon\Big(t-\frac{|x|}{b}\Big)-\varepsilon\Big(t-\frac{|x|}{a_1}\Big)\Big]\\
&+A_9t\Big(\frac{\delta_{jk}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\Big[\varepsilon\Big(t-\frac{|x|}{b}\Big)-\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\Big]\Big]\\
&+A_{10}\frac{x_j}{|x|^3}\Big[\varepsilon\Big(t-\frac{|x|}{a_1}\Big)-\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\Big]\delta_{4j}(1-\delta_{kj})\\
&+A_{10}\frac{x_j}{|x|^3}\Big[\delta\Big(t-\frac{|x|}{a_1}\Big)-\delta\Big(t-\frac{|x|}{a_2}\Big)\Big]\delta_{4j}(1-\delta_{kj})\Big\},\qquad j,k=1,2,3,4,
\end{aligned}\tag{3.21}$$

where A_j, $j=1,\ldots,10$, and B are constants given by

$$\begin{aligned}
A_1 &= \frac{\kappa}{(a_1^2-b^2)^2}[a_1^6\varrho^3\tau-a_1^4(\varrho^2k^2+\varrho^2\tau(\lambda+3\mu)+\varrho\beta^2)+a_1(\mu\varrho\tau(\lambda+2\mu)\\
&\quad+\varrho k(\lambda+3\mu)+\mu\beta^2)-k\mu(\lambda+2\mu)],\\
A_2 &= \frac{\kappa}{(a_2^2-b^2)^2}[a_2^6\varrho^3\tau-a_2^4(\varrho^2k^2+\varrho^2\tau(\lambda+3\mu)+\varrho\beta^2)+a_2(\mu\varrho\tau(\lambda+2\mu)\\
&\quad+\varrho k(\lambda+3\mu)+\mu\beta^2)-k\mu(\lambda+2\mu)],\\
A_3 &= \frac{\kappa}{(a_1^2-b^2)^2}[a_1^6-a_1^4a^2-2a_1^4b^2+2a_1^2a^2b^2+a_1^2b^4-a^2b^4],\\
A_4 &= \frac{\kappa}{(a_2^2-b^2)^2}[a_2^6-a_2^4a^2-2a_2^4b^2+2a_2^2a^2b^2+a_2^2b^4-a^2b^4],\\
A_5 &= \frac{\kappa}{(a_1^2-b^2)^2}[a_1^4(\varrho^2\tau(\lambda+\mu)+\varrho\beta^2)
\end{aligned}\tag{3.22}$$

$$
\begin{aligned}
& \quad - a_1^2(k\varrho(\lambda+\mu) + \varrho\tau\mu(\lambda+\mu) + \mu\beta^2) + \mu(\lambda+\mu)k],\\
A_6 &= \frac{\kappa}{(a_2^2-b^2)^2}[a_2^4(\varrho^2\tau(\lambda+\mu) + \varrho\beta^2)\\
& \quad - a_2^2(k\varrho(\lambda+\mu) + \varrho\tau\mu(\lambda+\mu) + \mu\beta^2) + \mu(\lambda+\mu)k],\\
A_7 &= \frac{\kappa}{(a_1^2-a_2^2)(a_1^2-b^2)^2(a_2^2-b^2)^2}\\
& \quad \times [[a_2^4a_1^4 - (a_1^2+a_2^2)(a_1^2a_2^2 - a_1^2a_2^2b^2)][\varrho\tau\mu(\lambda+\mu) + \mu\beta^2]\\
& \quad + [b^4 - (a_1^2+a_2^2)b^2][\mu(\lambda+\mu)k]],\\
A_8 &= \frac{\kappa}{(a_1^2-a_2^2)(a_1^2-b^2)^2(a_2^2-b^2)^2}\\
& \quad \times [[-a_2^4a_1^4 + a_1^4b^2 + a_1^2a_2^2b^2 - a_1^2b^4][\varrho\tau\mu(\lambda+\mu) + \mu\beta^2]\\
& \quad + (a_1^2a_2^2b^2 - a_2^2b^4 + a_2^4b^2 + a_2^4a_1^2)[k\varrho(\lambda+\mu)]],\\
A_9 &= \frac{\kappa}{(a_1^2-a_2^2)(a_1^2-b^2)^2(a_2^2-b^2)^2}\\
& \quad \times [[a_1^2a_2^4 - a_2^4b^2 - a_1^2a_2^2b^2 + a_2^2b^4][\varrho\tau\mu(\lambda+\mu) + \mu\beta^2]\\
& \qquad \quad + (a_1^2a_2^2b^2 - a_1^4b^2 + a_1^2b^4 + a_1^4a_2^2)[k\varrho(\lambda+\mu)]],\\
A_{10} &= \kappa\beta\varrho^2,\\
B &= \frac{\kappa}{(a_1^2-b^2)(a_2^2-b^2)}[\varrho\tau(\lambda+\mu) - k\varrho(\lambda+\mu) + \beta^2\mu],
\end{aligned}
\tag{3.22 [cont.]}
$$

where $\kappa = 336 \cdot 120$.

As already mentioned $a = \sqrt{(\lambda+2\mu)/\varrho}$ is the velocity of longitudinal elastic waves, $b = \sqrt{\mu/\varrho}$ the velocity of transversal elastic waves and $\vartheta = \sqrt{k/(\varrho\tau)}$ the velocity of thermal waves (cf. [138]).

It follows from (3.5) that the velocity of transversal elastic waves is unperturbed; the propagation of thermal waves affects the velocity of longitudinal elastic waves. We thus have three velocities of waves in hyperbolic thermoelasticity: a_1^2, a_2^2, b^2.

3.2. $\mathbb{L}^p$-$\mathbb{L}^q$ decay estimates for linear hyperbolic thermoelasticity. We consider the linearized problem associated with the nonlinear problem (1.10)–(1.13):

$$\varrho\,\partial_t^2 u - \mu\Delta u - (\lambda+\mu)\operatorname{grad}\operatorname{div} u + \beta\operatorname{grad}\partial_t T = \mathbf{0}, \tag{3.23}$$

$$\varrho\tau\partial_t^2 T - k\Delta T + \beta\operatorname{div}\partial_t u = 0 \tag{3.24}$$

with the initial conditions

$$u(+0,x) = u^0(x), \qquad (\partial_t u)(+0,x) = u^1(x), \tag{3.25}$$

$$T(+0,x) = T^0(x), \qquad (\partial_t T)(+0,x) = T^1(x). \tag{3.26}$$

Under the assumption that the Cauchy data u^0, u^1, T^0, T^1 are smooth enough (cf. [24], [28]) a solution of the problem (3.23)–(3.26) is given by convolutions:

$$U(t,x) = (u(t,x), T(t,x))^* = H(t,\cdot) * \overline{g}(x) + \partial_t H(t,\cdot) * \widetilde{h}(x), \tag{3.27}$$

where $\overline{g}(\cdot) = \widetilde{g}(\cdot) + \mathbb{D}(\partial)\widetilde{h}(\cdot)$, $\widetilde{g}(x) = (u^1(x), T^1(x))^*$, $\widetilde{h}(x) = (u^0(x), T^0(x))^*$,

$$\mathbb{D}(\partial) = \begin{pmatrix} 0 & 0 & 0 & \beta\partial_1 \\ 0 & 0 & 0 & \beta\partial_2 \\ 0 & 0 & 0 & \beta\partial_3 \\ \beta\partial_1 & \beta\partial_2 & \beta\partial_3 & 0 \end{pmatrix}. \tag{3.28}$$

The star on the right hand side of (3.27) denotes the convolution in $\mathbb{R}^3$ and $H(t,x)$ is the matrix of fundamental solutions of the system (3.23)–(3.24) given by (3.1), (3.21).

We shall prove the following

THEOREM 3.1 ($\mathbb{L}^\infty$-$\mathbb{L}^1$ time decay estimate). *Let the Cauchy data u^0, u^1, T^0, T^1 be functions vanishing at infinity. Moreover, let*

$$(u^1, Du^0, T^1, DT^0)^* \in \mathbb{L}^1_{,3}(\mathbb{R}^3).$$

Then the solution u^0, u^1, T^0, T^1 of the problem (3.23)–(3.26) *given by* (3.27) *satisfies the estimate*

$$\begin{aligned} &\|(\nabla u(t,\cdot), \nabla T(t,\cdot))\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \\ &\qquad \le C(1+t)^{-1}\|(u^1, Du^0, T^1, DT^0)^*\|\mathbb{L}^1_{,3}(\mathbb{R}^3) \quad \textit{for } t \ge 0, \end{aligned} \tag{3.29}$$

where C is a constant independent of u^0, u^1, T^0, T^1 and t.

Proof. We write

$$U_j(t,x) = \sum_{k=1}^4 H_{jk}(t,\cdot) * \overline{g}^k(x) + \sum_{k=1}^4 \partial_t H_{jk}(t,\cdot) * \widetilde{h}^k(x), \quad j = 1,2,3,4, \tag{3.30}$$

where $U_j(t,x) = u_j(t,x)$, $j = 1,2,3$, $U_4(t,x) = T(t,x)$, and differentiate (3.30) with respect to t and x_l (for $l = 1,2,3$) to get

$$\begin{aligned} \partial_t U_j(t,x) = &\sum_{k=1}^4 \partial_t H_{jk}(t,\cdot) * \overline{g}^k(x) \\ &+ \sum_{k,l=1}^{k=4,l=3} \left[\frac{\mu}{\varrho}\partial_l H_{jk}(t,\cdot) + \frac{\lambda+\mu}{\varrho}\partial_j H_{lk}(t,\cdot)\right] * \widetilde{h}^k_l(x) \\ &- \sum_{k=1}^4 \frac{\beta}{\varrho}\partial_t H_{jk}(t,\cdot) * \widetilde{h}^k_j(x), \quad j = 1,2,3, \end{aligned} \tag{3.31}$$

$$\begin{aligned} \partial_t U_4(t,x) = &\sum_{k=1}^4 \partial_t H_{4k}(t,\cdot) * \overline{g}^k(x) + \sum_{k,l=1}^{k=4,l=3} \frac{k}{\varrho}\partial_l H_{4k}(t,\cdot) * \widetilde{h}^k_l(x) \\ &- \sum_{k,l=1}^{k=4,l=3} \frac{\beta}{\varrho}\partial_t H_{lk}(t,\cdot) * \widetilde{h}^k_l(x), \end{aligned} \tag{3.32}$$

$$\begin{aligned} \partial_l U_j(t,x) = &\sum_{k=1}^4 \partial_l H_{jk}(t,\cdot) * \overline{g}^k(x) \\ &+ \sum_{k=1}^4 \partial_t H_{jk}(t,\cdot) * \widetilde{h}^k_l(x), \quad j = 1,2,3,4, \end{aligned} \tag{3.33}$$

where

$$\widetilde{h}_l^k = \partial_l \widetilde{h}^k.$$

We can write (3.30)–(3.33) in vector form as

$$V(t,x) = R(t,\cdot) * V^0(x), \tag{3.34}$$

where

$$V(t,x) = (\nabla u, \nabla T)^*, \qquad V^0(x) = (u^1, Du^0, T^1, DT^0)^* \tag{3.35}$$

and $R(t,x)$ is a 16×16 matrix with elements which are linear combinations of the terms $\partial_l H_{jk}(t,x)$ and $\partial_t H_{jk}(t,x)$ (cf. (3.30)–(3.33)).

It follows from (3.31)–(3.33) and (3.34) that in order to prove the estimate (3.29) it is sufficient to prove that

$$\|\partial_t H_{jk}(t,\cdot) * f\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C(1+t)^{-1} \|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)}, \tag{3.36}$$

$$\|\partial_l H_{jk}(t,\cdot) * f\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C(1+t)^{-1} \|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)} \tag{3.37}$$

for $j,k = 1,2,3,4$ and any scalar function $f(x) \in \mathbb{L}^1_{,3}(\mathbb{R}^3)$ vanishing at infinity. Taking into account the form of the matrix $H(t,x)$ (cf. (3.21)) we get

$$\begin{aligned}
H_{jk}(t,\cdot) * f(x) = {} & (32\pi 7!\, \varrho^4 \tau)^{-1} \Bigg\{ \delta_{jk}(1-\delta_{k4}) \Bigg[\frac{A_1}{a_1} \int\limits_{|z|=a_1} \frac{t}{a_1} f(x+tz)\, dS_z \\
& - \frac{A_2}{a_2} \int\limits_{|z|=a_2} \frac{t}{a_2} f(x+tz)\, dS_z - \frac{B}{b} \int\limits_{|z|=b} \frac{t}{b} f(x+tz)\, dS_z \Bigg] \\
& + \delta_{jk}\delta_{k4} \Bigg[\frac{A_3}{a_1} \int\limits_{|z|=a_1} \frac{t}{a_1} f(x+tz) dS_z - \frac{A_4}{a_2} \int\limits_{|z|=a_2} \frac{t}{a_2} f(x+tz)\, dS_z \Bigg] \\
& + \delta_{jj}(1-\delta_{k4}) \Bigg[\frac{A_5}{a_1} \int\limits_{|z|=a_1} \frac{t z_j z_k}{a_1^3} f(x+tz) dS_z - \frac{A_6}{a_2} \int\limits_{|z|=a_2} \frac{t z_j z_k}{a_2^3} f(x+tz)\, dS_z \\
& + \frac{B}{b} \int\limits_{|z|=b} \frac{t z_j z_k}{b^3} f(x+tz)\, dS_z \Bigg] \\
& - \delta_{jj}(1-\delta_{k4}) \Bigg[A_7 \int\limits_{a_1 \le |z| \le a_2} t \left(\frac{\delta_{jk}}{|z|^3} - \frac{3 z_j z_k}{|z|^5} \right) f(x+tz)\, dz \\
& + A_8 \int\limits_{b \le |z| \le a_1} \left(\frac{\delta_{jk}}{|z|^3} - \frac{3 z_j z_k}{|z|^5} \right) f(x+tz)\, dz \\
& + A_9 \int\limits_{b \le |z| \le a_2} t \left(\frac{\delta_{jk}}{|z|^3} - \frac{3 z_j z_k}{|z|^5} \right) f(x+tz)\, dz \Bigg] \\
& + \delta_{4j}(1-\delta_{kj}) A_{10} \int\limits_{a_1 \le |z| \le a_2} \frac{z_j}{|z|^3} f(x+tz)\, dz \\
& + \delta_{4j}(1-\delta_{kj}) A_{10} \Bigg[\int\limits_{|z|=a_1} \frac{z_j}{a_1^2} f(x+tz)\, dS_z - \int\limits_{|z|=a_2} \frac{z_j}{a_2^2} f(x+tz)\, dS_z \Bigg] \Bigg\},
\end{aligned} \tag{3.38}$$

where dS_z is the area element of the sphere $|z| = a_m$, $m = 1, 2$ or $|z| = b$, respectively.

For simplicity, we consider two typical integrals occurring on the right hand side of (3.38) (other integrals in (3.38) are estimated similarly):

$$I^1 = \int\limits_{|y|=b} f(x+ty)\,dS_y, \qquad I^2 = \int\limits_{b\le|y|\le a} t\,\frac{f(x+ty)}{|y|^3}\,dy.$$

Differentiating the integrals I^1 and I^2 with respect to t and x_l ($l = 1, 2, 3$) we obtain

$$I^1_t = \int\limits_{|y|=b} f(x+ty)\,dS_y + \int\limits_{|y|=b} t\,\partial_t f(x+ty)\,dS_y,$$

$$I^2_t = \int\limits_{b\le|y|\le a} \frac{f(x+ty)}{|y|^3}\,dy + \int\limits_{b\le|y|\le a} \frac{t}{|y|^3}\partial_t f(x+ty)\,dy,$$

$$I^1_{x_l} = \int\limits_{|y|=b} t\,\partial_l f(x+ty)\,dS_y, \qquad I^2_{x_l} = \int\limits_{b\le|y|\le a} \frac{t}{|y|^3}\partial_l f(x+ty)\,dy,$$

where $I^m_t = \partial_t I^m$ and $I^m_{x_l} = \partial_l I^m$ for $l = 1, 2, 3$, $m = 1, 2$.

Following S. Klainerman (cf. [79], [28]) we get

$$f(x+ty) = -\int\limits_t^\infty \partial_s f(x+sy)\,ds = \int\limits_t^\infty (s-t)\partial_s^2 f(x+sy)\,ds \tag{3.39}$$
$$= -\frac{1}{2}\int\limits_t^\infty (s-t)^2\partial_s^3 f(x+sy)ds,$$

$$\partial_t f(x+ty) = -\int\limits_t^\infty \partial_s^2 f(x+sy)\,ds = \int\limits_t^\infty (s-t)\partial_s^3 f(x+sy)\,ds. \tag{3.40}$$

In view of (3.39) and (3.40) we have

$$I^1_t = \int\limits_{|y|=b}\int\limits_t^\infty (s-t)\partial_s^2 f(x+sy)\,ds\,dS_y - \int\limits_{|y|=b} t\int\limits_t^\infty \partial_s^2 f(x+sy)\,ds\,dS_y \tag{3.41}$$
$$= t^{-1}\Big[\int\limits_{|y|=b}\int\limits_t^\infty t(s-t)\partial_s^2(x+sy)\,ds\,dS_y - \int\limits_{|y|=b}\int\limits_t^\infty t^2\partial_s^2 f(x+sy)\,ds\,dS_y\Big]$$

for $t > 0$.

Since

$$|\partial_s^2 f(x+sy)| = \Big|\sum_{j,k=1}^3 \partial^2_{x_jx_k} f(x+sy)y_jy_k\Big| \le \frac{1}{2}b^2|D_x^2 f(x+sy)|$$

for $|y| = b$ and $t(s-t) \le s^2, t^2 \le s^2$, where $0 \le t \le s \le \infty$, we get

$$|I^1_t| \le t^{-1}b^2\int\limits_{|y|=b}\int\limits_t^\infty s^2|D_x^2 f(x+sy)|\,ds\,dS_y.$$

Using spherical coordinates we have

$$|I^1_t| \le bt^{-1}\|D_x^2 f\|_{\mathbb{L}^1(\mathbb{R}^3)} \quad \text{for } t > 0. \tag{3.42}$$

Acting in the same way we get

$$I^1_{x_l} = -\int\limits_{|y|=b} t \int\limits_t^\infty \partial_s[\partial_{x_l} f(x+sy)]\, ds\, dS_y.$$

Since

$$|\partial_s[\partial_{x_l} f(x+sy)]| = \Big|\sum_{j=1}^3 \partial^2_{x_j x_j} f(x+sy) y_j\Big| \le b|D^2_x f(x+sy)| \quad \text{for } |y| = b,$$

we have

$$\begin{aligned} |I^1_{x_l}| &\le t^{-1} b \int\limits_{|y|=b} \int\limits_t^\infty s^2 |D^2_x f(x+sy)|\, ds\, dS_y \\ &\le t^{-1} \|D^2_x f\|_{\mathbb{L}^1(\mathbb{R}^3)} \quad \text{for } t > 0. \end{aligned} \tag{3.43}$$

Similarly

$$\begin{aligned} |I^2_t| &\le b^{-3}\Big[\int\limits_{b\le|y|\le a} |f(x+ty)|\, dy + \int\limits_{b\le|y|\le a} t|\partial_t f(x+ty)|\, dy\Big] \\ &\le b^{-3}\Big[\int\limits_{b\le|y|\le a} |f(x+ty)| dy + a\int\limits_{b\le|y|\le a} t|D^1_x f(x+ty)|\, dy\Big] \\ &\le b^{-3} \int\limits_{b\le|y|\le a} t|D^1_x f(x+ty)|\, dy. \end{aligned} \tag{3.44}$$

Changing the variable ty to z in the above integrals we derive

$$\begin{aligned} |I^2_t| &\le b^{-3}\left[\frac{1}{t^3}\int\limits_{tb\le|y|\le at} f(x+z)\, dz + \frac{a}{t^2}\int\limits_{tb\le|y|\le at} |D^1_x f(x+z)|\, dz\right] \\ &\le b^{-3}\left[\frac{1}{t^3}\|f\|_{\mathbb{L}^1(\mathbb{R}^3)} + \frac{a}{t^2}\|D^1_x f\|_{\mathbb{L}^1(\mathbb{R}^3)}\right], \\ |I^2_{x_l}| &\le \frac{b^{-3}}{t^2}\int\limits_{bt\le|y|\le at} |D^1_x f(x+z)|\, dz \le \frac{b^{-3}}{t^2}\|D^1_x f\|_{\mathbb{L}^1(\mathbb{R}^3)}. \end{aligned}$$

Noting that $1/t^3 \le 1/t^2 \le 1/t$ for $t \ge 1$, we get

$$|I^2_t| + |I^2_{x_l}| \le Ct^{-1}\|D^1_x f\|_{\mathbb{L}^1(\mathbb{R}^3)} \quad \text{for } t \ge 1. \tag{3.45}$$

From (3.42), (3.43) and (3.45) we obtain

$$\|\nabla H_{jk}(t,\cdot) * f(\cdot)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le Ct^{-1}\|f\|_{\mathbb{L}^1_{,2}(\mathbb{R}^3)} \tag{3.46}$$

for $t \ge 1$ and $j,k = 1,2,3,4$.

In order to obtain an estimate analogous to (3.45) for $0 \le t \le 1$ we proceed as above expressing the integrals in I^1_t and I^2_t (cf. (3.39), (3.40)) in the following form:

$$I^1_t = -\frac{1}{2}\int\limits_{|y|=b}\int\limits_t^\infty (s-t)^2\partial^3_s + \int\limits_{|y|=b} t\int\limits_t^\infty (s-t)\partial^3_s f(x+sy)\, ds\, dS_y, \tag{3.47}$$

$$(3.48)\qquad I^1_{x_l} = \int\limits_{|y|=b} t \int\limits_t^\infty (s-t)\partial_s^2[\partial_{x_l} f(x+sy)]\,ds\,dS_y.$$

After some calculations we get

$$(3.49)\qquad |I^1_t| + |I^1_{x_l}| \le C\|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)} \quad \text{for } t \ge 0.$$

It is easy to see that for $0 \le t \le 1$ we have

$$(3.50)\qquad \Big|\int\limits_{b\le|y|\le a} \frac{f(x+ty)}{|y|^3}\,dy\Big| \le b^{-3}\int\limits_{b\le|y|\le a} |f(x+ty)|\,dy \le b^{-3}\|f\|_{\mathbb{L}^1(\mathbb{R}^3)},$$

$$(3.51)\qquad \Big|\int\limits_{b\le|y|\le a} \frac{f(x+ty)}{|y|^3}\,dy\Big| \le b^{-3}\int\limits_{b\le|y|\le a} \Big|\sum_{j=1}^3 \partial_{x_j} f(x+ty)y_j\Big|\,dy \le b^{-3}a\|D^1_x f\|_{\mathbb{L}^1(\mathbb{R}^3)},$$

$$(3.52)\qquad \Big|\int\limits_{b\le|y|\le a} \frac{t}{|y|^3}\partial_{x_l} f(x+ty)\,dy\Big| \le b^{-3}\int\limits_{b\le|y|\le a} |\partial_{x_l} f(x+ty)|\,dy \le b^{-3}\|D^1_x f\|_{\mathbb{L}^1(\mathbb{R}^3)}.$$

Hence

$$(3.53)\qquad |I^2_t| + |I^2_{x_l}| \le C\|f\|_{\mathbb{L}^1_{,2}(\mathbb{R}^3)} \quad \text{for } 0 \le t \le 1.$$

We deduce from (3.49) and (3.53) that

$$(3.54)\qquad \|\nabla H_{jk}(t,\cdot) * f(\cdot)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)} \quad \text{for } 0 \le t \le 1.$$

Now in view of $1 \le 2(1+t)^{-1}$ for $0 \le t \le 1$ and $t^{-1} \le 2(1+t)^{-1}$ for $t \ge 0$ and taking into account (3.46), (3.51) we conclude that

$$(3.55)\qquad \|\nabla H_{jk}(t,\cdot) * f(\cdot)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C(1+t)^{-1}\|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)} \quad \text{for } t \ge 0. \ \blacksquare$$

Now we derive the $\mathbb{L}^2$-$\mathbb{L}^2$ time decay estimates for solution of the Cauchy problem (3.23)–(3.26):

THEOREM 3.2 ($\mathbb{L}^2$-$\mathbb{L}^2$ time decay estimates). *Let the Cauchy data u^0, u^1, T^0, T^1 be functions vanishing at infinity. Moreover, let*

$$(u^1, Du^0, T^1, DT^0)^* \in \mathbb{L}^2(\mathbb{R}^3).$$

Then the solution (u,T) of the problem (3.23)–(3.26), *given by* (3.27), *satisfies the estimate*

$$(3.56)\qquad \|(\nabla u(t,\cdot), \nabla T(t,\cdot))\|_{\mathbb{L}^2(\mathbb{R}^3)} \le C\|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{L}^2(\mathbb{R}^3)} \quad \textit{for } t \ge 0,$$

where C is a constant independent of u^0, u^1, T^0, T^1 and t.

Sketch of proof. Following Yu. V. Egorov (cf. [18]. pp. 320–322, 326–333) we reduce the Cauchy problem (3.23)–(3.27) to an equivalent one for a linear symmetric hyperbolic system of the first order. Next, applying the existence and uniqueness theorems (cf. [18], Theorem 3.2, p. 329) we obtain the estimate (3.56).

Below, we express the $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates for the solution of the Cauchy problem (3.23)–(3.26) in terms of its gradient.

We consider the operator Π_* defined by

$$\Pi_* f(x) = R(\cdot) * f(x). \tag{3.57}$$

It follows from Theorems 3.1 and 3.2 that

$$\Pi_*^0 : \mathbb{L}^1_{,3}(\mathbb{R}^3) \to \mathbb{L}^\infty(\mathbb{R}^3), \quad \|\Pi_*^0\| \le C(1+t)^{-1},$$
$$\Pi_*^1 : \mathbb{L}^2(\mathbb{R}^3) \to \mathbb{L}^2(\mathbb{R}^3), \quad \|\Pi_*^1\| \le C.$$

By interpolation (cf. [122], [79]) we have

$$\Pi_*^\Theta : [\mathbb{L}^1_{,3}, \mathbb{L}^2]_\Theta \to [\mathbb{L}^\infty, \mathbb{L}^2]_\Theta, \quad \|\Pi_*^\Theta\| = \|\Pi_*^0\|^{1-\Theta}\|\Pi_*^1\|^\Theta \quad \text{with } 0 \le \Theta \le 1,$$

where $[X,Y]_\Theta$, $0 \le \Theta \le 1$, denotes the complex interpolation space (cf. [88], [143]).

In order to obtain $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates for the solution of the Cauchy problem (3.23)–(3.26) (where $q = 2k_0+2$, $p = (2k_0+2)/(2k_0+1)$, $1/p+1/q = 1$, k_0 is a nonnegative integer) we notice that for $\Theta = 1/(k_0+1)$,

$$[\mathbb{L}^1_{,3}, \mathbb{L}^2]_{1/(k_0+1)} = \mathbb{L}^p_{,s_0},$$

where $s_0 = \left\lfloor \frac{3k_0}{k_0+1} \right\rfloor$, and $[\mathbb{L}^\infty, \mathbb{L}^2]_{1/(k_0+1)} = \mathbb{L}^{2k_0+2}$. Hence we have

$$\begin{gathered}\Pi_*^{(k_0)} : \mathbb{L}^p_{,s_0}(\mathbb{R}^3) \to \mathbb{L}^{2k_0+2}(\mathbb{R}^3),\\ \|\Pi_*^{(k_0)}\| = \|\Pi_*^0\|^{1-1/(k_0+1)}\|\Pi_*^1\|^{1/(k_0+1)} \le C(1+t)^{-k_0/(k_0+1)}.\end{gathered} \tag{3.58}$$

This way we have proved the following (cf. [28])

THEOREM 3.3 ($\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates). *Let the Cauchy data u^0, u^1, T^0, T^1 be functions vanishing at infinity. Moreover, let*

$$(u^1, Du^0, T^1, DT^0)^* \in \mathbb{L}^p_{,s_0}(\mathbb{R}^3) \quad \text{for } p = \frac{2k_0+2}{2k_0+1}, s_0 = [3k_0/(k_0+1)],$$

where k_0 comes from (1.16)–(1.18). *Then the solution of the problem* (3.23)–(3.26) *given by* (3.27) *satisfies the estimate*

$$\begin{aligned}&\|(\nabla u(t,\cdot), \nabla T(t,\cdot))\|_{\mathbb{L}^{2k_0+2}(\mathbb{R}^3)}\\ &\qquad \le C(1+t)^{-k_0/(k_0+1)} \times \|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{L}^p_{,s_0}(\mathbb{R}^3)} \quad \text{for } t \ge 0,\end{aligned} \tag{3.59}$$

where C is a constant independent of u^0, u^1, T^0, T^1 and t.

Remark 3.1. In Section 6 we shall apply Theorem 3.3 to the proof of global-in-time existence of solutions of the Cauchy problem for a nonlinear hyperbolic PDE system describing a thermoelastic medium.

3.3. Fundamental solution of the linear hyperbolic heat equation. In this section, we apply the Hörmander method (cf. [43], Theorem 12.5.3, p. 142) to construct fundamental solutions of the linear hyperbolic heat equation. This approach based on the Hörmander theorem is a new one and can be applied to other hyperbolic operators with constant coefficients.

THEOREM 3.4. *The linear hyperbolic heat operator $\partial_t^2 - a^2 + 2m\partial_t$ has a fundamental solution of the form of the convergent series*

$$\mathbb{E}(t,x) = \sum_{k=0}^{\infty} e^{-mt} m^{2k} (E_0)^{*(k+1)}(t,x)$$

or equivalently,

$$\mathbb{E}(t,x) = \frac{\varepsilon(t)}{4\pi a^2 t} e^{-mt}\delta(at-|x|) + me^{-mt}\varepsilon(at-|x|)\sum_{k=0}^{\infty}\frac{m^{2k+2}(a^2t^2-|x|^2)^k}{\pi k!(k+1)!(2a)^{(}2k+3)}, \tag{3.60}$$

$$\mathbb{E}(t,x) = \frac{\varepsilon(t)e^{-mk}}{4\pi a^2 t}\delta(at-|x|) - \frac{me^{-mt}\varepsilon(at-|x|)}{4\pi a^2}\frac{I_1\left(\frac{m}{a}\sqrt{a^2t^2-|x|^2}\right)}{\sqrt{a^2t^2-|x|^2}}, \tag{3.61}$$

where E_0 is a fundamental solution of the wave operator $\Box_a = \partial_t^2 - a^2\Delta$, $I_1(\cdot)$ denotes the modified Bessel function (cf. [95], [16]),

$$I_1(\xi) = -iJ_1(i\xi),$$

and $E_0^{*(k+1)} = E_0 * E_0 * \ldots * E_0$ ($k+1$ times), where $*$ is convolution with respect to t and x.

Proof. Set $\mathbb{E}(t,x) = W(t,x)e^{-mt}$. It is not difficult to verify that $\mathbb{E}(t,x)$ is a fundamental solution of $\Box_a + 2m\partial_t$ if $W(t,x)$ is a fundamental solution of $\Box_a + m^2$. To construct the latter we apply the Hörmander theorem, stating that we can take the convergent series (cf. [43])

$$W(t,x) = \sum_{k=0}^{\infty}[P_2(D) - P(D)]^k E_k(t,x), \tag{3.62}$$

where $P(D) = \Box_a - m^2$, $P_2(D) = \Box_a$ and $E_k(t,x)$ is a fundamental solution of $(P_2(D))^{k+1}$, $k = 0,1,\ldots$

It is easy to verify that if E_0 is a fundamental solution of $\Box_a$ then $E_k(t,x) = (E_0)^{*(k+1)}(t,x)$ is a fundamental solution of the operator $(\Box_a)^{k+1}$, $k = 0,1,2,\ldots$

Thus the fundamental solution for the operator $(\Box_a + m^2)$ is given by following formula (cf. (3.62)):

$$W(t,x) = \sum_{k=0}^{\infty}(m^2)^k (E_0)^{*(k+1)}(t,x). \tag{3.63}$$

LEMMA 3.1. *Let*

$$E_0(t,x) = \frac{\varepsilon(t)}{4\pi a^2 t}\delta(at-|x|), \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \tag{3.64}$$

a fundamental solution of the wave operator $\Box_a$. Then

$$(E_0)^{*n}(t,x) = \frac{(a^2t^2-|x|^2)^{n-2}\varepsilon(at-|x|)}{\pi(n-2)!(n-1)!(2a)^{2n-1}} \quad \text{for } n\ge 2. \tag{3.65}$$

Before proving Lemma 3.1 we prove

LEMMA 3.2. *The Fourier transform with respect to x of the function*

$$f_n(t,x) = (a^2t^2 - |x|^2)^n \varepsilon(at - |x|), \qquad n = 0,1,2,\ldots,$$

is given by

$$\mathcal{F}_{x\to\xi} f_n(t,x) = a\pi(-1)^{n+1}2^{n+2}n!\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^{n+1}\frac{\sin a|\xi|t}{a|\xi|}. \tag{3.66}$$

Proof (by induction on n). For $n = 0$ we have

$$\mathcal{F}_{x\to\xi}\, \varepsilon(at - |x|) = \int\limits_{|x|\le at} e^{-i(x,\xi)}\, dx. \tag{3.67}$$

In local spherical coordinates the integral is

$$I = \int_0^{at}\int_0^{2\pi}\int_0^{\pi} e^{-i|\xi|\varrho\cos\theta}\varrho^2 \sin\theta\, d\theta\, d\varphi\, d\varrho.$$

Integrating by parts, we get

$$I = -4\pi a\left(\frac{t\cos a|\xi|t}{|\xi|^2} - \frac{\sin a|\xi|t}{a|\xi|^3}\right) = -4\pi a\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\frac{\sin a|\xi|t}{a|\xi|}. \tag{3.68}$$

For $n = 1$, proceeding exactly as above we have

$$\begin{aligned}\mathcal{F}_{x\to\xi}[(a^2t^2 - |x|^2)\varepsilon(at - |\xi|)] &= \int_0^{at}\int_0^{2\pi}\int_0^{\pi} e^{-i|\xi|\varrho\cos\theta}(a^2t^2 - \varrho^2)\varrho^2 \sin\theta\, d\theta\, d\varphi\, d\varrho \\ &= 4\pi\int_0^{at}(a^2t^2 - \varrho^2)\varrho\frac{\sin|\xi|\varrho}{|\xi|}d\varrho.\end{aligned}$$

Integrating by parts shows this is

$$8\pi a\left(-\frac{at^2\sin a|\xi|t}{|\xi|^3} - \frac{3t\cos a|\xi|t}{|\xi|^4} + \frac{3\sin a|\xi|t}{|\xi|^5}\right) = 8\pi a\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^2\frac{\sin a|\xi|t}{a|\xi|},$$

which proves (3.66) for $n = 1$.

Similarly, we compute

$$\mathcal{F}_{x\to\xi}[(a^2t^2 - |x|^2)^k\varepsilon(at - |\xi|)] = 4\pi\int_0^{at}(a^2t^2 - \varrho^2)^k\varrho\frac{\sin|\xi|\varrho}{|\xi|}\, d\varrho.$$

Now, we shall prove that

$$4\pi\int_0^{at}(a^2t^2 - \varrho^2)^k\varrho\frac{\sin|\xi|\varrho}{|\xi|}d\varrho = a\pi(-1)^{k+1}2^{k+2}k!\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^{k+1}\frac{\sin a|\xi|t}{a|\xi|} \tag{3.69}$$

assuming that

$$4\pi\int_0^{at}(a^2t^2 - \varrho^2)^{k-1}\varrho\frac{\sin|\xi|\varrho}{|\xi|}d\varrho = a\pi(-1)^{k}2^{k+1}(k-1)!\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^{k}\frac{\sin a|\xi|t}{a|\xi|}. \tag{3.70}$$

Integrating by parts shows that the last integral is

$$-8\pi k \int_0^{at} (a^2t^2-\varrho^2)^{k-1}\varrho\left[\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\frac{\sin|\xi|\varrho}{|\xi|}\right]d\varrho = -8\pi\varrho\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\left[\int_0^{at}(a^2t^2-\varrho^2)^{k-1}\varrho\frac{\sin|\xi|\varrho}{|\xi|}d\varrho\right]. \tag{3.71}$$

Applying (3.71) and (3.70) we get

$$4\pi\int_0^{at}(a^2t^2-\varrho^2)^k\varrho\frac{\sin|\xi|\varrho}{|\xi|}\,d\varrho = -2k\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\left[a\pi(-1)^k2^{k+1}(k-1)!\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^k\frac{\sin a|\xi|t}{a|\xi|}\right],$$

which proves (3.69) and completes the proof of Lemma 3.2.

Now, we prove Lemma 3.1. The formula (3.65) is proved by induction on n ($n\ge 2$).

For $n=2$

$$(E_0 * E_0)(t,x) = \int_{\mathbb{R}^1}[E_0(t-s,\cdot) *_3 E_0(s,\cdot)](x)\,ds,$$

where $*_3$ denotes convolution with respect to x. Applying the Fourier transform with respect to x, we obtain

$$\begin{aligned}\mathcal{F}_{x\to\xi}\,E_0^{*2}(t,x) &= \int_{\mathbb{R}^1}\widehat{E_0}(t-s,\xi)\widehat{E_0}(s,\xi)\,ds = \int_0^t\frac{\sin a|\xi|(t-s)}{a|\xi|}\frac{\sin a|\xi|s}{a|\xi|}\,ds\\ &= \frac{-1}{2a^2}\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\frac{\sin a|\xi|t}{a|\xi|} = \mathcal{F}_{x\to\xi}\left[\frac{\varepsilon(at-|x|)}{8\pi a^3}\right].\end{aligned}$$

Applying now the inverse Fourier transform we obtain (3.65) for $n=2$.

Now, assume that (3.65) holds for all $2\le n\le k$. Then by Lemma 3.2 we have

$$\mathcal{F}_{x\to\xi}\,E_0^{*n}(t,x) = \frac{(-1)^{n-1}}{(n-1)2^{n-1}a^{2(n-1)}}\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^{n-1}\frac{\sin a|\xi|t}{a|\xi|} \tag{3.72}$$

for $2\le n\le k$.

We shall prove that

$$E_0^{*k+1}(t,x) = \frac{(a^2t^2-|x|^2)^{k-1}\varepsilon(at-|x|)}{\pi(k-1)!k!(2a)^{2k+1}}. \tag{3.73}$$

By Lemma 3.2,

$$\mathcal{F}_{x\to\xi}\left[\frac{(a^2t^2-|x|^2)^{k-1}\varepsilon(at-|x|)}{\pi(k-1)!k!(2a)^{2k+1}}\right] = \frac{(-1)^k}{k!2^ka^{2k}}\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^k\frac{\sin a|\xi|t}{a|\xi|}.$$

By the hypothesis (3.72) the right hand side equals

$$-\frac{1}{2ka^2}\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)[\mathcal{F}_{x\to\xi}E_0^{*k}(t,x)] = -\frac{1}{2ka^2}\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\int_0^t\widehat{E_0}(t-s,\xi)\widehat{E_0^{*(k-1)}}(s,\xi)\,ds.$$

Using again (3.72) we get

$$\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\widehat{E_0}(t-s,\xi) = -2a^2\widehat{E_0^{*2}}(t-s,\xi),$$
$$\left(\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)\widehat{E_0^{*(k-1)}}(s,\xi) = -2a^2(k-1)\widehat{E_0^{*k}}(s,\xi).$$

and finally

$$\mathcal{F}_{x\to\xi}\left[\frac{(a^2t^2-|x|^2)^{k-1}\varepsilon(at-|x|)}{\pi(k-1)!k!(2a)^{2k+1}}\right] = \frac{1}{k}\Big\{\int_0^t \widehat{E_0^{*2}}(t-s,\xi)\widehat{E_0^{*(k-1)}}(s,\xi)\,ds + \int_0^t \widehat{E_0}(t-s,\xi)(k-1)\widehat{E_0^{*k}}(s,\xi)\,ds\Big\} = \widehat{E_0^{*(k-1)}}(t,\xi).$$

which proves (3.73). This finishes the proof of Lemma 3.1 and Theorem 3.4.

3.4. $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates for the linear hyperbolic heat equation. We now consider the linearized problem associated with the nonlinear hyperbolic heat equation (cf. formulae (1.21), (1.22)) of the form

$$\partial_t^2\Theta + 2m\partial_t\Theta - a^2\Delta\Theta = 0, \tag{3.74}$$

$$\Theta(+0,x) = \Theta^0(x), \qquad (\partial_t\Theta)(+0,x) = \Theta^1(x). \tag{3.75}$$

Under the assumption that the Cauchy data $\Theta^0(x)$, $\Theta^1(x)$ are smooth enough a solution of the problem (3.74), (3.75) is given by (cf. [16])

$$\Theta(t,x) = \mathbb{E}(t,\cdot)*\Theta^1(x) + 2m\mathbb{E}(t,\cdot)*\Theta^0(x) + \partial_t\mathbb{E}(t,\cdot)*\Theta^0(x), \tag{3.76}$$

where $\mathbb{E}(t,x)$ is the fundamental solution of (3.74) (cf. [16], given by (3.61) or (3.60).

Using the formulae for $\mathbb{E}(t,x)$ we prove

THEOREM 3.4 ($\mathbb{L}^\infty$-$\mathbb{L}^1$ time decay estimate). *Let the Cauchy data Θ^0,Θ^1 be functions vanishing at infinity. Moreover, let*

$$(\Theta^1, D\Theta^0)^* \in \mathbb{L}^1_{,3}(\mathbb{R}^3).$$

Then the solution Θ of the problem (3.74)–(3.75) *given by* (3.76) *satisfies the estimate*

$$\|\nabla\Theta(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C(1+t)^{-3/2}\|(\Theta^1,D\Theta^0)^*\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)} \quad \text{for } t\ge 0, \tag{3.77}$$

where C is a constant independent of Θ^0,Θ^1 and t.

Proof. Differentiating (3.76) with respect to t and x_l (for $l=1,2,3$) we get

$$\partial_t\Theta(t,x) = \partial_t\mathbb{E}(t,\cdot)*(\Theta^1+2m\Theta^0)(x) + \partial_t^2\mathbb{E}(t,\cdot)*\Theta^0(x),$$
$$\partial_l\Theta(t,x) = \partial_l\mathbb{E}(t,\cdot)*(\Theta^1+2m\Theta^0)(x) + \partial_t\mathbb{E}(t,\cdot)*\partial_l\Theta^0(x)$$

for $l=1,2,3$. Since $\mathbb{E}(t,x)$ is a fundamental solution of (3.74) (cf. (3.60)), we have

$$\partial_t\Theta(t,x) = \partial_t\mathbb{E}(t,\cdot)*\Theta^1(x) + \partial_l\mathbb{E}(t,\cdot)*\partial_l\Theta^0(x), \tag{3.78}$$

$$\partial_l\Theta(t,x) = \partial_l\mathbb{E}(t,\cdot)*\Theta^1(x) + 2m\mathbb{E}(t,\cdot)*\partial_l\Theta^0(x) + \partial_l\mathbb{E}(t,\cdot)*\partial_l\Theta^0(x). \tag{3.79}$$

The relations (3.78)–(3.79) can be written in vector form as

$$W(t,x) = B(t,x) * W^0(x), \tag{3.80}$$

where $W^0(x) = (\Theta^1(x), D\Theta^0(x))^*$ and

$$B(t,\cdot) = \begin{bmatrix} \partial_t\mathbb{E} & a^2\partial_1\mathbb{E} & a^2\partial_2\mathbb{E} & a^2\partial_3\mathbb{E} \\ \partial_1\mathbb{E} & 2m\mathbb{E}+\partial_t\mathbb{E} & 0 & 0 \\ \partial_2\mathbb{E} & 0 & 2m\mathbb{E}+\partial_t\mathbb{E} & 0 \\ \partial_3\mathbb{E} & 0 & 0 & 2m\mathbb{E}+\partial_t\mathbb{E} \end{bmatrix}. \tag{3.81}$$

From (3.78)–(3.80) it follows that in order to prove (3.77) it is sufficient that

$$\|\mathbb{E}(t,\cdot) * f\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le (1+t)^{-3/2}\|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)}, \tag{3.82}$$

$$\|\partial_t\mathbb{E}(t,\cdot) * f\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le (1+t)^{-3/2}\|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)}, \tag{3.83}$$

$$\|\partial_l\mathbb{E}(t,\cdot) * f\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le (1+t)^{-3/2}\|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)}. \tag{3.84}$$

We prove the estimate (3.82).

By (3.60),

$$\mathbb{E}(t,\cdot) * f(x) = \frac{te^{-mt}}{4\pi a^2}\int\limits_{|y|=a} f(x+ty)\,dS_y + \sum_{k=0}^{\infty}\frac{m^{2k+2}t^{2k+3}e^{-mt}}{\pi k!(k+1)!(2a)^{2k+3}}\int\limits_{|y|\le a}(a^2-|y|^2)^k f(x+ty)\,dy. \tag{3.85}$$

Since $(a^2-|y|^2)^k \le a^{2k}$ for $|y|\le a$, it is sufficient to consider

$$\begin{aligned} J^1 &= \frac{te^{-mt}}{4\pi a^2}\int\limits_{|y|=a} f(x+ty)\,dS_y + \sum_{k=0}^{\infty}\frac{m^{2k+2}t^{2k+3}e^{-mt}}{\pi a^3 k!(k+1)!2^{2k+3}}\int\limits_{|y|\le a} f(x+ty)\,dy \\ &= \frac{te^{-mt}}{4\pi a^2}\int\limits_{|y|=a} f(x+ty)\,dSy + \frac{mt^2e^{-mt}}{4\pi a^2}I_1(mt)\int\limits_{|y|\le a} f(x+ty)\,dy. \end{aligned} \tag{3.86}$$

Integrating by parts (cf. Section 3.2) gives

$$\Big|\int\limits_{|y|=a} f(x+ty)\,dS_y\Big| \le \begin{cases} Ct^{-2}\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)}, \\ Ct^{-1}\|f\|_{\mathbb{W}^{2,1}(\mathbb{R}^3)}, \\ C\|f\|_{\mathbb{W}^{3,1}(\mathbb{R}^3)}, \end{cases} \tag{3.87}$$

for t near 0.

Hence

$$\Big|te^{-mt}\int\limits_{|y|=a} f(x+ty)\,dS_y\Big| \le \begin{cases} Ct^{-1}e^{-mt}\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)} & \text{for } t>1, \\ C\|f\|_{\mathbb{W}^{2,1}} & \text{for } 0\le t\le 1. \end{cases} \tag{3.88}$$

Here and elsewhere, we use C to denote various constants which need not be the same throughout.

To estimate the second term in (3.86) we use the following properties of Bessel functions (cf. [16]):

$$I_v \le \begin{cases} Ce^t/\sqrt{2\pi t} & \text{for } t>1, \\ C(t/2)^v & \text{for } 0\le t\le 1, \end{cases} \tag{3.89}$$

together with the following estimate obtained by a change of variable (cf. Section 3.2):

$$\Big|\int_{|y|=a} f(x+ty)\,dS_y\Big| \le t^{-3}\|f\|_{\mathbb{L}^1(\mathbb{R}^3)} \quad \text{for } t>0. \tag{3.90}$$

Thus

$$\Big|t^2e^{-mt}I_1(mt)\int_{|y|\le a} f(x+ty)\,dy\Big| \le \begin{cases} Ct^{-3/2}\|f\|_{\mathbb{L}^1(\mathbb{R}^3)} & \text{for } t>1,\\ C\|f\|_{\mathbb{L}^1(\mathbb{R}^3)} & \text{for } 0\le t\le 1. \end{cases} \tag{3.91}$$

Finally,

$$|J^1| \le \begin{cases} Ct^{-3/2}\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)} & \text{for } t>1,\\ C\|f\|_{\mathbb{W}^{2,1}(\mathbb{R}^3)} & \text{for } 0\le t\le 1. \end{cases} \tag{3.92}$$

or

$$|J^1| \le C(1+t)^{-3/2}\|f\|_{\mathbb{W}^{2,1}(\mathbb{R}^3)} \quad \text{for } t\ge 0.$$

In order to estimate $\partial_t\mathbb{E}(t,\cdot)*f(x)$ we compute the t-derivative of (3.86). Hence

$$\begin{aligned} J^2 = \frac{1}{4\pi a^2}\Big\{&e^{-mt}\int_{|y|=a} f(x+ty)\,dS_y - mte^{-mt}\int_{|y|=a} f(x+ty)\,dS_y \\ &+ te^{-mt}\int_{|y|=a}\partial_t f(x+ty)\,dS_y\Big\} \\ &+\frac{m}{4\pi a^3}\Big\{2te^{-mt}I_1(mt)\int_{|y|=a} f(x+ty)\,dy - mt^2e^{-mt}I_1(mt)\int_{|y|=a} f(x+ty)\,dy \\ &+ t^2e^{-mt}\frac{I_1(mt)+I_2(mt)}{2}\int_{|y|=a} f(x+ty)\,dy + t^2e^{-mt}I_1(mt)\int_{|y|=a}\partial_t f(x+ty)\,dy\Big\}. \end{aligned} \tag{3.93}$$

By (3.87) the first three terms in (3.93) can be estimated as follows:

$$\begin{aligned} |J_1^2+J_2^2+J_3^2| \le{}& Ct^{-2}e^{-mt}\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)} \\ &+ Ct^{-1}e^{-mt}\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)} + Ct^{-1}e^{-mt}\|f\|_{\mathbb{W}^{2,1}(\mathbb{R}^3)} \end{aligned}$$

for $t>1$ and

$$|J_1^2+J_2^2+J_3^2| \le C\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)} + C\|f\|_{\mathbb{W}^{2,1}(\mathbb{R}^3)} + C\|f\|_{\mathbb{W}^{3,1}(\mathbb{R}^3)} \quad \text{for } 0\le t\le 1.$$

Hence

$$\Big|\partial_t\Big\{\frac{te^{-mt}}{4\pi a^2}\int_{|y|=a} f(x+ty)\,dS_y\Big\}\Big| \le Ct^{-1}e^{-mt}\|f\|_{\mathbb{W}^{2,1}(\mathbb{R}^3)} \quad \text{for } t>1 \tag{3.94}$$

and

$$\Big|\partial_t\Big\{\frac{te^{-mt}}{4\pi a^2}\int_{|y|=a} f(x+ty)\,dS_y\Big\}\Big| \le C\|f\|_{\mathbb{W}^{3,1}(\mathbb{R}^3)} \quad \text{for } 0\le t\le 1. \tag{3.95}$$

Using (3.89) and acting as in (3.90) we obtain

$$\begin{aligned} \Big|t^2e^{-mt}(I_1(mt)+I_2(mt))\int_{|y|=a} f(x+ty)\,dy + t^2e^{-mt}I_1(mt)\int_{|y|=a}\partial_t f(x+ty)\,dy\Big| \\ \le Ct^{-3/2}\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)} \quad \text{for } t>1 \end{aligned}$$

and

$$\Big|t^2e^{-mt}(I_1(mt)+I_2(mt))\int\limits_{|y|=a} f(x+ty)\,dy + t^2e^{-mt}I_1(mt)\int\limits_{|y|=a}\partial_t f(x+ty)\,dy\Big| \le C\|f\|_{\mathbb{W}^{1,1}(\mathbb{R}^3)} \quad \text{for } 0\le t\le 1. \tag{3.96}$$

To estimate the remaining terms in (3.93) we use a stronger estimate (3.90) for t near 0:

$$\Big|\int\limits_{|y|\le a} f(x+ty)\,dy\Big| \le \|f\|_{\mathbb{L}^1(\mathbb{R}^3)} \quad \text{for } 0\le t\le 1,$$

to obtain

$$\Big|\{te^{-mt}I_1(mt)+t^2e^{-mt}I_2(mt)\}\int\limits_{|y|=a} f(x+ty)\,dy\Big| \le C\|f\|_{\mathbb{L}^1(\mathbb{R}^3)} \quad \text{for } 0\le t\le 1 \tag{3.97}$$

while for $t>1$, using again (3.89) and (3.90), we get

$$\Big|[te^{-mt}I_1(mt)+t^2e^{-mt}I_2(mt)]\int\limits_{|y|=a} f(x+ty)\,dy\Big| \le C(t^{-5/2}+t^{-3/2})\|f\|_{\mathbb{L}^1(\mathbb{R}^3)}. \tag{3.98}$$

Summarizing (3.94), (3.95), (3.97) and (3.98) we conclude that

$$\|B(t,\cdot)*f(\cdot)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C(1+t)^{-3/2}\|f\|_{\mathbb{L}^1_{,3}(\mathbb{R}^3)}.$$

R e m a r k 3.1. The proof of the estimate (3.84) follows in the same way as that of the estimate for J^1.

Now, we derive the $\mathbb{L}^2$-$\mathbb{L}^2$ time decay estimate for the solution of the Cauchy problem (3.74)–(3.75):

THEOREM 3.6 ($\mathbb{L}^2$-$\mathbb{L}^2$ time decay estimate). *Let the Cauchy data Θ^0, Θ^1 be functions vanishing at infinity. Moreover, let*

$$(\Theta^1, D\Theta^0)^* \in \mathbb{L}^1_{,3}(\mathbb{R}^3). \tag{3.99}$$

Then the solution Θ of the problem (3.74)–(3.75) *given by* (3.76) *satisfies the estimate*

$$\|\nabla\Theta(t,\cdot)\|_{\mathbb{L}^2(\mathbb{R}^3)} \le C\|(\Theta^1, D\Theta^0)^*\|_{\mathbb{L}^2_{,3}(\mathbb{R}^3)} \quad \textit{for } t\ge 0, \tag{3.100}$$

where C is a constant independent of Θ^0, Θ^1 and t.

S k e t c h o f p r o o f. Introducing the vector $V=\nabla\Theta=(\partial_t\Theta, D\Theta)^*$ we reduce the Cauchy problem (3.74)–(3.75) to an equivalent one for some linear symmetric hyperbolic system of the first order. Next, applying the existence and uniqueness theorem (cf. [18], Th. 3.2, p. 329) we get the estimate (3.100).

Now, to obtain $\mathbb{L}^p$-$\mathbb{L}^q$ estimates we apply interpolation between the cases $\mathbb{L}^2$-$\mathbb{L}^2$ and $\mathbb{L}^\infty$-$\mathbb{L}^1$. Set

$$T_*f(x) = B(t,\cdot)*f(x). \tag{3.101}$$

Theorems 3.5 and 3.6 show that T_* yields operators

$$T_*^0 : \mathbb{L}^1_{,3}(\mathbb{R}^3) \to \mathbb{L}^\infty(\mathbb{R}^3) \quad \text{with} \quad \|T_*^0\| \le C(1+t)^{-3/2}, \tag{3.102}$$

and

$$T_*^1 : \mathbb{L}^2(\mathbb{R}^3) \to \mathbb{L}^2(\mathbb{R}^3) \quad \text{with} \quad \|T_*^1\| \le C. \tag{3.103}$$

Using interpolation and acting as in Section 3.2 we get

THEOREM 3.7 ($\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimate). *Let the Cauchy data Θ^0, Θ^1 be functions vanishing at infinity. Moreover, let*

$$(\Theta^1, D\Theta^0)^* \in \mathbb{L}^p_{,s_0}(\mathbb{R}^3),$$

for $p = (2k_0+2)/(2k_0+1)$, $s_0 = [3k_0/(k_0+1)]$, where $k_0 \ge 2$. Then the solution of the problem (3.74)–(3.75) *given by* (3.76) *satisfies*

$$\|\nabla\Theta(t,\cdot)\|_{\mathbb{L}^{2k_0+2}(\mathbb{R}^3)} \le C(1+t)^{-3k_0/(2(k_0+1))}\|(\Theta^1, D\Theta^0)^*\|_{\mathbb{L}^p_{,s_0}(\mathbb{R}^3)} \quad \textit{for } t \ge 0,$$

where C is a constant independent of Θ^0, Θ^1 and t.

We apply Theorem 3.7 in Section 6 to prove the existence of a global-in-time solution of the Cauchy problem for a nonlinear hyperbolic heat equation.

4. Local existence of solutions

4.1. Local existence of solutions to the initial value problem for nonlinear hyperbolic thermoelasticity. We can write the nonlinear hyperbolic system of thermoelasticity theory (cf. [29]) as follows (cf. (1.10), (1.11)):

$$\partial_t^2 u^j = \sum_{k,n=1}^{3} c_{jn}^k(\nabla u, \nabla T)\partial_k\partial_t u^n + \sum_{k,m,n=1}^{3} c_{jkmn}(\nabla u, \nabla T)\partial_k\partial_m u^n + \sum_{m=1}^{3} c_{jm}(\nabla u, \nabla T)\partial_m\partial_t T, \tag{4.1}$$

$$\partial_t^2 T = \sum_{\substack{j,k=0\\ j+k\neq 0}}^{3} a_{jk}(\nabla u, \nabla T)\partial_j\partial_k T + \sum_{j,m=1}^{3} c_{jm}(\nabla u, \nabla T)\partial_m\partial_t u^j, \tag{4.2}$$

where

$$\begin{aligned}
&a_{jk}(\nabla u, \nabla T) = \widetilde{a}_{jk}(\nabla u, \nabla T) + k\delta_{jk},\\
&c_{jn}^k(\nabla u, \nabla T) = a_{jk0n}(\nabla u, \nabla T) + a_{j0kn}(\nabla u, \nabla T),\\
&c_{jkmn}(\nabla u, \nabla T) = \mu\delta_{jn}\delta_{km} + (\lambda+\mu)\delta_{jk}\delta_{mn} + a_{jkmn}(\nabla u, \nabla T),\\
&c_{jm}(\nabla u, \nabla T) = \tilde{c}_{jm}(\nabla u, \nabla T) - \beta\delta_{jm},
\end{aligned} \tag{4.3}$$

and for simplicity we assume that $\tau = 1$, $\varrho = 1$. Introducing the vector

$$U = (\nabla u, \nabla T)^* = (u_0^1, u_0^2, u_0^3, u_1^1, u_1^2, u_1^3, u_2^1, u_2^2, u_2^3, u_3^1, u_3^2, u_3^3, T_0, T_1, T_2, T_3)^*, \tag{4.4}$$

we obtain from (4.1)–(4.3) the first order system

$$\partial_0 U = \sum_{k=1}^{3} \widetilde{A}^k(U)\partial_k U, \tag{4.5}$$

where $\widetilde{A}^k(U)$ $(k = 1, 2, 3)$ are the 16×16 matrices

$$(4.6) \qquad \widetilde{A}^k(U) = \begin{pmatrix} (c^k_{jm}(U))^*_{3\times 3} & (c_{kjmn}(U))_{3\times 9} & (c_{jk}(U))_{3\times 1} \quad \mathbf{0}_{3\times 3} \\ (I\delta_{kj})_{9\times 3} & \mathbf{0}_{9\times 9} & \mathbf{0}_{9\times 4} \\ \begin{matrix} ((c_{jk}(U))^*_{1\times 3} \\ \mathbf{0}_{3\times 3} \end{matrix} & \mathbf{0}_{4\times 9} & (\widetilde{\alpha}^k_{ij}(U))_{4\times 4} \end{pmatrix},$$

$(c^k_{jm}(U))$ are 3×3 matrices $(j = 1, 2, 3,\ m = 1, 2, 3)$, $(c_{kjmn}(U))$ are 3×9 matrices $(j = 1, 2, 3, m, n = 11, 12, 13, 21, 22, 23, 31, 32, 33)$, $I\delta_{kj}$ denote the 9×9 matrices

$$I\delta_{kj} = \left\{ \begin{pmatrix} I_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \end{pmatrix} \text{ for } k = 1, \quad \begin{pmatrix} \mathbf{0}_{3\times 3} \\ I_{3\times 3} \\ \mathbf{0}_{3\times 3} \end{pmatrix} \text{ for } k = 2, \quad \begin{pmatrix} \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ I_{3\times 3} \end{pmatrix} \text{ for } k = 3 \right\},$$

$I_{3\times 3}$ is the 3×3 unit matrix, $\mathbf{0}$ denotes the null matrices, and $(\widetilde{\alpha}^k_{ij}(U))$ are nonsymmetric 4×4 matrices with elements

$$(4.7) \qquad \widetilde{\alpha}^k_{ij}(U) = \delta_{ik}\delta_{0j} + 2\delta_{0i}\delta_{0j}a_{0k}(U) + \delta_{0i}(1 - \delta_{0j})(k\delta_{kj} + a_{kj}(U)).$$

From (4.6) and (4.7) it follows that (4.5) is not a symmetric hyperbolic system. In order to convert it to a symmetric hyperbolic system we set

$$(4.8) \quad A(U) = \begin{pmatrix} I_{3\times 3} & & \mathbf{0}_{3\times 9} & & \mathbf{0}_{3\times 4} \\ \mathbf{0}_{9\times 3} & (c_{l1mn}(U))^*_{9\times 3} & (c_{l2mn}(U))^*_{9\times 3} & (c_{l3mn}(U))^*_{9\times 3} & \mathbf{0}_{9\times 4} \\ \mathbf{0}_{4\times 3} & & \mathbf{0}_{4\times 9} & & (\alpha_{ij}(U))_{4\times 4} \end{pmatrix},$$

where $\alpha_{ij}(U) = \delta_{0i}\delta_{0j} + (1 - \delta_{0i})(1 - \delta_{0j})(k\delta_{ij} + a_{ij}(U))$, $i, j = 0, 1, 2, 3,\ l, m, n = 1, 2, 3$. It is easy to see that $A(\cdot)$ is symmetric and positive definite in some neighbourhood $\mathcal{N}_0$ of $\mathbf{0} \in \mathbb{R}^{16}$ (because $\langle A(\mathbf{0})\xi, \xi\rangle \geq c|\xi|^2$, where c is a constant). After some calculations we see that

$$(4.9) \quad A(U)\widetilde{A}^k(U) = A^k(U) = \begin{pmatrix} (c^k_{ln}(U))_{3\times 3} & (c_{lkmn}(U))_{3\times 9} & (c_{lk}(U))_{3\times 1} \quad \mathbf{0}_{3\times 3} \\ (c_{lkmn}(U))^*_{9\times 3} & \mathbf{0}_{9\times 9} & \mathbf{0}_{9\times 4} \\ \begin{matrix} (c_{lk}(U))^*_{1\times 3} \\ \mathbf{0}_{3\times 3} \end{matrix} & \mathbf{0}_{4\times 9} & (\alpha^k_{ij}(U))_{4\times 4} \end{pmatrix},$$

where

$$\alpha^k_{ij}(U) = 2\delta_{0i}\delta_{0j}a_{0k}(U) + \delta_{0i}(1 - \delta_{0j})(k\delta_{kj} + a_{kj}(U))$$

and $A^k(\cdot)$ are symmetric 16×16 matrices. Therefore we can write (4.5) as a quasilinear symmetric hyperbolic system

$$(4.10) \qquad A(U)\partial_t U = \sum_{k=1}^{3} A^k(U)\partial_k U$$

under the assumption that $U(t, x) \in \mathcal{N}_0 \subset \mathbb{R}^{16}$ for $(t, x) \subset [0, \vartheta] \times \mathbb{R}^3$.

Thus, we have converted the initial value problem (1.1)–(1.3), under the conditions (1.5)–(1.8), to one for a quasilinear symmetric hyperbolic system of the first order. Taking into account the relations (4.4), (1.12), (1.13) we obtain the following initial conditions for $U = (\nabla u, \nabla T)^*$:

$$U(0,x) = (u^1, Du^0, T^1, DT^0)^*. \tag{4.11}$$

Moreover, the initial value problem (1.1)–(1.3), under the conditions (1.5)–(1.8), is equivalent to (4.10)–(4.11). Now, applying the results of Klainerman and Ponce [79] and Majda [90] we have the following

THEOREM 4.1 (Local existence). *Assume that* $(u^1, Du^0, T^1, DT^0)^* \in \mathbb{W}^{s,2}(\mathbb{R}^3)$ *with an integer* $s > 3/2 + 1$. *Then for* $\delta_0 > 0$ *small enough, if*

$$\|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} < \delta_0$$

then there exists a finite time interval $[0, \vartheta]$ *with* $\vartheta > 1$ *such that the initial value problem* (4.10)–(4.11) *has a unique local smooth solution with*

$$\begin{aligned}
&(\nabla u, \nabla T) \in C^1([0,\vartheta] \times \mathbb{R}^3),\\
&(\nabla u, \nabla T) \in C^1([0,\vartheta], \mathbb{W}^{s-1}(\mathbb{R}^3)) \cap C([0,\vartheta], \mathbb{W}^{s,2}(\mathbb{R}^3)),\\
&\|(\nabla u, \nabla T)(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} < 1 \quad \textit{for all } t \in [0,\vartheta].
\end{aligned}$$

4.2. Local existence of solutions to the initial value problem for the nonlinear hyperbolic heat equation. In this section we prove the local-in-time existence of solution to the initial value problem (1.21)–(1.22). We convert this problem to an equivalent one for the quasilinear first-order symmetric hyperbolic system

$$A^0(V)\partial_t V = \sum_{k=1}^{3} A^k(V)\partial_k V + BV, \tag{4.12}$$

$$V^0(+0,x) = V^0(x), \tag{4.13}$$

where

$$V \equiv V(t,x) = (\nabla\Theta(t,x))^* = (\partial_t\Theta, D\Theta)^* = (\partial_t\Theta, \partial_1\Theta, \partial_2\Theta, \partial_3\Theta)^*,$$
$$V^0(x) = (\Theta^1(x), \partial_1\Theta^0(x), \partial_2\Theta^0(x), \partial_3\Theta^0(x))^*,$$

$A^0(V), A^k(V), k = 1,2,3$, are symmetric 4×4 matrices and B is a constant 4×4 matrix defined as follows:

$$A^0(V) = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & c_{11} & c_{12} & c_{13}\\ 0 & c_{21} & c_{22} & c_{23}\\ 0 & c_{31} & c_{32} & c_{33}\end{bmatrix}, \qquad A^1(V) = \begin{bmatrix} 2a_{10} & c_{11} & c_{12} & c_{13}\\ c_{11} & & & \\ c_{21} & & \mathbf{0}_{3\times3} & \\ c_{31} & & & \end{bmatrix},$$

$$A^2(V) = \begin{bmatrix} 2a_{20} & c_{21} & c_{22} & c_{23}\\ c_{21} & & & \\ c_{22} & & \mathbf{0}_{3\times3} & \\ c_{23} & & & \end{bmatrix}, \quad A^1(V) = \begin{bmatrix} 2a_{30} & c_{31} & c_{32} & c_{33}\\ c_{13} & & & \\ c_{23} & & \mathbf{0}_{3\times3} & \\ c_{33} & & & \end{bmatrix},$$

$$B = \begin{bmatrix} -2m & 0 & 0 & 0\\ 0 & & & \\ 0 & & \mathbf{0}_{3\times3} & \\ 0 & & & \end{bmatrix},$$

where $c_{jk}(V) = a_{jk}(V) + a^2\delta_{jk}$ for $j,k = 1,2,3$, and $\mathbf{0}_{3\times3}$ is the null matrix of order 3.

From (1.24) it follows that the matrix $A^0(V)$ is positive definite for $\|V\|_{\mathbb{L}^\infty(\mathbb{R}^3)} < R_0$. Using the method of [79] and [90] we get the following

THEOREM 4.2 (Local existence for N.H.H.E.). *Assume that* $(\Theta^1, D\Theta^0)^* \in \mathbb{W}^{s,2}(\mathbb{R}^3)$ *with an integer* $s > 3/2 + 1$. *Then for* $\delta > 0$ *small enough*, *if*

$$\|(\Theta^1, D\Theta^0)^*\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} < \delta,$$

then there exists a finite time interval $[0, \zeta]$ *with* $\zeta > 1$ *such that the initial value problem* (4.12)–(4.13) *has a unique smooth solution with*

$$\nabla\Theta \in C^0([0,\zeta], \mathbb{W}^{s,2}(\mathbb{R}^3)) \cap C^1([0,\zeta], \mathbb{W}^{s-1,2}(\mathbb{R}^3)),$$
$$\|\nabla\Theta(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} < R_0 \quad \text{for } t \in [0,\zeta].$$

5. High energy estimates

5.1. High energy estimates for nonlinear hyperbolic thermoelasticity. We now establish a priori estimates for higher-order derivatives of solutions to (4.10)–(4.11) using an energy method.

THEOREM 5.1. *Let* $s \geq 4$ *be an integer and let* $U(t,x) = (\nabla u, \nabla T)^*$ *be a solution of the initial value problem* (4.10)–(4.11) *on some interval* $[0, \vartheta]$ *guaranteed by Theorem* 4.1 *with*

$$\|(\nabla u, \nabla T)(t)\|_{\mathbb{L}^\infty_{,\lfloor s/2 \rfloor}(\mathbb{R}^3)} < 1 \quad \text{for all } t \in [0, \vartheta]. \tag{5.1}$$

Then, *for some* $k_0 \geq 2$,

$$\|(\nabla u, \nabla T)(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \leq C_s \|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{H}^s(\mathbb{R}^3)} \times \exp\Big(C_s \int_0^t \|(\nabla u, \nabla T)(\tau)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2 \rceil}(\mathbb{R}^3)}\, d\tau\Big) \tag{5.2}$$

for $t \in [0, \vartheta]$ *and* $k_0 \geq 2$, *where the constant* C_s *depends on* s *and not on* ϑ.

Proof. We use the standard energy method with the help of mollification. Let $J_\varepsilon *$ denote the Friedrichs mollifier (cf. [90]). We put $U_\varepsilon = J_\varepsilon * U$. Then $U_\varepsilon \in C^\infty$ and it is not difficult to verify that $|U_\varepsilon(t,x)| < 1$ when $|U(t,x)| < 1$ for $(t,x) \in [0,\vartheta] \times \mathbb{R}^3$. Taking it into account, we get

$$A(U_\varepsilon(t))\partial_t U_\varepsilon^\alpha(t) - \sum_{k=1}^3 A^k(U_\varepsilon(t))\partial_k U_\varepsilon(t) = A(U_\varepsilon(t)) D_x^\alpha G_\varepsilon(t) + A(U_\varepsilon(t)) H_\varepsilon^{(\alpha)}(t),$$

where

$$G_\varepsilon(t) = \sum_{k=1}^3 (J_\varepsilon * (\widetilde{A}^k(U)\partial_k U) - \widetilde{A}^k(U_\varepsilon)\partial_k U_\varepsilon)(t),$$
$$H_\varepsilon^{(\alpha)}(t) = \sum_{k=1}^3 (D_x^\alpha(\widetilde{A}^k(U_\varepsilon)\partial_k U_\varepsilon) - \widetilde{A}^k(U_\varepsilon) D_x^\alpha(\partial_k U_\varepsilon^\alpha))(t);$$

here $U_\varepsilon^\alpha(t)$ stands for $D_x^\alpha U_\varepsilon(t)$, $|\alpha| \leq s$ and the matrix $\widetilde{A}^k(U)$ has the form given by (4.6).

Taking the scalar product (in $\mathbb{R}^{16}$) with $U_\varepsilon^\alpha(t)$, integrating the resulting equality over $x \in \mathbb{R}^3$, summing over α ($|\alpha| \le s$) and applying the Cauchy–Schwarz inequality we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U_\varepsilon(t)\|^2_{\mathbb{E},s} \le{} & \frac{1}{2}\|\partial_t A(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)}\|U_\varepsilon(t)\|^2_{\mathbb{H}^s(\mathbb{R}^3)} \\
& + \frac{1}{2}\sum_{k+1}^{3}\|\partial_k A^k(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)}\|U_\varepsilon(t)\|^2_{\mathbb{H}^s(\mathbb{R}^3)} \\
& + \sum_{|\alpha|\le s}\|A(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)}\|H_\varepsilon^{(\alpha)}(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}\|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \\
& + \sum_{|\alpha|\le s}\|A(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)}\|D_x^\alpha G_\varepsilon(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}\|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)},
\end{aligned}$$

where $\|\cdot\|_{\mathbb{E},s}$ denotes the following energy norm:

$$\|V(t)\|^2_{\mathbb{E},s} = \sum_{|\alpha|\le s}\int_{\mathbb{R}^3}\langle A(U)D_x^\alpha V(t),\, D_x^\alpha V(t)\rangle\, dx, \tag{5.3}$$

which is equivalent to the norm $\|\cdot\|^2_{\mathbb{H}^s}$. Using Taylor's formula (cf. [43]) and recalling that the matrices A, A^k are defined as in (4.8)–(4.9) with the coefficients given by (4.3), we can represent the coefficients $a(U_\varepsilon)$ (where for simplicity of $a(U_\varepsilon)$ stands for $a_{jkmn}(U_\varepsilon)$, $a_{jk}(U_\varepsilon)$ and $c_{jm}(U_\varepsilon)$, respectively) as follows:

$$a(U_\varepsilon) = a_1(U_\varepsilon) + a_2(U_\varepsilon),$$

where

$$a_1(U_\varepsilon) = \sum_{|\alpha|=k_0}\frac{D^\alpha a(0)}{\alpha!}U_\varepsilon^\alpha$$

is a homogeneous polynomial of order k_0 with respect to U_ε and

$$a_2(U_\varepsilon) = \sum_{|\alpha|=k_0+1}\frac{k_0+1}{\alpha!}\int_0^1(1-z)^{k_0}D^\alpha a_2(zU_\varepsilon)U_\varepsilon^\alpha\, dz$$

is a $\mathbb{C}^\infty$-function. Taking this into account and assuming $\|U_\varepsilon(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le 1$ we obtain

$$\|\partial_t A(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\|U_\varepsilon(t)\|^{k_0-1}_{\mathbb{L}^\infty(\mathbb{R}^3)}\|\partial_t U_\varepsilon(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)}, \tag{5.4}$$

$$\|\partial_k A^k(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\|U_\varepsilon(t)\|^{k_0-1}_{\mathbb{L}^\infty(\mathbb{R}^3)}\|\partial_k U_\varepsilon(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)}. \tag{5.5}$$

To estimate the term $\|H_\varepsilon^{(\alpha)}(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}$ it is enough to consider the following term (here $D^1 \equiv \partial_k$).

$$\|[D^\alpha(a(U_\varepsilon)D^1(U_\varepsilon)) - a(U_\varepsilon)D^\alpha D^1 U_\varepsilon](t)\|_{\mathbb{L}^2(\mathbb{R}^3)} \le I_1(t) + I_2(t),$$

where

$$\begin{aligned}
I_1(t) &= \|[D^\alpha(a_1(U_\varepsilon)D^1(U_\varepsilon)) - a_1(U_\varepsilon)D^\alpha D^1 U_\varepsilon](t)\|_{\mathbb{L}^2(\mathbb{R}^3)}, \\
I_2(t) &= \|[D^\alpha(a_2(U_\varepsilon)D^1(U_\varepsilon)) - a_2(U_\varepsilon)D^\alpha D^1 U_\varepsilon](t)\|_{\mathbb{L}^2(\mathbb{R}^3)}.
\end{aligned}$$

In view of the expression for $a_1(U_\varepsilon)$, acting in the same way as in [29], we get

$$I_1(t) \le C\|U_\varepsilon(t)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)}.$$

In order to estimate $I_2(t)$ we use Lemma A.1 of [78] to get

$$\begin{aligned} I_2(t) &= \|[D^\alpha(a_2(U_\varepsilon)D^1(U_\varepsilon)) - a_2(U_\varepsilon)D^\alpha D^1 U_\varepsilon](t)\|_{\mathbb{L}^2(\mathbb{R}^3)} \\ &\le C_s(\|D^1 a_2(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)}\|D^s(U_\varepsilon(t))\|_{\mathbb{L}^2(\mathbb{R}^3)} \\ &\quad + \|D^1(U_\varepsilon(t))\|_{\mathbb{L}^\infty(\mathbb{R}^3)}\|D^s a_2(U_\varepsilon(t))\|_{\mathbb{L}^2(\mathbb{R}^3)}). \end{aligned}$$

Using the formula for higher-order derivatives of the composite function $a_2(U_\varepsilon(t))$ (cf. (2.37)) we conclude that the expression for the derivative $D^\alpha_x a_2(U_\varepsilon)$ ($|\alpha| = s$) contains derivatives of order $l > \lfloor s/2\rfloor$ of $U_\varepsilon(t)$ with exponent one. Taking this into account and assuming (5.1) we get

$$|D^\alpha_x a_2(U_\varepsilon(t))| \le C\Big(\sum_{|\gamma|\le s}|D^\gamma_x U_\varepsilon(t)|\Big)\Big(\sum_{|\gamma|\le\lceil s/2\rceil}|D^\gamma_x U_\varepsilon(t)|\Big)^{k_0}.$$

Hence we have

$$I_2(t) \le C\|U_\varepsilon(t)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)}.$$

Thus

$$I_1(t) + I_2(t) \le C\|U_\varepsilon(t)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)}. \tag{5.6}$$

Therefore, using (5.4)–(5.5) and (5.6), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U_\varepsilon(t)\|^2_{\mathbb{E},s} &\le C\|U_\varepsilon(t)\|^{k_0-1}_{\mathbb{L}^\infty(\mathbb{R}^3)}(\|\partial_t U_\varepsilon(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \\ &\quad + \|U_\varepsilon(t)\|_{\mathbb{L}^\infty_{,1}(\mathbb{R}^3)})\|U_\varepsilon(t)\|^2_{\mathbb{H}^s(\mathbb{R}^3)} \\ &\quad + C\|U_\varepsilon(t)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\|U_\varepsilon(t)\|^2_{\mathbb{H}^s(\mathbb{R}^3)} + CG^s_\varepsilon(t)\|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)}, \end{aligned} \tag{5.7}$$

where

$$G^s_\varepsilon(t) = \sum_{|\alpha|\le s}\|D^\alpha G_\varepsilon(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}.$$

Noting that $\|U_\varepsilon(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)}$ and $\|U_\varepsilon(t)\|_{\mathbb{L}^\infty_{,1}(\mathbb{R}^3)}$ can be majorized by $\|U_\varepsilon(t)\|_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}$ and that the energy norm (5.3) is equivalent to $\|\cdot\|_{\mathbb{H}^s(\mathbb{R}^3)}$, we derive from (5.7) the inequality

$$\begin{aligned} \frac{1}{2}\|U_\varepsilon(t)\|^2_{\mathbb{E},s} &\le C(\|U_\varepsilon(t)\|^{k_0-1}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\|\partial_t U_\varepsilon(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \\ &\quad + \|U_\varepsilon(t)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)})\|U_\varepsilon(t)\|_{\mathbb{E},s} + CG^s_\varepsilon(t). \end{aligned}$$

Applying Gronwall's inequality we find

$$\begin{aligned} \|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} &\le C\|U_\varepsilon(0)\|_{\mathbb{H}^s(\mathbb{R}^3)} \\ &\quad\times\exp\Big(C\int_0^t(\|U_\varepsilon(\tau)\|^{k_0-1}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\|\partial_t U_\varepsilon(\tau)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} + \|U_\varepsilon(\tau)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)})\,d\tau\Big) \end{aligned} \tag{5.8}$$

$$+ C \int_0^t G_\varepsilon^s(\tau)$$

$$\times \exp\Big(C \int_\tau^t (\|U_\varepsilon(\Theta)\|^{k_0-1}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)} \|\partial_t U_\varepsilon(\Theta)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} + \|U_\varepsilon(\Theta)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)})\, d\Theta \Big)\, d\tau.$$

Note that $s \geq 4$, hence there is an imbedding of $\mathbb{H}^s(\mathbb{R}^3)$ in $\mathbb{C}^{\lceil s/2\rceil}(\mathbb{R}^3)$ and $\|\cdot\|_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)} \leq C_s\|\cdot\|_{\mathbb{H}^s(\mathbb{R}^3)}$. Moreover, $\mathbb{H}^{s-1}(\mathbb{R}^3)$ is imbedded in $\mathbb{C}^0(\mathbb{R}^3)$ and $\|\cdot\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \leq C_s\|\cdot\|_{\mathbb{H}^{s-1}(\mathbb{R}^3)}$. Thus, as $\varepsilon \to 0$, we have

$$\left.\begin{aligned} \|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} &\to \|U(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \\ \|\partial_t U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} &\to \|\partial_t U(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \end{aligned}\right. \quad \text{for } t \in [0,\vartheta]$$

(we have used the mollification property, cf. [90]), and

$$\left.\begin{aligned} \|U_\varepsilon(t)\|_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)} &\to \|U(t)\|_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)} \\ \|\partial_t U_\varepsilon(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} &\to \|\partial_t U(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \end{aligned}\right. \quad \text{for } t \in [0,\vartheta].$$

Thus, by taking the limit of both sides of (5.8) as $\varepsilon \to 0$, we observe that

$$\left.\begin{aligned} \|U_\varepsilon(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} &\to \|U(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \\ \|U_\varepsilon(0)\|_{\mathbb{H}^s(\mathbb{R}^3)} &\to \|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \end{aligned}\right. \quad \text{for } t \in [0,\vartheta]$$

and using Lebesgue's dominated convergence theorem the terms under the exponent tend to

$$C \int_0^t \|U(\tau)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\, d\tau, \qquad C \int_\tau^t \|U(\eta)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\, d\eta,$$

respectively. Therefore, to obtain (5.2) (i.e. the fact that $G_\varepsilon^s(t) \to \infty$) it remains to consider the term $G_\varepsilon^s(t)$ in (5.8) as ε tends to zero. The proof of this fact follows from the properties of mollification, the Leibniz formula and Taylor's theorem. This completes the proof of Theorem 5.1.

5.2. High energy estimates for the nonlinear hyperbolic heat equation. In this section we present a priori energy estimates for higher-order derivatives of solutions of the initial value problem (1.21), (1.22):

THEOREM 5.2. *Let $s \geq 4$ be an integer and let*

$$V(t,x) = (\nabla\Theta(t,x)) = (\partial_t\Theta(t,x), D\Theta(t,x))^*$$

be a solution to the initial value problem (4.12), (4.13) *on some interval* $[0,\zeta]$ *guaranteed by Theorem* 4.2 *with*

$$\|(\nabla\Theta)(t)\|_{\mathbb{L}^\infty_{,\lfloor s/2\rfloor}(\mathbb{R}^3)} < 1 \quad \textit{for all } t \in [0,\zeta].$$

Then, for some $k_0 \geq 0$,

$$(5.9) \quad \|(\nabla\Theta)(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \leq C_s\|(\Theta^1, D\Theta^0)^*\|_{\mathbb{H}^s(\mathbb{R}^3)} \exp\Big(C_s \int_0^t \|(\nabla\Theta)^*(\tau)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\, d\tau\Big)$$

for $t \in [0,\zeta]$, *where the constant* C_s *depends on* s *and not on* ζ.

Sketch of proof. We proceed similarly to the proof of Theorem 5.1 (instead of U_ε we put $V_\varepsilon = J_\varepsilon * V$, where $V = (\nabla\Theta)^*$).

6. Global solutions in nonlinear hyperbolic thermoelasticity

6.1. Proof of the main Theorem 1.1. First, we introduce the notation

$$(6.1) \qquad M_{s_1}(\vartheta) = \sup_{t\in[0,\vartheta]} (1+t)^{k_0/(k_0+1)}\|U(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)},$$

where $s_1 = \lceil s/2\rceil + 1$, $s \geq 10$ is an integer, and $k_0 \geq 2$ is the integer given by (1.5)–(1.7). Let $U(t,x)$ be the solution of (4.10)–(4.11) provided by Theorem 4.1 on $[0,\vartheta]$, where $s \geq 10$. Then, by the imbedding theorem (cf. (2.35)), $U(t,\cdot)$ also belongs to $\mathbb{W}^{s_1,2k_0+2}(\mathbb{R}^3)$ for $t \in [0,\vartheta]$. In the following theorem we show that for this solution $M_{s_1}(\vartheta)$ is bounded by a constant independent of ϑ provided that a smallness assumption on $\|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} + \|U^0\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)}$ is satisfied.

Theorem 6.1. *Let* $U(t,x) = (\nabla u(t,x), \nabla T(t,x))$ *be a solution to the initial value problem* (4.10)–(4.11) *on some interval* $[0,\vartheta]$ *and* $s_1 = \lceil s/2\rceil + 1$, $s \geq 10$. *If*

$$U^0 = (u^1, Du^0, T^1, DT^0)^* \in \mathbb{H}^s(\mathbb{R}^3) \cap \mathbb{W}^{s,p}(\mathbb{R}^3)$$

(*where* $p = (2k_0+2)/(2k_0+1)$, $k_0 \geq 2$ *is an integer*) *and*

$$\|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{H}^s(\mathbb{R}^3)} + \|(u^1, Du^0, T^1, DT^0)^*\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)} < \delta$$

with $\delta > 0$ *sufficiently small, then* $M_{s_1}(\vartheta) \leq M_0 < \infty$, *where the constant* M_0 *is independent of* ϑ.

Proof. We can treat (4.10)–(4.11) as a linear system with nonlinear right hand side of the form

$$(6.2) \qquad A(0)\partial_t U - \sum_{k=1}^{3} A^k(0)\partial_k U = F(U, DU)$$

with

$$(6.3) \qquad U(0,x) = U^0(x) = (u^1, Du^0, T^1, DT^0)^*,$$

where $A(0)$, $A^k(0)$ $(k = 1,2,3)$ are constant matrices given by (4.8)–(4.9) and $F = (F_1, F_2, F_3, 0,0,0,0,0,0,0,0,0, Q, 0,0,0)^*$ with F_j $(j = 1,2,3)$ and Q given by (1.4). For $t \in [0,\vartheta]$ we express the solution $U(t,x) = (\nabla u(t,x), \nabla T(t,x))$ to the initial value problem

(6.2)–(6.3) using the matrix of fundamental solutions of the linearized hyperbolic system of equations derived from (1.10)–(1.11) (cf. (3.21)). Thus ([1])

$$U(t,x) = R(t,\cdot) * U^0(x) + \int_0^t R(t-\tau,\cdot) * F(U,DU)(\tau,x)\,d\tau, \tag{6.4}$$

where $R(t,x)$ is a 16×16 matrix with elements being linear combinations of the terms $\partial_l H_{jk}(t,x)$ and $\partial_t H_{jk}(t,x)$ (cf. (3.30–3.34)). From (6.4) we have

$$\begin{aligned}\|U(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} &\le \|R(t,\cdot) * U^0(\cdot)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \\ &\quad + \int_0^t \|R(t-\tau,\cdot) * F(\tau,\cdot)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)}\,d\tau.\end{aligned} \tag{6.5}$$

Using Theorem 3.3 we get

$$\begin{aligned}\|U(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} &\le C(1+t)^{-k_0/(k_0+1)}\|U^0\|_{\mathbb{L}^{p}_{,s_1+s_0}(\mathbb{R}^3)} \\ &\quad + C\int_0^t (1+t-\tau)^{-k_0/(k_0+1)}\|F(\tau,\cdot)\|_{\mathbb{L}^{p}_{,s_1+s_0}(\mathbb{R}^3)}\,d\tau.\end{aligned} \tag{6.6}$$

By familiar calculus inequalities (cf. [78, 79, 29]) and our specific assumptions about F, we have

$$\|F(U,DU)(\tau)\|_{\mathbb{L}^{p}_{,s_1+s_0}(\mathbb{R}^3)} \le C\|U(\tau)\|^{k_0}_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)}\|U(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)}. \tag{6.7}$$

Inserting (6.7) into (6.6) we obtain

$$\begin{aligned}\|U(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} &\le C(1+t)^{-k_0/(k_0+1)}\|U^0\|_{\mathbb{L}^{p}_{,s_1+s_0}(\mathbb{R}^3)} \\ &\quad + C\int_0^t (1+t-\tau)^{-k_0/(k_0+1)}\|U(\tau)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)}\|U(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)}\,d\tau.\end{aligned} \tag{6.8}$$

Using Theorem 5.1 we get

$$\begin{aligned}&\|U(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)} \\ &\quad \le C_s\|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \exp\Big(C_s \int_0^\tau \|U(\Theta)\|^{k_0}_{\mathbb{L}^{\infty}_{,\lceil s/2\rceil}(\mathbb{R}^3)}\,d\Theta\Big) \quad \text{for } 0 \le \tau \le t \le \vartheta.\end{aligned} \tag{6.9}$$

Since $s_1 = \lceil s/2\rceil + 1 > [3/2] + 3/(2k_0+2)$ $(k_0 \ge 2)$, there is an imbedding of $\mathbb{W}^{s_1,2k_0+2}(\mathbb{R}^3)$ in $C^{\lceil s/2\rceil}(\mathbb{R}^3)$ and then

$$\|U(\Theta)\|_{\mathbb{L}^{\infty}_{,\lceil s/2\rceil}(\mathbb{R}^3)} \le C_s\|U(\Theta)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \quad \text{for } \Theta \in [0,\vartheta]. \tag{6.10}$$

By the notation (6.1) and taking into account (6.10) we can estimate (6.9) as follows:

$$\|U(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)} \le C_s\|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \exp\Big(C_s M^{k_0}_{s_1}(\vartheta) \int_0^\tau (1+\Theta)^{-k_0^2/(k_0+1)}\,d\Theta\Big). \tag{6.11}$$

Since $\int_0^\tau (1+\Theta)^{-k_0^2/(k_0+1)}\,d\Theta \le 3$ for any $\tau \ge 0$ and $k_0 \ge 2$, we obtain

$$\|U(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)} \le C_s\|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \exp(C_s M^{k_0}_{s_1}(\vartheta)) \quad \text{for } \tau \in [0,\vartheta]. \tag{6.12}$$

([1]) In (6.4)–(6.5) the asterisk denotes convolution with respect to $x \in \mathbb{R}^3$.

Inserting (6.12) into (6.8) with $\|U^0\|_{\mathbb{L}^p_{,s_1+s_0}(\mathbb{R}^3)}$ and $\|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)}$ replaced by δ we obtain

$$\begin{aligned} \|U(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} &\le C_s\delta(1+t)^{-k_0/(k_0+1)} + C_s\delta\exp\big(C_s M^{k_0}_{s_1}(\vartheta)\big) \\ &\quad\times\int_0^t (1+t-\tau)^{-k_0/(k_0+1)}\|U(\tau)\|^{k_0}_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)}\,d\tau. \end{aligned} \tag{6.13}$$

Since

$$\|U(\tau)\|^{k_0}_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \le M^{k_0}_{s_1}(\vartheta)(1+\tau)^{-k_0^2/(k_0+1)}$$

multiplying both sides of (6.13) by $(1+t)^{k_0/(k_0+1)}$ we derive

$$\begin{aligned} (1+t)^{k_0/(k_0+1)}&\|U(\tau)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \\ &\le C_s\delta + C_s\delta M^{k_0}_{s_1}(\vartheta)\exp(C_s M^{k_0}_{s_1}(\vartheta))(1+t)^{k_0/(k_0+1)} \\ &\quad\times\int_0^t (1+t-\tau)^{-k_0/(k_0+1)}(1+\tau)^{-k_0^2/(k_0+1)}\,d\tau. \end{aligned} \tag{6.14}$$

It is easy to see that

$$(1+t)^{k_0/(k_0+1)}\int_0^t (1+t-\tau)^{-k_0/(k_0+1)}(1+\tau)^{-k_0^2/(k_0+1)}d\tau \le 12$$

for $t\ge 0$ and $k_0\ge 2$. Thus from (6.14) we have

$$(1+t)^{k_0/(k_0+1)}\|U(\tau)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \le C_s\delta + C_s\delta M^{k_0}_{s_1}(\vartheta)\exp\big(C_s M^{k_0}_{s_1}(\vartheta)\big). \tag{6.15}$$

Therefore,

$$M_{s_1}(\vartheta) \le C_s\delta + C_s\delta M^{k_0}_{s_1}(\vartheta)\exp(C_s M^{k_0}_{s_1}(\vartheta)), \tag{6.16}$$

where C_s is a constant depending only on s and not on ϑ. Now, consider the function

$$f(x) = C\delta(1 + x^{k_0}\exp Cx^{k_0}) - x.$$

Note that if δ is sufficiently small, then the equation $f(x)=0$ has the first positive root M_0 and

$$M_0 > \delta > 0,\qquad \delta = \frac{M}{C(1+M_0^{k_0}\exp(CM_0^{k_0}))}.$$

Since $f(x)$ is continuous and $f(0)=C\delta>0$, we have

$$\begin{aligned} &f(x)\ge 0 \quad \text{for } x\in[0,M_0], \\ &f(x)<0 \quad \text{for } x\in(M_0,M_0+\varepsilon) \quad \text{for some } \varepsilon>0. \end{aligned} \tag{6.17}$$

Moreover, if δ is sufficiently small and $\delta<M_0$, then

$$M_{s_1}(0) = \|U^0\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \le C\|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \le C\delta < M_0,$$

that is,

$$M_{s_1}(0)\in[0,M_0]. \tag{6.18}$$

From (6.16)–(6.18) it follows that $M_{s_1}(\vartheta)\le M_0$. ∎

We now prove a result which guarantees that the local existence theorem can be re-applied to extend the local solution to a global one.

THEOREM 6.2. *Let* $s \geq 0$ *be an integer and let*

$$U^0 = (u^1, Du^0, T^1, DT^0)^* \in \mathbb{W}^{s,2}(\mathbb{R}^3) \cap \mathbb{W}^{s,2}(\mathbb{R}^3)$$

(*where* $p = (2k_0 + 2)/(2k_0 + 1)$, $k_0 \geq 2$ *is given by* (1.5)–(1.7)) *and let*

$$U(t, x) = (\nabla u(t, x), \nabla T(t, x))$$

be a solution of the problem (4.10)–(4.11) *on some interval* $[0, \vartheta]$. *Then there exists a constant* $\delta_1 > 0$ *sufficiently small such that if*

$$\|U^0\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} + \|U^0\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)} < \delta_1,$$

then

$$\|U(t)\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} \leq K_{s,k_0} \|U^0\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} \quad \textit{for} \ \ t \in [0, \vartheta],$$

where the constant K_{s,k_0}, *depends only on* s *and* k_0 *and does not depend on* ϑ.

P r o o f. It follows from Theorem 4.1 that

$$\|U(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \leq C_s \|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \exp\Big(C_s \int_0^t \|U(\tau)\|_{\mathbb{L}^\infty_{,\lceil s/2 \rceil}(\mathbb{R}^3)} \, d\tau\Big) \quad \text{for } t \in [0, \vartheta].$$

In order to estimate the integral under the exponent we repeat the argument of (6.10)–(6.12) to obtain

$$\|U(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \leq C \|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \exp(C_s M_{s_1}^{k_0}(\vartheta)) \quad \text{for } t \in [0, \vartheta],$$

provided that $\|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} + \|U^0\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)}$ is small enough. Setting

$$K_{s,k_0} = C_s \exp(C_s M_0^{k_0}),$$

we have $\|U(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \leq K_{s,k_0} \|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)}$ for $t \in [0, \vartheta]$. ■

In order to prove the main Theorem 1.1 we proceed as follows.

We take $\delta = \min(\delta_0, \delta_1, \delta/K_{s,k_0})$ with $\delta_0, \delta_1, \delta/K_{s,k_0}$ given by Theorems 6.1 and 6.2. Let the initial data U^0 satisfy

$$U^0 \in \mathbb{H}^s(\mathbb{R}^3) \cap \mathbb{W}^{s,p}(\mathbb{R}^3), \qquad \|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)} + \|U^0\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)} < \delta.$$

Then by Theorem 4.1 there is a constant $\vartheta_0 > 1$ such that the solution $U(t, x)$ exists on the interval $[0, \vartheta_0]$ and satisfies $\|U(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \leq K_{s,k_0} \|U^0\|_{\mathbb{H}^s(\mathbb{R}^3)}$ for $t \in [0, \vartheta_0]$, by Theorem 6.2. Since $\|U(\vartheta_0)\|_{\mathbb{H}^s(\mathbb{R}^3)} < \delta_0$ we can again apply Theorem 4.1 by taking $t = \vartheta_0$ as the new initial time. Then we have a solution $U(t, x)$ on $[0, 2\vartheta_0]$ and it satisfies the assumptions of Theorem 6.2. Therefore $\|U(2\vartheta_0)\|_{\mathbb{H}^s(\mathbb{R}^3)} < \delta_0$, etc. Thus we can extend the local solution to a global one.

Furthermore, we notice from (6.15) and the conclusion of Theorem 6.1 that the global solution $U(t, x)$ admits the estimate

$$\|U(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \leq C(1 + t)^{-k_0/(k_0+1)} \quad \text{for all } t \geq 0, \tag{6.19}$$

where the constant C is independent of t.

Next, the asymptotic decay of the global solution in the $\mathbb{L}^\infty$-norm follows directly from Sobolev's imbedding inequality (6.10) and the estimate (6.19):

$$\|U(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C(1+t)^{-k_0/(k_0+1)} \quad \text{for all } t \ge 0.$$

Finally, as an immediate consequence of Theorem 6.2 we have $\|U(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} \le C$ for all $t \ge 0$. Therefore, we conclude that the time-asymptotic behaviour of the solution for large t is described by

$$\|U(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} = O(t^{-k_0/(k_0+1)}), \quad \|U(t)\|_{\mathbb{L}^{2k_0+2}(\mathbb{R}^3)} = O(t^{-k_0/(k_0+1)}),$$
$$\|U(t)\|_{\mathbb{L}^{2k_0+2}(\mathbb{R}^3)} = O(1). \ \blacksquare$$

6.2. Proof of main Theorem 1.2. In order to prove the estimate of a global solution to the initial value problem for the nonlinear hyperbolic heat equation we act as in Section 6.1.

First, we introduce the notation

$$\mathcal{M}_{s_1}(\zeta) = \sup_{0 \le t \le \zeta} (1+t)^{3k_0/(2(k_0+1))} \|V(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)},$$

where $s_1 = [s/2]+1, s \ge 10$ is an integer and $k_0 \ge 2$ is the integer appearing in Theorem 1.2.

THEOREM 6.3. *Let $V(t,x) = (\nabla\Theta(t,x))$ be a solution to the initial value problem* (4.12)–(4.13) *on some interval $[0,\zeta]$ and $s_1 = [s/2]+1, s \ge 10$. If*

$$V^0 = (\Theta^1, D\Theta^0)^* \in \mathbb{H}^s(\mathbb{R}^3) \cap \mathbb{W}^{s,p}(\mathbb{R}^3)$$

(*where $p = (2k_0+2)/(2k_0+1), k_0 \ge 2$ is an integer*) *and*

$$\|(\Theta^1, D\Theta^0)^*\|_{\mathbb{H}^s(\mathbb{R}^3)} + \|(\Theta^1, D\Theta^0)^*\|_{\mathbb{W}^s(\mathbb{R}^3)} < \delta$$

with $\delta > 0$ sufficiently small, then $\mathcal{M}_{s_1}(\zeta) \le \mathcal{M}_0 < \infty$, where the constant $\mathcal{M}_0$ is independent of ζ.

Proof. For $t \in [0,\zeta]$ we express the solution $V(t) = (\nabla\Theta(t,\cdot))^*$ by the fundamental solution of the hyperbolic heat equation given by (3.60). Thus

$$V(t,x) = B(t,\cdot) * V^0(x) + \int_0^t B(t-\tau,\cdot) * [F(V,dV)](\tau,x)\,d\tau, \tag{6.20}$$

where $B(t,x)$ is the matrix defined by (3.80). From (6.20) we have

$$\|V(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \le \|B(t,\cdot) * V^0(\cdot)\|_{\mathbb{L}^{2k_0+2}_{s_1}(\mathbb{R}^3)} + \int_0^t \|B(t-\tau,\cdot) * F(\tau,\cdot)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)}\,d\tau. \tag{6.21}$$

Applying Theorem 3.7 we obtain

$$\|V(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \le C(1+t)^{-3k_0/(2(k_0+1))} \|V^0\|_{\mathbb{L}^p_{s_1+s_0}(\mathbb{R}^3)} + C\int_0^t (1+t-r)^{-k_0/(2(k_0+1))} \|F(\tau,\cdot)\|_{\mathbb{L}^p_{,s_1+s_0}(\mathbb{R}^3)}, \tag{6.22}$$

where

$$p = \frac{2k_0+2}{2k_0+1}, \qquad s_0 = \left[\frac{3k_0}{k_0+1}\right],$$

$$F = \sum_{j,k=1}^{3} a_{jk}(\nabla\Theta)\partial_j\partial_k u + 2\sum_{j=1}^{3} a_{j0}(\nabla\Theta)\partial_j\partial_t u.$$

By familiar calculus inequalities (cf. [79]) and using the Leibniz formula, the Hölder inequality and taking into account our specific assumption on F, we have

$$\|F(V,DV)(\tau)\|_{\mathbb{L}^p_{,s_1+s_0}(\mathbb{R}^3)} \le C\|V(\tau)\|^{k_0}_{\mathbb{L}^{2k_0+2}(\mathbb{R}^3)}\|V(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)}. \tag{6.23}$$

In view of (6.23) and (6.22) we get

$$\begin{aligned}\|V(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} &\le C(1+t)^{-3k_0/(2(k_0+1))}\|V^0\|_{\mathbb{L}^p_{,s_1+s_0}(\mathbb{R}^3)}\\ &\quad + C\int_0^t (1+t-\tau)^{-3k_0/(2(k_0+1))}\|V(\tau,\cdot)\|^{k_0}_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)}\|V(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)}\,d\tau.\end{aligned} \tag{6.24}$$

Applying the energy estimate (5.9) and taking into account the fact that

$$\|V(\tau)\|_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)} < \|V(\tau)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \quad \text{for } \tau\in[0,\zeta], \tag{6.25}$$

which follows from the imbedding $\mathbb{W}^{s_1,2k_0+2}(\mathbb{R}^3) \subseteq \mathbb{C}^{\lceil s/2\rceil}(\mathbb{R}^3)$, we get

$$\|V(\tau)\|_{\mathbb{H}^s(\mathbb{R}^3)} \le C_s\|V^0\|_{\mathbb{H}^s(\mathbb{R}^3)}\exp\{C_s\mathcal{M}^{k_0}_{s_1}(T)\} \quad \text{for } 0\le\tau\le T. \tag{6.26}$$

Inserting (6.26) into (6.24) with $\|V^0\|_{\mathbb{L}^p_{,s_1+s_0}(\mathbb{R}^3)}$ and $\|V^0\|_{\mathbb{H}^s(\mathbb{R}^3)}$ replaced by δ, we obtain

$$\begin{aligned}\|V(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} &\le C_s\delta(1+t)^{-3k_0/(2(k_0+1))}\\ &\quad + C_s\delta\exp\{C_s\mathcal{M}^{k_0}_{s_1}(T)\}\int_0^t (1+t-\tau)^{-3k_0/(2(k_0+1))}\|V(\tau)\|^{k_0}_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)}\,d\tau.\end{aligned} \tag{6.27}$$

Using once more the fact that

$$\|V(\tau)\|_{\mathbb{L}^\infty_{,\lfloor s/2\rfloor}(\mathbb{R}^3)}\|V(\tau)\|^{k_0}_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \le \mathcal{M}^{k_0}_{s_1}(T)(1+\tau)^{-3k_0^2/(2(k_0+1))}, \tag{6.28}$$

going back to (6.27), multiplying by $(1+t)^{3k_0/(2(k_0+1))}$, after some calculations we get

$$\mathcal{M}_{s_1}(\zeta) \le C_s\delta + C_s\delta\mathcal{M}^{k_0}_{s_1}(\zeta)\exp\{C_s\mathcal{M}^{k_0}_{s_1}(\zeta)\}.$$

Acting in the same way as in the proof of Theorem 6.1 we have $\mathcal{M}_{s_1}(\zeta) \le \mathcal{M}_0$.

The last step concludes the a priori estimate in the norm $\|\cdot\|_{\mathbb{H}^s(\mathbb{R}^3)}$.

THEOREM 6.4. *Let* $s\ge 10$ *be an integer*, *let*

$$V^0 = (\Theta^1, D\Theta^0)^* \in \mathbb{H}^s(\mathbb{R}^3)\cap\mathbb{W}^{s,p}(\mathbb{R}^3)$$

(*where* $p = (k_0+2)/(k_0+1)$, $k_0\ge 2$) *and let* $V(t,x) = (\nabla\Theta(t,x))$ *be a solution to the problem* (4.12), (4.13) *on some interval* $[0,\zeta]$. *Then there exists a positive constant* $\delta_1 > 0$ *sufficiently small such that if*

$$\|V^0\|_{\mathbb{H}^s(\mathbb{R}^3)} + \|V^0\|_{\mathbb{W}^{s,p}(\mathbb{R}^3)} \le \delta,$$

then

$$\|V(t)\|_{\mathbb{H}^s(\mathbb{R}^3)} \le K_{s_0,k_0}\|V^0\|_{\mathbb{H}^s(\mathbb{R}^3)} \tag{6.29}$$

for all $t \in [0, \zeta]$.

Sketch of proof. Applying the energy estimate to the solution $V(t,x)$ of the problem (4.10), (4.11) and carrying out the same calculations as in the proof of Theorem 6.2 we get (6.29).

Therefore, we have an a priori estimate in the $\mathbb{H}^s$-norm of the solution to the problem (4.12)–(4.13); thus we can re-apply the local existence theorem to obtain the desired global solution. From (6.28) and the conclusion of Theorem 6.3 it follows that the global solution $V(t,x)$ has the estimate

$$\|V(t)\|_{\mathbb{L}^{2k_0+2}_{,s_1}(\mathbb{R}^3)} \le C(1+t)^{-3k_0/(2(k_0+1))} \quad \text{for all } t \ge 0. \tag{6.30}$$

From the Sobolev imbedding inequality (6.25) and (6.30) it follows that

$$\|V(t)\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C(1+t)^{-3k_0/(2(k_0+1))} \quad \text{for all } t \ge 0.$$

Applying Theorem 6.3 we have $\|V(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} \le C$ for all $t \ge 0$. This completes the proof of Theorem 6.4.

7. General remarks

In this paper we have proved (assuming a special structure of nonlinearities F) that the global solutions to initial value problems of the nonlinear hyperbolic partial differential equations describing thermoelastic medium exist when the initial data are small enough. The method which we have used consists in combining the classical local existence theorem in $\mathbb{L}^2$-norms with a priori estimates in appropriate $\mathbb{L}^p$-norms. Here, interpolations allow us to obtain decay estimates of solution to the linearized equation (with constant coefficients) in an appropriate $\mathbb{L}^p$-norm.

These $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates play a key role in discussing the global existence of solutions of the corresponding nonlinear equations.

The general scheme consists of the following steps $\mathbf{I}^\circ$–$\mathbf{V}^\circ$.

$\mathbf{I}^\circ$ Reduction of the initial value problem for nonlinear hyperbolic partial differential equations to an equivalent one for quasilinear symetric hyperbolic system of the first order:

$$A(U)\partial_0 U = \sum_{k=1}^{3} A^k(U)\partial_k U, \tag{7.1}$$

$$U(0,x) = U^0(x) \tag{7.2}$$

and applying the local existence and uniqueness results.

There is a local solution U of the quasilinear symmetric hyperbolic system of the first order on some time interval $[0,\vartheta]$, $\vartheta > 1$, with the following regularity:

$$U \in \mathbb{C}^0([0,\vartheta], \mathbb{W}^{s,2}) \cap \mathbb{C}^1([0,\vartheta], \mathbb{W}^{s-1,2}), \tag{7.3}$$

where $s \in \mathbb{N}$ is sufficiently large to guarantee a classical solution. The proof of local existence theorem is always a problem in itself. Here we have presented one method of the proof of a corresponding theorem for a nonlinear hyperbolic heat equation and a nonlinear hyperbolic system of partial differentail equations describing thermoelastic medium in three-dimensional space.

II° *High energy estimates*. The local solution U satisfies

$$\|U(t)\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} \leq C_s \|U^0\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} \exp\Big\{C_s \int_0^t \|U(\tau)\|^{k_0}_{\mathbb{L}^\infty_{,\lceil s/2\rceil}(\mathbb{R}^3)}\, d\tau\Big\}, \quad t \in [0,\vartheta], \tag{7.4}$$

where C_s depends only on s but not on ϑ or U^0 and $k_0 \geq 2$.

This inequality is proved by using a general formula for the derivative of order α ($|\alpha| = s$) for composite functions (cf. (2.37)).

III° *Decay of solutions to the linearized problem*. A solution U to the associated linear problem:

$$A(0)\partial_t U = \sum_{k=1}^{3} A^k(0)\partial_k U, \tag{7.5}$$

$$U(0,x) = U^0, \tag{7.6}$$

satisfies

$$\|U(t)\|_{\mathbb{L}^q(\mathbb{R}^3)} \leq c(1+t)^{-d}\|U^0\|_{\mathbb{L}^p_{,s_0}(\mathbb{R}^3)}, \tag{7.7}$$

where

$$2 \leq q \leq \infty, \quad 1/p + 1/q = 1, \quad c, d > 0$$

and $s_0 \in \mathbb{N}$ depend on q and on the space dimension n.

This $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimate was obtained using the matrix of fundamental solutions constructed with the help of the Radon transform (or with the help of fundamental solutions obtained by the Hörmander method).

IV° *Weighted a priori estimates*. The local solution satisfies

$$\sup(1+t)^{d_1}\|U(t)\|_{\mathbb{L}^q_{,s_1}(\mathbb{R}^3)} \leq M_0 < \infty, \tag{7.8}$$

where M_0 is independent of T, s_1 is sufficiently large, $q = q_1(k_0)$ is chosen appropriately for each problem and $d_1 = d(q)$ according to **II°**, provided that U^0 is sufficiently small (i.e. the norm of U^0 in Sobolev spaces is small enough).

In this step the information obtained in **III°** is exploited with the help of the classical formula

$$U(t,x) = R(t,\cdot) * U^0(x) + \int_0^t R(t-\tau,\cdot) * F(U,DU)(\tau,x)\, d\tau, \tag{7.9}$$

where $R(t,x)$ is a matrix with elements being linear combinations of the terms $\partial_l H_{jk}(t,x)$, $\partial_t H_{jk}(t,x)$ or $(\partial_t E(t,x))$, $(\partial_l E(t,x))$, respectively ($H_{jk}(t,x)$ is the matrix of fundamental solutions of the linearized hyperbolic system describing thermoelastic medium and $E(t,x)$ is a fundamental solution of the linearized hyperbolic heat equation) (cf. (3.21)).

V° *Final energy estimate*. The results in **II**° and **IV**° lead to the following a priori bound:

$$\|U(t)\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)} \leq K\|U^0\|_{\mathbb{W}^{s,2}(\mathbb{R}^3)}, \quad 0 \leq t \leq \vartheta, \tag{7.10}$$

$s \in \mathbb{N}$ being sufficiently large, U^0 being sufficiently small (in the sense of Sobolev norms) and K being independent of ϑ. Such an a priori estimate allows us to apply now the standard continuation argument and to continue the local solution obtained in step **I**° to one defined for all $t \in [0, \infty)$.

The method presented above gives us some information on the asymptotic behaviour of the global solution as $t \to \infty$ in steps **IV**° and **V**°.

As we have shown above the main role in the proof of existence of global-in-time solutions of the nonlinear systems of partial differential equations is played by the $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimate for the solution to the initial value problem for the corresponding linear hyperbolic system of partial differential equations.

We obtained such $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimates constructing the matrix of fundamental solution with the help of the Radon transform or constructing the fundamental solution with the help of Hörmander's theorem.

The Radon transform can be applied to construct matrices of fundamental solutions to other systems of partial differential equations with constant coefficients, for example:

1) a system of partial differential equations describing microelasticity theory,

2) a system of partial differential equations describing elasticity theory,

3) a hyperbolic system of partial differential equations describing the so-called nonsimple thermoelastic mediums.

Remark 7.1. The matrix of fundamental solutions constructed with the help of the Radon transform can be applied to prove the existence of "almost global solutions" to the initial value problem for nonlinear hyperbolic thermoelasticity. This is very important in view of the fact that in this case the nonlinear right hand side F describes the so-called "geometrical nonlinearity" (cf. [34]).

The method of construction of fundamental solutions basing on Hörmander's theorem can be applied to construct fundamental solutions for equations describing mediums in continuum mechanics. Basing on the constructed fundamental solutions we can get an $\mathbb{L}^p$-$\mathbb{L}^q$ time decay estimate for solutions to the linearized problem associated with the nonlinear problem.

Many mediums in continuum mechanics are also described by coupled hyperbolic-parabolic systems of partial differential equations. Such is the case of thermodiffusion in a solid body (cf. [36], [31], [99], [100]). In order to obtain a matrix of fundamental solutions of a linear hyperbolic-parabolic system we can apply the Hilbert–Levy method (cf. [32], [30]).

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