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On the reflexivity of multigenerator algebras

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1. Introduction

In what follows we will deal with N-tuples of commuting linear bounded Hilbert space operators. The main problem which will be considered is:

(RP) Reflexivity Problem: is the lattice of all common invariant subspaces for the N-tuple so rich that it determines the algebra generated by the N-tuple in the sense that any operator leaving invariant all subspaces which are invariant for the N-tuple necessarily belongs to the smallest closed (in the weak operator topology) algebra containing the N-tuple and the identity?

It is straightforward that if the Reflexivity Problem has an affirmative answer then the N-tuple has a great deal of invariant subspaces. Hence the following problem also has an affirmative answer.

(ISP) Existence of a Common Non-trivial Invariant Subspace: is there a non-trivial closed subspace invariant for all operators of the N-tuple?

The second motivation for studying the Reflexivity Problem comes from von Neumann algebras. The commutant of any von Neumann algebra is generated by all projections in the commutant. Since a von Neumann algebra is self-adjoint, its commutant is generated by all its reducing projections. Thus, considering an N-tuple of operators and the smallest (non-self-adjoint) weak operator topology closed algebra generated by them, it is natural to consider the set of all invariant subspaces (invariant projections) instead of the set of reducing projections. For a von Neumann algebra, we consider the double commutant, the set of all operators which commute with all operators from the commutant, in other words, which commute with all projections that reduce all operators from the given von Neumann algebra. Moreover, the well-known double commutant theorem shows that the double commutant of a von Neumann algebra is equal to the algebra itself.

Thus, in the non-self-adjoint case, we can consider all operators which leave invariant all subspaces which are invariant for a given N-tuple. Now one can ask whether such an operator belongs to the algebra generated by the given N-tuple. In some sense we are asking whether this algebra satisfies the non-self-adjoint version of the double commutant theorem.

On the other hand, it is common to investigate an algebra by analyzing the properties of the commutant. Generally, the algebra generated by a commuting N-tuple (the smallest, closed (in the weak operator topology) algebra containing the N-tuple and the identity) is contained in the commutant of the given N-tuple. In some cases it is equal to its commutant. Thus, the following natural, more general problem arises:

(HRP) Hyporeflexivity Problem: is the intersection of the commutant of a commuting N-tuple and the set of operators leaving invariant all subspaces which are invariant for the N-tuple, equal to the smallest, closed (in the weak operator topology) algebra containing the N-tuple and the identity?

We can also extend the Reflexivity Problem from algebras generated by N-tuples of operators to subspaces of operators. For example, it is possible to study the reflexivity of some subspaces of von Neumann algebras generated by N-tuples.

1.1. Basic definitions: invariant subspace, reflexivity, hyporeflexivity. Throughout this paper we mostly deal with bounded operators on a finite-dimensional or separable infinite-dimensional complex Hilbert space $\mathcal{H}$. Let $\mathcal{S}$ be a commutative family of operators acting on a common Hilbert space $\mathcal{H}$. Then we denote by $\mathcal{W}(\mathcal{S})$ (respectively, $\mathcal{A}(\mathcal{S})$) the WOT(= weak operator topology)-closed (respectively, weak-star closed) subalgebra of $L(\mathcal{H})$ generated by $\mathcal{S}$ and the identity I. By $\mathcal{S}'$ we denote the commutant of $\mathcal{S}$. A subspace $L \subset \mathcal{H}$ is called *invariant* (respectively, *hyperinvariant*) for the family $\mathcal{S}$ if $TL \subset L$ for all $T \in \mathcal{S}$ (respectively, for all operators $T \in \mathcal{S}'$). $\operatorname{Lat}\mathcal{S}$ will be the lattice of all (closed) invariant subspaces for $\mathcal{S}$, and $\operatorname{Alg}\operatorname{Lat}\mathcal{S}$ is as usual the algebra of all $T \in L(\mathcal{H})$ such that $\operatorname{Lat}\mathcal{S} \subset \operatorname{Lat} T$. The inclusion $\mathcal{W}(\mathcal{S}) \subset \operatorname{Alg}\operatorname{Lat}\mathcal{S}$ is obvious.

Let $\mathcal{W}$ be a WOT-closed algebra of operators containing the identity I. The algebra $\mathcal{W}$ is said to be *reflexive* if it is determined by its lattice of invariant subspaces in the sense that $\mathcal{W} = \operatorname{Alg}\operatorname{Lat}\mathcal{W}$. A single operator T is called *reflexive* if the operator algebra $\mathcal{W}(T)$ generated by T is reflexive. A family $\mathcal{S}$ of operators is said to be *reflexive* if the algebra $\mathcal{W}(\mathcal{S})$ is reflexive.

Let us recall an extension of the reflexivity concept originally due to A. I. Loginov and V. I. Sulman [LSu]. The *reflexive closure* of an operator space $\mathcal{S} \subset L(\mathcal{H})$ is defined by $\operatorname{Ref}\mathcal{S} = \{T \in L(\mathcal{H}) : Tx \in \overline{\mathcal{S}x} \text{ for all } x \in \mathcal{H}\}$. The space $\mathcal{S}$ is called *reflexive* if $\mathcal{S} = \operatorname{Ref}\mathcal{S}$. When $\mathcal{S}$ is an algebra with identity, then $\operatorname{Ref}\mathcal{S} = \operatorname{Alg}\operatorname{Lat}\mathcal{S}$, and the above notion of reflexivity reduces to the classical concept.

If $\mathcal{W}$ is a commutative WOT-closed algebra of operators containing the identity I, it is clear that $\mathcal{W} \subset \mathcal{W}' \cap \operatorname{Alg}\operatorname{Lat}\mathcal{W}$. If $\mathcal{W} = \mathcal{W}' \cap \operatorname{Alg}\operatorname{Lat}\mathcal{W}$ the algebra $\mathcal{W}$ is called *hyporeflexive*. An operator T (a commutative family $\mathcal{S} \subset L(\mathcal{H})$, respectively) is called *hyporeflexive* if the operator algebra $\mathcal{W}(T)$ ($\mathcal{W}(\mathcal{S})$, respectively) is hyporeflexive. Note that in the commutative case reflexivity implies hyporeflexivity.

Denote by $\mathcal{C}_1(\mathcal{H})$ the ideal of trace-class operators on $\mathcal{H}$. Recall that $L(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$ and the duality is given by the form $\langle T, t\rangle := \operatorname{tr}(Tt)$ for $T \in L(\mathcal{H})$ and $t \in \mathcal{C}_1(\mathcal{H})$. If $\mathcal{S} \subset L(\mathcal{H})$, then $\mathcal{S}_\perp = \{t : \langle T, t\rangle = 0 \text{ for all } T \in \mathcal{S}\} \subset \mathcal{C}_1(\mathcal{H})$ denotes the preannihilator of $\mathcal{S}$. We denote by $\mathbf{F}_k$ the set of all operators of rank k.

For $x, y \in \mathcal{H}$, we denote by $x \otimes y$ the operator $(x \otimes y)(z) = (z, y)x$. This is the zero operator if $x = 0$ or $y = 0$. Otherwise $x \otimes y$ has rank one, and it is clear that x spans its range, while its kernel consists of all vectors orthogonal to y. Moreover, every rank-one operator has this form. One can easily see that $\langle A, x \otimes y\rangle = (Ax, y)$ for $A \in L(\mathcal{H})$ and $x, y \in \mathcal{H}$.

A weak-star closed subspace of operators $\mathcal{S}$ *has property* $\mathbb{A}_1(1)$ if for every weak-star continuous linear functional $t \in \mathcal{C}_1(\mathcal{H})$ and $\varepsilon > 0$, there are $x, y \in \mathcal{H}$ such that $\|x\| \cdot \|y\| \le (1+\varepsilon)\|t\|$ and $\langle A, t\rangle = (Ax, y)$ for all $A \in \mathcal{S}$. A family $\mathcal{S}$ of operators is said to have *property* $\mathbb{A}_1(1)$ if the algebra $\mathcal{W}(\mathcal{S})$ has property $\mathbb{A}_1(1)$.

We say that a commutative family $\mathcal{S} \subset L(\mathcal{H})$ is *doubly commuting* if $ST^* = T^*S$ for all $S, T \in \mathcal{S}$ with $S \neq T$. In particular, families consisting of a single operator are doubly commuting.

Recall also that an operator $A \in L(\mathcal{H})$ is called *nilpotent* if there is a positive integer n such that $A^n = 0$. By the *order* of A, we then mean the smallest positive integer n such that A^n is the zero operator.

We will also need some notation from function theory in the polydisc $\mathbb{D}^N$ and the unit ball $\mathbb{B}^N$ in $\mathbb{C}^N$. We denote by $A(\mathbb{D}^N)$ the algebra of all functions holomorphic on $\mathbb{D}^N$ and continuous on $\overline{\mathbb{D}^N}$, and call it the *polydisc algebra*. We denote by $H^2(\mathbb{D}^N)$ and $H^2(\mathbb{B}^N)$ the Hardy spaces. $H^\infty(\mathbb{D}^N)$ (respectively, $H^\infty(\mathbb{B}^N)$) stands for the algebra of all bounded holomorphic functions on $\mathbb{D}^N$ (respectively, on $\mathbb{B}^N$). If $N = 1$ we write A, H^2, H^∞ for $A(\mathbb{D}^1), H^2(\mathbb{D}^1), H^\infty(\mathbb{D}^1)$, respectively. By $\mathbb{T}^N$ we denote the polytorus.

1.2. Basic theorems and examples In this section we present some basic theorems and examples concerning reflexivity. The first result was proved in [Sa].

THEOREM 1.2.1. *Any commutative algebra of normal operators is reflexive.*

The following example shows that even in the finite-dimensional case the problem is non-trivial.

EXAMPLE 1.2.2. Consider the algebras

$$\mathcal{W}_1 = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{C} \right\}, \quad \mathcal{W}_2 = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus [\, a\,] : a, b \in \mathbb{C} \right\}.$$

One can easily see that $\mathcal{W}_1$ is not reflexive, but $\mathcal{W}_2$ is reflexive.

The reflexive nilpotents in finite-dimensional spaces are completely characterized in [DF]:

THEOREM 1.2.3. *A nilpotent operator in a finite-dimensional space is reflexive if and only if the two largest blocks in its Jordan decomposition differ no more than one in size.*

In [BF] the following was shown:

THEOREM 1.2.4. *Let T be a linear operator in a finite-dimensional space. Then T is hyporeflexive, i.e.,* $\mathcal{A}(T) = \{T\}' \cap \operatorname{Alg}\operatorname{Lat} T$.

In [Sa] the following was also shown:

THEOREM 1.2.5. *The shift operator is reflexive* ($(Sf)(z) = zf(z)$ *for* $f \in H^2$).

This result was extended in [De].

THEOREM 1.2.6. *Every isometry is reflexive.*

The following example shows that the question of reflexivity for N-tuples of operators is non-trivial.

Example 1.2.7. There is a pair $\{T_1, T_2\} \subset L(\mathcal{H})$ of commuting operators such that both T_1 and T_2 are reflexive, but $\mathcal{W}(T_1, T_2)$ is not reflexive.

Consider $\mathcal{H} = \mathbb{C}^2 \oplus \mathbb{C}^2$ and define

$$T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is straightforward that T_1 and T_2 commute. Moreover, T_1 is reflexive by Theorem 1.2.3 and T_2 is reflexive as a normal operator (see Theorem 1.2.1). It is easy to see that

$$\mathcal{W}(T_1, T_2) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\}.$$

On the other hand, one can check that

$$\begin{aligned} \operatorname{Lat}(T_1T_2, I - T_2) = \{L_1 \oplus L_2 : L_1 \in \{\{0\} \oplus \{0\}, \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C}\} \\ \text{and } L_2 \text{ is any subspace of } \mathbb{C}^2\}. \end{aligned}$$

Similarly,

$$\begin{aligned} \operatorname{Lat}(T_1(I - T_2), T_2) = \{L_1 \oplus L_2 : L_1 \text{ is any subspace of } \mathbb{C}^2 \\ \text{and } L_2 \in \{\{0\} \oplus \{0\}, \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C}\}\}. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Lat}(T_1, T_2) &\subset \operatorname{Lat}(T_1T_2, I - T_2) \cap \operatorname{Lat}(T_1(I - T_2), T_2) \\ &= \{\{0\} \oplus \{0\} \oplus \{0\} \oplus \{0\}, \{0\} \oplus \{0\} \oplus \mathbb{C} \oplus \{0\}, \{0\} \oplus \{0\} \oplus \mathbb{C} \oplus \mathbb{C}, \\ &\quad \mathbb{C} \oplus \{0\} \oplus \{0\} \oplus \{0\}, \mathbb{C} \oplus \{0\} \oplus \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \{0\} \oplus \mathbb{C} \oplus \mathbb{C}, \\ &\quad \mathbb{C} \oplus \mathbb{C} \oplus \{0\} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\}. \end{aligned}$$

The reverse inclusion is trivial. Now it is easy to see that

$$\operatorname{Alg}\operatorname{Lat}(T_1, T_2) = \left\{ \begin{bmatrix} a & b \\ 0 & s \end{bmatrix} \oplus \begin{bmatrix} c & d \\ 0 & t \end{bmatrix} : a, b, c, d, s, t \in \mathbb{C} \right\}.$$

Hence $\operatorname{Alg}\operatorname{Lat}(T_1, T_2)$ is larger than $\mathcal{W}(T_1, T_2)$.

We finish this section with a proposition from [Az], which will have substantial applications later.

Proposition 1.2.8. *Every one-dimensional operator subspace $\mathcal{S} \subset L(\mathcal{H})$ $(\dim \mathcal{S} = 1)$ is reflexive.*

Note. In Section 1.2, basic well known results were quoted. Example 1.2.7 is new.

2. N-tuples of linear transformations in a finite-dimensional space

In this chapter we present some results for N-tuples of operators in a finite-dimensional Hilbert space (it will usually be denoted by V). The term "linear transformation" instead of the term "operator" will be used. We use small letters to denote linear transformations. In this chapter we deal with doubly commuting N-tuples of linear transformations. We quote some results for a single linear transformation. Although, we deal with Hilbert spaces most of them remain true for any finite-dimensional space.

In Section 2.1 we present a model for doubly commuting N-tuples of nilpotents which will be used in the whole chapter.

The next section deals with the hyporeflexivity problem. The hyporeflexivity of a linear transformation in a finite-dimensional space was proved in [BF]. We show that it holds for N-tuples of doubly commuting linear transformations in a finite-dimensional Hilbert space.

In Section 2.3 some results on the reflexivity of direct sums are presented. As quoted in Theorem 1.2.3, Deddens and Fillmore completely characterized reflexive nilpotents, namely: A nilpotent, in a finite-dimensional space is reflexive if and only if the two largest blocks in the Jordan decomposition differ no more than one in size.

The drawback of this characterization is that it depends on block sizes. In Section 2.5 we present a necessary and sufficient condition for the reflexivity of doubly commuting N-tuples of nilpotents using ranks of operators. The condition also holds for a single linear transformation. Section 2.6 gives a block size characterization analogous to the case of single linear transformations. The non-nilpotent case is presented in Section 2.7.

2.1. Model for doubly commuting nilpotents In this section we construct a model for N-tuples of doubly commuting nilpotents in a finite-dimensional Hilbert space. The model will be helpful in investigating properties of algebras generated by such N-tuples. We start with the following well-known fact which will be a motivation for the definition of simple N-tuples.

PROPOSITION 2.1.1. *Let $a \in L(V)$ be nilpotent. Then the following are equivalent*:

(1) *the trivial operators* 0 *and* I *are the only idempotents commuting with* a,
(2) *the Jordan form of* a *is a single block*,
(3) $\mathcal{A}(a)$ *has a cyclic vector.*

An N-tuple $\mathbf{a} = (a_1, \ldots, a_N)$ of doubly commuting nilpotents is called *simple* if no non-trivial idempotent commutes with all of them.

EXAMPLE 2.1.2. Let $\widetilde{a}_i$ act on a Hilbert space V_i for $i = 1, \ldots, N$ and form the tensor product space $V = V_1 \otimes \ldots \otimes V_N$. Define $a_i = I \otimes \ldots \otimes \widetilde{a}_i \otimes \ldots \otimes I$. Then $\mathbf{a} = (a_1, \ldots, a_N)$ is doubly commuting.

In order for $\mathbf{a} = (a_1, \ldots, a_N)$ to be simple, it is necessary that the von Neumann algebras generated by the $\widetilde{a}_i$ are factors. The condition fails to be sufficient, even when $N = 1$, not only because commuting projections may fail to be central, but also because non-self-adjoint idempotents must be taken into account.

PROPOSITION 2.1.3. *Let $\mathbf{a} = (a_1, \ldots, a_N)$ be a doubly commuting N-tuple of nilpotents acting on a Hilbert space V. Then the following are equivalent*:

(1) $\mathbf{a} = (a_1, \ldots, a_N)$ *is simple*,
(2) $\mathbf{a} = (a_1, \ldots, a_N)$ *takes the form of Example* 2.1.2 *with each* $\widetilde{a}_i$ *being simple*,
(3) $\mathcal{A}(a_1, \ldots, a_N)$ *has a cyclic vector.*

Proof. (1)⇒(2). The precise meaning of (2) involves a unitary map between the underlying Hilbert spaces. We argue by induction. If $N = 1$, there is nothing to prove. If $N \geq 2$, then the von Neumann algebra $\mathcal{N}(a_1)$ generated by a_1 must be a type I factor. Thus, there are Hilbert spaces V_1 and K, and a unitary map $U : V \to V_1 \otimes K$ such that $U\mathcal{N}(a_1)U^{-1} = L(V_1) \otimes \mathbb{C}I_K$. There is no harm in suppressing U and assuming $\mathcal{N}(a_1) = L(V_1) \otimes \mathbb{C}I_K$, whence $a_1 = \widetilde{a}_1 \otimes I_K$. By double commutativity, the von Neumann algebra $\mathcal{N}(a_2, \dots, a_N)$ generated by $a_2, \dots, a_N$ is contained in $I_{V_1} \otimes L(K)$. In particular, $\widetilde{a}_1$ must be simple since $q \otimes I_K$ will commute with each a_i whenever q commutes with $\widetilde{a}_1$. The decomposition is completed by applying the inductive hypothesis to $\mathcal{N}(a_2, \dots, a_N)$.

(2)⇒(3). For each $i = 1, \dots, N$, choose a cyclic vector x_i for $\mathcal{A}(\widetilde{a}_i)$. Then $x = x_1 \otimes \dots \otimes x_N$ is a cyclic vector for $\mathcal{A}(a_1, \dots, a_N)$.

(3)⇒(1). Suppose q is an idempotent commuting with $a_1, \dots, a_N$ and x is a cyclic vector for $\mathcal{A}(\mathbf{a})$. Then there is $c \in \mathcal{A}(\mathbf{a})$ with $qx = cx$. But then $\ker(q - c)$ contains all of $\mathcal{A}(\mathbf{a})x$, so $q = c$ belongs to $\mathcal{A}(\mathbf{a})$. Since 0 is the only operator which is simultaneously idempotent and nilpotent, we conclude that q is either 0 or I, as desired.

Suppose $\mathbf{a} = (a_1, \dots, a_N)$ and $\mathbf{b} = (b_1, \dots, b_N)$ are N-tuples of operators acting on V, W, respectively. Then we say that $\mathbf{a}$ is *similar* to $\mathbf{b}$ if there is an invertible operator $s \in L(V, W)$ satisfying $b_i = s a_i s^{-1}$ for $i = 1, \dots, N$.

Recall the well-known Jordan Theorem.

Theorem 2.1.4. *Every nilpotent is similar to an orthogonal sum of simple nilpotents.*

Now we give a multioperator version of the above result. In general, similarities destroy double commutativity. Note, however, that tensor products of similarities on the underlying spaces V_i preserve the double commutativity of the operators a_i of Example 2.1.2. This will be important in the following proof.

Proposition 2.1.5. *Every N-tuple $\mathbf{a} = (a_1, \dots, a_N)$ of doubly commuting nilpotents is similar to an orthogonal direct sum of simple N-tuples. More precisely, there is a doubly indexed family $\{a_{ij} : i = 1, \dots, N; j = 1, \dots, k\}$ of nilpotents such that*

(1) *$(a_{1j}, \dots, a_{Nj})$ is a simple N-tuple for each fixed j, and*

(2) *the original operators a_i are simultaneously similar to the orthogonal direct sums $\bigoplus_{j=1}^{k} a_{ij}$.*

Proof. The von Neumann algebras $\mathcal{N}(a_1), \dots, \mathcal{N}(a_N)$ commute, so any self-adjoint projection in the center of one of them will automatically commute with all of them. Making a preliminary orthogonal decomposition we may thus assume all the $\mathcal{N}(a_i)$ to be factors. But then the proof of Proposition 2.1.3((1)⇒(2)) allows us to write $V = V_1 \otimes \dots \otimes V_N$ and $a_i = I \otimes \dots \otimes \widetilde{a}_i \otimes \dots \otimes I$ except that the $\widetilde{a}_i$ need not be simple. Now the Jordan Canonical Form Theorem tells us that each $\widetilde{a}_i$ is similar to an orthogonal direct sum of simple operators. Putting these similarities together, we can assume that the $\widetilde{a}_i$ are themselves orthogonal direct sums of simple operators. The proof is completed by "splitting" these direct sums.

2.2. Hyporeflexivity. If $\mathcal{A}$ is any commutative algebra then $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A} \subset \operatorname{Alg}\operatorname{Lat}\mathcal{A}$. Generally, we cannot expect equality in either inclusion even if the algebra

$\mathcal{A}$ is singly generated. In [BF], it was shown that $\mathcal{A}(a) = \operatorname{Alg}\operatorname{Lat}\mathcal{A}(a) \cap \mathcal{A}(a)'$ for any linear transformation $a \in L(V)$ in a finite-dimensional space V. It was proved in [Hw] that $\mathcal{A}(a_1, a_2) = \mathcal{A}(a_1, a_2)' \cap \operatorname{Alg}\operatorname{Lat}\mathcal{A}(a_1, a_2)$ for any pair of linear transformations $a_1, a_2 \in L(V)$ if $\dim V \leq 3$. By 2×2 and 3×3 matrix manipulations the result can be shown for an N-tuple of linear transformations instead of a pair. If $\dim V \geq 4$ then the above equality is not true in general (see [Hw]). Here we consider a doubly commuting N-tuples and prove

THEOREM 2.2.1. *Assume that* $\mathbf{a} = (a'_1, \ldots, a'_N) \subset L(V)$ *is an* N*-tuple of doubly commuting nilpotents on the finite-dimensional Hilbert space* V*. Then* $\mathbf{a}$ *is hyporeflexive*, *i.e.*, $\mathcal{A}(\mathbf{a}) = \mathcal{A}(\mathbf{a})' \cap \operatorname{Alg}\operatorname{Lat}\mathcal{A}(\mathbf{a})$.

PROOF. For simplicity of notation we present the proof for $N = 2$. Let $\mathbf{a} = (a, b)$. By Proposition 2.1.5 we represent as $a = \bigoplus_{i=1}^k a_i$, $b = \bigoplus_{i=1}^k b_i$, where each pair (a_i, b_i) is a simple pair on V_i, $i = 1, \ldots, k$ and $V = V_1 \oplus \ldots \oplus V_k$. Denote by m_i and n_i the orders of a_i and b_i, respectively. Take $c \in \operatorname{Alg}\operatorname{Lat}\mathcal{A}(a, b) \cap \mathcal{A}(a, b)'$. Then V_i and $V \ominus V_i$ are invariant for a and b, hence also invariant for c. Therefore $c = \bigoplus_{i=1}^k c_i$.

Now, for any i, the pair (a_i, b_i) is simple, thus the algebra $\mathcal{A}(a_i, b_i)$ has a cyclic vector x_i by Proposition 2.1.3. Hence, for some polynomial q_i we have $c_i x_i = q_i(a_i, b_i)x_i = \sum_{s=0}^{m_i-1}\sum_{t=0}^{n_i-1} \alpha_{st} a_i^s b_i^t x_i$. Since any vector $y \in V_i$ can be represented as $y = r(a_i, b_i)x_i$ for some polynomial r, by commutativity we have

$$\begin{aligned} cy &= cr(a_i, b_i)x_i = cr(a, b)x_i = r(a, b)cx_i = r(a, b)q_i(a_i, b_i)x_i \\ &= q_i(a_i, b_i)r(a_i, b_i)x_i = q_i(a_i, b_i)y. \end{aligned}$$

Take i and j with $i \neq j$, and write $c_i x_i = q_i(a_i, b_i)x_i = \sum_{s=0}^{m_i-1}\sum_{t=0}^{n_i-1} \alpha_{st} a_i^s b_i^t x_i$, and also $c_j x_j = q_j(a_j, b_j)x_j = \sum_{s=0}^{m_j-1}\sum_{t=0}^{n_j-1} \beta_{st} a_j^s b_j^t x_j$. To finish the proof we need to show that $\alpha_{st} = \beta_{st}$ for $s < \min(m_i, m_j)$ and $t < \min(n_i, n_j)$. The space generated by $x_i + x_j$ is invariant for a and b, thus also invariant for c. Hence, for some polynomial s, we can write

$$c(x_i + x_j) = s(a, b)(x_i + x_j) = \sum_{s=0}^{\overline{m}-1}\sum_{t=0}^{\overline{n}-1} \gamma_{st} a^s b^t (x_i + x_j),$$

where $\overline{m} = \max(m_i, m_j)$ and $\overline{n} = \max(n_i, n_j)$. Since V_i, V_j are orthogonal and invariant for a, b, c, we get

$$c_i x_i = cx_i = s(a, b)x_i = \sum_{s=0}^{\overline{m}-1}\sum_{t=0}^{\overline{n}-1} \gamma_{st} a^s b^t x_i = \sum_{s=0}^{m_i-1}\sum_{t=0}^{n_i-1} \gamma_{st} a_i^s b_i^t x_i,$$

since $a_i^s = 0$ for $s \geq m_i$ and $b_i^t = 0$ for $t \geq n_i$. Thus

$$\sum_{s=0}^{m_i-1}\sum_{t=0}^{n_i-1} \alpha_{st} a_i^s b_i^t x_i = \sum_{s=0}^{m_i-1}\sum_{t=0}^{n_i-1} \gamma_{st} a_i^s b_i^t x_i.$$

By Proposition 2.1.3 and Example 2.1.2 it is easy to see that the vectors in $\{a_i^s b_i^t x_i : s = 0, 1, \ldots, m_i - 1, t = 0, 1, \ldots, n_i - 1\}$ are linearly independent. Thus $\alpha_{st} = \gamma_{st}$ for $s < m_i$ and $t < n_i$. In the same way we can prove that $\beta_{st} = \gamma_{st}$ for $s < m_j$ and $t < n_j$. Thus $\alpha_{st} = \beta_{st}$ for $s < \min(m_i, m_j)$ and $t < \min(n_i, n_j)$, hence $c \in \mathcal{A}(a, b)$.

Now consider an algebra $\mathcal{A}$ generated by doubly commuting linear transformations in a finite-dimensional Hilbert space. If $\mathcal{A}$ does not contain any non-trivial idempotent, then we are exactly in the situation of the above theorem. If $\mathcal{A}$ contains some non-trivial idempotents, then we decompose the algebra as an orthogonal sum of algebras which contain only trivial idempotents. We have the desired equality for each component by Theorem 2.2.1. Since $\mathcal{A}$ splits into a sum of algebras, the commutant $\mathcal{A}'$, the lattice $\operatorname{Lat}\mathcal{A}$ of invariant subspaces and $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$ also split. Thus we have shown

THEOREM 2.2.2. *Assume that $\mathcal{A} \subset L(V)$ is an algebra generated by doubly commuting linear transformations in a finite-dimensional Hilbert space V. Then $\mathcal{A} = \mathcal{A}' \cap \operatorname{Alg}\operatorname{Lat}\mathcal{A}$.*

2.3. Reflexivity of subdirect sums. It is easy to see that a full direct sum $\mathcal{S} = \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_k$ of operator spaces is reflexive if and only if each summand is reflexive, but in general there is no relationship between the reflexivity of $\mathcal{S}$ and of its various subspaces. This is unfortunate, since algebras generated by direct sums of operators are usually not full direct sums of these algebras. Proposition 2.3.2 below provides a tool for dealing with this situation.

A vector $x \in V$ is called *separating* for a subspace $\mathcal{S} \subset L(V)$ if the map $s \mapsto sx$ is injective on $\mathcal{S}$. It is easy to see that the existence of separating vectors survives the taking of direct sums. It follows that each singly generated algebra has a separating vector. In particular, the existence of separating vectors does not guarantee the reflexivity. We do, however, have the following basic result; see [Az, Propositions 2.9, 3.2] for a proof.

PROPOSITION 2.3.1. *Suppose a subspace $\mathcal{S}$ of $L(V)$ is reflexive and has a separating vector. Then every subspace of $\mathcal{S}$ is reflexive.*

The following proposition will be used in the next section. An operator $a = a_1 \oplus \ldots \oplus a_k$ in $L(V_1) \oplus \ldots \oplus L(V_k)$ is said to be *supported* on V_i if $a_j = 0$ for all $j \neq i$.

PROPOSITION 2.3.2. *For each $i = 1, \ldots, k$, let $\mathcal{S}_i \subset L(V_i)$ be an operator space with a separating vector x_i. Suppose $\mathcal{T}$ is a subspace of $\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_k$, and for each i, write $\mathcal{T}_i = \{a \in \mathcal{T} : a \text{ is supported on } V_i\}$. Then $\mathcal{T}$ is reflexive if and only if each $\mathcal{T}_i$ is reflexive.*

PROOF. We assume $k \geq 2$ to avoid trivialities. Note first that $x = x_1 \oplus \ldots \oplus x_k$ is a separating vector for $\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_k$. For the necessity, observe that x must also separate $\mathcal{T}$, hence the reflexivity of each $\mathcal{T}_i$ follows from that of $\mathcal{T}$ by Proposition 2.3.1.

For the sufficiency, suppose $c = c_1 \oplus \ldots \oplus c_k \in \operatorname{Ref}\mathcal{T}$. Since $cx \in \mathcal{T}(x)$, by subtracting an appropriate member of $\mathcal{T}$ from c if necessary, we may assume that $cx = 0$. We will show that $c_1 = 0$, whence, by symmetry, we can conclude that $c = 0$ and the proof will be completed.

Write $\overline{c} = c_2 \oplus \ldots \oplus c_k$ and $\overline{x} = x_2 \oplus \ldots \oplus x_k$. Given $y \in V_1$, we must have $(c_1 \oplus \overline{c})(y \oplus \overline{x}) = (s_y \oplus t_y)(y \oplus \overline{x})$ for some operator $s_y \oplus t_y \in \mathcal{T}$. This is equivalent to the two conditions

$$0 = \overline{cx} = t_y \overline{x} \quad \text{and} \quad c_1 y = s_y y.$$

Since $\overline{x}$ is separating, we first see that $t_y = 0$, which means that $s_y \oplus 0 \in \mathcal{T}_1$. The arbitrariness of y thus yields $c_1 \oplus 0 \in \operatorname{Ref}\mathcal{T}_1$. Since $\mathcal{T}_1$ is reflexive, we have $c_1 \oplus 0 \in \mathcal{T}_1$. Since $x_1 \oplus 0$ separates $\mathcal{T}_1$ and $c_1 x_1 = 0$, we get $c_1 = 0$ as desired.

We conclude this section with a trivial consequence of Proposition 2.3.2.

COROLLARY 2.3.3. *If* a_1, a_2 *are nilpotents of the same order*, *then* $a_1 \oplus a_2$ *is reflexive.*

PROOF. Take $\mathcal{S}_i$ to be the algebra generated by a_i and $\mathcal{T}$ the algebra generated by $a_1 \oplus a_2$. Applying Proposition 2.3.2, note that $\mathcal{T}_1 = \mathcal{T}_2 = \{0\}$.

2.4. The rank-two operators In this section, we prove a necessary and sufficient condition for reflexivity of N-tuples of doubly commuting nilpotents in a finite-dimensional Hilbert space. An important role will be played by rank-two operators. We first recall two basic properties of nilpotent operators, which are true even in an infinite-dimensional space.

LEMMA 2.4.1. *Let* a *be nilpotent and suppose* c *is a non-zero operator of finite rank commuting with* a. *Then* $\operatorname{rank}(ac) < \operatorname{rank}(c)$.

PROOF. We know that $\operatorname{ran}(ac) \subseteq \operatorname{ran}(c)$. If this inclusion were not proper, we would have $\operatorname{ran}(a^n c) = \operatorname{ran}(c)$ for all n. But this is ruled out by the nilpotence of a.

Recall that every rank-one operator has the form $x \otimes y$ for non-zero $x, y \in V$.

LEMMA 2.4.2. *If* b *is a nilpotent commuting with the rank-one operator* $x \otimes y$, *then* $bx = 0$.

PROOF. Applying Lemma 2.4.1, we have $(bx) \otimes y = b(x \otimes y) = 0$ and the conclusion follows since $y \neq 0$.

Consider the simplest non-reflexive algebra $\left\{\left(\begin{smallmatrix}\lambda, & \mu \\ 0, & \lambda\end{smallmatrix}\right) : \lambda, \mu \in \mathbb{C}\right\}$. Note that the identity operator, which is of rank 2, does not generate a one-dimensional ideal.

PROPOSITION 2.4.3. *Let* $\mathcal{A} \subset L(V)$ *be a commutative algebra having a cyclic vector* x. *Then*

(1) x *is also a separating vector for* $\mathcal{A}$, *and*
(2) *for each* $c \in \mathcal{A}$, *the rank of* c *is equal to the dimension of the ideal* $c\mathcal{A}$.

PROOF. To see (1), note that by the commutativity of $\mathcal{A}$, if the cyclic vector x belongs to the kernel of an operator in $\mathcal{A}$, the whole space V is contained in that kernel. For (2), observe that the map $a \mapsto ax$ defines a vector space isomorphism between $\mathcal{A}$ and V; for each $c \in \mathcal{A}$ it maps the ideal $c\mathcal{A}$ to $\operatorname{ran}(c)$.

PROPOSITION 2.4.4. *Suppose* $\mathbf{a} = (a_1, \ldots, a_N)$ *is a simple* N*-tuple of doubly commuting nilpotents and* $c \in \mathcal{A}(\mathbf{a})$.

(1) *All rank-one members of* $\mathcal{A}(\mathbf{a})$ *are scalar multiples of one another.*
(2) *If* $\operatorname{rank}(c) = 2$, *then* $c\mathcal{A}(\mathbf{a})$ *is two-dimensional.*
(3) *If* $\operatorname{rank}(c) \geq 2$, *then* $c\mathcal{A}(\mathbf{a})$ *contains a member of rank two.*

PROOF. To establish (1), write n_i for the order of a_i, and set $\mathbf{a}^{\mathbf{k}} = a_1^{k_1} \ldots a_N^{k_N}$ for each N-tuple $\mathbf{k} = (k_1, \ldots, k_N)$ of natural numbers. Suppose $c = \sum \lambda_{\mathbf{k}} \mathbf{a}^{\mathbf{k}}$ has rank one. Lemma 2.4.1 tells us that $a_i c = 0$ for each i. But, in view of Proposition 2.1.3(2), the operators $\mathbf{a}^{\mathbf{k}}$ for all $\mathbf{k} = (k_1, \ldots, k_N)$ with $0 \leq k_i \leq n_i - 1$ are linearly independent. Thus, $\lambda_{\mathbf{k}} = 0$ whenever $k_i \leq n_i - 2$. This forces c to be a scalar multiple of $a_1^{n_1-1} \ldots a_N^{n_N-1}$.

Part (2) is a consequence of Propositions 2.1.3 and 2.4.3(2).

We prove (3) inductively. If $\operatorname{rank}(c) > 2$, Proposition 2.4.3(2) yields $\dim c\mathcal{A}(\mathbf{a}) > 2$. On the other hand, $\mathcal{A}(\mathbf{a})$ is spanned by its nilpotent members and I. Thus there are nilpotent members b, d of $\mathcal{A}(\mathbf{a})$ such that cb, cd are independent. By (1), at least one of these, say cd, has rank greater than one. Proposition 2.4.1 thus makes it possible to apply the inductive hypothesis to cd.

The following theorem gives a necessary and sufficient condition for the reflexivity of doubly commuting N-tuples of nilpotents in a finite-dimensional space.

THEOREM 2.4.5. *Suppose $\mathcal{A}$ is an operator algebra generated by a doubly commuting family of nilpotents in a finite-dimensional Hilbert space. Then, in order for $\mathcal{A}$ to be reflexive, it is necessary and sufficient that each rank-two member of $\mathcal{A}$ generates a one-dimensional ideal.*

PROOF. For the necessity assume that $c \in \mathcal{A}$ has rank 2, but fails to generate a one-dimensional ideal. Thus, there is some $b \in \mathcal{A}$ such that bc is independent of c. Subtracting a multiple of the identity from b if necessary, we can assume that b is a nilpotent, whence $\operatorname{rank}(cb) = 1$ by Lemma 2.4.1. Choose $x, y \in V$ with $cb = x \otimes y$. Write $c = x \otimes z + w \otimes v$ for appropriate w, z, v. Since b commutes with $cb = x \otimes y$, we have $bx = 0$ by Lemma 2.4.2. Hence $bc = bw \otimes v = x \otimes y$. This forces v to be a scalar multiple of y, so changing w if necessary, we can write $c = x \otimes z + w \otimes y$.

We complete the proof by showing that the rank-one operator $x \otimes z$ belongs to $\operatorname{Ref} \mathcal{A}$, but not to $\mathcal{A}$. For the first assertion, note that if u is not orthogonal to y, we have $(x \otimes z)u = \frac{(u,z)}{(u,y)} bcu$, while for u orthogonal to y, we get $(x \otimes y)u = cu$.

Suppose, on the other hand, that $x \otimes z \in \mathcal{A}$. Then b commutes with $x \otimes z$. Moreover, since we already know that b commutes with c, we also see that b commutes with $w \otimes y$. But then Lemma 2.4.2 yields $bx = bw = 0$, which leads to the contradiction $bc = 0$. Note that for the necessity we did not need double commutativity.

To prove the sufficiency we first apply Proposition 2.1.5 to write $a_i = \bigoplus_{j=1}^{k} a_{ij}$ where for each j, the N-tuple $(a_{1j}, \ldots, a_{Nj})$ acting on V_j is simple. For each j, take $\mathcal{S}_j$ to be $\mathcal{A}(a_{1j}, \ldots, a_{Nj})$ and set $\mathcal{T}_j = \{c \in \mathcal{A}(\mathbf{a}) : c \text{ is supported on } V_j\}$. Concentrating on $j = 1$ to simplify notation, observe that every member of $\mathcal{T}_1$ takes the form $c_1 \oplus 0$ with $c_1 \in \mathcal{S}_1$. Applying Proposition 2.4.4(2) and the hypothesis, we conclude that c_1 cannot have rank two. By Part (3) of the same proposition, the rank of c_1 cannot exceed one. Finally, Part (1) of that proposition implies that all such c_1 are scalar multiples of one another. By symmetry, each $\mathcal{T}_j$ is one-dimensional, and hence reflexive by Proposition 1.2.8. We complete the proof by applying Proposition 2.3.2 to $\mathcal{T} = \mathcal{A}(a_1, \ldots, a_N)$.

2.5. Jordan forms for doubly commuting pairs of nilpotents. In the present section, we show necessary and sufficient block size criteria for reflexivity of doubly commuting nilpotents. To simplify notation, we restrict attention to pairs of operators, but generalization to arbitrary N-tuples is routine. The *order* of a pair (a, b) of nilpotents is the pair of integers $(\operatorname{order}(a), \operatorname{order}(b))$.

We refer to the sequence of block sizes of a Jordan Canonical Form of an operator as a *Jordan sequence* for the operator; up to permutation, Jordan sequences provide a complete similarity invariant for single operators. Proposition 2.1.5 allows us to extend this notion to doubly commuting operator pairs.

Indeed, given a doubly commuting pair (a,b), apply Proposition 2.1.5 to obtain direct sums $\bigoplus_{i=1}^k a_i$, $\bigoplus_{i=1}^k b_i$ which are simultaneously similar to a,b such that for each i, the doubly commuting pair (a_i,b_i) is simple and acts on a Hilbert space V_i. Write (m_i,n_i) for the order of the simple pair (a_i,b_i). The finite sequence $(m_1,n_1),\dots,(m_k,n_k)$ is referred to as a *Jordan sequence* of (a,b). Up to permutation, these sequences provide a complete similarity invariant for doubly commuting pairs.

Lemma 2.5.1. *Let (a,b) be a simple pair of doubly commuting nilpotents with order (m,n) and suppose $c\in\mathcal{A}(a,b)$. Then*

(1) $\mathrm{rank}(c)\le 1$ *if and only if c is a scalar multiple of* $a^{m-1}b^{n-1}$,

(2) $\mathrm{rank}(c)\le 2$ *if and only if c is a linear combination of* $a^{m-2}b^{n-1}, a^{m-1}b^{n-2}$ *and* $a^{m-1}b^{n-1}$.

Proof. (1) is a consequence of Propositions 2.4.4 and 2.1.3(2).

For the sufficiency of (2), observe that if c takes the stated form, then ac and bc are both scalar multiples of $a^{m-1}b^{n-1}$, so $\dim(c\mathcal{A}(a,b))\le 2$. Thus $\mathrm{rank}(c)\le 2$ by Proposition 2.4.3(2).

For the converse, suppose $c=\sum_{i,j=0}^{m-1,n-1}\lambda_{ij}a^ib^j$ has rank two. By Lemma 2.4.1, we have $\mathrm{rank}(ac)<\mathrm{rank}(c)=2$. From (1), $ac=\sum_{i,j=0}^{m-1,n-1}\lambda_{ij}a^{i+1}b^j=\alpha a^{m-1}b^{n-1}$ for some $\alpha\in\mathbb{C}$. Thus $\lambda_{ij}=0$ for $i<m-2$ and $\lambda_{m-2,j}=0$ for $j<n-1$. By symmetry, we have the desired form.

Lemma 2.5.1 admits a partial generalization: every $c\in\mathcal{A}(a,b)$ of rank r or less must be a linear combination of $\{a^{m-i}b^{n-j}: 0<i,\ 0<j,\ i+j\le r+1\}$. To see that this condition is not sufficient, note that if (a,b) is a simple pair of order $(2,2)$ then $\mathcal{A}(a,b)$ does not contain any members of rank three.

The next theorem is a multinilpotent version of the Deddens–Fillmore result. When $b=0$, the first three conditions of the following theorem reduce to the corresponding conditions for a single nilpotent.

Theorem 2.5.2. *Suppose (a,b) is a doubly commuting pair with Jordan sequence $(m_1,n_1),\dots,(m_k,n_k)$. Then the following are equivalent.*

(1) *For each index i,*
- *if $m_i\ge 2$, we can find $j\ne i$ with $m_j\ge m_i-1$ and $n_j\ge n_i$, and*
- *if $n_i\ge 2$, we can find $j\ne i$ with $n_j\ge n_i-1$ and $m_j\ge m_i$.*

(2) *If $c=\bigoplus_{i=1}^k c_i\in\mathcal{A}(a,b)$ has rank 2, then $c_i\ne 0$ for two values of i.*

(3) *If $c\in\mathcal{A}(a,b)$ has rank 2, then $c\mathcal{A}(a,b)$ is one-dimensional.*

(4) *$\mathcal{A}(a,b)$ is reflexive.*

Proof. (1)$\Rightarrow$(2). Arguing by contradiction, assume for definiteness that $c=c_1\oplus 0$ in $\mathcal{A}(a,b)$ has rank two and it is supported on V_1. Write $c=\sum\lambda_{hl}a^hb^l$. On the one hand, $\sum\lambda_{hl}a_1^hb_1^l$ has rank two, so in view of Lemma 2.5.1(2), we may assume that

$\lambda_{m_1-2,n_1-1} \neq 0$. On the other hand, for $j \neq 1$ the vanishing of c_j forces $m_j \leq m_1 - 2$ or $n_j \leq n_1 - 1$. This means that (1) fails for $i = 1$, which completes the proof.

(2)⇒(3). Suppose that $c \in \mathcal{A}(a, b)$ has rank two. By (2), we may assume that $c = c_1 \oplus c_2 \oplus 0$ with c_1, c_2 of rank one. By Lemma 2.4.1, we have $a_1c_1 = a_2c_2 = 0$ so $ac = 0$. Similarly, $bc = 0$. Thus $c\mathcal{A}(a, b)$ is one-dimensional.

(3)⇔(4). This is Theorem 2.4.5.

(3)⇒(1). Suppose $m_1 \geq 2$. By Lemma 2.5.1, the rank of the operator $a_1^{m_1-2}b_1^{n_1-1}$ is precisely two and hence by Proposition 2.4.3(2), it generates a two-dimensional ideal. The assumption (3) rules out the possibility that $a^{m_1-2}b^{n_1-1}$ is supported on V_1. In other words, $m_j > m_1 - 2$ and $n_j > n_1 - 1$ for some $j \neq 1$. This establishes the first half of (1) when $i = 1$ and the rest follows by symmetry.

It is convenient to call the pair (m_i, n_i) *majorized* if Condition (1) of Theorem 2.5.2 is satisfied for the index i. The discussion of examples is also facilitated by calling a Jordan sequence *reflexive* if the corresponding operator algebra is reflexive. Our first application of Theorem 2.5.2 is

Example 2.5.3. Consider the following algebras:

$$(*)\qquad \left\{\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ & \alpha & & \gamma \\ & & \alpha & \beta \\ & & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta & \varepsilon \\ & \alpha & \beta \\ & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \gamma \\ & \alpha \end{pmatrix} : \alpha, \beta, \gamma, \delta, \varepsilon, \in \mathbb{C}\right\}$$

and

$$(**)\qquad \left\{\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ & \alpha & & \gamma \\ & & \alpha & \beta \\ & & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta & \varepsilon \\ & \alpha & \beta \\ & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta \\ & \alpha \end{pmatrix} : \alpha, \beta, \gamma, \delta, \varepsilon, \in \mathbb{C}\right\}$$

(the missing entries are assumed to be zero). The algebra denoted by $(*)$ is generated by a pair with Jordan sequence $(2, 2)$, $(3, 1)$, $(1, 2)$. It is easy to check that each term in the sequence is majorized, hence, the algebra is reflexive. But the algebra $(**)$ has Jordan sequence $(2, 2)$, $(3, 1)$, $(2, 1)$. The term $(2, 2)$ has no majorant, thus the second algebra is not reflexive.

Proposition 2.5.4. *Let a and b be nilpotents. Then $\mathcal{A}(a \otimes b)$ is reflexive.*

Proof. Suppose $c = p(a \otimes b) \in \mathcal{A}(a \otimes b)$ has rank two. Factor the polynomial p as $p(X) = X^k q(X)$, with $q(0) \neq 0$. Then $q(a \otimes b)$ is invertible, so in fact $(a \otimes b)^k$ has rank two. But then $(\mathrm{rank}(a^k)) \cdot (\mathrm{rank}(b^k)) = 2$. In particular, either a^k or b^k has rank one. In either case, $c(a \otimes b) = (a^{k+1} \otimes b^{k+1})q(a \otimes b) = 0$. Therefore, c generates a one-dimensional ideal and Theorem 2.4.5 implies the conclusion.

The operators c, d appearing in the next result are not assumed to be simple.

Corollary 2.5.5. *Suppose c, d are nilpotent operators. If c and d are reflexive, then the algebra $\mathcal{A}(c \otimes I, I \otimes d)$ is reflexive. If the two largest members of the Jordan sequence of c or of d are equal, then $\mathcal{A}(c \otimes I, I \otimes d)$ is reflexive. In all other cases, $\mathcal{A}(c \otimes I, I \otimes d)$ is not reflexive.*

Proof. The Jordan sequence of $(c \otimes I, I \otimes d)$ is the Cartesian product of the Jordan sequences of c and d. Let $m_1 \geq m_2$ and $n_1 \geq n_2$ denote the two largest terms in the Jordan sequences of c and d, respectively. All elements of the Jordan sequence of $(c \otimes I, I \otimes d)$ except (m_1, n_1) are majorized. The remaining term (m_1, n_1) is majorized in precisely the following situations:

(1) $m_1 = m_2$ or
(2) $n_1 = n_2$ or
(3) $m_1 = m_2 + 1$ and $n_1 = n_2 + 1$.

These correspond to the cases listed in the statement of our corollary.

Example 2.5.6. In the single linear transformation case, there are always at least three singleton sequences whose concatenations with a given Jordan sequence produce reflexive sequences. For example, the singleton sequence 5 can be lengthened to the reflexive sequences 5, 4; 5, 5; and 5, 6. On the other hand, the only two-term reflexive extension of the Jordan sequence $(5,7)$ is $(5,7)$, $(5,7)$. Even the three-term reflexive extensions of $(5,7)$ are limited. For example, only the first of the following four extensions of $(5,7)$ is reflexive:

$$(4,7),(5,7),(5,6); \quad (4,6),(5,7),(5,6); \quad (5,8),(5,7),(5,6); \quad (6,6),(5,7),(8,4).$$

Example 2.5.7. The Jordan sequence $(1,10),(2,9),(3,8),(4,7),(4,6)$ represents a reflexive pair and it is minimal in the sense that none of its proper subsequences is reflexive. This contrasts with the single operator case, where discarding all but the two largest terms of a Jordan sequence does not affect the reflexivity.

2.6. Non-nilpotent case. The first topic of this section is a Hilbert space free version of Theorems 2.4.5 and 2.5.2.

The reader is referred to [AF] for the background in ring theory, in particular for the Wedderburn Structure Theory used below. We recall the relevant definitions. A left module over a ring is said to be *simple* if it has no non-trivial submodules; it is *semisimple* if it can be expressed as a direct sum of simple modules. A ring is *simple* if it has no non-trivial two-sided ideals; it is *semisimple* if it is semisimple when regarded as a left module over itself.

We need the following well-known fact.

Proposition 2.6.1. *An operator algebra is semisimple if and only if it is similar to a von Neumann algebra.*

Proof. Let $\mathcal{B}$ be a subalgebra of $L(V)$. From the ring-theoretic point of view, the underlying vector space V is a (faithful, left) module over $\mathcal{B}$. It is clear from the definitions that simplicity and semisimplicity are invariant under similarity. Since we are restricting attention to finite-dimensional vector spaces, $\mathcal{B}$ is semisimple if and only if it is a direct sum of simple operator algebras.

It is easy to check that the full algebra $L(V)$ is simple. The proof of the sufficiency is completed by appealing to the known structure of von Neumann algebras as direct sums of factors.

For the converse, recall that the only finite-dimensional division algebra over the complex numbers is $\mathbb{C}$ itself. Suppose first that $\mathcal{B}$ is a simple operator algebra. The Wedderburn Structure Theorem then tells us that $\mathcal{B}$ is ring isomorphic to some full operator algebra $L(W)$; in fact, $\mathcal{B}$ is spatially isomorphic (i.e. similar) to $L(W) \otimes \mathbb{C}I_K$ for some auxiliary vector space K. The last algebra can be made into a von Neumann algebra by introducing an appropriate inner product on the underlying space $W \otimes K$. To complete the proof for semisimple $\mathcal{B}$, apply the preceding construction to its direct summands, taking care to define the inner product so as to make the corresponding direct summands of the underlying space mutually orthogonal.

An N-tuple $\mathbf{a} = (a_1, \dots, a_N)$ of operators, acting on a common vector space, is called *semisimple* if the a_i belong to mutually commuting semisimple algebras.

A semisimple N-tuple $\mathbf{a} = (a_1, \dots, a_N)$ of nilpotents is called *simple* if only the trivial idempotents commute with all of them.

The following are immediate consequences of Proposition 2.6.1.

COROLLARY 2.6.2. *An N-tuple $\mathbf{a} = (a_1, \dots, a_N)$ of nilpotents is semisimple if and only if it is similar to a doubly commuting N-tuple.*

PROPOSITION 2.6.3. *Theorems* 2.4.5 *and* 2.5.2 *remain valid when the assumption of double commutativity is replaced by semisimplicity.*

THEOREM 2.6.4. *In order for a commutative operator algebra $\mathcal{A}$ to be reflexive, it is necessary that for each rank-two member c, there is an idempotent $q \in \mathcal{A}$ such that qc generates a one-dimensional ideal. If the underlying vector space is finite-dimensional and $\mathcal{A}$ has a set of generators belonging to mutually commuting semisimple algebras, then this condition is also sufficient.*

PROOF. All properties mentioned in this theorem hold for a full direct sum of operator algebras if and only if they hold for each direct summand. Thus, we may assume that $\mathcal{A}$ contains only the trivial idempotents. In the latter situation, however, the theorem reduces to the remark made in Proposition 2.6.3.

NOTE. The model shown in Section 2.1 first appeared in [AP1]. The hyporeflexivity results from Section 2.2 are new. The reflexivity results from the next sections were proved in [AP1].

3. Toeplitz operators on the polydisc and the unit ball.

In this section we study the reflexivity and other properties of subspaces of Toeplitz operators on the polydisc and the unit ball. We generalize the results of [P1], [P3] and [J], where the reflexivity of the analytic Toeplitz operators on the polydisc and the unit ball was shown. On the other hand, the reflexivity of subspaces of Toeplitz operators on the classical Hardy space H^2 of the unit disc $\mathbb{D}$ was investigated in [AP3]. This section deal with the multivariable version of [AP3]. Namely we will consider the Toeplitz operators on the Hardy spaces $H^2(\mathbb{D}^N)$ and $H^2(\mathbb{B}^N)$.

Let $\mathcal{S} \subset L(\mathcal{H})$ be a closed subspace. Recall that $\operatorname{Ref}\mathcal{S} = \{A \in L(\mathcal{H}) : Ax \in \overline{\mathcal{S}x}$ for all $x \in \mathcal{H}\}$ and $\mathcal{S}$ is called reflexive if $\mathcal{S} = \operatorname{Ref}\mathcal{S}$. For a positive integer k we write $\mathcal{H}^{(k)}$ for a Hilbert space which is the direct sum of k copies of $\mathcal{H}$. For $A \in L(\mathcal{H})$ we write $A^{(k)}$ for the operator on $\mathcal{H}^{(k)}$ which is the direct sum of k copies of A. If $\mathcal{S} \subset L(\mathcal{H})$ is a subspace then $\mathcal{S}^{(k)} = \{A^{(k)} \in L(\mathcal{H}^{(k)}) : A \in \mathcal{S}\}$. A closed subspace $\mathcal{S} \subset L(\mathcal{H})$ is called *k-reflexive* if $\mathcal{S}^{(k)}$ is reflexive. For $k = 1$, this notion reduces to reflexivity. In [Az] the following characterization was given.

Lemma 3.1. *Let $\mathcal{S} \subset L(\mathcal{H})$ be a weak-star subspace and k be a positive integer. Then $\mathcal{S}$ is k-reflexive if and only if $\mathcal{S}_\perp \cap \mathbf{F}_k$ spans $\mathcal{S}_\perp$.*

Now we recall the definition of Toeplitz operators. If $\varphi \in L^\infty(\mathbb{T}^N)$ (or $L^\infty(\partial\mathbb{B}^N)$) we define an operator

$$T_\varphi f = P(\varphi f) \quad \text{for all } f \in H^2(\mathbb{D}^N) \text{ (or } H^2(\mathbb{B}^N)),$$

where P is the orthogonal projection from $L^2(\mathbb{T}^N)$ to $H^2(\mathbb{D}^N)$ (or from $L^2(\partial\mathbb{B}^N)$ to $H^2(\mathbb{B}^N)$, respectively). The basic properties of Toeplitz operators are summarized in

Proposition 3.2. *Let $\varphi, \psi \in L^\infty(\mathbb{T}^N)$ (or $L^\infty(\partial\mathbb{B}^N)$) and suppose $f \in H^\infty(\mathbb{D}^N)$ (or $H^\infty(\mathbb{B}^N)$). Then*

(1) $T_{\varphi+\psi} = T_\varphi + T_\psi$, $T_{\lambda\varphi} = \lambda T_\varphi$ *where* $\lambda \in \mathbb{C}$,
(2) $T_{\overline{\varphi}} = (T_\varphi)^*$,
(3) $T_1 = I$,
(4) $\|T_\varphi\| = \|\varphi\|_\infty$,
(5) $T_\varphi T_f = T_{\varphi f}$.
(6) $T_{\overline{f}} T_\varphi = T_{\overline{f}\varphi}$.

The only non-trivial result is the equality in (4). For the polydisc the proof can be done in the same way as for the unit disc (see [Do]). For the ball case it was shown in [DJ].

Define the symbol maps

$$\mathcal{T} : L^\infty(\mathbb{T}^N) \to L(H^2(\mathbb{D}^N)) \quad \text{by} \quad \mathcal{T}(\varphi) = T_\varphi \quad \text{for } \varphi \in L^\infty(\mathbb{T}^N)$$

and

$$\mathcal{T} : L^\infty(\partial\mathbb{B}^N) \to L(H^2(\mathbb{B}^N)) \quad \text{by} \quad \mathcal{T}(\varphi) = T_\varphi \quad \text{for } \varphi \in L^\infty(\partial\mathbb{B}^N).$$

Proposition 3.3. *The symbol map $\mathcal{T} : L^\infty(\mathbb{T}^N) \to L(H^2(\mathbb{D}^N))$ has the following properties*:

(1) *$\mathcal{T}$ is a linear isometry*,

(2) *$\mathcal{T}(L^\infty(\mathbb{T}^N))$ consists of those $A \in L(H^2(\mathbb{D}^N))$ such that $S_i^* A S_i = A$, where $(S_i f)(z) = z_i f(z)$ for $f \in H^2(\mathbb{D}^N)$, $i = 1, \ldots, N$.*

(3) *$\mathcal{T}(H^\infty(\mathbb{D}^N))$ consists of those $A \in L(H^2(\mathbb{D}^N))$ such that $AS_i = S_i A$ for $i = 1, \ldots, N$.*

The statement (1) follows from Proposition 3.2. The statements (2) and (3) can be shown in the same way as for the unit disc (see [Ha]).

PROPOSITION 3.4. *The symbol map* $\mathcal{T} : L^\infty(\partial\mathbb{B}^N) \to L(H^2(\mathbb{B}^N))$ *has the following properties*:

(1) $\mathcal{T}$ *is a linear isometry.*

(2) $\mathcal{T}(L^\infty(\partial\mathbb{B}^N))$ *consists of those* $A \in L(H^2(\mathbb{B}^N))$ *which satisfy* $\sum_{i=1}^N S_i^* A S_i = A$, *where* $(S_i f)(z) = z_i f(z)$ *for* $f \in H^2(\mathbb{D}^N)$, $i = 1, \ldots, N$.

(3) $\mathcal{T}(H^\infty(\mathbb{B}^N))$ *consists of those* $A \in L(H^2(\mathbb{B}^N))$ *such that* $AS_i = S_i A$.

The statement (1) is a consequence of Proposition 3.2. The statement (2) was proved in [DJ] and (3) was shown in [J].

COROLLARY 3.5. *The spaces* $\mathcal{T}(L^\infty(\mathbb{T}^N))$, $\mathcal{T}(L^\infty(\partial\mathbb{B}^N))$, $\mathcal{T}(H^\infty(\mathbb{D}^N))$, $\mathcal{T}(H^\infty(\mathbb{B}^N))$ *are weak and weak-star closed.*

PROOF. Using the properties of the trace-class operators and rank-one operators, we can show that the sets of operators satisfying (2) and (3) of Propositions 3.3 and 3.4 are weak and weak-star closed.

THEOREM 3.6. (1) *The symbol map* $\mathcal{T} : L^\infty(\mathbb{T}^N) \to \mathcal{T}(L^\infty(\mathbb{T}^N)) \subset L(H^2(\mathbb{D}^N))$ *is a weak-star homeomorphism.*

(2) *The symbol map* $\mathcal{T} : L^\infty(\partial\mathbb{B}^N) \to \mathcal{T}(L^\infty(\partial\mathbb{B}^N)) \subset L(H^2(\mathbb{B}^N))$ *is a weak-star homeomorphism.*

PROOF. Let G be either $\mathbb{D}^N$ or $\mathbb{B}^N$ and denote by G_∂ the set $\mathbb{T}^N$ in the polydisc case and $\partial\mathbb{B}^N$ in the ball case. To see the continuity of $\mathcal{T}$ it is enough to show that each weak-star continuous functional Λ on $\mathcal{T}(L^\infty(G_\partial))$ gives a function $\varphi \mapsto \Lambda(T_\varphi)$ which is also weak-star continuous. Since Λ must be given by a trace-class operator, we have

$$\Lambda(T_\varphi) = \sum_{i=1}^{\infty} \lambda_i (T_\varphi f_i, g_i),$$

where f_i, g_i are unit vectors in $H^2(G)$ and $\sum_{i=1}^\infty |\lambda_i| < \infty$.

Since the predual of $L^\infty(G_\partial)$ is $L^1(G_\partial)$, it is enough to show that the functional $\varphi \mapsto \Lambda(T_\varphi)$ is given by some function h in $L^1(G_\partial)$. Define $h = \sum_{i=1}^\infty \lambda_i f_i \overline{g}_i$. Then $h \in L^1(G_\partial)$ and $\Lambda(T_\varphi) = \sum_{i=1}^\infty \lambda_i (T_\varphi f_i, g_i) = \int T_\varphi \sum_{i=1}^\infty \lambda_i f_i \overline{g}_i \, d\sigma = \int T_\varphi h \, d\sigma$, where σ denotes the Lebesgue measure on G_∂. Thus T is continuous.

Since $\mathcal{T}$ is an isometry, it has trivial kernel and norm closed range. Hence, by [BCP1, Theorem 2.7], $\mathcal{T}(L^\infty(G_\partial))$ is weak-star closed and $\mathcal{T} : L^\infty(G_\partial) \to \mathcal{T}(L^\infty(G_\partial))$ is a weak-star homeomorphism.

Now we show that the spaces $\mathcal{T}(L^\infty(\mathbb{T}^N))$ and $\mathcal{T}(L^\infty(\partial\mathbb{B}^N))$ are far from being reflexive.

Recall that a subalgebra $\mathcal{A} \subset L(\mathcal{H})$ is *transitive* if $\operatorname{Lat}\mathcal{A} = \{0, \mathcal{H}\}$. This is a classical definition. In [Az] it was shown that $\mathcal{A}$ is transitive if and only if $\mathcal{A}_\perp \cap \mathbf{F}_1 = \{0\}$. Thus a subspace $\mathcal{S} \subset L(\mathcal{H})$ can be called *transitive* if $\mathcal{S}_\perp \cap \mathbf{F}_1 = \{0\}$. Equivalently, $\mathcal{S}$ is transitive if $\overline{\mathcal{S}x} = \mathcal{H}$ for any $x \in \mathcal{H}$. For more details see [Az].

THEOREM 3.7. *The subspace* $\mathcal{T}(L^\infty(\mathbb{T}^N)) \subset L(H^2(\mathbb{D}^N))$ *is transitive*, *thus it is not reflexive.*

Proof. Let $f, g \in H^2(\mathbb{D}^N)$ and $f \otimes g \in \mathcal{T}(L^\infty(\mathbb{T}^N))_\perp$. Then, for each $\varphi \in L^\infty(\mathbb{T}^N)$,

$$0 = \langle T_\varphi, f \otimes g\rangle = (T_\varphi f, g) = (\varphi f, g) = \int \varphi f \overline{g}\, dm_N,$$

where m_N is the Lebesgue measure on $\mathbb{T}^N$. Hence $f\overline{g} \equiv 0$.

Now we will show that either $f \equiv 0$ or $g \equiv 0$. To see this it is enough to note that if $f \in H^2(\mathbb{D}^N)$ and $m_N\{z \in \mathbb{T}^N : f(z) = 0\} > 0$ then $f \equiv 0$. But the above statement is a consequence of [Ru1, Theorem 3.3.5].

Theorem 3.8. *The subspace $\mathcal{T}(L^\infty(\partial\mathbb{B}^N) \subset L(H^2(\mathbb{B}^N))$ is transitive, thus it is not reflexive.*

Proof. Let $f, g \in H^2(\mathbb{B}^N)$ and $f \otimes g \in \mathcal{T}(L^\infty(\partial\mathbb{B}^N))_\perp$. Then, as in the polydisc case, we have $f\overline{g} \equiv 0$ on $\partial\mathbb{B}^N$. Let σ_N denote the Lebesgue measure on $\partial\mathbb{B}^N$. To finish the proof as in the polydisc case, it is enough to show that if $f \in H^2(\mathbb{B}^N)$ and $\sigma_N\{z \in \partial\mathbb{B}^N : f(z) = 0\} > 0$ then $f \equiv 0$. But [Ru2, Theorem 5.6.4] makes it straightforward.

On the other hand, we have

Proposition 3.9. (1) *The subspace $\mathcal{T}(L^\infty(\mathbb{T}^N))$ is 2-reflexive.*

(2) *The subspace $\mathcal{T}(L^\infty(\partial\mathbb{B}^N))$ is $(N+1)$-reflexive.*

Proof. (1) Let S_i be multiplication by the independent variable z_i in $H^2(\mathbb{D}^N)$ $(i = 1, \ldots, N)$. Following [J, Section 2, Example 1], $S = (S_1, \ldots, S_N) \subset L(H^2(\mathbb{D}^N))$ is an N-variable unilateral weighted shift with all weights equal to 1. (For definition see Section 4.2.) Let $\{e_I : I \text{ is a multi-index}, I \geq 0\}$ be an orthonormal basis with respect to which S is a shift. By Proposition 3.3(2), $A \in \mathcal{T}(L^\infty(\mathbb{T}^N))$ if and only if $(Ae_I, e_J) = (Ae_{I+\varepsilon_i}, e_{J+\varepsilon_i})$ for all $I, J \geq 0$ and $i = 1, \ldots, N$. Hence $\mathcal{M}_1 = \{e_I \otimes e_J - e_{I+\varepsilon_i} \otimes e_{J+\varepsilon_i} : I, J \geq 0,\ i = 1, \ldots, N\} \subset \mathcal{T}(L^\infty(\mathbb{T}^N))_\perp \cap \mathbf{F}_2$. Moreover, "only if" means that $\mathcal{M}_1$ spans $\mathcal{T}(L^\infty(\mathbb{T}^N))_\perp$, thus it is 2-reflexive by Lemma 3.1.

(2) Denote by S_i multiplication by z_i in $H^2(\mathbb{B}^N)$ $(i = 1, \ldots, N)$. Following [J, Section 2, Example 2], $S = (S_1, \ldots, S_N) \subset L(H^2(\mathbb{B}^N))$ is an N-variable unilateral weighted shift with weights $w_{I,j} = (i_j + 1)^{1/2}/(|I| + N)^{1/2}$, where $I = (i_1, \ldots, i_j)$. Let $\{e_I : I \geq 0\}$ be an appropriate orthonormal basis. Proposition 3.4(2) implies that $A \in \mathcal{T}(L^\infty(\partial\mathbb{B}^N))$ if and only if

$$(Ae_I, e_J) = \sum_{i=1}^{N} (Ae_{I+\varepsilon_i}, e_{J+\varepsilon_i}) \quad \text{for } I, J \geq 0.$$

Hence $\mathcal{M}_2 = \{e_I \otimes e_J - \sum_{i=1}^N e_{I+\varepsilon_i} \otimes e_{J+\varepsilon_i} : I, J \geq 0\} \subset \mathcal{T}(L^\infty(\partial\mathbb{B}^N))_\perp \cap \mathbf{F}_{N+1}$. Thus, as above, we can show that $\mathcal{T}(L^\infty(\mathbb{B}^N))$ is $(N+1)$-reflexive.

In [P1] it was shown that the algebra $\mathcal{T}(H^\infty(\mathbb{D}^N))$ of analytic Toeplitz operators on the polydisc is reflexive. The reflexivity of $\mathcal{T}(H^\infty(\mathbb{B}^N))$ was proved in [P3] and [J]. In [AP3], the reflexivity of $\mathcal{T}(\overline{\psi} H^\infty)$, where ψ is an inner function in the disc, was shown. There are a lot of inner functions defined on the polydisc. In the ball case the existence of inner functions was proved by A. B. Aleksandrov (see [Ru3]). Hence we can ask about reflexivity of $\mathcal{T}(\overline{\psi} H^\infty(\mathbb{D}^N))$ and $\mathcal{T}(\overline{\varphi} H^\infty(\mathbb{B}^N))$, where ψ, φ are inner functions on the polydisc and the unit ball, respectively.

For convenience we will denote $\mathbb{D}^N$ or $\mathbb{B}^N$ by G. The following lemma is a consequence of the Schneider Lemma (see [Ru2], [Iz]).

LEMMA 3.10. *Assume that $\psi \in H^\infty(G)$ is a non-constant inner function. Then T_ψ is a pure (completely non-unitary) isometry on $H^2(G)$.*

PROOF. Since ψ is inner, the operator T_ψ is an isometry. By the Wold decomposition, the subspace where T_ψ is unitary has the form $\mathcal{M} = \bigcap_{n=0}^\infty \psi^n H^2(G)$. Since ψ is inner, we have $\overline{\psi}\mathcal{M} \subset \mathcal{M}$. Moreover $\overline{\psi}^k\mathcal{M} \subset \mathcal{M}$ for $k = 0, 1, 2, 3, \ldots$ Now assuming that $\mathcal{M} \neq \{0\}$, by Schneider's Lemma (for the polydisc case see [Iz, Corollary 1], for the ball case see [Ru2, Lemma 7.5.5]), we get $\overline{\psi} \in H^\infty(G)$. Since $\psi, \overline{\psi} \in H^\infty(G)$ we conclude that ψ is a constant. Hence $\mathcal{M} = \{0\}$ and T_ψ is pure.

The proposition below will be crucial for the reflexivity result.

PROPOSITION 3.11. *Assume that $\psi \in H^\infty(G)$ is inner. If $f \in H^2(G)$ and $h \in \ker T_{\overline{\psi}}$, then*

$$(\lambda - \psi^k) f \otimes \frac{\psi^m}{\overline{\lambda}\psi^k - 1} h \in \mathcal{T}(\overline{\psi} H^\infty(G))_\perp$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $m + 2 \le k$.

PROOF. Let σ be the normalized Lebesgue measure on the polytorus $\mathbb{T}^N$ or on the unit sphere $\partial\mathbb{B}^N$. Then for any $g \in H^\infty(G)$, we have

$$\begin{aligned}\left\langle T_{\overline{\psi}g}, (\lambda - \psi^k) f \otimes \frac{\psi^m}{\overline{\lambda}\psi^k - 1} h \right\rangle &= \left(\overline{\psi} g(\lambda - \psi^k) f, \frac{\psi^m}{\overline{\lambda}\psi^k - 1} h\right) \\ &= \int \overline{\psi}(\lambda - \psi^k) g f \frac{\overline{\psi}^m}{\lambda\overline{\psi}^k - 1} \overline{h}\, d\sigma = \int \frac{\overline{\psi}(\lambda - \psi^k) g f \overline{\psi}^m \overline{h}}{\overline{\psi}^k(\lambda - \psi^k)}\, d\sigma \\ &= \int g f \psi^{k-m-1} \overline{h}\, d\sigma = (gf, \overline{\psi}^{k-m-1} h) = (gf, T_{\overline{\psi}}^{k-m-1} h) = 0,\end{aligned}$$

since $h \in \ker T_\psi$ and $k - m - 1 > 0$.

The following theorem is the main result of this section.

THEOREM 3.12. *Let G denote $\mathbb{D}^N$ or $\mathbb{B}^N$. If $\psi \in H^\infty(G)$ is an inner function then $\mathcal{T}(\overline{\psi} H^\infty(G))$ is reflexive.*

PROOF. Let A be an operator in $\operatorname{Ref} \mathcal{T}(\overline{\psi} H^\infty(G))$. Then $T_\psi A \in \operatorname{Ref} \mathcal{T}(H^\infty(G))$. Since $\mathcal{T}(H^\infty(\mathbb{D}^N))$ is reflexive by [P1] and $\mathcal{T}(H^2(\mathbb{B}^N))$ is reflexive by [P3] and [J], there is $\varphi \in H^\infty(G)$ such that $AT_\psi = T_\varphi$ and $A|_{\psi H^2(G)} = T_{\overline{\psi}\varphi}|_{\psi H^2(G)}$. Subtracting $T_{\overline{\psi}\varphi}$ from A if necessary, we may assume that $A|_{\psi H^2(G)} = 0$.

Now since $A \in \operatorname{Ref} \mathcal{T}(\overline{\psi} H^\infty(G))$ and $(\lambda - \psi^k) f \otimes \frac{\psi^k}{\overline{\lambda}\psi^k - 1} h \in \mathcal{T}(\overline{\psi} H^\infty(G))_\perp$ for any $f \in H^2(G), h \in \ker T_{\overline{\psi}}, |\lambda| < 1$ and $m + 2 \le k$, we have

$$\left(A(\lambda - \psi^k) f, \frac{\psi^m}{\overline{\lambda}\psi^k - 1} h\right) = 0.$$

Since $A|_{\psi H^2(G)} = 0$, we obtain

$$\lambda\left(Af, \frac{\psi^m}{\overline{\lambda}^k\psi^k - 1} h\right) = 0.$$

Dividing by λ and taking the limit as $\lambda \to 0$, we get $(Af, \psi^m h) = 0$. Hence $\psi^m h \perp \operatorname{ran} A$ for all $m = 0, 1, 2, \dots$ and $h \in \ker T_{\overline{\psi}}$. Since ψ is inner the operator T_ψ is an isometry and T_ψ is pure by Lemma 3.10. Hence, by the Wold decomposition, $H^2(G) = \bigoplus_{m=0}^{\infty} T_\psi^m (H^2(G) \ominus T_\psi H^2(G))$. Note that $H^2(G) \ominus T_\psi(H^2(G)) = \ker T_{\overline{\psi}}$. Since $\operatorname{ran} A \perp T_\psi^m \ker T_{\overline{\psi}}$ for $m = 0, 1, \dots,$ we get $A = 0$.

Now we generalize the above result to more general $\psi \in H^\infty(\mathbb{D}^N)$. Let $\mathrm{RP}(\mathbb{D}^N)$ denote the class of all real functions in $\mathbb{D}^N$ which are the real parts of holomorphic functions. If $\psi \in H^\infty(\mathbb{D}^N)$ and $\psi \not\equiv 0$ then, by [Ru1, Theorem 3.2.4], $\log|f|$ has the least N-harmonic majorant, which will be denoted by $u[\psi]$.

Theorem 3.13. *Let $\psi \in H^\infty(\mathbb{D}^N)$ be bounded below a.e. with respect to the Lebesgue measure on $\mathbb{T}^N$. Assume also that $u[\psi] \in \mathrm{RP}(\mathbb{D}^N)$. Then the subspace $\mathcal{T}(\overline{\psi} H^\infty(\mathbb{D}^N))$ is reflexive.*

Proof. Since $u[\psi] \in \mathrm{RP}(\mathbb{D}^N)$, we have $\log|\psi| \le \operatorname{Re}\varphi_2$ on $\mathbb{D}^N$ for some holomorphic function φ_2. Set $\varphi_1 = \psi e^{-\varphi_2}$. Then $|\varphi_1| \le 1$ on $\mathbb{D}^N$, since $\log|\psi| \le \operatorname{Re}\varphi_2$. Moreover, $|\varphi_1| = 1$ a.e. on $\mathbb{T}^N$ since $\operatorname{Re}\varphi_2 = \log|\psi|$ a.e. on $\mathbb{T}^N$ (see [Ru1, Theorems 3.2.4, 2.3.1, 3.3.5]). Hence $\psi = \varphi_1 \cdot \varphi_3$, where φ_1 is inner and $\varphi_3 = e^{\varphi_2}$. Since ψ is holomorphic and bounded below, φ_3 is also bounded below and $1/\varphi_3 = e^{-\varphi_2}$ is bounded. It is easy to check that $T_{1/\overline{\varphi}_3} = T_{\overline{\varphi}_3}^{-1}$. On the other hand, $\mathcal{T}(\overline{\varphi}_1 H^\infty(\mathbb{D}^N))$ is reflexive by Theorem 3.12. Moreover, since $T_{\overline{\varphi}_3}$ is invertible, $T_{\overline{\varphi}_3}\mathcal{T}(\overline{\varphi}_1 H^\infty(\mathbb{D}^N)) = \mathcal{T}(\overline{\psi} H^\infty(\mathbb{D}^N))$ is also reflexive by [AP3, Lemma 4.4].

We turn our attention to functions from $H^\infty(\mathbb{B}^N)$. A real C^2-smooth function f defined on $\mathbb{B}^N$ is called *$\mathcal{M}$-harmonic* if $\Delta(f \circ \varphi_a) = 0$ for all $a \in \mathbb{B}^N$, where $\varphi_a(\lambda) = (a - \lambda)/(1 - \overline{a}\lambda)$ and Δ is the ordinary Laplacian. If $\psi \in H^\infty(\mathbb{B}^N)$ and $\psi \not\equiv 0$ then $\log|\psi|$ has the least $\mathcal{M}$-harmonic majorant, by [Ru2, Theorem 5.6.4], which will be denoted by $u[\psi]$. Let $\mathrm{RP}(\mathbb{B}^N)$ be the set of all real parts of holomorphic functions on $\mathbb{B}^N$. Recall from [Ru2, Theorem 4.4.9] that, if an $\mathcal{M}$-harmonic function f is also harmonic then $f \in \mathrm{RP}(\mathbb{B}^N)$ and conversely, if $f \in \mathrm{RP}(\mathbb{B}^N)$ then it is $\mathcal{M}$-harmonic and harmonic.

Theorem 3.14. *Let $\psi \in H^\infty(\mathbb{B}^N)$ be bounded below a.e. with respect to the Lebesgue measure on $\partial\mathbb{B}^N$. Assume that $u[\psi] \in \mathrm{RP}(\mathbb{B}^N)$. Then $\mathcal{T}(\overline{\psi} H^\infty(\mathbb{B}^N))$ is reflexive.*

Proof. The main idea is the same as in the proof of Theorem 3.13, but we apply the theory of $\mathcal{M}$-harmonic majorants for the ball $\mathbb{B}^N$ ([Ru2], Theorem 5.6.4) instead of the polydisc theory.

Note. The results in chapter are new and present a multivariable version of [AP3].

4. Subspaces of weighted shifts

In [Sh, Proposition 37] the reflexivity of a wide class of unilateral weighted shifts was shown. We show the reflexivity of a certain class of subspaces of operators connected with unilateral weighted shifts.

4.1. Single operator case. Assume that $\mathcal{H}$ is a separable Hilbert space with an orthonormal basis $\{e_n : n \geq 0\}$. An operator T is called a *unilateral weighted shift* if there is a sequence $\{w_n\} \subset \mathbb{C}$ such that

$$Te_n = w_n e_{n+1} \quad \text{for } n = 0, 1, \ldots$$

Recall some basic facts about unilateral weighted shifts from [Sh].

PROPOSITION 4.1.1. *If T is a unilateral weighted shift then*

(1) $T^* e_n = \overline{w}_{n-1} e_{n-1}$ *for* $n \geq 1$ *and* $T^* e_0 = 0$,
(2) $(T^*T)^{1/2} e_n = |w_n| e_n$ *for* $n = 0, 1, \ldots$,
(3) *T is injective if and only if none of the weights is zero*,
(4) *T is unitarily equivalent to the weighted shift operator with weights $\{|w_n|\}$.*

The statement (4) allows us to consider weighted shifts with real non-negative weights.

Recall from [Sh] an analytic function model for the weighted shift. Let $\{\beta(n) : n \geq 0\}$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space $H^2(\beta)$ of sequences $f = \{f(n)\}$ such that

$$\sum_{n=0}^{\infty} |f(n)|^n \beta(n)^2 < \infty.$$

Sometimes we write formally $f(z) = \sum_{n=0}^{\infty} f(n) z^n$. The space $H^2(\beta)$ is a Hilbert space with inner product

$$(f, g) = \sum_{n=0}^{\infty} f(n)\overline{g(n)}\beta(n)^2 \quad \text{for } f = \{f(n)\}, g = \{g(n)\} \in H^2(\beta).$$

We will also consider the operator M_z defined by $(M_z f)(n) = f(n-1)$, where $f(-1) = 0$. The operator M_z is a model for the unilateral weighted shift (see [Sh, Proposition 7]).

PROPOSITION 4.1.2. *The operator M_z on $H^2(\beta)$ is unitarily equivalent to the unilateral injective weighted shift with weights $w_n = \beta(n+1)/\beta(n)$ for $n \geq 0$.*

Conversely, every unilateral injective weighted shift is unitarily equivalent to M_z on $H^2(\beta)$ with $\beta(n) = w_0 \ldots w_{n-1}$ for $n > 0$ and $\beta(0) = 1$.

In what follows we only consider injective weighted shifts. We are going to use the above model and we always drop the unitary equivalence given in the above proposition. In particular, this means that we can assume that the weights are positive. If $f = \{f(n)\} \in H^2(\beta)$ and $\varphi = \{\varphi(n)\}$ is a sequence, we can define $h = \varphi f$ as $h(n) = \sum_{k=0}^{n} \varphi(k) f(n-k)$. Let $H^\infty(\beta)$ denote the set of all $\varphi = \{\varphi(n)\}$ such that $\varphi H^2(\beta) \subset H^2(\beta)$. Hence, for $\varphi = \{\varphi(n)\} \in H^\infty(\beta)$, we can define an operator $M_\varphi f = \varphi f$ for $f \in H^2(\beta)$. Sometimes $H^\infty(\beta)$ will stand for the set of such operators. Recall from [Sh]

THEOREM 4.1.3. *Let T be a unilateral injective weighted shift. Then $\mathcal{W}(T) = \{T\}' = H^\infty(\beta)$.*

For $|\lambda| < 1$, denote by

$$k_\lambda = \sum_{i=0}^{\infty} \frac{\overline{\lambda}^i}{[\beta(i)]^2} e_i$$

the reproducing kernel for the point evaluation at λ. Then, for all $f \in H^2(\beta)$, we have $f(\lambda) = (f, k_\lambda)$. Write

$$k_\lambda(m,p,n) = \sum_{i=0}^{\infty} \frac{\overline{\lambda}^{im+p+n}\beta(im+p)}{[\beta(im+p+n)]^3}\, e_{im+p} \quad \text{for } m > n+p,\ p = 0,1,2,\ldots$$

Assume that the weights are bounded below, $w_t > \delta > 0$ for $t = 0,1,\ldots$ Thus, if $k_\lambda \in H^2(\beta)$, then $k_\lambda(m,p,n) \in H^2(\beta)$ since $\beta(im+p)/\beta(im+p+n) \le \delta^n$. Let

$$h_\lambda(m,j,p+n) = \frac{\lambda^m\beta(m+j)}{\beta(m+p+n)^3} e_j - \frac{\beta(j)}{\beta(p+n)^3}\, e_{m+j}$$

for m,p,n as above and $0 \le j < n$.

Proposition 4.1.4. *Let T be an injective weighted unilateral shift. Assume that its weights are bounded below ($|w_n| > \delta > 0$). Then $h_\lambda(m,j,p+n) \otimes k_\lambda(m,p,n) \in (T^{*n}H^\infty(\beta))_\perp$ for $m > n+p$, $p = 0,1,2,\ldots$ and $0 \le j \le n$.*

Proof. We show that $\langle T^{*n}T^l, h_\lambda(m,j,p+n) \otimes k_\lambda(m,p,n)\rangle = 0$ for all $l \ge 0$. Note that

$$T^n k_\lambda(m,p,n) = \sum_{i=0}^{\infty} \frac{\overline{\lambda}^{im+p+n}}{\beta(im+p+n)^2}\, e_{im+p+n}$$

and denote this vector by $\widetilde{k}_\lambda(m,p+n)$. First consider l such that $l+j < n+p$. Then

$$\begin{aligned}
\langle T^{*n}T^l, h_\lambda(m,j,p+n) \otimes k_\lambda(m,p,n)\rangle &= (T^{*n}T^l h_\lambda(m,j,p+n), k_\lambda(m,p,n))\\
&= (T^l h_\lambda(m,j,p+n), T^n k_\lambda(m,p,n))\\
&= \Big(\frac{\lambda^m\beta(m+j)\beta(l+j)}{\beta(m+p+n)^3\beta(l)}\, e_{j+l} - \frac{\beta(j)\beta(m+j+l)}{\beta(p+n)^3\beta(m+j)}\, e_{m+j+l}, \widetilde{k}_\lambda(m,p+n)\Big)\\
&= \Big(\frac{\lambda^m\beta(m+j)\beta(l+j)}{\beta(m+p+n)^3\beta(l)}\, e_{j+l}, \sum_{i=0}^{\infty} \frac{\overline{\lambda}^{im+p+n}}{[\beta(im+p+n)]^2}\, e_{im+p+n}\Big)\\
&\quad - \Big(\frac{\beta(j)\beta(m+j+l)}{\beta(p+n)^3\beta(m+j)}\, e_{m+j+l}, \sum_{i=0}^{\infty} \frac{\overline{\lambda}^{im+p+n}}{[\beta(im+p+n)]^2}\, e_{im+p+n}\Big).
\end{aligned}$$

The first term is 0, since $j+l < n+p$, the second might be non-zero only for $i=0$ since $j+l < n+p$. Then $m+j+l = p+n$, but this contradicts $m > n+p$.

Now take l such that $l+j \ge n+p$. Then as above

$$\begin{aligned}
\langle T^{*n}T^l, h_\lambda(m,j,p+n) \otimes k_\lambda(m,p,n)\rangle &= (T^l h_\lambda(m,j,p+n), \widetilde{k}_\lambda(m,p+n))\\
&= \Big(\frac{\lambda^m\beta(m+j)\beta(l+j)}{\beta(m+p+n)^3\beta(l)}\, e_{j+l} - \frac{\beta(j)\beta(m+j+l)}{\beta(p+n)^3\beta(m+j)}\, e_{m+j+l}, \widetilde{k}_\lambda(m,p,n)\Big).
\end{aligned}$$

Let us start with the case when m does not divide $j+l-n-p$ ($j+l-n-p$ is not divisible by m). Then m does not divide $m+j+l-n-p$, so $(e_{j+l}, e_{im+p+n}) = 0 = (e_{m+j+l}, e_{im+p+n})$. Thus the above scalar product is 0.

Assume now that $j+l = ms+p+n$ for some $s \in \mathbb{N}$. Notice also that

$$T^{*p+n-j} e_{im+p+n} = \frac{\beta(im+j)}{\beta(im+p+n)} e_{im+j}.$$

Since $\widetilde{k}_\lambda(m,p+n)$ is the eigenvector for T^{*sm} with eigenvalue $\overline{\lambda}^{sm}$, we have

$$T^{*l}\widetilde{k}_\lambda(m,p+n) = T^{*sm+p+n-j}\widetilde{k}_\lambda(m,p+n) = \overline{\lambda}^{sm}T^{*p+n-j}\widetilde{k}_\lambda(m,p+n) = \overline{\lambda}^{sm}\sum_{i=0}^{\infty}\frac{\overline{\lambda}^{in+p+n}\beta(im+j)}{\beta(im+p+n)^3}e_{im+j}.$$

Thus

$$\begin{aligned}(T^{*n}T^l, h_\lambda(m,j,p+n)\otimes k_\lambda(m,p,n)) &= (T^l h_\lambda(m,j,p+n), \widetilde{k}_\lambda(m,p+n))\\ &= (h_\lambda(m,j,p+n), T^{*l}\widetilde{k}_\lambda(m,p+n))\\ &= \left(\frac{\lambda^m\beta(m+j)}{\beta(m+p+n)^3}\,e_j - \frac{\beta(j)}{\beta(p+n)^3}\,e_{m+j}, T^{*l}\widetilde{k}_\lambda(m,p+n)\right).\end{aligned}$$

Since

$$T^{*l}\widetilde{k}_\lambda(m,p+n) = \overline{\lambda}^{sm}\biggl(\frac{\overline{\lambda}^{p+n}\beta(j)}{\beta(p+n)^3}e_j + \frac{\overline{\lambda}^{m+p+n}\beta(m+j)}{\beta(m+p+n)^3}e_{m+j} + \sum_{i=2}^{\infty}\frac{\overline{\lambda}^{im+p+n}\beta(im+j)}{\beta(im+p+n)^3}e_{im+j}\biggr),$$

calculating the scalar product we obtain $(T^{*n}T^l, h_\lambda(m,j,p+n)\otimes k_\lambda(m,p,n)) = 0$.

Theorem 4.1.5. *Let T be an injective weighted unilateral shift. Assume that its weights are bounded below ($|w_n| > \delta > 0$). Then $T^{*n}H^\infty(\beta)$ is reflexive for every $n\in\mathbb{N}$.*

Proof. Take $A \in \operatorname{Ref} T^{*n}H^\infty(\beta)$. Then $Ax \in \overline{T^{*n}H^\infty(\beta)x}$ and also $AT^n x \in \overline{T^{*n}T^nH^\infty(\beta)x}$. Since the weights are bounded below, $T^{*n}T^n$ is invertible. Hence $(T^{*n}T^n)^{-1}AT^nx \in \overline{H^\infty(\beta)x}$. The algebra $H^\infty(\beta)$ is reflexive by [Sh, Proposition 37] since the weights $\{w_n\}$ are bounded below. Hence $(T^{*n}T^n)^{-1}AT^n \in H^\infty(\beta)$. Thus there is $g\in H^\infty(\beta)$ such that $(T^{*n}T^n)^{-1}AT^n = M_g$. Thus $AT^n = T^{*n}M_gT^n$ and $(A - T^{*n}M_g)T^n = 0$. Since $T^*M_g \in T^*H^\infty(\beta)$ we may assume that $AT^n = 0$. Thus $A|_{\operatorname{span}\{e_j:j\ge n\}} = 0$. So it is enough to show that $Ae_j = 0$ for $0\le j<n$. Applying Proposition 4.1.4 we know that $h_\lambda(m,j,p+n)\otimes k_\lambda(m,p,n) \in (T^{*n}H^\infty(\beta))_\perp$ for $m > n+p$, $p = 0,1,2,\ldots$, $0\le j<n$ and $|\lambda|<1$. Thus

$$0 = (Ah_\lambda(m,j), k_\lambda(m,p,n)) = \frac{\lambda^m\beta(m+j)}{\beta(m+p+n)^3}(Ae_j, k_\lambda(m,p,n)).$$

Dividing by λ we obtain

$$0 = (Ae_j, k_\lambda(m,p,n)) = \sum_{i=0}^{\infty}\frac{\overline{\lambda}^{im+n+p}\beta(im+p)}{[\beta(im+n+p)]^3}(Ae_j, e_{im+p}).$$

Since this holds for any λ such that $0<|\lambda|<1$, we get $(Ae_j, e_p) = 0$ for $p = 0,1,2,\ldots$ and $0\le j<n$. Since e_p, $p = 0,1,\ldots$, is an orthogonal basis we get $Ae_j = 0$ for $0\le j<n$ and $A\equiv 0$.

Corollary 4.1.6. *Let T be an injective hyponormal weighted unilateral shift. Then $T^{*n}H^\infty(\beta)$ is reflexive.*

Proof. Since T is hyponormal, the absolute values of its weights are increasing. Hence the weights are bounded below.

Corollary 4.1.7. *Let $A_2(\mathbb{D})$ be the Bergman space, the space of all holomorphic functions that are square integrable with respect to the area measure on $\mathbb{D}$. Denote by M_z multiplication by the independent variable in $A_2(\mathbb{D})$. Then $\{M_z^{*n}\mathcal{W}(M_z)\}$ is reflexive.*

Proof. The operator M_z is a hyponormal unilateral weighted shift.

4.2. Multivariable case. In this section we generalize the main result of the previous section to the multioperator case. First we recall some notation from [J] and [JL].

Let $\mathbb{Z}_+^N$ be the set of all N-tuples of non-negative integers. Write $\varepsilon_k = (0, \ldots, 1, \ldots, 0)$, with 1 in the kth place and 0 elsewhere.

Let $\mathcal{H}$ be the Hilbert space with orthonormal basis $\{e_I : I \in \mathbb{Z}_+^N\}$. Assume that $\{w_{I,j} : I \in \mathbb{Z}_+^N,\ j = 1, \ldots, N\}$ is a bounded net of complex numbers such that

$$w_{I,k}w_{I+\varepsilon_k,l} = w_{I,l}w_{I+\varepsilon_k,k} \quad \text{for } I \in \mathbb{Z}_+^N, 1 \le k, l \le N. \tag{4.2.1}$$

An *N-variable weighted shift* is a family $T = (T_1, \ldots, T_N)$ of N operators on $\mathcal{H}$ such that $T_j e_I = w_{I,j} e_{I+\varepsilon_j}$ for $j = 1, \ldots, N$. Condition (4.2.1) implies that T is a commuting family. In the same way as for a single variable unilateral weighted shift we can prove

Theorem 4.2.1. *Let $T = (T_1, \ldots, T_N)$ be an injective N-variable weighted shift with the weights bounded below. Then $T^{*I}\mathcal{W}(T)$ is reflexive for every multiindex $I \in \mathbb{Z}_+^N$.*

Remark. The system $T = (T_{z_1}, \ldots, T_{z_N})$ of multiplication operators by independent variables in the Hardy space $H^2(\mathbb{D}^N)$ or $H^2(\mathbb{B}^N)$ is an N-variable unilateral weighted shift (see [J]). Theorem 4.2.1 gives us the reflexivity of the spaces $T^{*I}H^\infty(\mathbb{D}^N)$ and $T^{*I}H^\infty(\mathbb{B}^N)$ for $I \in \mathbb{Z}_+^N$.

Let G be either $\mathbb{D}^N$ or $\mathbb{B}^N$. Denote by $A_2(G)$ the space of holomorphic functions which are square integrable with respect to the area measure on G. Let P_+ denote the orthonormal projection from $L^2(G)$ onto $A_2(G)$. For an essentially bounded function φ on G, define the Toeplitz operator $T_\varphi f = P_+(\varphi f)$. A natural question is the reflexivity of the set of all Toeplitz operators. This space is not closed. Its SOT closure is equal to $L(A_2(G))$ (see [En]), hence it is reflexive. Thus the situation is different from the Hardy spaces case, where the space was not reflexive. Note that $L(\mathcal{H})$ has property $\mathbb{A}_1(1)$ if and only if $\dim \mathcal{H} = 1$. Hence $L(A_2(G))$ does not have property $\mathbb{A}_1(1)$ and the question of reflexivity of any subset of Toeplitz operators is interesting; the corollary below is a result in this direction.

Corollary 4.2.2. *Let $T = (T_{z_1}, \ldots, T_{z_n})$ be a system of multiplication operators by independent variables in the Bergman space $A_2(\mathbb{D}^N)$ or $A_2(\mathbb{B}^N)$. Then $T^{*I}\mathcal{W}(T)$ is reflexive for all $I \in \mathbb{Z}_+^N$.*

Note. The results of this chapter are new.

5. Joint spectra for N-tuples of operators

In the next chapters we will need definitions of joint spectra. We start with recalling that $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a *joint eigenvalue* for operators $B_1, \ldots, B_N \in L(\mathcal{H})$ if there exists a non-zero vector $x \in \mathcal{H}$ called a *joint eigenvector* such that $(B_i - \lambda_i)x = 0$ for $i = 1, \dots, N$. The set of all joint eigenvalues is denoted by $\sigma_p(B_1, \dots, B_N)$.

5.1. Left and right spectra. Before stating definitions we recall some well-known facts. The first can be found in [Da] and the second can be proved using arguments similar to [FSW, Theorem 1.1] and [BP, Lemma 2.3], where the equivalence was shown for a single operator.

LEMMA 5.1.1. *If $T_1, \dots, T_N \in L(\mathcal{H})$ commute, then the following are equivalent:*

(1) *There exists $\delta > 0$ such that $\|T_1 x\| + \dots + \|T_N x\| \geq \delta \|x\|$ for all $x \in \mathcal{H}$.*

(2) *There exist $S_1, \dots, S_N \in L(\mathcal{H})$ such that $S_1 T_1 + \dots + S_N T_N = I$.*

(3) *There is no sequence $\{x_n\} \subset \mathcal{H}$ with $\|x_n\| = 1$ such that $\lim_{n\to\infty} \|T_i x_n\| = 0$ for $i = 1, \dots, N$.*

LEMMA 5.1.2. *If $T_1, \dots, T_N \in L(\mathcal{H})$ commute, then the following are equivalent:*

(1) *There exists $\delta > 0$ such that $\|T_1 x\| + \dots + \|T_N x\| \geq \delta \|x\|$ for all x in the orthogonal complement of some finite-dimensional subspace.*

(2) *There exist $S_1, \dots, S_N \in L(\mathcal{H})$ such that $S_1 T_1 + \dots + S_N T_N - I$ is a projection onto a finite-dimensional subspace.*

(3) *There exist $S_1, \dots, S_N$ such that $S_1 T_1 + \dots + S_N T_N - I$ is a compact operator.*

(4) *If P is a projection such that $T_1 P, \dots, T_N P$ are compact, then P is finite-dimensional.*

(5) *There is no orthonormal sequence $\{x_n\}$ with $\lim_{n\to\infty} \|T_i x_n\| = 0$ for $i = 1, \dots, N$.*

(6) *There is no sequence $\{x_n\}$ with $x_n \to 0$ weakly and $\|x_n\| = 1$ such that $\lim_{n\to\infty} \|T_i x_n\| = 0$ for $i = 1, \dots, N$.*

For commuting operators $T_1, \dots, T_N$ on $\mathcal{H}$, we call $T_1, \dots, T_N$ *jointly left invertible* if there exist $S_1, \dots, S_N$ such that $S_1 T_1 + \dots + S_N T_N = I$. Recall that $\lambda = (\lambda_1, \dots, \lambda_N)$ belongs to the *joint left spectrum* $\sigma_l(T_1, \dots, T_N)$ (sometimes called *joint approximate point spectrum*) if $\lambda - T = (\lambda_1 - T_1, \dots, \lambda_N - T_N)$ is not joint left invertible. The negations of the conditions in Lemma 5.1.1 give equivalent conditions for the left spectrum. If $N = 1$ then we obtain the classical *approximate point spectrum* denoted by $\sigma_{ap}(T)$.

Denote by $C(\mathcal{H})$ the Calkin algebra and by π the quotient map $\pi : L(\mathcal{H}) \to C(\mathcal{H})$. Recall that the *joint left essential spectrum* $\sigma_{le}(T_1, \dots, T_N)$ of $T_1, \dots, T_N$ is defined as the joint left spectrum of $\pi(T_1), \dots, \pi(T_N)$. The negations of the conditions in Lemma 5.1.2 above give equivalent conditions for the left essential spectrum. The most common definition is that $\lambda = (\lambda_1, \dots, \lambda_N) \in \sigma_{le}(T_1, \dots, T_N)$ if and only if there exists an orthonormal sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|(T_i - \lambda_i)x_n\| = 0$ for $i = 1, \dots, N$.

We call $T_1, \dots, T_N$ *jointly right invertible* if there exist $S_1, \dots, S_N$ such that $T_1 S_1 + \dots + T_N S_N = I$. Recall that $\lambda = (\lambda_1, \dots, \lambda_N) \in \sigma_r(T_1, \dots, T_N)$ if and only if $\lambda - T = (\lambda_1 - T_1, \dots, \lambda_N - T_N)$ is not joint right invertible. As above, the *joint right essential spectrum* $\sigma_{re}(T_1, \dots, T_N)$ of $T_1, \dots, T_N$ can be defined as the joint right spectrum of $\pi(T_1), \dots, \pi(T_N)$. Recall the well known equalities $\sigma_r(T_1, \dots, T_N) = \overline{\sigma_l(T_1^*, \dots, T_N^*)}$ and $\sigma_{re}(T_1, \dots, T_N) = \overline{\sigma_{le}(T_1^*, \dots, T_N^*)}$. The union $\sigma_l(T_1, \dots, T_N) \cup \sigma_r(T_1, \dots, T_N)$ is called the *Harte spectrum* and denoted by $\sigma_H(T_1,\ldots,T_N)$, and the union $\sigma_{le}(T_1,\ldots,T_N)\cup\sigma_{re}(T_1,\ldots,T_N)$ is called the *Harte essential spectrum* and denoted by $\sigma_{He}(T_1,\ldots,T_N)$.

5.2. Taylor spectrum. Recall from [T1], [T2] that the *Koszul* (*cochain*) *complex* $K(T,\mathcal{H})$ for an N-tuple $T=(T_1,\ldots,T_N)$ of commuting operators in $L(\mathcal{H})$ with respect to $\mathcal{H}$ is given by

$$0\to\Lambda^0(\mathcal{H})\xrightarrow{\delta^0(T)}\Lambda^1(\mathcal{H})\xrightarrow{\delta^1(T)}\ldots\xrightarrow{\delta^{N-1}(T)}\Lambda^N(\mathcal{H})\to 0,$$

where $\Lambda^p(\mathcal{H})$ denotes the set of all p-forms with coefficients in $\mathcal{H}$ and the cochain mapping $\delta^p(T):\Lambda^p(\mathcal{H})\to\Lambda^{p+1}(\mathcal{H})$ is defined by

$$\delta^p(T)\sideset{}{'}\sum_{|I|=p}x_Is_I:=\sum_{j=1}^N\ \sideset{}{'}\sum_{|I|=p}T_jx_Is_j\wedge s_I,$$

where $\{s_1,\ldots,s_N\}$ is a fixed basis of $\Lambda^1(\mathbb{C})$ and $\sum'_{|I|=p}$ denotes that the sum is taken over all $I=(i_1,\ldots,i_p)\in\mathbb{N}^p$ with $1\le i_1<\ldots<i_p\le N$, $s_I:=s_{i_1}\wedge\ldots\wedge s_{i_p}$. Note that $\Lambda^p(\mathcal{H})$ can be endowed with the natural scalar product

$$\Big(\sideset{}{'}\sum_{|I|=p}x_Is_I,\sideset{}{'}\sum_{|I|=p}y_Is_I\Big):=\sideset{}{'}\sum_{|I|=p}(x_I,y_I),$$

which gives us a canonical isomorphism with the direct sum of $\binom{N}{p}$ copies of $\mathcal{H}$. Following [T1], [T2], λ belongs to the *Taylor spectrum* $\sigma(T)\subset\mathbb{C}^N$ if the complex $K(\lambda-T,\mathcal{H})$ is not exact, and λ belongs to the *Taylor essential spectrum* $\sigma_e(T)\subset\mathbb{C}^N$ if at least one of the cohomology groups $H^p(\lambda-T):=\ker\delta^p(\lambda-T)/\operatorname{ran}\delta^{p-1}(\lambda-T)$ has infinite dimension.

It is known that $\sigma_l(T_1,\ldots,T_N)\cup\sigma_r(T_1,\ldots,T_N)\subset\sigma(T_1,\ldots,T_N)$, but these two sets are not always equal (see [Cu2]). The same holds for the essential spectra: $\sigma_{le}(T_1,\ldots,T_N)\cup\sigma_{re}(T_1,\ldots,T_N)\subset\sigma_e(T_1,\ldots,T_N)$. If a single operator T is considered, then $\sigma_{le}(T)\cup\sigma_{re}(T)=\sigma_e(T)$, the essential spectrum of T.

Following [AC], we can decompose $\sigma_e(T)=\bigcup_{p=0}^N\sigma_e^p(T)$, where $\sigma_e^p(T)$ is the set of all $\lambda\in\mathbb{C}^N$ such that the induced mapping

$$\widehat{\delta^p}(\lambda-T):\Lambda^p(\mathcal{H})/\overline{\operatorname{ran}\delta^{p-1}(\lambda-T)}\to\Lambda^{p+1}(\mathcal{H})$$

has either non-closed range or infinite-dimensional kernel.

Following [AP] one can observe that the points of $\sigma_e^p(T)$ have the following property:

LEMMA 5.2.1. *Let $T=(T_1,\ldots,T_N)$ be an N-tuple of commuting operators. If λ is a point in $\sigma_e^p(T)$, then there exists an orthonormal sequence $\{\eta_n\}_{n=1}^\infty$ in $\Lambda^p(\mathcal{H})$ such that*

$$\delta^{p-1}(\lambda-T)\delta^{p-1}(\lambda-T)^*\eta_n+\delta^p(\lambda-T)^*\delta^p(\lambda-T)\eta_n\to 0 \tag{5.2.1}$$

as $n\to\infty$.

PROOF. By the definition of $\sigma_e^p(T)$ and a standard argument, we can find an orthonormal sequence $\{\eta_n\}_{n=1}^\infty$ in $\Lambda^p(\mathcal{H})\ominus\operatorname{ran}\delta^{p-1}(\lambda-T)$ such that $\delta^p(\lambda-T)\eta_n\to 0$. Obviously, this sequence satisfies (5.2.1).

We will need the following fact (see [Cu1, Corollary 3.7]).

Lemma 5.2.2. *Let $T=(T_1,\ldots,T_N)$ be an N-tuple of doubly commuting operators. Then*

$$\delta^{p-1}(T)\delta^{p-1}(T)^* + \delta^p(T)^*\delta^p(T) = \bigoplus_F \sum_{j=1}^N T_F(j),$$

where the orthogonal sum runs over all functions $F:\{1,\ldots,N\}\to\{0,1\}$ with $\operatorname{card}\{j : F(j)=0\}=p$, *and $T_F(j)=T_jT_j^*$ for $F(j)=0$, while $T_F(j)=T_j^*T_j$ for* $F(j)=1$.

6. Algebras of operator weighted shifts

In this section we construct a model for N doubly commuting operator weighted shifts. We also describe the algebra generated by such a model. Section 6.3 presents reflexivity results. They generalize the results of [P4], where a pair was considered. Section 6.5 concerns hyporeflexivity of such operators, which is a new result.

6.1. Model for N doubly commuting operator weighted shifts. An operator $T\in L(\mathcal{K})$ is an *operator unilateral weighted shift* on a Hilbert space $\mathcal{K}$ if there is a wandering subspace H such that $\mathcal{K}=\bigoplus_{n=0}^\infty H_n$, where $H_n=H$ and there is also a family $\{A_n\in L(H): n\ge 0\}$ such that $Tf=\sum_{n=0}^\infty (A_nf_n)e_{n+1}$ for all $f=\sum_{n=0}^\infty f_ne_n$ in $\mathcal{K}$, where e_n means that f_ne_n is an element of H_n.

Denote, as previously, by $\mathbb{Z}_+^N$ the set of all N-tuples of non-negative integers and by $\mathbb{Z}^N$ the set of all N-tuples of integers. If $\phi=(\phi^{(1)},\ldots,\phi^{(N)})$ and $\psi=(\psi^{(1)},\ldots,\psi^{(N)})\in\mathbb{Z}^N$, then we write $\phi\le\psi$ if $\phi^{(i)}\le\psi^{(i)},\ldots,\phi^{(N)}\le\psi^{(N)}$. We write $\phi\not\le\psi$ if $\phi\le\psi$ does not hold. Let $\varepsilon_k\in\mathbb{Z}_+^N$ be the canonical basis vectors, and set $\mathbf{0}=(0,\ldots,0)$ and $\mathbf{n}=(n,\ldots,n)$.

Now, by analogy, we define an *operator weighted multishift* $T=(T_1,\ldots,T_N)$. Assume that there is a wandering subspace H such that $\mathcal{K}=\bigoplus_{\phi\in\mathbb{Z}_+^N} H_\phi$, where $H_\phi=H$ and $f=\sum_{\phi\in\mathbb{Z}_+^N} f_\phi e_\phi$, $f_\phi\in H$, where e_ϕ means that $f_\phi e_\phi$ is an element of H_ϕ. Assume also that we have families $\{A_\phi^{(i)}\in L(H):\phi\in\mathbb{Z}_+^N\}$ $(i=1,\ldots,N)$. Then we define

$$T_if = \sum_{\phi\in\mathbb{Z}_+^N} (A_\phi^{(i)} f_\phi)e_{\phi+\varepsilon_i} \quad \text{for } i=1,\ldots,N. \tag{6.1.1}$$

Lemma 6.1.1. *An operator weighted multishift $T=(T_1,\ldots,T_N)$ consists of commuting operators if and only if*

$$A_{\phi+\varepsilon_j}^{(i)}A_\phi^{(j)} = A_{\phi+\varepsilon_i}^{(j)}A_\phi^{(i)} \quad \textit{for all } \phi\in\mathbb{Z}_+^N,\ i,j=1,\ldots,N. \tag{6.1.2}$$

To prove the lemma, it is enough to compare T_iT_j with T_jT_i on an arbitrary element of $\mathcal{K}$.

From now on, we will assume that $T=(T_1,\ldots,T_N)$ consists of commuting operators. For $s=(s_1,\ldots,s_N)\in\mathbb{Z}_+^N$, $\phi\in\mathbb{Z}_+^N$, set

$$T^s = T_1^{s_1}\ldots T_N^{s_N} \quad\text{and}$$

$$S_\phi^s = A_{\phi+s-\varepsilon_N}^{(N)}\ldots A_{\phi+s-s_N\varepsilon_N}^{(N)} A_{\phi+s-s_N\varepsilon_N-\varepsilon_{N-1}}^{(N-1)}\ldots A_{\phi+\varepsilon_1}^{(1)}A_\phi^{(1)}. \tag{6.1.3}$$

Hence, for $f=\sum_{\phi\in\mathbb{Z}_+^N} f_\phi e_\phi$,

$$T^s f=\sum_{\phi\in\mathbb{Z}_+^N}(S_\phi^s f_\phi)e_{\phi+s}. \tag{6.1.4}$$

We also assume that all $A_\phi^{(i)}$ have dense ranges for $\phi\in\mathbb{Z}_+^N$ and $i=1,\ldots,N$. Then

$$\overline{T^s H_{\mathbf{0}}}=H_s \quad \text{for all } s\in\mathbb{Z}_+^N. \tag{6.1.5}$$

We use the following notation: if $D\in L(\mathcal{K})$, then there is an associated matrix $[D_{\phi\psi}]_{\phi,\psi\in\mathbb{Z}_+^N}$ of operators on H such that

$$Df=\sum_{\phi\in\mathbb{Z}_+^N}\Big(\sum_{\psi\in\mathbb{Z}_+^N} D_{\phi\psi}f_\psi\Big)e_\phi \quad \text{for } f=\sum_{\phi\in\mathbb{Z}_+^N} f_\phi e_\phi.$$

We need the following result:

LEMMA 6.1.2. *Let $T=(T_1,\ldots,T_N)$ be an operator weighted multishift, where the weights have dense ranges. Let $D\in L(\mathcal{K})$ be represented by the matrix $[X_{\phi\psi}]$. If D commutes with the operators $T_1,\ldots,T_N$, then $X_{\phi,\psi}=0$ for $\phi\not\leq\psi$.*

PROOF. Since $\phi\not\leq\psi$, there is i_0 such that $\phi_{i_0}>\psi_{i_0}$. Define $\widetilde{\phi}=(\widetilde{\phi}_1,\ldots,\widetilde{\phi}_N)$ by $\widetilde{\phi}_i=\phi_i$ for $i\neq i_0$, $\widetilde{\phi}_{i_0}=0$, and $\widetilde{\psi}=(\widetilde{\psi}_1,\ldots,\widetilde{\psi}_N)$ by $\widetilde{\psi}_i=\psi_i$ for $i\neq i_0$, $\widetilde{\psi}_{i_0}=0$. Let $x,y\in H$. Then

$$\begin{aligned}(DT^\phi(xe_0),T^\psi ye_0)&=(T_{i_0}^{\phi_{i_0}}DT^{\widetilde{\phi}}(xe_0),T^{\widetilde{\psi}}T_{i_0}^{\psi_{i_0}}(ye_0))\\&=(DT^{\widetilde{\phi}}(xe_0),(T_{i_0}^*)^{\phi_{i_0}}T_{i_0}^{\psi_{i_0}}T^{\widetilde{\psi}}(ye_0))=0\end{aligned}$$

since $(T_{i_0}^*)^{\phi_{i_0}}T_{i_0}^{\psi_{i_0}}(H_{\widetilde{\psi}})=\{0\}$, because $\phi_{i_0}>\psi_{i_0}$ and T_{i_0} is a weighted shift. Since the weights have dense ranges, we get $DH_\phi\perp H_\psi$ and $X_{\phi\psi}=0$.

The main result of the section shows that an operator weighted multishift is a model for N operators which are operator weighted shifts separately and fulfills some additional assumptions.

PROPOSITION 6.1.3. *Let $T_i\in L(\mathcal{K})$ be operator weighted shifts such that the weights have dense ranges for $i=1,\ldots,N$. Assume that $T_1,\ldots,T_N$ doubly commute. Then $T=(T_1,\ldots,T_N)$ is an operator weighted multishift with some operator weights $\{A_\phi^{(i)}\in L(H):\phi\in\mathbb{Z}_+^N\}$ $(i=1,\ldots,N)$ whose ranges are dense and action is given by* (6.1.1). *Moreover*, (6.1.2) *and* (6.1.5) *are satisfied, and also*

$$A_\phi^{(i)}A_\phi^{(j)*}=A_{\phi+\varepsilon_i}^{(j)*}A_{\phi+\varepsilon_j}^{(i)} \tag{6.1.6}$$

for all $\phi\in\mathbb{Z}_+^N$, $i,j=1,\ldots,N, i\neq j$.

PROOF. We reason by induction. Thus there is a subspace H such that $\mathcal{K}=\bigoplus_{\phi\in Z}H_\phi$ (Z denotes $\mathbb{Z}_+^{N-1}$) where $H_\phi=H$. There is a family of bounded operators $\{B_\phi^{(k)}\in L(H):\phi\in Z,k=1,\ldots,N-1\}$ such that $T_kf=\sum_{\phi\in Z}B_\phi^{(k)}f_\phi e_{\phi+\varepsilon_k}$ for $f=\sum_{\phi\in Z}f_\phi e_\phi\in\mathcal{K}$. The operator T_N can be represented as a matrix, say $[X_{\phi\psi}]$. Thus, by Lemma 6.1.2, we have $X_{\phi\psi}=0$ for $\phi\not\leq\psi$. The operator T_N^* is represented by $[X_{\psi\phi}^*]$. It also commutes with $T_1,\ldots,T_{N-1}$ so that $X_{\psi,\phi}^*=0$ for $\psi\not\leq\phi$. Hence $X_{\phi\psi}=0$ if $\phi\neq\psi$. Therefore H_ϕ reduces T_N for all $\phi\in Z$.

The operators $T_N|_{H_\phi} = X_{\phi\phi}$ are weighted shifts for all $\phi \in Z$. Thus there are subspaces $H^{(\phi)}$ such that $H_\phi = \bigoplus_{j=0}^\infty H_{(\phi,j)}$, where $H_{(\phi,j)} = H^{(\phi)}$. For $f^{(\phi)} = \sum_{j=0}^\infty f_j^{(\phi)} e_j^{(\phi)} \in H_\phi$, we have $T_N|_{H_\phi} f^{(\phi)} = \sum_{j=0}^\infty B_{(\phi,j)}^{(N)} f^{(\phi)} e_{j+1}^{(\phi)}$ for some families $\{B_{(\phi,j)}^{(N)} \in L(H^{(\phi)}) : j = 0, 1, 2, \ldots\}$ of bounded operators.

We prove that $T_k H_{(\phi,0)} \subset H_{(\phi+\varepsilon_k,0)}$. Let $x \in H_{(\phi,0)}$, $y \in H_{\phi+\varepsilon_k}$, and $j > 0$. Then

$$(T_k x, T_N^j|_{H_{\phi+\varepsilon_k}} y) = (T_N^{j*} T_k x, y) = (T_k T_N^{j*} x, y) = 0,$$

since $x \in H_{(\phi,0)}$ and $T_N^* x = 0$. Hence $T_k H_{(\phi,0)} \subset H_{(\phi+\varepsilon_k,0)}$ for $k = 1, \ldots, N-1$. Thus $T_k T_N^j H_{(\phi,0)} = \underline{T_N^j T_k H_{(\phi,0)}} \subset T_N^j H_{(\phi+\varepsilon_k,0)} \subset H_{(\phi+\varepsilon_k,j)}$. The weights of T_N have dense ranges, so that $\overline{T_N^j H_{(\phi,0)}} = H_{(\phi,j)}$. Hence $T_k H_{(\phi,j)} \subset H_{(\phi+\varepsilon_k,j)}$.

Since $H_\phi = H_\psi$, we have $H^{(\phi)} = H^{(\psi)} =: H$. Define $A_{(\phi,j)}^{(k)} = B_\phi^{(k)}|_{H(\phi,j)}$ for $k = 1, \ldots, N-1$ and $A_{(\phi,j)}^{(N)} = B_{(\phi,j)}^{(N)}$. Note also that $A_{(\phi,j)}^{(i)}$ has dense range for all $\phi \in Z$, $i = 1, \ldots, N$, since the weights of the operators T_i for $i = 1, \ldots, N$ have dense ranges.

It is easy to see that (6.1.1), (6.1.2) and (6.1.5) are satisfied. The operators $T_1, \ldots, T_N$ doubly commute, thus (6.1.6) also holds.

6.2. Joint eigenvalues of an operator weighted multishift. As for a single shift in [La], we can show

LEMMA 6.2.1. *Let $T = (T_1, \ldots, T_N)$ be an operator weighted multishift. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a non-zero joint eigenvalue for $T_1^*, \ldots, T_N^*$. Then $\lambda' = (\lambda'_1, \ldots, \lambda'_N)$ is a joint eigenvalue for $T_1^*, \ldots, T_N^*$ if $|\lambda'_i| \le |\lambda_i|$ for $i = 1, \ldots, N$.*

The following can be proved by easy calculation:

LEMMA 6.2.2. *Let $T = (T_1, \ldots, T_N)$ be an operator weighted multishift and let $f = \sum_{\psi \in \mathbb{Z}_+^N} f_\psi e_\psi \in \mathcal{K}$. Then $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a joint eigenvalue for $T_1^*, \ldots, T_N^*$ with joint eigenvector f if and only if*

$$A_\psi^{(i)*} f_{\psi+\varepsilon_i} = \lambda_i f_\psi \quad \text{for } \psi \in \mathbb{Z}_+^N,\ i = 1, \ldots, N.$$

An immediate consequence of the above is

LEMMA 6.2.3. *Let $T = (T_1, \ldots, T_N)$ be an operator weighted multishift. Let $f = \sum_{\psi \in \mathbb{Z}_+^N} f_\psi e_\psi \in \mathcal{K}$. Then $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a joint eigenvalue for $T_1^*, \ldots, T_N^*$ with joint eigenvector f if and only if*

$$(S_\psi^{\phi-\psi})^* f_\phi = \lambda^{\phi-\psi} f_\psi \quad \text{for } \psi \le \phi \in \mathbb{Z}_+^N.$$

6.3. Description of the algebra generated by an operator weighted multishift. In this section we work in the model (6.1.1) for an operator weighted multishift $T = (T_1, \ldots, T_N) \subset L(\mathcal{K})$ whose weights have dense ranges. We can obtain, as in [La], the following

LEMMA 6.3.1. *Let $B = [B_{\alpha\beta}]_{\alpha,\beta \in \mathbb{Z}_+^N}$ be an operator on $\mathcal{K}$ and $[\gamma_{\alpha\beta}]_{\alpha,\beta \in \mathbb{Z}_+^N}$ be a scalar matrix such that $[\gamma_{\alpha\beta}]_{\alpha,\beta=0}^n$ define positive operators on $\mathbb{C}^n \times \mathbb{C}^n$ and $\gamma = \sup_{\alpha \in \mathbb{Z}_+^N} \gamma_{\alpha\alpha} < \infty$. Then the matrix $[\gamma_{\alpha\beta} B_{\alpha\beta}]_{\alpha,\beta \in \mathbb{Z}_+^N}$ defines an operator D on $\mathcal{K}$ satisfying $\|D\| \le \gamma \|B\|$.*

We also have

LEMMA 6.3.2. *The matrix* $C_n = [\gamma_{\alpha\beta}]_{\alpha,\beta\in\mathbb{Z}_+^N}$ *with*

$$\gamma_{\alpha\beta} = \begin{cases} \prod_{i=1}^{N}\left(1-\frac{|\alpha_i-\beta_i|}{n+1}\right) & \text{for } |\alpha_i-\beta_i|\le n,\ \alpha=(\alpha_1,\dots,\alpha_N), \\ & \beta=(\beta_1,\dots,\beta_N),\ i=1,\dots,N, \\ 0 & \text{otherwise}, \end{cases}$$

is positive definite.

PROOF. The matrix $B_n = [b_{ij}], i, j \ge 0$, with

$$b_{ij} = \begin{cases} 1-\frac{|i-j|}{n+1} & \text{if } |i-j|\le n, \\ 0 & \text{otherwise}, \end{cases}$$

is positive definite by [SW]. The matrix C_n is the tensor product of N copies of B_n, thus it is also positive definite.

The next lemma is a consequence of (for example) Lebesgue's Dominated Convergence Theorem for a discrete measure.

LEMMA 6.3.3. *Let* $\lambda_\alpha \ge 0$ *for* $\alpha\in\mathbb{Z}_+^N$ *and* $\sum_{\alpha\in\mathbb{Z}_+^N}\lambda_\alpha<\infty$. *Then*

$$\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{Z}_+^N\\ 0\le\alpha_i\le n}} \lambda_\alpha\left(\prod_{i=1}^{N}\left(1-\frac{\alpha_i}{n+1}\right)-1\right)\to 0 \quad \text{as } n\to\infty.$$

Since we have a model for N doubly commuting operator weighted shifts by Proposition 6.1.3, the following theorem describes, in fact, the WOT-closed algebra generated by N doubly commuting operator weighted shifts.

THEOREM 6.3.4. *Let* $T=(T_1,\dots,T_N)$ *be an operator weighted multishift. An operator* $D\in L(\mathcal{K})$ *belongs to* $\mathcal{W}(T_1,\dots,T_N)$ *if and only if* $D_{\alpha\beta}=0$ *for* $\beta\not\le\alpha$, $\alpha,\beta\in\mathbb{Z}_+^N$ *and there is a sequence* $\{\lambda_\alpha\}_{\alpha\in\mathbb{Z}_+^N}$ *of scalars such that* $D_{\alpha\beta}=\lambda_{\alpha-\beta}S_\beta^{\alpha-\beta}$ *for* $\beta\le\alpha$.

PROOF. The statements (6.1.3) and (6.1.4) imply that $T^s, s\in\mathbb{Z}_+^N$, has the following matrix: $(T^s)_{\phi+s,\phi}=S_\phi^s$ for $\phi\in\mathbb{Z}_+^N$ and $(T^s)_{\phi\psi}=0$ otherwise. Hence, for each polynomial p, there is a sequence $\{\lambda_\alpha(p)\}_{\alpha\in\mathbb{Z}_+^N}$ of (finitely non-zero) scalars such that $p(T_1,\dots,T_N)_{\alpha\beta}=\lambda_{\alpha-\beta}(p)S_\beta^{\alpha-\beta}$ for $\beta\le\alpha$ and $p(T_1,\dots,T_N)_{\alpha\beta}=0$ otherwise. Let $D\in\mathcal{W}(T_1,\dots,T_N)$. Then there is a net $\{p_\omega(T_1,\dots,T_N)\}$ of polynomials in $T_1,\dots,T_N$ ($T_1,\dots,T_N$ commute), converging in the weak operator topology to D. Thus $(p_\omega(T_1,\dots,T_N))_{\alpha\beta}$ converges to $D_{\alpha\beta}$. Hence D has the required matrix.

Conversely, assume that there is a sequence $\{\lambda_\alpha\}_{\alpha\in\mathbb{Z}_+^N}$ of scalars such that $D_{\alpha\beta}=\lambda_{\alpha-\beta}S_\beta^{\alpha-\beta}$ for $\beta\le\alpha$. Consider the sequence of polynomials in $T_1,\dots,T_N$,

$$p_n(T_1,\dots,T_N)=\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{Z}_+^N\\ 0\le\alpha_i\le n}}\gamma_\alpha^{(n)}\lambda_\alpha T^\alpha, \text{ where } \gamma_\alpha^{(n)}=\prod_{i=1}^{N}\left(1-\frac{\alpha_i}{n+1}\right).$$

Lemmas 6.3.1 and 6.3.2 show that

$$\|p_n(T_1,\dots,T_N)\| \le \|D\|. \tag{6.3.1}$$

Let $x \in H$, $\zeta \in \mathbb{Z}_+^N$ and let $\delta_{\zeta\beta}$ be the Kronecker symbol. Then

$$\begin{aligned}
&\|p_n(T_1,\dots,T_N)xe_\zeta - Dxe_\zeta\|^2 \\
&\quad = \Big\| \sum_{\alpha\in\mathbb{Z}_+^N} \Big(\sum_{\beta\in\mathbb{Z}_+^N} \delta_{\zeta\beta}(p_n(T_1,\dots,T_N)_{\alpha\beta})x\Big)e_\alpha - \sum_{\alpha\in\mathbb{Z}_+^N} \Big(\sum_{\beta\in\mathbb{Z}_+^N} \delta_{\zeta\beta} D_{\alpha\beta}x\Big)e_\alpha \Big\|^2 \\
&\quad = \sum_{\alpha\in\mathbb{Z}_+^N} \|(p_n(T_1,\dots,T_N)_{\alpha\zeta})x) - D_{\alpha\zeta}x\|^2 \\
&\quad = \sum_{\alpha\in\mathbb{Z}_+^N} \|(p_n(T_1,\dots,T_N)_{\alpha+\zeta,\zeta})x) - D_{\alpha+\zeta,\zeta}x\|^2 \\
&\quad = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{Z}_+^N \\ 0\le\alpha_i\le n}} \|\gamma_\alpha^{(n)}\lambda_\alpha S_\zeta^\alpha x - \lambda_\alpha S_\zeta^\alpha x\|^2 + \sum_{\alpha\not\le\mathbf{n}} \|D_{\alpha+\zeta,\zeta}x\|^2 \\
&\quad = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{Z}_+^N \\ 0\le\alpha_i\le n}} (\gamma_\alpha^{(n)}-1)\|\lambda_\alpha S_\zeta^\alpha x\|^2 + \sum_{\alpha\not\le\mathbf{n}} \|D_{\alpha+\zeta,\zeta}x\|^2 \\
&\quad = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{Z}_+^N \\ 0\le\alpha_i\le n}} (\gamma_\alpha^{(n)}-1)\|D_{\alpha+\zeta,\zeta}x\|^2 + \sum_{\alpha\not\le\mathbf{n}} \|D_{\alpha+\zeta,\zeta}x\|^2.
\end{aligned}$$

Since $\sum_{\alpha\in\mathbb{Z}_+^N} \|D_{\alpha+\zeta,\zeta}x\|^2 = \|Dxe_\zeta\|^2 < \infty$, we get $\|p_n(T_1,\dots,T_N)xe_\zeta - Dxe_\zeta\| \to 0$ ($n \to \infty$) by Lemma 6.2.3. Finally, $p_n(T_1,\dots,T_N)f \to Df$ on a dense set, and by (6.3.1), $p_n(T_1,\dots,T_N) \to D$ in the strong operator topology.

6.4. Reflexivity results for tuples of operator weighted shifts. We state the following

THEOREM 6.4.1. *Let $T = (T_1,\dots,T_N)$ be an operator weighted multishift given by (6.1.1) such that $\ker A_\alpha^{(i)} = \{0\} = \ker A_\alpha^{(i)*}$ for $\alpha \in \mathbb{Z}_+^N$, $i = 1,\dots,N$. Assume also that $T_1^*,\dots,T_N^*$ have a non-zero joint eigenvalue. Then $\mathcal{W}(T_1,\dots,T_N)$ is reflexive.*

PROOF. The main idea of the proof is taken from [La] and [P4]. However, we present some parts of the proof because we have an operator weighted multishift instead of a single shift or a pair and we assume less about them. Let $D \in L(\mathcal{K})$ and $\operatorname{Lat}(T_1,\dots,T_N) \subset \operatorname{Lat} D$. The subspaces $L^2(\bigoplus_{\psi\le\alpha} H_\alpha)$ (for all $\psi \in \mathbb{Z}_+^N$) are invariant for $T_1,\dots,T_N$. Hence they are also invariant for D, so that $D_{\phi\psi} = 0$ for $\psi \not\le \phi$.

Let $f \in H_{\mathbf{0}} = H$ and let $[f]$ denote the one-dimensional subspace generated by f. Then $\bigoplus_{\phi\in\mathbb{Z}_+^N} S_0^\phi[f]$ is invariant for $T_1,\dots,T_N$. Thus, if $\Lambda = \{\lambda_\phi\}_{\phi\in\mathbb{Z}_+^N}$ is a sequence of scalars such that $\sum_{\phi\in\mathbb{Z}_+^N} |\lambda_\phi|^2\|S_0^\phi f\|^2 < \infty$, then there is a sequence $\{\gamma_\phi(f)\}_{\phi\in\mathbb{Z}_+^N}$

depending on Λ and f such that

$$D \bigoplus_{\phi\in\mathbb{Z}_+^N} \lambda_\phi S_0^\phi f = \bigoplus_{\phi\in\mathbb{Z}_+^N} \gamma_\phi(f) S_0^\phi f.$$

Let $\phi \in \mathbb{Z}_+^N$. For $\Lambda_\phi = \{\delta_{\phi\psi}\}_{\psi\in\mathbb{Z}_+^N}$, there is $\Gamma_\phi = \{\gamma_{\phi\psi}(f)\}$ defined as above. Similarly to [La], it can be shown that

$$D_{\phi\psi} S_{\mathbf{0}}^\psi f = \gamma_{\phi\psi}(f) S_{\mathbf{0}}^\phi f \quad \text{for } f \in H \text{ and } \psi \le \phi. \tag{6.4.1}$$

Let $f, g \in H$ be non-zero elements. Then using (6.4.1) we can prove as in [La] that

$$\gamma_{\phi\psi}(f+g) S_{\mathbf{0}}^\phi (f+g) = S_{\mathbf{0}}^\phi(\gamma_{\phi\psi}(f) f + \gamma_{\phi\psi}(g) g).$$

Since $\ker S_{\mathbf{0}}^\phi \ne \{0\}$, we get

$$\gamma_{\phi\psi}(f+g)(f+g) = \gamma_{\phi\psi}(f) f + \gamma_{\phi\psi}(g) g.$$

Hence, if f, g are linearly independent, then

$$\gamma_{\phi\psi}(f) = \gamma_{\phi\psi}(g). \tag{6.4.2}$$

If $f = \alpha g$, then using (6.4.1), we can show that

$$\gamma_{\phi\psi}(f) S_{\mathbf{0}}^\phi f = \gamma_{\phi\psi}(g) S_{\mathbf{0}}^\phi f.$$

Thus, in this case we also have (6.4.2). Hence $\gamma_{\phi\psi}(f)$ do not depend on f, so that $D_{\phi\psi} S_{\mathbf{0}}^\psi = \gamma_{\phi\psi} S_{\mathbf{0}}^\phi$.

Now we show that

$$\gamma_{\phi\psi} = \gamma_{\phi-\psi,0}. \tag{6.4.3}$$

Let $g = \sum_{\phi\in\mathbb{Z}_+^N} g_\phi e_\phi$ be a joint eigenvector for a certain non-zero eigenvalue $\lambda = (\lambda_1, \ldots, \lambda_N)$ for $T_1^*, \ldots, T_N^*$. Then, for $\psi \in \mathbb{Z}_+^N$ and $f \in H_{\mathbf{0}} = H$, using Lemma 6.2.3 and (6.4.1), we have

$$\begin{aligned}
(D^* g, S_{\mathbf{0}}^\psi f) &= \Big(\sum_{\psi\le\phi} D_{\phi\psi}^* g_\phi e_\phi, S_{\mathbf{0}}^\psi f\Big) = \sum_{\psi\le\phi} (D_{\phi\psi}^* g_\phi, S_{\mathbf{0}}^\psi f) \\
&= \sum_{\psi\le\phi} (g_\phi, D_{\phi\psi} S_{\mathbf{0}}^\psi f) = \sum_{\psi\le\phi} (g_\phi, \gamma_{\psi\phi} S_{\mathbf{0}}^\phi f) \\
&= \sum_{\psi\le\phi} (\overline{\gamma}_{\psi\phi} g_\phi, S_\psi^{\phi-\psi} S_{\mathbf{0}}^\psi f) = \sum_{\psi\le\phi} (\overline{\gamma}_{\psi\phi} (S_\psi^{\phi-\psi})^* g_\phi, S_{\mathbf{0}}^\psi f) \\
&= \sum_{\psi\le\phi} (\overline{\gamma}_{\psi\phi} \lambda^{\phi-\psi} g_\psi, S_{\mathbf{0}}^\psi f) = \sum_{\psi\le\phi} \overline{\gamma}_{\psi\phi} \lambda^{\phi-\psi} (g_\psi, S_{\mathbf{0}}^\psi f).
\end{aligned}$$

On the other hand, we know that $\operatorname{Lat}(T_1^*, \ldots, T_N^*) \subset \operatorname{Lat} D^*$. Hence $[g]$ is invariant for D^*, so there is $\gamma \in \mathbb{C}$ such that $D^* g = \gamma g$. Hence

$$(D^* g, S_{\mathbf{0}}^\psi f) = \gamma (g, S_{\mathbf{0}}^\psi f) = \gamma (g_\psi, S_{\mathbf{0}}^\psi f).$$

Since $\ker S_{\mathbf{0}}^{\psi *} = \{0\}$, we have $\overline{\operatorname{ran}(S_{\mathbf{0}}^\psi)} = H$. Hence $\sum_{\psi\le\phi} \overline{\gamma}_{\psi\phi} \lambda^{\phi-\psi} = \gamma$. This is true for every $\psi \in \mathbb{Z}_+^N$, hence also for $\psi = \mathbf{0}$. Hence, for all $\psi \in \mathbb{Z}_+^N$, we have

$$\sum_{\phi\in\mathbb{Z}_+^N} \overline{\gamma}_{\psi+\phi,\psi} \lambda^\phi = \sum_{\phi\in\mathbb{Z}_+^N} \overline{\gamma}_{\psi\mathbf{0}} \lambda^\phi.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a non-zero eigenvalue for $T_1^*, \ldots, T_N^*$, existing by assumption. Then Lemma 6.2.1 shows that the above equality holds for all $\lambda = (\lambda_1, \ldots, \lambda_N)$ with $|\lambda_i| \le |\alpha_i|,\ i = 1, \ldots, N$. Hence (6.4.3) is shown and

$$D_{\phi\psi} = \gamma_{\phi-\psi} S_{\psi}^{\phi-\psi}.$$

So, Theorem 6.3.4 implies that $D \in \mathcal{W}(T_1, \ldots, T_N)$.

A consequence of the above and Proposition 6.1.3 is

THEOREM 6.4.2. *Let each T_i be an operator weighted shift whose weights and their adjoints have trivial kernels, for $i = 1, \ldots, N$. Assume that $T_1, \ldots, T_N$ doubly commute and $\sigma_p(T_1^*, \ldots, T_N^*) \ne \{0\}$. Then $\mathcal{W}(T_1, \ldots, T_N)$ is reflexive.*

6.5. Hyporeflexivity results for tuples of operator weighted shifts. Let us recall some notations. An operator A is called *locally nilpotent* if, for any $x \in \mathcal{H}$, there is a positive integer $n(x)$ such that $A^{n(x)}x = 0$. Let $\mathcal{S} \subset L(\mathcal{H})$ and $\mathcal{X} \subset \mathcal{H}$. Then $\operatorname{span}(\mathcal{S}; \mathcal{X})$ denotes the smallest subspace of $\mathcal{H}$ containing $\mathcal{X}$ and invariant for all operators from $\mathcal{S}$.

LEMMA 6.5.1. *Let $T = (T_1, \ldots, T_N) \subset L(\mathcal{H})$ be a doubly commuting N-tuple. Suppose that $S \in \operatorname{Alg}\operatorname{Lat}(T_1, \ldots, T_N)$, $ST_i = T_iS$ and T_i is locally nilpotent for $i = 1, \ldots, N$. Then there is a sequence $\{a_\phi\}_{\phi \in \mathbb{Z}_+^N}$ such that $S = \sum_{\phi \in \mathbb{Z}_+^N} a_\phi T^\phi$. Moreover, $S \in \{T\}''$.*

Note first that the expression $\sum_{\phi \in \mathbb{Z}_+^N} a_\phi T^\phi$ makes sense since each T_i is locally nilpotent. To prove the above we need

LEMMA 6.5.2. *With the above assumptions there is a net $\{p_\lambda\}$ of polynomials such that $p_\lambda(T) \to S$ in the strict topology (i.e., pointwise in the discrete topology on $\mathcal{H}$).*

PROOF. Take $x_1, \ldots, x_k \in \mathcal{H}$ and let $V = \operatorname{span}(T, T^*; x_1, \ldots, x_k)$. Since T_i is locally nilpotent for $i = 1, \ldots, N$, it is locally algebraic (i.e., for any $w \in \mathcal{H}$ there is a polynomial p_w such that $p_w(T_i)w = 0$). Hence, by [K] (see also [Had]) it is algebraic (i.e., there is polynomial p such that $p(T_i) = 0$)). Therefore T_i^* is also algebraic. Hence V is finite-dimensional.

Moreover, by Proposition 2.1.5, V is a finite direct sum of cyclic subspaces. The subspace V is invariant for S, and $S|_V$ leaves invariant all linear subspaces that are invariant for $T_i|_V$, for $i = 1, \ldots, N$. Hence by [HK, Lemma 2] there is a polynomial p such that $p(T)|_V = S|_V$, thus $p(T)x_j = Sx_j,\ j = 1, \ldots, k$. Equivalently, there is a net $\{p_\lambda\}$ such that $p_\lambda(T) \to S$ in the strict topology.

PROOF OF LEMMA 6.5.1. For convenience of notation we give the proof for $N = 2$. If, for some $\psi \in \mathbb{Z}_+^2$, there is no x such that $T^\psi x \ne 0$ then it is obvious that we should take $a_\psi = 0$.

Let $\psi = (\psi_1, \psi_2) \in \mathbb{Z}_+^2$ and $x \in \mathcal{H}$ be such that $T^\psi x = T_1^{\psi_1} T_2^{\psi_2} x \ne 0$ and $T_1^{\psi_1+1} T_2^{\psi_2} = T_1^{\psi_1} T_2^{\psi_2+1} x = 0$. Then, by Lemma 6.5.2, for sufficiently large λ we have $p_\lambda(T)x = Sx$. To show that for sufficiently large λ the coefficients of X^ϕ in $p_\lambda(X)$ are constantly equal to a_ϕ for $\phi \le \psi$, it is sufficient to show that $p(T)x = 0$ implies that all coefficients a_ϕ of p are 0.

Consider $V = \operatorname{span}(T, T^*; x)$ and $V_0 = \operatorname{span}(T; x)$. The operators T_1, T_2 doubly commute on a finite-dimensional space, so that, by Proposition 2.1.5, the pair is simultaneously similar, via a similarity U, to a pair $\bigoplus_{j\in\mathcal{J}} A_j \otimes I_j$, $\bigoplus_{j\in\mathcal{J}} J_j \otimes B_j$ on $\bigoplus_{j\in\mathcal{J}} V_j \otimes W_j$ where A_j, B_j are nilpotents and I_j, J_j are the identity operators on V_j, W_j, respectively. Let $Ux = \bigoplus_{j\in\mathcal{J}} x_j \otimes y_j$. For some j (say $j = 1$), $(A_1^{\psi_1} \otimes B_1^{\psi_2})(x_1 \otimes y_1) \neq 0$ and $(A_1^{\psi_1+1} \otimes B_1^{\psi_2})(x_1 \otimes y_1) = 0 = (A_1^{\psi_1} \otimes B_1^{\psi_2+1})(x_1 \otimes y_1)$. Therefore

$$A_1^{\psi_1} x_1 \neq 0, \quad A_1^{\psi_1+1} x_1 = 0, \quad \text{and} \quad B_1^{\psi_2} y_1 \neq 0, \quad B_1^{\psi_2+1} y_1 = 0. \tag{6.5.1}$$

We also have $p(A_1 \otimes I_1, J_1 \otimes B_1)(x_1 \otimes y_1) = 0$. Thus

$$0 = (A_1^{\psi_1} \otimes B_1^{\psi_2}) p(A_1 \otimes I_1, J_1 \otimes B_1)(x_1 \otimes y_1) = a_{\mathbf{0}}(A_1^{\psi_1} \otimes B_1^{\psi_2})(x_1 \otimes y_1),$$

where $a_{\mathbf{0}}$ is the constant term of p. Since $(A_1^{\psi_1} \otimes B_1^{\psi_2})(x_1 \otimes y_1) \neq 0$, we have $a_{\mathbf{0}} = 0$. Let $\phi = (\phi_1, \phi_2) \leq \psi$ and assume that $a_\alpha = 0$ for all $\alpha \leq \phi$ and $\alpha \neq \phi$. Then, by (6.5.1),

$$0 = (A_1^{\psi_1-\phi_1} \otimes B_1^{\psi_2-\phi_2} p(A_1 \otimes I_1, J_1 \otimes B_1)(x_1 \otimes y_1) = a_\phi(A_1^{\psi_1} \otimes B_1^{\psi_2})(x_1 \otimes y_1)$$

and $a_\phi = 0$ by the same argument as above.

It is clear that $S \in \{T\}''$.

Lemma 6.5.3. *Let $T = (T_1, \ldots, T_N)$ be the operator weighted multishift on $\mathcal{K} = \bigoplus_{\psi\in\mathbb{Z}_+^N} H_\psi$ given by the model* (6.1.1). *Let $M_\phi = \bigoplus_{\psi\leq\phi} H_\psi$ and let P_ϕ be the projection onto M_ϕ. Then $\widetilde{T} = (P_\phi T_1|_{M_\phi}, \ldots, P_\phi T_N|_{M_\phi})$ is a doubly commuting N-tuple.*

The lemma follows immediately from the model (6.1.1) of an N-tuple of operator weighted shifts.

Theorem 6.5.4. *Let $T_i \in L(\mathcal{K})$ be an operator weighted shift whose weights have dense ranges for $i = 1, \ldots, N$. Assume that $T_1, \ldots, T_N$ doubly commute. Then $\{T_1, \ldots, T_N\}' \cap \operatorname{Alg}\operatorname{Lat}(T_1, \ldots, T_N) = \mathcal{W}(T_1, \ldots, T_N)$.*

Proof. By Theorem 6.1.3, $T = (T_1, \ldots, T_N)$ is an operator weighted multishift given by (6.1.1) with dense operator weights. Let $A \in \{T\}' \cap \operatorname{Alg}\operatorname{Lat} T$. Take $\phi \in \mathbb{Z}_+^N$ and consider the space $M_\phi = \bigoplus_{\psi\leq\phi} H_\psi$. Lemma 6.5.3 shows that $\widetilde{T} = (P_\phi T_1|_{M_\phi}, \ldots, P_\phi T|_{M_\phi})$ is a doubly commuting N-tuple. Define $\widetilde{A}_\phi = P_\phi A|_{M_\phi}$. Since $M_\phi^\perp$ is invariant for T, it is also invariant for A. Thus $\widetilde{A}_\phi \in \{\widetilde{T}\}'$. Further $\widetilde{A}_\phi \in \operatorname{Alg}\operatorname{Lat}(\widetilde{T})$. Hence we can apply Lemma 6.5.1. So $\widetilde{A}_\phi = \sum_{\alpha\leq\phi} a_\alpha \widetilde{T}^\alpha$.

First we note that a_α does not depend on ϕ. Let $\phi \leq \psi$ and $\widetilde{A}_\psi = \sum_{\eta\leq\psi} b_\eta \widetilde{T}^\eta$. Take $\alpha \leq \phi$. Then we have, for $x, y \in H$,

$$a_\alpha(T^\alpha x e_{\mathbf{0}}, y e_\alpha) = (\widetilde{A}_\phi x e_{\mathbf{0}}, y e_\alpha) = (\widetilde{A}_\psi x e_{\mathbf{0}}, y e_\alpha) = b_\alpha(T^\alpha x_{\mathbf{0}}, y e_\alpha).$$

If $T^\alpha x e_0 = 0$ for all $x \in H$ then $a_\alpha = 0 = b_\alpha$ as we mentioned in the proof of Lemma 6.5.1. If $T^\alpha x e_0 \neq 0$ for some x, then $a_\alpha = b_\alpha$. Moreover, $(Axe_\eta, ye_\psi) = a_{\psi\eta}(T^{\psi-\eta} x e_\eta, y e_\psi) = (a_{\psi\eta} S_\eta^{\psi-\eta} x, y)$ for $\eta \leq \psi, x, y \in H$. Now, by Theorem 6.3.4, we get $A \in \mathcal{W}(T_1, \ldots, T_N)$.

Note. Sections 6.1–6.4 generalize the results of [P4], where a pair was considered. The hyporeflexivity, in such a context, is for the first time considered here (in Section 6.5).

7. Functional calculus for N-tuples of contractions

A representation of the polydisc algebra $A(\mathbb{D}^N)$ generated, in the natural way, by an N-tuple of contractions satisfying certain assumptions will be constructed in Section 7.2. The assumptions arise from dilation theory. In this chapter we will equip $H^\infty(\mathbb{D}^N)$ with a weak-star topology, in such a way that the representation of $A(\mathbb{D}^N)$ can be extended to $H^\infty(\mathbb{D}^N)$ under certain conditions on the N-tuple. In the single contraction case it is enough to assume that the spectral measure of the unitary part of the contraction is absolutely continuous with respect to the Lebesgue measure on the unit circle. This is possible since the Lebesgue measure is, in some sense, the "unique" representing measure for $0 \in \mathbb{D}$ for the disc algebra A. The multivariable case is not so simple.

In [KP] one possible approach was presented. In Section 7.1 we take a simpler approach, which is a natural generalization of that of [BDO] and [Oc] where pairs of operators were considered. The approach here is simpler since we deal with a specific situation, but the approach of [KP] is more general.

7.1. H^∞**-type algebras as dual algebras.** We recall some definitions and notations. Let X be a compact set. Denote by $\mathcal{M}(X)$ the set of all complex Borel measures on X. Recall that $\mathcal{M}(X)$ is a Banach space with the total variation norm. If $\mathcal{F}$ is a subspace of measurable complex functions on X, then we denote by $\mathcal{F}^\perp$ the set of all measures in $\mathcal{M}(X)$ which are orthogonal to the functions from $\mathcal{F}$, i.e., $\mathcal{F}^\perp = \{\nu \in \mathcal{M}(X) : \int f\, d\nu = 0$ for all $f \in \mathcal{F}\}$. If E is a subset of $\mathcal{M}(X)$ then E^s will denote the set of all measures on X that are singular with respect to every measure in E. A subset $\mathcal{B}$ of $\mathcal{M}(X)$ is a *band* (see [Bk1]) if $(\mathcal{B}^s)^s = \mathcal{B}$.

Equivalently (see [KS]), $\mathcal{B}$ is a band if $\mathcal{B}$ satisfies the following conditions:

(1) if $\mu \in \mathcal{B}$ and ν is absolutely continuous with respect to $|\mu|$, then $\nu \in \mathcal{B}$,
(2) if $\mu_n \in \mathcal{B}$, $n = 1, 2, \ldots,$ and $\sum_{n=1}^\infty \|\mu_n\| < \infty$, then $\sum_{n=1}^\infty \mu_n \in \mathcal{B}$.

Note that any band is a closed subspace of $\mathcal{M}(X)$. It is also almost elementary that, for every $E \subset \mathcal{M}(X)$, E^s is a band and $(E^s)^s \supset E$.

It is easy to see that $(E^s)^s$ is the smallest band containing E. We call $(E^s)^s$ the band *generated* by E. For further details on bands, we refer to [CG, Section 20]. In what follows we consider measures on the polytorus $\mathbb{T}^N$, hence $X = \mathbb{T}^N$.

As a function algebra we will consider the polydisc algebra $A(\mathbb{D}^N)$. A measure $\nu \in \mathcal{M}(\mathbb{T}^N)$ is called a *representing measure* for $z \in \mathbb{D}^N$ if

$$\int u\, d\nu = u(z) \quad \text{for } u \in A(\mathbb{D}^N).$$

Set $I_N = \{1, \ldots, N\}$. Denote by Ω the set of all functions $\omega : I_N \to \{0, 1\}$ with $\omega \not\equiv 0$. For $\omega \in \Omega$ define $k_\omega = \operatorname{card}\{\omega^{-1}(1)\}$ and let $P_\omega : \mathbb{T}^N \to \mathbb{T}^{k_\omega}$ be the projection onto those variables where ω has value 1.

For all $\omega \in \Omega$, we define a special band of measures $\mathcal{B}_\omega = (P_\omega^{-1}(A(\mathbb{D}^{k_\omega})^\perp))^s$, where $(P_\omega^{-1}\mu)(E) = \mu(P_\omega(E))$. We also denote by $\mathcal{B}_0$ the band of measures on $\mathbb{T}^N$ generated by all representing measures for points in $\mathbb{D}^N$. The facts below were shown in [BDO] for $N = 2$ and generalized in [K1] for arbitrary N.

Proposition 7.1.1. *Let μ be a complex measure on $\mathbb{T}^N$ and assume that $\mu\in\bigcap_{\omega\in\Omega}\mathcal{B}^s_\omega$. Let $f_n\in A(\mathbb{D}^N)$ be a bounded sequence and $f_n(z)\to 0$ for all $z\in\mathbb{D}^N$. Then $f_n\to 0$ weak-star in $L^\infty(|\mu|)$.*

Proposition 7.1.2. *Let μ be a complex measure on $\mathbb{T}^N$ and assume that μ has the following property*:

(∗) *If $f_n\subset A(\mathbb{D}^N)$ is a bounded sequence and $f_n(z)\to 0$ for all $z\in\mathbb{D}^N$ then $f_n\to 0$ weak-star in $L^\infty(|\mu|)$.*

Then $\mu\in\mathcal{B}_0$.

The following lemma is also well known; we show it for completeness.

Lemma 7.1.3. *If $\mu\in\mathcal{M}(\mathbb{T}^N)$ is a representing measure for a certain point in $\mathbb{D}^N$, then μ is singular with respect to $\mathcal{B}_\omega$ for all $\omega\in\Omega$.*

Proof. Let $\mu\in\mathcal{M}(\mathbb{T}^N)$ be a representing measure for a point $z^0=(z^0_1,\ldots,z^0_N)\in\mathbb{D}^N$ for the algebra $A(\mathbb{D}^N)$. Take $\omega\in\Omega$. Then μ is also representing for z^0 for every function $f\circ P_\omega$, where $f\in A(\mathbb{D}^{k_\omega})$. Let i_0 be the smallest number such that $\omega(i_0)=1$ (it exists since $\omega\in\Omega$). Let $((\cdot-z^0_{i_0})\mu)(F)=\int_F(z_{i_0}-z^0_{i_0})\,d\mu(z)$ for all Borel sets $F\subset\mathbb{T}^N$. Then $\mu\ll(\cdot-z^0_{i_0})\mu$ and $(\cdot-z^0_{i_0})\mu$ is orthogonal to $f\circ P_\omega$ for $f\in A(\mathbb{D}^{k_\omega})$. Hence μ is singular with respect to $\mathcal{B}_\omega$.

Summing up we obtain

Proposition 7.1.4. *The band $\mathcal{B}_0$ generated by the representing measures for points in $\mathbb{D}^N$ is equal to $\bigcap_{\omega\in\Omega}\mathcal{B}^s_\omega$. Moreover*, *the property* (∗) *is satisfied for all $\mu\in\mathcal{B}_0$.*

We recall some facts on boundary values following [Ru1]. Let $w=(w_1,\ldots,w_n)\in\mathbb{D}^N$ and $(z_1,\ldots,z_N)\in\mathbb{T}^N$, where $w_j=r_je^{i\theta_j}$ and $z_j=e^{i\phi_j}$. Recall that the Poisson kernel for the polydisc is $P_w(z)=\prod_{j=1}^N P_{r_j}(\theta_j-\phi_j)$, where $P_r(\theta)=(1-r^2)/(1-2r\cos(\theta)+r^2)$ is the classical Poisson kernel. If $f\in L^\infty(m_N)$ (m_N is, as usual, the normalized Lebesgue measure on $\mathbb{T}^N$), then $P[f](w)=\int_{\mathbb{T}^N}P_w(z)f(z)\,dm_N(z)$, $w\in\mathbb{D}^N$, is its Poisson integral and it is easy to check that $P[f]$ is N-harmonic. We denote by $H^\infty(\mathbb{T}^N)$ the set of all functions $f\in L^\infty(m_N)$ with Poisson integral $P[f]$ holomorphic on $\mathbb{D}^N$. If μ is any complex measure in $\mathcal{B}_0$ then $H^\infty(\mu+m_N)$ denotes the weak-star closure of $A(\mathbb{D}^N)$ in $L^\infty(\mu+m_N)$. Now we recall some facts from [Ru1] and sum up what we have proved.

Proposition 7.1.5. *The map $\Psi:H^\infty(\mathbb{T}^N)\ni f\mapsto P[f]\in H^\infty(\mathbb{D}^N)$ is an isometric linear algebra isomorphism. Moreover*,

(1) *$H^\infty(\mathbb{T}^N)$ is a weak-star closed subspace of $L^\infty(m_N)$.*

(2) *A bounded sequence $f_n\subset H^\infty(\mathbb{T}^N)$ converges weak-star to f if and only if $P[f_n]$ converges pointwise to $P[f]$ on $\mathbb{D}^N$.*

(3) *The polynomials are weak-star dense in $H^\infty(\mathbb{T}^N)$.*

The linearity of Ψ and equality of the supremum norms $\|f\|_\infty$ and $\|P[f]\|_\infty$ were shown in [Ru1, Chapter 2]. The map is onto since we can recover the function from $H^\infty(\mathbb{T}^N)$ by taking the radial limits [Ru1, Theorem 2.3.1]. The statement (1) is a consequence of [Ru1, Theorem 2.1.4]. The statement (2) is a consequence of Proposition 7.1.1. The statement (3) follows from [Ru1, Theorem 2.1.4]. The multiplicativity follows from (3).

The main difference between the one-variable and the multivariable case is that, in the latter, it is not enough to consider the algebra $H^\infty(\mathbb{T}^N)$; it is also necessary to consider the algebras $H^\infty(m_N+\mu)$ for all $\mu\in\mathcal{B}_0$. We now show that they are all isomorphic.

PROPOSITION 7.1.6. *Suppose that $\mu\in\mathcal{B}_0$ is a positive measure. Then there is a weak-star homeomorphism $\Psi : H^\infty(\mathbb{T}^N)\to H^\infty(m_N+\mu)$ such that Ψ is the identity on $A(\mathbb{D}^N)$. Moreover, Ψ is an algebra isomorphism.*

PROOF. Define $\Psi(f)=f$ on $A(\mathbb{D}^N)$. If $f\in H^\infty(\mathbb{T}^N)$, by Proposition 7.1.5 we can find a bounded sequence $A(\mathbb{D}^N)\ni f_n\to f$ weak-star. Since m_N is a representing measure, $P[f_n]$ converges pointwise to $P[f]$. Hence, by Proposition 7.1.3, f_n converges weak-star in $L^\infty(m_n+\mu)$ to an element denoted by $\widetilde{f}$. Define $\Psi(f)=\widetilde{f}$. That Ψ is a weak-star homeomorphism follows from Proposition 7.1.1. The multiplicativity follows from the multiplicativity of Ψ on $A(\mathbb{D}^N)$.

Propositions 7.1.5 and 7.1.6 show that the band $\mathcal{B}_0$ gives a unique weak-star topology and we can equip the algebra $H^\infty(\mathbb{D}^N)$ with this structure. From now on, when we consider a weak-star topology of $H^\infty(\mathbb{D}^N)$ we think about the topology given by $\mathcal{B}_0$.

7.2. Representations of $A(\mathbb{D}^N)$. Recall that an algebra homomorphism $\Phi : A(\mathbb{D}^N)\to L(\mathcal{H})$ is a *representation* if $\|\Phi(f)\|\le\|f\|$ for $f\in A(\mathbb{D}^N)$. Consider a pair of commuting contractions T_1,T_2. For every polynomial p in two variables, define the operator $p(T_1,T_2)$ in the natural way. Thus we have the polynomial functional calculus

$$\Phi : p\mapsto p(T_1,T_2). \tag{7.2.1}$$

By Ando's Theorem ([SNF, Theorem I.6.4]), the pair T_1,T_2 has a unitary dilation. More precisely, there is a space $\mathcal{K}\supset\mathcal{H}$ and a pair of unitary operators $U_1,U_2\in L(\mathcal{K})$ such that

$$T_1^nT_2^mx=P_{\mathcal{H}}U_1^nU_2^mx\quad\text{for } x\in\mathcal{H},\ n,m=0,1,2,\dots$$

The pair U_1,U_2 has a spectral measure E on the two-dimensional torus $\mathbb{T}^2$ (see [Berb] for a product of spectral measures) such that

$$U_1^nU_2^m=\int_{\mathbb{T}^2} z_1^nz_2^m\,dE(z_1,z_2)\quad\text{for } n,m=0,1,2,\dots$$

Hence, for any polynomial p in two variables and for $x,y\in\mathcal{H}$, we have

$$|(p(T_1,T_2)x,y)|=|(p(U_1,U_2)x,y)|=\Big|\int_{\mathbb{T}^2} p(z_1z_2)\,dE(z_1,z_2)\Big|\le\|p\|_\infty\|x\|\,\|y\|.$$

Thus we have the von Neumann inequality $\|p(T_1,T_2)\|\le\|p\|_\infty$. Since the polynomials are dense in the bidisc algebra $A(\mathbb{D}^2)$, by the von Neumann inequality, we can extend Φ to $A(\mathbb{D}^2)$ by $\Phi : A(\mathbb{D}^2)\ni h\mapsto h(T_1,T_2)\in L(\mathcal{H})$. Then we have

$$\|h(T_1,T_2)\|\le\|h\|_\infty\quad\text{for } h\in A(\mathbb{D}^2). \tag{7.2.2}$$

The multiplication property $\Phi(uv)=\Phi(u)\Phi(v)$ is easy to see for pairs of polynomials, and by (7.2.2) it also holds for all pairs of functions from $A(\mathbb{D}^2)$. Hence Φ is a representation.

Now consider an N-tuple $T = (T_1, \ldots, T_N) \subset L(\mathcal{H})$ of doubly commuting contractions. It has a unitary dilation by [SNF]. Thus, the representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ can be constructed as above, and

$$\|h(T_1, \ldots, T_N)\| \le \|h\|_\infty \quad \text{for } h \in A(\mathbb{D}^N). \tag{7.2.3}$$

Let $T = (T_1, \ldots, T_N) \subset L(\mathcal{H})$ be a commuting N-tuple of contractions. The polydisc $\mathbb{D}^N$ is said to be a (common) *spectral set* for $T = (T_1, \ldots, T_N)$ if for every polynomial p of N variables the naturally defined operator $p(T_1, \ldots, T_N)$ satisfies

$$\|p(T_1, \ldots, T_N)\| \le \sup\{|p(\lambda_1, \ldots, \lambda_N)| : (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N\}. \tag{7.2.4}$$

Assume that $\mathbb{D}^N$ is a spectral set for given contractions $T = (T_1, \ldots, T_N)$. Using (7.2.4) we can extend the natural representation to the representation

$$\Phi : A(\mathbb{D}^N) \ni u \mapsto u(T_1, \ldots, T_N) \in L(\mathcal{H}).$$

Then (7.2.3) is also satisfied. Our aim is to extend the above representations.

7.3. Extensions to H^∞-type algebras. Having a representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$, it is a consequence of standard techniques that for every $x, y \in \mathcal{H}$ there exists a complex, Borel, regular measure $\mu_{x,y}$ on $\mathbb{T}^N$ such that $(\Phi(u)x, y) = \int u\, d\mu_{x,y}$ for $u \in A(\mathbb{D}^N)$. The measures $\mu_{x,y}$ are called *elementary measures*. We say that Φ is *absolutely continuous* (*a.c.*) if it has a system of elementary measures $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$ such that each $\mu_{x,y}$ is absolutely continuous with respect to some (positive) representing measure ν_z, $z \in \mathbb{D}^N$. By [G, VI.1.2, II.7.5], the above definition is equivalent to one in terms of bands of measures. Namely, we can say that Φ is absolutely continuous if it has a system of elementary measures $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$ which belong to $\mathcal{B}_0$. We say that the N-tuple $T = (T_1, \ldots, T_N)$ is *absolutely continuous* (*a.c.*) if the representation generated by T (provided it exists) is a.c.

Having the representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ constructed in Section 7.2 and using [K1, Proposition] (see also [K1, Sec. 2], [K2, Sec. 3]), we can decompose Φ and $\mathcal{H}$ into orthogonal sums as follows:

$$\Phi = \bigoplus_{i=0}^{M} \Phi_i, \quad \mathcal{H} = \bigoplus_{i=0}^{M} \mathcal{H}_i, \tag{7.3.1}$$

where Φ_i $(i = 0, \ldots, M)$ is the restriction of Φ to $\mathcal{H}_i$ which reduces all the values of Φ. Moreover:

(1) Φ_0 is an absolutely continuous representation,
(2) $T_i|_{\mathcal{H}_i}$ $(i = 1, \ldots, N)$ is a unitary operator with singular spectral measure,
(3) $T_{k_i}|_{\mathcal{H}_i}, T_{l_i}|_{\mathcal{H}_i}$ $(i = N+1, \ldots, M)$ are unitary operators for some k_i, l_i $(k_i \ne l_i)$.

So we get

LEMMA 7.3.1. *If T_1 is a c.n.u.* (*completely non-unitary*) *contraction and T_2 is an a.c. contraction, then the pair $\{T_1, T_2\}$ is a.c.*

LEMMA 7.3.2. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of completely non-unitary contractions which generates a representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ as in Section* 7.2. *Then $T = (T_1, \ldots, T_N)$ is a.c.*

Now we show an extension property of absolutely continuous representations.

PROPOSITION 7.3.3. *If $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ is an absolutely continuous representation then*

(1) *Φ can be extended to a homomorphism, denoted also by Φ, from the algebra $H^\infty(\mathbb{D}^N)$ into $L(\mathcal{H})$ such that*

$$\|\Phi(h)\| \le \|h\|_\infty \quad \text{for } h \in H^\infty(\mathbb{D}^N), \text{ and} \tag{7.3.2}$$

(2) *Φ is continuous with respect to the weak-star topologies.*

PROOF. (1) By Section 7.1, the algebra $H^\infty(\mathbb{D}^N)$ can be equipped with the weak-star topology. For $h \in H^\infty(\mathbb{D}^N)$ and x, y we choose an elementary measure μ_{xy} on $\mathbb{T}^N$ in $\mathcal{B}_0$ and define $(\Phi(h)x, y) = \int h\, d\mu_{xy}$. We check that the definition of $\Phi(h)$ is independent of the choice of μ_{xy}. Take another representing measure $\mu'_{xy} \in \mathcal{B}_0$. Then, for $h \in A(\mathbb{D}^N)$, we have $\int h\, d\mu_{xy} = \int h\, d\mu'_{xy}$. Since $A(\mathbb{D}^N)$ is weak-star dense in $H^\infty(\mathbb{D}^N)$ and $\mu_{xy}, \mu'_{xy} \in \mathcal{B}_0$, we have $\int h\, d\mu_{xy} = \int h\, d\mu'_{xy}$ for $h \in H^\infty(\mathbb{D}^N)$.

Hence the extension Φ is well defined, linear, and we have

$$|(\Phi(h)x, y)| = |\langle h, \mu_{xy}\rangle| \le \|h\|_\infty \|\mu_{xy}\| \le \|h\|_\infty \|x\|\,\|y\|,$$

which gives the inequality (7.3.2).

(2) To show that Φ is weak-star continuous, take $h_n \in H^\infty(\mathbb{D}^N)$ converging weak-star to h. Since m_N is a representing measure and since $A(\mathbb{D}^N)$ is dense in $H^\infty(\mathbb{D}^N)$, $h_n \to h$ pointwise in $\mathbb{D}^N$ and $(h_n - h)$ is bounded. Since the weak-star and weak operator topologies coincide on bounded sets, it is enough to show that $\Phi(h_n - h) \to 0$ in the weak operator topology. Take $x, y \in \mathcal{H}$. Then $\mu_{xy} \in \mathcal{B}_0$. Since $A(\mathbb{D}^N)$ is dense in $H^\infty(m_N + \mu_{xy})$, by Proposition 7.1.1 we get

$$|\Phi(h_n - h)| \le \int |h_n - h|\, d|\mu_{xy}| \to 0.$$

The multiplicativity of Φ for polynomials is straightforward. Thus it can be extended to any functions in $H^\infty(\mathbb{D}^N)$ by weak-star continuity.

Let us sum up our results on the functional calculus:

THEOREM 7.3.4. *Assume that $T = (T_1, \dots, T_N) \subset L(\mathcal{H})$ is an absolutely continuous N-tuple of commuting contractions and that either*

(i) *$T = (T_1, T_2)$ is a pair of contractions ($N = 2$), or*
(ii) *$T = (T_1, \dots, T_N)$ is a doubly commuting N-tuple, or*
(iii) *$\mathbb{D}^N$ is a spectral set for $T = (T_1, \dots, T_N)$.*

Then there is an algebra homomorphism $\Phi : H^\infty(\mathbb{D}^N) \ni h \mapsto h(T) \in \mathcal{A}(T)$ such that

(1) *$\Phi(1) = I$ and $\Phi(p_i) = T_i$ for $i = 1, \dots, N$, where $p_i(z_1, \dots, z_N) = z_i$,*
(2) *$\|h(T)\| \le \|h\|_\infty$ for all $h \in H^\infty(\mathbb{D}^N)$,*
(3) *Φ is weak-star continuous,*
(4) *the range of Φ is weak-star dense in $\mathcal{A}(T)$, and*
(5) *if Φ is an isometry, then it is a weak-star homeomorphism onto $\mathcal{A}(T)$.*

PROOF. We only need to show (4) and (5). To see (4), it is enough to notice that the polynomials in T are weak-star dense in $\mathcal{A}(T)$. The claim (5) is a consequence of [BCP1, Theorem 2.7].

Having an N-tuple $T=(T_1,\dots,T_N)$ and the representation $\Phi : A(\mathbb{D}^N)\to\mathcal{A}(T)$ constructed in Section 7.2, we can consider the adjoint representation Φ^* of $A(\mathbb{D}^N)$ generated by $T^*=(T_1^*,\dots,T_N^*)$. Moreover, in [KOP] the following was proved:

LEMMA 7.3.5. (1) *If the representation Φ is a.c., then Φ^* is a.c. and we can extend Φ^* to $H^\infty(\mathbb{D}^N)$.*

(2) *If $T=(T_1,\dots,T_N)$ is an a.c. N-tuple of commuting contractions and $f\in H^\infty(\mathbb{D}^N)$, then $f(T_1,\dots,T_N)^*=f^\sim(T_1^*,\dots,T_N^*)$, where $f^\sim(z)=\overline{f(\overline{z})}$ for $z\in\mathbb{D}^N$.*

PROOF. Let $u\in A(\mathbb{D}^N)$, and let $\mu_{y,x}$ be an elementary measure of Φ for $y,x\in\mathcal{H}$ absolutely continuous with respect to a representing measure ν_z for some $z\in\mathbb{D}^N$. It is obvious that $u^\sim\in A(\mathbb{D}^N)$. Note also that (2) is trivial for polynomials and it holds for all functions from $A(\mathbb{D}^N)$ since they are norm limits of polynomials.

For a complex measure μ on $\overline{\mathbb{D}}^N$, denote by $\mu^\sim$ the Borel measure $\mu^\sim(\cdot)=\overline{\mu(\Pi(\cdot))}$, where $\Pi : z\mapsto\overline{z}$ is the complex conjugation applied to each coordinate. Then, for $u\in A(\mathbb{D}^N)$, we have

$$\begin{aligned}(u(T_1^*,\dots,T_N^*)x,y)=(x,u(T_1^*,\dots,T_N^*)^*y)&=(x,u^\sim(T_1,\dots,T_N)y)\\&=\overline{(u^\sim(T_1,\dots,T_N)y,x)}=\overline{\int u^\sim\,d\mu_{y,x}}\\&=\int\overline{u^\sim}\,d\overline{\mu_{y,x}}=\int u\,\mu_{y,x}^\sim.\end{aligned}$$

So $\eta_{x,y}:=\mu_{y,x}^\sim$ is an elementary measure of Φ^* for the vectors $x,y\in\mathcal{H}$. Since $\mu_{y,x}$ is absolutely continuous with respect to ν_z, $\eta_{x,y}$ is absolutely continuous with respect to $\nu_z^\sim$, and an easy calculation shows that $\nu_z^\sim$ is a representing measure for $\overline{z}\in\mathbb{D}^N$, which finishes the proof of (1).

Next, to see (2), by (1) and Proposition 7.3.3 we can extend Φ^* to $H^\infty(\mathbb{D}^N)$ and it is easy to see that $f\in H^\infty(\mathbb{D}^N)$ implies $f^\sim\in H^\infty(\mathbb{D}^N)$. Since Φ^* is continuous with respect to the weak-star topologies by Proposition 7.3.3(b) and the adjoint operation is WOT-continuous we can extend (2) from polynomials and functions from $A(\mathbb{D}^N)$ to all elements of $H^\infty(\mathbb{D}^N)$.

Now we find some conditions which imply that the functional calculus is isometric. Recall that a set E contained in the closed unit polydisc $\overline{\mathbb{D}}^N$ is *dominating* for the algebra $H^\infty(\mathbb{D}^N)$ of all bounded analytic functions on $\mathbb{D}^N$ if for all $h\in H^\infty(\mathbb{D}^N)$ we have $\|h\|_\infty=\sup_{z\in E\cap\mathbb{D}^N}|h(z)|$.

We will assume the dominance of some types of spectra. In various situations we will assume that various spectra are dominating for the algebra $H^\infty(\mathbb{D}^N)$: the Taylor spectrum $\sigma(T)$, Taylor essential spectrum $\sigma_e(T)$, left essential spectrum $\sigma_{le}(T)$, or right essential spectrum $\sigma_{re}(T)$.

The following known idea yields the isometry of functional calculus in various situations.

LEMMA 7.3.6. *Suppose that the assumptions of Theorem* 7.3.4 *are satisfied. Assume also that* $E \subset \sigma(T)$ *is dominating for* $H^\infty(\mathbb{D}^N)$*. If* $f \in H^\infty(\mathbb{D}^N)$*, then* $\|f\|_\infty \le \|f(T)\|$.

PROOF. Let $\lambda \in E \cap \mathbb{D}^N \subset \sigma(T) \cap \mathbb{D}^N$. Then, by [Rud], $f(\lambda) \in \sigma(f(T))$ and $\|f\|_\infty \le r(f(T)) \le \|f(T)\|$, which completes the proof.

NOTE. In Section 7.1 a new (simpler than in [KP]) approach was presented to equip $H^\infty(\mathbb{D}^N)$ with the weak-star topology on. Section 7.2 uses some standard techniques based on dilation theory. Theorem 7.3.3 is based on the approach from Section 7.1. Lemma 7.3.5 is a simple generalization of [KOP], where pairs were considered.

8. Dual algebras, invariant subspace problem and reflexivity

Recall that $L(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$, where $\mathcal{C}_1(\mathcal{H})$ is the ideal of trace-class operators and the duality is given by the form $\langle T, S\rangle := \operatorname{tr}(TS)$ for $T \in L(\mathcal{H})$, and $S \in \mathcal{C}_1(\mathcal{H})$. Hence every weak-star closed subspace $\mathcal{M}$ of $L(\mathcal{H})$ is a dual Banach space with predual $\mathcal{Q}_{\mathcal{M}} \cong \mathcal{C}_1(\mathcal{H})/\mathcal{M}_\perp$ via $\langle T, [S]\rangle := \operatorname{tr}(TS)$ for $T \in \mathcal{M}$, and $[S] \in \mathcal{C}_1(\mathcal{H})/\mathcal{M}_\perp$. (Usually we write $\mathcal{Q}$ instead of $\mathcal{Q}_{\mathcal{M}}$.) Thus, for a rank-one operator $x \otimes y$ ($z \mapsto (z, y)x$), we have $\langle T, [x \otimes y]\rangle = (Tx, y)$. Any weak-star closed algebra $\mathcal{A} \subset L(\mathcal{H})$ of operators is usually called a *dual algebra*, since it is a dual space to $\mathcal{C}_1(\mathcal{H})/\mathcal{A}_\perp$.

8.1. The role of rank-one operators. We start by recalling a well-known result [BCP1, Proposition 2.5].

PROPOSITION 8.1.1. *Let* X, Y *be Banach spaces. A linear mapping* $\Phi : X^* \to Y^*$ *is continuous in the weak-star topologies in both spaces if and only if there exists a map* $\phi : Y \to X$ *such that* $\langle \Phi\alpha, y\rangle = \langle \alpha, \phi(y)\rangle$ *for all* $\alpha \in X^*$ *and* $y \in Y$*. Moreover,* $\phi^* = \Phi$.

Until the end of this chapter we assume that the hypothesis of Theorem 7.3.4 are satisfied and that our functional calculus $\Phi : H^\infty(\mathbb{D}^N) \ni h \mapsto h(T) \in \mathcal{A}(T)$ is isometric. Since Φ is weak-star continuous and onto $\mathcal{A}(T)$, there is a mapping $\phi : \mathcal{C}_1(\mathcal{H})/\mathcal{A}(T)_\perp \to L^1(\mathbb{T}^N)/H^\infty(\mathbb{D}^N)_\perp$ such that $\langle h(T), t\rangle = \langle h, \phi(t)\rangle$ for $h \in H^\infty(\mathbb{D}^N)$ and $t \in \mathcal{C}_1(\mathcal{H})/\mathcal{A}(T)_\perp$. Let P_λ be a reproducing kernel for a point $\lambda \in \mathbb{D}^N$. Then $P_\lambda \in L^1(\mathbb{T}^N)$ and $\langle h, P_\lambda\rangle = h(\lambda)$ for $h \in H^\infty(\mathbb{D}^N)$. We write $[C_\lambda] = \phi^{-1}([P_\lambda])$. Moreover, $\langle h(T), [C_\lambda]\rangle = \langle h, [P_\lambda]\rangle = h(\lambda)$ for $\lambda \in \mathbb{D}^N$ and $h \in H^\infty(\mathbb{D}^N)$.

To explain the role of rank-one operators, we restrict ourselves to a very simple situation. We consider only a single operator instead of an N-tuple and we aim at showing the existence of a non-trivial invariant subspace instead of reflexivity.

Assume that $\Phi : H^\infty \to \mathcal{A}(T)$ is an isometry and a weak-star homeomorphism. Thus we have a mapping $\phi : \mathcal{C}_1(\mathcal{H})/\mathcal{A}(T)_\perp \to L^1/H^\infty_\perp$. Assume also that for any $[L] \in \mathcal{Q}$ there are $x, y \in \mathcal{H}$ such that $[L] = [x \otimes y]$. We now show that this gives an invariant subspace for T. Consider $[C_0] = \phi^{-1}([P_0])$ and assume that $[C_0] = [x \otimes y]$ for some $x, y \in \mathcal{H}$. Thus, for any $h \in H^\infty$, we have

$$h(0) = \langle h, [P_0]\rangle = \langle h(T), [C_0]\rangle = \langle h(T), [x \otimes y]\rangle = (h(T)x, y).$$

If $h(\lambda) \equiv 1$, then $1 = (x, y)$ and $x \neq 0$, $y \neq 0$. Now take $h(\lambda) = \lambda g(\lambda)$ for $g \in H^\infty$. Then $0 = (Tg(T)x, y)$, which means that $y \perp \operatorname{span}\{T^n Tx : n \geq 0\} := M$. Thus $M \neq \mathcal{H}$ since $y \neq 0$, and if $\ker T = \{0\}$, then $M \neq \{0\}$. If $\ker T \neq \{0\}$, then either $\ker T$ is a non-trivial invariant subspace or T is a zero operator. Hence either $\ker T$ or M is a non-trivial invariant subspace (assuming that $\dim \mathcal{H} > 1$).

8.2. Approximation in predual spaces. In this section we state some sufficient approximation conditions for reflexivity. We denote by $\overline{\operatorname{aco}}\,\Omega$ the closure of the absolutely convex hull of $\Omega \subset \mathbb{C}^N$. With the assumptions and notations of the previous section we have

LEMMA 8.2.1. *Assume that E is dominating for $H^\infty(\mathbb{D}^N)$. Then $\overline{\operatorname{aco}}\{[C_\lambda]_{\mathcal{Q}} : \lambda \in E \cap \mathbb{D}^N\}$ contains the closed unit ball in $\mathcal{Q}$ about the origin.*

PROOF. If $f \in H^\infty(\mathbb{D}^N)$, then

$$\|\Phi(f)\| \leq \|f\|_\infty = \sup_{\lambda \in E} |f(\lambda)| = \sup_{\lambda \in E} \langle \Phi(f), [C_\lambda] \rangle.$$

Now the result follows from the next proposition (see [BCP1, Proposition 2.8]).

PROPOSITION 8.2.2. *Let X be a complex Banach space, and let E be a subset of the closed unit ball $\mathcal{B}$ of X such that for all ϕ in X^*, $\|\phi\| = \sup_{x \in E} |\langle x, \phi \rangle|$. Then the closure of the absolutely convex hull of E is the entire unit ball $\mathcal{B}$.*

Suppose that $\mathcal{M}$ is a weak-star closed subspace of $L(\mathcal{H})$. Recall that $\mathcal{X}_0(\mathcal{M})$ denotes the set of all $[L] \in \mathcal{Q}_{\mathcal{M}}$ such that there exist sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ of vectors in $\mathcal{H}$ with $\|x_n\| \leq 1, \|y_n\| \leq 1$ for all n, satisfying

$$\lim_{n \to \infty} \|[x_n \otimes y_n] - [L]\|_{\mathcal{Q}} = 0, \tag{8.2.1}$$

$$\lim_{n \to \infty} \|[x_n \otimes w]\|_{\mathcal{Q}} = 0 \quad \text{for all } w \in \mathcal{H}, \tag{8.2.2}$$

$$\lim_{n \to \infty} \|[w \otimes y_n]\|_{\mathcal{Q}} = 0 \quad \text{for all } w \in \mathcal{H}. \tag{8.2.3}$$

In [BFP2] it is shown that $\mathcal{X}_0(\mathcal{M})$ is absolutely convex and closed. Recall also that $\mathcal{M}$ has *property* $X_{0,1}$ if the unit ball of $\mathcal{Q}_{\mathcal{M}}$ is contained in $\mathcal{X}_0(\mathcal{M})$.

An important observation from [BFP2] (see Theorems 3.4, 9.20 and Remark 9.14) is

PROPOSITION 8.2.3. *Every dual algebra $\mathcal{A}$ with property $X_{0,1}$ is reflexive.*

We will use the above to show reflexivity.

9. Reflexivity of jointly quasinormal operators and spherical isometries

In view of Deddens's result on the reflexivity of an isometry (Theorem 1.2.6) it is natural to ask about the reflexivity of any commuting N-tuple of isometries. A first result in that direction was shown in [P1]: pairs of doubly commuting isometries are reflexive. There were some other partial results in [P3]. The following recent result based on [HM] was shown in [Ber].

THEOREM 9.1. *Every commuting family of isometries* $\mathbf{V} = (V_\alpha)_{\alpha\in J}$ *is reflexive and has property* $\mathbb{A}_1(1)$.

Now we can turn our attention to quasinormal operators. Recall that an operator T is called *quasinormal* if T commutes with T^*T. Wogen [W] proved that individual quasinormal operators are reflexive. Following Lubin [Lu], we call a commutative family $\mathcal{S}$ of operators *jointly quasinormal* if S and T^*T commute for any $S, T \in \mathcal{S}$. Commutative families of normal operators or isometries are automatically jointly quasinormal, as are doubly commuting families of quasinormal operators. The result below was shown in [AP2].

THEOREM 9.2. *Every jointly quasinormal family* $\mathcal{S}$ *of operators in a separable Hilbert space* $\mathcal{H}$ *is reflexive and has property* $\mathbb{A}_1(1)$.

PROOF. Since $\mathcal{H}$ is separable, we may assume $\mathcal{S}$ to be countable. Write $\mathcal{Z}$ for the commutative von Neumann algebra generated by $\{T^*T : T \in \mathcal{S}\}$. By the direct integral theory (see [Sw]), $\mathcal{Z}$ is a diagonal algebra corresponding to a direct integral decomposition $\mathcal{H} = \int_\Lambda^\oplus H(\lambda)\, d\mu(\lambda)$. Here μ is a finite regular Borel measure on Λ and $\mathcal{Z} = \int_\Lambda^\oplus \mathcal{Z}(\lambda)\, d\mu(\lambda)$, where each $\mathcal{Z}(\lambda)$ consists of scalar multiples of $I_{H(\lambda)}$. For simplicity of notation, we assume that $H(\lambda) \equiv H$ and that the corresponding field of measurable vectors is the constant field. Each $T \in \mathcal{S}$ is decomposable, and by our choice of $\mathcal{Z}$, we know that $T^*(\lambda)T(\lambda)$ is a scalar multiple of the identity for almost all λ. Discarding a set of measure zero if necessary, we can assume $T(\lambda)$ to be a scalar multiple of an isometry for each $T \in \mathcal{S}$ and $\lambda \in \Lambda$.

For each fixed λ, the algebra $\mathcal{W}(T(\lambda) : T \in \mathcal{S})$ is generated by a family of commuting isometries. By Theorem 9.1, it is reflexive and has property $\mathbb{A}_1(1)$. Hence $\mathcal{W}(\mathcal{Z} \cup \mathcal{S})$ is reflexive by [AFG, Proposition 5.6] and has property $\mathbb{A}_1(1)$ by [HN1, Theorem 3.6]. Thus its subalgebra $\mathcal{W}(\mathcal{S})$ is reflexive and has property $\mathbb{A}_1(1)$ by [HN1, Proposition 2.5].

An N-tuple $V = (V_1, \ldots, V_N)$ is called a *spherical isometry* if $\sum_{i=1}^N V_i^* V_i = I$. Using Theorem 9.2 we can prove the following.

PROPOSITION 9.3. *Every N-tuple $V = (V_1, \ldots, V_N)$ of doubly commuting spherical isometries is reflexive.*

PROOF. First, we show that each V_i, $i = 1, \ldots, N$, is quasinormal. This follows from the equalities

$$\begin{aligned} V_i V_i^* V_i &= V_i\Big(I - \sum_{j\neq i} V_j^* V_j\Big) = V_i - \sum_{j\neq i} V_i V_j^* V_j \\ &= V_i - \sum_{j\neq i} V_j^* V_j V_i = \Big(I - \sum_{j\neq i} V_j^* V_j\Big) V_i = V_i^* V_i V_i. \end{aligned}$$

Now since V is a doubly commuting set of quasinormal operators, it is also jointly quasinormal and the conclusion follows from Theorem 9.2

NOTE. Theorem 9.2 was first shown in [AP2]. Proposition 9.3 is a new result.

10. Reflexivity and existence of invariant subspaces for tuples of contractions

The dual algebra technique has had great achievements in showing the existence of a non-trivial invariant subspace or reflexivity for a single operator. Recall some of them: [BFP1], [BFP2], [BCFP], [BCP1], [BCP2], [CP1], [CP2], [CEP], [CPS], [We], [OT], [Ro1], [Ro2]. The most striking result will be discussed in Chapter 11. Let us present the following two facts which give the inspiration for the results for N-tuples of contractions presented in this chapter. Instead of presenting them in their full strength, we show them in the form that permits the theorems below to be seen as their N-tuple generalization. The first one is from [Ro1].

THEOREM 10.0.1. *Let T be a $\mathbf{C}_{0\cdot}$ contraction. If $\sigma_{le}(T) \cap \mathbb{D}$ is dominating for H^∞, then $\mathcal{W}(T)$ is reflexive.*

The next statement from [BFLP] is stronger.

THEOREM 10.0.2. *Let T be a completely non-unitary contraction. If $\sigma_e(T) \cap \mathbb{D}$ is dominating for H^∞, then $\mathcal{W}(T)$ is reflexive.*

10.1. Results with dominance of left and right spectra. We start with the following

THEOREM 10.1.1. *Let $\{T_1, T_2\} \subset L(\mathcal{H})$ be a pair of commuting contractions. Assume also that $T_1 \in \mathbf{C}_{0\cdot}$. If $\sigma_l(T_1, T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then $\{T_1, T_2\}$ has a common non-trivial invariant subspace.*

If T_2 is not a.c., we can decompose T_2 into an absolutely continuous part and a singular unitary part. If the subspace where T_2 is a singular unitary operator is not $\{0\}$, then this subspace is hyperinvariant for T_2. Hence we have a non-trivial invariant subspace for the pair $\{T_1, T_2\}$, unless the space $\mathcal{H}$ is one-dimensional. But $\mathcal{H}$ cannot be one-dimensional, since $\sigma_l(T_1, T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$. Thus we can assume that T_2 is a.c.

Assume now that $\sigma_l(T_1, T_2) \setminus \sigma_{le}(T_1, T_2) \neq \emptyset$ and take $\lambda = (\lambda_1, \lambda_2) \in \sigma_l(T_1, T_2) \setminus \sigma_{le}(T_1, T_2)$. Then, by Lemmas 5.1.1 and 5.1.2, there is a sequence x_n in some finite-dimensional subspace $\mathcal{F}$ of $\mathcal{H}$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(\lambda_i - T_i)x_n\| = 0$ for $i = 1, 2$. Since the ball in a finite-dimensional space is compact, there is $x \in \mathcal{H}$ with $\|x\| = 1$ and $\|(\lambda_i - T_i)x\| = 0$ for $i = 1, 2$. Then either $\ker(\lambda_i - T_i)$ is a non-trivial hyperinvariant subspace for T_1, T_2, or $T_1 = \lambda_1 I$ and $T_2 = \lambda_2 I$. Thus (T_1, T_2) has a common non-trivial invariant subspace.

Hence we can also assume that $\sigma_l(T_1, T_2) = \sigma_{le}(T_1, T_2)$. In this case Theorem 10.1.1 reduces to the following reflexivity result.

THEOREM 10.1.2. *Let $\{T_1, T_2\} \subset L(\mathcal{H})$ be a pair of commuting contractions. Assume also that $T_1 \in \mathbf{C}_{0\cdot}$ and T_2 is absolutely continuous. If $\sigma_{le}(T_1, T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then the algebra $\mathcal{W}(T_1, T_2)$ is reflexive.*

Now let us discuss the related results. First Theorem 10.1.1 generalizes for pairs of operators the following result of [Ap]:

Theorem 10.1.3. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of $\mathbf{C}_{00}$ contractions and suppose $\mathbb{D}^N$ is a spectral set for T. If $\sigma_{le}(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then T has a common non-trivial invariant subspace.*

See [Ap] or [AC] for the definition of the class $\mathbf{C}_{00}$ for N-tuples, which is different from the classical one for a single operator.

By the symmetry of the conditions for left and right spectra and of the $\mathbf{C}_{0\cdot}$ and $\mathbf{C}_{\cdot 0}$ conditions, Theorem 10.1.2 can be stated in the following way.

Theorem 10.1.2′. *Let $\{T_1, T_2\} \subset L(\mathcal{H})$ be a pair of commuting contractions. Assume also that $T_1 \in \mathbf{C}_{\cdot 0}$ and T_2 is absolutely continuous. If $\sigma_{re}(T_1, T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then the algebra $\mathcal{W}(T_1, T_2)$ is reflexive.*

Thus the above results overlap with a result of [E2]:

Theorem 10.1.4. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of commuting completely non-unitary contractions having $\mathbb{D}^N$ as a spectral set. Assume also that there are $i, j \in \{1, \ldots, N\}$ such that $T_i \in \mathbf{C}_{0\cdot}$ and $T_j \in \mathbf{C}_{\cdot 0}$. If $\sigma_H(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $\mathcal{W}(T)$ is reflexive.*

Finally, let us quote a related result of [KP] with dominance of the left essential spectrum.

Theorem 10.1.5. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of doubly commuting completely non-unitary contractions. If $\sigma_{le}(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $\mathcal{W}(T)$ is reflexive.*

Now we can start

Proof of Theorem 10.1.2. We can construct the representation $\Phi : A(\mathbb{D}^2) \to L(\mathcal{H})$ generated by $T = (T_1, T_2)$ as in Section 7.2. Moreover, by Lemma 7.3.1, T is a.c., and thus we can extend Φ to $H^\infty(\mathbb{D}^2)$. Since $\sigma_{le}(T_1, T_2)$ is dominating for $H^\infty(\mathbb{D}^2)$, we see that Φ is an isometry and a weak-star homeomorphism by Theorem 7.3.4 and by Lemma 7.3.6. Recall that the set $\mathcal{X}_0(\mathcal{A}(T))$ is absolutely convex and closed. By Lemma 8.2.1 and Proposition 8.2.3, to show the reflexivity of $\mathcal{A}(T_1, T_2)$, it is enough to check that $[C_\lambda] \in \mathcal{X}_0(\mathcal{A}(T))$ for $\lambda \in \sigma_{le}(T_1, T_2) \cap \mathbb{D}^2$.

Let $\lambda \in \sigma_{le}(T) \cap \mathbb{D}^2$. Then there exists an orthonormal sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|(T_i - \lambda_i)x_n\| = 0$ for $i = 1, 2$. To show that $[C_\lambda] \in \mathcal{X}_0(\mathcal{A}(T))$ we need two sequences which satisfy (8.2.1)–(8.2.3). Take both of them equal to $\{x_n\}$. Then Lemmas 10.1.6, 10.1.7 and 10.1.9 below finish the proof of Theorem 10.1.2.

The first two lemmas are true not only for $N = 2$ but also for an arbitrary N and since they might be of independent interest we present them as follows.

Lemma 10.1.6. *Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \sigma_{le}(T_1, \ldots, T_N) \cap \mathbb{D}^N$ and let $\{x_n\}$ be an orthonormal sequence with $\lim_{n\to\infty} \|(T_i - \lambda_i)x_n\| = 0$ for all $i = 1, \ldots, N$. Then*

$$\lim_{n\to\infty} \|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = 0.$$

Proof. The Hahn–Banach theorem implies that there is $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\|_\infty = 1$ and $\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = \langle f_n(T),\ [x_n \otimes x_n] - [C_\lambda]\rangle$. The Gleason property for the polydisc shows that there are g_i^n such that $f_n(z) = f_n(\lambda) + \sum_{i=1}^N (z_i - \lambda_i) g_i^n(z)$ for $z \in \mathbb{D}^N$ and $\|g_i^n\|_\infty \le M_\lambda \|f_n\|_\infty = M_\lambda$. Hence

$$\begin{aligned}
\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} &= \Big\langle f_n(\lambda) + \sum_{i=1}^N g_i^n(T)(T_i - \lambda_i), [x_n \otimes x_n] - [C_\lambda] \Big\rangle \\
&= \langle f_n(\lambda), [x_n \otimes x_n]\rangle - \langle f_n(\lambda), [C_\lambda]\rangle \\
&\quad + \sum_{i=1}^N \langle g_i^n(T)(T_i - \lambda_i), [x_n \otimes x_n]\rangle + \sum_{i=1}^N \langle g_i^n(T)(T_i - \lambda_i), [C_\lambda]\rangle \\
&= f_n(\lambda)(x_n, x_n) - f_n(\lambda) + \sum_{i=1}^N (g_i^n(T)(T_i - \lambda_i)x_n, x_n) \\
&\quad + \sum_{i=1}^N g_i^n(\lambda)(\lambda_i - \lambda_i) \\
&\le \sum_{i=1}^N \|g_i^n(T)\| \, \|(T_i - \lambda_i)x_n\| \le M_\lambda \sum_{i=1}^N \|(T_i - \lambda_i)x_n\| \to 0.
\end{aligned}$$

Thus, the proof is complete.

LEMMA 10.1.7. *Let* $\lambda = (\lambda_1, \ldots, \lambda_N) \in \sigma_{le}(T_1, \ldots, T_N) \cap \mathbb{D}^N$ *and let* x_n *be an orthonormal sequence such that* $\lim_{n\to\infty} \|(T_i - \lambda_i)x_n\| = 0$ *for all* $i = 1, \ldots, N$. *Then* $\lim_{n\to\infty} \|[x_n \otimes y]\|_{\mathcal{Q}} = 0$ *for any fixed* $y \in \mathcal{H}$.

PROOF. Via the Hahn–Banach theorem, there is $f_n \in H^\infty(\mathbb{D}^2)$ such that $\|f_n(T)\| = \|f_n\|_\infty = 1$ and $\|[x_n \otimes y]\|_{\mathcal{Q}} = \langle f_n(T), x_n \otimes y\rangle = (f_n(T)x_n, y)$. As above, the Gleason property shows that there are g_i^n such that $f_n(z) = f_n(\lambda) + \sum_{i=1}^N (z_i - \lambda_i) g_i^n(z)$ for $z \in \mathbb{D}^N$ and $\|g_i^n\|_\infty \le M_\lambda \|f_n\|_\infty = M_\lambda$. Now

$$\begin{aligned}
\|[x_n \otimes y]\|_{\mathcal{Q}} &= f_n(\lambda)(x_n, y) + \sum_{i=1}^N (g_i^n(T)(T_i - \lambda_i)x_n, y) \\
&\le \|f_n\|_\infty |(x_n, y)| + \sum_{i=1}^N \|g_i^n(T)\| \, \|(T_i - \lambda_i)x_n\| \, \|y\| \\
&\le |(x_n, y)| + \sum_{i=1}^N \|g_i^n\|_\infty \|(T_i - \lambda_i)x_n\| \, \|y\| \\
&\le |(x_n, y)| + M_\lambda \|y\| \sum_{i=1}^N \|(T_i - \lambda_i)x_n\| \to 0
\end{aligned}$$

since x_n is an orthonormal sequence and $\|(T_i - \lambda_i)x_n\| \to 0$ for $i = 1, \ldots, N$.

It is an easy observation that in [Ro1, Lemma 3.4] the c.n.u. assumption is not essential, only the absolute continuity of T is needed. Moreover, weaker assumptions on the sequence $\{x_n\}$ are sufficient. So we have

LEMMA 10.1.8. *Assume that* $T \in L(\mathcal{H})$ *is an a.c. contraction generating an isometric functional calculus. If* $\{x_n\}$ *is a sequence such that* $x_n \to 0$ *weakly*, $\|x_n\| = 1$ *and* $\|(T - \lambda)x_n\| \to 0$, *then* $\|[y \otimes x_n]\|_{\mathcal{Q}_T} \to 0$ *for all* $y \in \mathcal{H}$.

The last approximation lemma will be proved for $N = 2$.

LEMMA 10.1.9. *Assume that* T_1, T_2 *generate an a.c. isometric representation and* $T_1^n \to 0$ *strongly. If* $\{x_n\}$ *is a sequence such that* $x_n \to 0$ *weakly*, $\|x_n\| = 1$ *and* $\|(T_2 - \lambda_2)x_n\| \to 0$, *then* $\|[y \otimes x_n]\|_{\mathcal{Q}} \to 0$ *for all* $y \in \mathcal{H}$.

PROOF. By [SNF, Theorem II.2.1], choose a minimal isometric dilation $V_1 \in L(\mathcal{K})$ of T_1^*. Then V_1 is a unilateral shift of a certain multiplicity and $T_1 = V_1^*|_{\mathcal{H}}$. On the other hand, by the commutant lifting theorem of Sz.-Nagy and Foiaş (see [SNF], [Pa, p. 484]), there is an operator W_2 (not necessarily an isometry) preserving the norm of T_2^* such that the pair $\{V_1, W_2\}$ dilates $\{T_1^*, T_2^*\}$. Let $\varepsilon > 0$. Choose $M > 0$ such that $\|(I - P_{\ker V_1^{*M}})y\| \le \varepsilon/3$ and set $y_1 = (P_{\ker V_1^{*M}})y$, $y_2 = (I - P_{\ker V_1^{*M}})y$. By the Hahn–Banach theorem, for each n, there is $f_n \in H^\infty(\mathbb{D}^2)$ such that

$$\|[y \otimes x_n]\|_{\mathcal{Q}} = \langle f_n(T_1, T_2), [y \otimes x_n]\rangle = (f_n(T_1, T_2)y, x_n), \quad \|f_n\| = 1.$$

For each n we can decompose f_n as follows:

$$f_n(z_1, z_2) = \sum_{k=0}^{M-1} a_{nk}(z_2)z_1^k + z_1^M h_n(z_1, z_2).$$

Note first that the functions a_{nk} are measurable. Moreover, since $a_{nk}(z_2)$ is the kth Fourier coefficient of $f_n(\cdot, z_2)$, we have $|a_{nk}(z_2)| \le 1$ for $z_2 \in \mathbb{D}$. Thus $\|a_{nk}\| \le 1$ and consequently $\|h_n\| \le M + 1$. It is easy to check that the negative Fourier coefficients of every a_{nk} vanish. Hence, for every n, k, we get $a_{nk} \in H^\infty(\mathbb{D})$ and $h_n \in H^\infty(\mathbb{D}^2)$.

Applying the Lebesgue-type decomposition (7.3.1) to the space $\mathcal{K}$, we get $\mathcal{H} \subset \mathcal{K}_0$ via an easy calculation on elementary measures. So, by the minimality of V_1, we have $\mathcal{K} = \mathcal{K}_0$, and hence the pair $\{V_1, W_2\}$ is a.c. . By Lemma 7.3.5, $\{V_1^*, W_2^*\}$ is also a.c. By [K1, Proposition], the representation generated by T_2 is a.c. and the same is true for W_2^*.

The above facts give us the existence of the functional calculus for all the above mentioned pairs and single operators. So, applying Lemma 7.3.5 to the pairs $\{T_1, T_2\}$ and $\{V_1^*, W_2^*\}$, we have

$$\begin{aligned} \|[y \otimes x_n]\|_{\mathcal{Q}} &= |(f_n(T_1, T_2)y, x_n)| = |(y, f_n^{\sim}(T_1^*, T_2^*)x_n)| \\ &\le |(y, f_n^{\sim}(V_1, W_2)x_n)| = |(f_n(V_1^*, W_2^*)y, x_n)| \\ &\le |(f_n(V_1^*, W_2^*)y_1, x_n)| + |(f_n(V_1^*, W_2^*)y_2, x_n)| \\ &\le \sum_{k=0}^{M-1} |(a_{nk}(W_2^*)V_1^{*k}y_1, x_n)| \\ &\quad + |(V_1^{*M}h_n(V_1^*, W_2^*)y_1, x_n)| + \|f_n\|\,\|y_2\|\,\|x_n\|. \end{aligned} \tag{10.1.1}$$

The vector y_1 satisfies $V_1^{*M}y_1 = 0$, and thus the second term is 0. Observe that for all k,

$$|(a_{nk}(W_2^*)V_1^{*k}y_1, x_n)| \le \|[V_1^{*k}y_1 \otimes x_n]\|_{\mathcal{Q}_{W_2^*}}.$$

On the other hand, by [SNF, Theorem II.2.3], W_2^* is an extension of T_2. Thus not only $\|(W_2^* - \lambda_2)x_n\| \to 0$, but also W_2^* generates an isometric representation, since T_2 does. Hence, Lemma 10.1.8 shows that $\|[V_1^{*k}y_1 \otimes x_n]\|_{\mathcal{Q}_{W_2^*}} \to 0$ $(n \to \infty)$. Hence we can choose n_0 such that $\|[V_1^{*k}y_1 \otimes x_n]\|_{\mathcal{Q}_{W_2^*}} \le \varepsilon/(3M)$ for all $n > n_0$ and $k = 0, 1, \ldots, M - 1$.

Coming back to the estimation of $\|[y\otimes x_n]\|_{\mathcal{Q}}$, using (10.1.1) and the estimation of $\|y_2\|$, we obtain

$$\|[y\otimes x_n]\|_{\mathcal{Q}} \le \sum_{k=0}^{M-1} \|[V_1^{*k} y_1 \otimes x_n]\|_{\mathcal{Q}_{W_2^*}} + \|f_n\|\,\|y_2\|\,\|x_n\| \le \varepsilon$$

for $n > n_0$. The proof of the lemma is finished.

10.2. Results with dominance of Taylor spectrum. In this section we present the result of [AP]. The main result is

THEOREM 10.2.1. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of doubly commuting contractions. If $\sigma(T)\cap\mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T = (T_1,\dots,T_N)$ has a common non-trivial invariant subspace.*

We reduce the proof of Theorem 10.2.1 to

THEOREM 10.2.2. *Let $T = (T_1,\dots,T_N)$ be an N-tuple of doubly commuting completely non-unitary contractions. If $\sigma_e(T)\cap\mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T = (T_1,\dots,T_N)$ is reflexive.*

Of course, reflexivity is a much stronger property than the existence of a non-trivial common invariant subspace. Theorem 10.2.2 is a generalization of the reflexivity result for a single contraction mentioned in Theorem 10.0.2. It also improves a theorem of [KP] stated as Theorem 10.1.5.

Let us also quote another related result from [AC].

THEOREM 10.2.3. *Let $T = (T_1,\dots,T_N)$ be an N-tuple of commuting completely non-unitary contractions of class $\mathbf{C}_{00}$ and suppose $\mathbb{D}^N$ is a spectral set for T. If the intersection of the Taylor essential spectrum with the open polydisc $\sigma_e(T)\cap\mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$ then T has a common non-trivial invariant subspace.*

To reduce Theorem 10.2.1 to Theorem 10.2.2, note first that if one of the contractions $T_1,\dots,T_N$, say T_{i_0}, has a non-trivial unitary part, then from the formula for the subspace $\mathcal{H}_{ui_0}$ where the contraction T_{i_0} is unitary (cf. [SNF, Theorem I.3.2]), we see that $\mathcal{H}_{ui_0}$ is invariant for all operators S doubly commuting with T_{i_0}, and $\mathrm{Lat}(T)$ will be non-trivial unless the space $\mathcal{H}$ is one-dimensional. But $\mathcal{H}$ cannot be one-dimensional, since $\sigma(T)\cap\mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$. Hence we may assume that $T_1,\dots,T_N$ are all c.n.u.

The proposition below allows us to assume that $\sigma(T) = \sigma_e(T)$ and we are in the situation of Theorem 10.2.2.

PROPOSITION 10.2.4. *Let $T = (T_1,\dots,T_N)$ be an N-tuple of doubly commuting operators in $L(\mathcal{H})$. If $\sigma(T)\setminus\sigma_e(T)\ne\emptyset$, then either T_{i_0} has a non-trivial hyperinvariant subspace for some $i_0\in\{1,\dots,N\}$ or the underlying Hilbert space is one-dimensional.*

This proposition is a consequence of

LEMMA 10.2.5. *Let $T = (T_1,\dots,T_N)$ be an N-tuple of doubly commuting operators in $L(\mathcal{H})$. If, for some $\lambda\in\mathbb{C}^N$ and some $p\in\{1,\dots,N\}$, we have*

$$\ker\delta^p(\lambda - T)\cap\operatorname{ran}\delta^{p-1}(\lambda-T)^{\perp}\ne\{0\},$$

then either there is $i_0 \in \{1,\dots,N\}$ such that T_{i_0} has a non-trivial hyperinvariant subspace or the tuple consists of scalar multiples of the identity operator.

PROOF. By our assumptions, there exists a non-zero $\omega \in \Lambda^p(\mathcal{H})$ with $\delta^p(\lambda - T)\omega = 0 = \delta^{p-1}(\lambda - T)^*\omega$. Hence

$$\delta^{p-1}(\lambda - T)\delta^{p-1}(\lambda - T)^*\omega + \delta^p(\lambda - T)^*\delta^p(\lambda - T)\omega = 0.$$

By Lemma 5.2.2, there are disjoint sets $\mathcal{S},\mathcal{T}$ with $\mathcal{S}\cup\mathcal{T} = \{1,\dots,N\}$ and a vector $0 \neq x \in \mathcal{H}$ such that

$$\sum_{i\in\mathcal{S}}(\lambda_i - T_i)^*(\lambda_i - T_i)x + \sum_{k\in\mathcal{T}}(\lambda_k - T_k)(\lambda_k - T_k)^*x = 0.$$

Hence $x \in \ker(\lambda_i - T_i)$ for $i \in \mathcal{S}$ and $x \in \ker(\lambda_k - T_k)^*$ for $k \in \mathcal{T}$. From this we obtain a non-trivial hyperinvariant subspace or else $T_j = \lambda_j$ for all $j \in \{1,\dots,N\}$.

The following lemma is crucial in the proof of Theorem 10.2.2.

LEMMA 10.2.6. *Assume that $T = (T_1,\dots,T_N)$ is an N-tuple of doubly commuting completely non-unitary contractions. Let $\lambda \in \sigma_e(T)\cap\mathbb{D}^N$. Then there are disjoint sets $\mathcal{S},\mathcal{T}$ such that $\mathcal{S}\cup\mathcal{T} = \{1,\dots,N\}$, a point $\lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{D}^N$ and a sequence $\{x_n\}$ with $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n such that $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$.*

PROOF. By Lemma 5.2.1, there is $p \in \{1,\dots,N\}$ and an orthonormal sequence $\{\eta_n\}_{n=1}^{\infty}$ in $\Lambda^p(\mathcal{H})$ such that (5.2.1) holds. Passing to a subsequence if necessary, we may assume that for some $I = (i_1,\dots,i_p) \in \mathbb{N}^p$ the coefficients x_n of $s_{i_1}\wedge\ldots\wedge s_{i_p}$ in η_n satisfy $\|x_n\| \ge \alpha$ for all $n \in \mathbb{N}$ and for some $\alpha > 0$. By Lemma 5.2.2, there are disjoint sets $\mathcal{S},\mathcal{T}$ with $\mathcal{S}\cup\mathcal{T} = \{1,\dots,N\}$ such that

$$\sum_{i\in\mathcal{S}}(\lambda_i - T_i)^*(\lambda_i - T_i)x_n + \sum_{k\in\mathcal{T}}(\lambda_k - T_k)(\lambda_k - T_k)^*x_n \to 0.$$

Taking the scalar product with x_n, we get $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_k^* - \overline{\lambda}_k)x_n\| \to 0$ for all $k \in \mathcal{T}$. Since the sequence $\{\eta_n\}_{n=1}^{\infty}$ is orthonormal, it follows that $x_n \to 0$ weakly. Moreover, since the numbers $\|x_n\|$ are bounded below, we can assume without loss of generality that $\|x_n\| = 1$.

Now we can construct the representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ generated by $T = (T_1,\dots,T_N)$ as in Section 7.2. Moreover, by Lemma 7.3.2, the N-tuple T is a.c., since each T_i is c.n.u.. Thus we can extend Φ to $H^\infty(\mathbb{D}^N)$ and show that it is a weak-star homeomorphism by Theorem 7.3.4. Since $\sigma_e(T)$ is dominating for $H^\infty(\mathbb{D}^N)$, we see that Φ is an isometry by Lemma 7.3.6. Recall that the set $\mathcal{X}_0(\mathcal{A}(T))$, defined at the end of Section 8.2, is absolutely convex and closed. By Lemma 8.2.1 and Proposition 8.2.3, it is enough to check that $[C_\lambda] \in \mathcal{X}_0(\mathcal{A}(T))$ for $\lambda \in \sigma_e(T)\cap\mathbb{D}^N$ to show the reflexivity of $\mathcal{A}(T)$.

Let $\lambda \in \sigma_e(T)\cap\mathbb{D}^N$. To show that $[C_\lambda] \in \mathcal{X}_0(\mathcal{A}(T))$ we need two sequences which satisfy (8.2.1)–(8.2.3). Take both of them equal to the sequence $\{x_n\}$ given by Lemma 10.2.6. Then Lemmas 10.2.7–10.2.9 below finish the proof of Theorem 10.2.2.

We start with the approximation of the point evaluation (8.2.1).

LEMMA 10.2.7. *Let $T = (T_1, \ldots, T_N)$ be an a.c. N-tuple of commuting contractions, $\mathcal{S}, \mathcal{T}$ be disjoint sets such that $\mathcal{S} \cup \mathcal{T} = \{1, \ldots, N\}$, and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N$. Assume that $x_n \to 0$ weakly and $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$ then $\lim_{n\to\infty} \|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = 0$.*

PROOF. By the Hahn–Banach theorem, for each n, there exists some $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\| = 1$ and $\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = |\langle f_n(T), [x_n \otimes x_n] - [C_\lambda]\rangle|$. Since the polydisc has the Gleason property, there are $g_i^n \in H^\infty(\mathbb{D}^N)$ satisfying $\|g_i^n\| \le M_\lambda$ for $i = 1, \ldots, N$ and $f_n(z) = f_n(\lambda) - \sum_{i=1}^N (z_i - \lambda_i) g_i^n(z)$, where $z = (z_1, \ldots, z_N) \in \mathbb{D}^N$. Hence

$$\begin{aligned}
\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} &= \Big|\langle f_n(\lambda) + \sum_{i=1}^N (T_i - \lambda_i) g_i^n(T), [x_n \otimes x_n] - [C_\lambda]\rangle\Big| \\
&= \Big|\Big(\sum_{i=1}^N (T_i - \lambda_i) g_i^n(T) x_n, x_n\Big)\Big| \\
&\le \sum_{i\in\mathcal{S}} |(g_i^n(T)(T_i - \lambda_i)x_n, x_n)| + \sum_{i\in\mathcal{T}} |(g_i^n(T)x_n, (T_i^* - \overline{\lambda}_i)x_n)| \\
&\le \sum_{i\in\mathcal{S}} \|g_i^n(T)\|\, \|(T_i - \lambda_i)x_n\| + \sum_{i\in\mathcal{T}} \|g_i^n(T)\|\ \|(T_i^* - \overline{\lambda}_i)x_n\| \\
&\le M_\lambda \Big(\sum_{i\in\mathcal{S}} \|(T_i - \lambda_i)x_n\| + \sum_{i\in\mathcal{T}} \|(T_i^* - \overline{\lambda}_i)x_n\|\Big).
\end{aligned}$$

Thus, the proof is finished since $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$.

Next, the approximate orthogonality (8.2.2) will be shown.

LEMMA 10.2.8. *Let $T = (T_1, \ldots, T_N)$ be an a.c. N-tuple of commuting contractions. Let $\mathcal{S}, \mathcal{T}$ be disjoint sets such that $\mathcal{S} \cup \mathcal{T} = \{1, \ldots, N\}$. Assume that $\{T_i : i \in \mathcal{S}\}$ is doubly commuting. Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N$ and $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$, then $\lim_{n\to\infty} \|[y \otimes x_n]\|_{\mathcal{Q}} = 0$ for all $y \in \mathcal{H}$.*

PROOF. Without loss of generality, we can assume that $\mathcal{S} = \{1, \ldots, K\}$. There are $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\| = 1$ and $\|[y \otimes x_n]\|_{\mathcal{Q}} = |(f_n(T)y, x_n)|$. Notice that

$$\begin{aligned}
\|[y \otimes x_n]\|_{\mathcal{Q}} \le{} & |(f_n(T_1, \ldots, T_K, \lambda_{K+1}, \ldots, \lambda_N)y, x_n)| \\
& + |((f_n(T_1, \ldots, T_N) - f_n(T_1, \ldots, T_K, \lambda_{K+1}, \ldots, \lambda_N))y, x_n)|.
\end{aligned}$$

By an obvious modification of the Gleason property for polydomains, there are functions $g_{K+1}^n, \ldots, g_N^n \in H^\infty(\mathbb{D}^N)$ such that $\|g_i^n\| \le M_\lambda$ for $i = K+1, \ldots, N$ and

$$f_n(z) - f_n(z_1 \ldots, z_K, \lambda_{K+1}, \ldots, \lambda_N) = \sum_{i=K+1}^N (\lambda_i - z_i) g_i^n(z),$$

where $z = (z_1, \ldots, z_N)$. For a fixed point λ, denote by $h_n \in H^\infty(\mathbb{D}^K)$ the function $h_n(z_1, \ldots, z_K) = f_n(z_1 \ldots, z_K, \lambda_{K+1}, \ldots, \lambda_N)$ and write $\widetilde{T} = (T_1, \ldots, T_K)$. Obviously, we have a natural functional calculus for $\widetilde{T} = (T_1, \ldots, T_K)$.

Hence, for any $\varepsilon > 0$, we have

$$\begin{aligned}\|[y \otimes x_n]\|_{\mathcal{Q}} &\le |(f_n(T_1, \ldots, T_K, \lambda_{K+1}, \ldots, \lambda_N)y, x_n)| \\ &\quad + \Big|\Big(\sum_{i=K+1}^{N} (\lambda_i - T_i) g_i^n(T) y, x_n\Big)\Big| \\ &\le |(h_n(\widetilde{T})y, x_n)| + \sum_{i=K+1}^{N} \|g_i^n(T)\|\,\|y\|\,\|(\overline{\lambda}_i - T_i^*)x_n\| \\ &\le |(h_n(\widetilde{T})y, x_n)| + \varepsilon\end{aligned}$$

for n sufficiently large, since $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for $i = K+1, \ldots, N$.

By the same construction as in [Sł1, p. 1234], we can produce a doubly commuting K-tuple of isometries $V = (V_1, \ldots, V_K) \subset L(\mathcal{K})$ which is a minimal isometric dilation of the K-tuple $\widetilde{T}^* = (T_1^*, \ldots, T_K^*)$. Then $V^* = (V_1^*, \ldots, V_K^*)$ is an extension of $\widetilde{T} = (T_1, \ldots, T_K)$. By the minimality and the decomposition (7.3.1) the K-tuple $V = (V_1, \ldots, V_K)$ is a.c. and so is V^* by Lemma 7.3.5. Moreover, by Theorem 7.3.4 we can construct a functional calculus for each of them.

In [Sł1, Theorem 3], the Wold type decomposition for a pair of doubly commuting isometries was shown. One can easily see that this theorem remains true for K-tuples of doubly commuting isometries with a similar proof as the inductive step. Hence, for the doubly commuting K-tuple of isometries $V = (V_1, \ldots, V_K)$ there are subspaces $\mathcal{K}_s$ and $\mathcal{K}_\omega$ indexed by all non-empty subsets $\omega \subset \{1, \ldots, K\}$ such that

(1) $\mathcal{K} = \mathcal{K}_s \oplus \bigoplus_\omega \mathcal{K}_\omega$, where the orthogonal sum runs over all non-empty subsets $\omega \subset \{1, \ldots, K\}$,

(2) $\mathcal{K}_s$ and $\mathcal{K}_\omega$ reduce the isometries V_i for $i = 1, \ldots, K$,

(3) $V_i^s = V_i|_{\mathcal{K}_s}$ is a shift operator for all $i = 1, \ldots, K$, and

(4) for all non-empty subsets $\omega \subset \{1, \ldots, K\}$, the operator $V_i^\omega = V_i|_{\mathcal{K}_\omega}$ is unitary for $i \in \omega$ and a shift operator for $i \notin \omega$.

Form the orthogonal sums $x_n = x_n^s \oplus \bigoplus_\omega x_n^\omega$ and $y = y^s \oplus \bigoplus_\omega y^\omega$ with respect to the above decomposition. Fix a non-empty $\omega \subset \{1, \ldots, K\}$. Then there is i_0 such that $V_{i_0}^\omega$ is a unitary operator. Then

$$\|(T_{i_0} - \lambda_{i_0})x_n\| = \|(V_{i_0}^* - \lambda_{i_0})x_n\| \ge \|((V_{i_0}^\omega)^* - \lambda_{i_0})x_n^\omega\|.$$

Since $\|(T_{i_0} - \lambda_{i_0})x_n\| \to 0$, we have $\|(V_{i_0}^* - \lambda_{i_0})x_n^\omega\| \to 0$. Moreover, since the spectrum of the unitary operator $(V_{i_0})^*$ is contained in the unit circle and $\lambda_{i_0} \in \mathbb{D}$ we get $\|x_n^\omega\| \to 0$ by the spectral theorem.

There is also a natural functional calculus for $V_s^* = (V_1^{s*}, \ldots, V_K^{s*})$ and for $V_\omega^* = (V_1^{\omega*}, \ldots, V_K^{\omega*})$, since V_s^* and V_ω^* are restrictions of V^*. Hence, because $\|x_n^\omega\| \to 0$ for all non-empty $\omega \subset \{1, \ldots, K\}$, we have

$$\begin{aligned}|(h_n(\widetilde{T})y, x_n)| &= |(h_n(V^*)y, x_n)| \\ &\le |(h_n(V_s^*)y^s, x_n^s)| + \sum_{\omega} |(h_n(V_\omega^*)y^\omega, x_n^\omega)| \\ &\le |(h_n(V_s^*)y^s, x_n^s)| + \sum_{\omega} \|h_n\| \, \|y^\omega\| \, \|x_n^\omega\| \\ &\le |(h_n(V_s^*)y^s, x_n^s)| + \varepsilon\end{aligned}$$

for n sufficiently large. Since $\|x_n^\omega\| \to 0$ for all ω, we have $\|x_n^s\| \to 1$ and hence we may assume that $\|x_n^s\| = 1$.

Now we know that V_i^{s*} is a unilateral shift for $i = 1, \ldots, K$. For $M \in \mathbb{N}$ let R_M be the orthogonal projection onto $\bigvee_{i=1}^K \operatorname{ran} V_i^{sM}$. The model for a pair of doubly commuting isometries was shown in [Sł2, Theorem 1]. It can be easily extended in an obvious way to K-tuples of doubly commuting shifts. By this model, there is M so large M that $\|R_M y^s\| \le \varepsilon/2$. Let $y_1 = (I - R_M)y^s$ and $y_2 = R_M y^s$. Then $V_i^{s*M} y_1 = 0$ for $i = 1, \ldots, K$. We can write

$$h_n(z) = \sum_{|I|=0}^{M-1} a_I^n z^I + \sum_{i=1}^K z_i^M q_i^n(z), \quad \text{where } z = (z_1, \ldots, z_K), a_I^n \in \mathbb{C}, q_i^n \in H^\infty(\mathbb{D}^K).$$

Moreover, $|a_I^n| \le 1$ since a_I^n is a Fourier coefficient.

Since $x_n^s \to 0$ weakly, we have, for n sufficiently large, $|(V_I^{*I} y_1, x_n^s)| \le \varepsilon/(2M^K)$ for all I such that $|I| \le M - 1$. Hence

$$\begin{aligned}|(h_n(V_s^*)y^s, x_n^s)| &\le |(h_n(V_s^*)y_2, x_n^s)| + \sum_{|I|=0}^{M-1} |a_I^n| \, |(V_s^{*I} y_1, x_n^s)| \\ &\quad + \sum_{i=1}^K |(q_i^n(V_s^*) V_i^{s*M} y_1, x_n^s)| \\ &\le \|h_n\| \, \|y_2\| \, \|x_n^s\| + \sum_{|I|=1}^{M-1} \frac{\varepsilon}{2M^K} + 0 \le \varepsilon.\end{aligned}$$

Thus, $\|[y \otimes x_n]\|_{\mathcal{Q}} \le 3\varepsilon$ for n sufficiently large.

The second orthogonality condition (8.2.3) turns out to be symmetric to the previous one.

LEMMA 10.2.9. *Let $T = (T_1, \ldots, T_N)$ be an a.c. N-tuple of commuting contractions. Let $\mathcal{S}', \mathcal{T}'$ be disjoint sets such that $\mathcal{S}' \cup \mathcal{T}' = \{1, \ldots, N\}$. Assume that $\{T_i : i \in \mathcal{T}'\}$ is doubly commuting. Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N$ and $x_n \to 0$ weakly, with $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}'$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}'$, then $\lim_{n\to\infty} \|[x_n \otimes y]\|_{\mathcal{Q}} = 0$ for all $y \in \mathcal{H}$.*

PROOF. Since $\{T_i^* : i \in \mathcal{T}'\}$ is also doubly commuting, we can apply Lemma 10.2.8 to $\mathcal{S} = \mathcal{T}'$ and $\mathcal{T} = \mathcal{S}'$. Hence $\|[x_n \otimes y]\|_{\mathcal{Q}} = \|[y \otimes x_n]\|_{\mathcal{Q}(\mathcal{A}(T^*))} \to 0$.

NOTE. Theorem 10.1.1 is stated here for the first time and it is a generalization of Theorem 10.1.2 first shown in [KOP]. The results in Section 10.2 are from [AP].

11. Questions and open problems

In Chapter 2, we considered the finite-dimensional case. In fact, we have completely characterized the reflexive algebras generated by N-tuples of doubly commuting linear transformations on finite-dimensional Hilbert spaces. A natural further generalization will be to drop the double commutativity assumption. The condition 2.5.2(1) seems unsuitable, since it depends on the Jordan sequence, whose definition is based on double commutativity. In Theorem 2.4.5 we proved without assuming the double commutativity that a necessary condition for the reflexivity of an algebra generated by commuting linear transformations is that each rank-two member generates a one-dimensional ideal. This condition is also sufficient in the doubly commuting case, thus our first conjecture is

CONJECTURE 11.1. *Suppose $\mathcal{A}$ is an operator algebra in a finite-dimensional Hilbert space generated by a commuting family of linear transformations. Then $\mathcal{A}$ is reflexive if and only if each rank-two member of $\mathcal{A}$ generates a one-dimensional ideal.*

We can also search for different conditions which completely characterize reflexive families of linear transformations.

Restricting ourselves to the nilpotent case in an infinite-dimensional Hilbert space one can propose

CONJECTURE 11.2. *Suppose $\mathcal{A}$ is an operator algebra in a Hilbert space generated by a commuting family of nilpotents. Then $\mathcal{A}$ is reflexive if and only if each rank-two member of $\mathcal{A}$ generates a one-dimensional ideal.*

Considering hyporeflexivity we can ask

QUESTION 11.3. *Suppose $\mathcal{A}$ is an operator algebra in a finite-dimensional Hilbert space generated by a commuting family of* (*nilpotent*) *linear transformations. Find a necessary and sufficient condition on $\mathcal{A}$ for the equality $\mathcal{A} = \mathcal{A}' \cap \operatorname{Alg}\operatorname{Lat}\mathcal{A}$ to hold.*

The next problems concern subspaces of Toeplitz operators. As a natural question after Theorem 3.12 we can ask

QUESTION 11.4. *Let G be either $\mathbb{D}^N$ or $\mathbb{B}^N$ and let $\mathcal{J}$ be a set of inner functions. Characterize those $\mathcal{J}$ with the property that the space $\{T_{\overline{\psi}\phi} : \psi \in \mathcal{J},\ \phi \in H^\infty(G)\}$ is reflexive.*

We can state a more general problem:

CONJECTURE 11.5. *Let G be either $\mathbb{D}^N$ or $\mathbb{B}^N$ and let $\mathcal{S}$ be a weak-star closed subspace of the space of all Toeplitz operators $\mathcal{T}(L^\infty(G))$. Then $\mathcal{S}$ is either reflexive or transitive.*

Considering N commuting operator weighted shifts, we do not have such a nice model as for doubly commuting ones. Thus we can ask about reflexivity and hyporeflexivity without the double commutativity assumption.

QUESTION 11.6. *Can we drop the double commutativity assumption in Theorems* 6.4.2 *and* 6.5.4?

The next questions concern Theorem 9.2. Is it sufficient to assume only that each operator is quasinormal instead of the joint quasinormality?

Conjecture 11.7. *Every family $\mathcal{S}$ of commuting quasinormal operators is reflexive and has property* $\mathbb{A}_1(1)$.

Now we can try to drop the double commutativity assumption in Proposition 9.3.

Conjecture 11.8. *Every N-tuple $V = (V_1, \ldots, V_N)$ of commuting spherical isometries is reflexive.*

The next set of questions concern N-tuples of contractions. Recall the most striking results for a single contraction from [BCP2].

Theorem 11.9. *Let T be a contraction. If $\sigma(T)$ contains the unit circle $\mathbb{T}$, then T has a non-trivial invariant subspace.*

Another one is from [CEP]:

Theorem 11.10. *Let T be an a.c. contraction. If $\sigma_e(T)$ contains the unit circle, then T is reflexive.*

By analogy, the natural conjectures for N-tuples should be:

Conjecture 11.11. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of commuting contractions. If $\sigma(T)$ (Taylor spectrum) contains the polytorus $\mathbb{T}^N$, then T has a common non-trivial invariant subspace.*

Conjecture 11.12. *Let $T = (T_1, \ldots, T_N)$ be an a.c. N-tuple of commuting contractions. If $\sigma_e(T)$ (essential Taylor spectrum) contains the polytorus $\mathbb{T}^N$, then $\mathcal{W}(T)$ is reflexive.*

These problems seem to be out of reach nowadays. Hence, let us state the following, easier problems which also remain open. We want to drop the double commutativity assumption from Theorems 10.2.1 and 10.2.2:

Conjecture 11.13. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of commuting contractions. If $\sigma(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then T has a common non-trivial invariant subspace.*

Question 11.13′. *We can ask about the reflexivity of $T = (T_1, \ldots, T_N)$ assuming the dominance of the Taylor essential spectrum $\sigma_e(T)$.*

The answer for the above is not known even if we consider the Harte spectrum $\sigma_H(T)$ instead of the Taylor spectrum $\sigma(T)$.

The above conjectures were stated for a polydisc. The unit ball $\mathbb{B}^N$ is also a natural generalization of the unit disc $\mathbb{D}$. We also have the notion of a spherical contraction. Namely, a commuting N-tuple $T = (T_1, \ldots, T_N)$ is called a *spherical contraction* if $\sum_{i=1}^N \|T_i x\|^2 \le \|x\|^2$ for all x. Hence all the above conjectures given for N-tuples of contractions can be stated for spherical contractions. For example:

Conjecture 11.14. *Let $T = (T_1, \ldots, T_N)$ be a spherical contraction. If $\sigma(T) \cap \mathbb{B}^N$ is dominating for $H^\infty(\mathbb{B}^N)$, then $T = (T_1, \ldots, T_N)$ has a common non-trivial invariant subspace.*

We finish by recalling the famous result of [OT] that every subnormal operator is reflexive. It is a generalization of the result of [Br] that every subnormal operator has a non-trivial invariant subspace. In [Ya], the existence of common non-trivial invariant subspaces was shown for jointly subnormal families. Thus we can state

CONJECTURE 11.15. *Jointly subnormal families are reflexive.*

ADDENDUM. Recently, the proof of Conjecture 11.5 for $N = 1$ was given in [AP4]. During the conference "Deuxièmes Journées Lilloises de Théorie des Opérateurs", March '97, B. Chevreau announced a proof for Conjecture 11.13 assuming the existence of a unitary dilation for an N-tuple of contractions and the dominance of the Taylor essential spectrum. Also, the hyporeflexivity of spherical isometries has been shown in [MP]. This gives a partial result towards proving Conjecture 11.8.

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