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Invariant measures and ideals on discrete groups

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0. Introduction

Shortly after Lebesgue measure had been defined, it was realized that not all sets of reals are measurable. Two classical examples of non-measurable sets are: Vitali's set (a selector of the family of cosets R/Q) and Bernstein's set (which intersects every perfect set and contains none of them). In the proofs that those sets are non-measurable, different properties of Lebesgue measure are used: invariance with respect to addition, i.e. a group theoretic property of Lebesgue measure in the case of Vitali's set and regularity of measure, i.e. a topological property in the case of Bernstein's set. Hence the question arose of what is the influence of various features enjoyed by Lebesgue measure on the existence of non-measurable sets. Historically the first approach was to investigate the case when invariance and regularity are not assumed.

The question was formulated as follows. Does there exist a σ-additive σ-finite measure vanishing on singletons defined on all subsets of the reals? In this setting the problem becomes purely set theoretic, i.e. depends exclusively on the cardinality of the underlying set. It is known in the literature as the general measure problem and its survey is given in Chapter 1.

The question of eliminating non-measurable sets and simultaneously keeping regularity of the measure is still open, more exactly the following problem remains unsolved: does there exist a σ-additive σ-finite measure vanishing on singletons, defined on all sets of the reals and such that every set of reals differs from a Borel set by a set of measure zero.

Another possibility is trying to keep invariance. This is the approach chosen in the present paper. We do not restrict ourselves to the additive group of the reals and consider σ-additive left-shift invariant measures on arbitrary groups. The aim is always to find measures defined on possibly large σ-algebras of subsets of a group, preferably on all subsets. The latter is impossible for σ-finite measures by a classical result of Harazišvili, Erdős and Mauldin. Hence we consider measures satisfying a somewhat weaker condition, semiregularity. In Chapter 2 we obtain an exact characterization of those abelian groups on which there exist such universal measures.

The search for invariant measures defined on large σ-algebras leads to the problem of extending measures which Chapter 3 is devoted to. We show that every σ-finite invariant measure on an abelian group has a proper

invariant extension thus generalizing a result of Harazišvili. We also consider the problem of extending H-invariant measures on a group G, where H is a subgroup of G. In this situation the interesting case turns out to be that of countable subgroups H and the answer strictly depends on their algebraic structure.

Let us note here the fundamental difference between analytic considerations connected with Haar measures on groups and our approach. For analytic purposes the existence of non-measurable sets doesn't seem to be harmful, a Haar measure being defined on a very small σ-algebra, that of Borel subsets of a locally compact group. On the other hand good topological properties of a Haar measure are often used, whereas in our case we neglect the topological structure of the group considering all groups as discrete. Instead we are concerned with enlarging invariant measures, the inspiration coming from the classical general measure problem. The aim remains still the same: to eliminate non-measurable sets as far as possible.

A set theoretic notion tightly connected with estimating the size of a measure is the saturation of an ideal. Ideals are in one-to-one correspondence with two-valued measures assuming value 0 on sets from the ideal and value 1 on sets from the dual filter. The saturation of an ideal is an index estimating the maximal size of a pairwise almost disjoint family of sets non-measurable with respect to the appropriate measure. The smaller is this index, the larger is the measure. Hence adopting our aim of investigating large invariant measures it seems interesting to look for stronly saturated invariant ideals on groups. Results connected with this subject are contained in Chapter 4. We obtain an exact estimate of the saturation of sufficiently complete, invariant ideals on abelian groups.

Notice that both for measures and ideals we usually assume countable additivity. The finitely additive case turns out to be completely different. The existence of invariant finitely additive measures (called here quasi-measures) defined on all subsets of a group is tightly connected with the existence of invariant means widely investigated in analysis. In Chapters 2 and 3 we give a brief survey of classical results in this area. At the end of Chapter 4 we discuss saturation of invariant ideals which are not σ-complete.

Let us also remark that the majority of our results are proved for abelian or solvable groups. We hope that at least some of them are true for arbitrary groups and would like to state this as the main problem suggested in this paper. Unfortunately the tools developped here do not seem to work in the general case.

Finally a remark should be made about our set theoretic framework. All the results are proved in usual set theory with the axiom of choice (ZFC). However, many of them have the form: "if a large cardinal exists then..." and sometimes even show the equivalence of this kind of assumptions with a measure theoretic problem. This important feature situates our dissertation

on the border of measure theory and set theory. The whole set theoretic background needed in the main body of the paper is contained in Chapter 1 and the techniques we use are combinatorial, hence no knowledge of metamathematical tools like forcing is necessary. As it will be seen "purely mathematical" results and those engaging large cardinals are strongly interrelated, the need of additional axioms often coming naturally during the investigation of various measure theoretic questions.

1. Preliminaries

In this chapter we fix the basic terminology used throughout the paper, establish some easy facts about the introduced notions and state several classical theorems both from general measure theory and set theory. Notation and terminology concerning groups is postponed until the end of the chapter.

Our set theoretic notation is standard. For a set X, $|X|$ denotes the cardinality of X and $P(X)$ the family of all subsets of X. Symmetric difference of sets A and B is denoted by $A \triangle B$. If $f\colon X \to Y$ is a function and $A \subset X$ then $f[A]$ denotes the image of the subset A. According to the general habit in modern set theory, an ordinal is identified with the set of its predecessors and cardinals with initial ordinals. R denotes the set of reals, Q the set of rationals, Z the set of integers and ω the set of natural numbers. 0 is considered to be a natural number.

The key notion of the paper is the notion of measure.

DEFINITION 1.1. A *measure* on a set X is a function m defined on a σ-algebra $\mathfrak{M}$ of subsets of X containing all singletons, such that:

(a) the values of m are non-negative reals or $+\infty$,

(b) $m(\{x\}) = 0$ for any $x \in X$,

(c) $m(X) > 0$,

(d) $m(\bigcup_{n\in\omega} A_n) = \sum_{n\in\omega} m(A_n)$ for pairwise disjoint sets A_n from the σ-algebra $\mathfrak{M}$

A *quasi-measure* is defined as above, except that its domain is an algebra $\mathfrak{M}$ of sets and condition (d) is replaced by a weaker condition

(d′) $m(A \cup B) = m(A) + m(B)$ for disjoint sets A and B from $\mathfrak{M}$ Elements of the domain of a measure or a quasi-measure are called *measurable* sets.

Next we define a few general properties of measures. The same notions can be considered for quasi-measures.

DEFINITION 1.2. A measure m on X is

a *probability measure* iff $m(X) = 1$,

finite iff $m(X) < \infty$,

σ-finite iff X is a countable union of sets of finite measure,

semiregular iff every set of positive measure contains a subset of positive finite measure,

uniform iff $m(A) = 0$ whenever A is a subset of X such that $|A| < |X|$,

$\varkappa$-additive iff every union of less than $\varkappa$ sets of measure zero has measure zero ($\varkappa$ is any cardinal),

universal iff m is defined on $P(X)$,

complete iff all subsets of sets of measure zero are measurable.

Clearly, every $|X|$-additive measure on X is uniform. Let us also remark that for universal measures semiregularity is a natural generalization of σ-finiteness. This can be easily seen using the following equivalent versions of the definitions of these notions. A measure is σ-finite iff every set of positive measure can be partitioned into countably many sets of positive finite measure. A universal measure is semiregular iff every set of positive measure can be partitioned into sets of positive finite measure (without any specific restriction on the cardinality of the partition).

Let us state the following easy fact interrelating the existence of universal measures of various kinds.

PROPOSITION 1.3. *The following are equivalent*:

(a) *there exists a universal probability measure on X*,

(b) *there exists a universal finite measure on X*,

(c) *there exists a universal σ-finite measure on X*,

(d) *there exists a universal semiregular measure on X*,

(e) *there exists a universal measure on X assuming at least one positive finite value.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) is trivial. Assume (e) and let m be a universal measure on X such that $0 < m(A) < \infty$ for some $A \subset X$. We define a universal probability measure m_1 on X by putting

$$m_1(B) = \frac{m(B \cap A)}{m(A)} \quad \text{for any } B \subset X. \ \blacksquare$$

It is clear that the existence of universal measures satisfying one of the above properties depends only on the cardinality of X. Using the introduced notions and Proposition 1.3, the general measure problem mentioned in the introduction can be formulated as follows.

Does there exist a universal probability measure on the cardinal 2^{ω}? or more generally

Does there exist a universal probability measure on any cardinal?

The following notions are useful in this context:

DEFINITION 1.4. A *real valued measurable* cardinal (RVM) is a cardinal $\varkappa$ on which there exists a universal $\varkappa$-additive probability measure.

An *Ulam real valued measurable* cardinal (URVM) is a cardinal $\varkappa$ on which there exists a universal probability measure.

A *measurable* cardinal (MC) is a cardinal $\varkappa$ on which there exists a universal $\varkappa$-additive measure assuming only values 0 and 1.

An *Ulam measurable* cardinal (UMC) is a cardinal $\varkappa$ on which there exists a universal measure assuming only values 0 and 1.

An *inaccessible* cardinal is a regular cardinal $\varkappa$ for which $2^\alpha < \varkappa$ whenever $\alpha < \varkappa$.

A *weakly inaccessible* cardinal is a regular limit cardinal.

Hence the general measure problem is that of the existence of an URVM.

The first partial answer to this question was given by Banach and Kuratowski [3]: if $2^\omega = \omega_1$ then 2^ω is not URVM. Ulam made an important contribution in this area. He proved the following

THEOREM 1.5 (Ulam [37]). (a) *The least* URVM *is* RVM.

(b) *The least* UMC *is* MC.

(c) *Every* RVM *is a weakly inaccessible cardinal.*

(d) *Every* MC *is an inaccessible cardinal.*

(e) *A* RVM *is either less or equal to* 2^ω *or is a* MC.

Since the existence of inaccessible and weakly inaccessible cardinals cannot be proved in ZFC, Ulam's results show that the general measure problem cannot have a positive solution in usual mathematics. Nevertheless the possibility of disproving the existence of (weakly) inaccessible cardinals or RVM's and MC's is not excluded. In spite of this danger the assumptions that such cardinals exist are commonly used in modern set theory as additional axioms. We are going to do the same and establish some consequences of and statements equivalent to the existence of those large cardinals.

Let us note that the positive answer to the original version of the general measure problem, i.e. the statement "2^ω is URVM" is in a sense strictly stronger than the statement "there exists an URVM". Levy and Solovay [22] proved that if the existence of MC is consistent with set theory then the statement "there exists a MC$+2^\omega = \omega_1$" is also consistent with ZFC and hence in view of the above mentioned result of Banach and Kuratowski the statement "there exists an URVM$+2^\omega$ is not URVM" is consistent.

On the other hand the consistency strength of those statements is the same. Solovay proved

THEOREM 1.6 (Solovay [35]). *The following are equivalent*:

(a) *The statement "A* MC *exists" is consistent with* ZFC.

(b) *The statement "*2^ω *is* RVM*" is consistent with* ZFC.

In view of Theorem 1.5 this shows that the consistency of "there exists an URVM" is equivalent to the consistency of "2^ω is an URVM".

A different problem tightly connected with investigation of measures defined on large σ-algebras is that of extending measures. In the general

setting only few results are known. The classical one is that of Łoś and Marczewski. Before stating it we fix more notation. If m is a measure, m_i- denotes the respective inner measure and m_e the outer measure.

THEOREM 1.7 (Łoś, Marczewski [23]). *Let m be a measure defined on a σ-algebra $\mathfrak{M}$ of subsets of X and $A \in P(X) \setminus \mathfrak{M}$. Then the measure m can be extended to the σ-algebra $\tilde{\mathfrak{M}}$ generated by $\mathfrak{M} \cup \{A\}$.*

Proof. The elements of $\tilde{\mathfrak{M}}$ have the form $E = (M_1 \cap A) \cup (M_2 \setminus A)$ for $M_1, M_2 \in \mathfrak{M}$. For a set E as above we define

$$m'(E) = m_i(M_1 \cap A) + m_e(M_2 \setminus A),$$
$$m''(E) = m_e(M_1 \cap A) + m_i(M_2 \setminus A).$$

It is easy to see that for any $c \in [0, 1]$ the measure defined on $\tilde{\mathfrak{M}}$ by the formula

$$\tilde{m}(E) = cm'(E) + (1-c)m''(E)$$

is an extension of m. ∎

Hence in the general setting every non-universal measure on X has a proper extension. It will be seen that in the invariant case this problem becomes much more complicated.

Some generalizations of the above result were established later. Ascherl and Lehn [1] proved that if we consider any partition $\mathscr{A}$ of the set X instead of the two element partition $\{A, X \setminus A\}$ then a probability measure m can be extended to the σ-algebra generated by $\mathfrak{M} \cup \mathscr{A}$. Weber [38] proved the same for $\mathscr{A}$ being a family of subsets of X well ordered by inclusion. Notice that well-ordering cannot be weakened to linear ordering because it is easy to find a measure which cannot be extended over some countable chain of sets.

Let us finally remark that both the general measure problem and the extension problem have a very strong affirmative answer in the finitely additive version, i.e. in the case of quasi-measures. It easily follows from Zorn's lemma that every quasi-measure on an infinite set X can be extended to a universal quasi-measure. In the invariant setting this problem is briefly discussed in the following chapters.

We are now going to fix the terminology and mention basic facts concerning the saturation of ideals.

DEFINITION 1.8. An *ideal on X* is a family $I \subset P(X)$ such that:

(a) $A \cup B \in I$ whenever $A, B \in I$,

(b) $A \in I$ whenever $A \subset B$, $B \in I$,

(c) $X \notin I$,

(d) $\{x\} \in I$ for any $x \in X$.

If I is an ideal on X then the family of complements of sets from I is called the *dual filter* and is denoted by I^*.

DEFINITION 1.9. An ideal I on X is

– *$\varkappa$-complete* iff $\bigcup A \in I$ whenever $A \subset I$ and $|A| < \varkappa$,

– *uniform* iff $A \in I$ whenever $A \subset X$ and $|A| < |X|$,

– *$\varkappa$-saturated* iff for any family $A \subset P(X) \setminus I$ of cardinality $\varkappa$ there exist two sets $a, b \in A$ such that $a \cap b \notin I$,

– *prime* iff every subset of X is either in the ideal or in the dual filter.

The largest cardinal $\varkappa$ for which an ideal I is $\varkappa$-complete is called its *degree of completeness*. The least cardinal λ for which I is λ-saturated is called its *degree of saturation*. Instead of ω_1-complete and ω_1-saturated we shall write σ-complete and σ-saturated thus following traditional terminology.

A family A of sets outside of the ideal I and such that $a \cap b \in I$ for all distinct $a, b \in A$ is called an *I-almost disjoint family* or a *family of I-almost disjoint sets*. Hence an ideal I is $\varkappa$-saturated iff there is no I-almost disjoint family of cardinality $\varkappa$.

Let us state some easy facts about ideals. Any ideal is ω-complete. A $|X|$-complete ideal on X is uniform. An ideal is prime iff it is 2-saturated. For $\varkappa$-complete ideals $\varkappa$-saturation can be equivalently defined as follows: there is no family of cardinality $\varkappa$ of pairwise disjoint sets outside of the ideal.

A typical example of a σ-complete σ-saturated ideal is the ideal of measure zero sets of any universal σ-finite measure. Similarly as Theorem 1.5 the following well-known facts can be established (cf. e.g. Jech [18]).

THEOREM 1.10. (a) *The least cardinal $\varkappa$ on which there is a σ-complete σ-saturated ideal carries a $\varkappa$-complete σ-saturated ideal.*

(b) *If a cardinal $\varkappa$ carries a $\varkappa$-complete $\varkappa$-saturated ideal then $\varkappa$ is weakly inaccessible.*

Hence the existence of σ-complete σ-saturated ideals cannot be proved in ZFC. The situation here is similar to that of the existence of universal probability measures. Kunen (cf. Jech [18]) proved the following

THEOREM 1.11 (Kunen). *The following are equivalent*:

(a) *The statement "A* MC *exists" is consistent with* ZFC.

(b) *The statement "There exists a $\varkappa$-complete λ-saturated ideal on a cardinal $\varkappa$" (where $\omega_1 \leqslant \lambda \leqslant \varkappa^+$) is consistent with* ZFC.

Hence the consistency strenght of assuming a σ-complete σ-saturated ideal on any cardinal is the same as that of assuming a measurable cardinal. On the other hand the first assumption is in a sense weaker than even the assumption of URVM. Prikry [31] proved that if the existence of a measurable cardinal is assumed then it is consistent with ZFC that there exists a σ-complete σ-saturated ideal on some cardinal but no cardinal is URVM.

Strongly saturated ideals (as well as two-valued measures associated with them) can be certainly considered as large. The decreasing indices of

saturation determine a scale of larger and larger ideals. A different hierarchy of ideals, also connected with their size was introduced by Grzegorek and Węglorz [9].

DEFINITION 1.12. Let I be a uniform ideal on a set X of cardinality $\varkappa$ and $\delta \geqslant 2$. The ideal I has property $\mathrm{U}(\delta)$ iff there is a partition of X into sets of cardinalities at least δ without any selector in I. Ideals without property $\mathrm{U}(2)$ are called *Ulam ideals*.

It was proved in [9] that for any cardinal δ such that $2 \leqslant \delta < \varkappa$ and any regular $\lambda \leqslant \varkappa$ there is a λ-complete uniform ideal on $\varkappa$ with the property $\mathrm{U}(\delta)$ and without the property $\mathrm{U}(\delta^+)$. Hence those properties determine a strict hierarchy of ideals. Ideals enjoying property $\mathrm{U}(\varkappa)$ are small and Ulam ideals are large. It is easy to see that a δ-saturated ideal cannot have property $\mathrm{U}(\delta)$. Węglorz proved the following

THEOREM 1.13 (Węglorz [40]). *Let λ be a regular uncountable cardinal. Every λ-complete ideal on $\varkappa$ can be extended to a λ-complete Ulam ideal.*

We close these preliminary remarks by fixing notation and terminology concerning groups, invariant measures and ideals.

A group is usually identified with the set of its elements. In particular we use letters R, Q and Z for the additive groups of the reals, rationals and integers, respectively. The result of the group operation on elements a and b is simply denoted by ab and for abelian groups we often use the traditional additive notation thus writing $a+b$. If G is a group and A, B are subsets of G then AB denotes the set $\{ab\colon a \in A, b \in B\}$. We write aB instead of $\{a\}B$. A similar convention applies to the additive notation. Instead of a product $a \ldots a$ (n times) we write a^n and instead of a sum $a + \ldots + a$ (n times) we write na. Similarly we use a^{-n} for $(a^{-1}) \ldots (a^{-1})$ (n times) and $(-n)a$ for $(-a) + \ldots + (-a)$ (n times).

By the left shift associated with an element g of a group G we mean the bijection $\varphi_g\colon G \to G$ given by the formula $\varphi_g(h) = gh$. We often identify elements with the associated left shifts thus speaking e.g. about left shifts from a subgroup H of G.

We now introduce the definition of a G-invariant measure in the most general case.

DEFINITION 1.14. Let G be a group of bijections of a set X. A measure m defined on a σ-algebra $\mathfrak{M}$ of subsets of X is *G-invariant* iff $g[A] \in \mathfrak{M}$ and $m(g[A]) = m(A)$ for any $A \in \mathfrak{M}$ and $g \in G$. If G is a group and H a subgroup of G then a measure on G is *H-invariant* iff it is invariant with respect to the group of left shifts from H. G-invariant measures on G are simply called *invariant*.

In this dissertation we shall be mainly interested in H-invariant measures on a group G, where H is a subgroup of G and particularly in invariant measures. This approach is justified by classical considerations of

Haar measures. Some results concerning the more general case of measures invariant with respect to a group of bijections of a set will be also mentioned. Since according to our definition there are no measures on countable sets, we may always assume that the underlying set is uncountable. It is easy to see that the measure completion of a G-invariant measure on X is also G-invariant. Hence when looking for proper G-invariant extensions of a given G-invariant measure m on X we shall assume without loss of generality that m is complete.

Let us finally introduce the notion of an invariant ideal which is crucial for Chapter 4.

DEFINITION 1.15. An ideal I on a group G is *invariant* iff $gA \in I$ whenever $A \in I$ and $g \in G$.

Notice that an ideal on a group is invariant iff the associated two valued measure is invariant. Hence the investigation of invariant ideals which Chapter 4 is devoted to can be considered as a part of the theory of invariant measures.

2. Universal invariant measures

This chapter is devoted to the invariant version of the general measure problem. More precisely, we are going to study the following question:

Given a group G and a subgroup H of G, does there exist a universal H-invariant measure on G?

Since the pathological measure assuming value 0 on countable and value ∞ on uncountable subsets of a group G is clearly invariant, some restrictions must be imposed on the measure in order to avoid trivialities. We start with the relatively narrow class of σ-finite measures. In this setting our problem turns out to have a complete solution.

The following theorem is due to Harazišvili and was later proved independently by Erdös and Mauldin [6].

THEOREM 2.1 (Harazišvili [11]). *Let G be an arbitrary group. There does not exist a universal invariant σ-finite measure on G.*

Proof. Suppose m is such a measure and let $\{K_n\colon n \in \omega\}$ be a family of sets of finite measure whose union is G. Since G is uncountable it contains a subgroup H of cardinality ω_1. Let S be any selector of the family of right cosets of H in G. The family $\{hS\colon h \in H\}$ is a partition of G. For any $n \in \omega$, $m(K_n \cap hS) > 0$ only for finitely many elements $h \in H$. Hence for some $h_0 \in H$ we get $m(K_n \cap h_0 S) = 0$ for every natural n. This implies $m(h_0 S) = 0$ by σ-additivity and $m(hS) = 0$ for any $h \in H$, by invariance. Hence the measure $\tilde{m}$ on H defined by the formula

$$\tilde{m}(A) = m(AS) \quad \text{for any } A \subset H$$

is a universal σ-finite measure on H which contradicts Ulam's Theorem 1.5. ∎

Notice that the same argument gives a more general result which also follows from a theorem of Ryll–Nardzewski and Telgarsky [32]:

THEOREM 2.2. *If H is any uncountable subgroup of a group G, there does not exist a universal H-invariant σ-finite measure on G.*

Hence the only remaining case for σ-finite measures is that of a countable subgroup H. In this situation the easy answer is given by the following

PROPOSITION 2.3. *If H is a countable subgroup of a group G, the following conditions are equivalent:*

(a) $|G|$ *is* URVM,

(b) *there exists a universal H-invariant σ-finite measure on G.*

Moreover a universal H-invariant measure on G may be finite iff H is finite.

Proof. (b) $\Rightarrow$ (a) obvious.

(a) $\Rightarrow$ (b). Let S be a selector of the family of left cosets of H in G. Since $|S|$ is URVM, there exists a universal probability measure μ on S. We define a universal measure m on G putting for any $A \subset G$:

$$m(A) = \sum_{h \in H} \mu(hA \cap S).$$

We check that m is H-invariant. Let $A \subset G$ and $g \in H$.

$$m(gA) = \sum_{h \in H} \mu(hgA \cap S) = \sum_{h \in H} \mu(hA \cap S) = m(A),$$

which proves (b).

If H is finite, the above constructed measure m is finite. If H is infinite then Vitali's argument shows that an H-invariant finite measure cannot be universal. (Selectors of the family of cosets cannot be measurable). ∎

The above results provide a complete solution of our problem in the σ-finite case. We are now going to look at some of its generalizations. Theorem 2.1 can be considered as the positive answer to the following question:

(a) Given a σ-finite invariant measure m on a group G, does there exist a set non-measurable with respect to m?

The aim of the following generalizations of problem (a) is to get a deeper insight in the properties of the non-measurable sets. In particular it seems interesting to know how "bad" those sets can be with respect to invariant measures on the group. Consider the following questions:

(b) Given a σ-finite invariant measure m on a group G, does there exist a set non-measurable with respect to any invariant extension of m?

(c) Does every group G contain a subset non-measurable with respect to any σ-finite invariant measure on G?

Problems (a) and (b) have their "local" counterparts:

(a′) Given a σ-finite invariant measure m on a group G, does every set of positive measure contain a set non-measurable with respect to m?

(b′) Given a σ-finite invariant measure m on a group G, does every set of positive measure contain a subset non-measurable with respect to any invariant extension of m?

The argument used in the proof of Theorem 2.1. gives a positive answer to (a′) as well (cf. Harazišvili [13]). Harazišvili [15] gave a positive answer to (c) for abelian groups. As for (b′) he observed in [13] that the answer is positive for m being any invariant extension of the Lebesgue measure on the Euclidean space E^n considered as the group of translations. We generalize his result as follows:

PROPOSITION 2.3 ([1]). *Let m be any invariant extension of a regular σ-finite Haar measure on a topological group G. If $m(A) > 0$ then there exists a subset $B \subset A$ non-measurable with respect to any invariant extension of m.*

Proof. Let E_i: $i \in \omega$ be Borel sets of positive finite measure such that $\bigcup_{i\in\omega} E_i = G$. We shall use the following

LEMMA 2.4 (cf. Halmos [10]). *If E is a Borel set of finite measure then the function f defined on G by the formula $f(x) = m(xE \triangle E)$ is continuous.*

Let U_n be such a neighbourhood of the neutral element for which $m(xE_n \triangle E_n) < 1/2^n$ whenever $x \in U_n$. Take a sequence $\{x_k\colon k \in \omega\}$ of elements of the group such that for any natural k:

$$x_k \in \Big(\bigcap_{i\leqslant k} U_i\Big) \setminus \{x_1, \ldots, x_{k-1}\}.$$

We claim that for any natural i

$$m\Big(\bigcup_{k\in\omega} x_k E_i\Big) < \infty.$$

Fix i. Since for $k > i$ we have $x_k \in U_i$, we get

$$m(x_k E_i \triangle E_i) < \frac{1}{2^i} \quad \text{for} \quad k > i.$$

([1]) This is the only place where we drop our convention of neglecting the topological structure of the group. For terminology and basic results concerning Haar measures see Halmos [10].

Hence

$$m\Big(\bigcup_{k\in\omega} x_k E_i\Big) \leqslant m\Big(\bigcup_{k=0}^{i} x_k E_i\Big)+m\Big(\bigcup_{k=i+1}^{\infty} x_k E_i\Big) \leqslant$$

$$\leqslant m\Big(\bigcup_{k=0}^{i} x_k E_i\Big)+m(E_i)+1 \leqslant (i+2)\,m(E_i)+1 < \infty.$$

Let now H be the subgroup of G generated by the family $\{x_k\colon k\in\omega\}$. Clearly, $|H| = \omega$. Take any set A of positive measure. Hence for some natural j the set $A^* = A\cap E_j$ has positive measure. Let C be the set of those right cosets of H in G which intersect A^*. Take any selector B of C, contained in A^*. We claim that B is as required. Suppose it is measurable with respect to some invariant extension m_1 of m. If $m_1(B) = 0$ then $m_1\big(\bigcup_{x\in H} xB\big) = 0$ and $A^* \subset \bigcup_{x\in H} xB$, contradiction. If $m_1(B) > 0$ then $m_1\big(\bigcup_{k\in\omega} x_k B\big) = \infty$ and $\bigcup_{k\in\omega} x_k B \subset \bigcup_{k\in\omega} x_k E_j$ the latter set having finite measure, contradiction. Hence B is non-measurable with respect to m_1, which finishes the proof. ∎

The general versions of problems (b), (c) and (b′) remain open. The latter was stated for the Euclidean space E^n (considered as the group of translations) by Harazišvili [13].

Let us also remark that all the above questions have easy positive answers in the case of finite measures. For problems (a), (b) and (c) it is enough to take any selector of the family of left cosets of a countable subgroup H of G. For (a′) and (b′), given a set A of positive measure it is sufficient to take a selector $S\subset A$ of those cosets of H which intersect A. In both cases Vitali's argument works.

Since the invariant version of the general measure problem has a complete solution for σ-finite measures we shall consider it in a more general setting. That of semiregular measures seems appropriate: on the one hand it excludes the trivial 0–∞ example mentioned before, on the other hand no argument of the type used in the proof of Theorem 2.1 is valid in this case.

The following problem was stated by Kannan and Raju [20]: Does there exist a universal semiregular invariant measure on any group G?

In view of Proposition 1.3 the necessary condition for the existence of such a measure is that $|G|$ is URVM. Our result shows that for abelian groups it is also a sufficient condition, thus giving a solution to the above problem.

THEOREM 2.5 (Pelc [25]). *Let G be an abelian group such that $|G|$ is* URVM. *Then there exists a universal semiregular invariant measure on G.*

Proof. Let us first assume that $|G| = \varkappa$ is RVM and denote by H the torsion subgroup of G. Consider two cases.

Case 1. $|H| < \varkappa$. The group G contains a free abelian subgroup G' of cardinality $\varkappa$.

We shall first construct the required measure on G'.

Let μ be a $\varkappa$-additive universal probability measure on $\varkappa$. Fix a well-ordering z_i: $i \in \omega$ of all non-zero integers.

Let $\{x_\alpha: \alpha < \varkappa\}$ be an enumeration of the base of G'. Every $g \in G'$ has exactly one representation as $z_{i_1} x_{\alpha_1} + \ldots + z_{i_n} x_{\alpha_n}$. Denote by $\varphi(g)$ the set $\{\alpha_1, \ldots, \alpha_n\}$ and put

$$G_g^i = \{h \in G': \exists \beta > \max(\varphi(g))[h = z_i x_\beta + g]\} \quad \text{for } g \in G' \text{ and } i \in \omega.$$

Notice that the sets G_g^i together with $\{0\}$ form a partition of G'. Now define the measure m for any $A \subset G'$ by the formula

$$m(A) = \sum_{i \in \omega} \sum_{g \in G'} \mu\big(\bigcup(\varphi[G_g^i \cap A])\big).$$

Clearly $m(\{g\}) = 0$ for any $g \in G'$. If $m(A) > 0$ then there are $i \in \omega$ and $g \in G'$ for which $\mu\big(\bigcup(\varphi[G_g^i \cap A])\big) > 0$. Of course $A \cap G_g^i \subset A$ and

$$0 < m(A \cap G_g^i) = \mu\big(\bigcup(\varphi[G_g^i \cap A])\big) < \infty.$$

Hence m is semiregular.

We check that m is $\varkappa$-additive. Suppose that $\{A_\gamma: \gamma < \beta\}$ is a pairwise disjoint family of subsets of G', with $\beta < \varkappa$. Then for every $i \in \omega$ and $g \in G'$ the sets $\bigcup(\varphi[G_g^i \cap A_\gamma])$: $\gamma < \beta$ are pairwise almost disjoint (in the sense that any two of them have finite intersection) and we get:

$$\begin{aligned} m\Big(\bigcup_{\gamma<\beta} A_\gamma\Big) &= \sum_{i \in \omega} \sum_{g \in G'} \mu\Big(\bigcup\Big(\varphi\Big[G_g^i \cap \Big(\bigcup_{\gamma<\beta} A_\gamma\Big)\Big]\Big)\Big) \\ &= \sum_{i \in \omega} \sum_{g \in G'} \mu\Big(\bigcup_{\gamma<\beta} \bigcup(\varphi[G_g^i \cap A_\gamma])\Big) \\ &= \sum_{i \in \omega} \sum_{g \in G'} \sum_{\gamma<\beta} \mu\big(\bigcup(\varphi[G_g^i \cap A_\gamma])\big) \\ &= \sum_{\gamma<\beta} \sum_{i \in \omega} \sum_{g \in G'} \mu\big(\bigcup(\varphi[G_g^i \cap A_\gamma])\big) \\ &= \sum_{\gamma<\beta} m(A_\gamma). \end{aligned}$$

We now check that m is invariant. Let $h \in G'$ and $A \subset G'$.

$$\begin{aligned} m(h+A) &= \sum_{i \in \omega} \sum_{g \in G'} \mu\big(\bigcup(\varphi[G_g^i \cap (h+A)])\big) \\ &= \sum_{i \in \omega} \sum_{g \in G'} \mu\big(\bigcup(\varphi[G_{h+g}^i \cap (h+A)])\big). \end{aligned}$$

For every $i \in \omega$, $g \in G'$ and $\beta > \max(\varphi(g) \cup \varphi(h))$ we have

$$\beta \in \bigcup(\varphi[G_g^i \cap A]) \equiv z_i x_\beta + g \in A \equiv z_i x_\beta + g + h \in h + A$$
$$\equiv \beta \in \bigcup(\varphi[G_{h+g}^i \cap (h+A)]).$$

Hence in view of uniformity of μ we get

$$\mu\big(\bigcup(\varphi[G_g^i \cap A])\big) = \mu\big(\bigcup(\varphi[G_{h+g}^i \cap (h+A)])\big)$$

and finally

$$m(h+A) = \sum_{i \in \omega} \sum_{g \in G'} \mu\big(\bigcup(\varphi[G_g^i \cap A])\big) = m(A).$$

This proves that m is a $\varkappa$-additive semiregular universal invariant measure on G'.

In order to get the required measure on G it is enough to prove the following

LEMMA 2.6. *Let G be an abelian group. If there exists a universal semiregular invariant measure on a subgroup H of G then there is one on G as well.*

Proof. Let S be a selector of the family of cosets of H in G and m a universal semiregular invariant measure on H. We define a measure m_1 on G by the formula

$$m_1(A) = \sum_{s \in S} m\big((s+A) \cap H\big) \quad \text{for} \quad A \subset G.$$

It is easy to see that m_1 is a universal measure on G which coincides with m on subsets of H. We prove that m_1 is invariant. Fix an element $g \in G$. There exist a bijection $f\colon S \to S$ and a function $h\colon S \to H$ such that for any $s \in S$

$$s + g = f(s) + h(s).$$

Hence we get for any $A \subset G$:

$$\begin{aligned} m_1(g+A) &= \sum_{s \in S} m\big((s+g+A) \cap H\big) \\ &= \sum_{s \in S} m\big((f(s)+h(s)+A) \cap H\big) \\ &= \sum_{s \in S} m\big((f(s)+A) \cap (-h(s)+H)\big) \\ &= \sum_{s \in S} m\big((f(s)+A) \cap H\big) \\ &= \sum_{s \in S} m\big((s+A) \cap H\big) = m_1(A). \end{aligned}$$

In order to prove semiregularity of m_1 let $m_1(A) > 0$. Hence for some $s \in S$, $m\big((s+A) \cap H\big) > 0$. By semiregularity of m we find a set $B \subset (s+A) \cap H$

such that $0 < m(B) < \infty$. Putting $C = -s+B$ we get $C \subset A$ and $m(B) = m_1(B) = m_1(-s+B) = m_1(C)$ and hence $0 < m_1(C) < \infty$ which proves semiregularity of m_1 and finishes the proof of the lemma.

Case 2. $|H| = \varkappa$. For some $n \in \omega$ the group H_n of elements of orders dividing n has cardinality $\varkappa$. Let n_0 be the smallest n with this property. We claim that n_0 is prime. If not, let $n_0 = n_1 n_2$, $1 < n_1, n_2$. There exist $\varkappa$ elements a_α: $\alpha < \varkappa$ of order exactly n_0. Consider the family $\{n_1 a_\alpha: \alpha < \varkappa\}$. All those elements have order n_2 hence there are less than $\varkappa$ of them. By regularity of $\varkappa$ it follows that there are $\varkappa$ distinct elements which have equal n_1-multiples. Thus there are $\varkappa$ elements with order dividing n_1 contrary to the minimality of n_0.

This proves that n_0 is a prime number and hence the set H_{n_0} consisting of elements of order n_0 and the neutral element is a group of cardinality $\varkappa$. It is also a linear space over the field Z_{n_0}. Hence we can construct a universal semiregular invariant measure on H_{n_0} similarly as on the free abelian group in Case 1. Next we produce such a measure on G using Lemma 2.6.

This finishes the proof of our theorem in the case when $|G|$ is RVM. In the general case when $|G|$ is URVM, we take a subgroup H of G such that $|H|$ is RVM. We construct the required measure on H as above and then get a universal semiregular invariant measure on G using again Lemma 2.6. ∎

We do not know if the above theorem is true for arbitrary groups and hence the problem of whether universal semiregular invariant measures exist on any group G of cardinality URVM remains unsolved. On the other hand the argument of Proposition 2.3. shows that if H is a subgroup of G whose set of left cosets is of URVM cardinality then there exist universal semiregular H-invariant measures on G. Hence our problem has a positive solution in the semiregular case e.g. for arbitrary groups G of cardinality URVM and for subgroups H of G such that $|H| < |G|$.

We close this chapter with a brief review of the problem stated at the beginning, considered in a different setting, that of invariant quasi-measures on groups. Now we restrict ourselves to infinite groups, just as before we assumed they were uncountable. In this finitely additive case the invariant version of the general measure problem may be formulated as follows:

Does there exist a universal invariant quasi-measure on a group G?

Notice that dropping countable additivity enables us to require even finiteness of the quasi-measure. It turns out that precisely those groups on which there exist universal invariant finite quasi-measures are interesting from the point of view of analysis. It is easy to show that those are exactly amenable groups (see Greenleaf [7] for the definition of amenability, the proof of the above fact and other related results). Hence all solvable groups have this property and no group containing a non-abelian free subgroup can enjoy it. It was conjectured by von Neumann that amenable groups are

exactly those which do not contain such a subgroup. This has been recently refuted by Grigorčuk [8].

The last result of this chapter shows that the class of groups carrying a universal invariant σ-finite quasi-measure is strictly larger than that of groups carrying a universal invariant finite quasi-measure (i.e. of amenable groups).

THEOREM 2.7. *Every free countable group carries a universal invariant σ-finite quasi-measure.*

Proof. Let μ be any universal invariant finite quasi-measure on the additive group of integers and let S be the free basis of a countable free group G. Every element x of G can be uniquely represented as a reduced word of the form $s_1^{z_1}\ldots s_n^{z_n}$. For this representation of x let:

$$r(x) = z_n, \quad a(x) = s_n, \quad O(x) = s_1^{z_1}\ldots s_{n-1}^{z_{n-1}}.$$

We define a universal quasi-measure on G as follows:

$$m(A) = \sum_{s\in S}\sum_{g\in G}\mu(\{r(x)\colon x\in A \ \&\ O(x) = g \ \&\ a(x) = s\})$$

for any subset A of G.

In order to show that m is a quasi-measure notice that if A and B are disjoint subsets of G then for any $s\in S$ and $g\in G$ the sets

$$\{r(x)\colon x\in A \ \&\ O(x) = g \ \&\ a(x) = s\}$$

and

$$\{r(x)\colon x\in B \ \&\ O(x) = g \ \&\ a(x) = s\}$$

are also disjoint. Hence

$$\begin{aligned}\mu(\{r(x)\colon x\in A\cup B \ \&\ O(x) = g \ \&\ a(x) = s\}) \\ = \mu(\{r(x)\colon x\in A \ \&\ O(x) = g \ \&\ a(x) = s\}) + \\ + \mu(\{r(x)\colon x\in B \ \&\ O(x) = g \ \&\ a(x) = s\})\end{aligned}$$

and we get $m(A\cup B) = m(A)+m(B)$ by definition.

It immediately follows from the definition of m that it is a universal σ-finite quasi-measure. Hence it remains to prove its invariance. Fix $A\subset G$. It is clearly sufficient to show that $m(sA) = m(s^{-1}A) = m(A)$ for any $s\in S$. We shall prove only $m(sA) = m(A)$, the argument for $s^{-1}A$ being similar. Fix $s_0\in S$ and let

$$A_1 = \{x\in A\colon O(x)\neq e \vee a(x)\neq s_0\} \quad (e \text{ denotes the neutral element}),$$

$$A_2 = \{x\in A\colon x = s_0^{r(x)}\}.$$

Since A_1 and A_2 form a disjoint partition of A, it suffices to show that $m(s_0 A_1) = m(A_1)$ and $m(s_0 A_2) = m(A_2)$.

For any $s \in S$ and $g \in G$ we have

$$\{r(x)\colon\ x \in A_1\ \&\ a(x) = s\ \&\ O(x) = g\}$$
$$= \{r(x)\colon\ x \in s_0 A_1\ \&\ a(x) = s\ \&\ O(x) = s_0 g\};$$

hence $m(s_0 A_1) = m(A_1)$.

In order to prove the second equality notice that

$$\{r(x)\colon\ x \in s_0 A_2\} = 1 + \{r(x)\colon\ x \in A_2\}.$$

Hence $\mu(\{r(x)\colon\ x \in s_0 A_2\}) = \mu(\{r(x)\colon\ x \in A_2\})$ by invariance of μ and consequently

$$m(s_0 A_2) = m(A_2).$$

This finishes the proof. ∎

It would be interesting to have an exact characterization of those groups which carry a σ-finite universal invariant quasi-measure, similarly as of amenable groups. We do not even know if any infinite group fails to carry such a quasi-measure.

3. Extensions of invariant measures

A matter closely connected with the invariant version of the general measure problem is that of extending invariant measures. Universal measures – if they exist – clearly do not have any proper extensions. Nevertheless one can ask if those are the only possible examples of invariant measures maximal with respect to extension. The following will be referred to as the invariant extension problem:

Given a group G and a subgroup H of G, does every non-universal H-invariant measure on G have a proper H-invariant extension?

For groups G of cardinality smaller than the first URVM this is the question about the existence of maximal H-invariant measures.

Let us first notice that even if there exists a universal H-invariant measure on G, it is easy to show such H-invariant measures on G which cannot be extended to a universal H-invariant measure. Hence this version of the invariant extension problem has an immediate solution. Indeed, if H is uncountable then any σ-finite H-invariant measure will do in view of Theorem 2.2. (The two-valued measure defined on countable and co-countable sets provides an example.) If H is countable then any selector S of its left cosets in G is uncountable. Take a probability measure μ on S which cannot be extended to a universal measure (such a measure exists on every

uncountable set) and define an H-invariant measure on G by the formula

$$m(A) = \sum_{h \in H} \mu(hA \cap S)$$

for such $A \subset G$ that $hA \cap S$ is μ-measurable for any $h \in H$. The measure m cannot be extended to a universal measure.

As in the previous chapter we start the investigation of our problem in the σ-finite case. In the proof of the first result we need the following two easy lemmas. The first one is essentially due to Szpilrajn [36] and the second one is its straightforward consequence, hence we omit the proofs. We state both lemmas in a more general setting which will be useful in further considerations.

LEMMA 3.1. *Let G be a group of bijections of a set X and m a semiregular G-invariant measure on X. If there exists a set $Y \subset X$ of positive outer measure such that for any $\{g_n\colon n \in \omega\} \subset G$ the set $\bigcup_{n \in \omega} g_n[Y]$ has inner measure zero then m has a proper semiregular G-invariant extension.*

LEMMA 3.2. *Let G be a group of bijections of a set X and m a σ-finite G-invariant measure on X. If there exists a set $E \subset X$ such that:*

1. *E has positive outer measure,*
2. *for every countable set $K \subset G$ there exists an uncountable set $L \subset G$ such that for distinct $l_1, l_2 \in L$*

$$m\Big(l_1\Big[\bigcup_{k \in K} k[E]\Big] \cap l_2\Big[\bigcup_{k \in K} k[E]\Big]\Big) = 0,$$

then the measure m has a proper G-invariant extension.

The following theorem provides a solution of the invariant extension problem in the case when G is abelian and $H = G$.

THEOREM 3.3 (Pelc [26]). *Every σ-finite invariant measure on an abelian group has a proper invariant extension.*

Proof. Let G be an abelian group and m a σ-finite invariant measure on G. In order to find a proper invariant extension of m we shall show a set E satisfying the conditions of Lemma 3.2 for the group G acting on iself by left shifts.

Case 1. Additive group of a linear space over a countable field.

Let V be a linear space over a countable field K and m any measure on V invariant with respect to addition. Fix a linear basis $\mathscr{B} = \{v_\alpha\colon \alpha < \varkappa\}$ of V over K and let V_n denote the set of those elements of V which have n summands in the basis $\mathscr{B}$ representation.

Hence $V = \bigcup_{n \in \omega} V_n$ and there exists the least number n_0 for which V_{n_0} has positive outer measure. We claim that V_{n_0} also satisfies condition 2 of Lemma 3.2. Let $\{g_n\colon n \in \omega\}$ be a countable sequence of elements of V and D

$= \bigcup_{n\in\omega} g_n + V_{n_0}$. As $L = \{l_\alpha\colon \alpha < \omega_1\}$ from the lemma take any subset of $\mathscr{B}$ of cardinality ω_1 whose elements do not appear in the $\mathscr{B}$-representation of any element g_n. We show that for distinct α, β, $m[(l_\alpha + D) \cap (l_\beta + D)] = 0$.

Suppose that for $d_1, d_2 \in D$ we have $w = l_\alpha + d_1 = l_\beta + d_2$. Then $d_1 = g_n + w_1$ and $d_2 = g_k + w_2$, $w_1, w_2 \in V_{n_0}$. Let W_i (for $i = 1, 2$) denote the set of those elements of $\mathscr{B}$ which appear with non-zero coefficients in the representation of w_i. By definition of V_{n_0} the sets W_i have n_0 elements.

If $l_\alpha \in W_1$ then w is of the form $k' l_\alpha + g_n + w'$, where $w' \in V_{n_0-1}$, $k' \in K$.

If $l_\alpha \notin W_1$ then, since it does not appear in the $\mathscr{B}$-representation of g_n or g_k, we get $l_\alpha \in W_2$ and w is of the form $l_\beta + g_k + k'' l_\alpha + w''$, where $w'' \in V_{n_0-1}$, $k'' \in K$. Hence in any case the set of elements w of the form $l_\alpha + d_1 = l_\beta + d_2$ is contained in the union of countably many translates of the set V_{n_0-1}, hence by definition of n_0 it has measure 0. It follows that $m((l_\alpha + D) \cap (l_\beta + D)) = 0$ and hence the set V_{n_0} satisfies the conditions of the lemma.

Case 2. Torsion-free abelian group.

Let G be a torsion free abelian group. There exists a homomorphic embedding of G into the additive group of a linear space V over the field Q of rationals, such that a certain basis $\mathscr{B} = \{v_\alpha\colon \alpha < \varkappa\}$ of V consists of elements of G. Let m be any invariant measure on G.

For any finite sequence $s = (q_1, \ldots, q_n)$ of non-zero rationals let V_s be the set of elements of V of the form $q_1 v_{\alpha_1} + \ldots + q_n v_{\alpha_n}$ where $\alpha_1 > \ldots > \alpha_n$ and $v_{\alpha_i} \in \mathscr{B}$. Let $s_0 = (r_1, \ldots, r_n)$ be a sequence for which the set $E = G \cap V_{s_0}$ has positive outer measure. In order to check that E also satisfies condition 2 of Lemma 3.2, let $\{g_n\colon n \in \omega\}$ be any sequence of elements in G. Take any uncountable set of elements w_α of $\mathscr{B}$ which do not appear in the $\mathscr{B}$-representation of any element g_n. Let k be a natural number different from all $\pm r_i$, $r_i - r_j$ $(i, j \leqslant n)$ and let $l_\alpha = k w_\alpha$ for $\alpha < \omega_1$. We claim that $[l_\alpha + \bigcup_{n\in\omega} (g_n + E)] \cap [l_\beta + \bigcup_{n\in\omega} (g_n + E)] = \emptyset$, for $\alpha \neq \beta$. Indeed suppose x is an element of the set on the left side. Then $x = k w_\alpha + g_n + r_1 v_{\alpha_1} + \ldots + r_n v_{\alpha_n} = k w_\beta + g_m + r_1 v_{\beta_1} + \ldots + r_n v_{\beta_n}$. Since $\alpha \neq \beta$ and w_α, w_β do not appear in the representation of g_n, g_m, we get that either $k = \pm r_i$ or $k + r_i = r_j$ for some $i, j \leqslant n$, contradiction.

3. The general case. Let G be now an arbitrary abelian group. By H denote the torsion subgroup of G. If $m(H) = 0$ then we define a measure m_1 on G/H putting $m_1(\{a + H\colon a \in A\}) = m(A + H)$ for such $A \subset G$ that $A + H$ is m-measurable. The measure m_1 is clearly invariant (it vanishes on singletons by $m(H) = 0$).

The group G/H is torsion-free and hence by case 2 there exists a set $E_1 \subset G/H$ satisfying both conditions from Lemma 3.2 for G/H and m_1. It is

not hard to see that the set $E = \bigcup E_1$ satisfies the condition from the lemma for G and m.

If H is a set of positive outer measure, let H_n (for $n \geqslant 1$) denote the subgroup of H consisting of those elements whose orders divide n. Clearly $H = \bigcup_{n \geqslant 1} H_n$ and let n_0 be the least natural number for which H_{n_0} has positive outer measure. Call H_{n_0} the minimal group for G and m. We shall prove the existence of a subset $E \subset H_{n_0}$ satisfying the conditions of our lemma, by induction on the number k of prime divisors of n_0 (counting multiple divisors many times).

If $k = 1$ then n_0 is prime and H_{n_0} is the additive group of a linear space over the finite field F_{n_0}. By case 1 we get a subset $E \subset H_{n_0}$ of positive outer measure such that for any countable $K \subset H_{n_0}$ there exists an uncountable $L \subset H_{n_0}$ satisfying $m\left((l_1+K+E) \cap (l_2+K+E)\right) = 0$ for distinct $l_1, l_2 \in L$. We prove that the set E satisfies condition 2 of Lemma 3.2 for the group G as well. Let K^* be any countable subset of G. There exist elements s_n: $n \in \omega$ belonging to distinct cosets of H_{n_0} in G and subsets K_n: $n \in \omega$ of the group H_{n_0} such that $K^* = \bigcup_{n \in \omega} s_n + K_n$. Consider the set $K = \bigcup_{n \in \omega} K_n$. Since it is a countable subset of H_{n_0} we can find an appropriate uncountable $L \subset H_{n_0}$ as above. We claim that L is also good for K^*. Indeed take distinct $l_1, l_2 \in L$. We have $m\left((l_1+K+E) \cap (l_2+K+E)\right) = 0$ and hence for any $n \in \omega$:

$$m\left((l_1+K_n+E) \cap (l_2+K_n+E)\right) = 0$$

which gives in view of the invariance of m:

$$m\left((l_1+s_n+K_n+E) \cap (l_2+s_n+K_n+E)\right) = 0$$

and finally:

$$m\left((l_1+K^*+E) \cap (l_2+K^*+E)\right) = 0.$$

This finishes the proof for $k = 1$.

Suppose that for n_0 having k prime divisors there exists a set $E \subset H_{n_0}$ satisfying the lemma. Let now $n_0 = p_1 \dots p_{k+1}$ (p_i-primes, $k \geqslant 1$) and H' be the subgroup of H_{n_0} consisting of elements of order p_1.

Since $m(H') = 0$, we can define an invariant measure m' on G/H' just as before. H_{n_0}/H' is a subgroup of G/H' all of whose elements have orders dividing the number $p_2 \dots p_{k+1}$. By definition H_{n_0}/H' is the minimal group for G/H' and m'. Hence by the inductive hypothesis there exists a set $E' \subset H_{n_0}/H'$ which satisfies both conditions of Lemma 3.2 for the group G/H' and the measure m'.

It is easy to see that the set $E = \bigcup E'$ is now good for G and m which finishes the proof in the general case. ∎

The above theorem admits the following easy self-refinement which is also a strengthening of Theorem 2.2 for abelian groups. Its importance for our purpose consists in the fact that – at least for abelian groups – it enables to restrict the σ-finite case of the invariant extension problem only to countable subgroups H.

PROPOSITION 3.4. *If H is an uncountable abelian subgroup of a group G then every σ-finite H-invariant measure on G has a proper H-invariant extension.*

Proof. Let H be an uncountable abelian subgroup of G, m an H-invariant σ-finite measure on G and S a selector of the family of right cosets of H in G. We consider two cases.

1. S has positive outer measure.

It is enough to show that S satisfies condition 2 of Lemma 3.2 for H acting on G by left shifts. Let K be any countable subset of H. We construct the set $L = \{l_\alpha\colon \alpha < \omega_1\}$ by induction. If $\{l_\alpha\colon \alpha < \beta\}$ are already defined take l_β to be any element of H outside of the group generated by $K \cup \{l_\alpha\colon \alpha < \beta\}$. It is easy to see that for distinct $\alpha, \beta < \omega_1$ we have

$$l_\alpha KS \cap l_\beta KS = \emptyset,$$

hence condition 2 of Lemma 3.2 is satisfied.

2. $m(S) = 0$.

We define a measure m' on H putting $m'(A) = m(AS)$ if AS is m-measurable. This is an invariant measure on the abelian group H hence by Theorem 3.3 it has a proper invariant extension m'_1. It is easy to see that the measure m_1 on G defined by the formula

$$m_1(AS) = m'_1(A) \quad \text{for } m'_1\text{-measurable sets } A \subset H,$$

is the required H-invariant extension of m. ∎

We do not know if it is necessary to assume commutativity of the subgroup H in the above proposition. We conjecture it is not.

Leaving this as an open question we now restrict our attention to the σ-finite case of the invariant extension problem for countable subgroups H. Our further results show that in this situation the solution can vary depending on the algebraic structure of H and the cardinality of G. Theorem 3.6 provides examples for which the invariant extension problem has positive solution, whereas Theorem 3.10. shows than in some other cases the answer must be negative. First we need the following.

LEMMA 3.5. *Let H be a subgroup of G and m a σ-finite H-invariant measure on G defined on a σ-algebra $\mathfrak{M}$ Let $\{A_1, \ldots, A_n\}$ be a disjoint*

partition of G such that

$$\forall h \in H \ \forall i \leqslant n \ \exists j \leqslant n \ (hA_i = A_j).$$

Then there exists an H-invariant extension μ of m defined on the σ-algebra $\mathfrak{N}$ generated by $\mathfrak{M} \cup \{A_1, \ldots, A_n\}$.

Proof. Let S_n denote the group of permutations of $\{1, \ldots, n\}$. For any permutation $\tau \in S_n$ and any $k \in \{0, \ldots, n-1\}$ we construct σ-algebras $\mathfrak{M}_k^\tau$ and measures $m^{\tau,k}$ defined on $\mathfrak{M}_k^\tau$, by induction on k. Let $\mathfrak{M}_0^\tau = \mathfrak{M}$, $m^{\tau,0} = m$. If the required objects are already constructed for an integer $k < n-1$, we take as $\mathfrak{M}_{k+1}^\tau$ the σ-algebra generated by $\mathfrak{M}_k^\tau \cup A_{\tau(k+1)}$. For any $Z \in \mathfrak{M}_{k+1}^\tau$ we have $Z = (M_1 \cap A_{\tau(k+1)}) \cup (M_2 \setminus A_{\tau(k+1)})$ for some $M_1, M_2 \in \mathfrak{M}_k^\tau$. For such Z we put

$$m^{\tau,k+1}(Z) = m_e^{\tau,k}(M_1 \cap A_{\tau(k+1)}) + m_i^{\tau,k}(M_2 \setminus A_{\tau(k+1)}).$$

Clearly the σ-algebra $\mathfrak{M}_{n-1}^\tau$ does not depend on τ. Call it $\mathfrak{N}$. We define $m^\tau = m^{\tau,n-1}$ and finally put

$$\mu(X) = \frac{1}{n!} \sum_{\tau \in S_n} m^\tau(X) \quad \text{for} \quad X \in \mathfrak{N}.$$

For any $\tau \in S_n$ and $k \in \{0, \ldots, n-1\}$ the measure $m^{\tau,k}$ is an extension of m (cf. the proof of Theorem 1.7). Hence any m^τ and consequently also μ is such an extension. We will show that μ is H-invariant. Fix $h \in H$ and $X \in \mathfrak{N}$. Let $\varrho \in S_n$ be such that $hA_l = A_{\varrho(l)}$ for any $l \in \{1, \ldots, n\}$. It is easy to prove by induction on k that for any $\tau \in S_n$ and any $Y \in \mathfrak{M}_k^\tau$ the following holds:

$$hY \in \mathfrak{M}_k^{\varrho\circ\tau} \quad \text{and} \quad m^{\tau,k}(Y) = m^{\varrho\circ\tau,k}(hY).$$

Hence taking $k = n-1$ we get

$$m^\tau(X) = m^{\varrho\circ\tau}(hX)$$

and finally

$$\mu(X) = \frac{1}{n!} \sum_{\tau \in S_n} m^\tau(X) = \frac{1}{n!} \sum_{\tau \in S_n} m^{\varrho\circ\tau}(hX) = \frac{1}{n!} \sum_{\tau \in S_n} m^\tau(hX) = \mu(hX).$$

THEOREM 3.6. *Let H be a countable subgroup of a group G.*

(a) *If there exists a descending sequence $\{H_n\colon n \in \omega\}$ of subgroups of H of finite indices such that $\bigcap_{n \in \omega} H_n$ is trivial then every non universal σ-finite H-invariant measure on G has a proper H-invariant extension.*

(b) *If H contains subgroups of arbitrarily large finite indices then every finite H-invariant measure on G has a proper H-invariant extension.*

Proof. (a) Let m be a non universal H-invariant σ-finite measure on G defined on a σ-algebra $\mathfrak{M}$ and S a selector of the family of right cosets of H

in G containing the neutral element e. For any natural n the family $\{hH_nS\colon h\in H\}$ is a finite disjoint partition of G satisfying conditions from Lemma 3.5. Hence the measure m can be extended H-invariantly over H_nS for any $n\in\omega$.

If one of the sets H_nS does not belong to $\mathfrak{M}$ then m has a proper H-invariant extension. If all of them are elements of $\mathfrak{M}$ then the set

$$S = \bigcap_{n\in\omega} H_nS$$

is also an element of $\mathfrak{M}$.

The measure m is non-universal, hence there exists a subset $T\subset S$ which is non measurable. We extend m to an H-invariant measure m' defined on the σ-algebra generated by $\mathfrak{M}\cup\{hT\colon h\in H\}$. Elements of this σ-algebra have form

$$X = \bigcup_{h\in H} h\big[(M'_h\cap T)\cup(M''_h\cap(S\setminus T))\big],$$

where $M'_h, M''_h\in\mathfrak{M}$.

We put

$$m'(X) = \sum_{h\in H}\big[m_e(M'_h\cap T)+m_i(M''_h\cap(S\setminus T))\big].$$

It is easy to check that m' is an H-invariant extension of m.

(b) Let m be an H-invariant measure on G defined on a σ-algebra $\mathfrak{M}$, suppose that $m(G)=c$, $0<c<\infty$ and let S be a selector of the family of right cosets of H in G. Denote by H_n a subgroup of H of index k_n, $k_n\to\infty$. For a fixed n let $H_n^1,\ldots,H_n^{k_n}$ be the left cosets of H_n in H. Consider the sets $A_n^i = H_n^iS$ for $1\leqslant i\leqslant k_n$. These sets satisfy conditions of Lemma 3.5, and hence there exists an H-invariant extension of m defined on the σ-algebra generated by $\mathfrak{M}\cup\{A_n^1,\ldots,A_n^{k_n}\}$. Clearly each of the sets A_n^i must have measure c/k_n.

If the measure m didn't have proper H-invariant extensions, all sets H_n^iS would have to be m-measurable. Hence the set $W=\bigcap_{n\in\omega} H_nS$ would also be measurable and $m(W)\leqslant c/k_n$ for any natural n, thus $m(W)=0$. By completeness we would get $m(S)=0$ and hence $m(G)=0$, contradiction.

It follows that for some natural n the measure μ^* defined on the σ-algebra generated by $\mathfrak{M}\cup\{A_n^1,\ldots,A_n^{k_n}\}$ as in Lemma 3.5 is a proper H-invariant extension of m. ∎

COROLLARY 3.7. *Every non-universal Z-invariant σ-finite measure on the reals has a proper Z-invariant extension.*

PROBLEM 3.8. Can Z be replaced by Q in the above corollary?

Our next result provides examples of non-universal H-invariant measures on a group G which do not have proper H-invariant extensions. The class of groups H for which the construction works is fairly wide, in

particular it contains the additive group of rationals, but our argument is valid only for extremely large groups G, i.e., of measurable cardinality. It seems interesting to know if such an example is possible on smaller groups (cf. Problem 3.8).

First we need the following

LEMMA 3.9. *If a group H doesn't have proper subgroups of finite index then for every non-empty proper subset $A \subset H$ there exist infinitely many distinct left shifts of A.*

Proof. Let H_0 be the subgroup of H consisting of those elements h for which $hA = A$. H_0 is proper and hence has infinite index. Take any selector S of the family of left cosets of H_0 in H. The set S is infinite and for distinct $s_1, s_2 \in S$ we have $s_1 A \neq s_2 A$.

THEOREM 3.10. *If $\varkappa$ is a measurable cardinal, G a group of cardinality $\varkappa$ and H a countable subgroup of G without proper subgroups of finite index, then there exists a maximal probability H-invariant measure on G.*

(Such a measure must be non-universal if H is non-trivial.)

Proof. Let S be any selector of the family of right cosets of H in G. We may assume that H is non-trivial. Since S has cardinality $\varkappa$, there exists a $\varkappa$-additive universal two-valued measure μ on S. We define a measure m on G putting

$$m(HA) = \mu(A) \quad \text{for} \quad A \subset S$$

and consider its measure completion $\bar{m}$ defined on $\mathfrak{M}$ (1). Clearly $\bar{m}$ is a $\varkappa$-additive H-invariant probability measure. It is non-universal because H is non-trivial and hence $S \notin \mathfrak{M}$. Assume that m_1 is an H-invariant extension of $\bar{m}$. If $m_1(B) = 0$ then $m_1(HB) = 0$. However $HB \in \mathfrak{M}$, hence $m(HB) = 0$ which implies $\bar{m}(B) = 0$.

This reasoning shows that $\bar{m}$ has the following property: none of its H-invariant extensions adds new sets of measure zero. We will show that this measure doesn't have proper H-invariant extensions whatsoever.

First notice that any H-invariant extension of m is $\varkappa$-additive. Indeed, it doesn't add new sets of measure zero, hence the ideal of null sets of the extension doesn't change and remains $\varkappa$-complete.

Suppose that the measure $\bar{m}$ can be properly extended to an H-invariant measure $\tilde{m}$ defined on a σ-algebra $\tilde{\mathfrak{M}}$ and take any set $X \in \tilde{\mathfrak{M}} \setminus \mathfrak{M}$. Without loss of generality we may assume that neither X nor $G \setminus X$ contain right cosets of H in G. Indeed, suppose that X contains a non-empty union A of cosets. Hence $\bar{m}(A) = 0$ (otherwise $\bar{m}(A) = 1$ and hence $X \in \mathfrak{M}$) and we may consider $X \setminus A$ instead of X. The same argument applies to $G \setminus X$.

Consider the σ-algebra generated by the family $\{hX\colon h \in H\}$. It is a

(1) Strictly speaking m is not a measure according to our definition (singletons are not measurable) but $\bar{m}$ is a measure.

countably generated H-invariant σ-algebra contained in $\mathfrak{M}$. Look at its atoms. Each of them is determined by a function $f\colon H \to \{0, 1\}$ in the following way:

$$\bigcap_{h\in H} (hX)^{f(h)}, \quad \text{where } Z^0 = Z,\ Z^1 = G\setminus Z.$$

The constant functions do not determine atoms because

$$\bigcap_{h\in H} hX = \bigcap_{h\in H} (G\setminus hX) = \emptyset.$$

Notice that if an atom C is determined by the characteristic function of a subset $A \subset H$ then the atom determined by the characteristic function of the set hA is equal to hC. It follows from Lemma 3.9 that for any atom of the σ-algebra in question there exist infinitely many of its distinct left shifts (all of them are clearly pairwise disjoint).

Since $\tilde{m}$ is a probability measure, each of the atoms must have $\tilde{m}$ measure zero. However G is the union of at most 2^ω atoms and $\tilde{m}$ is $\varkappa$-additive. The cardinal $\varkappa$ is measurable, hence $\varkappa > 2^\omega$. It follows that $\tilde{m}(G) = 0$, contradiction. ∎

COROLLARY 3.11. *If G is any group of measurable cardinality then there exists a maximal Q-invariant probability measure on $G\oplus Q$.*

Proof. It suffices to notice that Q doesn't have proper subgroups of finite index. ∎

This closes our discussion of the invariant extension problem in the σ-finite setting. Apart from the case of non-abelian uncountable groups H which remains unsolved, the situation for countable subgroups H is also far from being completely settled. The examples given in Theorems 3.6 and 3.10 do not provide any complete characterization of those subgroups for which the solution of the invariant extension problem is positive. It would be interesting to have a necessary and sufficient condition for the positive solution in terms of algebraic properties of the countable subgroup H and possibly of the cardinality of G.

Similarly as in Chapter 2 we now switch to semiregular measures. It turns out that even in this more general setting the solution of the invariant extension problem is positive for small groups G, no matter what subgroup H we choose.

We shall need the following lemma due to Hulanicki (cf. also Pkhakadze [30]).

LEMMA 3.12 (Hulanicki [17]). *Let G be a group of bijections of a set X such that $|G| \leqslant |X| <$ the first* RVM. *If m is any uniform measure on X assuming at least one positive finite value, there exists a non-measurable subset $Z \subset X$ such that for any $g \in G$*

$$m(g[Z] \triangle Z) = 0.$$

Proof. Let $|X| = \varkappa$ and $\{x_\alpha\colon \alpha < \varkappa\}$, $\{g_\alpha\colon \alpha < \varkappa\}$ be enumerations of X and G respectively. For any $\alpha < \varkappa$ denote by X_α the set $\{x_\beta\colon \beta < \alpha\}$ and by G_α the group generated by $\{g_\beta\colon \beta < \alpha\}$. Let also $P_\alpha = \bigcup_{g \in G_\alpha} g[X_\alpha]$ and $Q_\alpha = P_\alpha \setminus \bigcup_{\beta < \alpha} P_\beta$. Clearly every set P_α and hence also Q_α has cardinality smaller than $\varkappa$. The sets Q_α form a partition of X.

We claim that for any $A \subset \varkappa$ the set $\bar{Q} = \bigcup_{\alpha \in A} Q_\alpha$ has the following property:

$$|g[\bar{Q}] \triangle \bar{Q}| < \varkappa \quad \text{for any } g \in G.$$

Indeed, let $g = g_\alpha$. Then for every $\beta > \alpha$, $g[Q_\beta] \subset Q_\beta$, hence $g[\bar{Q}] \setminus \bar{Q} \subset \bigcup_{\beta \leqslant \alpha} Q_\beta$ and we conclude that $|g[\bar{Q}] \setminus \bar{Q}| < \varkappa$. Similarly, taking g^{-1} instead of g we get $|\bar{Q} \setminus g[\bar{Q}]| < \varkappa$, which proves the claim.

Let m be any uniform measure assuming at least one positive finite value, defined on a σ-algebra $\mathfrak{M}$ of subsets of X. By uniformity $m(Q_\alpha) = 0$ for any $\alpha < \varkappa$. If every set of the form $\bigcup_{\alpha \in A} Q_\alpha (A \subset \varkappa)$ were measurable then the measure μ on $\varkappa$ defined by the formula

$$\mu(A) = m\Big(\bigcup_{\alpha \in A} Q_\alpha\Big) \quad \text{for} \quad A \subset \varkappa,$$

would be a universal measure on $\varkappa$ assuming at least one positive finite value, contrary to the assumption that $\varkappa$ is smaller than the first RVM and thus is not URVM.

Hence there exists a set $Z = \bigcup_{\alpha \in A} Q_\alpha$, for some $A \subset \varkappa$, which is not measurable. In view of uniformity of m we have

$$m(g[Z] \triangle Z) = 0 \quad \text{for any } g \in G.$$

By use of Łoś and Marczewski's extension technique (see Theorem 1.7) Hulanicki got the following proposition which is a simple consequence of the above lemma.

PROPOSITION 3.13 (Hulanicki [17]). *Let G be a group of bijections of a set X such that $|G| \leqslant |X|$ < the first* RVM. *Then every uniform G-invariant measure on X assuming at least one positive finite value has a proper G-invariant extension.*

For our purpose however this argument does not work because Theorem 1.7 applied to semiregular measures does not necessarily give a semiregular extension and hence using Proposition 3.13 we would risk to loose semiregularity. Fortunately a different argument allows us to keep it.

PROPOSITION 3.14. *Let G be a group of bijections of a set X such that*

$|G| \leqslant |X| <$ *the first* RVM. *Then every uniform G-invariant semiregular measure on X has a proper G-invariant semiregular extension.*

Proof. Let Z be the set obtained in Lemma 3.12, i.e. Z is non-measurable and

$$m(g[Z] \triangle Z) = 0 \quad \text{for any } g \in G.$$

Let $Y \subset Z$ be measurable and such that the set $Z_1 = Z \setminus Y$ has inner measure zero. It is easy to see that $m(g[Y] \triangle Y) = 0$ for any $g \in G$. Hence the set Z_1 has the following properties (cf. Pkhakadze [30], the proof of Theorem 3.22):

(a) Z_1 has positive outer measure,

(b) Z_1 has inner measure zero,

(c) $m(g[Z_1] \triangle Z_1) = 0$ for any $g \in G$.

Let $\{g_n\colon n \in \omega\}$ be any countable subset of G. In view of property (c) we get

$$m\Big(\big(\bigcup_{n\in\omega} g_n[Z_1]\big) \triangle Z_1\Big) = 0$$

and hence $\bigcup_{n\in\omega} g_n[Z_1]$ has inner measure zero. Now the conclusion follows from Lemma 3.1. ∎

We do not know if the assumption of uniformity can be omitted in the general formulation of the above proposition. Our next result shows that in the special situation when a group acts on itself by left shifts, this assumption is not necessary.

We now state the above mentioned theorem giving a positive answer to the invariant extension problem for small groups.

THEOREM 3.15. *Let G be any group of cardinality smaller than the first* RVM *and H any subgroup of G. Then every semiregular H-invariant measure on G has a proper semiregular H-invariant extension.*

Proof. Let m be any semiregular H-invariant measure defined on a σ-algebra $\mathfrak{M}$ of subsets of G. Denote by λ the smallest cardinality of a subset of G of positive outer measure and let $A \subset G$ be such a subset of cardinality λ. If for every countable set $E \subset G$, EA has inner measure zero then we are done by Lemma 3.1. Hence assume that E is a countable subset of G for which EA has positive inner measure. Denote by G_2 the group generated by EA. It has cardinality λ and positive inner measure. Let $H_2 = H \cap G_2$ and let $C \subset G_2$ be a measurable set for which $G_2 \setminus C$ has inner measure zero. It is easy to see that for any $h \in H_2$, $m(hC \triangle C) = 0$.

Consider the sets Q_α from the proof of Lemma 3.12. for H_2 acting on G_2. For any $A \subset \lambda$ and $h \in H_2$ we have

$$\Big|h\big(\bigcup_{\alpha\in A} Q_\alpha\big) \triangle \bigcup_{\alpha\in A} Q_\alpha\Big| < \lambda,$$

hence

$$m\big(h(C \cap \bigcup_{\alpha \in A} Q_\alpha) \triangle (C \cap \bigcup_{\alpha \in A} Q_\alpha)\big) = 0$$

in view of the properties of C.

Using the fact that λ is not URVM we get

$$C \cap \bigcup_{\alpha \in A} Q_\alpha \notin \mathfrak{M} \quad \text{for some } A \subset \lambda \text{ (cf. the proof of Lemma 3.12).}$$

Next, by the argument from the proof of Proposition 3.14. we get a set $Z \subset C \cap \bigcup_{\alpha \in A} Q_\alpha$ which has the following properties:

(a) $Z \notin \mathfrak{M}$,

(b) Z has inner measure zero,

(c) $m((hZ) \triangle Z) = 0$ for any $h \in H_2$.

By Lemma 3.1 it is enough to show that for any countable set $K \subset H$ the set KZ has inner measure zero.

Suppose that $T \subset KZ$, $m(T) > 0$ and let $\{T_n\colon n \in \omega\}$ be a partition of T into subsets of distinct left cosets of G_2 in G. Each of those cosets contains elements from K hence for any $n \in \omega$ there exists $k_n \in K$ and a countable subset $L_n \subset H_2$ such that $T_n \subset k_n L_n C$. Since $m(L_n C \triangle C) = 0$ we get $k_n L_n C \in \mathfrak{M}$ for any natural n and hence $m(T_{n_0}) > 0$ for some n_0. It follows that $S = k_{n_0}^{-1} T_{n_0} \subset L_{n_0} Z$ and $m(S) > 0$. By property (c) we have $m((L_{n_0} Z) \triangle Z) = 0$ and by property (b) the set Z cannot contain any subset of positive measure, contradiction. ∎

Theorems 2.5 and 3.15 imply the following corollary which shows the connection between the invariant version of the general measure problem and the invariant extension problem for semiregular measures on abelian groups.

COROLLARY 3.16. *Let G be an abelian group. The following are equivalent*:

(a) $|G|$ *is* URVM,

(b) *there exists a universal semiregular invariant measure on G,*

(c) *there exists a maximal semiregular invariant measure on G.*

Nevertheless the following particular case of the invariant extension problem remains open even for abelian groups: Does every non-universal semiregular invariant measure on a group G of cardinality URVM have a proper semiregular invariant extension?

This closes our discussion concerning extensions of invariant measures on groups. A related classical problem should be mentioned in this context as it remains in the same stream of ideas and engages similar tools.

Sierpiński (quoted in Szpilrajn [36]) asked if there exists a maximal extension of the Lebesgue measure in the Euclidean space E^n, invariant with

respect to the group of all isometries of this space? He also proved (see [34]) that some proper invariant extensions do exist.

Several partial solutions of this problem have been first obtained.

Pkhakadze [30] proved that if 2^{ω} is smaller than the first RVM then the answer to Sierpiński's question is negative. This also follows from Hulanicki's result (see Proposition 3.13). Indeed, it is enough to notice that if G is the group of all isometries of the Euclidean space E^n then every σ-finite G-invariant measure on E^n has a uniform G-invariant extension. Hence Proposition 3.13 can be applied to this extension.

Further partial results concerning Sierpiński's problem have been gotten by Harazišvili [12]. He gave e.g. the negative answer in the one dimensional case.

The final solution has been recently obtained by Ciesielski and Pelc in the following theorem:

THEOREM 3.17 (Ciesielski, Pelc [5]). *Let G be any group of isometries of the Euclidean space E^n, which contains all translations. Then every σ-finite G-invariant measure on E^n has a proper G-invariant extension.*

Similarly as for the invariant extension problem on groups, the results turn out to be different in the case of semiregular measures. The following theorem is also proved in [5], using our Theorem 2.5 and techniques developped in the proof of Theorem 3.15.

THEOREM 3.18 (Ciesielski, Pelc [5]). *Let G be any group of isometries of the Euclidean space E^n. The following are equivalent:*

(a) 2^{ω} *is* URVM,

(b) *there exists a universal semiregular G-invariant measure on E^n,*

(c) *there exists a maximal semiregular G-invariant measure on E^n.*

The invariant extension problem considered in this chapter might be generalized to the context of G-invariant measures on a set X, for an arbitrary group of bijections of X. The history and the final solution of Sierpiński's problem suggest however that in full generality this question can be very difficult. In the case of the group of isometries on Euclidean spaces specific geometric properties were used in order to extend an invariant measure. It seems unlikely that the tools appropriate for invariant measures on groups or those developped in [5] for the case of Euclidean spaces were sufficient in the general setting. Until now the only result concerning the case of measures invariant with respect to an arbitrary group of bijections is essentially that of Hulanicki (Proposition 3.13 above).

Similarly as before we close this chapter pointing out how the invariant extension problem changes in the finitely additive setting, i.e. when measures are replaced by quasi-measures. It turns out that – similarly as for the general measure problem – the finitely additive invariant extension problem also reduces to the question of amenability of the group. Indeed, it follows

from Zorn's lemma that there exist maximal invariant probability quasi-measures on every group, moreover every invariant quasi-measure can be extended to a maximal one. A result of von Neumann (cf. e.g. [14]) says that every invariant probability quasi-measure on an amenable group can be extended to a universal invariant quasi-measure. Hence every non-universal invariant probability quasi-measure on a group G has a proper invariant extension iff G is amenable.

The above remarks show that in the finitely additive case both problems considered in the previous and present chapters can usually be reduced to classical analytic questions. This is in contrast to the countably additive setting actually discussed in this dissertation, where combinatorial tools are used in most of the arguments and the main techniques are rather distant from analysis.

4. Saturation of ideals on groups

In this chapter we shall investigate the existence of large invariant measures on groups from another point of view. Our attention will be focused on measures assuming only values 0 and 1. The investigation of such measures can be reduced to that of σ-complete ideals and thus we are going to look for large invariant ideals on groups. As it was mentioned in the introduction, saturation of an ideal seems one of the reasonable indices of its size, which justifies our approach. On the other hand the problem of saturation of ideals in connection with their completeness has been extensively investigated in set theory. The relationship of this topic with the theory of large cardinals was reviewed in Chapter 1. Now we are going to study the invariant counterpart of this well-known combinatorial question showing how good behaviour of an ideal with respect to the group structure restricts its possible degree of saturation. As in the previous chapters the most comprehensive results are obtained for abelian groups.

In order to get a positive result about the existence of a strongly saturated invariant ideal on a group we must obviously assume that the cardinality of the group admits ideals with this degree of saturation. As it was mentioned in Chapter 1 this depends on set theoretic assumptions.

We first prove two positive results showing that for $\lambda > |G|$, $|G|$-complete λ-saturated invariant ideals often exist on G, provided that $|G|$ admits such ideals. The assumptions that we impose on the group G are of two kinds: in the first theorem we require that G be abelian, in the second we want the cardinality of the group to be a successor cardinal.

The set theoretic hypothesis which has to be made is not contradictory even in the latter case. If $\lambda > \varkappa$, $\varkappa$-complete λ-saturated ideals can exist on a successor cardinal $\varkappa$ as well. Kunen [21] proved e.g. that the existence of an

ω_1-complete ω_2-saturated ideal on ω_1 is consistent with ZFC provided the so called huge cardinal is assumed. Hence the following Theorem 4.1 makes sense also for successor cardinals. Moreover it turns out to be true for all groups of such cardinality (see Theorem 4.5).

THEOREM 4.1. *Let $\lambda > \varkappa \geqslant \omega_1$ be cardinals and G an abelian group of cardinality $\varkappa$. If there exists a $\varkappa$-complete λ-saturated ideal on $\varkappa$, there also exists an invariant $\varkappa$-complete λ-saturated ideal on G.*

Proof. We split the proof into a few lemmas.

LEMMA 4.2. *Theorem 4.1. is true for the case of additive groups of linear spaces over a countable field K and for free abelian groups.*

Proof. We prove the first part of the lemma. The case of free abelian groups is analogous ([1]).

Let V be a linear space of cardinality $\varkappa$ over a countable field K and let $\mathscr{B} = \{b_\alpha\colon \alpha < \varkappa\}$ be its linear basis. For $x \in V$ we denote:

$$r(x) = \max\{\alpha < \varkappa\colon b_\alpha \text{ enters with a non-zero coefficient in the representation of } x \text{ in basis } \mathscr{B}\},$$

$$l(x) = \text{the coefficient at } b_{r(x)} \text{ in the representation of } x \text{ in } \mathscr{B},$$

$$o(x) = x - l(x)\cdot b_{r(x)}.$$

Let I be a $\varkappa$-complete λ-saturated ideal on $\varkappa$. We define an ideal J on V by the formula:

$$A \in J \quad \text{iff} \quad \forall y \in V \quad \{r(x)\colon x \in A \ \&\ o(x) = y\} \in I.$$

It is obvious that J is a $\varkappa$-complete ideal. In order to prove that J is invariant, take any $A \in J$ and $v \in V$. The set $A' = \{a \in A\colon r(a) \leqslant r(v)\}$ has cardinality $< \varkappa$ and hence by uniformity of J, $v + A' \in J$. Since for any $a \in A \setminus A'$ we have $r(a) > r(v)$, the following holds for every $y \in V$:

$$\{r(x)\colon x \in v + (A \setminus A') \ \&\ o(x) = y\} = \{r(x)\colon x \in A \setminus A' \ \&\ o(x) = -v + y\}.$$

Hence, in view of $A \setminus A' \in J$, we get $v + (A \setminus A') \in J$ and finally $v + A \in J$, which proves invariance.

It remains to show that J is λ-saturated. Assume the contrary and let $\{A_\xi\colon \xi < \lambda\}$ be a family of pairwise J-almost disjoint sets outside of J. For every $\xi < \lambda$ there exists $y_\xi \in V$ such that $B_\xi = \{r(x)\colon x \in A_\xi \ \&\ o(x) = y_\xi\} \notin I$. Let $Z \subset \lambda$ be a set of cardinality λ such that for all $\xi \in Z$, the elements y_ξ are equal. By λ-saturation of I we get that for some distinct $\xi', \xi'' \in Z$, $B_{\xi'} \cap B_{\xi''} \notin I$. This however implies that $A_{\xi'} \cap A_{\xi''} \notin J$, contradiction. Hence J is as required.

([1]) We include a simplified version of the proof of Lemma 4.2 pointed out by J. Cichoń.

LEMMA 4.3. *Let G be an uncountable abelian group of cardinality $\varkappa$ and let $\lambda > \varkappa$. If there exists an invariant $\varkappa$-complete λ-saturated ideal on a subgroup H of G then such an ideal exists on G as well.*

Proof. Let I be a $\varkappa$-complete λ-saturated ideal on H. Take any selector S of the family of cosets of H in G and define J on G as follows:

$$A \in J \quad \text{iff} \quad \forall s \in S\,[(s+A) \cap H \in I].$$

Clearly J is a $\varkappa$-complete ideal. In order to prove invariance take any $A \in J$ and $g \in G$. Fix an element $s \in S$. Clearly

$$g+s = s'+h \quad \text{for some } s' \in S \text{ and } h \in H.$$

Hence in view of the invariance of I we get:

$$(s'+A) \cap H \in I \Rightarrow h+[(s'+A) \cap H] \in I \Rightarrow (s'+h+A) \cap H \in I$$
$$\Rightarrow (s+g+A) \cap H \in I.$$

This proves $g+A \in J$ because s was chosen arbitrarily.

Next we prove λ-saturation of J. Assume the contrary and let C_α: $\alpha < \lambda$ be a family of J-almost disjoint sets. By definition of J for every α there exists $s \in S$ such that $C_\alpha \cap (s+H) \notin J$. Since $|S| \leqslant \varkappa$ we get λ J-almost disjoint subsets of a fixed coset. This yields such a family of subsets of H which contradicts λ-saturation of I.

LEMMA 4.4. *Theorem 4.1 is true for the case of abelian torsion groups.*

Proof. Let G be an abelian torsion group of cardinality $\varkappa$. Then, by regularity of $\varkappa$ there exists a prime p such that the subgroup H of G consisting of elements of order p has cardinality $\varkappa$. H is the additive group of a linear space over the field F_p and hence the theorem is true for H by Lemma 4.2. Finally we use Lemma 4.3.

Now we are ready to finish the proof of Theorem 4.1. Let G be any abelian group of cardinality $\varkappa$ and H the torsion subgroup of G. If $|H| = \varkappa$ then our theorem is true for H by Lemma 4.4 and hence for G by Lemma 4.3. If $|H| < \varkappa$ then G has a free abelian subgroup of cardinality $\varkappa$ and hence we are done by Lemmas 4.2 and 4.3. ∎

THEOREM 4.5. *Let $\varkappa$ be an infinite successor cardinal and G any group of cardinality $\varkappa$. If for some $\lambda > \varkappa$ there exists a $\varkappa$-complete λ-saturated ideal on $\varkappa$, there also exists a $\varkappa$-compltete λ-saturated invariant ideal on G.*

Proof. Let $\varkappa = \varrho^+$ and $\{g_\alpha\colon \alpha < \varkappa\}$ be an enumeration of G. For any $\alpha < \varkappa$ let G_α denote the subgroup of G generated by the family $\{g_\beta\colon \beta \leqslant \alpha\}$ and $Q_\alpha = G_\alpha \setminus \bigcup_{\beta<\alpha} G_\alpha$. Similarly as in the proof of Lemma 3.12 we show that for any $g \in G$ and any $A \subset \varkappa$

$$\Big|g\big(\bigcup_{\alpha\in A} Q_\alpha\big) \triangle \bigcup_{\alpha\in A} Q_\alpha\Big| < \varkappa.$$

Let I be any $\varkappa$-complete λ-saturated ideal on $\varkappa$. We define an ideal J on G by putting for any $B \subset G$

$$B \in J \quad \text{iff} \quad B \subset \bigcup_{\alpha \in A} Q_\alpha \text{ for some } A \in I.$$

In view of $\varkappa$-completeness of I, the ideal J is $\varkappa$-complete and invariant. We check its λ-saturation. Clearly $|Q_\alpha| \leqslant \varrho$ for any α and $|Q_\alpha| = \varrho$ for $\alpha > \varrho$. Let for every $\alpha > \varrho$, $\{q_\alpha^\beta\colon \beta < \varrho\}$ be an enumeration of Q_α and define for $\beta < \varrho$:

$$S_\beta = \{q_\alpha^\beta\colon \varrho < \alpha < \varkappa\}.$$

Suppose that J is not λ-saturated and let $C_\xi\colon \xi < \lambda$ be J-almost disjoint sets. Hence the sets $D_\xi\colon \xi < \lambda$ defined by the formula

$$D_\xi = C_\xi \setminus \bigcup_{\alpha \leqslant \varrho} Q_\alpha$$

are also J-almost disjoint (because $\bigcup_{\alpha \leqslant \varrho} Q_\alpha \in J$). By $\varkappa$-completeness of J for any $\xi < \lambda$ there is $\beta < \varrho$ such that

$$D_\xi \cap S_\beta \notin J.$$

Hence for a subfamily $\mathscr{D} \subset \{D_\xi\colon \xi < \lambda\}$ of cardinality λ this index β must be the same. Call it β_0 and define for any $D \in \mathscr{D}$:

$$\varphi(D) = \{\alpha < \varkappa\colon q_\alpha^{\beta_0} \in D\}.$$

It is easy to see that the family $\{\varphi(D)\colon D \in \mathscr{D}\}$ is I-almost disjoint which contradicts λ-saturation of I. This finishes the proof. ∎

The following unpublished result of J. Cichoń (1) shows that in both previous theorems it is necessary to assume that the degree of saturation of the ideal exceeds its degree of completeness.

THEOREM 4.6 (Cichoń [4]). *Let λ be an uncountable regular cardinal. There do not exist λ-complete λ-saturated invariant ideals on any group G.*

Proof. We may of course assume that $\lambda \leqslant |G|$. Suppose I is a λ-complete λ-saturated invariant ideal on G and let J be a λ-complete Ulam ideal containing I (see Theorem 1.13). Take a maximal family $\mathscr{A}$ of I-almost disjoint sets contained in $J \setminus I$. Then $|\mathscr{A}| < \lambda$ and hence $A = \bigcup \mathscr{A} \in J$. Let $B = G \setminus A$. Clearly $B \notin I$ (otherwise $B \in J$ and $A \in J$, hence $G \in J$) and

$$J = \{X \subset G\colon X \cap B \in I\}.$$

Fix any element $a \in G$ different from the neutral element and define the following equivalence relation $\sim$ on the set B:

$$x \sim y \text{ iff } \exists n \in \omega\ [(a^n x = y \text{ and } a^i x \in B \text{ for every } 0 \leqslant i \leqslant n)$$
$$\text{or } (a^n y = x \text{ and } a^i y \in B \text{ for every } 0 \leqslant i \leqslant n)].$$

(1) We are grateful to J. Cichoń for his kind permission to publish this theorem.

Let

$$B_0 = \{x \in B\colon |[x]_\sim| \geqslant 2\}$$

and

$$P = \{[x]_\sim\colon x \in B_0\} \cup \{G \setminus B_0\}.$$

Hence P is a partition of G into sets of cardinalities at least 2. Since J is an Ulam ideal, there exists a selector S of the family $\{[x]_\sim\colon x \in B_0\}$ belonging to J. On the other hand $S \subset B$, hence $S = S \cap B \in I$. Moreover we have

$$B \cap aB \subset \bigcup_{z \in Z} a^z S,$$

hence $B \cap aB \in I$. Since the element a was chosen arbitrarily we get $gB \cap hB \in I$ for every pair of distinct elements $h, g \in G$. For any $g \in G$ we have $gB \notin I$ because $B \notin I$. Those two facts contradict λ-saturation of I, which finishes the proof. ∎

In the case of solvable groups a much stronger negative result is valid: for all σ-complete invariant ideals the degree of saturation must exceed not only the degree of completeness, but the cardinality of the underlying group.

THEOREM 4.7. *Let G be a solvable group and I a σ-complete invariant ideal on G. Then I is not $|G|$-saturated.*

Proof. We split the proof into a sequence of lemmas.

LEMMA 4.8. *Theorem 4.7 is true for the additive group of any linear space over a countable field K.*

The proof is an easy modification of the argument used in case 1 of Theorem 3.3; hence we omit it.

LEMMA 4.9. *Theorem 4.7 is true for torsion-free abelian groups.*

Here the proof is similar to that of case 2 in Theorem 3.3 and hence we also leave it to the reader.

LEMMA 4.10. *Let G be any group and H a normal subgroup of G. Suppose that Theorem 4.7 is true for the groups H and G/H. Then the theorem holds for the group G as well.*

Proof. Let $|G| = \varkappa$ and I be a σ-complete invariant ideal on G. We consider two cases:

Case 1. $|G/H| = \varkappa$.

If $H \notin I$ then by invariance of I $aH \notin I$ for every left coset from G/H and in view of the assumption $|G/H| = \varkappa$ we get that I is not $\varkappa$-saturated. If $H \in I$ then we construct a σ-complete ideal I_1 on G/H by putting $\{aH\colon a \in A\} \in I_1$ iff $AH \in I$. Since $H \in I$, all singletons belong to I_1. The ideal I_1 is invariant by the invariance of I. Since the theorem is true for G/H, the ideal I_1 is not $\varkappa$-saturated. Let $\{\{aH\colon a \in A_\alpha\}, \alpha < \varkappa\}$ be a family of I_1-almost disjoint sets. Then $A_\alpha H\colon \alpha < \varkappa$ are I-almost disjoint sets in G.

Case 2. $|G/H| < \varkappa$.

In this case $|H| = \varkappa$. Take any selector S of the family of right cosets and consider the set $\mathscr{S} = \{hS\colon h \in H\}$ of its translates. Clearly $|\mathscr{S}| = \varkappa$.

If $S \notin I$, I is not $\varkappa$-saturated because the elements of $\mathscr{S}$ form a large family of pairwise disjoint sets outside of I.

If $S \in I$, then we define a σ-complete ideal I_2 on H with the formula: $A \in I_2$ iff $AS \in I$, for $A \subset H$. Again singletons belong to I_2 because $S \in I$ and I_2 is invariant because I was such. A large I_2-almost disjoint family of subsets of H yields such an I-almost disjoint family of subsets of G, which proves the lemma.

LEMMA 4.11. *Let n be a natural number and G be an abelian group all of whose elements have orders dividing n. Then Theorem* 4.7 *is true for the group G.*

Proof. We proceed by induction on the number k of prime divisors of n. (We count multiple divisors many times.) If $k = 1$, all elements in G have the same prime order n. Then G is a linear space over the field F_n and hence by Lemma 4.8 the theorem is true for G.

Assume that the lemma holds for k. Let n have $k+1$ prime divisors $p_1, \ldots, p_{k+1}$. Denote by H the subgroup of G consisting of all elements of order dividing $p_1 \ldots p_k$. By the inductive hypothesis the theorem is true for H. Let $G' = G/H$. Hence every element of G has order p_{k+1} and G satisfies the theorem as well. Now our lemma follows immediately from Lemma 4.10.

LEMMA 4.12. *Let G be an abelian torsion group. Then Theorem* 4.7 *is true for G.*

Proof. Let I be a σ-complete invariant ideal on G. G is the union of groups G_n, where G_n consists of elements whose orders divide n. Let n_0 be the least natural number for which $G_{n_0} \notin I$.

If there exists a natural number k for which $|G_k| = |G|$ then clearly $|G_{n_0 k}| = |G|$ and $G_{n_0 k} \notin I$. Then by Lemma 4.11 we get a family of cardinality $|G|$ of I-almost disjoint subsets of $G_{n_0 k}$ outside of I. This is the required family violating $|G|$-saturation of I on G.

If for every natural k, $|G_k| < |G|$ then there exists an increasing sequence $G_{n_0} \subset G_{n_1} \subset G_{n_2} \subset \ldots$ of groups, such that the cardinalities $\varkappa_i = |G_{n_i}|$ form a strictly increasing sequence with supremum $|G|$. Since for every i, $G_{n_i} \notin I$, all cosets belonging to groups $G_{n_{i+1}}/G_{n_i}$ are outside of I by invariance. Also $|G_{n_{i+1}}/G_{n_i}| = \varkappa_{i+1}$. Hence the set $\bigcup_{i \in \omega} (G_{n_{i+1}}/G_{n_i} \setminus G_{n_i})$ is a pairwise disjoint family of cardinality $|G|$ outside of I, which again violates $|G|$-saturation of this ideal.

Now we are ready to conclude the proof of Theorem 4.7. Let G be any solvable group and I and σ-complete invariant ideal on G. Let

$G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ be a sequence of normal subgroups such that the groups G_0 and G_{i+1}/G_i (for any $i < n$) are abelian. We use induction on n.

If $n = 0$ then G is an abelian group. Denote by H its torsion subgroup. Then G/H is a torsion-free abelian group and hence the theorem is true for H (Lemma 4.12) and for G/H (Lemma 4.9). By Lemma 4.10 we get the theorem for G.

Assume we are done for towers of normal subgroups of length k and let this tower for $G = G_{k+1}$ have length $k+1$. Then our theorem is true for G_k by the inductive hypothesis and for G/G_k because this group is abelian (see the argument above). Hence Theorem 4.7 is true for G by Lemma 4.10. This finishes the proof. ∎

For $|G|$-complete ideals on an uncountable abelian group G Theorem 4.7 can be strengthened by showing that not only such ideals fail to be invariant whenever they are $|G|$-saturated but in fact that their non-invariance is then particularly strong. In order to express it we need the following natural definitions.

DEFINITION 4.13. Let I be an ideal on a group G and $g \in G$. Then the ideal

$$I_g = \{gA\colon A \in I\}$$

is called a *translate* of I.

DEFINITION 4.14. Ideals I and J on X are called *incompatible* iff there is no ideal containing both of them (or equivalently iff $I \cap J^* \neq \emptyset$).

Using Definition 4.13, Theorem 4.7 can be reformulated as follows: every σ-complete $|G|$-saturated ideal on a solvable group G is different from one of its translates. This should be compared to the following

THEOREM 4.15. *Let G be an uncountable abelian group of cardinality $\varkappa$. Then every $\varkappa$-complete $\varkappa$-saturated ideal on G is incompatible with one of its translates.*

Proof. Without loss of generality we may assume that $\varkappa$ is regular.

LEMMA 4.16. *Let V be linear space of cardinality $\varkappa$ over a countable field K. Let $\mathscr{B} = \{v_\alpha\colon \alpha < \varkappa\}$ be its linear basis and I any $\varkappa$-complete $\varkappa$-saturated ideal on V. There exists $\beta < \varkappa$ such that for every $\alpha > \beta$ there exists a set $A \in I$ for which $-v_\alpha + A \in I^*$.*

Let V_n denote the set of those elements in V which have n summands in basis $\mathscr{B}$ representation. For every $n \geqslant 1$, consider the sets

$$V_n^\xi = \{v \in V_n\colon \exists k \in k\ \exists v' \in V_{n-1}\,[v = kv_\xi + v']\}.$$

We claim that for every $n \geqslant 1$ only less than $\varkappa$ sets V_n^ξ can be outside of I. Suppose not and let n be the first integer such that for $\varkappa$ indices ξ, $V_n^\xi \notin I$. Hence by $\varkappa$-saturation of I, for $\varkappa$ pairs (ξ_1, ξ_2) the sets $V_n^{\xi_1} \cap V_n^{\xi_2}$ are outside

of I. However

$$V_n^{\xi_1} \cap V_n^{\xi_2} = \{v \in V_n\colon \exists k_1, k_2 \in K\ \exists v' \in V_{n-2}[v = k_1 v_{\xi_1} + k_2 v_{\xi_2} + v']\}.$$

Next we use $\varkappa$-saturation of I to show that for $\varkappa$ pairs of pairs $((\xi_1, \xi_2), (\zeta_1, \zeta_2))$ the sets $V_n^{\xi_1} \cap V_n^{\xi_2} \cap V_n^{\zeta_1} \cap V_n^{\zeta_2}$ are outside of I. After at most n steps we will produce a set $V_n^{\xi_1} \cap \ldots \cap V_n^{\xi_n}$ outside of I. However

$$V_n^{\xi_1} \cap \ldots \cap V_n^{\xi_n} = \{v \in V_n\colon \exists k_1, \ldots, k_n \in K\,[v = k_1 v_{\xi_1} + \ldots + k_n v_{\xi_n}]\},$$

hence this set is countable and therefore must be in I.
This contradiction proves our claim and hence there exists an ordinal $\beta < \varkappa$ such that for every $n \geqslant 1$ and every $\alpha > \beta$ the set V_n^α belongs to I. Fix $\alpha > \beta$.

For every $n \geqslant 1$ let W_n^α be the set of those elements in V_n in whose representation the vector v_α doesn't appear. Consider the set $A = \bigcup_{n \geqslant 1} v_\alpha + W_n^\alpha$. $A \in I$ since $A \subset \bigcup_{n \geqslant 1} V_n^\alpha$. However $-v_\alpha + A = \bigcup_{n \geqslant 1} W_n^\alpha = V \setminus \bigcup_{n \geqslant 1} V_n^\alpha$, hence $-v_\alpha + A \in I^*$. This proves the lemma.

Let G be an arbitrary abelian group of cardinality $\varkappa$ and J a $\varkappa$-complete $\varkappa$-saturated ideal on G. Denote by H the torsion subgroup of G and by H_n (for $n \geqslant 1$) the subgroup consisting of those elements of H whose orders divide n.

Case 1. $|G/H| = \varkappa$.

Since G/H is a torsion-free abelian group of cardinality $\varkappa$, it can be homomorphically embedded into the additive group of a linear space V over Q in such a way that a basis $\mathscr{B} = \{v_\alpha\colon \alpha < \varkappa\}$ of V is contained in G/H. (cf. the proof of Theorem 3.3, Case 2).

Given any $\varkappa$-complete $\varkappa$-saturated ideal I on G/H we define an ideal I_1 on V by putting:

$$A \in I_1 \quad \text{iff} \quad A \cap G/H \in I.$$

This is also a $\varkappa$-complete $\varkappa$-saturated ideal. It follows from Lemma 4.16 that there exists an ordinal $\beta < \varkappa$ such that for $\alpha > \beta$ there is a set $A \in I_1$ for which $-v_\alpha + A \in I_1^*$. Taking $A' = A \cap G/H$ we have $A' \in I$ and $-v_\alpha + A' \in I^*$.

Consider now the ideal J on G. By $\varkappa$-saturation only $< \varkappa$ cosets from G/H are outside of J. Let $\mathscr{F}$ denote the family of these cosets. If $\bigcup(G/H \setminus \mathscr{F}) \in J$ then taking any coset g outside of the group generated by $\mathscr{F}$ we get $\bigcup_{f \in \mathscr{F}} g + f \in J$ (by $\varkappa$-completeness) and $\bigcup \mathscr{F} \in J^*$, hence we are done.

If $\bigcup(G/H \setminus \mathscr{F}) \notin J$, define the following ideal I on G/H:

$$\{c + H\colon c \in C\} \in I \quad \text{iff} \quad [(C + H) \setminus \bigcup \mathscr{F}] \in J.$$

I is a $\varkappa$-complete $\varkappa$-saturated ideal and hence for $\alpha > \beta$ there exists a set $A' \in I$ for which $-v_\alpha + A' \in I^*$.

We may assume that A' and $-v_\alpha + A'$ are disjoint from $\mathscr{F}$. Take any $\alpha > \beta$ such that v_α doesn't belong to the group generated by $\mathscr{F}$. Then denoting $B = \bigcup A'$ we get:

$$(-v_\alpha + B) \cup \bigcup \mathscr{F} \in J^* \quad \text{and} \quad v_\alpha + [(-v_\alpha + B) \cup \bigcup \mathscr{F}] \in J.$$

The latter follows from the fact that the set $v_\alpha + \bigcup \mathscr{F}$ is a union of less than $\varkappa$ cosets outside of $\mathscr{F}$ and hence it belongs to J by $\varkappa$-completeness.

Case 2. $|G/H| < \varkappa$.

In this case $|H| = \varkappa$ and hence by regularity of $\varkappa$, $|H_p| = \varkappa$ for some prime number p. Since H_p is a linear space over the field F_p, Lemma 4.16 can be applied. Let S be any selector of the family G/H_p and denote by $\mathscr{S}$ the family $\{h+S\colon h \in H_p\}$. Let M denote the subset of H_p consisting of those h for which $h+S \notin J$. By $\varkappa$-saturation of J, $|M| < \varkappa$. Now we argue similarly as in Case 1. If $(H_p \setminus M)+S \in J$ then taking any $\bar{h} \in H_p$ outside of the group generated by M we get $M+S \in J^*$ and $\bar{h}+M+S \in J$. If $(H_p \setminus M)+S \notin J$, we consider the ideal I on H_p defined by the formula:

$$A \in I \quad \text{iff} \quad (A \setminus M)+S \in J.$$

I is $\varkappa$-complete and $\varkappa$-saturated.

If $\mathscr{H} = \{h_\alpha\colon \alpha < \varkappa\}$ is a basis of H_p over F_p then by Lemma 4.16, for some $\beta < \varkappa$ and every $\alpha > \beta$ there is a set $D \in I$ such that $-h_\alpha + D \in I^*$. We may assume that D and $-h_\alpha + D$ are disjoint from M. Taking $\alpha > \beta$ such that h_α is outside of the group generated by M we get

$$(M \cup (-h_\alpha + D))+S \in J^* \quad \text{and} \quad h_\alpha + (M \cup (-h_\alpha + D))+S \in J.$$

This finishes the proof of our theorem. ∎

The results of this chapter and in particular Theorem 4.7 show that σ-complete invariant ideals cannot enjoy strong saturation properties and hence they can be considered as small from this point of view.

A different approach to the problem of size of invariant ideals on groups is due to Węglorz. He exploited the $\mathrm{U}(\delta)$ scale of ideals (see Definition 1.12). The following result of his shows that invariant ideals on groups must be small from this point of view as well.

THEOREM 4.17 (Węglorz [39]). *Every λ-complete invariant ideal on a group G has property $\mathrm{U}(\delta)$ for any $1 < \delta < \lambda$. In particular every invariant ideal on a group has property $\mathrm{U}(n)$ for any natural $n > 1$.*

A related result concerning saturation of finitely additive invariant ideals will be proved at the end of this chapter.

The fact that σ-complete invariant ideals usually cannot be strongly saturated inspired the search for a property weaker than invariance and not

refuting small degrees of saturation. Instead of requiring invariance with respect to all shifts one may investigate ideals invariant with respect to almost all of them.

DEFINITION 4.18. An ideal I on a group G is *idempotent* iff $\{g \in G\colon Ag^{-1} \notin I\} \in I$ for every $A \in I$. A filter is idempotent iff its dual ideal is idempotent.

Let us remark that considering the shift Ag^{-1} instead of gA is due to tradition coming from analysis. The two-valued measure associated with a σ-complete idempotent ideal is idempotent with respect to convolution. Idempotent measures on groups were investigated in connection with Haar measures (cf. e.g. Parthasarathy [24]). This justifies the interest in idempotent ideals as well. Another motivation comes from combinatories where idempotent ultrafilters turned out to be an important tool. Glazer (cf. Hindman [16]) proved the existence of idempotent ultrafilters on the integers (with addition or multiplication) and his argument works for arbitrary groups. In Pelc [29] countably complete idempotent ideals on abelian groups were investigated. The following results were proved:

THEOREM 4.19 (Pelc [29]). *Let G be any abelian group. There does not exist a σ-complete idempotent ultrafilter on G.*

THEOREM 4.20 (Pelc [29]). *Let G be an abelian group of cardinality $\varkappa$ and assume that $\varkappa$ carries a $\varkappa$-complete λ-saturated ideal for some uncountable regular cardinal $\lambda < \varkappa$. Then there exists a $\varkappa$-complete λ-saturated idempotent ideal on G.*

It is easy to see that both Theorems 4.19 and 4.20 remain valid for solvable groups. These results (compared to Theorem 4.7) show that by weakening invariance to idempotency we may gain an important amount of saturation.

Let us close this chapter with some remarks on invariant ideals which are not σ-complete. Similarly as for invariant quasi-measures the situation turns out to be much different from the σ-complete setting. As it was mentioned at the end of chapter 2, universal invariant probability quasi-measures do exist on every amenable group. The ideal of null sets of such a quasi-measure is an invariant σ-saturated ideal and hence such ideals exist e.g. on every solvable group.

The next proposition shows that invariant σ-saturated ideals can also exist on some groups which are not amenable. As for universal invariant σ-finite quasi-measures we construct such ideals on countable free groups closely following the idea from Theorem 2.7. Notice that the ideal of null sets of a universal σ-finite quasi-measure need not be σ-saturated (contrary to the σ-complete setting) hence our present proposition is not a straightforward consequence of Theorem 2.7 though a similar idea works in both arguments.

PROPOSITION 4.21. *Let G be any countable free group. There exists a σ-saturated invariant ideal on G.*

Proof. Let I be any σ-saturated invariant ideal on the additive group of integers and let S be the free basis of G. Let $r(x)$, $a(x)$ and $O(x)$ have the same meaning as in the proof of Theorem 2.7.

We define an ideal J on G as follows:

$$A \in J \quad \text{iff} \quad \forall s \in S\ \forall g \in G\,\{r(x)\colon\ x \in A\ \&\ a(x) = s\ \&\ O(x) = g\} \in I.$$

In order to show that J is σ-saturated suppose that $\{A_\xi\colon\ \xi < \omega_1\}$ is a J-almost disjoint family of subsets of G. For every $\xi < \omega_1$ there exist $g_\xi \in G$ and $s_\xi \in S$ such that

$$B_\xi = \{r(x)\colon\ x \in A_\xi\ \&\ O(x) = g_\xi\ \&\ a(x) = s_\xi\} \notin I.$$

Let $T \subset \omega_1$ be such an uncountable set that for $\xi \in T$ all elements g_ξ are equal and all elements s_ξ are equal. By σ-saturation of I we get that for some distinct $\xi', \xi'' \in T$, $B_{\xi'} \cap B_{\xi''} \notin I$. This however implies that $A_{\xi'} \cap A_{\xi''} \notin J$, contradiction.

The proof that J is invariant is similar to the argument used to show the invariance of the quasi-measure constructed in Theorem 2.7, hence we omit it. ∎

Ryll-Nardzewski and Telgarsky [32] pointed out that there is no prime invariant ideal on any group (cf. the above mentioned result of Glazer on idempotent ultrafilters). A strengthening of this remark follows immediately from Theorem 4.17: invariant ideals on a group cannot be n-saturated for any natural n. Our last proposition gives an even stronger statement, this time the best possible in this direction since ω_1-saturated invariant ideals do exist on many groups.

PROPOSITION 4.22. *There do not exist ω-saturated invariant ideals on groups.*

Proof. Suppose I is an ω-saturated invariant ideal on a group G. Hence G is infinite. Let J be a prime ideal extending I and let A, B, a, $\sim$ and B_0 have the same meaning as in the proof of Theorem 4.6. Take any selector S of the family $\{[x]_\sim\colon\ x \in B_0\}$ and let $X = \bigcup_{k \in Z} a^{2k} S$, $Y = \bigcup_{k \in Z} a^{2k+1} S$. Clearly $aX = Y$ and X, Y are disjoint. Hence one of these sets is an element of J. Without loss of generality let $X \in J$. Hence $X \cap B \in I$ and also $Y \cap aB = aX \cap aB = a(X \cap B) \in I$ in view of the invariance of I. It follows that

$$B \cap aB = B \cap aB \cap \bigcup_{z \in Z} a^z S = B \cap aB \cap (X \cup Y) \subset (B \cap X) \cup (aB \cap Y) \in I.$$

This implies $gB \cap hB \in I$ for distinct $g, h \in G$ which contradicts ω-saturation of I because $gB \notin I$ for any $g \in G$. ∎

The above remarks show that invariant ideals admit much more saturation in the finitely additive case than in the σ-complete one. Since σ-saturated invariant ideals exist on a large class of groups –strictly larger than

that of amenable groups – and ω-saturated invariant ideals are excluded by Proposition 4.22, it would be interesting to find an exact characterization of groups carrying σ-saturated invariant ideals. Similarly as for the question of carrying a universal invariant σ-finite quasi-measure we do not even know if any infinite group fails to have this property.

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