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DISSERTATIONES MATHEMATICAE

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Noncharacteristic mixed problems for hyperbolic systems of the first order

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CONTENTS

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1. Introduction

This paper concerns the existence and uniqueness of solution of the following mixed problem for a hyperbolic system:

$$(1.1) \quad E(v)u_t - \sum_{j=1}^{m} A_j(v)u_{x_j} - C(v)u = F \quad \text{in } \Omega^T \equiv [0,T] \times \Omega,$$

$$(1.2) \quad\quad\quad [I, -S(v)]u = g \quad\quad\quad \text{on } \partial\Omega^T \equiv [0,T] \times \partial\Omega,$$

$$(1.3) \quad\quad\quad u(0,\cdot) = f(\cdot) \quad\quad\quad \text{in } \Omega,$$

where $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$ or Ω is a bounded domain in $\mathbf{R}^m$ with boundary $\partial\Omega$ of class C^∞; $T > 0$; $G(v) = G(t,x,v)$ for $G = E, A_j (j = 1,\ldots,m), C, S$; v is either a given n-vector function of class $\Pi_0^k(\Omega^T) \cap H^k(\partial\Omega^T)$ for $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$ ($\Pi_0^k(\Omega^T)$ denotes a certain Sobolev space defined in Chapter 2, $H^k(\partial\Omega^T)$ is the well-known Sobolev space) or $v = u$ (the nonlinear case); $E, A_j (j = 1,\ldots,m), C$ are $n \times n$-matrix-valued functions; S is an $l \times (n-l)$-matrix-valued function (where $1 \leq l \leq n$); F and f are n-vector-valued functions; g is an l-vector-valued function. All the above functions are real. Moreover, $E, A_j (j = 1,\ldots,m), C, S \in C^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$ and are constant matrices outside a compact set $K_{t,x,v} \subset \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$.

We assume that the above problem is noncharacteristic, i.e.

$$(1.4) \quad\quad\quad \forall(t,x,v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n \quad \det A_\nu(t,x,v) \neq 0,$$

where $A_\nu = -\sum_{j=1}^{m} A_j \cdot \nu_j$ and $\nu = (\nu_1,\ldots,\nu_m)$ is the unit outward vector normal to the boundary $\partial\Omega$.

We first examine the existence and uniqueness of a solution of (1.1)–(1.3) in the linear case (Theorems 4.1.2, 4.1.3, 4.2.1, 4.2.2). We consider separately the cases of $f \neq 0$ and $f = 0$. When $f \neq 0$ we assume the strict hyperbolicity of system (1.1), while in the case of the zero initial condition both the structure and regularity assumptions we impose on the system are weaker. The strict hyperbolicity is replaced by the condition that the matrix

$$A(t,x,v,\omega) = iE^{-1}(t,x,v) \sum_{j=1}^{m} A_j(t,x,v)\omega_j$$

has simple structure on $\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n \times \mathbf{R}^m \setminus \{0\}$ (see Definition 2.3.1) and the matrix

$$M_\nu(t, x, v, s, i\omega') = A_\nu^{-1}(t, x, v)\Big(E(t, x, v)s - i\sum_{j=1}^{m} A_j(t, x, v)\omega_j\Big)$$

(where $s = \eta + i\xi$, $\eta > 0$, $\xi \in \mathbf{R}$, $\omega' = (\omega_2, \ldots, \omega_m)$) is of a certain special form (see assumption (3.1.24)). Moreover, for $f = 0$ it is sufficient to assume that $k > m/2 + 5/2$ and $v \in H^k(\partial \Omega^T)$ (see Theorem 4.2.2).

In the last chapter we apply the results proved for the linear problem to obtain the existence and uniqueness of a classical local-in-time solution of the nonlinear problem (1.1)–(1.3) (see Theorem 6.1 and Remark 6.1).

The noncharacteristic linear problem (1.1)–(1.3) was previously examined only in the case of smooth coefficients E, A_j, C, S independent of v (see [13], [21]–[23]) and in the case when the coefficients were the sums of smooth functions and sufficiently small perturbations of class H_η^k (see the definition in Chapter 2), where $k > [(m+1)/2] + 3$ (see [16]). More exactly, Li Dening [16] considers the problem

$$(1.5) \quad u_t - \sum_{j=1}^{m}(A_j(t, x, 0) + a_j(t, x))u_{x_j}$$

$$-(C(t, x, 0) + c(t, x))u = F \quad \text{for } t > 0, x_1 > 0,$$

$$(1.6) \qquad\qquad (P(t, x, 0) + p(t, x))u|_{x_1 = 0} = g \quad \text{for } t > 0,$$

$$(1.7) \qquad\qquad\qquad u|_{t=0} = 0 \quad \text{for } x_1 > 0,$$

where $P(t, x, v)$ is an $l \times n$-matrix-valued function; $A_j (j = 1, \ldots, m), C, P$ are smooth matrix-valued functions which are constant for $|t| + |x| \geq 2R_0$ ($R_0 > 0$); $a_j (j = 1, \ldots, m), c$ and p are functions of class H_η^k satisfying

$$\sum_{j=1}^{m} \|a_j\|_{k,\eta,\mathbf{R}_+ \times \Omega} + \|c\|_{k,\eta,\mathbf{R}_+ \times \Omega} + \|p\|_{k,\eta,\mathbf{R}_+ \times \Omega} \leq \varepsilon \quad \text{for some } \varepsilon > 0$$

($\| \cdot \|_{k,\eta,\mathbf{R}_+ \times \Omega}$ denotes the norm in $H_\eta^k(\mathbf{R}_+ \times \Omega)$). He proves the existence and uniqueness of a solution $u \in H^k$ under the assumption that problem (1.5)–(1.7) with $a_j = c = p = 0$ (the case of smooth coefficients) is well posed in the sense of Kreiss (see [13]).

The nonlinear noncharacteristic system (1.1) was already considered in [11], [16], [18], [22], [25], [26]–[32].

T. Kato [11] and A. Majda [18] (Chapter 2) discuss the Cauchy problem for this system in the case of symmetric equations. Also, one section of M. Taylor's book [25] (Section 5 of Chapter IV) is devoted to the Cauchy problem, however, the author does not assume the symmetry of the system.

The papers of W. M. Zajączkowski [28]–[32], J. Rauch and F. Massey

[22] and Li Dening [16] concern nonlinear mixed problems for system (1.1) (the case when $v = u$). In [28] and [29] W. M. Zajączkowski considers a noncharacteristic mixed problem for a symmetric hyperbolic system (1.1). He proves the existence and uniqueness of a local-in-time solution of a mixed problem for system (1.1) assuming that the norms $\| \cdot \|_{k,\eta,V}$ (where $V = \Omega^t, \partial\Omega^t, \Omega$) of $F, A_j f_{x_j}, (j = 1, \ldots, m), Cf, g$ and the norms $| \cdot |_{k-1,0,\Omega}$ (see the definition in Chapter 2) of the functions F, $A_j f_{x_j}$, Cf are sufficiently small (for $k > m/2 + 1$).

In [30]–[32] W. M. Zajączkowski proves the existence and uniqueness of a global-in-time solution of the same problem as in [28] and [29], but under some additional assumption imposed on C.

In [22] J. Rauch and F. Massey examine the differentiability of a solution of a mixed problem for a linear hyperbolic system and then apply the obtained results to the proof of the existence of a local-in-time smooth solution to the nonlinear problem (1.1)–(1.3). They only consider the case when $g = 0$, F and f are smooth and satisfy certain compatibility conditions, and the coefficients in (1.1)–(1.3) are of class C^∞.

Finally, Li Dening [16] considers the nonlinear system (1.1) (where $E = I$) together with the conditions

(1.8) $P(u)u|_{\partial\Omega} = g, \quad \text{where } P(u) = P(t, x, u),$

and (1.7).

He proves the existence and uniqueness of a local-in-time solution of class H^k of problem (1.1), (1.8), (1.7) under the assumption that $F \in H^k(\mathbf{R}_+ \times \Omega)$, $g \in H^k(\mathbf{R}_+ \times \partial\Omega)$ (where $k \geq [(m+1)/2] + 3$ and Ω is a bounded domain in $\mathbf{R}^m$), $\forall 0 \leq j \leq k-1$, $\partial_t^j F|_{t=0} = \partial_t^j g|_{t=0} = 0$, and $A_j (j = 1, \ldots, m), C, P$ are smooth functions which are constant matrices for $t + |x| \geq 2R_0$ ($R_0 > 0$). Li Dening obtains the above result using his theorem for the linear problem (1.5)–(1.7).

In contrast to [28]–[32] we do not assume the symmetry of system (1.1), while unlike [22] and [16], we assume that $E, A_j, C, S \in C^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$ (where $k > m/2 + 5/2$) and we consider general g and f (not only zero).

The main results of this paper are presented in a simplified form in [26] and [27].

The paper consists of six chapters. Chapter 2 contains the definitions of the function spaces used in the paper and some relevant lemmas (Section 2.1), details on perturbations of matrix-valued functions of class C_B^k (Section 2.2) and some properties of matrices (Section 2.3).

In Chapter 3 the boundary value problem

(1.9) $E(v)u_t - \sum_{j=1}^{m} A_j(v)u_{x_j} - C(v)u = F \quad \text{in } \mathbf{R} \times \Omega,$

(1.10) $[I, -S(v)]u = g$ on $\mathbf{R} \times \partial\Omega$

(where $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$, $v \in \Pi_0^k(\mathbf{R} \times \Omega)$, $k > m/2 + 5/2$) is considered. Section 3.1 contains the conditions which we impose on problem (1.1)–(1.3). Later in this section we transform system (1.9) into a system of pseudodifferential operators which is more convenient for further investigations. So (1.9) turns into

(1.11) $e^{-\eta t}u_{x_1} = M(v, iD_t + \eta, iD_{x'})e^{-\eta t}u - A_1^{-1}(v)C(v)e^{-\eta t}u$
          $\qquad - A_1^{-1}(v)Fe^{-\eta t}$,

where $\eta > 0$, $M(v, iD_t + \eta, iD_{x'})\,(x' = (x_2, \ldots, x_m))$ is the pseudodifferential operator with symbol

(1.12) $$M(v, s, i\omega') = A_1^{-1}(v)\Big(E(v)s - i\sum_{j=2}^{m} A_j(v)\omega_j\Big)$$

($s = \eta + i\xi$, $\omega' = (\omega_2, \ldots, \omega_m)$; ξ and ω_i ($i = 2, \ldots, m$) are the dual variables to t and x_i). Lemma 3.1.1 of Section 3.1, concerning the algebraic properties of the matrix $M(t, x, v, s, i\omega')$ for $(t, x, v, s, \omega') \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times \mathbf{C} \times \mathbf{R}^{m-1}$ with $\operatorname{Re} s > 0$, is analogous to [8, Lemma 1], where a noncharacteristic mixed problem with constant coefficients is considered.

   Section 3.2 is devoted to the construction of a so-called symmetrizer for problem (1.9)–(1.10), i.e. a pseudodifferential operator which we apply in Section 3.3 to get an estimate of a solution $u \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$ (see the definition in Chapter 2) of (1.9)–(1.10) in L_η^2-norm. The idea of the construction of a matrix-valued function $R(t, x, v, s, \omega')$ of class C_B^k such that the function $\boldsymbol{R}(t, x, s, \omega') = R(t, x, v(t, x), s, \omega')$ is the symbol of the symmetrizer comes from the paper of Kreiss [13], concerning a noncharacteristic mixed problem for a strictly hyperbolic system with constant coefficients. Lemmas 3.2.1–3.2.7, concerning certain properties of the matrix-valued function $M(t, x, v, s, i\omega')$ (of class C^k), are modifications of the appropriate lemmas of [13]. Also Theorem 3.2.1, which yields the construction of $R(t, x, v, s, \omega')$, is a suitable modification of [13, Theorem 3]. The proofs of Lemmas 3.2.1–3.2.7 and of Theorem 3.2.1 are given in more detail in comparison with analogous proofs of [13].

   In Section 3.3 we get an estimate of a smooth solution of problem (1.9)–(1.10) in L_η^2-norm under the assumption that $E, A_j, C, S \in C^k$, where $k > m/2 + 5/2$. To obtain this estimate we apply the symmetrizer which is a pseudodifferential operator with symbol of class $\Pi_0^k(\mathbf{R} \times \Omega)$. In the proofs of all lemmas in this section we use [2, Lemma 1.4]. Moreover, in the proofs of Lemmas 3.3.3 and 3.3.4 we apply some ideas of Yu. V. Egorov [4] and P. D. Lax and L. Nirenberg [14].

   In Section 3.4 we obtain an estimate of a solution of the formally adjoint

problem to problem (1.9)–(1.10) using the fact that the formally adjoint problem is of the same type as (1.9)–(1.10).

In Section 3.5 we prove the existence and uniqueness of a strong solution $u \in L^2_\eta(\mathbf{R} \times \Omega)$ of (1.9)–(1.10) (see Definition 3.5.1) under the assumption that $F \in L^2_\eta(\mathbf{R} \times \Omega)$ and $g \in L^2_\eta(\mathbf{R} \times \partial\Omega)$ (see Theorem 3.5.1) and next the existence and uniqueness of a solution $u \in C^\infty_{(0)}(\mathbf{R} \times \bar\Omega)$ of the same problem assuming that $F \in C^\infty_{(0)}(\mathbf{R} \times \bar\Omega)$, $g \in C^\infty_{(0)}(\mathbf{R} \times \partial\Omega)$ and additionally that $v \in C^\infty_{(0)}(\mathbf{R} \times \bar\Omega)$ and $E, A_j, C, S \in C^\infty(\mathbf{R} \times \bar\Omega \times \mathbf{R}^n)$ (see Theorem 3.5.2). To prove Theorems 3.5.1 and 3.5.2 we use the papers of K. O. Friedrichs and P. D. Lax [5] and D. S. Tartakoff [24]. Furthermore, for the solution $u \in C^\infty_{(0)}(\mathbf{R} \times \bar\Omega)$ of (1.9)–(1.10) we get the estimate

$$(1.13) \quad \eta \|u\|^2_{s,\eta,\mathbf{R}\times\Omega} + \|u\|^2_{s,\eta,\mathbf{R}\times\partial\Omega}$$
$$\leq \bar{f}_s(\varepsilon_1, \varepsilon_2, \varepsilon_3)(p_s(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f})\eta^{-1}\|F\|^2_{s,\eta,\mathbf{R}\times\Omega}$$
$$+ q_s(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f})\|g\|^2_{s,\eta,\mathbf{R}\times\partial\Omega}) \quad \forall 0 \leq s \leq k,$$

where $k > m/2 + 5/2$; $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ are defined by (3.3.13); $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the constants from Theorem 3.2.1; $\bar{f}_s$ are continuous functions; for $1 \leq s < k$, p_s and q_s are polynomials, while $p_k = \hat{p}_k + p_{k-1}e^{2\eta T_2^*}$, $q_k = \hat{q}_k + q_{k-1}e^{2\eta T_2^*}$, where $\hat{p}_k$ and $\hat{q}_k$ are polynomials, $T_2^* > 0$ is such that s is constant outside $[T_1^*, T_2^*]$; and $\eta > 0$ is sufficiently large.

For $s = 0$ the estimate (1.13) coincides with the result of Section 3.3. We obtain the estimates for $0 < s \leq k$ using the case $s = 0$ and the proof of [28, Lemma 3.2]. For the restriction of u to the set $J \times \Omega$ (where $J \subset \mathbf{R}$ is an interval) we obtain an inequality analogous to (1.13).

In Section 3.6 we consider the Cauchy problem

$$(1.14) \qquad E(v)u_t - \sum_{j=1}^m A_j(v)u_{x_j} - C(v)u = F \quad \text{in } [0, T] \times \Omega,$$

$$(1.15) \qquad u(0, \cdot) = 0 \qquad \text{in } \Omega,$$

where $v \in \Pi_0^k(\mathbf{R} \times \Omega)$; $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$ or Ω is a bounded domain in $\mathbf{R}^m$ with smooth boundary. We prove that if $u \in U_{[0,T]}$ is a solution of (1.14)–(1.15), then for $0 \leq t \leq T$ and $0 \leq s \leq k$ ($k > m/2 + 5/2$),

$$(1.16) \qquad\qquad \eta\|u\|^2_{s,\eta,\Omega^t} \leq p^s(\bar{a}, \bar{b})\eta^{-1}\|F\|^2_{s,\eta,\Omega^t},$$

where $U_{[0,T]} = \{u \in C^\infty_{(0)}([0, T] \times \mathbf{R}^m) : u = 0 \text{ in } (\mathbf{R}^m \setminus \Omega) \cup \Omega_\varepsilon \text{ for some } \varepsilon > 0\}$, $\Omega_\varepsilon = \{x \in \mathbf{R}^m : \text{dist}(x, \partial\Omega) < \varepsilon\}$, p^s are polynomials, $\eta > q(\bar{a}, \bar{b})$ (q is a polynomial).

The results of this section (Theorems 3.6.1 and 3.6.2) are used in the proofs of Remarks 3.5.3, 4.1.4 and Theorems 5.1, 5.4.

In Chapter 4 we consider the linear mixed problem (1.1)–(1.3), where $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$. Section 4.1 is concerned with the case of a nonzero initial condition. In this section we apply the methods of J. Rauch [21] and the results of Chapter 3. Similarly to [21] we prove that problem (1.1)–(1.3) (where Ω is a halfspace) has a unique solution $u \in C_{(0)}^\infty([0,T] \times \bar{\Omega})$ under the assumption that $v \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$; $E, A_j, C, S \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$; $F \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$ and $F = 0$ for $t < 0$; $g \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$ and $g = 0$ for $t < 0$; $f \in C_0^\infty(\Omega)$, and under the remaining assumptions from Section 3.1 (see Theorem 4.1.1).

Next we find estimates for a solution u of (1.1)–(1.3) in L_η^2- and H_η^s-norms, where $0 \le s \le k$, $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$. We then assume additionally that system (1.1) is strictly hyperbolic. Using Lemmas 4.1.1–4.1.7, which are slight modifications of the lemmas from [21], and next applying [28, Lemma 3.2] we obtain the following estimate for the smooth solution u of problem (1.1)–(1.3):

$$
\begin{aligned}
(1.17) \quad & |u(t)|_{s,0,\Omega}^2 e^{-2\eta t} + \eta \|u\|_{s,\eta,\Omega^t}^2 + \|u\|_{s,\eta,\partial\Omega^t}^2 \\
& \le \tilde{f}_s((\varepsilon_i)_{i=1}^3, (\varepsilon_i^*)_{i=1}^3)[p_s(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k,0,\infty,K}) \|F\|_{s,\eta,\Omega^t}^2 \\
& \quad + q_s(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k,0,\infty,K}) \|g\|_{s,\eta,\partial\Omega^t}^2 \\
& \quad + r_s(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k,0,\infty,K}) \|f\|_{s,\Omega}^2]
\end{aligned}
$$

for $0 \le t \le T$, $0 \le s \le k$ and η sufficiently large. In (1.17), p_s and q_s are of the same form as in (1.13), r_s are polynomials, K is a compact set outside of which E, A_j, C and S are constant (see assumption (3.1.7)).

Assuming further that $F \in H^k(\Omega^T)$, $g \in H^k(\partial\Omega^T)$, $f \in \mathring{H}^k(\Omega)$, where $\forall 0 \le j \le k - 1$, $\partial_t^j F|_{t=0} = \partial_t^j g|_{t=0} = 0$, and $E, A_j, C, S \in C^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$ ($k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$) we prove (by means of appropriate smooth approximations of v, E, A_j, C, S) the existence of a unique strong solution $u \in \Pi_0^k(\Omega^T) \cap H^k(\partial\Omega^T)$ of problem (1.1)–(1.3) (see Definition 4.1.1) satisfying the estimate (1.17) (see Theorems 4.1.2–4.1.3).

In Section 4.2 we consider separately the case when $f = 0$.

In Chapter 5 we obtain results analogous to those of Chapters 3 and 4, but in the case of a bounded domain Ω with smooth boundary. Appropriate existence and uniqueness theorems and estimates of the types (1.13) and (1.17) are obtained by means of the local straightening of $\partial\Omega$, using a suitable partition of unity and applying the results of Chapters 3 and 4.

Finally, Chapter 6 is devoted to proving the existence and uniqueness of a local-in-time solution $u \in \Pi_0^k(\Omega^t) \cap H^k(\partial\Omega^t)$ (where t is sufficiently small) of the nonlinear problem (1.1)–(1.3), where $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$ ($k > m/2 + 5/2$ if $f = 0$). This is accomplished by applying the

method of successive approximations (similarly to W. M. Zajączkowski [28]) and using the results of Chapters 4 and 5.

I would like to thank Prof. W. M. Zajączkowski for his helpful advice during my work on this paper. I also thank Prof. B. Ziemian for his remarks.

2. Notations and preliminaries

2.1. Function spaces and spaces of distributions. Let $Q \subset \mathbb{R}^\nu$ ($\nu \geq 1$) be a domain in $\mathbb{R}^\nu$ and $z = (z_1, \ldots, z_\nu)$ a point in Q.

Below, $V = Q$ or $V = \overline{Q}$.

Let us introduce the following function spaces:

$C^k(V)$ (where $k \in \mathbb{N} \cup \{0\}$) — the space of functions defined on V which have continuous derivatives up to order k;

$C^\infty(V)$ — the space of functions defined on V which have derivatives of any order;

$C_0^\infty(V)$ ($C_0^k(V)$, $k \in \mathbb{N} \cup \{0\}$) — the space of functions of class $C^\infty(V)$ ($C^k(V)$) whose supports are compact subsets of V;

$C_{(0)}^\infty(V)$ ($C_{(0)}^k(V)$, $k \in \mathbb{N} \cup \{0\}$) — the space of restrictions to V of functions in $C_0^\infty(\mathbb{R}^\nu)$ ($C_0^k(\mathbb{R}^\nu)$);

$C_B^\infty(V)$ ($C_B^k(V)$, $k \in \mathbb{N} \cup \{0\}$) — the space of functions of class $C^\infty(V)$ ($C^k(V)$) which have bonuded derivatives of any order (up to order k) on V.

Notice that $C_B^\infty(V) = C^\infty(\overline{Q})$ and $C_B^k(V) = C^k(\overline{Q})$ if $V = \overline{Q}$ and Q is a bounded domain.

In $C_B^k(V)$ we introduce the norm

$$\|v\|_{C_B^k(V)} = \|v\|_{C_B^k} = \sum_{|\alpha| \leq k} \sup_{z \in V} |\partial^\alpha v(z)|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_\nu)$ is a multiindex and $\partial^\alpha v = \partial^{|\alpha|} v / \partial z_1^{\alpha_1} \ldots \partial z_\nu^{\alpha_\nu}$, $|\alpha| = \alpha_1 + \ldots + \alpha_\nu$.

We next introduce Sobolev spaces:

$L^2(V)$ — the space of (Lebesgue) measurable functions v which are square integrable, with the norm $\|v\|_{0,V} = \left(\int_V |v(z)|^2 \, dz \right)^{1/2}$;

$H^k(V)$ (where $k \in \mathbb{N} \cup \{0\}$) — the space of measurable functions v such that $\forall |\alpha| \leq k$, $\partial^\alpha v \in L^2(V)$, with the norm

$$\|v\|_{k,V} = \left(\sum_{|\alpha| \leq k} \int_V |\partial^\alpha v(z)|^2 \, dz \right)^{1/2}$$

(where $\partial^\alpha v$ is the distributional derivative);

$\overset{\bullet}{H}{}^k(V)$ (where $k \in \mathbb{N} \cup \{0\}$) — the closure of $C_0^\infty(V)$ in the norm of $H^k(V)$;

$S(\mathbb{R}^\nu)$ – the space of functions $v \in C^\infty(\mathbb{R}^\nu)$ such that $\sup_{z \in \mathbb{R}^\nu} |z^\beta \partial^\alpha v(z)| < \infty$ for any multiindices α and β;

$S'(\mathbb{R}^\nu)$ — the space of slowly increasing distributions;

$H^s(\mathbb{R}^\nu)$ (where $s \in \mathbb{R}$) — the space of distributions $v \in S'(\mathbb{R}^\nu)$ such that $\int_{\mathbb{R}^\nu} |\hat{v}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty$ (here $\hat{v}$ denotes the Fourier transform of v).

The norm in $H^s(\mathbb{R}^\nu)$ is defined by

$$\|v\|'_{s,\mathbb{R}^\nu} = \left(\int\limits_{\mathbb{R}^\nu} |\hat{v}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2}.$$

In the case when $s = k \in \mathbb{N} \cup \{0\}$, the space $H^s(\mathbb{R}^\nu)$ is identical with the space $H^k(\mathbb{R}^\nu)$ introduced earlier and the norms $\|v\|_{k,\mathbb{R}^\nu}$ and $\|v\|'_{k,\mathbb{R}^\nu}$ are equivalent.

Next we define the spaces $H^s(V)$ ($s \geq 0$). To do this introduce first the space $W = W(2, \nu, H^1(V), L^2(V))$ of measurable functions $v : [0, \infty) \to H^1(V) + L^2(V)$ (where $H^1(V) + L^2(V)$ is the vector sum of the spaces $H^1(V)$ and $L^2(V)$) such that $t^\nu \tilde{v} \in L^2(0, \infty; H^1(V))$ and $t^\nu \tilde{v}' \in L^2(0, \infty; L^2(V))$ ($\tilde{v}'$ denotes the distributional derivative of $\tilde{v}$, $\nu \in \mathbb{R}$).

Denote next by $T^{\theta,2}(V)$ ($\theta = \nu + 1/2$) the space of functions $v = \tilde{v}(0) = \lim_{t \to 0+} \tilde{v}(t)$, where $\tilde{v} \in W$ (the above limit exists by [1, Lemma 7.13]), with the norm

$$\|v; T^{\theta,2}(V)\|$$
$$= \inf_{\substack{\tilde{v} \in W \\ v = \tilde{v}(0)}} \max \left\{ \left(\int\limits_0^\infty t^{2\nu} \|\tilde{v}(t)\|_{1,V}^2 \, dt \right)^{1/2}, \left(\int\limits_0^\infty t^{2\nu} \|\tilde{v}'(t)\|_{0,V}^2 \, dt \right)^{1/2} \right\}.$$

Take now any $s \geq 0$. If $s = k \in \mathbb{N} \cup \{0\}$ then we define $H^s(V) = H^k(V)$. If $s = k + \sigma$ (where $0 < \sigma < 1$), then $H^s(V)$ is defined to be the space of those $v \in H^k(V)$ for which the distributional derivatives $\partial^\alpha v \in T^{1-\sigma,2}(V)$ for $|\alpha| = k$. This is a Banach space with the norm

$$\|v\|''_{s,v} = \left\{ \|v\|_{k,V}^2 + \sum_{|\alpha|=k} \|\partial^\alpha v; T^{1-\sigma,2}(V)\|^2 \right\}^{1/2}.$$

In the case when $V = \mathbb{R}^\nu$, $H^s(V)$ is identical with the space $H^s(\mathbb{R}^\nu)$ introduced earlier and the norms $\| \cdot \|'$ and $\| \cdot \|''$ are equivalent ([1, Chap. VII] and [17]).

Let further $\Omega \subset \mathbb{R}^m$ ($m \geq 1$) denote a domain in $\mathbb{R}^m$, and $[0, T] \subset \mathbb{R}$ (where $T > 0$) an arbitrary interval. Set

$$\Omega^T = [0, T] \times \Omega, \quad \partial\Omega^T = [0, T] \times \partial\Omega,$$

where $\partial\Omega$ is the boundary of Ω. Let J denote a finite or infinite interval (in particular, J may be $[0,T]$). Denote further by t a point of J and by $x = (x_1, \ldots, x_m)$ a point of Ω.

Let now $V = J \times \Omega$ or $V = \Omega$. By $H_\eta^k(V)$ (where $k \in \mathbb{N} \cup \{0\}$, $\eta \in \mathbb{R}$) we denote the completion of $C_{(0)}^\infty(V)$ in the norm

$$(2.1.1) \qquad \|v\|_{H_\eta^k(V)} = \|v\|_{k,\eta,V} = \Big(\sum_{|\alpha| \le k} \int_V |\partial^\alpha v|^2 e^{-2\eta t} \Big)^{1/2},$$

where in the sum on the right-hand side, we take the t and x derivatives if $V = J \times \Omega$, and the x derivatives if $V = \Omega$.

In the case when $V = J \times \partial\Omega$ or $V = J \times K$ (where $\Omega = \{x \in \mathbb{R}^m : x_1 > 0\}$ and K is a subset of $\partial\Omega$) the space $H_\eta^k(V)$ is similarly defined, but in the sum on the right-hand side, we then take the t and $x' = (x_2, \ldots, x_m)$ derivatives only. In the case when Ω is a bounded domain with C^k boundary we cover $\partial\Omega$ with a finite number of balls K_{R_q} ($q = 1, \ldots, \mu$) with centres $x^{0q} \in \overline{\Omega}$ with the property that there exists $N_0 < \mu$ such that the intersection of any $N_0 + 1 \le \mu$ of these balls is empty. Set

$$\Omega_q = K_{R_q} \cap \Omega, \quad \Gamma_q = K_{R_q} \cap \partial\Omega.$$

Then with x^{0q} we can associate the coordinates

$$(2.1.2) \qquad z_i = c_{ij}^q(x_j - x_j^{0q}) \quad \text{for } i = 1, \ldots, m$$

(where $[c_{ij}^q]$ is an orthogonal matrix), such that Γ_q is given in (z_i) by the equation

$$z_1 = h_q(z'), \quad z' = (z_2, \ldots, z_m),$$

where $h_q \in C_B^k(z' : |z'| \le R_q)$ and the z_1 axis is normal to $\partial\Omega$ and points in the opposite direction to ν, the unit outward normal to the boundary at x^{0q}.

Next introduce the coordinates

$$(2.1.3) \qquad y_1 = z_1 - h_q(z'), \quad y' = z'.$$

Then from (2.1.2) and (2.1.3) we obtain

$$(2.1.4) \qquad x = \Psi_q(y),$$

where $\Psi_q = (\Psi_{q1}, \ldots, \Psi_{qm})$,

$$\Psi_{qj}(y) = x_j^{0q} + c_{1j}^q[y_1 + h_q(y')] + \sum_{i=2}^m c_{ij}^q y_i \quad (j = 1, \ldots, m),$$

and

$$(2.1.5) \qquad y = \Psi_q^{-1}(x),$$

where

$$\Psi_{q1}^{-1}(x) = c_{1j}^q(x_j - x_j^{0q}) - h_q(c_{2j}^q(x_j - x_j^{0q}), \ldots, c_{mj}^q(x_j - x_j^{0q})),$$
$$\Psi_{qi}^{-1}(x) = c_{ij}^q(x_j - x_j^{0q}) \quad (i = 2, \ldots, m).$$

Under the change of variables (2.1.4), Ω_q is the image of the set $\Sigma_{R_q} = \{y : [y_1 + h_q(y')]^2 + |y'|^2 < R_q^2, y_1 > 0\}$, while Γ_q is the image of $\sigma_{R_q} = \{y : [y_1 + h_q(y')]^2 + |y'|^2 < R_q^2, y_1 = 0\}$.

With the above covering $\{K_{R_q}\}_{q=1}^{\mu}$ of $\partial\Omega$ we associate a partition of unity $\{\psi_q\}_{q=1}^{\mu}$ satisfying the properties:

(a) $\psi_q \in C_0^{\infty}(\mathbf{R}^m)$ and $\operatorname{supp}\psi_q \subset K_{R_q}$, $\forall 1 \le q \le \mu$,
(b) $0 \le \psi_q \le 1$, $\forall 1 \le q \le \mu$,
(c) $\sum_{q=1}^{\mu} \psi_q = 1$.

Let now v be a function defined on $J \times \partial\Omega$. Define $\theta_q v$ on $J \times \mathbf{R}^{m-1}$ by

$$\theta_q v = \begin{cases} (\psi_q v)(t, \Psi_q(0, y')) & \text{if } |y'| < R_q, \\ 0 & \text{otherwise}. \end{cases}$$

The space $H_\eta^k(J \times \partial\Omega)$ is understood to be the set of functions v such that $\theta_q v \in H_\eta^k(J \times \mathbf{R}^{m-1})$, $\forall 1 \le q \le \mu$. The norm in this space is given by

$$(2.1.6) \qquad \|v\|_{H_\eta^k(J \times \partial\Omega)} = \|v\|_{k,\eta,J \times \partial\Omega} = \left\{ \sum_{q=1}^{\mu} \|\theta_q v\|_{k,\eta,J \times \mathbf{R}^{m-1}}^2 \right\}^{1/2},$$

where the norms on the right-hand side are given by (2.1.1).

It can be proved ([17]) that the same space with an equivalent norm results if we carry out this construction for other collections $\{\Psi_q\}$ and $\{\psi_q\}$.

In the same way one can define the space $H_\eta^k(J \times K)$, where K is a subset of $\partial\Omega$, and the space $H^s(J \times \partial\Omega)$, where $s \ge 0$.

Notice that for $\eta = 0$, $H_\eta^k(V) = H^k(V)$.

Let now $V = \Omega$, $V = J \times \Omega$ or $V = J \times \partial\Omega$.

We introduce the following spaces:

$\mathring{H}_\eta^k(V)$ (where $k \in \mathbf{N} \cup \{0\}$, $\eta \ge 0$) — the closure of $C_0^{\infty}(V)$ in the norm of $H_\eta^k(V)$;

$H_{-\eta}^{-k}(V)$ (where $k \in \mathbf{N} \cup \{0\}$, $\eta \ge 0$) — the closure of $L^2(V)$ in the norm

$$\|v\|_{-k,-\eta,V} = \sup_{\varphi \in C_0^{\infty}(V)} |(\varphi, v)_{0,V}| / \|\varphi\|_{k,\eta,V},$$

where $(\varphi, v)_{0,V}$ denotes the scalar product in $L^2(V)$;

$\mathring{H}_{-\eta}^{-k}(V)$ (where $k \in \mathbf{N} \cup \{0\}$, $\eta \ge 0$) — the closure of $L^2(V)$ in the norm

$$\|v\|_{(-k)^0,-\eta,V} = \sup_{\varphi \in C_{(0)}^{\infty}(V)} |(\varphi, v)_{0,V}| / \|\varphi\|_{k,\eta,V}.$$

For all integers s the space $\overset{\circ}{H}{}^{-s}_{-\eta}(V)$ is dual to $H^s_\eta(V)$ and similarly $H^s_\eta(V)$ is dual to $\overset{\circ}{H}{}^{-s}_{-\eta}(V)$.

Further, denote by $L^k_\infty(J; H^i(\Omega))$ and $L^k_p(J; H^i(\Omega))$ (where $k \in \mathbb{N} \cup \{0\}$, $i \in \mathbb{N} \cup \{0\}$, $1 \le p < \infty$) the spaces of functions u such that

$$\underset{t \in J}{\mathrm{ess\ sup}} \left\| \frac{\partial^k}{\partial t^k} u(t) \right\|_{i,\Omega} < \infty \quad \text{and} \quad \int_J \left\| \frac{\partial^k}{\partial t^k} u(t) \right\|^p_{i,\Omega} dt < \infty$$

respectively.

Using the above definitions we also introduce the spaces

$$\Pi^k_0(J \times \Omega) = \bigcap_{i=0}^k L^{k-i}_\infty(J; H^i(\Omega))$$

with the norm

$$|u|_{k,0,\infty,J \times \Omega} = \sum_{i=0}^k \underset{t \in J}{\mathrm{ess\ sup}} \left\| \frac{\partial^{k-i}}{\partial t^{k-i}} u(t) \right\|_{i,\Omega},$$

and the space

$$\Pi^k_{0,p}(J \times \Omega) = \bigcap_{i=0}^k L^{k-i}_p(J; H^i(\Omega))$$

with the norm

$$|u|_{k,0,p,J \times \Omega} = \sum_{i=0}^k \left(\int_J \left\| \frac{\partial^{k-i}}{\partial t^{t-i}} u(t) \right\|^p_{i,\Omega} dt \right)^{1/p}.$$

Finally, denote by $\Gamma^k_0(\Omega)$ (where $k \in \mathbb{N} \cup \{0\}$) the space of functions v such that

$$\left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha_{t,x} v(t,x)|^2 \, dx \right)^{1/2} < \infty,$$

with the norm

$$\|u\|_{\Gamma^k_0(\Omega)} = |u|_{k,0,\Omega} = \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha_{t,x} v(t,x)|^2 \, dx \right)^{1/2}.$$

Let X be whichever of the function spaces mentioned above. We will say that a vector-valued function $u = (u_1, \dots, u_\nu)$ or a matrix-valued function $A = [a_{ij}]_{1 \le i,j \le \nu}$ belongs to X if $\forall 1 \le i \le \nu$, $u_i \in X$, resp. $\forall 1 \le i,j \le \nu$, $a_{ij} \in X$.

Now we formulate some lemmas which are useful in our paper.

By induction one can prove

LEMMA 2.1.1 ([28, Lemma A.1]). *Let* $\theta = \theta(z, v) \in C^k(\mathbf{R}^\nu \times \mathbf{R}^{\nu'})$, *where* $k \geq 1$, $\nu, \nu' \geq 1$. *Then for any* ν'-*vector-valued function* $v \in C^k(Q)$ (*where* $Q \subset \mathbf{R}^\nu$), *we have*

$$(2.1.7)\quad \partial_z^r \theta = \sum_{1 \leq s \leq r} \sum_{\substack{\alpha_1 + \ldots + \alpha_s = r+1-s \\ \alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = r}} c_{\alpha,s} \partial_v^{r+1-s} \theta \cdot (\partial_z^1 v)^{\alpha_1} \ldots (\partial_z^s v)^{\alpha_s}$$

$$+ \sum_{\substack{\sigma \geq 1 \\ 1 \leq s \leq r-\sigma}} \sum_{\substack{\alpha_1' + \ldots + \alpha_s' = r+1-s-\sigma \\ \alpha_1' + 2\alpha_2' + \ldots + s\alpha_s' = r-s}} c_{\alpha,s,\sigma}' \partial_z^\sigma \partial_v^{r+1-s-\sigma} \theta \cdot (\partial_z^1 v)^{\alpha_1'} \ldots (\partial_z^s v)^{\alpha_s'},$$

where $1 \leq r \leq k$; $\alpha_j, \alpha_j' \in \mathbf{N} \cup \{0\}$, $\forall 1 \leq j \leq s$; $c_{\alpha,s}$ *and* $c_{\alpha,s,\sigma}'$ *are some constants; and* $(\partial_z^j v)^{\alpha_j}$ *denotes the tensor product of* α_j *vectors* $\partial_z^j v$ *composed of all derivatives of order* j *of* v.

Using Lemma 2.1.1 one can prove

LEMMA 2.1.2 ([28, Lemma 6.1]). *Let* $\theta = \theta(t, x, v) \in C_{(0)}^k(J \times \bar{\Omega} \times G)$, *where* $\Omega \subset \mathbf{R}^m$, $J \subset \mathbf{R}$, $G \subset \mathbf{R}^n$, $k > m/2 + 1$. *If* $v(t,x) \in \Pi_0^k(J \times \Omega)$, *then* $\bar{\theta}(t,x) = \theta(t, x, v(t,x)) \in \Pi_0^k(J \times \Omega)$ *and*

$$|\bar{\theta}|_{k,0,\infty,J\times\Omega} \leq p(\|\theta\|_{C_B^k(J\times\bar{\Omega}\times G)}, \|v\|_{k,0,\infty,J\times\Omega}),$$

where p *is a polynomial.*

Similarly, if $v \in H^k(J \times \partial\Omega)$, *then* $\bar{\theta} \in H^k(J \times \partial\Omega)$ *and*

$$\|\bar{\theta}\|_{k,J\times\partial\Omega} \leq q(\|\theta\|_{C_B^k(J\times\bar{\Omega}\times G)}, \|v\|_{k,J\times\partial\Omega}),$$

where q *is a polynomial.*

The next lemma concerns extending functions in $\Pi_0^k([T_1, T_2] \times \Omega) \cap H^{k+1/2}([T_1, T_2] \times \Omega)$ (where $T_1 < T_2$, $\Omega \subset \mathbf{R}^m$) to $\mathbf{R} \times \mathbf{R}^m$.

LEMMA 2.1.3. *Let* $v \in \Pi_0^k([T_1, T_2] \times \Omega) \cap H^{k+1/2}([T_1, T_2] \times \Omega)$, *where* $T_1 < T_2$, $k \geq 1$, *and either*

1° $\Omega = \{x \in \mathbf{R}^m : a_i < x_i < b_i, \ a_i, b_i \in \mathbf{R}, \ i = 1, \ldots, m\}$, *or*

2° Ω *is a domain in* $\mathbf{R}^m$ *with bounded* C^k *boundary.*

Then there exists $\tilde{v} \in \Pi_0^k(\mathbf{R} \times \mathbf{R}^m) \cap H^{k+1/2}(\mathbf{R} \times \mathbf{R}^m)$ *such that* $\tilde{v} = v$ *in* $[T_1, T_2] \times \Omega$, $\tilde{v} = 0$ *outside a compact set* $K \subset \mathbf{R} \times \mathbf{R}^m$ *and*

$$(2.1.8)\quad \|\tilde{v}\|_{k+1/2,\mathbf{R}\times\mathbf{R}^m} + |\tilde{v}|_{k,0,\infty,\mathbf{R}\times\mathbf{R}^m}$$
$$\leq c(\|v\|_{k+1/2,[T_1,T_2]\times\Omega} + |v|_{k,0,\infty,[T_1,T_2]\times\Omega}),$$

where $c > 0$ *is a constant.*

P r o o f. Consider first case 1°. We apply the method of Hestenes (see [1, Chap. IV] and [3, p. 135]). Assume first that $v \in C_{(0)}^{k+1}([T_1, T_2] \times \bar{\Omega})$ and

define

(2.1.9)
$$v_1^t(t,x) = \begin{cases} v(t,x) & \text{for } T_1 \leq t \leq T_2, \\ \sum_{i=1}^{k+2} \lambda_i v(-it + T_1(i+1), x) & \text{for } \frac{T_1-T_2}{k+2} + T_1 \leq t \leq T_1, \\ 0 & \text{for } t < \frac{T_1-T_2}{k+2} + T_1, \end{cases}$$

where the numbers λ_i, $i = 1, \ldots, k+2$, fulfil the system of equations

$$\sum_{i=1}^{k+2} (-i)^\mu \lambda_i = 1, \quad \mu = 0, 1, \ldots, k+1.$$

Let E_1^t denote the operator

(2.1.10)
$$E_1^t v = v_1^t.$$

Notice that (2.1.9) implies

(2.1.11) $$\partial^\alpha E_1^t v = \sum_{|\gamma| \leq |\alpha|} E_{1\alpha\gamma}^t \partial^\gamma v \quad \forall |\alpha| \leq k \quad \text{if } t \neq \frac{T_1 - T_2}{k+2} + T_1,$$

where $E_{1\alpha\gamma}^t = 0$ for $\alpha \neq \gamma$ and

(2.1.12)
$$E_{1\alpha\alpha}^t \partial^\alpha v = \begin{cases} \partial^\alpha v(t,x) & \text{for } T_1 \leq t \leq T_2, \\ \sum_{i=1}^{k+2} (-i)^{\alpha_0} \lambda_i \partial^\alpha v(-it + T_1(i+1), x) & \\ & \text{for } \frac{T_1-T_2}{k+2} + T_1 < t \leq T_1, \\ 0 & \text{for } t < \frac{T_1-T_2}{k+2} + T_1 \end{cases}$$

(here $\partial^\gamma = \partial^{|\gamma|}/\partial t^{\gamma_0} \partial x_1^{\gamma_1} \ldots \partial x_m^{\gamma_m}$, $|\gamma| = \gamma_0 + \gamma_1 + \ldots + \gamma_m$).

Next, define

(2.1.13)
$$v^t(t,x) = \varphi(t) v_1^t(t,x),$$

where $\varphi \in C_{(0)}^\infty((-\infty, T_2])$ and $\varphi = 1$ for $T_1 \leq t \leq T_2$ and $\varphi = 0$ for $t \leq \frac{T_1-T_2}{k+2} + T_1 + \varepsilon$ ($\varepsilon > 0$ is such that $\frac{T_1-T_2}{k+2} + T_1 + \varepsilon < T_1$).

Futher, set

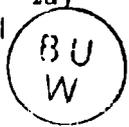(2.1.14) $$v_2^t(t,x) = \begin{cases} v^t(t,x) & \text{for } t \leq T_2, \\ \sum_{j=1}^{k+2} \lambda_j v^t(-jt + T_2(j+1), x) & \text{for } t > T_2. \end{cases}$$

Denote by E_2^t the operator

(2.1.15)
$$E_2^t v^t = v_2^t.$$

Then (2.1.14) implies that

(2.1.16) $$\partial^\alpha E_2^t v^t = \sum_{|\gamma| \leq |\alpha|} E_{2\alpha\gamma}^t \partial^\gamma v \quad \forall |\alpha| \leq k,$$

where $E_{2\alpha\gamma}^t = 0$ for $\alpha \neq \gamma$ and

(2.1.17)
$$E_{2\alpha\alpha}^t \partial^\alpha v = \begin{cases} \partial^\alpha v^t(t, x) & \text{for } t \geq T_2, \\ \sum_{j=1}^{k+2} (-j)^{\alpha_0} \lambda_j \partial^\alpha v^t(-jt + T_2(j+1), x) & \text{for } t < T_2. \end{cases}$$

Now denote by E^t the operator given by

(2.1.18) $E^t v = v_2^t, \quad$ i.e. $E^t = E_2^t \varphi E_1^t$.

By (2.1.11)–(2.1.13), (2.1.16) and (2.1.17) we have

(2.1.19) $\partial^\alpha E^t v = \sum_{|\gamma| \leq |\alpha|} E_{\alpha\gamma}^t \partial^\gamma v \quad \forall |\alpha| \leq k$,

where the $E_{\alpha\gamma}^t$ are continuous operators from $H^j([T_1, T_2] \times \Omega)$ into $H^j(\mathbf{R} \times \Omega)$ for $j = 0, 1$. Moreover, $E^t v = 0$ outside a compact set $K_t \subset \mathbf{R}$, and

$$\|E^t v\|_{r,\mathbf{R} \times \Omega} \leq c\|v\|_{r,[T_1,T_2] \times \Omega} \quad \forall 0 \leq r \leq k+1, \ r \in \mathbf{N},$$
$$|E^t v|_{k,0,\infty,\mathbf{R} \times \Omega} \leq c|v|_{k,0,\infty,[T_1,T_2] \times \Omega}.$$

Further, analogously to E^t define the operator E^{x_1} extending the function $E^t v$ to $\mathbf{R} \times \mathbf{R} \times \{x \in \mathbf{R}^{m-1} : a_i \leq x_i \leq b_i, i = 2, \ldots, m\}$, and next the operators $E^{x_2}, \ldots, E^{x_m}$. Thus we obtain the operator

$$E_{\mathbf{R} \times \mathbf{R}^m} = E^{x_m} \ldots E^{x_1} E^t$$

such that

(2.1.20) $E_{\mathbf{R} \times \mathbf{R}^m} v = 0 \quad$ outside a compact set $K \subset \mathbf{R} \times \mathbf{R}^m$,

(2.1.21) $\|E_{\mathbf{R} \times \mathbf{R}^m} v\|_{r,\mathbf{R} \times \mathbf{R}^m} \leq c\|v\|_{r,[T_1,T_2] \times \Omega} \quad \forall 0 \leq r \leq k+1, \ r \in \mathbf{N}$,

(2.1.22) $|E_{\mathbf{R} \times \mathbf{R}^m} v|_{k,0,\infty,\mathbf{R} \times \mathbf{R}^m} \leq c|v|_{k,0,\infty,[T_1,T_2] \times \Omega}$

(where $c > 0$ is a constant) and

(2.1.23) $\partial^\alpha E_{\mathbf{R} \times \mathbf{R}^m} v = \sum_{|\gamma| \leq |\alpha|} E_{\alpha\gamma} \partial^\gamma v \quad \forall |\alpha| \leq k$,

where the $E_{\alpha\gamma}$ are continuous operators from $H^j([T_1, T_2] \times \Omega)$ into $H^j(\mathbf{R} \times \mathbf{R}^m)$, $j = 0, 1$.

Now assume that $v \in \Pi_0^k([T_1, T_2] \times \Omega) \cap H^{k+1/2}([T_1, T_2] \times \Omega)$. Then there exists a sequence $v_j \in C^\infty([T_1, T_2] \times \overline{\Omega})$ (see [3, Section 19.2]) such that

(2.1.24) $v_j \to v \quad$ in $H^r([T_1, T_2]) \times \Omega) \quad \forall 0 \leq r \leq k+1, \ r \in \mathbf{N}$,

(2.1.25) $|v_j|_{k,0,\infty,[T_1,T_2] \times \Omega} \leq c|v|_{k,0,\infty,[T_1,T_2] \times \Omega}$,

where $c > 0$ is a constant. Using (2.1.24), (2.1.25) and the fact that v_j satisfies (2.1.20)–(2.1.23) we get

$$E_{\mathbf{R} \times \mathbf{R}^m} v_j \to E_{\mathbf{R} \times \mathbf{R}^m} v \quad \text{in } H^r([T_1, T_2] \times \Omega) \quad \forall 0 \leq r \leq k+1, \ r \in \mathbf{N},$$

and (2.1.20)–(2.1.23) are satisfied for $E_{\mathbf{R}\times\mathbf{R}^m}v$.

By (2.1.21) the operator (2.1.20) is a strong $(k+1)$-extension opera-tor for $(T_1, T_2) \times \Omega$ ([1, Definition 4.24]). Therefore (2.1.20)–(2.1.23) and [1, Theorem 7.41] imply that if $v \in \Pi_0^k([T_1, T_2] \times \Omega) \cap H^{k+1/2}([T_1, T_2] \times \Omega)$, then the lemma is satisfied with $\tilde{v} = E_{\mathbf{R}\times\mathbf{R}^m}v$.

In case 2°, the proof is analogous to that of [1, Theorem 4.26(ii)]. ■

R e m a r k 2.1.1. In the same way as Lemma 2.1.3 it can be proved that if $A \in C^k([T_1, T_2] \times \bar{\Omega} \times \bar{G})$ (where $k \geq 1$, $\Omega \subset \mathbf{R}^n$ fulfils one of assumptions 1° or 2°, $[T_1, T_2] \subset \mathbf{R}$, $G \subset \mathbf{R}^n$ satisfies assumption 1°), then there exists a function $\tilde{A} \in C^k(\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n)$ such that $\tilde{A} = A$ in $[T_1, T_2] \times \bar{\Omega} \times \bar{G}$, $\tilde{A} = 0$ outside a compact set $K \subset \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$ and

$$\|\tilde{A}\|_{C_B^k(\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^n)} \leq c\|A\|_{C_B^k([T_1,T_2]\times\bar{\Omega}\times G)} ,$$

where $c > 0$ is a constant.

We now formulate a lemma whose proof is in [21, p. 279].

LEMMA 2.1.4. *Assume that* $\eta \geq 0$, s *is a nonnegative integer,* $J = (-\infty, T]$, $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$.

1° *If* $\psi \in C^\infty(J \times \bar{\Omega})$ *and if*

(i) $\partial_{t,x}^\alpha \psi \in \overset{\circ}{H}{}_{-\eta}^{s-1}(J \times \Omega)$ *for* $|\alpha| \leq 1$,

(ii) $\psi(T) \in \overset{\circ}{H}{}_{-\eta}^{s}(\Omega)$,

(iii) $\psi|_{x_1=0} \in \overset{\circ}{H}{}_{-\eta}^{s}(J \times \partial\Omega)$,

then $\psi \in \overset{\circ}{H}{}_{-\eta}^{s}(J \times \Omega)$ *and*

$$\eta^{1/2}\|\psi\|_{s^0,-\eta,J\times\Omega} \leq c_s\Big(\eta^{1/2}\sum_{|\alpha|\leq 1}\|\partial_{t,x}^\alpha\psi\|_{(s-1)^0,-\eta,J\times\Omega} + \|\psi(T)\|_{s^0,-\eta,\Omega}$$
$$+ \|\psi\|_{s^0,-\eta,J\times\partial\Omega}\Big) ,$$

where $c_s > 0$ *is a constant independent of* ψ *and* η.

2° *If* $\theta \in C^\infty(J \times \partial\Omega)$ *and if*

(i′) $\partial_{t,x}^\beta\theta \in \overset{\circ}{H}{}_{-\eta}^{s-1}(J \times \partial\Omega)$ *for* $|\beta| \leq 1$,

(ii′) $\theta(T) \in \overset{\circ}{H}{}_{-\eta}^{s}(\partial\Omega)$,

then $\theta \in \overset{\circ}{H}{}_{-\eta}^{s}(J \times \partial\Omega)$ *and*

$$\|\theta\|_{s^0,-\eta,J\times\partial\Omega} \leq c_s\Big(\sum_{|\beta|\leq 1}\|\partial_{t,x}^\beta\theta\|_{(s-1)^0,-\eta,J\times\partial\Omega} + \eta^{-1/2}\|\theta(T)\|_{s^0,-\eta,\partial\Omega}\Big) .$$

2.2. Perturbations of linear operators. Let $M = M(z)$ (where $z = (z_1, \ldots, z_\nu) \in Q \subset \mathbf{R}^\nu$) be a matrix-valued function, $M : Q \to \mathbf{C}^{n^2}$. Take two unordered n-tuples $\mathfrak{S} = (\mu_1, \ldots, \mu_n)$ and $\mathfrak{S}' = (\mu'_1, \ldots, \mu'_n)$ of complex numbers and define their distance by

$$\mathrm{dist}(\mathfrak{S}, \mathfrak{S}') = \min \max |\mu_j - \mu'_j| \,,$$

where we take the minimum over all permutations of one of the tuples.

Next denote by $\mathfrak{S}(z)$ the complete set of eigenvalues of $M(z)$, i.e. the set composed of eigenvalues of $M(z)$ repeated according to their multiplicities.

DEFINITION 2.2.1 ([10, Chap. II]). We say that the complete set of eigenvalues $\mathfrak{S}(z)$ of the matrix-valued function $M(z)$ *depends continuously on z in Q* if

$$\forall z_0 \in Q \quad \mathrm{dist}(\mathfrak{S}(z), \mathfrak{S}(z_0)) \to 0 \quad \text{as } z \to z_0 \,.$$

DEFINITION 2.2.2 ([10, Chap. II]). The *resolvent* of $M(z)$ is the matrix-valued function

$$\widehat{R}(\beta, z) = (M(z) - \beta I)^{-1} \,,$$

where $\beta \in \mathbf{C}$ and I is the unit matrix.

Let $\kappa_\sigma = \kappa_\sigma(z)$, where $\sigma = 1, \ldots, r$ $(r \leq n)$, be all distinct eigenvalues of $M = M(z)$. Set

$$\mathfrak{N} = \{(\beta, z) \in \mathbf{C} \times Q : \beta \neq \kappa_\sigma(z), \ \sigma = 1, \ldots, r\} \,.$$

LEMMA 2.2.1. *If $M \in C_B^k(Q)$ (where $k \geq 1$), then all derivatives $\partial_z^\alpha \widehat{R} = \partial^{|\alpha|} \widehat{R} / \partial z_1^{\alpha_1} \ldots \partial z_\nu^{\alpha_\nu}$ (where $|\alpha| \leq k$) are defined and continuous on $\mathfrak{N}$.*

P r o o f. Take $(\beta_0, z_0) \in \mathfrak{N}$. Then $\widehat{R}(\beta_0, z_0)$ is defined. Hence for every $(\beta, z) \in \mathbf{C} \times Q$,

$$(2.2.1) \quad M(z) - \beta I = [I - (\beta I - \beta_0 I - A(z))\widehat{R}(\beta_0, z_0)][M(z_0) - \beta_0 I] \,,$$

where $A(z) = M(z) - M(z_0)$. Define

$$\widetilde{\mathfrak{N}}_\varepsilon = \{(\beta, z) \in \mathbf{C} \times Q : |\beta - \beta_0| + |z - z_0| \text{ is so small that}$$

$$|\beta - \beta_0| + |A(z)| \leq \varepsilon |\widehat{R}(\beta_0, z_0)|^{-1}\} \,,$$

where $\varepsilon < 1$ (here $|\cdot|$ denotes any matrix norm such that $|I| = 1$). Then $\widehat{R}(\beta, z)$ is defined for any $(\beta, z) \in \widetilde{\mathfrak{N}}_\varepsilon$ and by (2.2.1)

$$(2.2.2) \quad \widehat{R}(\beta, z) = \widehat{R}(\beta_0, z_0)[I - (\beta I - \beta_0 I - A(z))\widehat{R}(\beta_0, z_0)]^{-1}$$

$$= \sum_{n=0}^{\infty} [(\beta I - \beta_0 I - A(z))\widehat{R}(\beta_0, z_0)]^n \,,$$

where the above series is uniformly convergent in $\widetilde{\mathfrak{N}}_\varepsilon$.

Consider the function

$$\gamma_n(v(z)) = [v(z)]^n, \quad \text{where} \quad v(z) = (\beta I - \beta_0 - A(z))\widehat{R}(\beta_0, z_0).$$

Using (2.1.7) we get

$$\partial_z^r \gamma_n = \sum_{1 \leq s \leq r} \sum_{\substack{\alpha_1 + \ldots + \alpha_s = r + 1 - s \\ \alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = r}} c_{\alpha,s} \partial_v^{r+1-s} \gamma_n (\partial_z^1 v)^{\alpha_1} \ldots (\partial_z^s v)^{\alpha_s},$$

where $1 \leq r \leq k$. Fix $1 \leq s \leq r$ and consider $\partial_v^{r+1-s}\gamma_n$. We obtain

$$\partial_v^{r+1-s}\gamma_n = \begin{cases} 0 & \text{if } n < r+1-s, \\ n! & \text{if } n = r+1-s, \\ (-1)^{r+1-s} n \ldots (n-r+s) v^{n-r+s-1} & \text{if } n > r+1-s. \end{cases}$$

Therefore

$$\forall n > r+1 \quad |\partial_z^r \gamma_n| \leq C_r n^r \varepsilon^{n-1-r} \quad \text{on } \widetilde{\mathfrak{N}}_\varepsilon.$$

Hence we obtain the uniform convergence of $\sum_{n=1}^\infty \partial_z^r \gamma_n$, $\forall 1 \leq r \leq k$, and by (2.2.2), $\widehat{R}(\beta, z)$ has partial derivatives in $z_1, \ldots, z_\nu$ up to order k, which are continuous on $\widetilde{\mathfrak{N}}_\varepsilon$, and so in particular at (β_0, z_0). ∎

Now take any $z_0 = (z_{01}, \ldots, z_{0\nu}) \in Q$. Let $\kappa = \kappa(z_0)$ be an eigenvalue of $M = M(z_0)$ of multiplicity q. Let Γ denote a closed positively oriented circle enclosing κ but no other eigenvalues of $M(z_0)$.

DEFINITION 2.2.3 ([10, Chap. II]). The matrix

$$(2.2.3) \qquad P_{z_0}(z) = -\frac{1}{2\pi i} \int_\Gamma \widehat{R}(\beta, z) \, dz$$

is called the *total projection matrix of the κ-group*. Define

$$\theta_{\delta, z_0} = \{z \in Q : |z - z_0| < \delta\}, \quad \delta > 0.$$

LEMMA 2.2.2. *If $M \in C^k(Q)$ (where $k \geq 1$), then there exists $\delta > 0$ such that $P_{z_0} \in C^k(\overline{\theta}_{\delta, z_0})$.*

Proof. Take $\beta \in \Gamma$. Then $(\beta, z_0) \in \mathfrak{N}$ and using (2.2.2) we obtain $\widehat{R}(\beta, z) = \widehat{R}(\beta, z_0)[I + A(z)\widehat{R}(\beta, z_0)]^{-1}$, where $z \in \widetilde{Q}_{\varepsilon, z_0} = \{z : |z - z_0|$ is so small that $|A(z)| \leq \varepsilon \min_{\beta \in \Gamma} |\widehat{R}(\beta, z_0)|^{-1}\}$, $\varepsilon < 1$. Therefore $\widehat{R}(\beta, z)$ is defined in $\Gamma \times \widetilde{Q}_{\varepsilon, z_0} \subset \mathfrak{N}$ and by Lemma 2.2.1 it has partial derivatives up to order k in $z_1, \ldots, z_\nu$ continuous on $\Gamma \times \widetilde{Q}_{\varepsilon, z_0}$, and hence on $\Gamma \times \theta_{\delta, z_0}$ provided $\delta > 0$ is so small that $\overline{\theta}_{\delta, z_0} \subset \widetilde{Q}_{\varepsilon, z_0}$. Hence $P_{z_0} \in C^k(\overline{\theta}_{\delta, z_0})$. ∎

Now consider a matrix-valued function $M \in C_B^k(Q)$ and a sequence $M_\mu \in C_B^k(Q)$ (where $k \geq 1$, $0 < \mu \leq 1$) such that

$$(2.2.4) \qquad M_\mu \to M \quad \text{in } C_B^k(Q) \quad \text{as } \mu \to 0.$$

Set

$$m_{z_0}(\mu) = \begin{cases} M_\mu(z_0) & \text{for } 0 < \mu \leq 1, \\ M(z_0) & \text{for } \mu = 0. \end{cases}$$

(2.2.4) implies the continuity of $m_{z_0}(\mu)$ at $\mu = 0$. Assume that $m_{z_0} : [0,1] \to$ $\mathbf{C}^{n^2}$ is continuous in $[0,1]$. Then by $[10, \text{Theorem 5.2}]$ there exist n sequences $\kappa_\sigma^\mu(z_0)$ $(\sigma = 1, \ldots, n, \ \mu \neq 0)$ which form for every $0 < \mu \leq \mu_0$ the set of all eigenvalues of $M_\mu(z_0)$ and satisfy the condition

$$\forall 1 \leq \sigma \leq n \quad \kappa_\sigma^\mu(z_0) \to \kappa_\sigma(z_0),$$

where $\kappa_\sigma(z_0)$ $(\sigma = 1, \ldots, n)$ are the eigenvalues of $M(z_0)$.

Take now an arbitrary eigenvalue $\kappa(z_0)$ of $M(z_0)$, i.e. $\kappa(z_0) = \kappa_\sigma(z_0)$ for some $1 \leq \sigma \leq n$, and write $\kappa^\mu(z_0)$ for $\kappa_\sigma^\mu(z_0)$ (with the above convergence). Further, denote by $P_{z_0}^\mu$ the total projection matrix of the $\kappa^\mu(z_0)$-group, i.e. the matrix given by (2.2.3) (with $\widehat{R}(\beta, z)$ replaced by the resolvent $\widehat{R}_\mu(\beta, z)$ of $M_\mu(z)$).

LEMMA 2.2.3. *If the multiplicity of the eigenvalue $\kappa^\mu(z_0)$ of $m_{z_0}(\mu)$ is constant in $[0, \mu_0]$, then there exist $\delta > 0$ and a subsequence of $(P_{z_0}^\mu)$, still denoted by $(P_{z_0}^\mu)$, such that*

1° $\forall \mu, \ P_{z_0}^\mu \in C^k(\overline{\theta}_{\delta, z_0})$;
2° $P_{z_0}^\mu \to P_{z_0}$ *in* $C^k(\overline{\theta}_{\delta, z_0})$ *as* $\mu \to 0$.

Proof. From the convergence $\kappa^\mu(z_0) \to \kappa(z_0)$ it follows that $\exists \mu_1 > 0$ $\forall \mu < \mu_1, \ P_{z_0}^\mu$ can be defined by $P_{z_0}^\mu(z) = -(2\pi i)^{-1} \int_\Gamma \widehat{R}_\mu(\beta, z) \, d\beta$, where Γ is a circle enclosing $\kappa(z_0)$ but no other eigenvalues of $M(z_0)$. Take $\beta \in \Gamma$. Then $(\beta, z_0) \in \mathfrak{N}_\mu$, where $\mathfrak{N}_\mu = \{(\beta, z) \in \mathbf{C} \times Q : \beta \neq \kappa_\sigma^\mu(z), \sigma = 1, \ldots, r\}$, $\kappa_\sigma^\mu = \kappa_\sigma^\mu(z)$ $(\sigma = 1, \ldots, r)$ are all distinct eigenvalues of $M_\mu(z)$.

Now consider

(2.2.5) $M_\mu(z) - \beta I = [I + A_\mu(z)\widehat{R}(\beta, z_0)][M(z_0) - \beta I],$

where $A_\mu(z) = M_\mu(z) - M(z_0)$. By (2.2.4) there exists $\mu_2 > 0$ such that for $\mu < \mu_2$ and for z sufficiently close to z_0 we have $|A_\mu(z)| \leq \varepsilon$ $\times \min_{\beta \in \Gamma} |\widehat{R}(\beta, z_0)|^{-1}$, where $0 < \varepsilon < 1$. Therefore for $(\beta, z) \in \Gamma \times \widetilde{Q}_{\varepsilon, z_0}$ (where $\widetilde{Q}_{\varepsilon, z_0} = \{z : |z - z_0|$ is so small that $\forall \mu < \mu_2, \ |A_\mu(z)| \leq \varepsilon$ $\times \min_{\beta \in \Gamma} |\widehat{R}(\beta, z_0)|^{-1}\}, \ 0 < \varepsilon < 1)$, by (2.2.5) we obtain

$$\widehat{R}_\mu(\beta, z) = \widehat{R}(\beta, z_0)[I + A_\mu(z)\widehat{R}(\beta, z_0)]^{-1}.$$

Hence using Lemma 2.2.2 we see that $P_{z_0}^\mu(z)$ (where $\mu < \min(\mu_1, \mu_2) = \mu_0$) is defined for $z \in \widetilde{Q}_{\varepsilon, z_0}$ and $P_{z_0}^\mu \in C^k(\overline{\theta}_{\delta, z_0}), \ \forall \mu < \mu_0$, where $\delta > 0$ is so small that $\overline{\theta}_{\delta, z_0} \subset \widetilde{Q}_{\varepsilon, z_0}$.

Now for $\mu < \mu_0$ we obtain

$$\widehat{R}_\mu(\beta, z) - \widehat{R}(\beta, z)$$

$$= \widehat{R}(\beta, z_0)\{[I + A_\mu(z)\widehat{R}(\beta, z_0)]^{-1} - [I + A(z)\widehat{R}(\beta, z_0)]^{-1}\}$$

$$= \widehat{R}(\beta, z_0) \sum_{n=0}^{\infty} [(-A_\mu(z))^n - (-A(z))^n][\widehat{R}(\beta, z_0)]^n$$

for $(\beta, z) \in \Gamma \times \overline{\theta}_{\delta, z_0}$. Therefore by (2.2.4), $\forall \beta \in \Gamma$, $\widehat{R}_\mu(\beta, \cdot) \to \widehat{R}(\beta, \cdot)$ in $C^k(\overline{\theta}_{\delta, z_0})$ and hence $P_{z_0}^\mu \to P_{z_0}$ in $C^k(\overline{\theta}_{\delta, z_0})$. ∎

R e m a r k 2.2.1. The $n \times n$-matrix $P_{z_0}(z)$ is the matrix of a projection of $X = \mathbf{C}^n$ onto its subspace which we denote by $\widehat{N}(z)$, i.e. $P_{z_0}(z)X = \widehat{N}(z)$, $\forall z \in \overline{\theta}_{\delta, z_0}$ ([10, Chap. I, Section 3]).

DEFINITION 2.2.4. 1° $\widehat{N}(z_0)$ is called the *algebraic eigenspace* of $M(z_0)$ corresponding to the eigenvalue $\kappa(z_0)$.

2° Each nonzero vector of $\widehat{N}(z_0)$ is called a *generalized eigenvector* corresponding to $\kappa(z_0)$.

3° The number $q = \dim \widehat{N}(z_0)$ is called the *algebraic multiplicity* of $\kappa(z_0)$.

It turns out that $\dim \widehat{N}(z) = \dim \widehat{N}(z_0)$, $\forall z \in \overline{\theta}_{\delta, z_0}$, where $\overline{\theta}_{\delta, z_0}$ is the set from Lemma 2.2.2. This results from the following lemma.

LEMMA 2.2.4. *Let X be a finite-dimensional normed space, and Q a domain in $\mathbf{R}^\nu$. Let further $P(z)$ for $z \in Q$ be a projection operator of X onto its subspace and assume that P is a continuous function of z. Then $\dim P(z)X$ is constant.*

P r o o f. By the continuity of P, $|P(z') - P(z'')| \leq 1$ for $z', z'' \in Q$ sufficiently close to each other. Hence by [10, Chap. II, §4.6] the assertion follows. ∎

Assume now, as before, that $M \in C^k(Q)$ ($k \geq 1$) and take $z_0 \in Q$. Let $\kappa_\sigma = \kappa_\sigma(z_0)$ (where $\sigma = 1, \ldots, r$, $1 \leq r \leq n$) be all distinct eigenvalues of $M(z_0)$. Let Γ_σ ($\sigma = 1, \ldots, r$) be closed positively oriented circles enclosing $\kappa_\sigma(z_0)$ but no other eigenvalues of $M(z_0)$. Let further $P_{z_0, \sigma}(z)$ ($\sigma = 1, \ldots, r$) be the total projection matrices given by (2.2.3) with Γ replaced by Γ_σ.

R e m a r k 2.2.2. By Lemma 2.2.2, $\exists \delta > 0$ $\forall 1 \leq \sigma \leq r$, $P_{z_0, \sigma} \in C^k(\overline{\theta}_{\delta, z_0})$.

R e m a r k 2.2.3. It can be proved ([10, Chap. I, §5]) that:

(a) $\sum_{\sigma=1}^r P_{z_0, \sigma}(z) = I$, $\forall z \in \overline{\theta}_{\delta, z_0}$;

(b) $P_{z_0, \sigma}(z)P_{z_0, \sigma'}(z) = \delta_{\sigma\sigma'} P_{z_0, \sigma'}(z)$, $\forall z \in \overline{\theta}_{\delta, z_0}$, where $\delta_{\sigma\sigma'}$ is the Kronecker symbol;

(c) $P_{z_0, \sigma}(z)M(z) = M(z)P_{z_0, \sigma}(z)$, $\forall z \in \overline{\theta}_{\delta, z_0}$.

R e m a r k 2.2.4. (a) Remark 2.2.3(a) yields that $X = \mathbb{C}^n$ can be written in the form

$$X = \widehat{N}_{z_0,1}(z) \oplus \ldots \oplus \widehat{N}_{z_0,r}(z) \quad \forall z \in \overline{\theta}_{\delta,z_0},$$

where $\widehat{N}_{z_0,\sigma} = P_{z_0,\sigma}(z)X$.

(b) Remark 2.2.3(c) implies that the subspaces $\widehat{N}_{z_0,\sigma}(z)$ $(\sigma = 1,\ldots,r)$ are invariant with respect to $M(z)$ for every $z \in \overline{\theta}_{\delta,z_0}$.

(c) By Lemma 2.2.4, $\dim \widehat{N}_{z_0,\sigma}(z_0) = \dim \widehat{N}_{z_0,\sigma}(z)$ for all $z \in \overline{\theta}_{\delta,z_0}$ and $1 \leq \sigma \leq r$.

2.3. Properties of matrices. We first formulate four lemmas (used in Chapter 3 in the construction of a symmetrizer); their proofs are found in [13].

LEMMA 2.3.1. *Consider the $s \times s$-matrix*

$$(2.3.1) \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

The real symmetric matrices

$$(2.3.2) \qquad D = \begin{bmatrix} 0 & \ldots\ldots & 0 & d_{1s} \\ 0 & \ldots & 0 & d_{1s} & d_{2s} \\ \multicolumn{5}{c}{\dotfill} \\ 0 & d_{1s} & \ldots\ldots\ldots & \\ d_{1s} & d_{2s} & \ldots\ldots & d_{ss} \end{bmatrix}$$

are precisely those which symmetrize C, i.e. $DC = C^*D$ (C^* is the adjoint matrix to C).

LEMMA 2.3.2. *Consider a real symmetric $s \times s$-matrix B of the form*

$$B = \begin{bmatrix} b_{11} & \ldots & b_{1,s-1} & 0 \\ \multicolumn{4}{c}{\dotfill} \\ b_{s-1,1} & \ldots & b_{s-1,s-1} & 0 \\ 0 & \ldots & 0 & 0 \end{bmatrix}.$$

Then $BC = C^*B$ implies $B = 0$.

LEMMA 2.3.3. *Let E be a general real $s \times s$-matrix and consider $C + \varepsilon E$, where C is given by (2.3.1). Then for sufficiently small $0 \leq \varepsilon \leq \varepsilon_0$ and every matrix D of the form (2.3.2) there exists a real symmetric matrix B such that $(D + \varepsilon B)(C + \varepsilon E) = S$ is a symmetric matrix, i.e. $S = S^*$. Moreover, the elements of εB are continuous functions of εE.*

LEMMA 2.3.4. *Let C be the matrix (2.3.1). Then for every $d \geq 2$, there exists a real antisymmetric matrix*

$$(2.3.3) \qquad F = \begin{bmatrix} 0 & -f_{12} & 0 & \ldots & 0 & 0 \\ f_{12} & 0 & -f_{23} & \ldots & 0 & 0 \\ 0 & f_{23} & 0 & \ldots & 0 & 0 \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0 & 0 & 0 & \ldots & 0 & -f_{s-1,s} \\ 0 & 0 & 0 & \ldots & f_{s-1,s} & 0 \end{bmatrix}$$

such that

$$(2.3.4) \qquad FC + C^*F^* > \operatorname{diag}\left(-1, \tfrac{1}{2}d, \ldots, \tfrac{1}{2}d\right) .$$

The next two lemmas will be used in Chapter 3 to obtain an L_η^2-estimate for a solution of the boundary value problem.

LEMMA 2.3.5 ([15]). *Let A and B be hermitian $n \times n$-matrices such that for all $v \in \mathbf{C}^n$*

$$|v^*Bv| \leq v^*Av$$

(where v^* is the adjoint vector to v, i.e. $v^* = \bar{v}^T$). Then for all $v, w \in \mathbf{C}^n$,

$$|\operatorname{Re} w^*Bv| \leq (v^*Av)^{1/2}(w^*Aw)^{1/2} .$$

In the way analogous to the proof of Lemma 2.3.5 we shall prove

LEMMA 2.3.6. *Let A and B be hermitian $n \times n$-matrices for which there exists a linear function $f : \mathbf{C}^n \to \mathbf{C}^l$ $(n, l \geq 1)$ such that for any $v \in \mathbf{C}^n$,*

$$(2.3.5) \qquad |v^*Bv| \leq v^*Av + |f(v)|^2 .$$

Then for all $v, w \in \mathbf{C}^n$ and for any $\varepsilon > 0$,

$$(2.3.6) \qquad |\operatorname{Re} w^*Bv| \leq \frac{w^*Aw}{2\varepsilon} + \frac{|f(w)|^2}{2\varepsilon} + \frac{\varepsilon v^*Av}{2} + \frac{\varepsilon|f(v)|^2}{2} .$$

Proof. For any $v, w \in \mathbf{C}^n$ in view of assumption (2.3.5) we obtain

$$(v^* + w^*)(A + B)(v + w) + |f(v + w)|^2 \geq 0 ,$$
$$(v^* - w^*)(A - B)(v - w) + |f(v - w)|^2 \geq 0 .$$

Adding the above inequalities and using the linearity of f we get

$$(2.3.7) \qquad v^*Av + 2\operatorname{Re} w^*Bv + w^*Aw + |f(v)|^2 + |f(w)|^2 \geq 0 .$$

Consider the following cases:

$1°$ Either $v^*Av + |f(v)|^2 > 0$ or $w^*Aw + |f(w)|^2 > 0$. Assume for example that $v^*Av + |f(v)|^2 > 0$. Replacing in (2.3.7) v by vt gives

$$(v^*Av + |f(v)|^2)t^2 + 2\operatorname{Re} w^*Bvt + w^*Aw + |f(w)|^2 \geq 0 .$$

Hence

$$\Delta = 4(\operatorname{Re} w^* B v)^2 - 4(v^* A v + |f(v)|^2)(w^* A w + |f(w)|^2) \leq 0 \,.$$

So the assertion of the lemma is fulfilled.

2° $v^* A v + |f(v)|^2 = 0$ and $w^* A w + |f(w)|^2 = 0$. Then

$$2 \operatorname{Re} w^* B v t s = (v^* A v + |f(v)|^2) t^2 + 2 \operatorname{Re} w^* B v t s$$
$$+ (w^* A w + |f(w)|^2) s^2 = 0 \quad \forall s, t \in \mathbf{R} \,.$$

This completes the proof of the lemma. ∎

Now we introduce the definition of a matrix-valued function of simple structure and formulate a lemma from [12] which will be used in Section 3.6.

DEFINITION 2.3.1 ([12]). Let $A(z)$ be a matrix-valued function with entries $a_{ij}(z)$ $(1 \leq i, j \leq n)$ which are continuous functions defined in an open set $Q \subset \mathbf{R}^\nu$. We say that $A(z)$ *has simple structure* if

1) for each $z \in Q$, $A(z)$ has n linearly independent eigenvectors;

2) the multiplicities k_r of the roots λ of the characteristic equation $f(\lambda) = \det(A - \lambda I) = \prod_r (\lambda^{k_r} + q_{r1} \lambda^{k_r - 1} + \ldots + q_{r k_r})^r = 0$ $(\sum_r r k_r = n)$ are independent of z.

LEMMA 2.3.7 ([12]). *Let $A(z) = [a_{ij}(z)]_{1 \leq i,j \leq m}$ be a matrix-valued function of simple structure and continuous on $\bar{Q}$ ($Q \subset \mathbf{R}^\nu$ is an open set which can be written as $Q_1 \times Q_2$, where $Q_1 \subset \mathbf{R}^\mu$, $Q_2 \subset \mathbf{R}^{\nu-\mu}$). Assume that for all $z = (z_1, \ldots, z_\nu) \in Q$ the eigenvalues λ of $A(z)$ are such that $\operatorname{Re} \lambda = 0$ and a_{ij} and q_{rs} are analytic functions in $z_{II} = (z_{\mu+1}, \ldots, z_\nu) \in Q_2$ for any $z_I = (z_1, \ldots, z_\mu) \in Q_1$. Then for any $z_I^{(0)} \in Q_1$ there exist $\varepsilon > 0$ and a matrix-valued function $H(z)$ defined in $\mathfrak{M}_k = \{(z_I, z_{II}) : |z_I - z_I^0| < \varepsilon, z_{II} \in K \subset Q_2\}$ (where K is a compact subset of Q_2), which is hermitian, positive definite and satisfies*

$$H(z) A(z) + A^*(z) H(z) = 0 \quad \text{for all } z = (z_I, z_{II}) \in \mathfrak{M}_k \,.$$

($A^*(z)$ is the adjoint matrix to $A(z)$, i.e. $A^* = \bar{A}^T$). Moreover, the entries of $H(z)$ are polynomials in a_{ij}, $\bar{a}_{ij}$ and q_{rs}.

3. A boundary value problem for a linear hyperbolic system of the first order in a halfspace

In this chapter we shall consider the boundary value problem

$$(3.1) \qquad E(v) u_t - \sum_{j=1}^m A_j(v) u_{x_j} - C(v) u = F \quad \text{in } \mathbf{R} \times \Omega \,,$$

(3.2) $[I, -S(v)]u = g$ on $\mathbf{R} \times \partial\Omega$,

where

(3.3) $G(v) = G(t, x, v)$, $G = E, A_j \, (j = 1, \ldots, m), C, S$;

$\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$ $(m \geq 1)$;

$u_t = \partial u / \partial t$, $u_{x_j} = \partial u / \partial x_j$ $(j = 1, \ldots, m)$.

In (3.1)-(3.2), $u = [u_1, \ldots, u_n]^T$ is an unknown function; $E, A_j \, (j = 1, \ldots, m), C$ are given $n \times n$-matrix-valued functions defined for $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$; I is the $l \times l$ unit matrix $(1 \leq l \leq n)$; S is an $l \times (n-l)$-matrix-valued function defined for $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$; v and F are n-vector-valued functions defined for $(t, x) \in \mathbf{R} \times \bar{\Omega}$; g is an l-vector-valued function defined for $(t, x) \in \mathbf{R} \times \bar{\Omega}$.

3.1. Assumptions. Assume the following properties:

(3.1.1)   all functions given in (3.1)-(3.2) are real;

(3.1.2)   system (3.1) is *hyperbolic*, i.e. $\forall \omega = (\omega_1, \ldots, \omega_m) \in \mathbf{R}^m \setminus \{0\}$ and $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ the equation

$$\det \left(\lambda E(t, x, v) - \sum_{j=1}^m A_j(t, x, v)\omega_j \right) = 0$$

has real roots only;

(3.1.3) $\det E(t, x, v) \neq 0$, $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$;

(3.1.4) $\det A_1(t, x, v) \neq 0$, $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$;

(3.1.5) $A(t, x, v, \omega) = iE^{-1}(t, x, v) \sum_{j=1}^m A_j(t, x, v)\omega_j$ has simple structure on $\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times (\mathbf{R}^m \setminus \{0\})$, i.e. $\forall (t, x, v, \omega) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times (\mathbf{R}^m \setminus \{0\})$ it has n linearly independent eigenvectors and the multiplicities of the roots of the equation in (3.1.2) are independent of (t, x, v, ω);

(3.1.6a) $E, A_j \, (j = 1, \ldots, m), S \in C^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$, where $k \geq 1$;

(3.1.6b) $C \in C^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$, where $k \geq 1$;

(3.1.7) $E, A_j \, (j = 1, \ldots, m), C$ and S are constant outside a compact set $K_{t,x,v} \subset \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$, i.e. there exist constant matrices E', A_j' $(j = 1, \ldots, m), C'$ and S' such that $E(t, x, v) = E'$, $A_j(t, x, v) = A_j' \, (j = 1, \ldots, m)$, $C(t, x, v) = C'$, $S(t, x, v) = S'$, $\forall (t, x, v) \in (\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n) \setminus K_{t,x,v}$.

In order to examine problem (3.1)-(3.2) and to adopt some additional assumptions let us multiply both sides of (3.1) and (3.2) by $e^{-\eta t}$ (where

$\eta > 0$) and rewrite this problem as a system of pseudodifferential operators of the form

(3.1.8) $[E(v)(iD_t + \eta) - iB(v, D_{x'})]e^{-\eta t}u - C(v)e^{-\eta t}u - Fe^{-\eta t}$
$$= A_1(v)e^{-\eta t}u_{x_1},$$

(3.1.9) $[I, -S(v)]\begin{bmatrix} e^{-\eta t}u^I \\ e^{-\eta t}u^{II} \end{bmatrix} = e^{-\eta t}g,$

where

$$x' = (x_2, \ldots, x_m), \quad D_t = \frac{1}{i}\frac{\partial}{\partial t}, \quad D_{x'} = \frac{1}{i}\left(\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m}\right),$$

$$iB(v, D_{x'}) = \sum_{j=2}^{m} A_j(v)\frac{\partial}{\partial x_j}, \quad S(v) = S(t, x', v(t, x')),$$

$$u^I = [u_1, \ldots, u_l]^T, \quad u^{II} = [u_{l+1}, \ldots, u_n]^T.$$

Now, transform system (3.1.8) to the form

(3.1.10) $e^{-\eta t}u_{x_1} = M(v, iD_t + \eta, iD_{x'})e^{-\eta t}u - A_1^{-1}(v)C(v)e^{-\eta t}u$
$$- A_1^{-1}(v)Fe^{-\eta t},$$

where

(3.1.11) $M(v, iD_t + \eta, iD_{x'}) = M(t, x, v(t, x), iD_t + \eta, iD_{x'})$

(analogously to notation (3.3)).

The symbol of the operator $M(v, iD_t + \eta, iD_{x'})$ is the matrix-valued function

(3.1.12) $M(v, s, i\omega') = A_1^{-1}(v)\left(E(v)s - i\sum_{j=2}^{m} A(v)\omega_j\right),$

where $s = \eta + i\xi$, $\omega' = (\omega_2, \ldots, \omega_m)$; the variables ξ and ω_i $(i = 2, \ldots, m)$ are dual to t and x_i.

Notice that $u \in C_{(0)}^{\infty}(\mathbf{R} \times \bar{\Omega})$ is a solution of (3.1)–(3.2) iff $ue^{-\eta t}$ is a solution of (3.1.10)–(3.1.9). Therefore we now examine (3.1.10)–(3.1.9). To do this we consider the properties of the function

(3.1.13) $M(t, x, v, s, i\omega') = A_1^{-1}(t, x, v)\left[E(t, x, v)s - i\sum_{j=2}^{m} A_j(t, x, v)\omega_j\right]$

where $(t, x, v, s, \omega') \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times \{s \in \mathbf{C}^n : \text{Re } s \geq 0\} \times \mathbf{R}^{m-1}$.

From (3.1.2)–(3.1.3) it results that the matrix $E^{-1}(t, x, v)A_1(t, x, v)$ has nonzero real eigenvalues $\lambda_1(t, x, v), \ldots, \lambda_n(t, x, v)$ for all $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$.

Assume the following property:

(3.1.14) $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ the matrix $E^{-1}(t, x, v)A_1(t, x, v)$ has l negative and $n - l$ positive eigenvalues.

LEMMA 3.1.1. $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ and $\forall (s, \omega') \in \mathbf{C} \times \mathbf{R}^{m-1}$: $\operatorname{Re} s > 0$, the matrix M defined by (3.1.13) has exactly l eigenvalues κ such that $\operatorname{Re} \kappa < 0$ and $n - l$ eigenvalues κ with $\operatorname{Re} \kappa > 0$.

P r o o f. The proof is analogous to that of [8, Lemma 1]. Take $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ and consider the equation

$$(3.1.15) \quad \det(M - \kappa I) = \det A_1^{-1} \det \left(Es - \left(\kappa A_1 + i \sum_{j=2}^{m} A_j \omega_j \right) \right) = 0 \,,$$

where $\operatorname{Re} s > 0$. Suppose that $\kappa = i\omega_1$, where $\omega_1 \in \mathbf{R}$. Then in view of assumption (3.1.2) the equation $\det(Es - (i\omega_1 A_1 + i \sum_{j=2}^{m} A_j \omega_j)) = 0$ yields that s is purely imaginary, contrary to the assumption that $\operatorname{Re} s > 0$. Hence κ is not purely imaginary.

Put now in (3.1.15) $s = 1$ and $\omega' = 0$. We obtain

$$(3.1.16) \quad \det(A_1^{-1}E - \kappa I) = \det A_1^{-1} E \det(I - \kappa E^{-1}A_1)$$

$$= \det A_1^{-1} E \det(I - \kappa \widetilde{F}) = \det A_1^{-1} E \cdot \prod_{\sigma=1}^{n} (1 - \kappa \lambda_\sigma) = 0 \,,$$

where $\widetilde{F}$ is the Jordan canonical form of $E^{-1}A_1$. Equation (3.1.16) yields that $\kappa_\sigma = 1/\lambda_\sigma$, $\forall \sigma = 1, \dots, n$. Hence $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$, $M(t, x, v, 1, 0)$ has exactly l eigenvalues κ such that $\operatorname{Re} \kappa < 0$ and $n - l$ eigenvalues κ such that $\operatorname{Re} \kappa > 0$.

Now take any $1 \leq \sigma \leq n$ and assume that

(3.1.17) $\kappa_\sigma(t, x, v, 1, 0) > 0$ for a fixed $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$.

M is a linear function of the variables η, ξ and ω_j $(j = 2, \dots, m)$ for any $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$. Therefore for all $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ the eigenvalues κ_σ $(\sigma = 1, \dots, n)$ of M are continuous functions of the variables η, ξ and ω_j $(j = 2, \dots, m)$ ([10, Chap. II]). Consequently, since $\operatorname{Re} \kappa_\sigma(t, x, v, s, i\omega) \neq 0$, $\forall s : \operatorname{Re} s > 0$ and $\forall \omega' \in \mathbf{R}^{m-1}$, by (3.1.17) we get

$$\operatorname{Re} \kappa_\sigma(t, x, v, s, i\omega) > 0 \quad \forall \omega' \in \mathbf{R}^{m-1} \text{ and } \forall s : \operatorname{Re} s > 0 \,.$$

Since σ and (t, x, v) are arbitrary, the assertion of the lemma follows. ∎

Consider now the system of ordinary differential equations

$$(3.1.18) \qquad \frac{d\widetilde{u}}{dx_1} = M(t, 0, x', v, s, i\omega')\widetilde{u} \,,$$

where $\widetilde{u} = [\widetilde{u}_1, \dots, \widetilde{u}_n]^T$. Lemma 3.1.1 implies that for $\operatorname{Re} s > 0$ this system has exactly l linearly independent solutions which decay exponentially as $x_1 \to \infty$.

Let vector-valued functions $\varphi_1(t, x_1, v, s, i\omega'), \ldots, \varphi_l(t, x_1, v, s, i\omega')$ be a basis of the set of solutions with the above property, orthonormalized at $x_1 = 0$. Take any n-vector-valued function $\alpha \in C^0(\mathbf{R} \times \bar{\Omega})$ and define

$$(3.1.19) \quad N(t, x', \alpha, v, s, \omega') = [I, -S(t, 0, x', \alpha(t, 0, x'))]$$
$$\times [\varphi_1(t, x_1, x', v, s, \omega'), \ldots, \varphi_l(t, x_1, x', v, s, \omega')]|_{x_1=0} .$$

Assume

$(3.1.20)$ $\det N(t, x', \alpha, v, s, \omega')$ is a continuous function for $\operatorname{Re} s \geq 0$ and $\forall (t, x', v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n$ $\forall (s, \omega') \in \mathbf{C} \times \mathbf{R}^{m-1}$: $\operatorname{Re} s \geq 0$, $|s| + |\omega| \neq 0$,

$$|\det N(t, x', \alpha, v, s, \omega')| > \varepsilon^* ,$$

where $\varepsilon^* > 0$ is independent of t, x', v and α.

In the sequel we shall consider M defined by $(3.1.13)$ as a function of the variables t, x, v, η and ζ (where $\zeta = (\xi, \omega')$) and we shall denote it still by M, i.e.

$$(3.1.21) \quad M = M(t, x, v, \eta, \zeta)$$
$$= A_1^{-1}(t, x, v) \left[(\eta + i\xi) E(t, x, v) - i \sum_{j=2}^{m} A_j(t, x, v)\omega_j \right] .$$

(Notice that M is defined for all $\eta \in \mathbf{R}$.)

Take now any $\zeta_0 = (\xi, \omega_0') \neq 0$ (where $\xi_0 \in \mathbf{R}$, $\omega_0' \in \mathbf{R}^{m-1}$), $\eta = 0$ and $(t_0, x_0, v_0) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$. Set

$$(3.1.22) \qquad\qquad z_0 = (t_0, x_0, v_0, 0, \zeta_0) ,$$
$$(3.1.23) \qquad\qquad z = (t, x, v, \eta, \zeta) .$$

Let $\kappa_\sigma = \kappa_\sigma(z_0)$ (where $\sigma = 1, \ldots, r$, $1 \leq r \leq n$) be all distinct eigenvalues of $M(z_0)$.

R e m a r k 3.1.1. Assumption $(3.1.6)$ and Lemmas 2.2.1 and 2.2.2 yield that there exists $\delta > 0$ such that in the set $\bar{\theta}_{\delta,z_0} = \{ z \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times [0, \infty) \times \mathbf{R}^m : |z - z_0| < \delta \}$ there is defined the family of $n \times n$ total projection matrices of the κ_σ-groups such that $P_{z_0,\sigma} \in C^k(\bar{\theta}_{\delta,z_0})$ for $\sigma = 1, \ldots, r$ $(k \geq 1)$ and Remarks 2.2.3 and 2.2.4 are valid.

Assume

$(3.1.24)$ there exists $\delta > 0$ such that for $z \in \theta_{\delta,z_0}$ the eigenvalues of $M(z)$ in $\widehat{N}_{z_0,\sigma}(z)$ (see Remark 2.2.4(a)) either have real parts separated from zero or all tend to one purely imaginary value as $z \to z_0$; in the second case the geometric eigenspace of $M(z_0)$ corresponding to $\kappa_\sigma(z_0)$ is one-dimensional.

3.2. Construction of a symmetrizer. Consider all distinct eigenvalues κ_σ ($\sigma = 1, \ldots, r$) of $M(z_0)$ (notation (3.1.22)). Assume that p_1 of these eigenvalues have real parts less than $-\varepsilon$ (say, $\operatorname{Re}\kappa_\sigma < -\varepsilon$ for $\sigma = 1, \ldots, p_1$), p_2 have real parts greater than ε (say, $\operatorname{Re}\kappa_\sigma > \varepsilon$ for $\sigma = p_1 + 1, \ldots, p_1 + p_2$), while the remaining p_3 eigenvalues are purely imaginary ($\operatorname{Re}\kappa_\sigma = 0$ for $\sigma = p_1 + p_2 + 1, \ldots, p_1 + p_2 + p_3$, where $p_1 + p_2 + p_3 = r$).

LEMMA 3.2.1. *For any z_0 there exists a matrix T_0 such that*

$$(3.2.1) \qquad T_0 M T_0^{-1} = \begin{bmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \ldots & M_q \end{bmatrix} = \operatorname{diag}(M_1, \ldots, M_q)$$

where the matrices M_j for $j = 1, \ldots, q = 2 + p_3$ have the following properties:

(a) *the eigenvalues κ of M_1 and M_2 satisfy $\operatorname{Re}\kappa < -\varepsilon$ and $\operatorname{Re}\kappa > \varepsilon$ respectively; moreover,*

$$M_j = \operatorname{diag}(J_{1j}(\kappa_{1j}), \ldots, J_{r_j j}(\kappa_{r_j j})), \quad j = 1, 2 \,,$$

where

$$(3.2.2) \qquad J_{kj}(\kappa_{kj}) = \begin{bmatrix} \kappa_{kj} & \frac{\varepsilon}{2(n+1)^3}i & 0 & \ldots & 0 \\ 0 & \kappa_{kj} & \frac{\varepsilon}{2(n+1)^3}i & \ldots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & \kappa_{kj} \end{bmatrix} \,,$$

$k = 1, \ldots, r_j$, $r_j \geq p_j$ *for $j = 1, 2$ (i.e. the κ_{ij} in (3.2.2) can recur);*

(b) *the eigenvalues κ^j of M_j ($3 \leq j \leq q$) have zero real parts and*

$$(3.2.3) \qquad M_j = \begin{bmatrix} \kappa^j & i & 0 & \ldots & 0 \\ 0 & \kappa^j & i & \ldots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & \kappa^j \end{bmatrix} \,,$$

where $\kappa^j \neq \kappa^{j'}$ for $j \neq j'$ (we write here $\kappa^j = \kappa_{p_1+p_2+j-2}$ for $j = 3, \ldots, q$).

P r o o f. For T_0 we can take V_0^{-1}, where V_0 is the matrix formed by the coordinates of suitably chosen generalized eigenvectors of $M(z_0)$. Assumption (3.1.24) yields that $\kappa^j \neq \kappa^{j'}$ for $j \neq j'$ if $j, j' \geq 3$. ∎

R e m a r k 3.2.1. We may assume that the generalized eigenvectors of M corresponding to the eigenvalues κ with $\operatorname{Re}\kappa = 0$ are real.

LEMMA 3.2.2. *For any z_0 there exists $\delta > 0$ and a matrix-valued function $T = T(z)$ such that $T \in C_B^k(\theta_{\delta, z_0})$ (where $k \geq 1$) and*

$$(3.2.4) \qquad\qquad TMT^{-1} = \operatorname{diag}(M_1, \ldots, M_q) \,,$$

where $T(z_0) = T_0$ (T_0 is the matrix from Lemma 3.2.1).

Proof. Let $\kappa_\sigma = \kappa_\sigma(z_0)$ $(\sigma = 1, \ldots, r)$ be all distinct eigenvalues of $M(z_0)$, and let $P_{z_0,\sigma}(z)$ be the total projection matrices of the κ_σ-groups. By Lemma 3.2.1, κ_σ $(\sigma = 1, \ldots, p_1)$ are the eigenvalues of $M_1(z_0)$; κ_σ $(\sigma = p_1 + 1, \ldots, p_2)$ are the eigenvalues of $M_2(z_0)$; and $\kappa^j = \kappa_{p_1+p_2+j-2}$ $(j = 3, \ldots, q)$ are the eigenvalues of $M_j(z_0)$.

Next, define

$$P_{z_0}^1 = \sum_{\sigma=1}^{p_1} P_{z_0,\sigma}, \qquad P_{z_0}^2 = \sum_{\sigma=p_1+1}^{p_1+p_2} P_{z_0,\sigma},$$

$$P_{z_0}^j = P_{z_0,p_1+p_2+j-2}, \quad j = 3, \ldots, q \ (q = 2 + p_3).$$

By Remark 3.1.1, $\exists \delta > 0, P_{z_0}^j \in C^k(\overline{\theta}_{\delta,z_0})$ (where $k \geq 1$, $j = 1, \ldots, q$). Further, set $\widehat{N}_{z_0}^j(z) = P_{z_0}^j(z)\mathbf{C}^n$, $j = 1, \ldots, q$. Lemma 2.2.4 implies that $\dim \widehat{N}_{z_0}^j(z) = \dim \widehat{N}_{z_0}^j(z_0) = n_j$, $\forall z \in \overline{\theta}_{\delta,z_0}$, while by Remark 2.2.4(a) we get

$$\mathbf{C}^n = \widehat{N}_{z_0}^1(z_0) \oplus \ldots \oplus \widehat{N}_{z_0}^q(z_0), \quad \text{where} \sum_{j=1}^q n_j = n.$$

Choose in each $\widehat{N}_{z_0}^j(z_0)$ $(j = 1, \ldots, q)$ a basis $\{\widetilde{\varphi}_\nu^j : \nu = 1, \ldots, n_j\}$ such that in the basis $\{\widetilde{\varphi}_\nu^j : \nu = 1, \ldots, n_j, \ j = 1, \ldots, q\}$ of $\mathbf{C}^n$ the matrix $M(z_0)$ has the form (3.2.1).

Take now any $1 \leq j \leq q$. We shall find a basis $\{\widetilde{\varphi}_\nu^j(z) : \nu = 1, \ldots, n_j\}$ of $\widehat{N}_{z_0}^j(z)$ such that $\widetilde{\varphi}_\nu^j(z_0) = \widetilde{\varphi}_\nu^j$ and $\widetilde{\varphi}_\nu^j \in C^k(\overline{\theta}_{\delta,z_0}^j)$ $(\nu = 1, \ldots, n_j; k \geq 1)$, where $\overline{\theta}_{\delta,z_0}^j = \{z \in \overline{\theta}_{\delta,z_0} : |z - z_0| \text{ is so small that } |P_{z_0}^j(z) - P_{z_0}^j(z_0)| \leq \varepsilon\}$, $0 < \varepsilon < 1$. (As in Chapter 2, $|\cdot|$ denotes a matrix norm such that $|I| = 1$.) To do this let us apply the method from [10, Chap. II, §4, and Chap. I, §4].

First we shall find a matrix-valued function $U_j \in C^k(\overline{\theta}_{\delta,z_0}^j)$ (where $k \geq 1$) satisfying the following conditions:

1° $U_j(z)$ is invertible for each $z \in \overline{\theta}_{\delta,z_0}^j$ (then $U_j^{-1} \in C^k(\overline{\theta}_{\delta,z_0}^j)$);

2° $U_j(z)P_{z_0}^j(z_0)U_j^{-1}(z) = P_{z_0}^j(z)$, $\forall z \in \overline{\theta}_{\delta,z_0}^j$.

To do this consider $Q_j(z) = (P_{z_0}^j(z) - P_{z_0}^j(z_0))^2$, which satisfies

(3.2.5) $$P_{z_0}^j(z)Q_j(z) = Q_j(z)P_{z_0}^j(z).$$

We set

(3.2.6) $$\widetilde{U}_j(z) = P_{z_0}^j(z)P_{z_0}^j(z_0) + (I - P_{z_0}^j(z))(I - P_{z_0}^j(z_0)),$$

(3.2.7) $$\widetilde{V}_j(z) = P_{z_0}^j(z_0)P_{z_0}^j(z) + (I - P_{z_0}^j(z_0))(I - P_{z_0}^j(z)).$$

Then

(3.2.8) $$\widetilde{U}_j\widetilde{V}_j = \widetilde{V}_j\widetilde{U}_j = I - Q_j,$$

(3.2.9) $\widetilde{U}_j Q_j = Q_j \widetilde{U}_j\,,$

(3.2.10) $\widetilde{V}_j Q_j = Q_j \widetilde{V}_j\,.$

Next, define

(3.2.11) $U_j(z) = \widetilde{U}_j(z)(I - Q_j(z))^{-1/2} = (I - Q_j(z))^{-1/2}\widetilde{U}_j(z)\,,$

(3.2.12) $V_j(z) = \widetilde{V}_j(z)(I - Q_j(z))^{-1/2} = (I - Q_j(z))^{-1/2}\widetilde{V}_j(z)\,.$

They are defined and continuous in $\overline{\theta}_{\delta,z_0}^j$ which follows from the continuity in this set of the functions $P_{z_0}^j(z)$ and $(I - Q_j(z))^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-Q_j(z))^n$ (this series is uniformly convergent in $\overline{\theta}_{\delta,z_0}^j$). Moreover, the right-hand equalities in (3.2.11) and (3.2.12) result from (3.2.9) and (3.2.10).

Now, consider the function

$$\gamma_n(Q_j(z)) = \binom{-1/2}{n}(-1)^n(Q_j(z))^n\,.$$

Using (2.1.7) we obtain, for all $1 \le \rho \le k$,

$$\partial_z^\rho \gamma_n = \sum_{1 \le s \le \rho}\ \sum_{\substack{\alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = \rho \\ \alpha_1 + \ldots + \alpha_s = \rho + 1 - s}} c_{\alpha,s}\partial_{Q_j}^{\rho+1-s}\gamma_n \cdot (\partial_z^1 Q_j)^{\alpha_1}\cdot\ldots\cdot(\partial_z^s Q_j)^{\alpha_s}\,.$$

Fix $1 \le s \le \rho$. Then

$$\partial_{Q_j}^{\rho+1-s}\gamma_n = \begin{cases} 0 & \text{if } n < \rho + 1 - s, \\ n!(-1)^n\binom{-1/2}{n} & \text{if } n = \rho + 1 - s, \\ (-1)^n\binom{-1/2}{n}n\ldots(n-\rho+s)(Q_j(z))^{n-\rho+s-1} & \\ & \text{if } n > \rho + 1 - s. \end{cases}$$

Hence for all $n > \rho + 1$,

$$\left|\sum_{1 \le s \le \rho}\ \sum_{\substack{\alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = \rho \\ \alpha_1 + \ldots + \alpha_s = \rho + 1 - s}} c_{\alpha,s}\partial_{Q_j}^{\rho+1-s}\gamma_n \cdot (\partial_z^1 Q_j)^{\alpha_1}\cdot\ldots\cdot(\partial_z^s Q_j)^{\alpha_s}\right|$$

$$\le c_\rho\binom{-1/2}{n}n_\rho \varepsilon^{2(n-1-\rho)} \quad \text{in } \overline{\theta}_{\delta,z_0}^j\,.$$

The above inequality implies the uniform convergence of $\sum_{n=1}^{\infty}\partial_z^\rho\gamma_n$, $\forall 1 \le \rho \le k$. Therefore the sum of $\sum_{n=0}^{\infty}\binom{-1/2}{n}(-Q_j(z))^n$ has partial derivatives up to order k with respect to $t, x_1, \ldots, \xi_m$, which are continuous in $\overline{\theta}_{\delta,z_0}^j$. Hence by (3.2.6)–(3.2.7) and (3.2.11)–(3.2.12) we find that $U_j, V_j \in C^k(\overline{\theta}_{\delta,z_0}^j)$ (where $k \ge 1$).

We now check that U_j satisfies $1°$ and $2°$. Using (3.2.11), (3.2.12) and

(3.2.8) we get

$$U_j(z)V_j(z) = (I - Q_j(z))^{-1/2}\widetilde{U}_j(z)\widetilde{V}_j(z)(I - Q_j(z))^{-1/2}$$
$$= (I - Q_j(z))^{-1/2}(I - Q_j(z))(I - Q_j(z))^{-1/2} = I\,.$$

Similarly $V_j(z)U_j(z) = I$. Therefore $V_j = U_j^{-1}$, so $1°$ is fulfilled.

In order to check $2°$ notice that $\widetilde{U}_j(z)P_{z_0}^j(z_0) = P_{z_0}^j(z)\widetilde{U}_j(z)$. Hence using (3.2.5) and (3.2.11) we obtain

$$U_j(z)P_{z_0}^j(z_0) = (I - Q_j(z))^{-1/2}\widetilde{U}_j(z)P_{z_0}^j(z_0)$$

$$= (I - Q_j(z))^{-1/2}P_{z_0}^j(z)\widetilde{U}_j(z) = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-Q_j(z))^n P_{z_0}^j(z)\widetilde{U}_j(z)$$

$$= P_{z_0}^j(z)\sum_{n=0}^{\infty}\binom{-1/2}{n}(-Q_j(z))^n\widetilde{U}_j(z) = P_{z_0}^j(z)(I - Q_j(z))^{-1/2}\widetilde{U}_j(z)$$

$$= P_{z_0}^j(z)U_j(z) \quad \forall z \in \overline{\theta}_{\delta,z_0}^j\,.$$

Therefore $2°$ is fulfilled.

Conditions $1°$ and $2°$ imply that $U_j(z)$ maps one-to-one $\widehat{N}_{z_0}^j(z_0)$ onto $\widehat{N}_{z_0}^j(z)$. Hence the vectors

(3.2.13) $$\widetilde{\varphi}_\nu^j(z) = U_j(z)\widetilde{\varphi}_\nu^j, \quad \nu = 1,\ldots,n_j,$$

are a basis in $\widehat{N}_{z_0}^j(z)$ and $\widetilde{\varphi}_\nu^j \in C^k(\overline{\theta}_{\delta,z_0}^j)$ $(k \geq 1)$. Therefore $\{\widetilde{\varphi}_\nu^j(z) : \nu = 1,\ldots,n_j, \ j = 1,\ldots,q\}$ is a basis in $\mathbf{C}^n$ such that $\widetilde{\varphi}_\nu^j \in C^k(\overline{\theta}_{\delta_0,z_0})$ (where $\theta_{\delta_0,z_0} = \bigcap_{j=1}^q \theta_{\delta,z_0}^j$) and $\widetilde{\varphi}_\nu^j(z_0) = \widetilde{\varphi}_\nu^j$. Since the subspaces $\widehat{N}_{z_0}^j(z)$ $(1 \leq j \leq q)$ are $M(z)$-invariant (Remark 2.2.4), this clearly finishes the proof. Moreover, $M\widetilde{\varphi}_\nu^j = \sum_{s=1}^{n_j} m_{\nu s}^j\widetilde{\varphi}_s^j$, $j = 1,\ldots,q$, $\nu = 1,\ldots,n_j$, $M_j = [m_{\nu s}^j]$, and T^{-1} is formed by the coordinates of the vectors $\widetilde{\varphi}_\nu^j$ $(\nu = 1,\ldots,n_j;$ $j = 1,\ldots,q)$. ∎

R e m a r k 3.2.2. Lemma 3.2.2 yields that $M_j \in C_B^k(\theta_{\delta,z_0})$.

R e m a r k 3.2.3. Let $M_\mu \in C_B^k(\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n \times [0,\widetilde{\eta}) \times \omega)$ (where $[0,\widetilde{\eta}) \times \omega$ is a bounded subset of $\mathbf{R} \times \mathbf{R}^m$) be a sequence such that $M_\mu \to M$ in $C^k(\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n \times [0,\widetilde{\eta}) \times \omega)$ as $\mu \to 0$, satisfying the conditions of Lemma 2.2.3 and having the same properties as M. Lemma 2.2.3 and the proof of Lemma 3.2.2 imply that there exists a subsequence of (M_μ), which we shall still denote by (M_μ), such that for any z_0 (of the form (3.1.22)) there exist $\delta > 0$ and a sequence $T_\mu = T_\mu(z)$ with the following properties:

$1°$ $T_\mu \in C_B^k(\theta_{\delta,z_0})$;
$2°$ $T_\mu M_\mu T_\mu^{-1} = \text{diag}(M_{1\mu},\ldots,M_{q\mu})$;

$3°$ $T_\mu(z_0) = T_{0\mu}$, where $T_{0\mu}$ is the matrix transforming $M_\mu(z_0)$ to the canonical form from Lemma 3.2.1;

$4°$ $T_\mu \to T$ in $C_B^k(\theta_{\delta,z_0})$; moreover, $T_\mu = \widetilde{T}(M_\mu)$, $T = \widetilde{T}(M)$, where $\widetilde{T}$ is a function of class C^∞.

LEMMA 3.2.3. $\exists \delta > 0 \; \forall z \in \theta_{\delta,z_0}$

$$(3.2.14) \qquad\qquad M_1(z) + M_1^*(z) \leq -\widetilde{\varepsilon} I \,,$$

$$(3.2.15) \qquad\qquad M_2(z) + M_2^*(z) \geq \widetilde{\varepsilon} I \,,$$

where $\widetilde{\varepsilon} > 0$ is a constant, and M_1^* and M_2^* are the matrices adjoint to M_1 and M_2 respectively.

Proof. Take any $y = [y_1, \dots, y_n]^T \in \mathbf{C}^n$. Then

$$y^*(M_1 + M_1^*)y = 2\,\mathrm{Re}\,m_{11}^1 |y_1|^2 + \dots + 2\,\mathrm{Re}\,m_{n_1 n_1}^1 |y_{n_1}|^2$$
$$+ (m_{21}^1 + \overline{m}_{12}^1)\overline{y}_2 y_1 + \dots + (m_{n_1 1}^1 + \overline{m}_{1 n_1}^1)\overline{y}_{n_1} y_1 + (m_{12}^1 + \overline{m}_{21}^1)\overline{y}_1 y_2$$
$$+ \dots + (m_{n_1 2}^1 + \overline{m}_{2 n_1}^1)\overline{y}_{n_1} y_2 + \dots + (m_{1 n_1}^1 + \overline{m}_{n_1 1}^1)\overline{y}_1 y_{n_1}$$
$$+ (m_{2 n_1}^1 + \overline{m}_{n_1 2}^1)\overline{y}_2 y_{n_1} + \dots + (m_{n_1-1,n_1}^1 + \overline{m}_{n_1,n_1-1}^1)\overline{y}_{n_1-1} y_{n_1}$$
$$= 2\sum_{j=1}^{n_1} \mathrm{Re}\,m_{jj}^1 |y_j|^2 + 2\sum_{\substack{j,k=1 \\ j \neq k}}^{n_1} (m_{jk}^{1\prime} y_j' y_k' + m_{kj}^{1\prime} y_j' y_k' + m_{jk}^{1\prime\prime\prime} y_j'' y_k'$$
$$- m_{jk}^{1\prime\prime\prime} y_j'' y_k' + m_{kj}^{1\prime} y_j'' y_k'' + m_{jk}^{1\prime} y_j'' y_k'' + m_{kj}^{1\prime\prime\prime} y_j' y_k'' - m_{jk}^{1\prime\prime\prime} y_j' y_k'') \,,$$

where we have set

$$\mathrm{Re}\,m_{jk}^1 = m_{jk}^{1\prime}, \quad \mathrm{Im}\,m_{jk}^1 = m_{jk}^{1\prime\prime\prime}, \quad \mathrm{Re}\,y_j = y_j', \quad \mathrm{Im}\,y_j = y_j''$$

for $j,k = 1, \dots, n_1$, $j \neq k$.

In view of the continuity of M_1 (Remark 3.2.2) and the form of $M_1(z_0)$ (Lemma 3.2.1) and since $\mathrm{Re}\,\kappa_{k1} < -\varepsilon$ ($\kappa_{k1} = \kappa_{k1}(z_0)$, $k = 1, \dots, r_1$) we conclude that $\exists \delta > 0 \; \forall z \in \theta_{\delta,z_0}$

$$\mathrm{Re}\,m_{jj}^1 < -\varepsilon + \frac{\varepsilon}{2(n+1)^3} \quad (j = 1, \dots, n_1)\,,$$

$$|m_{j,j+1}^{1\prime\prime\prime}| < \frac{\varepsilon}{2(n+1)^3} + \frac{\varepsilon}{2(n+1)^3} \quad (j = 1, \dots, n_1 - 1)\,,$$

$$|m_{j,j+1}^{1\prime}| < \frac{\varepsilon}{2(n+1)^3} \quad (j = 1, \dots, n_1 - 1)\,,$$

$$|m_{jk}^{1\prime}|, |m_{jk}^{1\prime\prime\prime}| < \frac{\varepsilon}{2(n+1)^3} \quad (j,k = 1, \dots, n_1; k \neq j, j+1)\,.$$

Using the above inequalities and the Young inequality we get

$$y^*(M_1 + M_1^*)y \leq -\widetilde{\varepsilon}|y|^2 \,.$$

Therefore inequality (3.2.14) is satisfied. The proof of (3.2.15) is similar. ∎

R e m a r k 3.2.4. $\exists \mu_0 \; \forall \mu > \mu_0 \; \forall z \in \theta_{\delta,z_0}, M_{1\mu}(z) + M_{1\mu}^*(z) \le -\tilde{\varepsilon} I$ and $M_{2\mu}(z) + M_{2\mu}^*(z) \ge \tilde{\varepsilon} I$, where $\tilde{\varepsilon} > 0$ is a constant.

LEMMA 3.2.4. *For each $j = 3, \ldots, q$ there exist $\delta > 0$ and a matrix-valued function $U_j \in C^0(\theta_{\delta,z_0})$ satisfying the following properties:*

(a) $U_j(z_0) = I$,
(b) $\forall z \in \theta_{\delta,z_0}$, $U_j(z)$ *is a nonsingular matrix,*
(c) $U_j^{-1}(z)$ *is a bounded function in θ_{δ,z_0} such that*

$$(3.2.16) \qquad U_j M_j U_j^{-1} = \begin{bmatrix} \kappa_1^j & \gamma_{j1} & 0 & \ldots & 0 & 0 \\ 0 & \kappa_2^j & \gamma_{j2} & \ldots & 0 & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & 0 & \ldots & \kappa_{n_j-1}^j & \gamma_{j,n_j-1} \\ 0 & 0 & 0 & \ldots & 0 & \kappa_{n_j}^j \end{bmatrix},$$

where $\kappa_\sigma^j(z)$ $(\sigma = 1, \ldots, n_j)$ are the eigenvalues of $M_j(z)$ ordered so that those with $\mathrm{Re}\,\kappa_\sigma^j < 0$ for $\eta > 0$ are put at the beginning; $\gamma_{j1}, \ldots, \gamma_{j,n_j-1}$ are certain functions.

P r o o f. Fix $3 \le j \le q$ and consider the eigenvalue $\kappa_1^j(z)$ of $M_j(z)$. Then

$$M_j(z)\varphi_1(z) = \kappa_1^j(z)\varphi_1(z),$$

where $\varphi_1 = [\varphi_{11}, \ldots, \varphi_{1n_j}]^T$ is an eigenvector corresponding to κ_1^j. Therefore using the form (3.2.3) of $M_j(z_0)$ we get

(3.2.17)
$$\begin{bmatrix} \kappa^j - \kappa_1^j & i & 0 & \ldots & 0 \\ 0 & \kappa^j - \kappa_1^j & i & \ldots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & 0 & \ldots & \kappa^j - \kappa_1^j \end{bmatrix} \varphi_1 + (M_j(z) - M_j(z_0))\varphi_1 = 0.$$

If $z \to z_0$, then $\kappa^j - \kappa_1^j \to 0$ $(\kappa^j = \kappa^j(z_0))$. Put $\varphi_{11} = 1$ and rewrite (3.2.17) in the form

(3.2.18)
$$\begin{cases} (\kappa^j - \kappa_1^j) + i\varphi_{12} = m_{11}^j(z) - m_{11}^j(z_0) + [m_{12}^j(z) - m_{12}^j(z_0)]\varphi_{12} \\ \qquad\qquad + \ldots + [m_{1n_j}^j(z) - m_{1n_j}^j(z_0)]\varphi_{1n_j}, \\ (\kappa^j - \kappa_1^j)\varphi_{12} + i\varphi_{13} = m_{21}^j(z) - m_{21}^j(z_0) + [m_{22}^j(z) - m_{22}^j(z_0)]\varphi_{12} \\ \qquad\qquad + \ldots + [m_{2n_j}^j(z) - m_{2n_j}^j(z_0)]\varphi_{1n_j}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\kappa^j - \kappa_1^j)\varphi_{1n_j} = m_{n_j1}^j(z) - m_{n_j1}^j(z_0) + [m_{n_j2}^j(z) - m_{n_j2}^j(z_0)]\varphi_{12} \\ \qquad\qquad + \ldots + [m_{n_jn_j}^j(z) - m_{n_jn_j}^j(z_0)]\varphi_{1n_j}. \end{cases}$$

Drop the last equation of (3.2.18) and consider the system consisting of the remaining equations. Then $W(z) \to i^{n_j-1} \ne 0$ as $z \to z_0$, where

$W(z)$ denotes the principal determinant of the resulting system. Hence $\exists \delta > 0 \ \forall z \in \theta_{\delta, z_0}$, $W(z) \neq 0$, and there exist unique solutions $\varphi_{12}, \ldots, \varphi_{1n_j}$ which are continuous in θ_{δ, z_0} and $\varphi_1 \to [1, 0, \ldots, 0]^T$ as $z \to z_0$. By means of φ_1 we construct a matrix-valued function $U_{j1} \in C^0(\theta_{\delta, z_0})$ such that $U_{j1}^{-1}(z)$ is bounded in θ_{δ, z_0}, $U_{j1}(z_0) = I$ and

$$U_{j1} M_{j1} U_{j1}^{-1} = \begin{bmatrix} \kappa_{j1} & m^j \\ 0 & M_{j1} \end{bmatrix},$$

where m^j is an $(n_j - 1) \times 1$-matrix-valued function, and M_{j1} is an $(n_j - 1) \times (n_j - 1)$-matrix-valued function (the coordinates of φ_1 form the first column of U_{j1}). Further, we apply the same process to the matrix M_{j1} obtaining a function U_{j2} and so on. After $n_j - 1$ steps we get a function $\widetilde{U}_j$ of the form

$$\widetilde{U}_j = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & U_{j,n_j-1} \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & U_{j2} \end{bmatrix} U_{j1}$$

satisfying (a), (b), (c) and such that

$$(3.2.19) \qquad \widetilde{U}_j M_j \widetilde{U}_j^{-1} = \begin{bmatrix} \kappa_1^j & \mu_{12}^j & \mu_{13}^j & \cdots & \mu_{1,n_j-1}^j & \mu_{1n_j}^j \\ 0 & \kappa_2^j & \mu_{23}^j & \cdots & \mu_{2,n_j-1}^j & \mu_{2n_j}^j \\ 0 & 0 & \kappa_3^j & \cdots & \mu_{3,n_j-1}^j & \mu_{3n_j}^j \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & \kappa_{n_j-1}^j & \mu_{n_j-1,n_j}^j \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{n_j}^j \end{bmatrix}.$$

The continuity of M_j, $\widetilde{U}_j$ and $\widetilde{U}_j^{-1}$ implies the continuity of μ_{lk}^j in θ_{δ, z_0}.

Now we shall find functions t_{lk}^j which satisfy in θ_{δ, z_0} the equation

$$\begin{bmatrix} \kappa_1^j & \mu_{12}^j & \cdots & \mu_{1n_j}^j \\ 0 & \kappa_2^j & \cdots & \mu_{2n_j}^j \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & \mu_{n_j-1,n_j}^j \\ 0 & 0 & \cdots & \kappa_{n_j}^j \end{bmatrix} \begin{bmatrix} 1 & t_{12}^j & \cdots & t_{1n_j}^j \\ 0 & 1 & \cdots & t_{2n_j}^j \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & t_{n_j-1,n_j}^j \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t_{12}^j & \cdots & t_{1n_j}^j \\ 0 & 1 & \cdots & t_{2n_j}^j \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & t_{n_j-1,n_j}^j \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \kappa_1^j & \gamma_{j1} & \cdots & 0 \\ 0 & \kappa_2^j & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & \gamma_{j,n_j-1} \\ 0 & 0 & \cdots & \kappa_{n_j}^j \end{bmatrix}.$$

The above equation can be rewritten in the form of the system

$$\kappa_k^j t_{k,k+1}^j + \mu_{k,k+1}^j = \gamma_{jk} + \kappa_{k+1}^j t_{k,k+1}^j, \quad k = 1,\dots,n_j - 1,$$

$$\kappa_k^j t_{k,k+1}^j + \sum_{\nu=0}^{l-2} \mu_{k,k+1+\nu}^j t_{k+1+\nu,k+1}^j + \mu_{k,k+1}^j$$

$$= \kappa_{k+l-1}^j t_{k,k+l-1}^j + \kappa_{k+1}^j t_{k,k+1}^j,$$
$$k = 1,\dots,n_j - 2;\ l = 2,\dots,n_j - k.$$

Considering successively the equations of the above system we conclude that it is fulfilled by the functions:

$$t_{12}^j = 0, \quad \gamma_{j1} = \mu_{12}^j,$$

$$t_{13}^j = 0, \quad t_{23}^j = -\frac{\mu_{13}^j}{\mu_{12}^j}, \quad \gamma_{j2} = (\kappa_2^j - \kappa_3^j)t_{23}^j + \mu_{23}^j,$$

$$t_{14}^j = 0, \quad t_{24}^j = \frac{\begin{vmatrix} -\mu_{14}^j & \mu_{13}^j \\ -\mu_{24}^j + \mu_{34}^j t_{23}^j & \mu_{23}^j - (\kappa_3^j - \kappa_4^j)t_{23}^j \end{vmatrix}}{\begin{vmatrix} \mu_{12}^j & \mu_{13}^j \\ \kappa_2^j - \kappa_4^j & \mu_{23}^j - (\kappa_3^j - \kappa_4^j)t_{23}^j \end{vmatrix}},$$

$$t_{34}^j = \frac{\begin{vmatrix} \mu_{12}^j & -\mu_{14}^j \\ \kappa_2^j - \kappa_4^j & -\mu_{24}^j + \mu_{34}^j t_{23}^j \end{vmatrix}}{\begin{vmatrix} \mu_{12}^j & \mu_{13}^j \\ \kappa_2^j - \kappa_4^j & \mu_{23}^j - (\kappa_3^j - \kappa_4^j)t_{23}^j \end{vmatrix}},$$

. . . .

$$t_{1n_j}^j = 0,$$

$$t_{2n_j}^j = \frac{\begin{vmatrix} -\mu_{1n_j}^j & \cdots & \mu_{1,n_j-1}^j \\ -\mu_{2n_j}^j + \mu_{n_j-1,n_j}^j t_{2,n_j-1}^j & \cdots & \mu_{2,n_j-1}^j - (\kappa_{n_j-1}^j - \kappa_{n_j}^j)t_{2,n_j-1}^j \\ \cdots\cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots \\ -\mu_{n_j-2,n_j}^j + \mu_{n_j-1,n_j}^j t_{n_j-2,n_j-1}^j & \cdots & \mu_{n_j-2,n_j-1}^j - (\kappa_{n_j-1}^j - \kappa_{n_j}^j)t_{n_j-2,n_j-1}^j \end{vmatrix}}{\begin{vmatrix} \mu_{12}^j & \cdots & \mu_{1,n_j-1}^j \\ \kappa_2^j - \kappa_{n_j}^j & \cdots & \mu_{2,n_j-1}^j - (\kappa_{n_j-1}^j - \kappa_{n_j}^j)t_{2,n_j-1}^j \\ \cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots \\ 0 & \cdots & \mu_{n_j-2,n_j-1}^j - (\kappa_{n_j-1}^j - \kappa_{n_j}^j)t_{n_j-2,n_j-1}^j \end{vmatrix}},$$

. . . .

$$t^j_{n_j-1,n_j} = \frac{\begin{vmatrix} \mu^j_{12} & \mu^j_{13} & \cdots & -\mu^j_{1n_j} \\ \kappa^j_2 - \kappa^j_{n_j} & \mu^j_{23} & \cdots & -\mu^j_{2n_j} + \mu^j_{n_j-1,n_j} t^j_{2,n_j-1} \\ \hdotsfor{4} \\ \kappa^j_{n_j-1} - \kappa^j_{n_j} & 0 & \cdots & -\mu^j_{n_j-2,n_j} + \mu^j_{n_j-1,n_j} t^j_{n_j-2,n_j-1} \end{vmatrix}}{\begin{vmatrix} \mu^j_{12} & \mu^j_{13} & \cdots & \mu^j_{1,n_j-1} \\ \kappa^j_2 - \kappa^j_{n_j} & \mu^j_{23} & \cdots & \mu^j_{2,n_j-1} - (\kappa^j_{n_j-1} - \kappa^j_{n_j}) t^j_{2,n_j-1} \\ 0 & \kappa^j_3 - \kappa^j_{n_j} & \cdots & \mu^j_{3,n_j-1} - (\kappa^j_{n_j-1} - \kappa^j_{n_j}) t^j_{3,n_j-1} \\ \hdotsfor{4} \\ 0 & 0 & \cdots & \mu^j_{n_j-2,n_j-1} - (\kappa^j_{n_j-1} - \kappa^j_{n_j}) t^j_{n_j-2,n_j-1} \end{vmatrix}},$$

$$\gamma_{j,n_j-1} = (\kappa^j_{n_j-1} - \kappa^j_{n_j}) t^j_{n_j-1,n_j} + \mu^j_{n_j-1,n_j}.$$

Thus we have found a function

$$\widetilde{T}_j = \begin{bmatrix} 1 & t^j_{12} & t^j_{13} & \cdots & t^j_{1n_j} \\ 0 & 1 & t^j_{23} & \cdots & t^j_{2n_j} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \cdots & t^j_{n_j-1,n_j} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

with $\widetilde{T}_j \in C^0(\theta_{\delta,z_0})$ and $\widetilde{T}_j(z_0) = I$ which transforms the matrix on the right-hand side of (3.2.19) to the form in (3.2.16). Hence $U_j(z) = \widetilde{T}_j(z)\widetilde{U}_j(z)$ satisfies the assertion of the lemma. ∎

R e m a r k 3.2.5. Let (M_μ) be a sequence as in Remark 3.2.3. The proof of Lemma 3.2.4 yields that $\exists \delta > 0$ $\exists \mu_0$ $\forall \mu > \mu_0$ and for each $j = 3,\ldots,q$ there exists a function $U_{j\mu} \in C^0(\theta_{\delta,z_0})$ satisfying (a)–(c) and (3.2.16) with M_j replaced by $M_{j\mu}$ and with κ^j_k, $\gamma_{j,k-1}$ (where $k = 1,\ldots,n_j$) replaced by $\kappa^j_{k\mu}$, $\gamma^\mu_{j,k-1}$, respectively. Moreover, $U_{j\mu} \to U_j$ in $C^0(\theta_{\delta,z_0})$ $(j = 3,\ldots,q)$.

LEMMA 3.2.5. *Fix* $(t_0,x_0,v_0) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$. *Then* $\exists K > 0$ $\forall \zeta = (\xi,\omega')$: $\xi \in \mathbf{R}, \omega' \in \mathbf{R}^{m-1}$ $\forall \eta > 0$ $\forall \rho \in \mathbf{R}$

$$|(M(t_0,x_0,v_0,\eta,\zeta) - i\rho I)^{-1}| \leq K\eta^{-1}|A_1(t_0,x_0,v_0)|.$$

P r o o f. Consider the system

$$(3.2.20) \quad E(t_0,x_0,v_0)u_t + \sum_{j=1}^m A_j(t_0,x_0,v_0)u_{x_j} = 0 \quad \text{in } [0,T] \times \mathbf{R}^m,$$

where $T > 0$ is a constant. By (3.1.2), (3.1.3) and (3.1.5) the Cauchy problem for the above system is well posed (see [12]), i.e.

$$\exists K > 0 \quad \|u(t,x)\|_{0,[0,T] \times \mathbf{R}^m} \leq K\|u(0,x)\|_{0,\mathbf{R}^m}$$

for a solution $u \in C_{(0)}^{\infty}([0,T] \times \mathbf{R}^m)$ of (3.2.20). Therefore the assertion of the lemma results from [13, Theorem 2.1.1 and Lemma 2.2]. ∎

Now consider the functions M_j $(3 \leq j \leq q)$ from Lemma 3.2.2. Fix $3 \leq j \leq q$ and apply to $M_j(t,x,v,\eta,\zeta)$ the Taylor formula with respect to $\eta \in \theta' = \{\eta : (t,x,v,\eta,\zeta) \in \theta_{\delta,z_0}\}$ (in a neighbourhood of $\eta = 0$) with remainder of the second order, and next to $M_j(t,x,v,0,\zeta)$ and $M_{j,\eta}(t,x,v,0,\zeta)$ (which appear in the resulting formula) the Taylor formula with respect to $(t,x,v) \in \theta'' = \{(t,x,v) : (t,x,v,0,\zeta) \in \theta_{\delta,z_0}\}$ (in a neighbourhood of (t_0,x_0,v_0)) with remainder of the first order. Thus we get

$$
\begin{aligned}
(3.2.21) \quad M_j(t,x,v,\eta,\zeta) = {}& M_j(t_0,x_0,v_0,0,\zeta) + \eta L_j(t_0,x_0,v_0,0,\zeta) \\
& + O(\eta^2) + O(|t-t_0|) + O(|x-x_0|) + O(|v-v_0|) \\
& + O(\eta|t-t_0|) + O(\eta|x-x_0|) + O(\eta|v-v_0|) \,,
\end{aligned}
$$

where $L_j = L_j(t_0,x_0,v_0,0,\zeta) = M_{j,\eta}(t_0,x_0,v_0,0,\zeta)$.

PROPOSITION 3.2.1. *The $n_j \times n_j$-matrix $L_j = [l_{\nu s}]$ is real for any $\zeta \in \theta''' = \{\zeta : (t_0,x_0,v_0,0,\zeta) \in \theta_{\delta,z_0}\}$.*

P r o o f. Recall that (Lemma 3.2.2)

$$
(3.2.22) \qquad M\widetilde{\varphi}_{\nu}^{j} = \sum_{s=1}^{n_j} m_{\nu s}^{j} \widetilde{\varphi}_{s}^{j}, \quad \nu = 1,\dots,n_j \,.
$$

Differentiating (3.2.22) with respect to η we get

$$
(3.2.23) \qquad M_{,\eta}\widetilde{\varphi}_{\nu}^{j} + M\widetilde{\varphi}_{\nu,\eta}^{j} = \sum_{s=1}^{n_j} (m_{\nu s,\eta}^{j}\widetilde{\varphi}_{s}^{j} + m_{\nu s}^{j}\widetilde{\varphi}_{s,\eta}^{j}) \,, \quad \nu = 1,\dots,n_j \,.
$$

To simplify notation, set $\widetilde{z}_0 = (t_0,x_0,v_0)$. Fix the functions occurring in (3.2.22) at $(\widetilde{z}_0,0,\zeta) = (t_0,x_0,v_0,0,\zeta)$. Using (3.2.6) and (3.2.11) we have

$$
\begin{aligned}
\widetilde{\varphi}_{\nu}^{j}(\widetilde{z}_0,0,\zeta) = {}& [I - (P_{z_0}^{j}(\widetilde{z}_0,0,\zeta) - P_{z_0}^{j}(z_0))^2]^{-1/2}[P_{z_0}^{j}(\widetilde{z}_0,0,\zeta)P_{z_0}^{j}(z_0) \\
& + (I - P_{z_0}^{j}(\widetilde{z}_0,0,\zeta))(I - P_{z_0}^{j}(z_0))]\widetilde{\varphi}_{\nu}^{j}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\varphi}_{\nu,\eta}^{j}(\widetilde{z}_0,0,\zeta) = {}& 2\sum_{n=0}^{\infty}\binom{-1/2}{n} n(-(P_{z_0}^{j}(\widetilde{z}_0,0,\zeta) - P_{z_0}^{j}(z_0))^2)^{n-1} \\
& \times (P_{z_0}^{j}(\widetilde{z}_0,0,\zeta) - P_{z_0}^{j}(z_0))P_{z_0,\eta}^{j}(\widetilde{z}_0,0,\zeta) \\
& \times [P_{z_0}^{j}(\widetilde{z}_0,0,\zeta)P_{z_0}^{j}(z_0) + (I - P_{z_0}^{j}(\widetilde{z}_0,0,\zeta))(I - P_{z_0}^{j}(z_0))] \\
& + [I - (P_{z_0}^{j}(\widetilde{z}_0,0,\zeta) - P_{z_0}^{j}(z_0))^2]^{-1/2}[2P_{z_0,\eta}^{j}(\widetilde{z}_0,0,\zeta)P_{z_0}^{j}(z_0) \\
& - P_{z_0,\eta}^{j}(\widetilde{z}_0,0,\zeta)]\widetilde{\varphi}_{\nu}^{j} \,.
\end{aligned}
$$

It can be proved ([13]) that the matrix

$$P_{z_0}^j(\tilde{z}_0, 0, \zeta) = -(2\pi i)^{-1} \int_{\Gamma_j} (M(\tilde{z}_0, 0, \zeta) - \beta I)^{-1} \, d\beta$$

is real for any $\zeta \in \theta'''$, while the matrix

$$P_{z_0, \eta}^j(\tilde{z}_0, 0, \zeta)$$
$$= (2\pi i)^{-1} \int_{\Gamma_j} (M(\tilde{z}_0, 0, \zeta) - \beta I)^{-1} A_1^{-1}(\tilde{z}_0) E(\tilde{z}_0)(M(\tilde{z}_0, 0, \zeta) - \beta I)^{-1} \, d\beta$$

is purely imaginary in θ''' (we have used here [10, formula (4.28)]). Therefore the vectors $\tilde{\varphi}_\nu^j(\tilde{z}_0, 0, \zeta)$ are real, while the vectors $\tilde{\varphi}_{\nu, \eta}^j(\tilde{z}_0, 0, \zeta)$ are purely imaginary. Moreover, the matrix $M_{,\eta}(\tilde{z}_0, 0, \zeta) = A_1^{-1}(\tilde{z}_0) E(\tilde{z}_0)$ is real and $M(\tilde{z}_0, 0, \zeta) = iA_1^{-1}(\tilde{z}_0)[E(\tilde{z}_0)\xi - \sum_{i=2}^m A_i(\tilde{z}_0)\omega_i]$ is purely imaginary.

We now prove that for any $\zeta \in \theta'''$ the $m_{il}^j(\tilde{z}_0, 0, \zeta)$ are purely imaginary. The left sides of equations (3.2.22) are purely imaginary. Hence

$$\sum_{s=1}^{n_j} \operatorname{Re} m_{\nu s}^j \tilde{\varphi}_s^j = 0, \quad \nu = 1, \ldots, n_j .$$

Therefore in view of the linear independence of $\tilde{\varphi}_\nu^j$ ($\nu = 1, \ldots, n_j$) we obtain

$$\operatorname{Re} m_{il}^j(\tilde{z}_0, 0, \zeta) = 0 \quad \text{for } i, l = 1, \ldots, n_j .$$

Now consider the system (3.2.23). Since the vectors $M_{,\eta} \tilde{\varphi}_\nu^j + M \tilde{\varphi}_{\nu, \eta}^j - m_{\nu 1}^j \tilde{\varphi}_{1, \eta}^j - \ldots - m_{\nu n_j}^j \tilde{\varphi}_{n_j, \eta}^j$ ($\nu = 1, \ldots, n_j$) are real, from (3.2.23) it results that $\operatorname{Im} m_{il, \eta}^j = 0$ for $i, l = 1, \ldots, n_j$. Therefore $L_j(\tilde{z}_0, 0, \zeta)$ is real $\forall \zeta \in \theta'''$. ∎

Introduce the following notations:

(3.2.24) $|\bar{\zeta}|^2 = |\xi|^2 + |\omega'|^2 + \eta^2 = |\zeta|^2 + \eta^2 ,$

(3.2.25) $\zeta' = \zeta/|\bar{\zeta}|, \eta' = \eta/|\bar{\zeta}| ,$

(3.2.26) $\tilde{z} = (t, x, v) ,$

and, as above,

(3.2.27) $\tilde{z}_0 = (t_0, x_0, v_0) .$

Next let θ_{δ, z_0} be the set from Lemma 3.2.2 (where $z_0 = (\tilde{z}_0, 0, \zeta_0')$, $|\zeta_0'| = 1$) and define

$$\mathfrak{M} = \{z \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times [\eta_1, \infty) \times \mathbf{R}^m : |\bar{\zeta}| \geq 1/2, (\tilde{z}, \eta', \zeta') \in \theta_{\delta, z_0}\}, \eta_1 > 0 .$$

Then Lemma 3.2.2 implies the existence of $\hat{T} \in C_B^k(\mathfrak{M})$ such that

(3.2.28) $\hat{T}(z) M(z) \hat{T}^{-1}(z) = \operatorname{diag}(M_1(z), \ldots, M_q(z))$

for $z \in \mathfrak{M}$ (z is defined by (3.1.23)), where $M_j(z) = M_j(\tilde{z}, \eta, \zeta) = |\bar{\zeta}| M_j(\tilde{z}, \eta', \zeta')$ ($j = 1, \ldots, q$), $\hat{T}(z) = T(\tilde{z}, \eta', \zeta')$. According to (3.2.21),

for each $3 \leq j \leq q$,

$$(3.2.29) \qquad M_j(z) = |\tilde{\zeta}|(M_j(\tilde{z}_0, 0, \zeta') + \eta' L_j(\tilde{z}_0, 0, \zeta') + O(\eta'^2)$$
$$+ O(|\tilde{z} - \tilde{z}_0|) + O(\eta'|\tilde{z} - \tilde{z}_0|)).$$

Fix $3 \leq j \leq q$ and consider $M_j(\tilde{z}, \eta', \zeta')$ for $(\tilde{z}, \eta', \zeta')$ belonging to a certain neighbourhood θ_{δ, z_0} of $z_0 = (\tilde{z}_0, 0, \zeta'_0)$, where $|\zeta'_0| = 1$.

LEMMA 3.2.6. *Let $n_j \geq 2$. Then*

1° $\exists c > 0 \ \exists \delta > 0 \ \forall (\tilde{z}, \eta', \zeta') \in \theta_{\delta, z_0}, \ |l_{n_j 1}| \geq c$, *where $l_{n_j 1}$ is the lower left corner entry of $L_j(\tilde{z}, \eta', \zeta') = M_{j, \eta'}(\tilde{z}, \eta', \zeta')$;*

2° *the eigenvalues of $M_j(\tilde{z}, \eta', \zeta'_0)$ have the form*

$$(3.2.30) \quad \kappa_\nu^j(\tilde{z}, \eta', \zeta'_0) = \kappa^j(\tilde{z}_0, 0, \zeta'_0) + (i^{n_j - 1} l_{n_j 1} \eta')^{1/n_j} + O(\eta'^{2/n_j})$$
$$+ O(|\tilde{z} - \tilde{z}_0|^{1/n_j}) + O(\eta'^{1/n_j} |\tilde{z} - \tilde{z}_0|^{1/n_j}), \quad \nu = 1, \ldots, n_j$$

(here $(\tilde{z}, \eta', \zeta'_0) \in \theta_{\delta, z_0}$);

3° $\exists \delta > 0 \ \forall (\tilde{z}, \eta', \zeta') \in \{(\tilde{z}, \eta', \zeta') \in \theta_{\delta, z_0} : \eta' > 0\}$ *there exist exactly*

$$(3.2.31) \qquad\qquad \rho_j = \begin{cases} \frac{1}{2} n_j, \\ \frac{1}{2}(n_j - 1), \\ \frac{1}{2}(n_j + 1) \end{cases}$$

eigenvalues κ_ν^j of $M_j(\tilde{z}, \eta', \zeta')$ with $\operatorname{Re} \kappa_\nu^j < 0$ according as

$$\begin{cases} n_j & \text{is even,} \\ n_j & \text{is odd and } l_{n_j 1} > 0, \\ n_j & \text{is odd and } l_{n_j 1} < 0. \end{cases}$$

P r o o f. 1° First we prove that $\exists c > 0$ such that $|l_{n_j 1}| \geq c$ for $(\tilde{z}, \eta', \zeta') = z_0 = (z'_0, 0, \zeta'_0)$. To do this notice that $\kappa^j = \kappa^j(\tilde{z}_0, 0, \zeta'_0)$ is an eigenvalue of $M_j(\tilde{z}_0, 0, \zeta'_0)$ iff $|\tilde{\zeta}| \kappa^j(\tilde{z}_0, 0, \zeta'_0) = \kappa^j(\tilde{z}_0, 0, \zeta)$ is an eigenvalue of $M_j(\tilde{z}_0, 0, \zeta)$ (where $\zeta = |\tilde{\zeta}| \zeta'_0, \ |\zeta'_0| = 1, \ |\tilde{\zeta}| \geq 1/2$). Hence using (3.2.29) with $\zeta' = \zeta'_0$ and (3.2.3) we get

$$(3.2.32) \quad \prod_{\nu=1}^{n_j} |\kappa_\nu^j(\tilde{z}, \eta, \zeta) - \kappa^j(z_0, 0, \zeta)| = |\det(M_j(\tilde{z}, \eta, \zeta) - \kappa^j(\tilde{z}_0, 0, \zeta))|$$

$$= |\det(|\tilde{\zeta}|(M_j(\tilde{z}_0, 0, \zeta'_0) - \kappa^j(\tilde{z}_0, 0, \zeta'_0) + \eta' L_j(\tilde{z}_0, 0, \zeta'_0)$$
$$+ O(\eta'^2) + O(|\tilde{z} - \tilde{z}_0|) + O(\eta'|\tilde{z} - \tilde{z}_0|)))|$$

$$= \left| \det \left(|\tilde{\zeta}| \begin{bmatrix} 0 & i & 0 & \ldots & 0 \\ 0 & 0 & i & \ldots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} + \eta \begin{bmatrix} l_{11} & \ldots & l_{1 n_j} \\ \multicolumn{3}{c}{\dotfill} \\ l_{n_j 1} & \ldots & l_{n_j n_j} \end{bmatrix} \right. \right.$$

$$\left. \qquad + |\tilde{\zeta}|O(\eta^2) + |\tilde{\zeta}|O(|\tilde{z} - \tilde{z}_0|) + |\tilde{\zeta}|O(\eta'|\tilde{z} - \tilde{z}_0|) \right)\right\|$$

$$= \eta|\tilde{\zeta}|^{n_j-1}|l_{n_j 1}| + O(\eta^2|\tilde{\zeta}|^{n_j-2})$$
$$+ O(|\tilde{z} - \tilde{z}_0||\tilde{\zeta}|^{n_j}) + O(\eta|\tilde{z} - \tilde{z}_0||\tilde{\zeta}|^{n_j-1}),$$

where $z = (\tilde{z}, \eta, \zeta) \in \mathfrak{M}$, $\zeta = |\tilde{\zeta}|\zeta_0'$. Using (3.2.28) we obtain

$$M(\tilde{z}, \eta, \zeta) - \kappa^j(\tilde{z}_0, 0, \zeta)I$$
$$= T^{-1}(\tilde{z}, \eta', \zeta_0') \operatorname{diag}(M_1(\tilde{z}, \eta, \zeta) - \kappa^j(\tilde{z}_0, 0, \zeta)I, \ldots,$$
$$M_q(\tilde{z}, \eta, \zeta) - \kappa^j(\tilde{z}_0, 0, \zeta)I)T(\tilde{z}, \eta', \zeta_0').$$

Hence

$$T^{-1}[\operatorname{diag}(M_1 - \kappa^j I, \ldots, M_q - \kappa^j I)]^{-1}T = (M - \kappa^j I)^{-1}$$

and further

$$(3.2.33) \quad \operatorname{diag}((M_1 - \kappa^j I)^{-1}, \ldots, (M_q - \kappa^j I)^{-1}) = T(M - \kappa^j I)^{-1}T^{-1}.$$

Putting now $\tilde{z} = \tilde{z}_0$ in the matrices on both sides of (3.2.33), denoting by $(M_j(\tilde{z}_0, \eta, \zeta) - \kappa^j I)^{-1}_{1n_j}$ the upper right corner entry of $(M_j(\tilde{z}_0, \eta, \zeta) - \kappa^j I)^{-1}$ and applying Lemma 3.2.5 we get

$$(3.2.34) \quad |(M_j(\tilde{z}_0, \eta, \zeta) - \kappa^j I)^{-1}_{1n_j}| \leq |T||(M(\tilde{z}_0, \eta, \zeta) - \kappa^j I)^{-1}||T^{-1}|$$
$$\leq K|T(\tilde{z}, \eta', \zeta_0')||T(\tilde{z}, \eta', \zeta_0')|^{-1}|A_1(\tilde{z}_0)|\eta^{-1} \leq c\eta^{-1}$$

for a constant $c > 0$ depending on δ from Lemma 3.2.2, on $\tilde{z}_0$ and on ζ_0'.

On the other hand, using (3.2.32) we have

$$|(M_j(\tilde{z}_0, \eta, \zeta) - \kappa^j I)^{-1}_{1n_j}| = \frac{|\tilde{\zeta}|^{n_j-1} + O(\eta|\tilde{\zeta}|^{n_j-1})}{\eta|\tilde{\zeta}|^{n_j-1}|l_{n_j 1}| + O(\eta^2|\tilde{\zeta}|^{n_j-2})}.$$

Hence by (3.2.34) $\exists c > 0$, $|l_{n_j 1}| \geq c$. From the continuity of $L_j(\tilde{z}, \eta', \zeta') = M_{j,\eta'}(\tilde{z}, \eta', \zeta')$ in θ_{δ, z_0} it follows that $\exists c > 0$ such that the above inequality is satisfied in a neighbourhood of z_0, which for simplicity we shall still denote by θ_{δ, z_0}.

2° The characteristic equation for $M_j(\tilde{z}, \eta, \zeta)$ (where $(\tilde{z}, \eta, \zeta) \in \mathfrak{M}$, $\zeta = |\tilde{\zeta}|\zeta_0'$) has the form

$$\det(M_j - \kappa) = \det(M_j - \kappa^j - (\kappa - \kappa^j)) = (\kappa - \kappa^j)^{n_j}$$
$$+ W_1(M_j)(\kappa - \kappa_j)^{n_j-1} + \ldots + W_{n_j-1}(M_j)(\kappa - \kappa_j) + \det(M_j - \kappa^j) = 0,$$

where $W_k(M_j)$ $(k = 1, \ldots, n_j - 1)$ are polynomials of degree k with respect to the entries of M_j. This equation is satisfied by $\kappa = \kappa^j_\nu(\tilde{z}, \eta', \zeta_0')|\tilde{\zeta}|$, where

$\kappa_\nu^j(\tilde{z}, \eta', \zeta_0')$ is given by (3.2.30). Hence the $\kappa_\nu^j(\tilde{z}, \eta', \zeta_0')$ given by (3.2.30) are the eigenvalues of $M_j(\tilde{z}, \eta', \zeta_0')$.

$3°$ First we prove (3.2.31) for the eigenvalues $\kappa_\nu^j(\tilde{z}_0, \eta', \zeta_0')$ of $M_j(\tilde{z}_0, \eta', \zeta_0')$ for η' sufficiently small.

We consider several cases:

I) n_j even: $n_j = 2r$, $r \in \mathbb{N}$. Then from (3.2.30) we get

$$\kappa_\nu^j = \kappa^j + |\tilde{\zeta}|^{-1/(2r)}[(-1)^{r+1} i l_{n_j 1} \eta]^{1/(2r)} + O(\eta^{1/r} |\tilde{\zeta}|^{-1/r}).$$

a) r odd and $l_{n_j 1} > 0$. Then

$$\kappa_\nu^j = \kappa^j + |\tilde{\zeta}|^{-1/(2r)} (i l_{n_j 1} \eta)^{1/(2r)} + O(\eta^{1/r} |\tilde{\zeta}|^{-1/r}).$$

We have

$$(i l_{n_j 1} \eta)^{1/(2r)} = \sqrt[2r]{l_{n_j 1} \eta} \left[\cos\left(\frac{k\pi}{r} + \frac{\pi}{4r} \right) + i \sin\left(\frac{k\pi}{r} + \frac{\pi}{4r} \right) \right],$$

where $k = 0, \ldots, 2r - 1$.

Suppose that there exists $0 \le k \le 2r - 1$ such that

$$\frac{k\pi}{r} + \frac{\pi}{4r} = 2p\pi + \frac{\pi}{2} \quad \vee \quad \frac{k\pi}{r} + \frac{\pi}{4r} = 2p\pi + \frac{3}{2}\pi$$

for some integer p. From the first equation we then obtain $k = 2pr + r/2 - 1/4 = p_1 - 3/4$, and from the second $k = 2pr + 3r/2 - 1/4 = p_2 - 9/4$ (where p_1 and p_2 are integers); that is impossible because k is an integer. Hence none of the n_jth roots of $i l_{n_j 1} \eta$ are purely imaginary. Since n_j is even, if $a + bi$ is such a root, then so is $-a - bi$. Hence there are exactly $\rho_j = n_j/2$ roots with negative real part.

Now write the eigenvalues of κ_ν^j in the form

$$\kappa_\nu^j = \kappa^j + \eta'^{1/(2r)} \left[l_{n_j 1}^{1/(2r)} \left(\cos\left(\frac{k\pi}{r} + \frac{\pi}{4r} \right) \right. \right.$$
$$\left. \left. + i \sin\left(\frac{k\pi}{r} + \frac{\pi}{4r} \right) \right) + O(\eta'^{1/(2r)}) \right]$$

(where $\nu = 1, \ldots, n_j$), and choose any $0 \le k \le 2r - 1$ with $\cos(k\pi/r + \pi/(4r)) < 0$. Since $\operatorname{Re} \kappa^j = 0$, for η' sufficiently small we get $\operatorname{Re} \kappa_\nu^j < 0$. Therefore in this case there exist $\rho_j = n_j/2$ eigenvalues κ_ν^j such that $\operatorname{Re} \kappa_\nu^j < 0$.

The cases: b) r odd and $l_{n_j 1} < 0$; c) r even and $l_{n_j 1} > 0$; and d) r even and $l_{n_j 1} < 0$, can be considered in the analogous way.

II) n_j odd: $n_j = 2r - 1$, $r \in \mathbb{N}$, and $l_{n_j 1} > 0$. Then

$$\kappa_\nu^j = \kappa^j + |\tilde{\zeta}|^{-\frac{1}{2r-1}}[(-1)^{r-1} l_{n_j 1} \eta]^{\frac{1}{2r-1}} + O(\eta^{\frac{2}{2r-1}} |\tilde{\zeta}|^{-\frac{2}{2r-1}}).$$

a) r odd. Then

$$\kappa_\nu^j = \kappa^j + |\tilde\zeta|^{-\frac{1}{2r-1}}(l_{n_j1}\eta)^{\frac{1}{2r-1}} + O(\eta^{\frac{2}{2r-1}}|\tilde\zeta|^{-\frac{2}{2r-1}}).$$

We have

$$(l_{n_j1}\eta)^{\frac{1}{2r-1}} = \sqrt[2r-1]{l_{n_j1}\eta}\left(\cos\frac{2k\pi}{2r-1} + i\sin\frac{2k\pi}{2r-1}\right), \quad k = 0, 1, \ldots, 2r-2.$$

Suppose that there exists $0 \le k \le 2r - 2$ such that

$$\frac{2k\pi}{2r-1} = 2p\pi + \frac{\pi}{2} \quad \vee \quad \frac{2k\pi}{2r-1} = 2p\pi + \frac{3}{2}\pi$$

for some integer p. From the first equation we get $k = (2r-1)p+r/2-1/4 = p_1 - 3/4$, and from the second $k = (2r-1)p+3r/2-3/4 = p_2 - 9/4$ (where p_1 and p_2 are integers); that is impossible because k is an integer. Hence none of the $(2r-1)$th roots of $l_{n_j1}\eta$ are purely imaginary.

Notice that if $a + bi$ is such a root, then so is $a - bi$.

We now find the number of roots with positive real part. For $k = 0$ we get a real positive root. Since

$$\frac{r-1}{2}\frac{2\pi}{2r-1} = \frac{r-1}{2r-1}\pi < \frac{\pi}{2} \quad \text{and} \quad \left(\frac{r-1}{2}+1\right)\frac{2\pi}{2r-1} = \frac{\pi(r+1)}{2r-1} > \frac{\pi}{2},$$

there are $(r-1)/2$ roots in the first quadrant, and in view of the earlier remark, there are the same number of roots in the fourth quadrant. Therefore, there are r roots with positive real part and $\rho_j = n_j - r = r - 1 = (n_j-1)/2$ roots with negative real part. Hence, if η' is sufficiently small, then there exist $\rho_j = (n_j - 1)/2$ eigenvalues κ_ν^j for which $\operatorname{Re}\kappa_\nu^j < 0$.

b) r even. Then

$$\kappa_\nu^j = \kappa^j + |\tilde\zeta|^{-\frac{1}{2r-1}}(-l_{n_j1}\eta)^{\frac{1}{2r-1}} + O(\eta^{\frac{2}{2r-1}}|\tilde\zeta|^{-\frac{2}{2r-1}}).$$

Consider

$$(-l_{n_j1}\eta)^{\frac{1}{2r-1}} = \sqrt[2r-1]{l_{n_j1}\eta}\left[\cos\frac{(2k+1)\pi}{2r-1} + i\sin\frac{(2k+1)\pi}{2r-1}\right],$$
$$k = 0, 1, \ldots, 2r-2.$$

As before it can be proved that no $(2r-1)$th root of $-l_{n_j1}\eta$ is purely imaginary. Moreover, the roots occur in conjugate pairs.

We first find the number of roots with positive real part. For $k = 0$,

$$z_0 = \sqrt[2r-1]{l_{n_j1}\eta}\left(\cos\frac{\pi}{2r-1} + i\sin\frac{\pi}{2r-1}\right).$$

The angle

$$\frac{\pi}{2} - \frac{\pi}{2r-1} = \frac{2r-3}{2(2r-1)}\pi$$

contains $(r-2)/2$ angles equal to $2\pi/(2r-1)$. Therefore in the first quadrant there are $(r-2)/2+1$ roots, and, by symmetry, the same number in the fourth quadrant. Consequently, there exist exactly r roots with positive real part and $\rho_j = r - 1 = (n_j - 1)/2$ roots with negative real part. Hence for η' sufficiently small there exist $\rho_j = (n_j - 1)/2$ eigenvalues κ_ν^j for which $\operatorname{Re}\kappa_\nu^j < 0$.

III) n_j odd: $n_j = 2r - 1$, $r \in \mathbb{N}$, and $l_{n_j 1} < 0$. In this case we use a similar argument to case II.

From the continuity of the eigenvalues $\kappa_\nu^j(\widetilde{z}, \eta', \zeta')$ with respect to each variable separately (the continuity with respect to $\widetilde{z}$ follows from (3.2.30)) we conclude that the formulas (3.2.31) hold in a neighbourhood θ_{δ,z_0} of $z_0 = (\widetilde{z}_0, 0, \zeta_0')$ for points z with $\eta' > 0$. ∎

Let now $z_0 = (t_0, x_0, v_0, \eta_0, \zeta_0) \in \widetilde{\mathfrak{M}} = \mathbb{R} \times \overline{\Omega} \times \mathbb{R}^n \times \{(\eta, \zeta) \in [\eta_1, \infty) \times \mathbb{R}^m : |\widetilde{\zeta}| \le 1\}$ (where $\eta_1 > 0$) and set $\theta_{\delta,z_0} = \{z \in \widetilde{\mathfrak{M}} : |z - z_0| < \delta\}$.

LEMMA 3.2.7. *There exist a matrix-valued function $U = U(z)$ and $\delta > 0$ such that $U \in C_B^k(\theta_{\delta,z_0})$ and*

$$(3.2.35) \qquad UMU^{-1} = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix},$$

where M_{11} is an $l \times l$-matrix-valued function whose eigenvalues have negative real part, and M_{22} is an $(n - l) \times (n - l)$-matrix-valued function whose eigenvalues have positive real part. Moreover,

$1°$ $M_{11}(z_0)$ *and* $M_{22}(z_0)$ *are formed of blocks of the form (3.2.2) with* $k' = \varepsilon\eta/(2(n + 1)^3)$;

$2°$ $\forall z \in \theta_{\delta,z_0}$

$$(3.2.36) \qquad M_{11}(z) + M_{11}^*(z) \le -\widetilde{\varepsilon}\eta I,$$

$$(3.2.37) \qquad M_{22}(z) + M_{22}^*(z) \ge \widetilde{\varepsilon}\eta I,$$

where $\widetilde{\varepsilon} > 0$ is a constant.

Proof. (3.2.35) and $1°$ can be proved similarly to Lemmas 3.2.1 and 3.2.2. By Lemma 3.1.1, assumption (3.1.7) and the continuity of the eigenvalues κ_j $(j = 1, \ldots, n)$ in the sense of Definition 2.2.1 (assumption (3.1.6a), formula (3.1.21) and [10, Chap. II]) we have

$$\exists \varepsilon > 0 \; \forall z \in \widetilde{\mathfrak{M}} \quad \text{either } \operatorname{Re}\kappa_j(z) < -\varepsilon < -\varepsilon\eta \text{ or } \operatorname{Re}\kappa_j(z) > \varepsilon > \varepsilon\eta.$$

Therefore by the same argument as in Lemma 3.2.3 we get $2°$. ∎

Remark 3.2.6. Let $M_\mu \in C_B^k(\widetilde{\mathfrak{M}})$ be a sequence with $M_\mu \to M$ in $\widetilde{\mathfrak{M}}$ as $\mu \to 0$, satisfying the conditions of Lemma 2.2.3. By Lemma 2.2.3 and the proof of Lemma 3.2.7 there exists a subsequence of (M_μ), which we still

denote by (M_μ), such that for any $z_0 \in \widetilde{\mathfrak{M}}$ there exist $\delta > 0$ and a sequence $U_\mu \in C_B^k(\theta_{\delta,z_0})$ satisfying the conditions of Lemma 3.2.7 with M replaced by M_μ and M_{11}, M_{22} replaced by $M_{11\mu}$, $M_{22\mu}$ respectively, and moreover such that $U_\mu = \widetilde{U}(M_\mu) \to U = \widetilde{U}(M)$ in $C_B^k(\widetilde{\mathfrak{M}})$ ($\widetilde{U}$ is a function of class C^k).

Now, define

$$Q = \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times [\eta_1, \infty) \times \mathbf{R}^m, \quad \eta_1 > 0.$$

THEOREM 3.2.1. *Let assumptions* (3.1.1)–(3.1.7), (3.1.14), (3.1.20) *and* (3.1.24) *be satisfied. Then there exists a matrix-valued function* $R = R(t, x, v, \eta, \zeta)$ *such that:*

$1°$ $R \in C_B^k(Q)$;

$2°$ $\forall (t, x, v, \eta, \zeta) \in Q$, $R(t, x, v, \eta, \zeta)$ *is a hermitian matrix;*

$3°$ $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$, $R(t, x, v, \cdot, \cdot)$ *is a homogeneous function of order zero for* $|\tilde{\zeta}| \geq 1$;

$4°$ $\exists \varepsilon_1 > 0 \; \forall (t, x, v, \eta, \zeta) \in Q$

$$\text{Re } R(t, x, v, \eta, \zeta) M(t, x, v, \eta, \zeta) \geq \varepsilon_1 \eta I,$$

where ε_1 *is a constant which is independent of* t, x, v, η, ζ, *and depends on* η_1 *and on the functions* E, A_j $(j = 1, \ldots, m)$, C;

$5°$ $\exists \varepsilon_2, \varepsilon_3 > 0 \; \forall (t, x', v, \eta, \zeta) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n \times \mathbf{R}^m$

$$y^* R(t, 0, x', v, \eta, \zeta) y \geq \varepsilon_2 |y|^2 - \varepsilon_3 |g(t, x')|^2$$

for all vectors y *satisfying the boundary condition* (3.1.9), *where* ε_2, ε_3 *are constants which are independent of* t, x, v, η, ζ, *and depend on* η_1 *and on the functions* E, A_j $(j = 1, \ldots, m)$, C, S.

P r o o f. We have $Q = Q_1 \cup Q_2$, where $Q_1 = \{(t, x, v, \eta, \zeta) \in Q : |\tilde{\zeta}| \leq 1\}$ and $Q_2 = \{(t, x, v, \eta, \zeta) \in Q : |\tilde{\zeta}| \geq 1/2\}$. Let us first construct a function R^1 satisfying $1°$, $2°$, $4°$ and $5°$ in Q_1. To do this take a compact set $\widetilde{K}_{t,x,v} \subset \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$ such that $K_{t,x,v} \subset \text{Int } \widetilde{K}_{t,x,v} \cap \bar{\Omega} = \Omega_{\widetilde{K}}$, where $K_{t,x,v}$ is the set from assumption (3.1.7). Take next any $z_0 = (t_0, x_0, v_0, \eta_0, \zeta_0) \in \bar{\Omega}_{\widetilde{K}} \cap \{\tilde{\zeta} : |\tilde{\zeta}| \leq 1\}$ and define

(3.2.38)
$$R_0^1 = U^* \begin{bmatrix} -cI & 0 \\ 0 & I \end{bmatrix} U,$$

where $c > 0$ is a constant to be chosen later, and U is the function from Lemma 3.2.7. Thus $R_0^1 \in C_B^k(\theta_{\delta,z_0})$ and from (3.2.38) it follows that $R_0^1(z)$ is a hermitian matrix $\forall z \in \theta_{\delta,z_0}$.

Further, using (3.2.35) we have

$$2\operatorname{Re} R_0^1 M = R_0^1 M + M^* R_0^1 = U^* \begin{bmatrix} -cI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} U$$

$$+ U^* \begin{bmatrix} M_{11}^* & 0 \\ 0 & M_{22}^* \end{bmatrix} \begin{bmatrix} -cI & 0 \\ 0 & I \end{bmatrix} U = U^* \begin{bmatrix} -c(M_{11} + M_{11}^*) & 0 \\ 0 & M_{22} + M_{22}^* \end{bmatrix} U.$$

Take $y = [y_1, \dots, y_n]^T \in \mathbf{C}^n$. By Lemma 3.2.7

$$y^*(2\operatorname{Re} R_0^1 M)y = (Uy)^* \begin{bmatrix} -c(M_{11} + M_{11}^*) & 0 \\ 0 & M_{22} + M_{22}^* \end{bmatrix} Uy$$

$$\geq \widetilde{\varepsilon}\eta |Uy|^2 \geq \frac{\widetilde{\varepsilon}\eta |y|^2}{|U^{-1}|^2} \geq \widetilde{\varepsilon}_{10}\eta |y|^2 ,$$

where $\widetilde{\varepsilon}_{10} > 0$ is a constant. Hence 4° is satisfied in θ_{δ,z_0}.

In order to check 5° assume that $\theta_{\delta,z_0} \cap \partial\Omega \neq \emptyset$, take y satisfying (3.1.9) and set $w = Uy$. Then (3.1.9) takes the form

(3.2.39) $S_I w^I + S_{II} w^{II} = g$ in $\theta_{\delta,z_0} \cap \partial\Omega$,

where

$$w = \begin{bmatrix} w^I \\ w^{II} \end{bmatrix}, \quad w^I = \begin{bmatrix} w_1 \\ \vdots \\ w_l \end{bmatrix}, \quad w^{II} = \begin{bmatrix} w_{l+1} \\ \vdots \\ w_n \end{bmatrix},$$

(3.2.40) $S_I =$

$$\begin{bmatrix} U_{11}^{-1} - S_{11}U_{1,l+1}^{-1} - S_{12}U_{1,l+2}^{-1} - \dots - S_{1,n-l}U_{1n}^{-1} \\ U_{12}^{-1} - S_{21}U_{1,l+1}^{-1} - S_{22}U_{1,l+2}^{-1} - \dots - S_{2,n-l}U_{1n}^{-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ U_{1l}^{-1} - S_{l1}U_{1,l+1}^{-1} - S_{l2}U_{1,l+2}^{-1} - \dots - S_{l,n-l}U_{1n}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \dots & U_{l1}^{-1} - S_{11}U_{l,l+1}^{-1} - S_{12}U_{l,l+2}^{-1} - \dots - S_{1,n-l}U_{ln}^{-1} \\ \dots & U_{l2}^{-1} - S_{21}U_{l,l+1}^{-1} - S_{22}U_{l,l+2}^{-1} - \dots - S_{2,n-l}U_{ln}^{-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots & U_{ll}^{-1} - S_{l1}U_{l,l+1}^{-1} - S_{l2}U_{l,l+2}^{-1} - \dots - S_{l,n-l}U_{ln}^{-1} \end{bmatrix} .$$

Here $S_{ij} = S_{ij}(t, x', v(t, x'))$, and $U_{ij}^{-1} = U_{ij}^{-1}(t, x', v, \eta, \zeta)$ denote the entries of the matrix U^{-1}. Moreover, the vectors

$$U_i^{-1} = U_i^{-1}(z_0) = [U_{i1}^{-1}(z_0), U_{i2}^{-1}(z_0), \dots, U_{in}^{-1}(z_0)]^T, \quad i = 1, \dots, l,$$

are the generalized eigenvectors of $M(z_0)$ corresponding to eigenvalues with negative real parts, which results from Lemma 3.2.7. Since the vectors $\varphi_i(t, 0, x', v, s, \omega')$ $(i = 1, \dots, l)$ from (3.1.19) (where $(t, x', v, s, \omega') \in \theta_{\delta,z_0} \cap \partial\Omega$) span the same subspace of $\mathbf{C}^n$ as $U_i^{-1}(t, 0, x', v, s, \omega')$ $(i = 1, \dots, l)$, i.e.

the negative eigenspace of $M(t, 0, x', v, s, i\omega')$, the vectors U_i^{-1} are linear combinations of the φ_i. Therefore by assumption (3.1.20),

$$(3.2.41) \qquad \exists \varepsilon^* > 0 \ \forall z \in \theta_{\delta, z_0} \cap \partial\Omega \quad |\det S_I| > \varepsilon^*.$$

Take now any y satisfying (3.1.9) and consider

$$y^* R_0^1 y = (Uy)^* \begin{bmatrix} -cI & 0 \\ 0 & I \end{bmatrix} Uy = [w^{I*}, w^{II*}] \begin{bmatrix} -cI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w^I \\ w^{II} \end{bmatrix}$$

$$= -c|w^I|^2 + |w^{II}|^2 = -2c|w^I|^2 + c|w^I|^2 + |w^{II}|^2.$$

By (3.2.39), $|w^I|^2 \leq C_1|g|^2 + C_2|w^{II}|^2$, where C_1 and C_2 are constants depending on z_0. Therefore

$$y^* R_0^1 y \geq -2cC_1|g|^2 - 2cC_2|w^{II}|^2 + |w^{II}|^2 + c|w^I|^2.$$

Assume that c is so small that $1 - 2cC_2 \geq C_3 > 0$. Then

$$y^* R_0^1 y \geq -C|g|^2 + C^*|w|^2$$

(where $C, C^* > 0$ are constants) and since $|y| \leq |U^{-1}||w|$ (hence $|w| \geq (1/|U^{-1}|)|y|$), we have

$$y^* R_0^1 y \geq -C|g|^2 + \varepsilon_{20}|y|^2$$

(where $C, \varepsilon_{20} > 0$ are constants depending on z_0).

Thus we have proved that R_0^1 fulfils 1°, 2°, 4° and 5° in θ_{δ, z_0}.

Take next into account the set $((\mathbf{R} \times \bar\Omega \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \{\tilde\zeta : |\tilde\zeta| \leq 1\}$. In view of assumption (3.1.7)

$$\forall(t, x, v, \eta, \zeta) \in ((\mathbf{R} \times \bar\Omega \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \{\tilde\zeta : |\tilde\zeta| \leq 1\}$$

$$M(t, x, v, \eta, \zeta) = M'(\eta, \zeta) = A_1'^{-1}\left((i\zeta + \eta)E' - i\sum_{j=2}^m A_j'\omega_j\right).$$

Lemma 3.2.7 yields that $\forall\tilde\zeta_0 \in \{\tilde\zeta : |\tilde\zeta| \leq 1\} \ \exists\delta > 0$ and $\exists U' \in C_B^k(\theta_{\delta, \tilde\zeta_0})$ such that in $((\mathbf{R} \times \bar\Omega \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \theta_{\delta, \tilde\zeta_0}$ (where $\tilde\zeta_0 = (\eta_0, \zeta_0)$)

$$U'M'U'^{-1} = \begin{bmatrix} M'_{11} & 0 \\ 0 & M'_{22} \end{bmatrix},$$

where M'_{11} and M'_{22} satisfy properties 1° and 2° from Lemma 3.2.7. Hence similarly to the case of the neighbourhood θ_{δ, z_0} of the point $z_0 \in \bar\Omega_{\tilde K} \cap \{\tilde\zeta : |\tilde\zeta| \leq 1\}$ it can be checked that the function

$$(3.2.42) \qquad R_0^{1'} = U'^* \begin{bmatrix} -cI & \\ 0 & I \end{bmatrix} U'$$

fulfils 1°, 2°, 4° and 5° in $((\mathbf{R} \times \bar\Omega \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \theta_{\delta, \tilde\zeta_0}$.

Choose now in $\Omega_{\widetilde{K}} \times \{\widetilde{\zeta} : |\widetilde{\zeta}| \leq 1\}$ a finite number of points $z_{(i)}$ ($i = 1, \ldots, r_1$) and neighbourhoods $\theta_{\delta_{(i)}, z_{(i)}}$ in such a way that these neighbourhoods cover $\bar{\Omega}_{\widetilde{K}} \times \{\widetilde{\zeta} : |\widetilde{\zeta}| \leq 1\}$ and in each of them the function R_0^1 defined by (3.2.38) can be constructed. Associate with the covering $\{\theta_{\delta_{(i)}, z_{(i)}}\}_{i=1}^{r_1}$ a partition of unity $\{\psi_{(i)}^1\}$, and with the covering $\{\theta_{\delta_{(i)}, \tilde{\zeta}_{(i)}}\}_{i=1}^{r_2}$ a partition of unity $\{\psi_{(i)}^{1\prime}\}$. Moreover, take functions $\overline{\psi}_1 = \overline{\psi}_1(t, x, v)$ and $\overline{\psi}_2 = \overline{\psi}_2(t, x, v)$ of class C^∞ in $\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ and such that $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$, $\overline{\psi}_1(t, x, v) + \overline{\psi}_2(t, x, v) = 1$, $\operatorname{supp} \overline{\psi}_1 \subset \Omega_{\widetilde{K}}$, $\operatorname{supp} \overline{\psi}_2 \subset (\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n) \setminus K_{t,x,v}$. Then the function

$$(3.2.43) \quad R^1(t, x, v, \eta, \zeta) = \overline{\psi}_1(t, x, v) \sum_{i=1}^{r_1} R_{(i)}^1(t, x, v, \eta, \zeta) \psi_{(i)}^1(t, x, v, \eta, \zeta)$$

$$+ \overline{\psi}_2(t, x, v) \sum_{i=1}^{r_2} R_{(i)}^{1\prime}(\eta, \zeta) \psi_{(i)}^{1\prime}(\eta, \zeta)$$

satisfies 1°, 2°, 4° and 5° in Q_1.

Now we construct a function R^2 satisfying 1°–5° in Q_2. To do this write $M(t, x, v, \eta, \zeta)$ as

$$M(t, x, v, \eta, \zeta) = |\tilde{\zeta}| M(t, x, v, \eta', \zeta')$$

(where η' and ζ' are defined by (3.2.25)) and construct a function $R^2 = R^2(t, x, v, \eta', \zeta')$ satisfying the conditions of the theorem in the set $Q_2' = \{(t, x, v, \eta', \zeta') : (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n, \sqrt{|\zeta'|^2 + \eta'^2} = 1, \eta' \geq 0\}$. Take any $z_0 = (t_0, x_0, v_0, 0, \zeta_0') \in \bar{\Omega}_{\widetilde{K}} \times \{(\eta', \zeta') : \eta' = 0, |\zeta'| = 1\}$ and $\delta > 0$ so small that in θ_{δ, z_0} the assertions of Lemmas 3.2.2–3.2.4 and 3.2.6 hold, and assume that $\theta_{\delta, z_0} \cap \partial\Omega \neq 0$. Consider the matrix-valued function T in $\theta_{\delta, z_0} \cap \partial\Omega$ and take any y fulfilling (3.1.9). Set $w = Ty$. According to the block form (3.2.4) of M we can write

$$w = \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(q)} \end{bmatrix}.$$

Denote by ρ_j ($j = 1, \ldots, q$) the number of eigenvalues κ of M_j such that $\operatorname{Re} \kappa < 0$ for $\eta' > 0$. We can then write

$$w^{(j)} = \begin{bmatrix} w_I^{(j)} \\ w_{II}^{(j)} \end{bmatrix},$$

where $w_I^{(j)}$ consists of the first ρ_j coordinates of $w^{(j)}$. Notice further that by Lemmas 3.2.2 and 3.2.4 the vectors $\{\widetilde{\varphi}_\nu^j : j = 1, \ldots, q, \ \nu = 1, \ldots, \rho_j\}$ from Lemma 3.2.2 (forming columns of $V = T^{-1}$) span the negative eigenspace

of M for $\eta' > 0$. Therefore the boundary condition (3.1.9) takes the form

(3.2.44) $$S_I w_I + S_{II} w_{II} = g \quad \text{in } \theta_{\delta,z_0} \cap \partial\Omega,$$

where

$$S_I = \begin{bmatrix} \widetilde{\varphi}_{11}^1 - S_{11}\widetilde{\varphi}_{1,l+1}^1 - S_{12}\widetilde{\varphi}_{1,l+2}^1 - \ldots - S_{1,n-l}\widetilde{\varphi}_{1n}^1 \\ \widetilde{\varphi}_{12}^1 - S_{21}\widetilde{\varphi}_{1,l+1}^1 - S_{22}\widetilde{\varphi}_{1,l+2}^1 - \ldots - S_{2,n-l}\widetilde{\varphi}_{1n}^1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \widetilde{\varphi}_{1l}^1 - S_{l1}\widetilde{\varphi}_{1,l+1}^1 - S_{l2}\widetilde{\varphi}_{1,l+2}^1 - \ldots - S_{l,n-l}\widetilde{\varphi}_{1n}^1 \end{bmatrix}$$

$$\begin{matrix} \ldots & \widetilde{\varphi}_{\rho_q 1}^q - S_{11}\widetilde{\varphi}_{\rho_q,l+1}^q - S_{12}\widetilde{\varphi}_{\rho_q,l+2}^q - \ldots - S_{1,n-l}\widetilde{\varphi}_{\rho_q n}^q \\ \ldots & \widetilde{\varphi}_{\rho_q 2}^q - S_{21}\widetilde{\varphi}_{\rho_q,l+1}^q - S_{22}\widetilde{\varphi}_{\rho_q,l+2}^q - \ldots - S_{2,n-l}\widetilde{\varphi}_{\rho_q n}^q \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots & \widetilde{\varphi}_{\rho_q l}^q - S_{l1}\widetilde{\varphi}_{\rho_q,l+1}^q - S_{l2}\widetilde{\varphi}_{\rho_q,l+2}^q - \ldots - S_{l,n-l}\widetilde{\varphi}_{\rho_q n}^q \end{matrix}.$$

Since the vectors $\{\widetilde{\varphi}_\nu^j : j = 1,\ldots,q, \ \nu = 1,\ldots,\rho_j\}$ span the same subspace as $\varphi_1,\ldots,\varphi_l$ from (3.1.19), assumption (3.1.20) implies

(3.2.45) $$\exists \varepsilon^* > 0 \ \forall z \in \theta_{\delta,z_0} \cap \partial\Omega \quad |\det S_I| > \varepsilon^*.$$

Assume now that for each block M_j $(j = 1,\ldots,q)$ we have constructed a matrix-valued function $\widetilde{R}_j^2$ with the following properties:

(3.2.46) $\quad \widetilde{R}_j^2 \in C_B^k(\theta_{\delta,z_0})$;

(3.2.47) $\quad \forall (t,x,v,\eta',\zeta') \in \theta_{\delta,z_0}, \ \widetilde{R}_j^2(t,x,v,\eta',\zeta')$ is a hermitian matrix;

(3.2.48) $\quad \exists \widetilde{\varepsilon}_1 > 0 \ \forall (t,x,v,\eta',\zeta') \in \theta_{\delta,z_0}$

$$\operatorname{Re} \widetilde{R}_j^2(t,x,v,\eta',\zeta') M_j(t,x,v,\eta',\zeta') \geq \widetilde{\varepsilon}_1 \eta' I;$$

(3.2.49) $\quad \exists \widetilde{\varepsilon}_2 > 0 \ \forall (t,x,v,\eta',\zeta') \in \theta_{\delta,z_0} \cap \partial\Omega$

$$w^{(j)*} \widetilde{R}_j^2(t,0,x',v,\eta',\zeta') w^{(j)} \geq 2\widetilde{\varepsilon}_2(-c|w_I^{(j)}|^2 + |w_{II}^{(j)}|^2),$$

where $c > 0$ is a sufficiently small constant.

Then the function

(3.2.50) $$R_0^2 = T^* \operatorname{diag}(\widetilde{R}_1^2,\ldots,\widetilde{R}_q^2) T$$

satisfies the assertion of the theorem in θ_{δ,z_0}. In fact, 1° follows from (3.2.46) and Lemma 3.2.2, and 2° is obviously also fulfilled. We now check 4°. Using Lemma 3.2.2 we obtain

$$2\operatorname{Re} R_0^2 M = R_0^2 M + M^* R_0^2 = T^* \operatorname{diag}(2\operatorname{Re} \widetilde{R}_1^2 M_1,\ldots,2\operatorname{Re} \widetilde{R}_q^2 M_q) T.$$

Hence and by (3.2.48) we get

$$y^* \operatorname{Re} R_0^2 M y = (Ty)^* \operatorname{diag}(2\operatorname{Re} \widetilde{R}_1^2 M_1,\ldots,2\operatorname{Re} \widetilde{R}_q^2 M_q) Ty \geq \varepsilon_1 \eta' |y|^2$$

for any $y \in \mathbf{C}^n$. Therefore 4° holds.

In order to check 5° take any y satisfying (3.1.9) and consider

$$
y^* R_0^2 y = (Ty)^* \begin{bmatrix} \widetilde{R}_1^2 & & 0 \\ & \ddots & \\ 0 & & \widetilde{R}_q^2 \end{bmatrix} Ty
$$

$$
= [w^{(1)*}, \ldots, w^{(q)*}] \begin{bmatrix} \widetilde{R}_1^2 & & 0 \\ & \ddots & \\ 0 & & \widetilde{R}_q^2 \end{bmatrix} \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(q)} \end{bmatrix}
$$

$$
= w^{(1)*} \widetilde{R}_1^2 w^{(1)} + \ldots + w^{(q)*} \widetilde{R}_q^2 w^{(q)} \geq 2\widetilde{\varepsilon}_2 (-c|w_I|^2 + |w_{II}|^2)
$$

(where we have used (3.2.49)). Making the same calculations as in the construction of R_0^1 in a neighbourhood of $z_0 \in Q_1$, we obtain 5°.

Let us now construct $\widetilde{R}_j^2$ for $j = 1, \ldots, q$. Set $\widetilde{R}_1^2 = -cI$ and $\widetilde{R}_2^2 = I$. Using Lemma 3.2.3 we find that $\widetilde{R}_1^2$ and $\widetilde{R}_2^2$ satisfy (3.2.46)–(3.2.49).

Take now $j \geq 3$. If $n_j = 1$ (i.e. the block M_j is one-dimensional) and $\mathrm{Re}\,\kappa^j > 0$ for $\eta' > 0$, then we put $\widetilde{R}_j^2 = -c_j I$, where $c_j > 0$ is sufficiently small. Consider now the case $n_j \geq 2$. We use Lemmas 2.3.1–2.3.4. According-ing to (3.2.21) write $M_j(t, x, v, \eta', \zeta')$ in the form

$$
\text{(3.2.51)} \qquad M_j(\widetilde{z}, \eta', \zeta') = \kappa^j I + i(C + \varepsilon E) + \eta' L_j(\widetilde{z}_0, 0, \zeta') + O(\eta'^2)
$$
$$
+ O(|\widetilde{z} - \widetilde{z}_0|) + O(\eta'|\widetilde{z} - \widetilde{z}_0|),
$$

where $(\widetilde{z}, \eta', \zeta') \in \theta_{\delta, z_0}$ (we have used here notations (3.2.26) and (3.2.27) for $\widetilde{z}$ and $\widetilde{z}_0$), and

$$
\varepsilon = |\zeta_0 - \zeta_0'| = \sqrt{(\xi' - \xi_0')^2 + \sum_{j=2}^{m}(\omega_j' - \omega_{0j}')^2},
$$
$$
\kappa^j I + iC = M_j(\widetilde{z}_0, 0, \zeta_0'),
$$
$$
\text{(3.2.52)} \qquad i\varepsilon E = M_j(\widetilde{z}_0, 0, \zeta' - \zeta_0')
$$

(C is given by (2.3.1), and E is defined in Lemma 2.3.3). Notice that the matrices $M_j(\widetilde{z}_0, 0, \zeta_0')$ and $M_j(\widetilde{z}_0, 0, \zeta' - \zeta_0')$ are purely imaginary, while $L_j(\widetilde{z}_0, 0, \zeta')$ is real by Proposition 3.2.1, and in view of Lemma 3.2.6, for some $c > 0$, $|l_{n,1}| \geq c$ in θ_{δ, z_0}.

Choose $\widetilde{R}_j^2$ in the form

$$
\text{(3.2.53)} \qquad \widetilde{R}_j^2 = (D + \varepsilon B) - i\eta' F,
$$

where D, B and F are the matrices from Lemmas 2.3.1, 2.3.3 and 2.3.4 respectively, and assume that the entry d_{1n_j} of D is chosen so that

$$
\text{(3.2.54)} \qquad d_{1n_j} l_{n_j 1} \geq 1 \quad \text{in } \theta_{\delta, z_0}.
$$

(Exact forms of D and F will be introduced later.) We see that $\widetilde{R}_j^2$ is

hermitian, and so satisfies (3.2.47). Next, using Lemmas 2.3.1–2.3.3 we obtain

$$(3.2.55) \quad 2\operatorname{Re}\widetilde{R}_j^2 M_j = \widetilde{R}_j^2 M_j + M_j^* \widetilde{R}_j^2 = [(D + \varepsilon B) - i\eta' F]$$

$$\times\, [\kappa^j I + i(C + \varepsilon E) + \eta' L_j + O(\eta'^2) + O(|\widetilde{z} - \widetilde{z}_0|) + O(\eta'|\widetilde{z} - \widetilde{z}_0|)]$$

$$+\, [\kappa^j I - i(C^* + \varepsilon E^*) + \eta' L_j^* + O(\eta'^2) + O(|\widetilde{z} - \widetilde{z}_0|) + O(\eta'|\widetilde{z} - \widetilde{z}_0|)]$$

$$\times\, [(D + \varepsilon B) - i\eta' F] = \eta'(DL_j + L_j^* D + FC + C^* F) + O(|\varepsilon\eta'| + \eta'^2)$$

$$+\, O(|\widetilde{z} - \widetilde{z}_0|) + O(\eta'|\widetilde{z} - \widetilde{z}_0|) + O(\varepsilon|\widetilde{z} - \widetilde{z}_0|) + O(\varepsilon\eta'|\widetilde{z} - \widetilde{z}_0|)\,.$$

Let us now calculate DL_j and $L_j^* D$. We obtain

$$DL_j = \begin{bmatrix} d_{1n_j} l_{n_j 1} & \cdots & d_{1n_j} l_{n_j, n_j - 1} & d_{1n_j} l_{n_j n_j} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n_j} d_{in_j} l_{i1} & \cdots & \sum_{i=1}^{n_j} d_{in_j} l_{i,n_j-1} & \sum_{i=1}^{n_j} d_{in_j} l_{in_j} \end{bmatrix}$$

and

$$L_j^* D = \begin{bmatrix} l_{n_j 1} d_{1n_j} & \cdots & \sum_{i=1}^{n_j} l_{i1} d_{in_j} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ l_{n_j n_j} d_{1n_j} & \cdots & \sum_{i=1}^{n_j} l_{in_j} d_{in_j} \end{bmatrix}\,.$$

Then for every $y \in \mathbf{C}^n$

$$y^*(DL_j + L_j^* D)y = 2l_{n_j 1} d_{1n_j} y_1^2 + \ldots + 2\sum_{i=1}^{n_j} l_{in_j} d_{in_j} y_{n_j}^2 + \sum_{\substack{i,k=1 \\ i \neq k}}^{n_j} A_{ik} y_i y_k\,,$$

where $A_{ik} = A_{ik}(L_j, D)$ are linear functions of the entries of L_j and D. Denote by W the sum of all components in which y_1 does not occur. Then $W \geq -|D|\widetilde{K}(|L_j|)\sum_{i=2}^{n_j} y_i^2$, where $\widetilde{K} = \widetilde{K}(|L_j|)$ is a constant. For components containing y_1 except $2l_{n_j 1} d_{1n_j} y_1^2$, i.e. components of the type $\sum_{k=2}^{n_j} A_{1k} y_1 y_k$, we get

$$-\sum_{k=2}^{n_j} A_{1k} y_1 y_k \leq \sum_{k=2}^{n_j} \frac{|A_{1k}| y_1^2}{2\rho} + \sum_{k=2}^{n_j} \frac{|A_{1k}| \rho y_k^2}{2} \leq K_1 |D| y_1^2 + K_2 |D| \sum_{k=2}^{n_j} y_k^2\,,$$

where K_1 and K_2 are positive constants depending only on z_0. Choose ρ so large that K_1 is small enough that $2l_{n_j 1} d_{1n_j} - K_1 |D| \geq 3/2$. Then we obtain

$$y^*(DL_j + L_j^* D)y \geq \frac{3}{2} y_1^2 - K|D| \sum_{k=2}^{n_j} y_k^2$$

$$= y^* \left(\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 0 \end{bmatrix} - K|D| \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) y\,,$$

where $K = \widetilde{K} + K_2$. By Lemma 2.3.4 we can choose a matrix F such that

(3.2.56) $DL_j + L_j^* D + FC + C^* F > 2\widetilde{\varepsilon}_1 I > 0$

for a constant $\widetilde{\varepsilon}_1 > 0$. In fact,

$$y^*(DL_j + L_j^* D + FC + C^* F)y > y^* \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2}d - K|D| \end{bmatrix} y,$$

so if we take $d \geq 2$ so large that $\frac{1}{2}d - K|D| \geq 2\widetilde{\varepsilon}_2 > 0$ for some constant $\widetilde{\varepsilon}_2 \leq 1/4$, then (3.2.56) will hold. Thus by (3.2.55) we obtain (3.2.48) for δ sufficiently small.

We now check (3.2.49). We get

(3.2.57) $w^{(j)*}\widetilde{R}_j^2 w^{(j)} = w^{(j)*}Dw^{(j)} + (O(\varepsilon) + O(\eta'))|w(j)|^2 .$

We shall choose elements of D in such a way that (3.2.49) holds. We consider several cases:

a) n_j even and $l_{n_j 1} > 0$. By Lemma 3.2.6 there exist then $\rho_j = n_j/2$ eigenvalues of $M_j(\widetilde{z}, \eta', \zeta')$ with negative real part for $\eta' > 0$. Thus

$$w_I^{(j)} = [w_1^{(j)}, \ldots, w_{n_j/2}^{(j)}]^T, \qquad w_{II}^{(j)} = [w_{n_j/2+1}^{(j)}, \ldots, w_{n_j}^{(j)}]^T .$$

We have

$$(3.2.58)\quad w^{(j)*}Dw^{(j)} = w_1^{(j)}\overline{w}_{n_j}^{(j)}d_{1n_j} + w_2^{(j)}\overline{w}_{n_j-1}^{(j)}d_{1n_j} + w_2^{(j)}\overline{w}_{n_j}^{(j)}d_{2n_j}$$
$$+ w_3^{(j)}\overline{w}_{n_j-2}^{(j)}d_{1n_j} + w_3^{(j)}\overline{w}_{n_j-1}^{(j)}d_{2n_j} + w_3^{(j)}\overline{w}_{n_j}^{(j)}d_{3n_j}$$
$$+ \ldots + w_{n_j-1}^{(j)}\overline{w}_3^{(j)}d_{1n_j} + \ldots + w_{n_j-1}^{(j)}\overline{w}_{n_j}^{(j)}d_{n_j-1,n_j}$$
$$+ w_{n_j}^{(j)}\overline{w}_1^{(j)}d_{1n_j} + w_{n_j}^{(j)}\overline{w}_2^{(j)}d_{2n_j} + w_{n_j}^{(j)}\overline{w}_3^{(j)}d_{3n_j} + \ldots + |w_{n_j}^{(j)}|^2 d_{n_j n_j}$$
$$= 2(\operatorname{Re} w_1^{(j)} \operatorname{Re} w_{n_j}^{(j)} + \operatorname{Im} w_1^{(j)} \operatorname{Im} w_{n_j}^{(j)})d_{1n_j}$$
$$+ 2(\operatorname{Re} w_2^{(j)} \operatorname{Re} w_{n_j-1}^{(j)} + \operatorname{Im} w_2^{(j)} \operatorname{Im} w_{n_j-1}^{(j)})d_{1n_j}$$
$$+ 2(\operatorname{Re} w_3^{(j)} \operatorname{Re} w_{n_j-2}^{(j)} + \operatorname{Im} w_3^{(j)} \operatorname{Im} w_{n_j-2}^{(j)})d_{1n_j}$$
$$+ \ldots + 2(\operatorname{Re} w_{n_j/2}^{(j)} \operatorname{Re} w_{n_j/2+1}^{(j)} + \operatorname{Im} w_{n_j/2}^{(j)} \operatorname{Im} w_{n_j/2+1}^{(j)})d_{1n_j}$$
$$+ 2(\operatorname{Re} w_2^{(j)} \operatorname{Re} w_{n_j}^{(j)} + \operatorname{Im} w_2^{(j)} \operatorname{Im} w_{n_j}^{(j)})d_{2n_j} + \ldots +$$
$$+ 2(\operatorname{Re} w_{n_j-1}^{(j)} \operatorname{Re} w_{n_j}^{(j)} + \operatorname{Im} w_{n_j-1}^{(j)} \operatorname{Im} w_{n_j}^{(j)})d_{n_j-1,n_j}$$
$$+ \sum_{k=0}^{n_j/2-1} d_{n_j-2k,n_j}|w_{n_j-k}^{(j)}|^2 .$$

Taking d_{1n_j} such that (3.2.54) is satisfied, and the other d_{kn_j} $(k = 2, \ldots, n_j)$ so that d_{kn_j} $(k = 1, \ldots, n_j)$ is a sufficiently fast increasing sequence of

positive numbers, and using Young's inequality, we get

$$w^{(j)*} \widetilde{R}_j^2 w^{(j)} \geq 2\widetilde{\varepsilon}_2(-c|w_1^{(j)}|^2 - \ldots - c|w_{n_j/2}^{(j)}|^2) + |w_{n_j/2+1}^{(j)}|^2 + \ldots + |w_{n_j}^{(j)}|^2$$
$$= 2\widetilde{\varepsilon}(-c|w_I^{(j)}|^2 + |w_{II}^{(j)}|^2),$$

so (3.2.49) holds.

b) n_j even and $l_{n_j1} < 0$. This case is analogous to a), but in view of (3.2.54) we take d_{1n_j} negative, and the other d_{kn_j} ($k = 2, \ldots, n_j$) positive sufficiently fast increasing.

c) n_j odd and $l_{n_j1} > 0$. In this case by Lemma 3.2.6 there exist exactly $\rho_j = (n_j - 1)/2$ eigenvalues of $M_j(\widetilde{z}, \eta', \zeta')$ with negative real part for $\eta' > 0$. Hence

$$w_I^{(j)} = [w_1^{(j)}, \ldots, w_{(n_j-1)/2}^{(j)}]^T, \qquad w_{II}^{(j)} = [w_{(n_j+1)/2}^{(j)}, \ldots, w_{n_j}^{(j)}]^T,$$

and in (3.2.58) there occur summands of the form $d_{n_j-2k,n_j}|w_{n_j-k}^{(j)}|^2$, where $k = 0, 1, \ldots, (n_j - 1)/2$. Taking, as before, d_{1n_j} such that (3.2.54) is satisfied, and the remaining d_{kn_j} so that the sequence d_{kn_j} ($k = 1, \ldots, n_j$) is a sufficiently fast increasing sequence of positive numbers and using Young's inequality we obtain (3.2.49).

d) n_j odd and $l_{n_j1} < 0$. By Lemma 3.2.6 there exist then exactly $\rho_j = (n_j + 1)/2$ eigenvalues of $M_j(\widetilde{z}, \eta', \zeta')$ with negative real part for $\eta' > 0$. Then

$$w_I^{(j)} = [w_1^{(j)}, \ldots, w_{(n_j+1)/2}^{(j)}]^T, \qquad w_{II}^{(j)} = [w_{(n_j+3)/2}^{(j)}, \ldots, w_{n_j}^{(j)}]^T,$$

and in (3.2.58) there occur summands of the form $d_{n_j-2k,n_j}|w_{n_j-k}|^2$, where $k = 0, 1, \ldots, (n_j+1)/2$. Taking d_{k,n_j} in such a way as before, we get (3.2.49).

Moreover, notice that $\widetilde{R}_j^2$ ($j = 3, \ldots, q$) satisfies (3.2.46), because D and F are constant matrices, while εB is of class C^∞ in the entries of εE.

Therefore we have proved that the function R_0^2 given by (3.2.50) fulfils the assertion of the theorem in θ_{δ,z_0}.

Take next $z_0 = (t_0, x_0, v_0, 0, \zeta_0') \in ((\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \{\widetilde{\zeta}' = (\eta', \zeta') : \eta' = 0, |\zeta'| = 1\}$. In the same way as for $((\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \{\widetilde{\zeta} : |\widetilde{\zeta}| \leq 1\}$ we can construct (using Lemmas 3.2.2–3.2.4 and 3.2.6) a function $R_0^{2\prime} = R_0^{2\prime}(\eta', \zeta')$ which satisfies the assertion of the theorem (except 3°) in $((\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \theta_{\delta,\widetilde{\zeta}_0'}$ (where $\theta_{\delta,\zeta_0'}$ is a neighbourhood of $(0, \zeta_0')$).

Choose now in $\overline{\Omega}_{\widetilde{K}} \times \{\widetilde{\zeta}' : \sqrt{|\zeta'|^2 + \eta'^2} = 1, \eta' \geq 0\}$ a finite number of points $z_{(i)} = (t_{(i)}, x_{(i)}, v_{(i)}, \eta_{(i)}', \zeta_{(i)}')$ ($i = 1, \ldots, r_3$) and neighbourhoods $\theta_{\delta_{(i)},z_{(i)}}$ associated with $z_{(i)}$ such that these neighbourhoods cover $\overline{\Omega}_{\widetilde{K}} \times \{\widetilde{\zeta}' : \sqrt{\eta'^2 + |\zeta'|^2} = 1, \eta' \geq 0\}$ (where we assume that p_1 of these points $z_{(i)}$, say

for $i = 1, \ldots, p_1$, are of the form $z_{(i)} = (t_{(i)}, x_{(i)}, v_{(i)}, 0, \zeta_{(i)})$, while the remaining $r_3 - p_1$ points $z_{(i)}$, for $i = p_1 + 1, \ldots, r_3$, are of the form $z_{(i)} = (t_{(i)}, x_{(i)}, v_{(i)}, \eta'_{(i)}, \zeta'_{(i)})$ with $\eta'_{(i)} > 0$) and in each of the neighbourhoods the function $R^2_{(i)}$ defined by (3.2.50) for $i = 1, \ldots, p_1$ and by (3.2.38) for $i = p_1 + 1, \ldots, r_3$ can be constructed. Similarly, in $\{\widetilde{\zeta}' : \sqrt{\eta'^2 + |\zeta'|^2} = 1, \eta' \geq 0\}$ choose a finite number of points $\widetilde{\zeta}'_{(i)}$ $(i = 1, \ldots, r_4)$ (where $\widetilde{\zeta}'_{(i)} = (0, \zeta'_{(i)})$ for $i = 1, \ldots, p_2$, and $\widetilde{\zeta}'_{(i)} = (\eta'_{(i)}, \zeta'_{(i)})$ with $\eta'_{(i)} > 0$ for $i = p_2 + 1, \ldots, r_4$) such that the sets $((\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \theta_{\delta_{(i)}, \widetilde{\zeta}'_{(i)}}$ cover $((\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \{\widetilde{\zeta}' : \sqrt{|\zeta'|^2 + \eta'^2} = 1, \eta' \geq 0\}$ and in each of them the function $R^{1\prime}_0$ when $i = 1, \ldots, p_2$ and the function $R^{2\prime}_0$ when $i = p_2 + 1, \ldots, r_4$ can be constructed. With the covering $\{\theta_{\delta_{(i)}, z_{(i)}}\}_{i=1}^{r_3}$ we associate a partition of unity $\{\psi^2_{(i)}\}$ and with the covering $\{\theta_{\delta_{(i)}, \widetilde{\zeta}_{(i)}}\}_{i=1}^{r_4}$ a partition of unity $\{\psi^{2\prime}_{(i)}\}$. Then the function

$$R^2(t, x, v, \eta', \zeta') = \overline{\psi}_1(t, x, v) \sum_{i=1}^{r_3} R^2_{(i)}(t, x, v, \eta', \zeta') \psi^2_{(i)}(t, x, v, \eta', \zeta')$$

$$+ \overline{\psi}_2(t, x, v) \sum_{i=1}^{r_4} R^{2\prime}_{(i)}(\eta', \zeta') \psi^{2\prime}_{(i)}(\eta', \zeta')$$

satisfies 1°, 2°, 4° and 5° in Q_2 (here $\overline{\psi}_1$ and $\overline{\psi}_2$ are the functions from (3.2.43)).

Take further C^∞ functions $\overline{\psi}_3 = \overline{\psi}_3(\widetilde{\zeta})$ and $\overline{\psi}_4 = \overline{\psi}_4(\widetilde{\zeta})$ in $[\eta_1, \infty) \times \mathbf{R}^m$ such that $\overline{\psi}_3 + \overline{\psi}_4 = 1$, $\mathrm{supp}\, \overline{\psi}_3 \subset \{\widetilde{\zeta} : |\widetilde{\zeta}| < 1\}$ and $\mathrm{supp}\, \overline{\psi}_4 \subset \{\widetilde{\zeta} : |\widetilde{\zeta}| > 1/2\}$. The function

(3.2.59) $R(t, x, v, \eta, \zeta')$

$$= \overline{\psi}_3(\eta, \zeta) \Big[\overline{\psi}_1(t, x, v) \sum_{i=1}^{r_1} R^1_{(i)}(t, x, v, \eta, \zeta) \psi^1_{(i)}(t, x, v, \eta, \zeta)$$

$$+ \overline{\psi}_2(t, x, v) \sum_{i=1}^{r_2} R^{1\prime}_{(i)}(\eta, \zeta) \psi^{1\prime}_{(i)}(\eta, \zeta) \Big]$$

$$+ \overline{\psi}_4(\eta, \zeta) \Big[\overline{\psi}_1(t, x, v) \sum_{i=1}^{r_3} R^2_{(i)} \Big(t, x, v, \frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \Big) \psi^2_{(i)} \Big(t, x, v, \frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \Big)$$

$$+ \overline{\psi}_2(t, x, v) \sum_{i=1}^{r_4} R^{2\prime}_{(i)} \Big(\frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \Big) \psi^2_{(i)} \Big(\frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \Big) \Big]$$

satisfies the assertion of the theorem. ∎

R e m a r k 3.2.7. From the construction of R (the proof of Theorem 3.2.1

and Lemmas 3.2.2, 3.2.7) it follows that the functions $R^1_{(i)}$, $R^{1\prime}_{(i)}$, $R^2_{(i)}$ and $R^{2\prime}_{(i)}$ depend on M, i.e. there exist C^∞ functions H'_1, H''_1, H'_2 and H''_2 such that R can be written as

$$(3.2.60) \quad R(t,x,v,\eta,\zeta)$$
$$= \overline{\psi}_3(\eta,\zeta) H''_1(\Psi_1(t,x,v,\eta,\zeta), M''(t,x,v,\eta,\zeta), M'(\eta,\zeta))$$
$$+ \overline{\psi}_3(\eta,\zeta) H'_1(\Psi'_1(\eta,\zeta), M'(\eta,\zeta))$$
$$+ \overline{\psi}_4(\eta,\zeta) H''_2\left(\Psi_2\left(t,x,v,\frac{\eta}{|\widetilde{\zeta}|},\frac{\zeta}{|\widetilde{\zeta}|}\right), M''\left(t,x,v,\frac{\eta}{|\widetilde{\zeta}|},\frac{\zeta}{|\widetilde{\zeta}|}\right), M'\left(\frac{\eta}{|\widetilde{\zeta}|},\frac{\zeta}{|\widetilde{\zeta}|}\right)\right)$$
$$+ \overline{\psi}_4(\eta,\zeta) H''_2\left(\Psi'_2\left(\frac{\eta}{|\widetilde{\zeta}|},\frac{\zeta}{|\widetilde{\zeta}|}\right), M'\left(\frac{\eta}{|\widetilde{\zeta}|},\frac{\zeta}{|\widetilde{\zeta}|}\right)\right),$$

where $\Psi_1 = (\overline{\psi}_1, (\psi^1_{(i)})^{r_1}_{i=1}, \overline{\psi}_2 - 1, (\psi^{1\prime}_{(i)})^{r_2}_{i=1})$, $\Psi'_1 = (\psi^{1\prime}_{(i)})^{r_2}_{i=1}$, $\Psi_2 = (\overline{\psi}_1, (\psi^2_{(i)})^{r_3}_{i=1}, \overline{\psi}_2 - 1, (\psi^{2\prime}_{(i)})^{r_4}_{i=1})$, $\Psi'_2 = (\psi^{2\prime}_{(i)})^{r_4}_{i=1}$, $M'(\eta,\zeta) = A'^{-1}_1((\eta + i\xi)E' - i\sum^m_{j=2} A'_j\omega_j)$, $M''(t,x,v,\eta,\zeta) = M(t,x,v,\eta,\zeta) - M'(\eta,\zeta)$. Moreover,

$$(3.2.61) \quad H''_i(\Psi_i(t,x,v,\eta,\zeta), M''(t,x,v,\eta,\zeta), M'(\eta,\zeta)) = 0 \quad (i = 1,2)$$

outside a compact set $\overline{\Omega}_{\widetilde{K}} \subset \mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n$, where $\widetilde{\zeta} = (\eta, \zeta) \in \{\widetilde{\zeta} : |\widetilde{\zeta}| \leq 1\}$.

R e m a r k 3.2.8. Let (E_μ), $(A_{j\mu})$ $(j = 1,\ldots,m)$, (C_μ) and (S_μ) be sequences satisfying assumptions (3.1.1)–(3.1.7), (3.1.14), (3.1.20), (3.1.24) and assume that condition (3.1.7) is fulfilled with the same constant matrices E', A'_j $(j = 1,\ldots,m)$, C' and S' (independent of μ) as for the functions E, A_j, C and S. Moreover, assume that $E_\mu, A_{j\mu}, C_\mu, S_\mu \to E, A_j, C, S$ in $C^k_B(\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n)$ as $\mu \to 0$. Then by (3.2.60), $R_\mu \to R$ in $C^k_B(Q)$ as $\mu \to 0$.

3.3. An estimate of a solution of the boundary value problem (3.1)–(3.2) in L^2_η-norm. In this chapter let

$$(3.3.1) \qquad z = (t,x'), \quad \text{where} \quad x' = (x_2,\ldots,x_m).$$

Take any $v = v(t,x) \in \Pi^k_0(\mathbf{R} \times \Omega)$ with $k > m/2 + 5/2$ and write

$$(3.3.2) \qquad G(z,x_1) = G(t,x,v(t,x)) \quad \forall (z,x_1) \in \mathbf{R}^m \times [0,\infty),$$

where $G = E, A_j, C, S$, i.e. $G = E, A_j, C, S$;

$$(3.3.3) \quad J(z,x_1,\eta,\zeta) = J(t,x,v(t,x),\eta,\zeta)$$
$$\forall (z,x_1,\eta,\zeta) \in \mathbf{R}^m \times [0,\infty) \times [\eta_1,\infty) \times \mathbf{R}^m,$$

where $J = M, R, \Psi_1, \Psi_2$ (R, Ψ_1, Ψ_2 are defined by (3.2.61)), i.e. $J = M, R, \Psi_1, \Psi_2$. Notice that in view of assumption (3.1.7) and (3.2.61) the functions $G = E, A_j, C, S$ and $J = M, R, \Psi_1, \Psi_2$ can be written in the form

$$(3.3.4) \qquad G(z,x_1) = G' + G''(z,x_1)$$

and

$$(3.3.5) \qquad J(z, x_1, \eta, \zeta) = J' + J''(z, x_1, \eta, \zeta)$$

respectively. Here the functions E'', A_j'', C'', S'', M'', R'', Ψ_i'' vanish outside a compact set $K_{t,x} \subset \mathbf{R} \times \bar{\Omega}$ and

$$(3.3.6) \qquad R'(\eta, \zeta) = \overline{\psi}_3(\eta, \zeta) H_1'(\psi_1'(\eta, \zeta) M'(\eta, \zeta))$$
$$+ \overline{\psi}_4(\eta, \zeta) H_2' \left(\Psi_2' \left(\frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \right), M' \left(\frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \right) \right),$$

$$(3.3.7) \qquad R''(z, x_1, \eta, \zeta) = \overline{\psi}_3(\eta, \zeta) H_1''(\Psi_1(z, x_1, \eta, \zeta),$$
$$M''(z, x_1, \eta, \zeta), M'(\eta, \zeta))$$
$$+ \overline{\psi}_4(\eta, \zeta) H_2'' \left(\Psi_2 \left(z, x_1, \frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \right), M'' \left(z, x_1, \frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \right), M' \left(\frac{\eta}{|\widetilde{\zeta}|}, \frac{\zeta}{|\widetilde{\zeta}|} \right) \right).$$

Notice further that

$$(3.3.8) \qquad M''(t, x, v, \eta, \zeta) \in K_{M''}$$
$$\forall (t, x, v, \eta, \zeta) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times \{(\eta, \zeta) \in [\eta_1, \infty) \times \mathbf{R}^m : |\widetilde{\zeta}| \le 1\},$$

$$(3.3.9) \qquad M'(\eta, \zeta) \in K_{M'} \quad \forall (\eta, \zeta) \in \{(\eta, \zeta) \in [\eta_1, \infty) \times \mathbf{R}^m : |\widetilde{\zeta}| \le 1\},$$

where $K_{M''}$ and $K_{M'}$ are compact subsets of $\mathbf{R}^{n^2}$. Moreover, define

$$(3.3.10) \qquad \widetilde{L} = (E, (A_j)_{j=1}^m, C),$$

$$(3.3.11) \qquad \widetilde{L} = (E, (A_j)_{j=1}^m, C),$$

$$(3.3.12) \qquad \widetilde{L}'' = (E'', (A_j'')_{j=1}^m, C), \ \widetilde{\mathcal{L}}'' = (\widetilde{\mathcal{E}}'', (\mathcal{A}_j'')_{j=1}^m, C'') \ \text{(where} \ \mathcal{G}''(t, x)$$
$$= G''(z, x_1) \ \ \forall (t, x) \in \mathbf{R} \times \Omega; \ \mathcal{G}'' = \mathcal{E}'', \mathcal{A}_j'', C'');$$

$$(3.3.13) \qquad \overline{a} = |\widetilde{L}''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)} = |\widetilde{\mathcal{L}}''|_{k,0,\infty,\mathbf{R} \times \Omega},$$
$$\overline{b} = \|\widetilde{L}\|_{C_B^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)} + \|S\|_{C_B^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)},$$
$$\overline{c} = \sum_{|\alpha| \le 2} \sup_{|\widetilde{\zeta}| \le 1} |\partial_\zeta^\alpha \Psi_1''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}$$
$$+ \sum_{|\alpha| \le 2} \sup_{|\widetilde{\zeta}|=1} |\partial_\zeta^\alpha \Psi_2''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)};$$
$$\overline{d} = \|H_1''\|_{C^{k+2}([0,1]^{r_1+r_2+2} \times K_{M''} \times K_{M'})}$$
$$+ \|H_2''\|_{C^{k+2}([0,1]^{r_1+r_2+2} \times K_{M''} \times K_{M'})}$$
$$+ \|\Psi_1\|_{C^{k+2}(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times \{\zeta : |\widetilde{\zeta}| \le 1\})}$$
$$+ \|\Psi_2\|_{C^{k+2}(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times \{\zeta : |\widetilde{\zeta}| \le 1\})},$$
$$\overline{e} = |S''|_{k,0,\infty,\mathbf{R} \times \mathbf{R}^{m-1} \times [0,\infty)},$$

$$\overline{f} = \|S''\|_{k,\eta,\mathbf{R}\times\mathbf{R}^{m-1}} .$$

COROLLARY 3.3.1.

$1°$ $\forall|\alpha| \leq 2$ $\forall\widetilde{\zeta} = (\eta,\zeta) \in [\eta_1,\infty) \times \mathbf{R}^m$, $\partial_\zeta^\alpha R''(\cdot,\cdot,\eta,\zeta) \in \Pi_0^k(\mathbf{R}^m \times [0,\infty))$ and

(3.3.14) $$|\partial_\zeta^\alpha R''|_{k,0,\infty,\mathbf{R}^m\times[0,\infty)} \leq p^\alpha(\overline{a},\overline{c},\overline{d})(1 + |\widetilde{\zeta}|)^{-|\alpha|} ,$$

where p^α is a polynomial;

$2°$ $R' \in C_B^k([\eta_1,\infty) \times \mathbf{R}^m)$ and $|\partial_\zeta^\alpha R'| \leq c(1 + |\widetilde{\zeta}|)^{-|\alpha|}$, $\forall|\alpha| \leq k$, where $c > 0$ is a constant;

$3°$ $R'' \in C_B^2(\mathbf{R}^m \times [0,\infty) \times [\eta_1,\infty) \times \mathbf{R}^m)$;

$4°$ $\forall(z,x_1,\eta,\zeta) \in \mathbf{R}^m \times [0,\infty) \times [\eta_1,\infty) \times \mathbf{R}^m$, $R(z,x_1,\eta,\zeta)$ is a hermitian matrix;

$5°$ $\exists\varepsilon_1 > 0$ $\forall(z,x_1,\eta,\zeta) \in \mathbf{R}^m \times [0,\infty) \times [\eta_1,\infty) \times \mathbf{R}^m$

$$\operatorname{Re} R(z,x_1,\eta,\zeta)M(z,x_1,\eta,\zeta) \geq \varepsilon_1\eta I ,$$

where ε_1 is independent of z, x_1, η and ζ, while it depends on η_1 and E, A_j $(j = 1,\ldots,m)$, C;

$6°$ $\exists\varepsilon_2,\varepsilon_3 > 0$ $\forall(z,\eta,\zeta) \in \mathbf{R}^m \times [\eta_1,\infty) \times \mathbf{R}^m$

$$y^* R(z,0,\eta,\zeta)y \geq \varepsilon_2|y|^2 - \varepsilon_3|g(z)|^2$$

for all vectors y satisfying the boundary condition (3.1.9) with the functions S and $g(z) = g(t,x')$, where ε_2 and ε_3 are independent of z, η and ζ, while they depend on η_1 and E, A_j $(j = 1,\ldots,m)$, C, S.

P r o o f. $4°$–$6°$ follow immediately from Theorem 3.2.1 ($2°$–$5°$), $2°$ follows from the proof of Theorem 3.2.1. By the assumption and the Sobolev lemma $\tilde{v} \in C_B^2(\mathbf{R}^m \times [0,\infty))$, where $\tilde{v}(z,x_1) = v(t,x)$, $\forall(z,x_1) \in \mathbf{R}^m \times [0,\infty)$. Therefore Theorem 3.2.1($1°$) and (3.3.5)–(3.3.6) imply $3°$.

We now check $1°$. By (3.3.8) and Lemma 2.1.2

$$\forall(\eta,\zeta) \in [\eta_1,\infty) \times \mathbf{R}^m \quad |R''(\cdot,\cdot,\eta,\zeta)|_{k,0,\infty,\mathbf{R}^m\times[0,\infty)}$$

$$\leq p^0(\|H_1''\|_{C_B^k} + \|H_2''\|_{C_B^k}, \sup_{\overline{\zeta}=(\eta,\zeta):|\zeta|\leq 1} |\Psi_1''|_{k,0,\infty,\mathbf{R}^m\times[0,\infty)}$$

$$+ \sup_{\overline{\zeta}=(\eta,\zeta):|\zeta|\leq 1} |\Psi_2''|_{k,0,\infty,\mathbf{R}^m\times[0,\infty)}, |\widetilde{L}''|_{k,0,\infty,\mathbf{R}^m\times[0,\infty)})$$

$$\leq p^0(\overline{a},\overline{c},\overline{d}) .$$

Take $|\alpha| = 1$. Suppose $|\widetilde{\zeta}| \leq 1/2$. Then (3.2.61) implies that

$$\partial_\zeta^\alpha R''(t,x,v,\eta,\zeta) = \partial_{\Psi_1}^\alpha H_1'' \partial_\zeta^\alpha \Psi_1(t,x,v,\eta,\zeta)$$

$$+ \partial_{M''}^\alpha H_1'' \partial_\zeta^\alpha M''(t,x,v,\eta,\zeta) + \partial_{M'}^\alpha H_1'' \partial_\zeta^\alpha M'(\eta,\zeta) .$$

Denote further by F_1 a linear function such that $\partial_\zeta^\alpha R'' = F_1(\partial_{\Psi_1}^\alpha H_1'', \partial_\zeta^\alpha \Psi_1,$ $\partial_{M''}^\alpha H_1'', \partial_\zeta^\alpha M'', \partial_{M'}^\alpha H_1'', \partial_\zeta^\alpha M')$ and let

$$G_1(z, x_1, \eta, \zeta) = \partial_{\Psi_1}^\alpha H_1''(\Psi_1(z, x_1, \eta, \zeta), M''(z, x_1, \eta, \zeta), M'(\eta, \zeta)),$$
$$G_2(z, x_1, \eta, \zeta) = \partial_{M''}^\alpha H_1''(\Psi_1(z, x_1, \eta, \zeta), M''(z, x_1, \eta, \zeta), M'(\eta, \zeta)),$$
$$G_3(z, x_1, \eta, \zeta) = \partial_{M'}^\alpha H_1''(\Psi_1(z, x_1, \eta, \zeta), M''(z, x_1, \eta, \zeta), M'(\eta, \zeta)).$$

Then by Lemma 2.2.1

$$|\partial_\zeta^\alpha R''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}$$
$$\leq q(\|F_1\|_{C_B^k}, |G_1|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}, |\partial_\zeta^\alpha \Psi_1''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)},$$
$$|G_2|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}, |\widetilde{L}''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}, |G_3|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)})$$
$$\leq \widetilde{q}(\|\partial_{\Psi_1}^\alpha H_1''\|_{C_B^k}, \|\partial_\zeta^\alpha \Psi\|_{C_B^k}, \|\partial_{M''}^\alpha H_1''\|_{C_B^k}, \|\partial_{M'}^\alpha H_1''\|_{C_B^k}, \|\widetilde{L}\|_{C_B^k},$$
$$|\Psi_1''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}, |\partial_\zeta^\alpha \Psi_1''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}, |\widetilde{L}''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)})$$
$$\leq \widetilde{p}^\alpha(\overline{a}, \overline{c}, \overline{d}) \leq p^\alpha(\overline{a}, \overline{c}, \overline{d})(1 + |\widetilde{\zeta}|)^{-1}.$$

For $1/2 \leq |\widetilde{\zeta}| \leq 1$ we take into account in (3.2.61) also a term associated with H_2'' and we use the same argument as before. Assume now that $|\widetilde{\zeta}| \geq 1$. Then by (3.2.61),

$$\partial_\zeta^\alpha R'' = \partial_{\Psi_2}^\alpha H_2'' \partial_\zeta^\alpha \Psi_1 \partial_\zeta^\alpha \widetilde{\zeta}' + \partial_{M''}^\alpha H_2'' \partial_\zeta^\alpha M'' \partial_\zeta^\alpha \widetilde{\zeta}',$$

where $\widetilde{\zeta}' = (\eta', \zeta')$. Hence in the same way as for $|\widetilde{\zeta}| \leq 1/2$ we get the estimate

$$|\partial_\zeta^\alpha R''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)} \leq p^\alpha(\overline{a}, \overline{c}, \overline{d})(1 + |\widetilde{\zeta}|)^{-1}.$$

The proof of (3.3.14) for $|\alpha| = 2$ is similar. ∎

Introduce now the following pseudodifferential operators:

$$(3.3.15) \qquad M(z, x_1, \eta, D_z) w(z, x_1)$$
$$= (2\pi)^{-m} \int M(z, x_1, \eta, \zeta) \widehat{w}(\zeta, x_1) e^{iz\zeta} \, d\zeta,$$

$$(3.3.16) \qquad R(z, x_1, \eta, D_z) w(z, x_1)$$
$$= (2\pi)^{-m} \int R(z, x_1, \eta, \zeta) \widehat{w}(\zeta, x_1) e^{iz\zeta} \, d\zeta,$$

where the symbols M and R are given by (3.3.5) and (3.3.4), $\widehat{w}(\cdot, x_1)$ is the Fourier transform of $w(\cdot, x_1)$, $\eta > 0$ is a parameter.

The operator R given by (3.3.16) is called the *symmetrizer* for the boundary value problem (3.1)–(3.2). We shall use the following lemma of [2].

LEMMA 3.3.1. *Assume that*

$$C_g^2 \equiv \sup_\kappa \int |g(\zeta, \kappa)|^2 \, d\zeta < \infty, \qquad C_G^2 \equiv \sup_\zeta \int |G(\zeta, \kappa)|^2 \, d\zeta < \infty.$$

For $h \in L^2$ define

$$(Th)(\zeta) = \int G(\zeta, \kappa) g(\zeta - \kappa, \kappa) h(\kappa) \, d\kappa.$$

Then $\|Th\|_{L^2} \leq C_g C_G \|h\|_{L^2}$.

LEMMA 3.3.2. *Define*

(3.3.17) $B(z, x_1, \eta, D_z) w(z, x_1)$
$$= (2\pi)^{-m} \int R(z, x_1, \eta, \zeta) M(z, x_1, \eta, \zeta) \widehat{w}(\zeta, x_1) e^{iz\zeta} \, d\zeta.$$

Then

(3.3.18) $\forall w \in C_{(0)}^\infty(\mathbf{R}^m \times [0, \infty))$
$$|(RM - B) w(\cdot, x_1), w(\cdot, x_1))_{0, \mathbf{R}^m}| \leq p(\overline{a}, \overline{b}, \overline{c}, \overline{d}) \|w(\cdot, x_1)\|_{0, \mathbf{R}^m}^2.$$

Proof. Let $R'(\eta, D_z)$, $R''(z, x_1, \eta, D_z)$, $M'(\eta, D_z)$, $M''(z, x_1, \eta, D_z)$, $B_1(\eta, D_z)$, $B_2(z, x_1, \eta, D_z)$, $B_3(z, x_1, \eta, D_z)$ and $B_4(z, x_1, \eta, D_z)$ be the pseudodifferential operators with the respective symbols $R'(\eta, \zeta)$, $R''(z, x_1, \eta, \zeta)$, $M'(\eta, \zeta)$, $M''(z, x_1, \eta, \zeta)$, $B_1 = R'M'$, $B_2 = R'M''$, $B_3 = R''M'$, $B_4 = R''M''$. Write

(3.3.19) $RM - B = (R'M' - B_1) + (R'M'' - B_2) + (R''M' - B_3)$
$$+ (R''M'' - B_4).$$

Take $w \in C_{(0)}^\infty(\mathbf{R}^m \times [0, \infty))$. Then

(3.3.20) $R'M'w(z, x_1) = (2\pi)^{-m} \int R'(\eta, \zeta) \widehat{M'w}(\zeta, x_1, \eta) e^{iz\zeta} \, d\zeta$
$$= (2\pi)^{-m} \int R'(\eta, \zeta) M'(\eta, \zeta) \widehat{w}(\zeta, x_1) e^{iz\zeta} \, d\zeta,$$

(3.3.21) $R'M''w(z, x_1) = (2\pi)^{-m} \int R'(\eta, \zeta)$
$$\times \left(\int \widehat{M''}(\zeta - \kappa, x_1, \eta, \kappa) \widehat{w}(\kappa, x_1) \, d\kappa \right) e^{iz\zeta} \, d\zeta,$$

(3.3.22) $R''M'w(z, x_1) = (2\pi)^{-m} \int R''(z, x_1, \eta, \zeta) M'(\eta, \zeta) \widehat{w}(\zeta, x_1) e^{iz\zeta} \, d\zeta,$

(3.3.23) $R''M''w(z, x_1) = (2\pi)^{-m} \int R''(z, x_1, \eta, \zeta)$
$$\times \left(\int \widehat{M''}(\zeta - \kappa, x_1, \eta, \kappa) \widehat{w}(\kappa, x_1) \, d\kappa \right) e^{iz\zeta} \, d\zeta.$$

Notice first that

(3.3.24) $((R'M' - B_1) w(\cdot, x_1), w(\cdot, x_1))_{0, \mathbf{R}^m} = 0,$

(3.3.25) $((R''M' - B_3) w(\cdot, x_1), w(\cdot, x_1))_{0, \mathbf{R}^m}$
$$= (((R''M' - B_3) w)^\wedge(\cdot, x_1) \widehat{w}(\cdot, x_1))_{0, \mathbf{R}^m} = 0,$$

for any $x_1 \in [0, \infty)$.

Since $v \in \Pi_0^k(\mathbf{R} \times \Omega)$ $(k > m/2 + 5/2)$, from Lemma 2.1.2 it follows that $\forall \kappa \in \mathbf{R}^m \, \forall \eta \in [\eta_1, \infty)$, $M''(\cdot, \cdot, \eta, \kappa) \in \Pi_0^k(\mathbf{R}^m \times [0, \infty)) \subset H^k(\mathbf{R}^m \times [0, \infty))$.

Therefore using the definition of Sobolev spaces (Chapter 2) and the trace theorem we get

$$(3.3.26) \int (1 + |\zeta|^2)^{k-1/2} |\widehat{M''}(\zeta, x_1, \eta, \kappa)|^2 \, d\zeta = \|M''(\cdot, x_1, \eta, \kappa)\|^2_{k-1/2, \mathbf{R}^m}$$

$$\leq c |M''(\cdot, \cdot, \eta, \kappa)|^2_{k,0,\infty, \mathbf{R}^m \times [0,\infty)} \leq [p_1(\overline{a})]^2 (\eta + |\kappa|)^2$$

for any $x_1 \in [0, \infty)$, $\eta \in [\eta_1, \infty)$ and $\kappa \in \mathbf{R}^m$ (p_1 is a polynomial, and so are all p_i appearing below). Using now the definition of the Fourier transformation in $\mathcal{S}'(\mathbf{R}^m)$ and the Taylor formula we obtain

$$(3.3.27) \quad ((R'M'' - B_2)w)^{\wedge}(\cdot, x_1)$$

$$= \int (R'(\eta, \zeta) - R'(\eta, \kappa)) \cdot \widehat{M''}(\zeta - \kappa, x_1, \eta, \kappa) \widehat{w}(\kappa, x_1) \, d\kappa$$

$$= \int \int_0^1 \sum_{|\gamma|=1} \partial_\zeta^\gamma R'(\eta, \kappa + \theta(\zeta - \kappa)) \cdot (\zeta - \kappa)^\gamma$$

$$\times \widehat{M''}(\zeta - \kappa, x_1, \eta, \kappa) \widehat{w}(\kappa, x_1) \, d\theta \, d\kappa$$

$$= \int \left(\int_0^1 \sum_{|\gamma|=1} (1 + |\zeta - \kappa|^2)^{-k/2+1/4} \partial_\zeta^\gamma R'(\eta, \kappa + \theta(\zeta - \kappa)) \right) \cdot (\zeta - \kappa)^\gamma \, d\theta$$

$$\times (\eta + |\kappa|)(\eta + |\kappa|)^{-1} (1 + |\zeta - \kappa|^2)^{k/2-1/4} \widehat{M''}(\zeta - \kappa, x_1, \eta, \kappa) \widehat{w}(\kappa, x_1) \, d\kappa \, .$$

Let us now estimate $\partial_\zeta^\gamma R'(\eta, \kappa + \theta(\zeta - \kappa))$ in the sets:

$$(3.3.28) \qquad \begin{aligned} \Omega_1 &= \{(\theta, \eta, \kappa, \zeta) : |\theta(\zeta - \kappa)| \leq |\kappa|/2\} \, , \\ \Omega_2 &= \{(\theta, \eta, \kappa, \zeta) : |\theta(\zeta - \kappa)| \geq |\kappa|/2\} \, . \end{aligned}$$

In Ω_1,

$$(3.3.29) \qquad \tfrac{1}{2}(\eta + |\kappa|) \leq 1 + \eta + |\kappa + \theta(\zeta - \kappa)| \, .$$

By Corollary 3.3.1(2°) and (3.3.29) we have in Ω_1

$$(3.3.30) \qquad |\partial_\zeta^\gamma R'(\eta, \kappa + \theta(\zeta - \kappa))| \leq c_{1\gamma}(\eta + |\kappa|)^{-1} \, .$$

In Ω_2 we have

$$(3.3.31) \qquad \tfrac{1}{2}(\eta + |\kappa|) \leq \eta + |\zeta - \kappa| \, .$$

By Corollary 3.3.1(2°) and (3.3.31) we obtain in Ω_2

$$(3.3.32) \quad |\partial_\zeta^\gamma R'(\eta, \kappa + \theta(\zeta - \kappa))| \leq \frac{c}{1 + \eta + |\kappa + \theta(\zeta - \kappa)|}$$

$$\leq \frac{c}{\eta^2}(\eta + |\kappa|)(\eta + |\kappa|)^{-1}$$

$$\leq c_{2\gamma}(1 + |\zeta - \kappa|)(\eta + |\kappa|)^{-1} \quad \text{for } \eta \geq 1 \, .$$

Now, (3.3.30) and (3.3.32) yield

$$(3.3.33) \qquad |\partial_\zeta^\gamma R'(\eta, \kappa + \theta(\zeta - \kappa))| \leq c(1 + |\zeta - \kappa|)(\eta + |\kappa|)^{-1} \, ,$$

where $c > 0$ is a constant. Applying (3.3.33) we obtain

$$(3.3.34) \quad C_G^2 \equiv \sup_\zeta \int (1 + |\zeta - \kappa|^2)^{-k+1/2} |\zeta - \kappa|^2$$

$$\times \int_0^1 \sum_{|\gamma|=1} |\partial_\zeta^\gamma R'(\eta, \kappa + \theta(\zeta - \kappa))|^2 \, d\theta \, (\eta + |\kappa|)^2 \, d\kappa$$

$$\leq \sup_\zeta \int (1 + |\zeta - \kappa|^2)^{-k+5/2} \, d\kappa < C \,,$$

where $C > 0$ is a constant. Using next (3.3.26) we have

$$(3.3.35) \quad C_g^2 \equiv \sup_\zeta \int (\eta + |\kappa|)^{-2} (1 + |\zeta|^2)^{k-1/2} |\widehat{M''}(\zeta, x_1, \eta, \kappa)|^2 \, d\zeta$$

$$\leq [p_1(\bar{a})]^2 \,.$$

By (3.3.34), (3.3.35) and Lemma 3.3.1 we obtain

$$\|((R'M'' - B_2)w)^\wedge(\cdot, x_1)\|_{0, \mathbf{R}^m} \leq p_2(\bar{a}) \|\widehat{w}(\cdot, x_1)\|_{0, \mathbf{R}^m} \,.$$

The Parseval equality now gives

$$(3.3.36) \quad |((R'M'' - B_2)w(\cdot, x_1), w(\cdot, x_1))_{0, \mathbf{R}^m}| \leq p_3(\bar{a}) \|w(\cdot, x_1)\|_{0, \mathbf{R}^m}^2 \,.$$

Next consider

$$(3.3.37) \quad ((R''M'' - B_4)w)^\wedge(\cdot, x_1) = \int \int [\widehat{R''}(\zeta - \kappa, x_1, \eta, \kappa)$$

$$- \widehat{R''}(\zeta - \kappa, x_1, \eta, \rho)] \, \widehat{M''}(\kappa - \rho, x_1, \eta, \rho) \widehat{w}(\rho, x_1) \, d\rho \, d\kappa$$

$$= \int \int \int_0^1 \sum_{|\gamma|=1} \partial_\rho^\gamma \widehat{R''}(\zeta - \kappa, x_1, \eta, \rho + \theta(\kappa - \rho)) \cdot (\kappa - \rho)^\gamma \, d\theta$$

$$\times \widehat{M''}(\kappa - \rho, x_1, \eta, \rho) \widehat{w}(\rho, x_1) \, d\rho \, d\kappa \,.$$

Using Corollary 3.3.1(1°), the definition of Sobolev spaces and the trace theorem we obtain

$$(3.3.38) \quad \int (1 + |\zeta|^2)^{k-1/2} |\partial_\kappa^\gamma \widehat{R''}(\zeta, x_1, \eta, \kappa)|^2 \, d\zeta$$

$$= \|\partial_\kappa^\gamma R''(\cdot, x_1, \eta, \kappa)\|_{k-1/2, \mathbf{R}^m}^2 \leq c |\partial_\kappa^\gamma R''(\cdot, \cdot, \eta, \kappa)|_{k, 0, \infty, \mathbf{R}^m \times [0, \infty)}^2$$

$$\leq [p_4(\bar{a}, \bar{c}, \bar{d})]^2 (1 + \eta + |\kappa|)^{-|\gamma|}$$

for $|\gamma| \leq 2$, $x_1 \in [0, \infty)$, $\eta \in [\eta_1, \infty)$ and $\kappa \in \mathbf{R}^m$ (where we have used the fact that $(\partial_\kappa^\gamma R'')^\wedge(\zeta, x_1, \eta, \kappa) = \partial_\kappa^\gamma \widehat{R''}(\zeta, x_1, \eta, \kappa)$ for $|\gamma| \leq 2$). Next define

$$F(\zeta, \rho) = (1 + |\zeta - \rho|^2)^{-k/2 + 5/4} F_1(\zeta, \rho)$$

$$= (1 + |\zeta - \rho|^2)^{-k/2 + 5/4} \int \int_0^1 \sum_{|\gamma|=1} \partial_\rho^\gamma \widehat{R''}(\zeta - \kappa, x_1, \eta, \rho + \theta(\kappa - \rho))$$

$$= (1 + |\kappa - \rho|^2)^{-k/2 + 5/4} \int \int_0^1 \sum_{|\gamma| = 1} (1 + |\zeta - \kappa|^2)^{-k/2 + 1/4}$$

$$\times \partial_\rho^\gamma \widehat{R''}(\zeta - \kappa, x_1, \eta, \rho + \theta(\kappa - \rho)) \cdot (\kappa - \rho)^\gamma \, d\theta (\eta + |\rho|)$$

$$\times (1 + |\kappa - \rho|^2)^{-k/2 + 5/4} (1 + |\kappa - \rho|^2)^{k/2 - 5/4} (\eta + |\rho|)^{-1}$$

$$\times (1 + |\zeta - \kappa|^2)^{k/2 - 1/4} \widehat{M''}(\kappa - \rho, x_1, \eta, \rho)(1 + |\zeta - \rho|^2)^{k/2 - 5/4} \, d\kappa,$$

$$G(\zeta, \kappa) = (1 + |\zeta - \kappa|^2)^{-k/2 + 1/4}(1 + |\kappa - \rho|^2)^{-k/2 + 1/4}$$

$$\times (1 + |\zeta - \rho|^2)^{k/2 - 5/4}(\eta + |\rho|)^{-1}(1 + |\kappa - \rho|^2),$$

$$g(\zeta, \kappa) = (1 + |\zeta|^2)^{k/2 - 1/4} \int_0^1 \sum_{|\gamma| = 1} \partial_\rho^\gamma \widehat{R''}(\zeta, x_1, \eta, \rho + \theta(\kappa - \rho))$$

$$\times (\kappa - \rho)^\gamma \, d\theta (\eta + |\rho|)(1 + |\kappa - \rho|^2)^{-1}$$

(where in the last two formulae we fix ρ). Let us calculate:

$$C_G^2 \equiv \sup_\zeta \int |G(\zeta, \kappa)|^2 \, d\kappa$$

$$= \sup_\zeta \int (1 + |\zeta - \rho|^2)^{k - 5/2}(1 + |\zeta - \kappa|^2)^{-k + 5/2}$$

$$\times (1 + |\kappa - \rho|^2)^{-k + 5/2}(\eta + |\rho|)^{-2} \, d\kappa \le (\eta + |\rho|)^{-2};$$

$$C_g^2 \equiv \sup_\kappa \int |g(\zeta, \kappa)|^2 \, d\zeta \le c \sup_\kappa \int (1 + |\zeta|^2)^{k - 1/2}(1 + |\kappa - \rho|^2)^{-2}$$

$$\times \sum_{|\gamma| = 1} |\partial_\rho^\gamma \widehat{R''}(\zeta, x_1, \eta, \rho + \theta(\kappa - \rho))|^2(\eta + |\rho|)^2(1 + |\kappa - \rho|^2) \, d\kappa$$

$$\le [p_5(\bar{a}, \bar{c}, \bar{d})]^2(1 + |\kappa - \rho|^2)^{-1}(\eta + |\rho|)^2(1 + \eta + |\rho + \theta(\kappa - \rho)|)^{-2}$$

$$\le [p_6(\bar{a}, \bar{c}, \bar{d})]^2,$$

where we have used (3.3.38) and the method applied earlier.

Applying Lemma 3.3.1 with $h(\kappa) = (1 + |\kappa - \rho|^2)^{k/2 - 1/4}\widehat{M''}(\kappa - \rho, x_1, \eta, \rho)$ and (3.3.26) we get

$$(3.3.39) \quad \forall \rho \in \mathbf{R}^m \quad \|F_1(\cdot, \rho)\|_{0, \mathbf{R}^m} \le p_7(\bar{a}, \bar{c}, \bar{d})\sqrt{(\eta + |\rho|)^{-2}}(\eta + |\rho|)$$
$$= p_7(\bar{a}, \bar{c}, \bar{d}).$$

Thus taking into account (3.3.37) we have

$$(3.3.40) \quad \|((R''M'' - B_4)w)^\wedge(\cdot, x_1)\|_{0, \mathbf{R}^m}$$

$$= \sup_{\|f\|_{0, \mathbf{R}^m} \le 1} \left| \int f(\zeta) \left[\int (1 + |\zeta - \rho|^2)^{-k/2 + 5/4} F_1(\zeta, \rho)\widehat{w}(\rho, x_1) \, d\rho \right] d\zeta \right|$$

$$\le \sup_{\|f\|_{0, \mathbf{R}^m} \le 1} \int |\widehat{w}(\rho, x_1)|(1 + |\zeta - \rho|^2)^{-k/2 + 5/4}|F_1(\zeta, \rho)||f(\zeta)| \, d\rho \, d\zeta$$

$$\leq \sup_{\|f\|_{0,\mathbf{R}^m} \leq 1} \left(\int \int |\widehat{w}(\rho, x_1)|^2 |F_1(\zeta, \rho)|^2 \, d\zeta \, d\rho \right)^{1/2}$$

$$\times \left(\int \int |f(\zeta)|^2 (1 + |\zeta - \rho|^2)^{-k+5/2} \, d\zeta \, d\rho \right)^{1/2}$$

$$\leq \sup_{\|f\|_{0,\mathbf{R}^m} \leq 1} \left(\sup_{\rho} \int |F_1(\zeta, \rho)|^2 \, d\zeta \right)^{1/2} \|\widehat{w}(\cdot, x_1)\|_{0,\mathbf{R}^m}$$

$$\times \sup_{\zeta} \int (1 + |\zeta - \rho|^2)^{-k+5/2} \, d\rho \|f\|_{0,\mathbf{R}^m} \Big)$$

$$\leq p_8(\overline{a}, \overline{c}, \overline{d}) \|\widehat{w}(\cdot, x_1)\|_{0,\mathbf{R}^m} \ ,$$

where we have used (3.3.39).

From (3.3.24), (3.3.25), (3.3.27) and (3.3.40) the assertion of the lemma follows. ■

LEMMA 3.3.3. *Let B be the operator given by (3.3.17). Then*

$$(3.3.41) \quad \forall w \in C_{(0)}^\infty(\mathbf{R}^m \times [0, \infty))$$

$$\mathrm{Re}(Bw(\cdot, x_1), w(\cdot, x_1))_{0,\mathbf{R}^m} \geq \varepsilon_1 \eta \|w(\cdot, x_1)\|_{0,\mathbf{R}^m}^2 - p(\overline{a}, \overline{c}, \overline{d}) \|w(\cdot, x_1)\|_{0,\mathbf{R}^m}^2 \ ,$$

where p is a polynomial and ε_1 is the constant from Theorem 3.2.1.

P r o o f. Define

$$(3.3.42) \quad Q(z, x_1, \eta, \zeta) = \mathrm{Re}\, R(z, x_1, \eta, \zeta) M(z, x_1, \eta, \zeta) - \varepsilon_1 \eta I,$$

$$(3.3.43) \quad Q'(\eta, \zeta) = \mathrm{Re}\, R'(\eta, \zeta) M'(\eta, \zeta) - \varepsilon_1 \eta I,$$

$$(3.3.44) \quad Q''(z, x_1, \eta, \zeta) = \mathrm{Re}\, R'(\eta, \zeta) M''(z, x_1, \eta, \zeta)$$
$$+ \mathrm{Re}\, R''(z, x_1, \eta, \zeta) M'(\eta, \zeta)$$
$$+ \mathrm{Re}\, R''(z, x_1, \eta, \zeta) M''(z, x_1, \eta, \zeta).$$

Let next $Q(z, x_1, \eta, D_z)$, $Q'(\eta, D_z)$ and $Q''(z, x_1, \eta, D_z)$ be the pseudodifferential operators with symbols $Q(z, x_1, \eta, \zeta)$, $Q'(\eta, \zeta)$ and $Q''(z, x_1, \eta, \zeta)$. As in the proof of [4, Theorem 5.1] consider the covering of $[\eta_1, \infty) \times \mathbf{R}^m$ by the balls $K_{\widetilde{\zeta}}$ ($\widetilde{\zeta} = (\eta, \zeta) \in [\eta_1, \infty) \times \mathbf{R}^m$) with centre at $\widetilde{\zeta}$ and radius $(1 + |\widetilde{\zeta}|^2)^{1/4}/4$. Choose its locally finite subcovering such that each point of $[\eta_1, \infty) \times \mathbf{R}^m$ is covered by no more than $c(m)$ balls. Number these balls as $K_1, \ldots, K_j, \ldots$ and construct functions $h_j(\widetilde{\zeta})$ such that $h_j \in C_{(0)}^\infty([\eta_1, \infty) \times \mathbf{R}^m)$, $h_j \geq 0$, $h_j(\widetilde{\zeta}) = 1$ for $\widetilde{\zeta} \in K_j$ and $h_j = 0$ outside the ball K_j^1 with the same centre $\widetilde{\zeta}^j$ as K_j and twice its radius. Moreover, assume that $|\partial^\alpha h_j(\widetilde{\zeta})| \leq c_\alpha (1 + |\widetilde{\zeta}^j|)^{-|\alpha|/2}$ and c_α is independent of j. Now define

$$\varphi_j(\widetilde{\zeta}) = h_j(\widetilde{\zeta}) \Big[\sum_{l=1}^\infty h_l^2(\widetilde{\zeta}) \Big]^{-1/2} \ .$$

Then $\sum_{j=1}^{\infty} \varphi_j^2(\widetilde{\zeta}) = 1$, $\varphi_j \in C_{(0)}^{\infty}([\eta_1, \infty) \times \mathbf{R}^m)$ and

$$(3.3.45) \qquad\qquad |\partial^{\alpha} \varphi_j| \leq c_{\alpha}'(1 + |\widetilde{\zeta^j}|)^{-|\alpha|/2},$$

where c_{α}' is independent of j. Introducing the operator $\varphi_j(\eta, D_z)$ with symbol $\varphi_j(\eta, \zeta)$ we can write

$$(3.3.46) \quad (\boldsymbol{Q}w, w)_{0, \mathbf{R}^m} = (Q'w, w)_{0, \mathbf{R}^m} + (Q''w, w)_{0, \mathbf{R}^m}$$
$$= \mathrm{Re} \sum_j \int \boldsymbol{Q}\varphi_j w \cdot \overline{\varphi_j w}\, dz + \mathrm{Re} \sum_j \int [\varphi_j(\eta, D_z), \boldsymbol{Q}'']w \cdot \overline{\varphi_j w}\, dz\,,$$

where $[\varphi_j, \boldsymbol{Q}''] = \varphi_j \boldsymbol{Q}'' - \boldsymbol{Q}'' \varphi_j$.

Using next (3.3.26) and (3.3.38) with $|\gamma| = 0$ and (3.3.44) we obtain

$$(3.3.47) \quad \int (1 + |\zeta|^2)^{k-1/2} |\widehat{Q''}(\zeta, x_1, \eta, \kappa)|^2\, d\zeta$$
$$= \|Q''(\cdot, x_1, \eta, \kappa)\|_{k-1/2, \mathbf{R}^m}^2 \leq c \|Q''(\cdot, \cdot, \eta, \kappa)\|_{k, \mathbf{R}^m \times [0, \infty)}^2$$
$$\leq p_1(\overline{a}, \overline{c}, \overline{d})(\eta + |\kappa|)^2$$

(p_1 and all p_i occurring below are some polynomials). Moreover (see [4, p. 110]),

$$(3.3.48) \quad \sum_j (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))^2 \leq c_1 (\eta + |\kappa|)^{-1}(1 + |\zeta - \kappa|^2)\,,$$

where $c_1 > 0$ is a constant.

Consider now

$$(3.3.49) \quad \mathrm{Re} \sum_j \int [\varphi_j, \boldsymbol{Q}'']w \cdot \overline{\varphi_j w}\, dz$$

$$= (2\pi)^{-m} \mathrm{Re} \sum_j \int \int (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))$$

$$\times \widehat{\boldsymbol{Q}''}(\zeta - \kappa, x_1, \eta, \kappa) \widehat{w}(\kappa, x_1) \varphi_j(\eta, \zeta) \overline{\widehat{w}}(\zeta, x_1)\, d\kappa\, d\zeta$$

$$= (2\pi)^{-m} \mathrm{Re} \sum_j \int \int (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))[\widehat{\boldsymbol{Q}''}(\zeta - \kappa, x_1, \widetilde{\kappa})$$

$$- \widehat{\boldsymbol{Q}''}(\zeta - \kappa, x_1, \widetilde{\zeta^j})]\widehat{w}(\kappa, x_1) \varphi_j(\eta, \zeta) \overline{\widehat{w}}(\zeta, x_1)\, d\kappa\, d\zeta$$
$$+ \tfrac{1}{2}(2\pi)^{-m} \mathrm{Re} \sum_j \int \int (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))^2 \widehat{\boldsymbol{Q}''}(\zeta - \kappa, x_1, \widetilde{\zeta^j})$$

$$\times \widehat{w}(\zeta, x_1) \overline{\widehat{w}}(\kappa, x_1)\, d\kappa\, d\zeta\,.$$

The first integral in (3.3.49) can be estimated in the same way as (3.3.27).

Estimate now

$$T\widehat{w}(\zeta, x_1) \equiv \frac{1}{2} \int \sum_j (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))^2 \widehat{Q''}(\zeta - \kappa, x_1, \widetilde{\zeta}^j) \widehat{w}(\kappa, x_1)\, d\kappa$$

$$= \frac{1}{2} \int \sum_j (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))^2 (\eta^j + |\zeta^j|)(1 + |\zeta - \kappa|^2)^{-k/2 + 1/2}$$

$$\times (\eta^j + |\zeta^j|)^{-1}(1 + |\zeta - \kappa|^2)^{k/2 - 1/4} \widehat{Q''}(\zeta - \kappa, x_1, \eta, \kappa) \widehat{w}(\kappa, x_1)\, d\kappa\,.$$

Taking

$$G(\zeta, \kappa) = \sum_j (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))^2 (\eta^j + |\zeta^j|)(1 + |\zeta - \kappa|^2)^{-k/2 + 1/4}\,,$$

$$g(\zeta, \kappa) = (\eta^j + |\zeta^j|)^{-1}(1 + |\zeta|^2)^{k/2 - 1/4} \widehat{Q''}(\zeta, x_1, \widetilde{\zeta}^j)\,,$$

and using Lemma 3.3.1, (3.3.47), (3.3.48) and the estimate

$$(3.3.50) \qquad c'(1 + \eta + |\kappa|) \leq 1 + |\widetilde{\zeta}^j| \leq c''(1 + \eta + |\kappa|) \quad (c', c'' > 0)$$

we get

$$(3.3.51) \qquad \|T\widehat{w}(\cdot, x_1)\|_{0, \mathbf{R}^m} \leq p_2(\overline{a}, \overline{c}, \overline{d}) \|w(\cdot, x_1)\|_{0, \mathbf{R}^m}\,.$$

By (3.3.51), (3.3.49) and (3.3.46) we obtain

$$(3.3.52) \qquad (Qw, w)_{0, \mathbf{R}^m} \geq \sum_j \operatorname{Re} \int Q\varphi_j w \cdot \overline{\varphi_j w}\, dz - p_3(\overline{a}, \overline{c}, \overline{d}) \|w\|^2_{0, \mathbf{R}^m}\,.$$

Now define

$$(3.3.53) \qquad S_j(z, x_1, \eta, D_z) = Q(z, x_1, \eta, D_z) - Q(z, x_1, \eta^j, \zeta^j)$$

$$+ \sum_{l=1}^m Q_{,\zeta_l}(z, x_1, \eta^j, \zeta^j)(D_l - \zeta_l^j) + Q_{,\zeta}(z, x_1, \eta^j, \zeta^j)(\eta - \eta^j)\,,$$

where $\widetilde{\zeta}^j = (\eta^j, \zeta^j)$ is the centre of K_j. Write the symbol $S_j(z, x_1, \eta, \zeta)$ of $S_j(z, x_1, \eta, D_z)$ as

$$(3.3.54) \qquad S_j(z, x_1, \eta, \zeta) = S_j''(z, x_1, \eta, \zeta) + S_j'(\eta, \zeta)\,,$$

where

$$(3.3.55) \qquad S_j'(\eta, \zeta) = Q'(\eta, \zeta) - Q'(\widetilde{\zeta}^j) + \sum_{r=1}^{m+1} Q_{,\widetilde{\zeta}_r}'(\widetilde{\zeta}^j)(\widetilde{\zeta}_r - \widetilde{\zeta}_r^j)\,,$$

$$(3.3.56) \qquad S_j''(z, x_1, \eta, \zeta) = Q''(z, x_1, \eta, \zeta) - Q''(z, x_1, \widetilde{\zeta}^j)$$

$$- \sum_{r=1}^{m+1} Q_{,\zeta_r}''(z, x_1, \widetilde{\zeta}^j)(\widetilde{\zeta}_r - \widetilde{\zeta}_r^j)$$

$(\widetilde{\zeta}_1 = \eta, \widetilde{\zeta}_r = \zeta_{r-1}$ for $r = 2, \ldots, m+1)$. Applying the Taylor formula with respect to ζ we can write

$$(3.3.57) \quad Q''(z, x_1, \eta, \zeta) = Q''(z, x_1, \widetilde{\zeta}^j) + \sum_{r=1}^{m+1} Q''_{,\widetilde{\zeta}_r}(z, x_1, \widetilde{\zeta}^j)(\widetilde{\zeta}_r - \widetilde{\zeta}^j_r)$$

$$+ \sum_{k,r=1}^{m+1} \int_0^1 Q''_{,\widetilde{\zeta}_r \widetilde{\zeta}_k}(z, x_1, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))(\widetilde{\zeta}_r - \widetilde{\zeta}^j_r)(\widetilde{\zeta}_k - \widetilde{\zeta}^j_k)\, d\theta.$$

We now estimate $\partial_\zeta^\alpha Q''(z, x_1, \eta, \zeta)$ for $|\alpha| \leq 2$. Taking into account (3.3.44), Corollary 3.3.1($1°$) and the fact that $\forall |\alpha| \leq 2$, $\partial_\zeta^\alpha M''(\cdot, \cdot, \eta, \zeta) \in \Pi_0^k(\mathbf{R}^m \times [0, \infty))$ for $(\eta, \zeta) \in [\eta_1, \infty) \times \mathbf{R}^m$ and

$$(3.3.58) \qquad |\partial_\zeta^\alpha M''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)} \leq p^\alpha(\overline{a}, \overline{c}, \overline{d})(1 + |\widetilde{\zeta}|)^{-|\alpha|+1}$$

we obtain

$$(3.3.59) \quad \int (1 + |\zeta|^2)^{k-1/2} |(\partial_{\widetilde{\kappa}}^\alpha Q'')^\wedge(\zeta, x_1, \eta, \kappa)|^2\, d\zeta$$

$$= \|\partial_{\widetilde{\kappa}}^\alpha Q''(\cdot, x_1, \eta, \kappa)\|^2_{k-1/2, \mathbf{R}^m} \leq c_2 \|\partial_{\widetilde{\kappa}}^\alpha Q''(\cdot, \cdot, \eta, \kappa)\|^2_{k, \mathbf{R}^m \times [0,\infty)}$$

$$\leq [p^\alpha(\overline{a}, \overline{c}, \overline{d})]^2 [(1 + |\widetilde{\kappa}|)^{-|\alpha|+1}]^2.$$

By (3.3.56), (3.3.57) and (3.3.59) with $|\alpha| = 2$ we get

$$(3.3.60) \quad \int (1 + |\zeta|^2)^{k-1/2} |\widehat{S''_j}(\zeta, x_1, \eta, \kappa)|^2\, d\zeta$$

$$\leq c_3 \sum_{|\alpha|=2} \int (1 + |\zeta|^2)^{k-1/2} |(\partial_{\widetilde{\kappa}}^\alpha Q'')^\wedge(\zeta, x_1, \widetilde{\zeta}^j + \theta(\widetilde{\kappa} - \widetilde{\zeta}^j))|^2$$

$$\times (1 + |\widetilde{\kappa} - \widetilde{\zeta}^j|^2)^2\, d\zeta$$

$$\leq [p_4(\overline{a}, \overline{c}, \overline{d})]^2 (1 + |\widetilde{\zeta}^j + \theta(\widetilde{\kappa} - \widetilde{\zeta}^j)|)^{-2}(1 + |\widetilde{\kappa} - \widetilde{\zeta}^j|^2)^2,$$

where $\widetilde{\kappa}_1 = \eta$, $\widetilde{\kappa}_r = \kappa_{r-1}$, $r = 2, \ldots, m+1$.

We now estimate $(1 + |\widetilde{\zeta}^j + \theta(\widetilde{\kappa} - \widetilde{\zeta}^j)|)^{-2}(1 + |\widetilde{\kappa} - \widetilde{\zeta}^j|^2)^2$ for $\widetilde{\kappa} \in \operatorname{supp} \varphi_j$. We consider two cases:

a) $|\widetilde{\zeta}^j| \geq 2$. Then

$$|\widetilde{\kappa} - \widetilde{\zeta}^j| \leq (1 + |\widetilde{\zeta}^j|^2)^{1/4}/2 \leq (1 + |\widetilde{\zeta}^j|)/2 \leq \tfrac{3}{4}|\widetilde{\zeta}^j|.$$

Hence $\frac{1}{4}|\widetilde{\zeta}^j| \leq |\widetilde{\zeta}^j + \theta(\widetilde{\kappa} - \widetilde{\zeta}^j)| \leq \frac{7}{4}|\widetilde{\zeta}^j|$. Therefore

$$(3.3.61) \qquad (1 + |\widetilde{\zeta}^j + \theta(\widetilde{\kappa} - \widetilde{\zeta}^j)|^2)^{-2}(1 + |\widetilde{\kappa} - \widetilde{\zeta}^j|^2)^2 \leq c,$$

where $c > 0$ is a constant.

b) $|\widetilde{\zeta}^j| < 2$. Then

$$|\widetilde{\kappa} - \widetilde{\zeta}^j|^2 \leq (1 + |\widetilde{\zeta}^j|^2)^{1/2}/2 \leq \sqrt{5}/2.$$

So (3.3.61) is also true.

Hence (3.3.60) yields

$$(3.3.62) \qquad \int \left(1 + |\zeta|^2\right)^{k-1/2} |\widehat{S_j''}(\zeta, x_1, \eta, \kappa)|^2 \, d\zeta \leq [p_5(\overline{a}, \overline{b}, \overline{c}, \overline{d})]^2$$

for $(\eta, \kappa) \in \mathrm{supp}\, \varphi_j$.

Similarly, by Corollary 3.3.1(2°), (3.3.43), (3.3.55) and the Taylor formula applied to $Q'(\eta, \zeta)$ we have

$$(3.3.63) \qquad\qquad\qquad |S_j'(\eta, \zeta)| \leq C$$

for $(\eta, \zeta) \in \mathrm{supp}\, \varphi_j$ ($C > 0$ is a constant), which implies

$$(3.3.64) \qquad \left| \mathrm{Re} \int S_j' \varphi_j w \cdot \overline{\varphi_j w} \, dz \right| \leq C \|\varphi_j w\|_{0,\mathbf{R}^m}^2 \,.$$

Applying further Lemma 3.3.1 and (3.3.62) we have

$$(3.3.65) \quad \left| \mathrm{Re} \int S_j'' \varphi_j w \cdot \overline{\varphi_j w} \, dz \right| \leq \|(S_j'' \varphi_j w)^\wedge\|_{0,\mathbf{R}^m} \|\varphi_j w\|_{0,\mathbf{R}^m}$$

$$\leq c \left\| \int \widehat{S_j''}(\zeta - \kappa, x_1, \eta, \kappa) \varphi_j(\eta, \kappa) \widehat{w}(\kappa, x_1) \, d\kappa \right\|_{0,\mathbf{R}^m} \|\varphi_j w\|_{0,\mathbf{R}^m}$$

$$= c \left\| \int \left(1 + |\zeta - \kappa|^2\right)^{-k/2+1/4} \left(1 + |\zeta - \kappa|^2\right)^{k/2-1/4} \right.$$

$$\left. \times \widehat{S_j''}(\zeta - \kappa, x_1, \eta, \kappa) \varphi_j(\eta, \kappa) \widehat{w}(\kappa, x_1) \, d\kappa \right\|_{0,\mathbf{R}^m} \|\varphi_j w\|_{0,\mathbf{R}^m}$$

$$\leq p_6(\overline{a}, \overline{c}, \overline{d}) \|\varphi_j w\|_{0,\mathbf{R}^m}^2 \,.$$

We will now estimate $\mathrm{Re} \int Q_{,\eta}(z, x_1, \eta^j, \zeta^j)(\eta - \eta^j) \varphi_j w \cdot \overline{\varphi_j w} \, dz$ and $\mathrm{Re} \int Q_{,\zeta_l}(z, x_1, \eta^j, \zeta^j)(D_l - \zeta_l^j) \varphi_j w \cdot \overline{\varphi_j w} \, dz$ $(l = 1, \ldots, m)$. To do this we use the Taylor formula (3.3.57) with Q'' replaced by Q. Put in this formula

$$\widetilde{\zeta}_1 - \widetilde{\zeta}_1^j = \widetilde{\zeta}_2 - \widetilde{\zeta}_2^j = \ldots = \widetilde{\zeta}_{r-1} - \widetilde{\zeta}_{r-1}^j = \widetilde{\zeta}_{r+1} - \widetilde{\zeta}_{r+1}^j = \ldots = \widetilde{\zeta}_{m+1} - \widetilde{\zeta}_{m+1}^j = 0 \,,$$
$$\widetilde{\zeta}_r - \widetilde{\zeta}_r^j = \pm c (1 + |\widetilde{\zeta}^j|^2)^{1/4} \quad (r = 1, \ldots, m+1) \,,$$

where $c > 0$ is a constant to be chosen later. Then by (3.3.42) and Corollary 3.3.1(6°) we have

$$(3.3.66) \quad Q(z, x_1, \widetilde{\zeta}^j) \pm c (1 + |\widetilde{\zeta}^j|^2)^{1/4} Q_{,\widetilde{\zeta}_r}(z, x_1, \widetilde{\zeta}^j)$$
$$+ c^2 (1 + |\widetilde{\zeta}^j|^2)^{1/2} Q_{,\widetilde{\zeta}_r \widetilde{\zeta}_r}(z, x_1, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j)) \geq 0 \,.$$

Applying Lemma 2.3.5 (in view of (3.3.66) its assumption is satisfied) and next the Young inequality with $\varepsilon = c(1 + |\widetilde{\zeta}^j|^2)^{1/4}/(m+1)$ we obtain for any $w_1, w_2 \in \mathbf{C}^n$

$$(3.3.67) \quad | \mathrm{Re}\, Q_{,\widetilde{\zeta}_r}(z, x_1, \widetilde{\zeta}^j) w_1 \cdot \overline{w}_2 |$$
$$\leq \frac{1}{2(m+1)} [Q(z, x_1, \widetilde{\zeta}^j) + Q_{,\widetilde{\zeta}_r \widetilde{\zeta}_r}(z, x_1, \widetilde{\zeta}^j + \theta(\zeta - \widetilde{\zeta}^j))$$

$$\times c^2 (1 + |\widetilde{\zeta^j}|^2)^{1/2}] w_1 \cdot \overline{w}_1$$

$$+ \frac{m+1}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} [Q(z, x_1, \widetilde{\zeta^j}) + Q_{,\zeta_r \zeta_r}(z, x_1, \widetilde{\zeta^j} + \theta(\widetilde{\zeta} - \widetilde{\zeta^j}))$$

$$\times c^2 (1 + |\widetilde{\zeta^j}|^2)^{1/2}] w_2 \cdot \overline{w}_2 \,.$$

Using (3.3.67) with $w_1 = \varphi_j w$ and $w_2 = (D_l - \zeta_l^j) \varphi_j w$ we get

$$(3.3.68) \quad \left| \operatorname{Re} \int Q_{,\zeta_l}(z, x_1, \widetilde{\zeta^j})(D_l - \zeta_l^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz \right|$$

$$\leq \frac{1}{2(m+1)} \int Q(z, x_1, \widetilde{\zeta^j})\varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{m+1}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} \int Q(z, x_1, \widetilde{\zeta^j})[(D_l - \zeta_l^j)\varphi_j w \cdot (D_l - \zeta_l^j)\overline{\varphi_j w}$$

$$- (D_l - \zeta_l^j)^2 \varphi_j w \cdot \overline{\varphi_j w}] \, dz$$

$$+ \frac{m+1}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} \int Q(z, x_1, \widetilde{\zeta^j})(D_l - \zeta_l^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{1}{2(m+1)} \int Q_{,\zeta_l \zeta_l}(z, x_1, \widetilde{\zeta^j} + \theta(\widetilde{\zeta} - \widetilde{\zeta^j}))c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}\varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{m+1}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} \int Q_{,\zeta_l \zeta_l}(z, x_1, \widetilde{\zeta^j} + \theta(\widetilde{\zeta} - \widetilde{\zeta^j}))c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}$$

$$\times (D_l - \zeta_l^j)\varphi_j w \cdot (D_l - \zeta_l^j)\overline{\varphi_j w} \, dz \,.$$

Let us estimate the components on the right-hand side of (3.3.68). Consider first

$$(3.3.69) \quad \frac{m+1}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} \int Q(z, x_1, \widetilde{\zeta^j})[|(D_l - \zeta_l^j)\varphi_j w|^2$$

$$- (D_l - \zeta_l^j)^2 \varphi_j w \cdot \overline{\varphi_j w}] \, dz$$

$$= \frac{(2\pi)^{-m}(m+1)}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} \operatorname{Re} \int \int \widehat{Q''}(\zeta - \kappa, x_1, \widetilde{\zeta^j})[(\zeta_l - \zeta_l^j) - (\kappa_1 - \zeta_l^j)]$$

$$\times (\kappa_l - \zeta_l^j)\varphi_j(\eta, \kappa)\widehat{w}(\kappa, x_1)\varphi_j(\eta, \zeta), \overline{\widehat{w}}(\zeta, x_1) \, d\kappa \, d\zeta \,.$$

If in the last integral (call it I) we change variables, we obtain

$$(3.3.70) \quad \operatorname{Re} \int \int \widehat{Q''}(\kappa - \zeta, x_1, \widetilde{\zeta^j})[(\kappa_l - \zeta_l^j) - (\zeta_l - \zeta_l^j)](\zeta_l - \zeta_l^j)$$

$$\times \varphi_j(\eta, \zeta)\widehat{w}(\zeta, x_1)\varphi_j(\eta, \kappa)\overline{\widehat{w}}(\kappa, x_1) \, d\kappa \, d\zeta$$

$$= \operatorname{Re} \int \int \widehat{Q''}^*(\zeta - \kappa, x_1, \widetilde{\zeta^j})[(\kappa_l - \zeta_l^j) - (\zeta_l - \zeta_l^j)]$$
$$\times (\zeta_l - \zeta_l^j)\varphi_j(\eta, \zeta)\widehat{w}(\zeta, x_1)\varphi_j(\eta, \kappa)\,\overline{\widehat{w}}(\kappa, x_1)\,d\kappa\,d\zeta.$$

Since

$$\operatorname{Re} \widehat{Q''}(\zeta - \kappa, x_1, \widetilde{\zeta^j})\widehat{w}(\kappa, x_1)\,\overline{\widehat{w}}(\zeta, x_1)$$
$$= \operatorname{Re} \widehat{Q''}^*(\zeta - \kappa, x_1, \widetilde{\zeta^j})\widehat{w}(\zeta, x_1)\,\overline{\widehat{w}}(\kappa, x_1),$$

adding the integral on the right-hand side of (3.3.69) to the one on the right-hand side of (3.3.70) we obtain

$$(3.3.71) \qquad 2I = -\operatorname{Re} \int \int \widehat{Q''}(\zeta - \kappa, x_1, \widetilde{\zeta^j})(\zeta_l - \kappa_l)^2$$
$$\times \varphi_j(\eta, \kappa)\widehat{w}(\kappa, x_1)\varphi_j(\eta, \zeta)\,\overline{\widehat{w}}(\zeta, x_1)\,d\kappa\,d\zeta.$$

Applying now Lemma 3.3.1 to the function

$$T\varphi_j\widehat{w}(\zeta, x_1) \equiv \tfrac{1}{2}\int \widehat{Q''}(\zeta - \kappa, x_1, \widetilde{\zeta^j})(\zeta_l - \kappa_l)^2 \cdot \varphi_j(\eta, \kappa)\widehat{w}(\kappa, x_1)\,d\kappa$$
$$= \int (1 + |\zeta - \kappa|^2)^{-k/2+1/4}(\zeta_l - \kappa_l)^2(1 + |\zeta - \kappa|^2)^{k/2-1/4}$$
$$\times \widehat{Q''}(\zeta - \kappa, x_1, \widetilde{\zeta^j})\,d\kappa$$

and taking into account (3.3.59) with $|\alpha| = 0$, and next (3.3.69) and (3.3.71) we get

$$(3.3.72) \quad \left| \frac{m+1}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} \int Q(z, x_1, \widetilde{\zeta^j})[|(D_l - \zeta_l^j)\varphi_j w|^2 \right.$$
$$\left. -(D_l - \zeta_l^j)^2\varphi_j w \cdot \overline{\varphi_j w}]\,dz \right| \le p_7(\overline{a}, \overline{c}, \overline{d})\|\varphi_j w\|_{0, \mathbf{R}^m}^2.$$

From (3.3.59) and the Sobolev lemma it follows that

$$(3.3.73) \qquad |Q_{,\zeta_r\zeta_r}(z, x_1, \widetilde{\zeta^j} + \theta(\widetilde{\zeta} - \widetilde{\zeta^j}))c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}| \le p_8(\overline{a}, \overline{c}, \overline{d})$$

(where we have used the estimate analogous to (3.3.61) of the form $c^2(1 + |\widetilde{\zeta^j} + \theta(\widetilde{\zeta} - \widetilde{\zeta^j})|)^{-2}(1 + |\widetilde{\zeta^j}|^2)^{1/2} \le \widetilde{c}$ for a constant $\widetilde{c} > 0$). Hence

$$(3.3.74) \quad \left| \frac{1}{2(m+1)} \int Q_{,\zeta_l\zeta_l}(z, x_1, \widetilde{\zeta^j} + \theta(\widetilde{\zeta} - \widetilde{\zeta^j}))c^2 \right.$$
$$\left. \times (1 + |\widetilde{\zeta^j}|^2)^{1/2}\varphi_j w \cdot \overline{\varphi_j w}\,dz \right| \le p_9(\overline{a}, \overline{c}, \overline{d})\|\varphi_j w\|_{0, \mathbf{R}^m}^2$$

and

$$(3.3.75) \quad \left| \frac{m+1}{2c^2(1 + |\widetilde{\zeta^j}|^2)^{1/2}} \int Q_{,\zeta_l\zeta_l}(z, x_1, \widetilde{\zeta^j} + \theta(\widetilde{\zeta} - \widetilde{\zeta^j}))c^2 \right.$$

$$\times (1 + |\widetilde{\zeta}^j|^2)^{1/2} |(D_l - \zeta_l^j)\varphi_j w|^2 \, dz \bigg|$$

$$\le p_{10}(\overline{a}, \overline{c}, \overline{d}) \frac{1}{(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \int (\zeta_l - \zeta_l^j)^2 |\widehat{\varphi_j w}(\zeta, x_1)|^2 \, d\zeta$$

$$\le p_{11}(\overline{a}, \overline{c}, \overline{d}) \|\varphi_j w\|_{0, \mathbf{R}^m}^2 \, .$$

Applying next (3.3.67) with $w_1 = \varphi_j w$ and $w_2 = (\eta - \eta^j)\varphi_j w$ we obtain

$$(3.3.76) \quad \bigg| \operatorname{Re} \int Q_{,\eta}(z, x_1, \widetilde{\zeta}^j)(\eta - \eta^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz \bigg|$$

$$\le \frac{1}{2(m+1)} \int Q(z, x_1, \widetilde{\zeta}^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{m+1}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \int Q(z, x_1, \widetilde{\zeta}^j)(\eta - \eta^j)^2 \varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{1}{2(m+1)} \int Q_{,\eta^2}(z, x_1, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2} \varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{m+1}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \int Q_{,\eta^2}(z, x_1, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}$$

$$\times (\eta - \eta^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz \, .$$

Similarly to (3.3.74) and (3.3.75) we can prove the estimates

$$(3.3.77) \quad \bigg| \frac{1}{2(m+1)} \int Q_{,\eta^2}(z, x_1, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}$$

$$\times \varphi_j w \cdot \overline{\varphi_j w} \, dz \bigg| \le p_{12}(\overline{a}, \overline{c}, \overline{d}) \|\varphi_j w\|_{0, \mathbf{R}^m}^2 \, ,$$

$$(3.3.78) \quad \bigg| \frac{m+1}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \int Q_{,\eta^2}(z, x_1, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))c^2(1 + |\widetilde{\zeta}^j|^2))^{1/2}$$

$$\times (\eta - \eta^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz \bigg| \le p_{13}(\overline{a}, \overline{c}, \overline{d}) \|\varphi_j w\|_{0, \mathbf{R}^m}^2 \, .$$

By (3.3.72), (3.3.74)–(3.3.78) and (3.3.68) we get

$$(3.3.79) \quad \bigg| \sum_j \sum_{l=1}^m \operatorname{Re} \int [Q_{,\zeta_l}(z, x_1, \widetilde{\zeta}^j)(D_l - \zeta_l^j)$$

$$+ Q_{,\eta}(z, x_1, \widetilde{\zeta}^j)(\eta - \eta^j)]\varphi_j w \cdot \overline{\varphi_j w} \, dz \bigg|$$

$$\le \frac{1}{2} \sum_j Q(z, x_1, \widetilde{\zeta}^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz + \sum_j \frac{m+1}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}}$$

$$\times \int Q(z, x_1, \widetilde{\zeta}^j) \Big[\sum_{l=1}^{m} (D_l - \zeta_l^j)^2 + (\eta - \eta^j)^2 \Big] \varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \, p_{14}(\overline{a}, \overline{c}, \overline{d}) \sum_j \|\varphi_j w\|_{0, \mathbf{R}^m}^2 \, .$$

Now inserting (3.3.64), (3.3.65) and (3.3.79) in (3.3.52) we have

$$(3.3.80) \quad (Qw, w)_{0, \mathbf{R}^m} \geq \frac{1}{2} \sum_j \int Q(z, x_1, \widetilde{\zeta}^j)$$

$$\times \left(1 - \frac{(m+1)[|D_z - \zeta^j|^2 + (\eta - \eta^j)^2]}{c^2 (1 + |\widetilde{\zeta}^j|^2)^{1/2}} \right) |\varphi_j w|^2 \, dz$$

$$- \, p_{15}(\overline{a}, \overline{c}, \overline{d}) \|w\|_{0, \mathbf{R}^m}^2 \, .$$

Take $c = \sqrt{2(m+1)}$ and define

$$h(\eta, \zeta) = \begin{cases} \left(1 - \frac{m+1}{c^2} \frac{|\zeta - \zeta^j|^2 + (\eta - \eta^j)^2}{(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \right)^{1/2} & \text{for } |\widetilde{\zeta} - \widetilde{\zeta}^j| \leq (1 + |\widetilde{\zeta}^j|^2)^{1/4}, \\ \sqrt{1/2} & \text{otherwise.} \end{cases}$$

Then for every (η, ζ) and (η, κ)

$$(3.3.81) \qquad |h(\eta, \zeta) - h(\eta, \kappa)| \leq \frac{|\zeta - \kappa|}{\sqrt{2}(1 + |\widetilde{\zeta}^j|^2)^{1/4}} \, .$$

We can now rewrite (3.3.80) as

$$(3.3.82) \; (Qw, w)_{0, \mathbf{R}^m} \geq \frac{1}{2} \sum_j \int Q(z, x_1, \widetilde{\zeta}^j) h^2(\eta, D_z) \varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$- \frac{1}{2} \sum_j \int Q(z, x_1, \widetilde{\zeta}^j) |h(\eta, D_z) \varphi_j w|^2 \, dz$$

$$+ \frac{1}{2} \sum_j \int Q(z, x_1, \widetilde{\zeta}^j) |h(\eta, D_z) \varphi_j w|^2 \, dz$$

$$- \, p_{16}(\overline{a}, \overline{c}, \overline{d}) \|w\|_{0, \mathbf{R}^m}^2 \, .$$

Using Lemma 3.3.1 and (3.3.81) we obtain

$$(3.3.83) \quad \left| \frac{1}{2} \sum_j \int Q(z, x_1, \widetilde{\zeta}^j) [h^2(\eta, D_z) \varphi_j w \cdot \overline{\varphi_j w} - |h(\eta, D_z) \varphi_j w|^2] \, dz \right|$$

$$= \frac{1}{2}\Big| \sum_j \int\int \widehat{Q''}(\zeta - \kappa, x_1, \widetilde{\zeta}^j)(h(\eta, \kappa) - h(\eta, \zeta))h(\eta, \kappa)$$

$$\times \varphi_j(\eta, \kappa)\widehat{w}(\kappa, x_1)\varphi_j(\eta, \zeta)\,\overline{\widehat{w}}(\zeta, x_1)\,d\kappa\,d\zeta\Big|$$

$$\leq p_{17}(\overline{a}, \overline{c}, \overline{d})\|w\|^2_{0, \mathbf{R}^m}\,.$$

From (3.3.82), (3.3.83), (3.3.42) and Corollary 3.3.1(6°) the assertion of the lemma follows. ∎

LEMMA 3.3.4.

(3.3.84) $\forall w \in C^\infty_{(0)}(\mathbf{R}^m \times [0, \infty))$

$\mathrm{Re}(Rw(\cdot, 0), w(\cdot, 0))_{0, \mathbf{R}^m}$

$$\geq \varepsilon_2\|w(\cdot, 0)\|^2_{0, \mathbf{R}^m} - \eta^{-1}(1 + \varepsilon_3)p(\overline{a}, \overline{c}, \overline{d}, \overline{e})\|w(\cdot, 0)\|^2_{0, \mathbf{R}^m}$$
$$- c_3\varepsilon_3\|w^I(\cdot, 0) - Sw^{II}(\cdot, 0)\|^2_{0, \mathbf{R}^m}\,,$$

where p is a polynomial, while ε_2 and ε_3 are the constants from Theorem 3.2.1.

Proof. The proof is the same as for (3.3.41). Define

(3.3.85) $T(z, \eta, \zeta) = R(z, 0, \eta, \zeta) - \varepsilon_3 I\,,$

(3.3.86) $T'(\eta, \zeta) = R'(\eta, \zeta) - \varepsilon_2 I\,,$

(3.3.87) $T''(z, \eta, \zeta) = R''(z, 0, \eta, \zeta)\,.$

Let $T(z, \eta, D_z)$, $T'(\eta, D_z)$ and $T''(z, \eta, D_z)$ be the pseudodifferential operators with symbols $T(z, \eta, \zeta)$, $T'(\eta, \zeta)$ and $T''(z, \eta, \zeta)$. Corollary 3.3.1(7°) together with (3.3.85) yield

(3.3.88) $y^* T(z, \eta, \zeta)y \geq -\varepsilon_3|y^I - Sy^{II}|^2$

for $y \in \mathbf{C}^n$.

Take $w \in C^\infty_{(0)}(\mathbf{R}^m \times [0, \infty))$. Using the partition of unity φ_j $(j = 1, 2, \ldots)$ from Lemma 3.3.3, as in the proof of Lemma 3.3.3 we show that

(3.3.89) $(Tw, w)_{0, \mathbf{R}^m} = \mathrm{Re}\sum \int T\varphi_j w \cdot \overline{\varphi_j w}\,dz$

$$+ \mathrm{Re}\sum [\varphi_j(\eta, D_z), T'']w \cdot \overline{\varphi_j w}\,dz$$

(where w means $w(\cdot, 0)$).

By (3.3.87) and Corollary 3.3.1(1°) we have

(3.3.90) $\int (1 + |\zeta|^2)^{k-1/2}|(\partial^\alpha_\zeta R'')^\wedge(\zeta, 0, \eta, \kappa)|^2\,d\zeta$

$$= \|\partial^\alpha_\zeta T''(\cdot, \eta, \kappa)\|^2_{k-1/2, \mathbf{R}^m} \leq c_1|\partial^\alpha_\zeta R''(\cdot, \cdot, \eta, \kappa)|^2_{k, 0, \infty, \mathbf{R}^m \times [0, \infty)}$$

$$\leq [p^\alpha(\overline{a}, \overline{c}, \overline{d})]^2[(1 + |\widetilde{\kappa}|)^{-|\alpha|}]^2$$

for $|\alpha| \le 2$ and every $\widetilde{\kappa} = (\eta, \kappa) \in [\eta_1, \infty) \times \mathbf{R}^m$. Using (3.3.90) and Lemma 3.3.1 we obtain

$$(3.3.91) \qquad \left| \mathrm{Re} \sum_j \int [\varphi_j, T''] w \cdot \overline{\varphi_j w} \, dz \right| \le \frac{p_1(\overline{a}, \overline{c}, \overline{d})}{\eta} \int |\widehat{w}(\kappa, 0)|^2 \, d\kappa$$

$$\le \eta^{-1} p_2(\overline{a}, \overline{c}, \overline{d}) \| w(\cdot, 0) \|_{0, \mathbf{R}^m}^2$$

(p_1, p_2 and all p_i occurring below are polynomials). By (3.3.89) and (3.3.91)

$$(3.3.92) \qquad (Tw, w)_{0, \mathbf{R}^m} \ge \mathrm{Re} \sum_j \int T \varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$- \eta^{-1} p_3(\overline{a}, \overline{c}, \overline{d}) \| w(\cdot, 0) \|_{0, \mathbf{R}^m}^2 .$$

Set now

$$(3.3.93) \qquad T_j(z, \eta, D_z) = T(z, \eta, D_z) - T(z, \widetilde{\zeta}^j)$$

$$- \sum_{j=1}^m T_{,\zeta_l}(z, \widetilde{\zeta}^j)(D_l - \zeta_l^j) + T(z, \widetilde{\zeta}^j)(\eta - \eta^j) ,$$

where $\widetilde{\zeta}^j = (\eta^j, \zeta^j)$ is the centre of K_j, as before. Write the symbol $T_j(z, \eta, \zeta)$ of $T_j(z, \eta, D_z)$ as

$$(3.3.94) \qquad T_j(z, \eta, \zeta) = T_j''(z, \eta, \zeta) + T_j'(\eta, \zeta) ,$$

where

$$(3.3.95) \qquad T_j'(\eta, \zeta) = T'(\eta, \zeta) - T'(\widetilde{\zeta}^j) - \sum_{r=1}^{m+1} T_{,\zeta_r}'(\widetilde{\zeta}_r - \widetilde{\zeta}_r^j) ,$$

$$(3.3.96) \qquad T_j''(z, \eta, \zeta) = T''(z, \eta, \zeta) - T''(z, \widetilde{\zeta}^j) - \sum_{r=1}^{m+1} T_{,\zeta_r}''(\widetilde{\zeta}_r - \widetilde{\zeta}_r^j)$$

(here we write $\widetilde{\zeta}_1 = \eta$, $\widetilde{\zeta}_r = \zeta_{r-1}$ for $r = 2, \ldots, m + 1$). Using the Taylor formula with respect to $\widetilde{\zeta}$ we have

$$(3.3.97) \quad T''(z, \eta, \zeta) = T''(z, \widetilde{\zeta}^j) + \sum_{r=1}^{m+1} T_{,\zeta_r}''(z, \widetilde{\zeta}^j)(\widetilde{\zeta}_r - \widetilde{\zeta}_r^j)$$

$$+ \sum_{k,r=1}^{m+1} \int_0^1 T_{,\widetilde{\zeta}_r \widetilde{\zeta}_k}''(z, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))(\widetilde{\zeta}_r - \widetilde{\zeta}_r^j)(\widetilde{\zeta}_k - \widetilde{\zeta}_k^j) \, d\theta .$$

From (3.3.96), (3.3.97) and (3.3.90) with $|\alpha| = 2$ it follows that

$$(3.3.98) \qquad \int (1 + |\zeta|^2)^{k-1/2} |\widehat{T_j''}(\zeta, \eta, \kappa)|^2 \, d\zeta$$

$$\leq c \sum_{|\alpha|=2} \int (1 + |\zeta|^2)^{k-1/2} |(\partial_\zeta^\alpha R'')^\wedge(\zeta, 0, \widetilde{\zeta}^j + \theta(\widetilde{\kappa} - \widetilde{\zeta}^j))|^2$$

$$\times (1 + |\widetilde{\kappa} - \widetilde{\zeta}^j|^2)^2 \, d\zeta$$

$$\leq [p_4(\overline{a}, \overline{c}, \overline{d})]^2 (1 + |\widetilde{\zeta}^j + \theta(\widetilde{\kappa} - \widetilde{\zeta}^j)|)^{-4} (1 + |\widetilde{\kappa} - \widetilde{\zeta}^j|^2)^2$$

$$\leq [p_5(\overline{a}, \overline{c}, \overline{d})]^2 (1 + |\widetilde{\kappa}|)^{-2} \,,$$

where $\widetilde{\kappa} = (\eta, \kappa) \in \operatorname{supp} \varphi_j$ (here (3.3.61) is used). Similarly

(3.3.99) $|T_j'(\eta, \zeta)| \leq c_3 (1 + |\widetilde{\zeta}|)^{-1}$ for $(\eta, \zeta) \in \operatorname{supp} \varphi_j$.

As in (3.3.65), using Lemma 3.3.1 and (3.3.98) and next (3.3.99) we obtain

(3.3.100) $\left| \operatorname{Re} \sum_j \int T_j' \varphi_j w \cdot \overline{\varphi_j w} \, dz \right| \leq \eta^{-1} p_6(\overline{a}, \overline{c}, \overline{d}) \|w(\cdot, 0)\|_{0, \mathbf{R}^m}^2$.

To estimate the integrals $\operatorname{Re} \int T_{,\zeta_l}(z, \widetilde{\zeta}^j)(D_l - \zeta_l^j) \varphi_j w \cdot \overline{\varphi_j w} \, dz$ ($l = 1, \ldots, m$) and $\operatorname{Re} \int T_{,\eta}(z, \widetilde{\zeta}^j)(\eta - \eta^j) \varphi_j w \cdot \overline{\varphi_j w} \, dz$ let us employ the Taylor formula (3.3.97) with T'' replaced by T. Put in that formula

$$\widetilde{\zeta}_1 - \widetilde{\zeta}_1^j = \widetilde{\zeta}_2 - \widetilde{\zeta}_2^j = \ldots = \widetilde{\zeta}_{r-1} - \widetilde{\zeta}_{r-1}^j = \widetilde{\zeta}_{r+1} - \widetilde{\zeta}_{r+1}^j = \ldots = \widetilde{\zeta}_{m+1} - \widetilde{\zeta}_{m+1}^j = 0 \,,$$
$$\widetilde{\zeta}_r - \widetilde{\zeta}_r^j = \pm c (1 + |\widetilde{\zeta}^j|^2)^{1/4} \,,$$

where $c > 0$ will be chosen later. Then we obtain

(3.3.101) $T(z, \eta, \zeta) = T(z, \widetilde{\zeta}^j) \pm c(1 + |\widetilde{\zeta}^j|^2)^{1/4} T_{,\widetilde{\zeta}_r}(z, \widetilde{\zeta}^j)$
$$+ c^2 (1 + |\widetilde{\zeta}^j|^2)^{1/2} T_{,\widetilde{\zeta}_r^2}(z, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j)) \,.$$

In view of (3.3.88) and (3.3.101) the assumption of Lemma 2.3.6 is satisfied for the matrices

$$B = \pm c(1 + |\widetilde{\zeta}^j|^2)^{1/4} T_{,\widetilde{\zeta}_r}(z, \widetilde{\zeta}^j) \,.$$
$$A = T(z, \widetilde{\zeta}^j) + c^2 (1 + |\widetilde{\zeta}^j|^2)^{1/2} T_{,\widetilde{\zeta}_r \widetilde{\zeta}_r}(z, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))$$
$$(r = 1, \ldots, m+1) \,.$$

Take $w_1 = \varphi_j w(z, 0)$ and $w_2 = (D_l - \zeta_l^j) \varphi_j w(z, 0)$ and notice that

(3.3.102) $(\varphi_j w)^I(z, 0) - S(\varphi_j w)^{II}(z, 0)$
$$= (\varphi_j S - S \varphi_j) w^{II}(z, 0) + \varphi_j (w^I(z, 0) - S w^{II}(z, 0)) \,,$$

(3.3.103) $[(D_l - \zeta_l^j) \varphi_j w]^I(z, 0) - S[(D_l - \zeta_l^j) \varphi_j w]^{II}(z, 0)$
$$= [(D_l - \zeta_l^j) \varphi_j S - S(D_l - \zeta_l^j) \varphi] w^{II}(z, 0)$$
$$+ (D_l - \zeta_l^j) \varphi_j (w^I(z, 0) - S w^{II}(z, 0)) \,.$$

By Lemma 2.3.6 (with $\varepsilon = (m+1)/[2c(1+|\widetilde{\zeta}^j|^2)^{1/4}]$) we have

$$|\operatorname{Re}(\varphi_j w)^* c(1+|\widetilde{\zeta}^j|^2)^{1/4} T_{,\zeta_l}(z,\widetilde{\zeta}^j)[(D_l - \zeta_l^j)\varphi_j w]|$$

$$\leq \frac{m+1}{2c(1+|\widetilde{\zeta}^j|^2)^{1/4}}[(D_l - \zeta_l^j)\varphi_j w]^*[T(z,\widetilde{\zeta}^j)$$

$$+ c^2(1+|\widetilde{\zeta}^j|^2)^{1/2} T_{,\zeta_l \zeta_l}(z,\widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))][(D_l - \zeta_l^j)\varphi_j w]$$

$$+ \frac{c(1+|\widetilde{\zeta}^j|^2)^{1/4}}{2(m+1)}(\varphi_j w)^*[T(z,\widetilde{\zeta}^j) + c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}$$

$$\times T_{,\zeta_l \zeta_l}(z,\widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))]\varphi_j w$$

$$+ \frac{(m+1)\varepsilon_3}{2c(1+|\widetilde{\zeta}^j|^2)^{1/4}}|(D_l - \zeta_l^j)\varphi_j g|^2 + \frac{c(1+|\widetilde{\zeta}^j|^2)^{1/4}}{2(m+1)}|\varphi_j g|^2$$

$$+ \frac{(m+1)\varepsilon_3}{2c(1+|\widetilde{\zeta}^j|^2)^{1/4}}|[(D_l - \zeta_l^j)\varphi_j S - S(D_l - \zeta_l^j)\varphi_j]w^{II}|^2$$

$$+ \frac{c(1+|\widetilde{\zeta}^j|^2)^{1/4}\varepsilon_3}{2(m+1)}|(\varphi_j S - S\varphi_j)w^{II}|^2,$$

and

(3.3.104) $$g = w^I - Sw^{II}.$$

Hence

(3.3.105) $$\left| \operatorname{Re} \int T_{,\zeta_l}(z,\widetilde{\zeta}^j)[(D_l - \zeta_l^j)\varphi_j w] \cdot \overline{\varphi_j w}\, dz \right|$$

$$\leq \frac{m+1}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \operatorname{Re} \int T(z,\widetilde{\zeta}^j)[|(D_l - \zeta_l^j)\varphi_j w|^2$$

$$- (D_l - \zeta_l^j)^2 \varphi_j w \cdot \overline{\varphi_j w}\, dz]$$

$$+ \frac{m+1}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \operatorname{Re} \int T(z,\widetilde{\zeta}^j)(D_l - \zeta_l^j)^2 \varphi_j w \cdot \overline{\varphi_j w}\, dz$$

$$+ \frac{1}{2(m+1)} \operatorname{Re} \int T(z,\widetilde{\zeta}^j)\varphi_j w \cdot \overline{\varphi_j w}\, dz + \frac{m+1}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}}$$

$$\times \operatorname{Re} \int T_{,\zeta_l \zeta_l}(z,\widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}|(D_l - \zeta_l^j)\varphi_j w|^2\, dz$$

$$+ \frac{1}{2(m+1)} \operatorname{Re} \int T_{,\zeta_l \zeta_l}(z,\widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}|\varphi_j w|^2\, dz$$

$$+ \frac{(m+1)\varepsilon_3}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \int |(D_l - \zeta_l^j)\varphi_j g|^2\, dz + \frac{\varepsilon_3}{2(m+1)} \int |\varphi_j g|^2\, dz$$

$$+ \frac{(m+1)\varepsilon_3}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \int |[(D_l - \zeta_l^j)\varphi_j S - S(D_l - \zeta_l^j)\varphi_j]w^{II}|^2\, dz$$

$$+ \frac{\varepsilon_3}{2(m+1)} \int |(\varphi_j S - S\varphi_j) w^{II}|^2 \, dz \, .$$

Using (3.3.90) and the Sobolev lemma we obtain

$$(3.3.106) \quad \left| \frac{m+1}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \operatorname{Re} \int T_{,\zeta_l \zeta_l}(z, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j)) c^2 \right.$$

$$\left. \times (1+|\widetilde{\zeta}^j|^2)^{1/2} |(D_l - \zeta_l^j) \varphi_j w|^2 \, dz \right| \le \eta^{-1} p_7(\overline{a}, \overline{c}, \overline{d}) \| \varphi_j w(\cdot, 0) \|^2_{0, \mathbf{R}^m} ,$$

$$(3.3.107) \quad \left| \frac{1}{2(m+1)} \operatorname{Re} \int T_{,\zeta_l \zeta_l}(z, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j)) c^2 \right.$$

$$\left. \times (1+|\widetilde{\zeta}^j|^2)^{1/2} |\varphi_j w|^2 \, dz \right| \le \eta^{-1} p_8(\overline{a}, \overline{c}, \overline{d}) \| \varphi_j w(\cdot, 0) \|^2_{0, \mathbf{R}^m} \, .$$

In the same way as for (3.3.72) we next obtain

$$(3.3.108) \quad \left| \frac{m+1}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \operatorname{Re} \int T(z, \widetilde{\zeta}^j) [|(D_l - \zeta_l^j) \varphi_j w|^2 \right.$$

$$\left. - (D_l - \zeta_l^j)^2 \varphi_j w \cdot \overline{\varphi_j w}] \, dz \right| \le \eta^{-1} p_9(\overline{a}, \overline{c}, \overline{d}) \| \varphi_j w(\cdot, 0) \|^2_{0, \mathbf{R}^m} ,$$

where we have used (3.3.50). Moreover,

$$(3.3.109) \quad \frac{\varepsilon_3}{2(m+1)} \sum_j \int |\varphi_j g|^2 \, dz = \frac{(2\pi)^{-m} \varepsilon_3}{2(m+1)} \| g \|^2_{0, \mathbf{R}^m} ,$$

$$(3.3.110) \quad \frac{(m+1) \varepsilon_3}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \sum_j \int |(D_l - \zeta_l^j) \varphi_j g|^2 \, dz$$

$$= \frac{(m+1)(2\pi)^{-m} \varepsilon_3}{2c^2(1+|\widetilde{\zeta}^j|^2)^{1/2}} \sum_j \int (\zeta_l - \zeta_l^j) |\varphi_j(\zeta)|^2 |g|^2 \, d\zeta$$

$$= \widetilde{c} \varepsilon_3 \int \sum_j |\varphi_j(\zeta)|^2 |g|^2 \, d\zeta = \widetilde{c} \varepsilon_3 \| g \|^2_{0, \mathbf{R}^m} \, .$$

From assumption (3.1.6), Lemma 2.1.2 and (3.3.5)–(3.3.6) it follows that $S'' \in \Pi_0^k(\mathbf{R}^m \times [0, \infty))$ and

$$(3.3.111) \quad \int (1+|\zeta|^2)^{k-1/2} |\widehat{S''}(\zeta, 0)|^2 \, d\zeta = \| S''(z, 0) \|^2_{k-1/2, \mathbf{R}^m}$$

$$\le c |S''|_{k, 0, \infty, \mathbf{R}^m \times [0, \infty)} \le [p_{10}(\overline{b}, \overline{c})]^2 \, .$$

Using now (3.3.111) and (3.3.48) we get

$$(3.3.112) \quad \frac{\varepsilon_3}{2(m+1)} \sum_j \int |(\varphi_j S - S\varphi_j) w^{II}|^2 \, dz$$

$$= \frac{(2\pi)^{-m} \varepsilon_3}{2(m+1)} \sum_j \int |((\varphi_j S - S\varphi_j) w^{II})^\wedge|^2 \, d\zeta$$

$$= \frac{(2\pi)^{-m}\varepsilon_3}{2(m+1)} \sum_j \int \left| \int \widehat{S''}(\zeta - \kappa, 0)(1 + |\zeta - \kappa|^2)^{k/2 - 1/4} \right.$$

$$\left. \times (1 + |\zeta - \kappa|^2)^{-k/2 + 1/4}(\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))w^{II}(\kappa, 0)\, d\kappa \right|^2 d\zeta$$

$$\leq c\varepsilon_3 \sum_j \int \left(\int |\widehat{S''}(\zeta - \kappa, 0)|^2 (1 + |\zeta - \kappa|^2)^{k - 1/2}\, d\kappa \right)$$

$$\times \left(\int (1 + |\zeta - \kappa|^2)^{-k + 1/2}(\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))^2 |w^{II}(\kappa, 0)|^2\, d\kappa \right) d\zeta$$

$$\leq \varepsilon_3 p_{11}(\bar{b}, \bar{e}) \int \int (1 + |\zeta - \kappa|^2)^{-k + 1/2} \sum_j (\varphi_j(\eta, \zeta) - \varphi_j(\eta, \kappa))^2$$

$$\times |w^{II}(\kappa, 0)|^2\, d\kappa\, d\zeta$$

$$\leq \varepsilon_3 p_{12}(\bar{b}, \bar{e}) \int \int (1 + |\zeta - \kappa|^2)^{-k + 1/2}(\eta + |\kappa|)^{-1}$$

$$\times (1 + |\zeta - \kappa|^2)|w^{II}(\kappa, 0)|^2\, d\kappa\, d\zeta$$

$$\leq \eta^{-1}\varepsilon_3 p_{12}(\bar{b}, \bar{e}) \int \left(\int (1 + |\zeta - \kappa|^2)^{-k + 3/2}\, d\zeta \right) |\widehat{w^{II}}(\kappa, 0)|^2\, d\kappa$$

$$\leq \eta^{-1}\varepsilon_3 p_{13}(\bar{b}, \bar{e}) \|w(\cdot, 0)\|^2_{0, \mathbf{R}^m} .$$

Estimate next

$$(3.3.113) \qquad \frac{(m+1)\varepsilon_3}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \sum_j \int \|[(D_l - \zeta_l^j)\varphi_j S$$

$$- S(D_l - \zeta_l^j)\varphi_j]w^{II}|^2\, dz$$

$$\leq \frac{(2\pi)^{-m}(m+1)\varepsilon_3}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \sum_j \int \left(\int |\widehat{S''}(\zeta - \kappa, 0)||\zeta_l - \kappa_l| \right.$$

$$\left. \times |\varphi_j(\eta, \zeta)||\widehat{w^{II}}(\kappa, 0)|\, d\kappa \right)^2 d\zeta$$

$$+ \frac{(2\pi)^{-m}(m+1)\varepsilon_3}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \sum_j \int \int |\widehat{S''}(\zeta - \kappa, 0)||\varphi_j(\eta, \zeta)$$

$$- \varphi_j(\eta, \kappa)||\widehat{w^{II}}(\kappa, 0)|^2\, d\kappa\, d\zeta$$

$$\leq \frac{c_1 \varepsilon_3}{(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \sum_j \int \left(\int |\widehat{S''}(\zeta - \kappa, 0)|^2 (1 + |\zeta - \kappa|^2)^{k - 1/2}\, d\kappa \right)$$

$$\times \left(\int (1 + |\zeta - \kappa|^2)^{-k + 3/2} |\varphi_j(\eta, \zeta)|^2 |\widehat{w^{II}}(\kappa, 0)|^2\, d\kappa \right) d\zeta$$

$$+ \frac{c_2 \varepsilon_3}{(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \sum_j \int \left(\int |\widehat{S''}(\zeta - \kappa, 0)|^2 (1 + |\zeta - \kappa|^2)^{k - 1/2}\, d\kappa \right)$$

$$\times \left(\int (1 + |\zeta - \kappa|^2)^{-k + 3/2}(1 + |\widetilde{\zeta}^j|^2)^{1/2} \right.$$

$$\left. \times (\eta + |\kappa|)^{-1} |\widehat{w^{II}}(\kappa, 0)|^2\, d\kappa \right) d\zeta \leq \eta^{-1}\varepsilon_3 p_{14}(\bar{b}, \bar{e}) \|w(\cdot, 0)\|^2_{0, \mathbf{R}^m} ,$$

where we have used (3.3.48), (3.3.111) and (3.3.108).

Take now $w_1 = \varphi_j w(z, 0)$, $w_2 = (\eta - \eta^j)\varphi_j w(z, 0)$ and the matrices A and B with $r = 1$ ($\zeta_1 = \eta$). Notice that

$$(3.3.114) \quad [(\eta - \eta^j)\varphi_j w]^I(z, 0) - S[(\eta - \eta^j)\varphi_j w]^{II}(z, 0)$$
$$= (\eta - \eta^j)(\varphi_j S - S\varphi_j)w^{II}(z, 0) + (\eta - \eta^j)\varphi_j g.$$

Applying, as before, Lemma 2.3.6 with $\varepsilon = (m + 1)/[2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/4}]$ we get

$$(3.3.115) \quad \left| \sum_j \operatorname{Re} \int T_{,\eta}(z, \widetilde{\zeta}^j)(\eta - \eta^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz \right|$$

$$\leq \sum_j \frac{m + 1}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \operatorname{Re} \int T(z, \widetilde{\zeta}^j)(\eta - \eta^j)^2 \varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{1}{2(m + 1)} \sum_j \int T(z, \widetilde{\zeta}^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \sum_j \frac{m + 1}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \operatorname{Re} \int T_{,\eta^2}(z, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))$$

$$\times c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2} |(\eta - \eta^j)\varphi_j w|^2 \, dz$$

$$+ \sum_j \frac{1}{2(m + 1)} \operatorname{Re} \int T_{,\eta^2}(z, \widetilde{\zeta}^j + \theta(\widetilde{\zeta} - \widetilde{\zeta}^j))$$

$$\times c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2} |\varphi_j w|^2 \, dz$$

$$+ \sum_j \frac{(m + 1)\varepsilon_3}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \int (\eta - \eta^j)^2 |\varphi_j g|^2 \, dz$$

$$+ \frac{\varepsilon_3}{2(m + 1)} \sum_j \int |\varphi_j g|^2 \, dz$$

$$+ \sum_j \frac{(m + 1)\varepsilon_3}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \int (\eta - \eta^j)^2 |(\varphi_j S - S\varphi_j)w^{II}|^2 \, dz$$

$$+ \frac{\varepsilon_3}{2(m + 1)} \sum_j \int |(\varphi_j S - S\varphi_j)w^{II}|^2 \, dz$$

$$\leq \sum_j \frac{m + 1}{2c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}} \operatorname{Re} \int T(z, \widetilde{\zeta}^j)(\eta - \eta^j)^2 \varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \frac{1}{2(m + 1)} \sum_j \operatorname{Re} \int T(z, \widetilde{\zeta}^j)\varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$+ \widetilde{c}\varepsilon_3 \|g\|_{0,\mathbf{R}^m}^2 + \eta^{-1}(\varepsilon_3 + 1)p_{15}(\overline{b}, \overline{e})\|w(\cdot, 0)\|_{0,\mathbf{R}^m}^2 \, .$$

From (3.3.92), (3.3.93), (3.3.100), (3.3.104)–(3.3.107), (3.3.109), (3.3.110), (3.3.112), (3.3.113) and (3.3.115) we obtain

$$(3.3.116) \quad (Tw, w)_{0,\mathbf{R}^m} \geq \frac{1}{2} \sum_j \int T(z, \widetilde{\zeta}^j)$$

$$\times \left(1 - \frac{(m+1)[|D_z - \zeta^j|^2 + (\eta - \eta^j)^2]}{c^2(1 + |\widetilde{\zeta}^j|^2)^{1/2}}\right) |\varphi_j w|^2 \, dz$$

$$- \eta^{-1}(\varepsilon_3 + 1)p_{16}(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e})\|w(\cdot, 0)\|_{0,\mathbf{R}^m}^2 - c_1\varepsilon_3 \|g\|_{0,\mathbf{R}^m}^2 \, .$$

Taking $c = \sqrt{2(m+1)}$ and introducing the function h from the proof of the previous lemma we get

$$(3.3.117) \quad (Tw, w)_{0,\mathbf{R}^m}$$

$$\geq \frac{1}{2} \sum_j \int T(z, \widetilde{\zeta}^j) h^2(\eta, D_z)\varphi_j w \cdot \overline{\varphi_j w} \, dz$$

$$- \frac{1}{2} \sum_j \int T(z, \widetilde{\zeta}^j)|h(\eta, D_z)\varphi_j w|^2 \, dz$$

$$+ \frac{1}{2} \sum_j \int T(z, \widetilde{\zeta}^j)|h(\eta, D_z)\varphi_j w|^2 \, dz$$

$$- \eta^{-1}(\varepsilon_3 + 1)p_{17}(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e})\|w(\cdot, 0)\|_{0,\mathbf{R}^m}^2 - c_1\varepsilon_3 \|g\|_{0,\mathbf{R}^m}^2$$

$$\geq - \eta^{-1}(\varepsilon_3 + 1)p_{17}(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e})\|w(\cdot, 0)\|_{0,\mathbf{R}^m}^2 - c_1\varepsilon_3 \|g\|_{0,\mathbf{R}^m}^2$$

$$+ \frac{1}{2} \sum_j \int T(z, \widetilde{\zeta}^j)|h(\eta, D_z)\varphi_j w|^2 \, dz \, .$$

In order to estimate $\sum_j \int T(z, \widetilde{\zeta}^j)|h(\eta, D_z)\varphi_j w|^2 \, dz$ notice that $h(\eta, D_z)\varphi_j w$ satisfies

$$(3.3.118) \quad [h(\eta, D_z)\varphi_j w]^I(z, 0) - S[h(\eta, D_z)\varphi_j w]^{II}(z, 0)$$

$$= h(\eta, D_z)\varphi_j g + [h(\eta, D_z)\varphi_j S - Sh(\eta, D_z)\varphi_j]w^{II} \, ,$$

where g is given by (3.3.104). Therefore by (3.3.88) we have

$$(3.3.119) \quad \sum_j \int T(z, \widetilde{\zeta}^j)|h(\eta, D_z)\varphi_j w|^2 \, dz$$

$$\geq -\varepsilon_2 \sum_j \int |h(\eta, D_z)\varphi_j g|^2 \, dz$$

$$\geq -\varepsilon_3 \sum_j \int |[h(\eta, D_z)\varphi_j S - Sh(\eta, D_z)\varphi_j]w^{II}|^2 \, dz$$

$$\geq -c_2\varepsilon_3\|g\|_{0,\mathbf{R}^m}^2 - \eta^{-1}\varepsilon_3 p_{18}(\overline{b},\overline{e})\|w(\cdot,0)\|_{0,\mathbf{R}^m}^2 .$$

Taking into account (3.3.117) and (3.3.119) and notation (3.3.104) we get the assertion of the lemma. ∎

LEMMA 3.3.5.

(3.3.120) $\forall w \in C_{(0)}^\infty(\mathbf{R}^m \times [0,\infty))$

$$\mathrm{Re}(\boldsymbol{R}_{x_1}w(\cdot,x_1),w(\cdot,x_1))_{0,\mathbf{R}^m} \leq p(\overline{a},\overline{b},\overline{c},\overline{d},|v|_{k,0,\infty,K})\|w(\cdot,x_1)\|_{0,\mathbf{R}^m}^2 ,$$

where p is a polynomial, $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$ are given by (3.3.13), and $K \subset \mathbf{R} \times \overline{\Omega}$ is a compact set outside which M'', Ψ_1'' and Ψ_2'' (given by (3.3.3) and (3.3.5)) vanish.

Proof. We have $\boldsymbol{R}_{x_1} = \boldsymbol{R}_{x_1}''$. Suppose $|\widetilde{\zeta}| \leq 1/2$. Then by (3.3.8) we obtain

$$\begin{aligned}
\boldsymbol{R}_{x_1}''(z,x_1,\eta,\zeta) &= \partial_{\boldsymbol{\Psi}_1}^1 H_1'' \cdot \widetilde{\Psi}_{1x_1}''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta) \\
&\quad + \partial_{M''}^1 H_1'' \cdot \widetilde{M}_{x_1}''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta),
\end{aligned}$$

$\partial_{\boldsymbol{\Psi}_1}^1 H_1''$ and $\partial_{M''}^1 H_1''$ denote the vectors of the first derivatives of H_1'' with respect to Ψ_1 and M'', and

$$\begin{aligned}
\widetilde{\Psi}_1''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta) &= \Psi_1''(t,x,v(t,x),\eta,\zeta), \\
\widetilde{M}''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta) &= M''(t,x,v(t,x),\eta,\zeta),
\end{aligned}$$

for $(z,x_1) \in \mathbf{R}^m \times [0,\infty)$.

Denote further by F_3 a linear function such that

$$\boldsymbol{R}_{x_1}'' = F_3(\partial_{\boldsymbol{\Psi}_1}^1 H_1'', \partial_{M''}^1 H_1'', \widetilde{\Psi}_{1x_1}'', \widetilde{M}_{x_1}'')$$

and let

$$\begin{aligned}
G_1(z,x_1,\eta,\zeta) &= \partial_{\boldsymbol{\Psi}_1}^1 H_1''(\Psi_1(z,x_1,\eta,\zeta),M''(z,x_1,\eta,\zeta),M'(\eta,\zeta)), \\
G_2(z,x_1,\eta,\zeta) &= \partial_{M''}^1 H_1''(\Psi_1(z,x_1,\eta,\zeta),M''(z,x_1,\eta,\zeta),M'(\eta,\zeta)), \\
G_3(z,x_1,\eta,\zeta) &= \widetilde{\Psi}_{1x_1}''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta)+\widetilde{\Psi}_{1\widetilde{v}}''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta)\widetilde{v}_{x_1}(z,x_1), \\
G_4(z,x_1,\eta,\zeta) &= \widetilde{M}_{x_1}''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta)+\widetilde{M}_{\widetilde{v}}''(z,x_1,\widetilde{v}(z,x_1),\eta,\zeta)\widetilde{v}_{x_1}(z,x_1).
\end{aligned}$$

Then Lemma 2.2.1 yields

$$\begin{aligned}
|\boldsymbol{R}_{x_1}''|&_{k-1,0,\infty,\mathbf{R}^m \times [0,\infty)} \\
&\leq q(\|F\|_{C^{k-1}}, |G_1|_{k-1,0,\infty,\mathbf{R}^m \times [0,\infty)}, |G_2|_{k-1,0,\infty,\mathbf{R}^m \times [0,\infty)}, \\
&\qquad\qquad |G_3|_{k-1,0,\infty,\mathbf{R}^m \times [0,\infty)}, |G_4|_{k-1,0,\infty,\mathbf{R}^m \times [0,\infty)}) \\
&\leq p(\overline{a},\overline{b},\overline{c},\overline{d},|v|_{k,0,\infty,K}) .
\end{aligned}$$

A similar estimate holds for $1/2 \leq |\widetilde{\zeta}| \leq 1$ and $|\widetilde{\zeta}| \geq 1$. Hence

$$\int (1 + |\zeta|^2)^{k-3/2} |\widehat{R''_{x_1}}(\zeta, x_1, \eta, \kappa)|^2 \, d\zeta = \|R''_{x_1}(\cdot, x_1, \eta, \kappa)\|^2_{k-3/2, \mathbf{R}^m}$$

$$\leq c |R''_{x_1}(\cdot, \cdot, \eta, \kappa)|^2_{k-1, 0, \infty, \mathbf{R}^m \times [0, \infty)} \leq [p(\overline{a}, \overline{b}, \overline{c}, \overline{d}, |v|_{k, 0, \infty, K})]^2$$

for $x_1 \in [0, \infty)$, $\eta \in [\eta_1, \infty)$ and $\kappa \in \mathbf{R}^m$. Applying the above inequality and Lemma 3.3.1 we obtain the assertion. ∎

THEOREM 3.3.1. *Let assumptions* (3.1.1)–(3.1.5), (3.1.7), (3.1.14), (3.1.20), (3.1.24) *and* (3.1.6a)–(3.1.6b) *with* $k > m/2 + 5/2$ *hold. Moreover, assume*

$$(3.3.121) \qquad\qquad v \in \Pi_0^k(\mathbf{R} \times \Omega).$$

Then, if $u \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$ *is a solution of problem* (3.1)–(3.2), *then for* η *sufficiently large (namely, for* $\eta > \widetilde{f}_1(\varepsilon_1)\widetilde{p}_1(\overline{a}, \overline{b}, \overline{c}, \overline{d}, |v|_{k,0,\infty,K}) + \widetilde{f}_2(\varepsilon_2, \varepsilon_3)$ $\times \widetilde{p}_2(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e})$, *where* $\widetilde{p}_1$ *and* $\widetilde{p}_2$ *are polynomials,* $\widetilde{f}_1$ *and* $\widetilde{f}_2$ *are continuous functions,* $K \subset \mathbf{R} \times \overline{\Omega}$ *is a compact set outside which* M'', Ψ_1'' *and* Ψ_2'' *vanish),*

$$(3.3.122) \quad \eta\|u\|^2_{0,\eta,\mathbf{R} \times \Omega} + \|u\|^2_{0,\eta,\mathbf{R} \times \partial\Omega}$$
$$\leq p(\overline{a}, \overline{b}, \overline{c}, \overline{d})\widetilde{f}(\varepsilon_1, \varepsilon_2, \varepsilon_3)(\eta^{-1}\|F\|^2_{0,\eta,\mathbf{R} \times \Omega} + \|g\|^2_{0,\eta,\mathbf{R} \times \partial\Omega}),$$

where p *is a polynomial and* $\widetilde{f}$ *is a continuous function.*

Proof. Set $w = e^{-\eta t}\widetilde{u}$, where $\widetilde{u}(z, x_1) = u(t, x)$, $\forall (z, x_1) \in \mathbf{R}^m \times [0, \infty)$. Then w is a solution of the problem

$$(3.3.123) \qquad w_{x_1} = M(z, x_1, \eta, D_z)w - A_1^{-1}(z, x_1)C(z, x_1)w$$
$$- A_1^{-1}(z, x_1)\widetilde{F}(z, x_1)e^{-\eta t}$$

(where $\widetilde{F}(z, x_1) = F(t, x)$, $\forall (z, x_1) \in \mathbf{R}^m \times [0, \infty)$),

$$(3.3.124) \qquad [I, -S(z, 0)] \begin{bmatrix} w^I(z, 0) \\ w^{II}(z, 0) \end{bmatrix} = e^{-\eta t}g(z).$$

This follows from (3.1.10)–(3.1.9) and notations (3.3.5). Applying the operator $R(z, x_1, \eta, D_z)$ given by (3.3.16) to both sides of (3.3.123) we get

$$(3.3.125) \quad \text{Re} \int_0^\infty (Rw_{x_1}, w)_{0,\mathbf{R}^m} \, dx_1 = \text{Re} \int_0^\infty (RMw, w)_{0,\mathbf{R}^m} \, dx_1$$

$$- \text{Re} \int_0^\infty (RA_1^{-1}Cw, w)_{0,\mathbf{R}^m} \, dx_1 - \text{Re} \int_0^\infty (RA_1^{-1}\widetilde{F}e^{-\eta t}, w)_{0,\mathbf{R}^m} \, dx_1.$$

Notice that since the matrix $R(z, x_1, \eta, \zeta)$ is hermitian (Corollary 3.3.1(4°))

we have

$$\text{Re} \int_0^\infty (Rw_{x_1}, w)_{0,\mathbf{R}^m} \, dx_1 = \frac{1}{2} \text{Re}(R(z,0,\eta,D_z)w(z,0), w(z,0))_{0,\mathbf{R}^m}$$

$$- \frac{1}{2} \text{Re} \int_0^\infty (R_{x_1} w, w)_{0,\mathbf{R}^m} \, dx_1 \,.$$

Therefore we can rewrite (3.3.125) in the form

$$(3.3.126) \quad \text{Re} \int_0^\infty ((RM - B)w, w)_{0,\mathbf{R}^m} \, dx_1 + \text{Re} \int_0^\infty (Bw, w)_{0,\mathbf{R}^m} \, dx_1$$

$$- \frac{1}{2} \text{Re} \int_0^\infty (R_{x_1} w, w)_{0,\mathbf{R}^m} \, dx_1 + \frac{1}{2} \text{Re}(R(z,0,\eta,D_z)w(z,0), w(z,0))_{0,\mathbf{R}^m}$$

$$= \text{Re} \int_0^\infty (RA_1^{-1}Cw, w)_{0,\mathbf{R}^m} \, dx_1 + \text{Re} \int_0^\infty (RA_1^{-1}\widetilde{F}e^{-\eta t}, w)_{0,\mathbf{R}^m} \, dx_1 \,.$$

The integrals on the right-hand side of (3.3.126) can be estimated by using Lemma 3.3.1 similarly to $\text{Re} \int_0^\infty ((RM - B)w, w)_{0,\mathbf{R}^m} \, dx_1$. We get

$$(3.3.127) \quad \left| \text{Re} \int_0^\infty (RA_1^{-1}Cw, w)_{0,\mathbf{R}^m} \, dx_1 \right|$$

$$\leq p_1(\overline{a}, \overline{b}, \overline{c}, \overline{d}) \|w\|_{0,\mathbf{R}^m \times [0,\infty)}^2 = p_1(\overline{a}, \overline{b}, \overline{c}, \overline{d}) \|u\|_{0,\eta,\mathbf{R} \times \Omega}^2 \,,$$

$$(3.3.128) \quad \left| \text{Re} \int_0^\infty (RA_1^{-1}\widetilde{F}e^{-\eta t}, w)_{0,\mathbf{R}^m} \, dx_1 \right|$$

$$\leq p_2(\overline{a}, \overline{b}, \overline{c}, \overline{d})(\varepsilon^{-1}\eta^{-1}\|F\|_{0,\eta,\mathbf{R} \times \Omega}^2 + \varepsilon\eta\|u\|_{0,\eta,\mathbf{R} \times \Omega}^2) \,.$$

Combining (3.3.127), (3.3.128) and (3.3.18), (3.3.41), (3.3.84), (3.3.120) with polynomials denoted by $p_3(\overline{a}, \overline{b}, \overline{c}, \overline{d})$, $p_4(\overline{a}, \overline{b}, \overline{c}, \overline{d})$, $p_5(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e})$ and $p_6(\overline{a}, \overline{b}, \overline{c}, \overline{d}, |v|_{k,0,\infty,K})$, we obtain from (3.3.126) the inequality

$$(3.3.129) \quad \varepsilon_1\eta\|u\|_{0,\eta,\mathbf{R} \times \Omega}^2 - p_4(\overline{a}, \overline{b}, \overline{c}, \overline{d})\|u\|_{0,\eta,\mathbf{R} \times \Omega}^2 + \varepsilon_2\|u\|_{0,\eta,\mathbf{R} \times \partial\Omega}^2$$

$$- \eta^{-1}(1 + \varepsilon_2)p_5(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e})\|u\|_{0,\eta,\mathbf{R} \times \partial\Omega}^2 - c_3\varepsilon_3\|g\|_{0,\eta,\mathbf{R} \times \partial\Omega}^2$$

$$\leq p_6(\overline{a}, \overline{b}, \overline{c}, \overline{d}, |v|_{k,0,\infty,K})\|u\|_{0,\eta,\mathbf{R} \times \Omega}^2 + (p_3(\overline{a}, \overline{b}, \overline{c}, \overline{d})$$

$$+ p_1(\overline{a}, \overline{b}, \overline{c}, \overline{d}))\|u\|_{0,\eta,\mathbf{R} \times \Omega}^2 + \varepsilon\eta p_7(\overline{a}, \overline{b}, \overline{c}, \overline{d})\|u\|_{0,\eta,\mathbf{R} \times \Omega}^2$$

$$+ \varepsilon^{-1}\eta^{-1}p_7(\overline{a}, \overline{b}, \overline{c}, \overline{d})\|F\|_{0,\eta,\mathbf{R} \times \Omega}^2 \,.$$

Take $\varepsilon = \varepsilon_1/[4p_7(\overline{a},\overline{b},\overline{c},\overline{d})]$ and assume that η is so large that

(3.3.130) $\varepsilon_1\eta/4 > p_4(\overline{a},\overline{b},\overline{c},\overline{d}) + p_6(\overline{a},\overline{b},\overline{c},\overline{d},|v|_{k,0,\infty,K})$
$$+ p_3(\overline{a},\overline{b},\overline{c},\overline{d}) + p_1(\overline{a},\overline{b},\overline{c},\overline{d}),$$

(3.3.131) $\varepsilon_2/2 > (1 + \varepsilon_3)p_5(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e})\eta^{-1}.$

Inequalities (3.3.130) and (3.3.131) are satisfied if we choose η such that

(3.3.132) $\eta > 4[p_4(\overline{a},\overline{b},\overline{c},\overline{d}) + p_6(\overline{a},\overline{b},\overline{c},\overline{d},|v|_{k,0,\infty,K}) + p_3(\overline{a},\overline{b},\overline{c},\overline{d})$
$$+ p_1(\overline{a},\overline{b},\overline{c},\overline{d})]/\varepsilon_1 + 2(1 + \varepsilon_3)p_5(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e})/\varepsilon_2$$
$$= \widetilde{f}_1(\varepsilon_1)\widetilde{p}_1(\overline{a},\overline{b},\overline{c},\overline{d},|v|_{k,0,\infty,K}) + \widetilde{f}_2(\varepsilon_2,\varepsilon_3)\widetilde{p}_2(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e}).$$

For such η we get from (3.3.129)

$$\eta\|u\|^2_{0,\eta,\mathbf{R}\times\Omega} + \|u\|^2_{0,\eta,\mathbf{R}\times\partial\Omega} \le \left(4p_7^2 + \frac{8p_7^2\varepsilon_1}{\varepsilon_2}\right)\eta^{-1}\|F\|^2_{0,\eta,\mathbf{R}\times\Omega}$$
$$+ \left(\frac{2c_3\varepsilon_3}{\varepsilon_1} + \frac{2c_3\varepsilon_3}{\varepsilon_2}\right)\|g\|^2_{0,\eta,\mathbf{R}\times\partial\Omega}.$$

Hence the theorem follows. ∎

R e m a r k 3.3.1. From the proofs of Lemmas 3.3.2–3.3.5, Corollary 3.3.1 and Theorem 3.3.1 it follows that it suffices to assume in all those assertions that $v \in H^k(\mathbf{R}\times\Omega)$. Then $\widetilde{L}'' \in H^k(\mathbf{R}^m\times[0,\infty))$, $\Psi_i'' \in H^k(\mathbf{R}^m\times[0,\infty))$ $(i = 1,2)$, $S'' \in H^k(\mathbf{R}^m\times[0,\infty))$ (the proof of this fact is analogous to the proof of Lemma 2.1.2) and $\overline{a}$, $\overline{c}$ and $\overline{e}$ are replaced by

$$\overline{a}' = \|L''\|_{k,\mathbf{R}^m\times[0,\infty)},$$
$$\overline{c}' = \sum_{|\alpha|\le 2}\sup_{\tilde{\zeta}=(\eta,\zeta):|\zeta|\le 1}\|\partial_\zeta^\alpha\Psi_1''\|_{k,\mathbf{R}^m\times[0,\infty)}$$
$$+ \sum_{|\alpha|\le 2}\sup_{\tilde{\zeta}=(\eta,\zeta):|\zeta|\le 1}\|\partial_\zeta^\alpha\Psi_2''\|_{k,\mathbf{R}^m\times[0,\infty)}.$$
$$\overline{e}' = \|S''\|_{k,\mathbf{R}^m\times[0,\infty)},$$

3.4. An estimate of a solution of the problem formally adjoint to (3.1)–(3.2) in $L^2_{-\eta}$-norm. Set

(3.4.1) $$L = E(v)\partial_t - \sum_{j=1}^m A_j(v)\partial_{x_j} - C(v),$$

(3.4.2) $$L_B = [I, -S(v)]$$

(where we use notation (3.3)). Then problem (3.1)–(3.2) can be written in the form

(3.4.3) $$Lu = F \quad \text{in } \mathbf{R}\times\Omega,$$

(3.4.4) $L_B u = g$ on $\mathbf{R} \times \partial\Omega$.

Take $u, w \in C_{(0)}^{\infty}(\mathbf{R} \times \bar{\Omega})$. Applying integration by parts we get

(3.4.5) $(Lu, w)_{0, \mathbf{R} \times \Omega} = (u, L^{(*)}w)_{0, \mathbf{R} \times \Omega} + (A_1 u, w)_{0, \mathbf{R} \times \partial\Omega}$,

where

(3.4.6) $L^{(*)} = -E^* \partial_t + \sum_{j=1}^{m} A_j^* \partial_{x_j} + \sum_{j=1}^{m} \widetilde{\partial}_{x_j} A_j^* - \widetilde{\partial}_t E^* - C^*$,

(3.4.7) $\widetilde{\partial}_{x_j} A_j^* = \partial_{x_j} A_j^* + \sum_{i=1}^{n} \partial_{v_i} A_j^* \partial_{x_j} v_i$,

(3.4.8) $\widetilde{\partial}_t E^* = \partial_t E^* + \sum_{i=1}^{n} \partial_{v_i} E^* \partial_{x_j} v_i$.

Define

$$P = \ker L_B .$$

In view of the form of L_B, dim $P = n - l$ in $\mathbf{R} \times \partial\Omega$. P is called the *boundary space* for problem (3.4.3)–(3.4.4), while $P^{(*)} = (A_1[P])^{\perp}$ is called the *adjoint boundary space* and dim $P^{(*)} = l$ on $\mathbf{R} \times \partial\Omega$.

If $u \in P$ and $w \in P^{(*)}$, then

(3.4.9) $A_1 u \cdot w \equiv w^T A_1 u = 0$ on $\mathbf{R} \times \partial\Omega$.

We now find a boundary operator $L_B^{(*)}$ such that $P^{(*)} = \ker L_B^{(*)}$. We calculate it from equation (3.4.9). To do this rewrite A_1 as

$$A_1 = \begin{bmatrix} A_1^I & B_1 \\ B_2 & A_1^{II} \end{bmatrix} ,$$

where A_1^I, A_1^{II}, B_1, B_2 are matrices of appropriate dimensions. We then write (3.4.9) as

(3.4.10) $A_1^I u^I w^I + B_1 u^{II} w^I + B_2 u^I w^{II} + A_1^{II} u^{II} w^{II} = 0$.

We look for $L_B^{(*)}$ in the form

(3.4.11) $L_B^{(*)} = [-S^{(*)}(v), I]$,

where $S^{(*)}$ is an $(n - l) \times l$-matrix, which we calculate putting $u^I = S u^{II}$ and $w^{II} = S^{(*)} w^I$ into (3.4.10). Then we obtain

$$A_1^{II} S u^{II} w^I + B_1 u^{II} w^I + B_2 S u^{II} S^{(*)} w^I + A_1^{II} u^{II} S^{(*)} w^I = 0$$

for $u \in P$ and $w \in P^{(*)}$. Hence

(3.4.12) $A_1^I S + B_1 + S^{(*)T}(B_2 S + A_1^{II}) = 0$.

Assume additionally that

(3.4.13) $\det(B_2 S + A_1^{II}) \neq 0$ on $\mathbf{R} \times \partial\Omega$

for every n-vector-valued function $v \in C^0(\mathbf{R} \times \bar{\Omega})$ occurring in the coefficients of problem (3.4.3)–(3.4.4). Under the above assumption equation (3.4.12) has a unique solution, and so there exists a unique boundary operator $L_B^{(*)}$ defined by (3.4.11).

The formally adjoint problem to (3.4.3)–(3.4.4) has the form

$$(3.4.14) \qquad L^{(*)}w = F_1 \quad \text{in } \mathbf{R} \times \Omega\,,$$

$$(3.4.15) \qquad L_B^{(*)}w = g_1 \quad \text{on } \mathbf{R} \times \partial\Omega\,.$$

From the form (3.4.6) of $L^{(*)}$ and assumption (3.1.2) it follows that system (3.4.14) is hyperbolic, while by (3.1.1), (3.1.3) and (3.1.4) we have

$$(3.4.16) \qquad \det E^{(*)}(t,x,v) \neq 0 \quad \forall (t,x,v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n\,,$$

$$(3.4.17) \qquad \det A_1^*(t,x,v) \neq 0 \quad \forall (t,x,v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n\,.$$

Write now problem (3.4.14)–(3.4.15) (as before (3.4.3)–(3.4.4)) by means of pseudodifferential operators. Multiplying both sides of (3.4.14) and of (3.4.15) by $e^{\eta t}$ (where $\eta > 0$) we get

$$(3.4.18) \quad e^{\eta t}w_{x_1} = M^{(*)}(t,x,v(t,x),-iD_t+\eta,iD_{x'})e^{\eta t}w$$

$$-A_1^{*-1}(v)\Big[\sum_{j=1}^m \tilde{\partial}_{x_j} A_j^*(v) - \tilde{\partial}_t E^*(v) - C^*(v)\Big]e^{\eta t}w + A_1^{*-1}(v)Fe^{\eta t}w\,,$$

where we have used notations (3.3), (3.4.7) and (3.4.8); moreover, $M^{(*)}(t,x,v(t,x),-iD_t+\eta,iD_{x'})$ is the pseudodifferential operator with symbol

$$(3.4.19) \quad M^{(*)}(t,x,v(t,x),s,i\omega') = -A_1^{*-1}(v)\Big(E^*(v)s + i\sum_{j=2}^m A_j^*(v)\omega_j\Big)\,,$$

where $s = \eta - i\xi$, $\omega' = (\omega_2,\ldots,\omega_m)$.

Now we shall consider the properties of the matrix-valued function

$$(3.4.20) \quad M^{(*)}(t,x,v,s,i\omega') = -A_1^{*-1}(t,x,v)\Big(E^*(t,x,v)s$$

$$+ i\sum_{j=2}^m A_j^*(t,x,v)\omega_j\Big)$$

for $(t,x,v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$.

Since $\det(E^{*-1}A_1^*) = \det(E^{-1}A_1)$, from assumption (3.1.14) it follows that $E^{*-1}A_1^*$ has l negative and $n-l$ positive eigenvalues.

LEMMA 3.4.1. $\forall (t,x,v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ and $\forall (s,\omega') \in \mathbf{C} \times \mathbf{R}^m : \operatorname{Re} s > 0$ the matrix $M^{(*)}$ defined by (3.4.20) has exactly l eigenvalues κ such that $\operatorname{Re}\kappa > 0$ and $n-l$ eigenvalues κ such that $\operatorname{Re}\kappa < 0$.

The proof of the lemma is analogous to that of Lemma 3.1.1.

Consider now the system of ordinary differential equations

$$(3.4.21) \qquad \frac{d\widetilde{u}}{dx_1} = M^{(*)}(t, 0, x', v, s, i\omega')\widetilde{u},$$

where $\widetilde{u} = [\widetilde{u}_1, \ldots, \widetilde{u}_n]^T$. By Lemma 3.4.1 system (3.4.21) has exactly $n - l$ linearly independent solutions which decay exponentially as $x_1 \to \infty$.

Let vector-valued functions $\psi_1(t, x_1, x', v, s, \omega'), \ldots, \psi_{n-l}(t, x_1, x', v, s, \omega')$ be a basis of the set of such solutions, orthonormalized at $x_1 = 0$. Define

$$(3.4.22) \qquad N^{(*)}(t, x', \alpha, v, s, \omega') = [-S^{(*)}(t, 0, x', \alpha(t, 0, x')), I]$$
$$\cdot [\psi_1(t, x_1, x', v, s, \omega'), \ldots, \psi_{n-l}(t, x_1, x', v, s, \omega')]|_{x_1=0},$$

where $\alpha \in C^0(\mathbf{R} \times \bar{\Omega})$ is an arbitrary vector-valued function and the entries of $S^{(*)}$ are solutions of system (3.4.12).

LEMMA 3.4.2.

$$(3.4.23) \qquad \det N^{(*)}(t, x', \alpha(t, x'), v, s, \omega') \neq 0$$
$$\forall (t, x', v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n \ \forall (s, \omega') : \operatorname{Re} s \geq 0, |s| + |\omega'| \neq 0.$$

P r o o f. Notice first that from assumption (3.1.20) it follows that there is no nonzero solution of system (3.1.18) which satisfies the boundary condition

$$(3.4.24) \qquad \widetilde{u}^I = S\widetilde{u}^{II} \quad \text{on } \mathbf{R} \times \partial\Omega$$

(where $S = S(t, 0, x', \alpha(t, 0, x'))$ and which belongs to $L^2(0, \infty)$ for all $(t, x', v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n$ and for all (s, ω') with $|s| + |\omega'| \neq 0$, $\operatorname{Re} s \geq 0$. In fact, fix (t, x', v, s, ω'). A general solution of system (3.1.18) which belongs to $L^2(0, \infty)$ is of the form

$$(3.4.25) \qquad \widetilde{u} = \sum_{i=1}^{l} C_i \varphi_i,$$

where C_i are arbitrary constants and φ_i are defined on p.30. Suppose that $\widetilde{u}$ satisfies (3.4.24). Then putting (3.4.25) into (3.4.24) we obtain

$$(3.4.26) \qquad ([I, -S][\varphi_1, \ldots, \varphi_l]) \begin{bmatrix} C_1 \\ \vdots \\ C_l \end{bmatrix} = 0.$$

Assumption (3.1.20) implies that the only solution of (3.4.26) is the zero vector, i.e. $C_i = 0$ for $i = 1, \ldots, l$. Hence the only solution of system (3.1.18) belonging to $L^2(0, \infty)$ and satisfying (3.4.24) is $\widetilde{u} \equiv 0$.

Denote by $J = J(t, x', v, s, \omega')$ the l-dimensional subspace of $\mathbf{C}^n$ spanned by the vectors $\varphi_1(t, 0, x', v, s, \omega'), \ldots, \varphi_l(t, 0, x', v, s, \omega')$ (for $\operatorname{Re} s > 0$, J is equal to the negative generalized eigenspace of M given by (3.1.13)). Take $(t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$. From the above considerations it follows that

$J(t, x', v, s, \omega') \cap P = \{0\}$ for all (s, ω') with $|s| + |\omega'| \neq 0$, Re $s \geq 0$ (where $P = \ker L_B$, $L_B = [I, -S(t, 0, x', \alpha(t, 0, x'))]$).

Denote next by $J^{(*)} = J^{(*)}(t, x', v, s, \omega')$ the $(n-l)$-dimensional subspace of $\mathbf{C}^n$ spanned by $\psi_1(t, 0, x', v, s, \omega'), \ldots, \psi_{n-l}(t, 0, x', v, s, \omega')$ (for Re $s > 0$, $J^{(*)}$ is equal to the negative generalized eigenspace of $M^{(*)}$). Then by [8, Lemma 7, p. 333], $J^{(*)}(t, x', v, s, \omega') \cap P^{(*)} = \{0\}$ for all (s, ω') with $|s| + |\omega'| \neq 0$, Re $s > 0$ (where $P^{(*)} = \ker L_B^{(*)}$, $L_B^{(*)} = [-S^{(*)}(t, 0, x', \alpha(t, 0, x')), I]$), and so problem (3.4.14)–(3.4.15) with coefficients in the boundary condition (3.4.15) fixed at (t, x, v) and $(t, 0, x', \alpha(t, 0, x'))$ is well posed in L_η^2. Hence by [13, Chap. 3], (3.4.23) is satisfied. ∎

THEOREM 3.4.1. *Let assumptions* (3.1.1)–(3.1.5), (3.1.7), (3.1.14), (3.1.20), (3.1.24), (3.4.13) *and* (3.3.121), (3.1.6a)–(3.1.6b) *with* $k > m/2 + 5/2$ *be satisfied. If* $w \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$ *is a solution of problem* (3.4.14)–(3.4.15) *formally adjoint to* (3.4.3)–(3.4.4), *then for* η *sufficiently large (namely, for*
$\eta > \tilde{f}_1^{(*)}(\varepsilon_1^{(*)})\tilde{p}_1^{(*)}(\overline{a}, \overline{b}, \overline{c}, \overline{d}, |v|_{k,0,\infty,K}) + \tilde{f}_2^{(*)}(\varepsilon_2^{(*)}, \varepsilon_3^{(*)})\tilde{p}_2^{(*)}(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e})$,
where $\tilde{p}_1^{(*)}$ *and* $\tilde{p}_2^{(*)}$ *are polynomials,* $\tilde{f}_1^{(*)}$ *and* $\tilde{f}_2^{(*)}$ *are continuous functions,* $\varepsilon_i^{(*)}$ $(i = 1, 2, 3)$ *are the constants from Theorem 3.2.1, where* R *is replaced by an analogous function constructed for problem* (3.4.14)–(3.4.15), *and* $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, |v|_{k,0,\infty,K}$ *are as in Theorem 3.3.1),*

$$(3.4.27) \quad \eta \|w\|_{0,\eta,\mathbf{R} \times \Omega}^2 + \|w\|_{0,-\eta,\mathbf{R} \times \Omega}^2$$
$$\leq p^{(*)}(\overline{a}, \overline{b}, \overline{c}, \overline{d}, |v|_{k,0,\infty,K})\tilde{f}^{(*)}(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*)$$
$$\times (\eta^{-1}\|F\|_{0,-\eta,\mathbf{R} \times \Omega}^2 + \|g\|_{0,-\eta,\mathbf{R} \times \partial\Omega}^2),$$

where $p^{(*)}$ *is a polynomial and* $\tilde{f}^{(*)}$ *is a continuous function.*

Proof. Conditions (3.1.1)–(3.1.6a) imply the analogous conditions for the functions $-E^*$, $-A_j^*$ and $S^{(*)}$ from problem (3.4.14)–(3.4.15). By (3.1.7), $E^*(t, x, v) = E'^*$, $A_j^*(t, x, v) = A_j'^*$ $(j = 1, \ldots, m)$, $C^*(t, x, v) = C'^*$, $S^*(t, x, v) = S'^*$ outside a compact set $K_{t,x,v}$. In view of Lemma 3.4.2, (3.1.20) implies (3.4.23).

We now check that (3.1.24) for the matrix M implies the analogous condition for $M^{(*)}$. To do this take $z_0 = (t_0, x_0, v_0, 0, \zeta_0)$, where $(t_0, x_0, v_0) \in \mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n$, $\zeta_0 \in \mathbf{R}^m$. Let $\kappa_\sigma^{(*)} = \kappa_\sigma^{(*)}(z_0)$ (where $\sigma = 1, \ldots, s$, $1 \leq s \leq n$) be all distinct eigenvalues of $M^{(*)}(z_0)$. Then by assumption (3.1.6a) and Lemmas 2.2.1, 2.2.2, there exists $\delta > 0$ such that in the set θ_{δ,z_0} there is defined the family $P_{z_0,\sigma}^{(*)}$ $(\sigma = 1, \ldots, s)$ of $n \times n$ total projection matrices of the $\kappa_\sigma^{(*)}$-groups satisfying the conditions of Remarks 2.2.3, 2.2.4 with $\widehat{N}_{z_0,\sigma}^{(*)}(z) \equiv P_{z_0,\sigma}^{(*)}(z)\mathbf{C}^n$ and $P_{z_0,\sigma}^{(*)} \in C^k(\overline{\theta}_{\delta,z_0})$. Take now any eigenvalue $\kappa(z)$ of $M(z)$ (where $z \in \theta_{\delta,z_0}$ with $\delta > 0$ so small that in θ_{δ,z_0} both the above family $P_{z_0,\sigma}^{(*)}$ and the family $P_{z_0,\sigma}$ of Remark 3.3.1 are defined). We shall

prove that $-\overline{\kappa}(z)$ is an eigenvalue of $M^{(*)}(z)$ for $z \in \theta_{\delta, z_0}$. In order to do this set $A_1^{-1} = D_1$, $E\xi - \sum_{j=2}^m A_j \omega_j = D_2$. Then

$$
\begin{aligned}
\det(M - \kappa I) &= \det(\eta D_1 E + i D_1 D_2 - \kappa I) \\
&= \det(-D_1) \det(-\eta E D_1 - i D_2 D_1 + \kappa I) \det D_1^{-1} \\
&= \det(-I) \det((-\eta D_1^* E^* + i D_1^* D_2^*)^* + \kappa I) \\
&= \det(-I) \det \overline{(M^{(*)T} + \overline{\kappa} I)} = \det(-I) \overline{\det(M^{(*)} + \overline{\kappa} I)}.
\end{aligned}
$$

Therefore κ is an eigenvalue of M iff $-\overline{\kappa}$ is an eigenvalue of $M^{(*)}$. Hence, if $\kappa_\sigma(z)$ is an eigenvalue of $M(z)$ in $\widehat{N}_{z_0,\sigma}(z)$ with real part separated from zero then $-\overline{\kappa}_\sigma(z)$ is an eigenvalue of $M^{(*)}(z)$ in $\widehat{N}_{z_0,\sigma}^{(*)}(z)$ with real part also separated from zero.

Take now $\kappa_\sigma(z)$ such that $\kappa_\sigma(z) \to \kappa_\sigma(z_0)$ as $z \to z_0$, where $\operatorname{Re} \kappa_\sigma(z_0) = 0$. Then $\kappa_\sigma = \kappa_\sigma(z_0)$ is an eigenvalue of $M(z_0)$, but in view of the previous considerations it is also an eigenvalue of $M^{(*)}(z_0)$. Consider the rank of $M^{(*)} - \kappa_\sigma I$, where $M^{(*)} = M^{(*)}(z_0)$, $\kappa_\sigma = \kappa_\sigma(z_0)$. We obtain

$$
\begin{aligned}
\operatorname{rank}(M^{(*)} - \kappa_\sigma I) &= \operatorname{rank}(i D_1^* D_2^* - \kappa_\sigma I) \\
&= \operatorname{rank}[(-D_1^*)(-i D_2^* D_1^* + \kappa_\sigma I) D_1^{*-1}] \\
&\le \min\{\operatorname{rank}(-D_1^*), \operatorname{rank}(-i D_2^* D_1^* + \kappa_\sigma I), \operatorname{rank} D_1^{*-1}\} \\
&\le \operatorname{rank}(i D_2^* D_1^* - \kappa_\sigma) = \operatorname{rank}(M - \kappa_\sigma I),
\end{aligned}
$$

where $M = M(z_0)$. On the other hand,

$$
\begin{aligned}
\operatorname{rank}(M - \kappa_\sigma I) &= \operatorname{rank}(-i D_2^* D_1^* + \kappa_\sigma I) \\
&\le \min\{\operatorname{rank}(-D_2^*), \operatorname{rank}(i D_1^* D_2^* - \kappa_\sigma I), \operatorname{rank} D_2^{*-1}\} \\
&\le \operatorname{rank}(i D_1^* d_2^* - \kappa_\sigma I) = \operatorname{rank}(M^{(*)} - \kappa_\sigma I).
\end{aligned}
$$

By assumption (3.1.24), $\operatorname{rank}(M - \kappa_\sigma I) = n - 1$. Hence $\operatorname{rank}(M^{(*)} - \kappa_\sigma I) = n - 1$, and so the geometric eigenspace of $M^{(*)}(z_0)$ corresponding to the eigenvalue $\kappa_\sigma^{(*)}(z_0) = \kappa_\sigma(z_0)$ is one-dimensional.

So we have checked that for $M^{(*)}$ assumption (3.1.24) is satisfied. Therefore by Theorem 3.2.1 there exists a matrix-valued function $R^{(*)} = R^{(*)}(t, x, v, \eta, \zeta)$ satisfying conditions $1°$-$5°$ of this theorem with $M^{(*)}$ in place of M, g_1 in place of g, and with constants $\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^* > 0$ in place of ε_1, ε_2, ε_3, depending on $M^{(*)}$ in the same way as ε_1, ε_2, ε_3 depend on M.

Using further an argument similar to the one in the proof of Theorem 3.3.1 we obtain the assertion of the theorem. ∎

R e m a r k 3.4.1. If instead of (3.1.2), (3.1.5) and (3.1.24) we assume that system (3.1) is *strictly hyperbolic*, i.e. $\forall \omega = (\omega_1, \ldots, \omega_m) \in \mathbf{R}^m \setminus \{0\}$ and $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ the equation $\det(\lambda E(t, x, v) - \sum_{j=1}^m A_j(t, x, v)\omega_j) = 0$ has n distinct real roots, then system (3.4.14) is also strictly hyperbolic and the estimates (3.3.122), (3.4.27) are fulfilled.

3.5. Existence and uniqueness of solution of the boundary value problem (3.1)–(3.2)

DEFINITION 3.5.1. A function $u \in L_\eta^2(\mathbf{R} \times \Omega)$ is called a *strong solution* of problem (3.1)–(3.2) if there exist a sequence $u_\mu \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$ such that $L_B u_\mu = g$ on $\mathbf{R} \times \partial\Omega$ and a function $u_0 \in L_\eta^2(\mathbf{R} \times \partial\Omega)$ satisfying, as $\mu \to \infty$,

(3.5.1) $$\|u_\mu - u\|_{0,\eta,\mathbf{R} \times \Omega} \to 0 \,,$$

(3.5.2) $$\|u_\mu - u_0\|_{0,\eta,\mathbf{R} \times \partial\Omega} \to 0 \,,$$

(3.5.3) $$\|Lu_\mu - F\|_{0,\eta,\mathbf{R} \times \Omega} \to 0 \,,$$

(3.5.4) $$L_B u_0 = g \quad \text{on } \mathbf{R} \times \partial\Omega \,.$$

THEOREM 3.5.1. *Let assumptions* (3.1.1)–(3.1.5), (3.1.7), (3.1.14), (3.1.20), (3.1.24), (3.4.13) *and* (3.3.121), (3.1.6a)–(3.1.6b) *with* $k > m/2 + 5/2$ *hold. Then for any* $F \in L_\eta^2(\mathbf{R} \times \Omega)$ *and* $g \in L_\eta^2(\mathbf{R} \times \partial\Omega)$ *problem* (3.1)–(3.2) *has a unique strong solution* $u \in L_\eta^2(\mathbf{R} \times \Omega)$ *for which the estimate* (3.3.122) *holds with* η *such as in Theorem 3.3.1.*

Proof. Notice first that $u \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$ is a solution of (3.1)–(3.2) iff $u_1 = e^{-\eta t}u$ is a solution of the problem

(3.5.5) $$L_\eta u_1 \equiv (L + \eta E)u_1 = Fe^{-\eta t} \quad \text{in } \mathbf{R} \times \Omega \,,$$

(3.5.6) $$L_B u_1 \equiv ge^{-\eta t} \quad \text{on } \mathbf{R} \times \partial\Omega \,.$$

In the sequel we shall consider problem (3.5.5)–(3.5.6). The formally adjoint problem to (3.5.5)–(3.5.6) is

(3.5.7) $$L_\eta^{(*)} w_1 \equiv (L^{(*)} + \eta E^*)w_1 = F_1 e^{\eta t} \quad \text{in } \mathbf{R} \times \Omega \,,$$

(3.5.8) $$L_B^{(*)} w_1 \equiv g_1 e^{\eta t} \quad \text{on } \mathbf{R} \times \partial\Omega \,,$$

where the operators $L^{(*)}$ and $L_B^{(*)}$ are given by (3.4.6) and (3.4.11). Then $w \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$ is a solution of (3.4.14)–(3.4.15) iff $w_1 = e^{\eta t}w$ is a solution of (3.5.7)–(3.5.8). Therefore by Theorems 3.3.1 and 3.4.1 we have

(3.5.9) $\eta\|u_1\|_{0,\mathbf{R} \times \Omega}^2 + \|u_1\|_{0,\mathbf{R} \times \partial\Omega}^2$
$$\leq p(\overline{a}, \overline{b}, \overline{c}, \overline{d})(\eta^{-1}\|L_\eta u_1\|_{0,\mathbf{R} \times \Omega}^2 + \|L_B u_1\|_{0,\mathbf{R} \times \partial\Omega}^2)$$

for all $u_1 \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$, and

(3.5.10) $\eta\|w_1\|_{0,\mathbf{R} \times \Omega}^2 + \|w_1\|_{0,\mathbf{R} \times \partial\Omega}^2$
$$\leq p^{(*)}(\overline{a}, \overline{b}, \overline{c}, \overline{d})(\eta^{-1}\|L_\eta^{(*)} w_1\|_{0,\mathbf{R} \times \Omega}^2 + \|L_B^{(*)} w_1\|_{0,\mathbf{R} \times \partial\Omega}^2)$$

for all $w_1 \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$. In the above two inequalities η is sufficiently small satisfying the inequalities of Theorems 3.3.1 and 3.4.1.

Set now

(3.5.11) $D(L_\eta) = \{u_1 \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega}) : L_B u_1 = 0 \quad \text{on } \mathbf{R} \times \partial\Omega\}$,

(3.5.12) $D(L_\eta^{(*)}) = \{w_1 \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega}) : L_B^{(*)} w_1 = 0 \text{ on } \mathbf{R} \times \partial\Omega\}$.

In view of (3.4.9), for all $u_1 \in D(L_\eta)$ and $w_1 \in D(L_\eta^{(*)})$

(3.5.13) $(L_\eta u_1, w_1)_{0, \mathbf{R} \times \Omega} = (u_1, L_\eta^{(*)} w_1)_{0, \mathbf{R} \times \Omega}$.

Moreover, L_η and $L_\eta^{(*)}$, being differential operators, are closable in $L^2(\mathbf{R} \times \Omega)$ (see [20, p. 32]). Denote their closures by $\bar{L}_\eta$ and $\bar{L}_\eta^{(*)}$. By [5, Theorem 1] they both map their domains one-to-one onto $L^2(\mathbf{R} \times \Omega)$. Therefore for any $F \in L_\eta^2(\mathbf{R} \times \Omega)$ there exists a unique $u_1 \in D(\bar{L}_\eta)$ such that $\bar{L}_\eta u_1 = Fe^{-\eta t}$. Hence there exists a sequence $(u_{1\mu}) \subset D(L_\eta)$ such that

(3.5.14) $u_{1\mu} \to u_1$ in $L^2(\mathbf{R} \times \Omega)$,

(3.5.15) $L_\eta u_{1\mu} \to Fe^{-\eta t}$ in $L^2(\mathbf{R} \times \Omega)$.

In view of (3.5.9) and (3.5.11) for every μ, ν

(3.5.16) $\eta \|u_{1\mu} - u_{1\nu}\|_{0, \mathbf{R} \times \Omega}^2 + \|u_{1\mu} - u_{1\nu}\|_{0, \mathbf{R} \times \partial\Omega}^2$
$$\leq p(\bar{a}, \bar{b}, \bar{c}, \bar{d}) \|L_\eta u_{1\mu} - L_\eta u_{1\nu}\|_{0, \mathbf{R} \times \Omega}^2.$$

From (3.5.15) and (3.5.16) it follows that $(u_{1\mu}|_{x_1=0})$ is a Cauchy sequence in $L^2(\mathbf{R} \times \partial\Omega)$, and hence there exists $u_{10} \in L^2(\mathbf{R} \times \partial\Omega)$ such that $u_{1\mu}|_{x_1=0} \to u_{10}$ in $L^2(\mathbf{R} \times \partial\Omega)$. Futher, since $u_{1\mu} \in D(L_\eta)$ we have

(3.5.17) $L_B u_{10} = 0$ on $\mathbf{R} \times \partial\Omega$.

Thus we have proved that there exists a unique u_1 for which there exist a sequence $u_{1\mu} \in D(L_\eta)$ and a function $u_{10} \in L^2(\mathbf{R} \times \partial\Omega)$ such that (3.5.1)–(3.5.3) and (3.5.4) with $g = 0$ are satisfied and the following inequality is true:

(3.5.18) $\eta \|u_1\|_{0, \mathbf{R} \times \Omega}^2 + \|u_{10}\|_{0, \mathbf{R} \times \partial\Omega}^2 \leq p(\bar{a}, \bar{b}, \bar{c}, \bar{d}) \eta^{-1} \|Fe^{-\eta t}\|_{0, \mathbf{R} \times \Omega}^2$.

Hence $u_1 \in L^2(\mathbf{R} \times \Omega)$ is the unique strong solution of problem (3.5.5)–(3.5.6) with $g = 0$.

Assume now that $g \in C_{(0)}^\infty(\mathbf{R} \times \partial\Omega)$. Let $\tilde{g} \in C_{(0)}^\infty(\mathbf{R} \times \partial\Omega)$ be such that

(3.5.19) $L_B \tilde{g} = ge^{-\eta t}$ on $\mathbf{R} \times \partial\Omega$.

Extend $\tilde{g}$ to a function $G \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$. Set

(3.5.20) $z = u_1 - G$.

From (3.5.5)–(3.5.6) we get

(3.5.21) $L_\eta z = Fe^{-\eta t} - L_\eta G$ in $\mathbf{R} \times \Omega$,

(3.5.22) $L_B z = 0$ on $\mathbf{R} \times \partial\Omega$.

By the above considerations there exists a unique $z \in L^2(\mathbf{R} \times \Omega)$ for which there exist a sequence $(z_\mu) \subset D(L_\eta)$ and a function $z_0 \in L^2(\mathbf{R} \times \partial\Omega)$ such that

(3.5.23) $\qquad \|z_\mu - z\|_{0,\mathbf{R} \times \Omega} \to 0 \,,$

(3.5.24) $\qquad \|z_\mu - z_0\|_{0,\mathbf{R} \times \partial\Omega} \to 0 \,,$

(3.5.25) $\qquad \|L_\eta z_\mu - Fe^{-\eta t} - L_\eta G\|_{0,\mathbf{R} \times \Omega} \to 0 \,,$

(3.5.26) $\qquad L_B z_0 = 0 \quad \text{on } \mathbf{R} \times \partial\Omega \,.$

Hence $u_1 = z + G \in L^2(\mathbf{R} \times \Omega)$ is the unique strong solution of problem (3.5.5)–(3.5.6) and satisfies the estimate

(3.5.27) $\quad \eta\|u_1\|_{0,\mathbf{R} \times \Omega}^2 + \|u_{10}\|_{0,\mathbf{R} \times \partial\Omega}^2$
$$\leq p(\overline{a},\overline{b},\overline{c},\overline{d})(\eta^{-1}\|Fe^{-\eta t}\|_{0,\mathbf{R} \times \partial\Omega}^2 + \|ge^{-\eta t}\|_{0,\mathbf{R} \times \partial\Omega}^2) \,.$$

Take now any $g \in L^2_\eta(\mathbf{R} \times \partial\Omega)$. Since $C^\infty_{(0)}(\mathbf{R} \times \partial\Omega)$ is dense in $L^2(\mathbf{R} \times \partial\Omega)$ it follows that there exists a sequence $(g_\nu) \subset C^\infty_{(0)}(\mathbf{R} \times \partial\Omega)$ such that

(3.5.28) $\qquad\qquad g_\nu \to g \quad \text{in } L^2_\eta(\mathbf{R} \times \partial\Omega) \,.$

Consider the problem

(3.5.29) $\qquad\qquad L_\eta u_{1\nu} = Fe^{-\eta t} \quad \text{in } \mathbf{R} \times \Omega \,,$

(3.5.30) $\qquad\qquad L_B u_{1\nu} = g_\nu e^{-\eta t} \quad \text{on } \mathbf{R} \times \partial\Omega \,.$

By the previous considerations there is a unique function $u_{1\nu} \in L^2(\mathbf{R} \times \Omega)$ for which there exist a sequence $(u_{1\nu_\mu}) \subset C^\infty_{(0)}(\mathbf{R} \times \overline{\Omega})$ with $L_B u_{1\nu_\mu} = ge^{-\eta t}$ and a function $u_{10\nu} \in L^2(\mathbf{R} \times \partial\Omega)$ such that, as $\mu \to \infty$,

(3.5.31) $\qquad \|u_{1\nu_\mu} - u_{1\nu}\|_{0,\mathbf{R} \times \Omega} \to 0 \,,$

(3.5.32) $\qquad \|u_{1\nu_\mu} - u_{10\nu}\|_{0,\mathbf{R} \times \partial\Omega} \to 0 \,,$

(3.5.33) $\qquad \|L_\eta u_{1\nu_\mu} - Fe^{-\eta t}\|_{0,\mathbf{R} \times \Omega} \to 0 \,,$

(3.5.34) $\qquad L_B u_{10\nu} = g_\nu e^{-\eta t} \quad \text{on } \mathbf{R} \times \partial\Omega \,.$

For the solution $u_{1\nu} - u_{1\sigma}$ of problem (3.5.29)–(3.5.30) with right-hand sides equal to zero and $(g_\nu - g_\sigma)e^{-\eta t}$ respectively, the estimate (3.5.27) has the form

$$\eta\|u_{1\nu} - u_{1\sigma}\|_{0,\mathbf{R} \times \Omega}^2 + \|u_{10\nu} - u_{10\sigma}\|_{0,\mathbf{R} \times \partial\Omega}^2 \leq p(\overline{a},\overline{b},\overline{c},\overline{d})\|g_\nu - g_\sigma\|_{0,\eta,\mathbf{R} \times \partial\Omega}^2 \,.$$

By (3.5.28) and the above inequality there exist unique $u_1 \in L^2(\mathbf{R} \times \Omega)$ and $u_{10} \in L^2(\mathbf{R} \times \partial\Omega)$ such that

(3.5.35) $\qquad\qquad u_{1\nu} \to u_1 \quad \text{in } L^2(\mathbf{R} \times \Omega) \,,$

(3.5.36) $\qquad\qquad u_{10\nu} \to u_{10} \quad \text{in } L^2(\mathbf{R} \times \partial\Omega) \,.$

By (3.5.31)–(3.5.36), $u = u_1 e^{\eta t} \in L^2_\eta(\mathbf{R} \times \Omega)$ is the unique strong solution of problem (3.1)–(3.2) and satisfies inequality (3.3.122). $\blacksquare$

R e m a r k 3.5.1. Using the above existence theorem one can derive (see [21]) the following estimate for the restriction of u to $J \subset \mathbf{R}$:

$$(3.5.37) \quad \eta\|u\|_{0,\eta,J\times\Omega}^2 + \|u_0\|_{0,\eta,J\times\partial\Omega}^2$$
$$\leq p(\overline{a},\overline{b},\overline{c},\overline{d})(\eta^{-1}\|F\|_{0,\eta,J\times\Omega}^2 + \|g\|_{0,\eta,J\times\partial\Omega}^2)\,.$$

This estimate will be used later, in particular in Chapter 4.

THEOREM 3.5.2. *Let assumptions* (3.1.1)–(3.1.5), (3.1.7), (3.1.14), (3.1.20), (3.1.24) *and* (3.4.13) *be satisfied. Moreover, assume that*

$$(3.5.38) \quad v \in C_{(0)}^\infty(\mathbf{R}\times\overline{\Omega})\,,$$

$$(3.5.39) \quad E, A_j\ (j=1,\ldots,m), C, S \in C^\infty(\mathbf{R}\times\overline{\Omega}\times\mathbf{R}^n)\,,$$

$$(3.5.40) \quad F \in C_{(0)}^\infty(\mathbf{R}\times\overline{\Omega})\,,$$

$$(3.5.41) \quad g \in C_{(0)}^\infty(\mathbf{R}\times\partial\Omega)\,.$$

Then problem (3.1)–(3.2) *has a unique strong solution* $u \in C_{(0)}^\infty(\mathbf{R}\times\overline{\Omega})$ *satisfying*

$$(3.5.42) \quad \eta\|u\|_{s,\eta,\mathbf{R}\times\Omega}^2 + \|u\|_{s,\eta,\mathbf{R}\times\partial\Omega}^2$$
$$\leq \tilde{f}_s(\varepsilon_1,\varepsilon_2,\varepsilon_3)(p_s(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f})\eta^{-1}\|F\|_{s,\eta,\mathbf{R}\times\Omega}^2$$
$$+ q_s(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f})\|g\|_{s,\eta,\mathbf{R}\times\partial\Omega}^2) \quad \forall 1 \leq s \leq k\,,$$

where $k > m/2 + 5/2$; $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{e}$, $\overline{f}$ *are given by* (3.3.13); ε_1, ε_2, ε_3 *are the constants from Theorem* 3.2.1; $\tilde{f}_s$ *are continuous functions;* p_s *and* q_s *are polynomials for* $1 \leq s < k$, *while* $p_k = \hat{p}_k + p_{k-1}e^{2\eta T_2^*}$ *and* $q_k = \hat{q}_k + q_{k-1}e^{2\eta T_2^*}$, *where* $\hat{p}_k$ *and* $\hat{q}_k$ *are polynomials, and* $T_2^* > 0$ *is such that the function* S *is constant outside an interval* $[T_1^*, T_2^*]$; *and* η *fulfils*

$$(3.5.43) \quad \eta > \tilde{f}(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K},\varepsilon_1,\varepsilon_2,\varepsilon_3)$$

($\tilde{f}$ *is a continuous function,* K *is the set from Theorem* 3.3.1).

P r o o f. By assumption $g \in C_{(0)}^\infty(\mathbf{R}\times\partial\Omega)$. Let (as in Theorem 3.5.1) $\tilde{g} \in C_{(0)}^\infty(\mathbf{R}\times\partial\Omega)$ satisfy (3.5.19), and let $G \in C_{(0)}^\infty(\mathbf{R}\times\overline{\Omega})$ be an extension of $\tilde{g}$ onto $\mathbf{R}\times\overline{\Omega}$. By Theorem 3.5.1 there exists a unique strong solution $u \in L_\eta^2(\mathbf{R}\times\Omega)$ of (3.1)–(3.2), and thus $u_1 = ue^{-\eta t} \in L^2(\mathbf{R}\times\Omega)$ is the unique strong solution of (3.5.5)–(3.5.6). Let $u_\mu \in C_{(0)}^\infty(\mathbf{R}\times\overline{\Omega})$ be a sequence as in Definition 3.5.1. Then $z_\mu = u_\mu - G$ satisfies (3.5.23)–(3.5.26) and $z_\mu \in D(L_\eta)$. Therefore by (3.5.13)

$$(L_\eta z_\mu, w)_{0,\mathbf{R}\times\Omega} = (z_\mu, L_\eta^{(*)}w)_{0,\mathbf{R}\times\Omega} \quad \forall w \in D(L_\eta^{(*)})\,,$$

and so

$$(L_\eta u_{1\mu} - L_\eta G, w)_{0,\mathbf{R}\times\Omega} = (u_{1\mu} - G, L_\eta^{(*)}w)_{0,\mathbf{R}\times\Omega} \quad \forall w \in D(L_\eta^{(*)})\,,$$

where $u_{1\mu} = u_\mu e^{-\eta t}$. (3.5.1) and (3.5.3) imply

$$(3.5.44) \quad (Fe^{-\eta t} - L_\eta G, w)_{0,\mathbf{R}\times\Omega} = (u_1 - G, L_\eta^{(*)}w)_{0,\mathbf{R}\times\Omega} \quad \forall w \in D(L_\eta^{(*)}).$$

By (3.5.19) and (3.5.43), u_1 is a weak solution of (3.5.5)–(3.5.6) (see the definition in [24, p. 1117]). Therefore, the assumptions concerning F and g, Remark 3.5.1 and [24, Theorems 1, 3] yield that $\forall s > 0 \; \exists \eta > 0$ such that $u \in H_\eta^s(\mathbf{R} \times \Omega)$ and $u = 0$ at infinity. Thus by the Sobolev lemma $u \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$. Repeating now the proof of [28, Lemma 3.2] we get the estimate (3.5.42). ∎

R e m a r k 3.5.2. Let $F \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$ and $g \in C_{(0)}^\infty(\mathbf{R} \times \partial\Omega)$ and let $J \subset \mathbf{R}$ be an interval. Using Remark 3.5.1 and repeating the proof of [28, Lemma 3.2] we obtain

$$(3.5.45) \quad \eta\|u\|_{s,\eta,J\times\Omega}^2 + \|u\|_{s,\eta,J\times\partial\Omega}^2$$
$$\leq \tilde{f}_s(\varepsilon_1,\varepsilon_2,\varepsilon_3)(p_s(\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f})\eta^{-1}\|F\|_{s,\eta,J\times\Omega}^2$$
$$+ q_s(\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f})\|g\|_{s,\eta,J\times\partial\Omega}^2) \quad \forall 0 \leq s \leq k,$$

where p_s, q_s and $\tilde{f}_s$ are as in Theorem 3.5.2.

R e m a r k 3.5.3. Theorems 3.5.1 and 3.5.2 still hold if we replace (3.1.4) by the following weaker condition:

$$(3.5.46) \qquad \det A_1(t,x,v) \neq 0 \quad \forall(t,x,v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n.$$

P r o o f. As in [21, p. 268], take $\psi \in C_{(0)}^\infty(\bar{\Omega})$ with $\psi \equiv 1$ in a neighbourhood of $\partial\Omega$ and $\det A_1(t,x,v) \neq 0$ for $x \in \operatorname{supp}\psi$. A solution u of (3.1)–(3.2) can be written as $u = \psi u + (1-\psi)u$. Since $\det A_1 \neq 0$ on $\operatorname{supp}\psi u$, for η sufficiently large the estimate (3.3.122) for ψu is satisfied. Since $(1-\psi)u$ is a solution of the Cauchy problem for the operator L, by Theorem 3.6.1 of the next section an estimate of type (3.3.122) for $(1-\psi)u$ is also fulfilled. Hence Theorem 3.5.1 is true. The estimates in H_η^s (where $1 \leq s \leq k$) are also satisfied: for ψu by Theorem 3.5.2, and for $(1-\psi)u$ by Theorem 3.6.1. ∎

3.6. An estimate of a solution of the Cauchy problem for system (3.1) in H_η^s-norm.

In this section we consider the problem

$$(3.6.1) \qquad E(v)u_t - \sum_{j=1}^m A_j(v)u_{x_j} - C(v)u = F \quad \text{in } [0,T] \times \Omega,$$

$$(3.6.2) \qquad u(0,\cdot) = 0 \qquad \qquad \text{in } \Omega,$$

where $v \in \Pi_0^k(\mathbf{R} \times \Omega)$ and either $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$ or Ω is a bounded domain in $\mathbf{R}^m$ with a smooth boundary $\partial\Omega$.

Theorems 3.6.1 and 3.6.2 given below concern the estimates of a solution of problem (3.6.1)–(3.6.2) in L_η^2- and H_η^s-norms and help us to prove Remarks 3.5.3, 4.1.4 and Theorems 5.1, 5.4.

We set

$$\Omega_\varepsilon = \left\{ x \in \mathbf{R}^m : \mathrm{dist}(x, \partial\Omega) < \varepsilon \right\},$$

$$U_{[0,T]} = \left\{ u \in C_{(0)}^\infty([0,T] \times \mathbf{R}^m) : u = 0 \text{ in } (\mathbf{R}^m \setminus \Omega) \cup \Omega_\varepsilon \text{ for some } \varepsilon > 0 \right\}.$$

THEOREM 3.6.1. *Let assumptions* (3.1.1)–(3.1.3), (3.1.5), (3.1.7) *and* (3.1.6a)–(3.1.6b) *with* $k > m/2 + 5/2$ *be satisfied. Then, if* $u \in U_{[0,T]}$ *is a solution of* (3.6.1)–(3.6.2), *then for* $0 \le t \le T$

$$(3.6.3) \qquad \eta\|u\|_{0,\eta,\Omega'}^2 + \|u(t,\cdot)\|_{0,\eta,\Omega}^2 \le \eta^{-1} p(\overline{a}, \overline{b})\|F\|_{0,\eta,\Omega'}^2$$

for η *satisfying*

$$(3.6.4) \qquad\qquad\qquad \eta > q(\overline{a}, \overline{b}),$$

where $\overline{a}$ *and* $\overline{b}$ *are given by* (3.3.13), *and* p, q *are continuous functions.*

P r o o f. Using Remark 2.1.1 extend the functions $E'' = E - E'$, $A_j'' = A_j - A_j'$ $(j = 1, \ldots, m)$ and $C'' = C - C'$ (where in the case when Ω is a halfspace, E', A_j', C' are the constant matrices from assumption (3.1.7), while in the case of bounded Ω they are some constant matrices satisfying assumptions (5.1)–(5.4), (5.6), (5.7), (5.10) in Chapter 5) to $\widetilde{E}''$, $\widetilde{A}_j''$, $\widetilde{C}'' \in C_{(0)}^k(\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n)$ such that

$$\|\widetilde{E}''\|_{C_B^k(\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^n)} \le c_1 \|E''\|_{C_B^k(\mathbf{R}\times\overline{\Omega}\times\mathbf{R}^n)},$$

$$(3.6.5) \quad \|\widetilde{A}_j''\|_{C_B^k(\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^n)} \le c_1 \|A_j''\|_{C_B^k(\mathbf{R}\times\overline{\Omega}\times\mathbf{R}^n)} \quad \text{for } j = 1, \ldots, m,$$

$$\|\widetilde{C}''\|_{C_B^k(\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^n)} \le c_1 \|C''\|_{C_B^k(\mathbf{R}\times\overline{\Omega}\times\mathbf{R}^n)},$$

where $c_1 > 0$ is a constant. Then the functions $\widetilde{E} = \widetilde{E}'' + E'$, $\widetilde{A}_j = \widetilde{A}_j'' + A_j'$, $\widetilde{C} = \widetilde{C}'' + C'$ are C^k-extensions of E, A_j, C to $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$ such that $\widetilde{E} = E'$, $\widetilde{A}_j = A_j'$, $\widetilde{C} = C'$ outside a compact set $\widetilde{K}_{t,x,v} \subset \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$. Similarly, using the Hestenes method extend v to $\widetilde{v} \in \Pi_0^k(\mathbf{R} \times \mathbf{R}^m)$ such that

$$(3.6.6) \qquad\qquad \|\widetilde{v}\|_{k,0,\infty,\mathbf{R}\times\mathbf{R}^m} \le c_2 \|v\|_{k,0,\infty,\mathbf{R}\times\Omega},$$

where $c_2 > 0$ is a constant.

Now instead of (3.6.1)–(3.6.2) we consider the problem

$$(3.6.7) \qquad \widetilde{E}(\widetilde{v})u_t - \sum_{j=1}^m \widetilde{A}_j(\widetilde{v})u_{x_j} - \widetilde{C}(\widetilde{v})u = F \quad \text{in } [0,T] \times \mathbf{R}^m,$$

$$(3.6.8) \qquad\qquad\qquad u(0,\cdot) = 0 \quad \text{in } \mathbf{R}^m,$$

where $\widetilde{G}(\widetilde{v}) = \widetilde{G}(t, x, \widetilde{v})$ $(\widetilde{G} = \widetilde{E}, \widetilde{A}_j, \widetilde{C})$.

Let $\widetilde{K}$ be a compact set such that $\widetilde{K}_{t,x,v} \subset \text{Int } \widetilde{K}$. Take $w_0 = (t_0, x_0, v_0)$ $\in \widetilde{K}$. Then in view of assumptions (3.1.5) and (3.1.2) Lemma 2.3.7 implies that there exists a neighbourhood $\theta_{w_0,\delta}$ of w_0 and a matrix-valued function $\widetilde{H}_0(t, x, v, \omega)$ defined in $\theta_{w_0,\delta} \times \{\omega : |\omega| = 1\}$ such that:

$1°$ the entries of $\widetilde{H}_0(t, x, v, \omega)$ are polynomials in the entries of $A(t, x, v, \omega)$. Hence by (3.1.6a), $\widetilde{H}_0 \in C_B^k(\theta_{w_0,\delta} \times \{\omega : |\omega| = 1\})$;

$2°$ $\forall(t, x, v) \in \widetilde{K}$, $\widetilde{H}_0(t, x, v, \cdot)$ is an analytic function in $\{\omega : |\omega| = 1\}$;

$3°$ $\exists \delta > 0$ $\forall(t, x, v, \omega) \in \theta_{w_0,\delta} \cap \overline{\Omega} \times \{\omega : |\omega| = 1\}$

$$\widetilde{H}_0(t, x, v, \omega) \geq \delta I \,;$$

$4°$ $\forall(t, x, v, \omega) \in \theta_{w_0,\delta} \cap \overline{\Omega} \times \{\omega : |\omega| = 1\}$

$$A^*(t, x, v, \omega)\widetilde{H}_0(t, x, v, \omega) + \widetilde{H}_0(t, x, v, \omega)A(t, x, v, \omega) = 0 \,;$$

$5°$ $\forall(t, x, v, \omega) \in \theta_{w_0,\delta} \times \{\omega : |\omega| = 1\}$, $\widetilde{H}_0(t, x, v, \omega)$ is a hermitian matrix.

Lemma 2.3.7 also gives the existence of a function $\widetilde{H}_1(t, x, v, \omega) = \widetilde{H}'(\omega)$ satisfying $1°-5°$ in $((\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n) \setminus K_{t,x,v}) \times \{\omega : |\omega| = 1\}$. As in Theorem 3.2.1, applying a partition of unity we obtain a function $\widetilde{H}_2(t, x, v, \omega)$ satisfying $1°, 2°, 5°$ in $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n \times \{\omega : |\omega| = 1\}$ and $3°, 4°$ in $\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n \times \{\omega : |\omega| = 1\}$. Next, the function $\widetilde{H}_3$ defined by

$$\widetilde{H}_3(t, x, v, \omega) = \widetilde{H}_2(t, x, v, \omega/|\omega|)$$

fulfils $1°, 2°, 5°$ in $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n \times \{\omega : |\omega| \geq 1\}$ and $3°, 4°$ in $\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n \times \{\omega : |\omega| \geq 1\}$. By (3.1.7) and $1°$, $\widetilde{H}_3$ can be written as $\widetilde{H}_3(t, x, v, \omega) = \widetilde{H}_3'(\omega) + \widetilde{H}_3''(t, x, v, \omega)$, where $\widetilde{H}_3'' = 0$ outside $\widetilde{K}_{t,x,v}$. Using the method of Hestenes extend $\widetilde{H}_3'(\omega)$ to $\widetilde{H}'(\omega) \in C_B^k(\mathbf{R}^m)$ and $\widetilde{H}_3''(t, x, v, \omega)$ to $\widetilde{H}''(t, x, v, \omega) \in C_B^k(\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^m)$ satisfying

$$\|\widetilde{H}'\|_{C_B^k(\mathbf{R}^m)} \leq c\|\widetilde{H}_3'\|_{C_B^k(\{\omega:|\omega|\geq 1\})} \,,$$

(3.6.9) $$\|\widetilde{H}''\|_{C_B^k(\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^m)} \leq c\|\widetilde{H}_3''\|_{C_B^k(\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n \times \{\omega:|\omega|\geq 1\})}$$

and $\widetilde{H}'' = 0$ outside $\widetilde{K}_{t,x,v}$. Then $\widetilde{H}(t, x, v, \omega) = \widetilde{H}'(\omega) + \widetilde{H}''(t, x, v, \omega)$ is an extension of $\widetilde{H}_3$ to $\mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^m$ and satisfies $2°, 5°$ for $|\omega| \geq 1$ and $3°, 4°$ in $\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n \times \{\omega : |\omega| \geq 1\}$.

Now replace v in $\widetilde{H}(t, x, v, \omega)$ by $\widetilde{v}$. We then get a function $\widetilde{\mathcal{H}}(t, x, \omega) = \widetilde{H}(t, x, \widetilde{v}(t, x), \omega)$ such that

(3.6.10) $$\widetilde{\mathcal{H}}(t, x, \omega) = \widetilde{H}'(\omega) + \widetilde{\mathcal{H}}''(t, x, \omega)$$

(where $\widetilde{\mathcal{H}}'' = 0$ outside a compact set $K_{t,x} \subset \mathbf{R} \times \mathbf{R}^m$) and

1) $\partial_\omega^\alpha \widetilde{\mathcal{H}}''(\cdot,\cdot,\omega) \in \Pi_0^k([0,T] \times \mathbf{R}^m)$, $\forall |\alpha| \leq 2 \ \forall \omega \in \mathbf{R}^m$, and

(3.6.11) $\qquad |\partial_\omega^\alpha \widetilde{\mathcal{H}}''|_{k,0,\infty,[0,T] \times \mathbf{R}^m} \leq p^\alpha(\overline{a},\overline{b})(1 + |\omega|)^{-|\alpha|}$

for $|\omega| \geq 1$, where p^α are positive polynomials;

2) $|\partial_\omega^\alpha \widetilde{H}'| \leq c(1 + |\omega|)^{-|\alpha|}$, $\forall |\alpha| \leq k \ \forall |\omega| \geq 1$, where $c > 0$ is a constant;

3) $\exists \delta > 0 \ \forall (t,x,\omega) \in [0,T] \times \overline{\Omega} \times \{\omega : |\omega| \geq 1\}$, $\widetilde{\mathcal{H}}(t,x,\omega) \geq \delta I$;

4) $\forall (t,x,\omega) \in [0,T] \times \overline{\Omega} \times \{\omega : |\omega| \geq 1\}$

$$\widetilde{A}^*(t,x,\omega)\widetilde{\mathcal{H}}(t,x,\omega) + \widetilde{\mathcal{H}}(t,x,\omega)\widetilde{A}(t,x,\omega) = 0 \,,$$

where $\widetilde{A}(t,x,\omega) = i\sum_{j=1}^m \widetilde{\mathcal{E}}^{-1}(t,x)\widetilde{A}_j(t,x)\omega_j$, $\widetilde{\mathcal{G}}(t,x) = \widetilde{G}(t,x,\widetilde{v}(t,x))$, where $\widetilde{G} = \widetilde{E},\widetilde{A}_j,\widetilde{C}$, thus $\widetilde{\mathcal{G}} = \widetilde{\mathcal{E}},\widetilde{A}_j,\widetilde{\mathcal{C}}$;

5) $\forall (t,x,\omega) \in [0,T] \times \mathbf{R}^m \times \{\omega : |\omega| \geq 1\}$, $\widetilde{\mathcal{H}}(t,x,\omega)$ is a hermitian matrix.

Introduce now the pseudodifferential operator

(3.6.12) $\qquad \widetilde{\mathcal{H}}(t,x,D_x)u = (2\pi)^{-m} \int \widetilde{\mathcal{H}}(t,x,\omega)e^{ix\omega}\widehat{u}(t,\omega)\,d\omega \,,$

where $u \in U_{[0,T]}$, and for any $t \in [0,T]$, $\widehat{u}(t,\cdot)$ is the Fourier transform of $u(t,\cdot)$. We shall apply the operator (3.6.12) to obtain the estimate (3.6.3). To do this write system (3.6.7) as

(3.6.13) $\qquad \dfrac{\partial u}{\partial t} = \widetilde{A}(t,x,D_x) + \widetilde{\mathcal{E}}^{-1}(t,x)\widetilde{\mathcal{C}}(t,x)u + \widetilde{\mathcal{E}}^{-1}(t,x)F(t,x) \,,$

where $\widetilde{A}(t,x,D_x) = i\sum_{j=1}^m \widetilde{\mathcal{E}}^{-1}(t,x)\widetilde{A}_j(t,x)D_{x_j}$ $(D_{x_j} = i^{-1}\partial/\partial x_j)$ is the operator with symbol $\widetilde{A}(t,x,\omega)$. After multiplying (3.3.13) by $e^{-\eta t}$ we obtain

(3.6.14) $\qquad \dfrac{\partial w}{\partial t} = \widetilde{A}(t,x,D_x)w + \widetilde{\mathcal{E}}^{-1}(t,x)\widetilde{\mathcal{C}}(t,x)w + \widetilde{\mathcal{E}}^{-1}Fe^{-\eta t} - \eta w \,,$

where $w = ue^{-\eta t}$. Using (3.6.14) we have

$$\frac{\partial}{\partial t}(\widetilde{\mathcal{H}}w,w)_{0,\mathbf{R}^m} = \left(\widetilde{\mathcal{H}}\frac{\partial w}{\partial t}, w\right)_{0,\mathbf{R}^m} + \left(\widetilde{\mathcal{H}}w, \frac{\partial w}{\partial t}\right)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}_t w, w)_{0,\mathbf{R}^m}$$

$$= (\widetilde{\mathcal{H}}\widetilde{A}w, w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}w, \widetilde{A}w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}_t w, w)_{0,\mathbf{R}^m}$$

$$+ (\widetilde{\mathcal{H}}\widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{C}}w, w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}w, \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{C}}w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}\widetilde{\mathcal{E}}^{-1}Fe^{-\eta t}, w)_{0,\mathbf{R}^m}$$

$$+ (\widetilde{\mathcal{H}}w, \widetilde{\mathcal{E}}^{-1}Fe^{-\eta t})_{0,\mathbf{R}^m} - 2\eta(\widetilde{\mathcal{H}}w, w)_{0,\mathbf{R}^m} \,.$$

Hence

(3.6.15) $\quad 2\eta(\widetilde{\mathcal{H}}w,w)_{0,\mathbf{R}^m} + \dfrac{\partial}{\partial t}(\widetilde{\mathcal{H}}w,w)_{0,\mathbf{R}^m}$

$$= ((\widetilde{\mathcal{H}}\widetilde{A} + \widetilde{A}^*\widetilde{\mathcal{H}})w, w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}\widetilde{\mathcal{K}}w, w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}_t w, w)_{0,\mathbf{R}^m}$$

$$+ (\widetilde{\mathcal{H}}\widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{C}}w, w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}}w, \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{C}}w)_{0,\mathbf{R}^m}$$

$$+ (\widetilde{\mathcal{H}}\widetilde{\mathcal{E}}^{-1} F e^{-\eta t}, w)_{0,\mathbf{R}^m} + (\widetilde{\mathcal{H}} w, \widetilde{\mathcal{E}}^{-1} F e^{-\eta t})_{0,\mathbf{R}^m} ,$$

where

$$\widetilde{\mathcal{K}}(t,x) = -\sum_{j=1}^{m} \left(\widetilde{\mathcal{A}}_j^*(t,x) \frac{\partial \widetilde{\mathcal{E}}^{*-1}(t,x)}{\partial x_j} + \frac{\partial \widetilde{\mathcal{A}}_j^*(t,x)}{\partial x_j} \widetilde{\mathcal{E}}^{*-1}(t,x) \right) ,$$

and $\widetilde{\mathcal{A}}^*(t,x,D_x)$ is the pseudodifferential operator with symbol $\widetilde{\mathcal{A}}^*(t,x,\omega) = -i \sum_{j=1}^{m} \widetilde{\mathcal{A}}_j^*(t,x) \widetilde{\mathcal{E}}^{*-1}(t,x) \omega_j$. Just as in Lemma 3.3.3 we can prove that property 3) implies

$$(3.6.16) \qquad \eta(\widetilde{\mathcal{H}} w, w)_{0,\mathbf{R}^m} = (\eta \widetilde{\mathcal{H}} w, w)_{0,\mathbf{R}^m}$$
$$\geq \eta \delta \|w\|_{0,\mathbf{R}^m}^2 - p_1(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 ,$$

$$(3.6.17) \qquad (\widetilde{\mathcal{H}} w, w)_{0,\mathbf{R}^m} \geq \delta \|w\|_{0,\mathbf{R}^m}^2 - p_2(\overline{a},\overline{b}) \|w\|_{-1/2,\mathbf{R}^m}^2 ,$$

where p_1 and p_2 are polynomials. Using next property 4) and proceeding in the same way as in Lemma 3.3.2 we get

$$(3.6.18) \qquad |((\widetilde{\mathcal{H}}\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}^*\widetilde{\mathcal{H}}) w, w)_{0,\mathbf{R}^m}| \leq p_3(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 .$$

Moreover, we have

$$(3.6.19) \qquad |(\widetilde{\mathcal{H}}\widetilde{\mathcal{K}} w, w)_{0,\mathbf{R}^m}| \leq p_4(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 ,$$

$$(3.6.20) \qquad |(\widetilde{\mathcal{H}}_t w, w)_{0,\mathbf{R}^m}| \leq p_5(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 ,$$

$$(3.6.21) \qquad |(\widetilde{\mathcal{H}}\widetilde{\mathcal{E}}^{-1}\widetilde{C} w, w)_{0,\mathbf{R}^m}| \leq p_6(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 ,$$

$$(3.6.22) \qquad |(\widetilde{\mathcal{H}} w, \widetilde{\mathcal{E}}^{-1}\widetilde{C} w)_{0,\mathbf{R}^m}| \leq p_7(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 ,$$

$$(3.6.23) \qquad |(\widetilde{\mathcal{H}}\widetilde{\mathcal{E}}^{-1} F e^{-\eta t}, w)_{0,\mathbf{R}^m}| \leq \varepsilon^{-1}\eta^{-1} p_8(\overline{a},\overline{b}) \|F\|_{0,\mathbf{R}^m}^2$$
$$+ \varepsilon\eta p_9(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 ,$$

$$(3.6.24) \qquad |(\widetilde{\mathcal{H}} w, \widetilde{\mathcal{E}}^{-1} F e^{-\eta t})_{0,\mathbf{R}^m}| \leq \varepsilon^{-1}\eta^{-1} p_{10}(\overline{a},\overline{b}) \|F\|_{0,\mathbf{R}^m}^2$$
$$+ \varepsilon\eta p_{11}(\overline{a},\overline{b}) \|w\|_{0,\mathbf{R}^m}^2 ,$$

where p_i $(i = 3, \ldots, 11)$ are polynomials and $\varepsilon > 0$ is a sufficiently small constant. From (3.6.18)–(3.6.24) and (3.6.15) we obtain

$$2\eta(\widetilde{\mathcal{H}} w, w)_{0,\mathbf{R}^m} + \frac{\partial}{\partial t}(\widetilde{\mathcal{H}} w, w)_{0,\mathbf{R}^m}$$
$$\leq \left[\sum_{i=3}^{7} p_i(\overline{a},\overline{b}) + p_9(\overline{a},\overline{b})\varepsilon\eta + p_{11}(\overline{a},\overline{b})\varepsilon\eta \right] \|w\|_{0,\mathbf{R}^m}^2$$
$$+ (p_8(\overline{a},\overline{b}) + p_{10}(\overline{a},\overline{b})\varepsilon^{-1}\eta^{-1}) \|F\|_{0,\mathbf{R}^m}^2 .$$

Hence using (3.6.15) and taking ε sufficiently small and η large enough we

get

$$\eta\delta\|w\|_{0,\mathbf{R}^m}^2 + \frac{\partial}{\partial t}(\widetilde{\mathcal{H}}w,w)_{0,\mathbf{R}^m} \leq \eta^{-1}p_{11}(\overline{a},\overline{b})\|F\|_{0,\mathbf{R}^m}^2 \,.$$

Next applying (3.6.17) we have

$$\eta\delta\int_0^t \|w(\tau,\cdot)\|_{0,\mathbf{R}^m}^2 \, d\tau + \delta\|w(t,\cdot)\|_{0,\mathbf{R}^m}^2 \leq (\widetilde{\mathcal{H}}(0,x,D_x)w(0,\cdot),w(0,\cdot))_{0,\mathbf{R}^m}$$

$$+ \eta^{-1}p_{11}(\overline{a},\overline{b})\int_0^t \|F(\tau,\cdot)\|_{0,\mathbf{R}^m}^2 \, d\tau\,,$$

where $0 \leq t \leq T$. Now using (3.6.2) we obtain (3.6.3). ∎

THEOREM 3.6.2. *Let the assumptions of Theorem* 3.6.1 *be satisfied. Then, if* $u \in U_{[0,T]}$ *is a solution of* (3.6.1)–(3.6.2), *then for* $0 \leq t \leq T$ *and* $0 \leq s \leq k$

$$(3.6.25) \qquad \eta\|u\|_{s,\eta,\Omega^t} \leq \eta^{-1}p^s(\overline{a},\overline{b})\|F\|_{s,\eta,\Omega^t}^2$$

for η *satisfying* (3.6.4) (p^s *are polynomials*).

P r o o f. The proof is analogous to that of [28, Lemma 3.2]. Let $u \in U_{[0,T]}$ be a solution of (3.6.1)–(3.6.2). Then u satisfies the system of problems

$$(3.6.26) \qquad \mathcal{L}\partial_x^\sigma u = \partial_x^\sigma \mathcal{L}u + (\mathcal{L}\partial_x^\sigma u - \partial_x^\sigma \mathcal{L}u) \quad \text{in } [0,T] \times \mathbf{R}^m\,,$$

$$(3.6.27) \qquad \partial_x^\sigma u|_{t=0} = 0\,,$$

where $|\sigma| = s$, $\mathcal{L} = \widetilde{\mathcal{E}}u_t - \sum_{j=1}^m \widetilde{A}_j u_{x_j} - \widetilde{C}u$. In the same way as in [28] one should first obtain (3.6.25) with $s = 1$. Thus we first estimate the derivative $\partial_x^1 u$ (which is the solution of (3.6.26)–(3.6.27)) in $L_\eta^2(\Omega^t)$-norm, applying Theorem 3.6.1. The derivative u_t is calculated from (3.6.1) and estimated in $L_\eta^2(\Omega^t)$-norm using the earlier estimate of $\partial_x^1 u$. The estimates for $s > 1$ are obtained by induction. ∎

COROLLARY 3.6.1. *If* $u \in U_{\mathbf{R},\varepsilon} = \{u \in C_{(0)}^\infty(\mathbf{R} \times \mathbf{R}^m) : u = 0$ *in* $(\mathbf{R}^m \setminus \Omega) \cup \Omega_\varepsilon\}$ (*where* $\varepsilon > 0$ *is a constant*) *is a solution of*

$$(3.6.28) \qquad E(v)u_t - \sum_{j=1}^m A_j(v)u_{x_j} - C(v)u = F \quad \text{in } \mathbf{R} \times \Omega\,,$$

then

$$(3.6.29) \qquad \eta\|u\|_{s,\eta,\mathbf{R}\times\Omega}^2 \leq p^s(\overline{a},\overline{b})\eta^{-1}\|F\|_{s,\eta,\mathbf{R}\times\Omega}^2$$

for η *satisfying* (3.6.4).

P r o o f. Since $u \in U_{\mathbf{R},\varepsilon}$, $u = 0$ outside some interval $[t_1, t_2]$. Hence u is

a solution of the problem

$$E(v)u_t - \sum_{j=1}^{m} A_j(v)u_{x_j} - C(v)u = F \quad \text{in } [t_1, t_2] \times \Omega \,,$$

$$u(t_1, \cdot) = 0 \quad \text{in } \Omega \,.$$

Then (3.6.29) follows from Theorem 3.6.2. ∎

4. A mixed problem for a system of linear hyperbolic equations of the first order in a halfspace

4.1. A mixed problem with nonzero initial condition. In this section we consider the problem

(4.1.1) $Lu = F \quad \text{in } [0, T] \times \Omega \,,$

(4.1.2) $L_B u = g \quad \text{on } [0, T] \times \partial\Omega \,,$

(4.1.3) $u(0) = f \quad \text{in } \Omega \,,$

where $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$, $T > 0$, L and L_B are given by (3.4.1) and (3.4.2).

Consider first the problem

(4.1.4) $Lu = F \quad \text{for } x_1 > 0, \ t \geq 0 \,,$

(4.1.5) $L_B u = g \quad \text{for } x_1 = 0, \ t \geq 0 \,,$

(4.1.6) $u(0) = f \quad \text{for } x_1 > 0 \,.$

THEOREM 4.1.1. *Let assumptions* (3.1.1)–(3.1.5), (3.1.7), (3.1.14), (3.1.20), (3.1.24), (3.4.13), (3.5.38), (3.5.39) *be satisfied. Furthermore, assume:*

(4.1.7) $F \in C_{(0)}^{\infty}(\mathbf{R} \times \overline{\Omega}), \quad F = 0 \quad \text{for } t < 0 \,,$

(4.1.8) $g \in C_{(0)}^{\infty}(\mathbf{R} \times \partial\Omega), \quad g = 0 \quad \text{for } t < 0 \,,$

(4.1.9) $f \in C_0^{\infty}(\Omega) \,.$

Then there exists a unique solution $u \in C_{(0)}^{\infty}([0, \infty) \times \overline{\Omega})$ *of problem* (4.1.4)–(4.1.6).

P r o o f. Consider the Cauchy problem

(4.1.10) $Lw_1 = 0 \quad \text{in } [0, \infty) \times \mathbf{R}^m \,,$

(4.1.11) $w_1(0) = f \quad \text{in } \mathbf{R}^m \,.$

There exists a unique smooth solution of (4.1.10)–(4.1.11), which for a sufficiently short time interval $[0, \delta]$ vanishes identically in a neighbourhood of $x_1 = 0$ (see [12], [4, Chap. IX, Section 3], [25, Chap. IV, Section 4]).

Take $\chi \in C_0^\infty(-\delta, \delta)$ such that $\chi \equiv 1$ in a neighbourhood of $t = 0$. Then $u = \chi w_1 + w_2$ is a smooth solution of (4.1.4)–(4.1.6), where w_1 is the smooth solution of (4.1.10)–(4.1.11), and w_2 is a smooth solution of the problem

$$(4.1.12) \qquad Lw_2 = \begin{cases} F - w_1 \partial_t \chi & \text{for } x_1 > 0, \ t \geq 0, \\ 0 & \text{for } x_1 > 0, \ t < 0, \end{cases}$$

$$(4.1.13) \qquad L_B w_2 = g \qquad \text{for } x_1 = 0, \ -\infty < t < \infty. \ \blacksquare$$

R e m a r k 4.1.1. Theorem 4.1.1 yields the existence of a smooth solution of problem (4.1.1)–(4.1.3) if the functions F, g and f satisfy respectively assumptions (4.1.7), (4.1.8) and (4.1.9).

Further we shall find an H_η^s-estimate of a smooth solution of (4.1.1)–(4.1.3) and we shall prove the existence and uniqueness of solution of this problem in Sobolev spaces. To do this we shall use the paper [21]. We repeat the proofs of some lemmas of that paper, but keeping track of the norms on which depend the constants appearing in the estimates obtained in the lemmas.

Similarly to [21] consider the differential operator

$$(4.1.14) \qquad P(t, x, \partial_t, \partial_x) = \sum_{|\alpha+\beta| \leq n} p_{\alpha\beta}(t, x) \partial_t^\alpha \partial_x^\beta$$

acting on scalar-valued functions defined in $\mathbf{R} \times \bar{\Omega}$. Let further

$$P_0(t, x, \partial_t, \partial_x) = \sum_{|\alpha+\beta|=n} p_{\alpha\beta}(t, x) \partial_t^\alpha \partial_x^\beta$$

be the principal part of P and let

$$P_0(t, x, \tau, \xi) = \sum_{|\alpha+\beta|=n} p_{\alpha\beta}(t, x) \tau^\alpha \xi^\beta$$

be the characteristic polynomial of P_0. Assume

(4.1.15) $p_{\alpha\beta} \in C_B^0(\mathbf{R} \times \Omega)$ and are constant outside a compact set $K \subset \mathbf{R} \times \bar{\Omega}$;

(4.1.16) P is strictly hyperbolic in the t direction, i.e. $p_{(1,0,\ldots,0),(1,0,\ldots,0)} \neq 0$ and P_0 has the representation

$$P_0(t, x, \tau, \xi) = p_{(1,0,\ldots,0),(1,0,\ldots,0)}(t, x) \prod_{j=1}^r (\tau - \lambda_j)$$

where the numbers $\lambda_j = \lambda_j(t, x, \xi)$ (here $\xi = (\xi_1, \ldots, \xi_m)$) are real and distinct for all $\xi \in \mathbf{R}^m \setminus \{0\}$ and for all $(t, x) \in \mathbf{R} \times \bar{\Omega}$.

Set further

$$(4.1.17) \qquad\qquad J = (-\infty, T].$$

LEMMA 4.1.1. *Let P be the operator defined by (4.1.14) and let assumptions (4.1.15), (4.1.16) be satisfied. Then there exists a constant $c > 0$ independent of*

$$(4.1.18) \quad |P|_0 = \sum_{\alpha,\beta} \sup_{t,x} |p_{\alpha\beta}(t,x)| + \delta^{-1} + \sup_{t,x} |p^{-1}_{(1,0,\dots,0),(1,0,\dots,0)}(t,x)|$$

(where $\delta = \min_{k \neq j} \inf_{t,x,|\xi|=1} |\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)|$) such that for all $T>0$, all $\varphi \in H^n(J \times \Omega)$ and all $\varepsilon > 0$ sufficiently small,

$$(4.1.19) \quad \|\varphi(T)\|_{n-1,\Omega} \leq c\Big(\varepsilon\|P\varphi\|_{0,J\times\Omega} + \varepsilon^{-1}\|\varphi\|_{n-1,J\times\Omega}$$
$$+ \sum_{j=0}^{n-1} \|\partial_{x_1}^j \varphi\|_{n-1,J\times\partial\Omega}\Big).$$

P r o o f. Denote by $P^\tau(t,x,\partial_t,\partial_x)$ the differential operator with symbol $(\partial/\partial t)P_0(t,x,\tau,\xi)$. Then by [7, Section 3, pp. 74–75]

$$(4.1.20) \quad \mathrm{Re}\, P^\tau\varphi \cdot \overline{P\varphi} = \partial_t Q^0(\varphi) + \sum_{j=1}^m \partial_{x_j} Q^j(\varphi) + Q(\varphi),$$

where Q^0 and $\{Q^j\}_{j=1}^m$ are quadratic forms (with bounded coefficients, constant outside a compact set $K \subset \mathbf{R} \times \overline{\Omega}$) in the derivatives of φ of order $n-1$, while Q is a quadratic form in the derivatives of φ of order less than n (with bounded coefficients, constant outside K). Moreover, by [7, Lemma 3.1 and Theorem 3.1]

$$(4.1.21) \quad \int_\Omega Q^0(\psi) \geq \widetilde{c}^{-1}\|\psi\|_{n-1,\Omega}^2 - \widetilde{c}\|\psi\|_{n-2,\Omega}^2 \quad \text{if } n > 1,$$

$$(4.1.22) \quad \int_\Omega Q^0(\psi) \geq \widetilde{\widetilde{c}}\|\psi\|_{0,\Omega}^2 \quad \text{if } n = 1,$$

for all $\psi \in C_{(0)}^{n-1}(\overline{\Omega})$ and $t \in J$ ($\widetilde{c}, \widetilde{\widetilde{c}} > 0$ are independent of $|P|_0$).

Integrating (4.1.20) over $J \times \Omega$ we get

$$\int_\Omega Q^0(\varphi(T)) = \int_{J\times\Omega} Q(\varphi) + \int_{J\times\Omega} \mathrm{Re}\, P^\tau\varphi \cdot \overline{P\varphi} - \int_{J\times\partial\Omega} Q^1(\varphi).$$

Applying further to $\int_{J\times\Omega} \mathrm{Re}\, P^\tau\varphi \cdot \overline{P\varphi}$ the Young inequality and the fact that P^τ is an operator of order $n-1$ (and similarly Q and Q^1) we get

$$(4.1.23) \quad \int_\Omega Q^0(\varphi(T)) \leq c\Big(\varepsilon^2\|P\varphi\|_{0,J\times\Omega}^2 + \varepsilon^{-2}\|\varphi\|_{n-1,J\times\Omega}^2$$
$$+ \sum_{j=0}^{n-1} \|\partial_{x_1}^j \varphi\|_{n-1-j,J\times\partial\Omega}^2\Big)$$

for small $\varepsilon > 0$.

If $n = 1$, then using (4.1.22) we obtain (4.1.19).

In order to prove the lemma for $n > 1$ we apply the inequalities:

(4.1.24) $\forall \theta \in H^{n-1}(\Omega) \ \forall T > 0$

$$\int_{\Omega} Q^0(\theta) \geq \tfrac{1}{2}\tilde{c}^{-1}\|\theta\|^2_{n-1,\Omega} - \tilde{c}\|\theta\|^2_{n-2,\Omega} - c^* \sum_{j=0}^{n-2} \|\partial_{x_1}^j \theta\|^2_{n-1-j-1/2,\partial\Omega},$$

where $c^* > 0$ is a constant;

(4.1.25) $\forall \psi \in H^{n-1}(J \times \partial\Omega) \ \forall 0 \leq j \leq n - 2$

$$\|\partial_{x_1}^j \psi(T)\|_{n-1-j-1/2,\partial\Omega} \leq c\|\partial_{x_1}^j \psi\|_{n-1-j,J\times\partial\Omega},$$

where $c > 0$ is a constant ([9, Theorem 2.5.6]);

(4.1.26) $\forall \varphi \in H^{n-1}(J \times \Omega) \quad \|\varphi(T)\|_{n-2,\Omega} \leq \sqrt{2}\|\varphi\|_{n-1,J\times\Omega}.$

In order to prove (4.1.26) integrate over t from $t = -\infty$ to $t = T$ the identity

$$\frac{d}{dt}\|\varphi(t)\|^2_{n-2,\Omega} = \sum_{|\alpha|\leq n-2} (\partial_x^\alpha \varphi, \partial_t(\partial_x^\alpha \varphi))_{0,\Omega} + \sum_{|\alpha|\leq n-2} (\partial_t(\partial_x^\alpha \varphi), \partial_x^\alpha \varphi)_{0,\Omega}.$$

Then we obtain

$$\|\varphi(T)\|^2_{n-2,\Omega} = \int_{-\infty}^{T} \sum_{|\alpha|\leq n-2} (\partial_x^\alpha \varphi, \partial_t(\partial_x^\alpha \varphi))_{0,\Omega}$$
$$+ \int_{-\infty}^{T} \sum_{|\alpha|\leq n-2} (\partial_t(\partial_x^\alpha \varphi), \partial_x^\alpha \varphi)_{0,\Omega} \leq 2\|\varphi\|^2_{n-1,J\times\Omega},$$

which is (4.1.26).

In order to obtain (4.1.24) write θ as $\theta = \theta_0 + \theta_1$, where $\theta_0 \in \mathring{H}^{n-1}(\Omega)$ and θ_1 satisfies

(4.1.27) $\displaystyle \|\theta_1\|_{n-1,\Omega} \leq C \sum_{j=0}^{n-2} \|\partial_{x_1}^j \theta\|_{n-1-j-1/2,\partial\Omega}$

($C > 0$ is a constant). This representation of θ follows from [9, Theorem 2.5.7].

Let now $Q^0(\cdot,\cdot)$ be the sesquilinear form associated with Q^0. Then

(4.1.28) $Q^0(\theta) = Q^0(\theta_0 + \theta_1, \theta_0 + \theta_1)$
$\qquad\qquad = Q^0(\theta_0,\theta_0) + Q^0(\theta_1,\theta_1) + Q^0(\theta_1,\theta_0) + Q^0(\theta_0,\theta_1).$

By (4.1.21) we have

$$\int\limits_{\Omega} Q^0(\theta_0, \theta_0) = \int\limits_{\Omega} Q(\theta_0) \geq \widetilde{c}^{-1} \|\theta_0\|^2_{n-1,\Omega} - \widetilde{c} \|\theta_0\|^2_{n-2,\Omega},$$

while the integrals of all the other terms on the right-hand side of (4.1.28) can be estimated by $C_1(\sigma\|\theta_0\|^2_{n-1,\Omega} + \sigma^{-1}\|\theta_1\|^2_{n-1,\Omega})$ (where $0 < \sigma < 1$, $C_1 > 0$ is a constant). Choosing σ sufficiently small we have

$$(4.1.29) \qquad \int\limits_{\Omega} Q^0(\theta) \geq \tfrac{1}{2}\widetilde{c}^{-1}\|\theta_0\|^2_{n-1,\Omega} - C_2\|\theta_1\|^2_{n-1,\Omega} - \widetilde{c}\|\theta\|^2_{n-2,\Omega}.$$

Further, since $\|\theta_0\|^2_{n-1,\Omega} \geq \|\theta\|^2_{n-1,\Omega} - \|\theta_1\|^2_{n-1,\Omega}$, using (4.1.27) in (4.1.29) we get (4.1.24). Combining now (4.1.23) with (4.1.24) for $\theta = \varphi(T)$, (4.1.25) for $\psi(T) = \varphi(T)$ and (4.1.26), we get (4.1.19). ∎

LEMMA 4.1.2. *Let assumptions* (3.1.1), (3.1.3), (3.1.4), (3.1.7), (3.1.14), (3.1.20), (3.4.13), (3.5.38)–(3.5.41) *be satisfied and assume that*

$$(4.1.30) \qquad \forall (t, x, v) \in \mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n \ \forall \omega = (\omega_1, \dots, \omega_m) \in \mathbf{R}^m \setminus \{0\} \text{ the equation}$$
$$\det(\lambda E(t, x, v) - \textstyle\sum_{j=1}^m A_j(t, x, v)\omega_j) = 0 \text{ has exactly } n \text{ distinct}$$
real roots (i.e. system (3.1) is strictly hyperbolic).

Then the restriction to $J \times \Omega$ of the solution u of problem (3.1)–(3.2) *satisfies the estimate*

$$(4.1.31) \qquad \|u(T)\|_{n-1,\eta,\Omega} \leq \widetilde{f}(\varepsilon_1, \varepsilon_2, \varepsilon_3) p(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f})$$
$$\times (\eta^{-1/2}\|F\|_{n-1,\eta,J\times\Omega} + \|g\|_{n-1,\eta,J\times\partial\Omega}),$$

where p is a polynomial with coefficients independent of F, g, T and η; $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}$ and $\overline{f}$ are given by (3.3.13) *with $k > \max(m/2 + 5/2, (m+1)/2 + n - 1)$; η is as in Theorem* 3.5.2, *i.e. η satisfies* (3.5.43); *$\widetilde{f}$ is a continuous function; and ε_1, ε_2, ε_3 are the constants from Theorem* 3.2.1.

P r o o f. Consider the operator L given by (3.4.1) as a matrix of differential operators and denote by L^{co} the matrix of differential operators whose entries are the corresponding cofactors of L.

Let $u \in C^\infty_{(0)}(\mathbf{R} \times \overline{\Omega})$ be the solution of (3.1)–(3.2). Then each component u_j of u satisfies

$$(4.1.32) \qquad (\det L)u_j = (L^{co}F)_j + ((\text{operator of order } n-1)u)_j$$

for $j = 1, \dots, n$ (see [21]).

Assumptions (3.1.7), (3.5.38), (3.5.39) and (4.1.30) yield that the scalar operator $\det L$ fulfils the assumptions of Lemma 4.1.1. Therefore by (4.1.32) we get

$$(4.1.33) \qquad \|u_j(T)\|_{n-1,\Omega} \leq c_*\Big(\varepsilon\|(L^{co}F)_j\|_{0,J\times\Omega} + p_1(h)\varepsilon^{-1}\|u\|_{n-1,J\times\Omega}$$

$$+ \sum_{i=0}^{n-1} \|\partial_{x_1}^i u\|_{n-1-i, J \times \partial \Omega} \Bigg),$$

where $c_* > 0$ is a constant, p_1 is a polynomial, $h = \|\widetilde{L}\|_{C_B^{n-1}(\mathbb{R} \times \overline{\Omega})}$, and $\widetilde{L}$ is given by (3.3.12).

Summing up inequalities (4.1.33) we get

(4.1.34) $\|u(T)\|_{n-1, \Omega} \leq p_2(h) \Big(\varepsilon \|F\|_{n-1, J \times \Omega} + \varepsilon^{-1} \|u\|_{n-1, J \times \Omega}$

$$+ \sum_{j=0}^{n-1} \|\partial_{x_1}^j u\|_{n-1-j, J \times \partial \Omega} \Big).$$

The identity

(4.1.35) $$\partial_{x_1} u = -A_1^{-1} \Big(Lu - E\partial_t u + \sum_{j>1} A_j \partial_{x_j} u + Cu \Big)$$

yields the estimate

(4.1.36) $\|\partial_{x_1} u\|_{n-1, J \times \partial \Omega} \leq p_3(h)(\|F\|_{n-2, J \times \partial \Omega} + \|u\|_{n-1, J \times \partial \Omega}).$

Applying next the Young inequality and the trace theorem we get

(4.1.37) $\|F\|_{n-2, J \times \partial \Omega} \leq c(\varepsilon \|F\|_{n-1, J \times \Omega} + \varepsilon^{-1} \|F\|_{n-2, J \times \Omega}).$

Further

(4.1.38) $\|F\|_{n-2, J \times \Omega} = \|Lu\|_{n-2, J \times \Omega} \leq p_4(h) \|u\|_{n-1, J \times \Omega}.$

By (4.1.36)–(4.1.38) for small ε we obtain

(4.1.39) $\|\partial_{x_1} u\|_{n-2, J \times \partial \Omega}$

$$\leq p_5(h)(\varepsilon \|F\|_{n-1, J \times \Omega} + \varepsilon^{-1} \|u\|_{n-1, J \times \Omega} + \|u\|_{n-1, J \times \partial \Omega}).$$

Now, for $1 \leq j \leq n-1$ using (4.1.35) we get

(4.1.40) $\partial_{x_1}^j u = \partial_{x_1}^{j-1} \Big[-A_1^{-1} \Big(Lu - E\partial_t u + \sum_{i>1} A_i \partial_{x_1} u + Cu \Big) \Big]$

$$= -\partial_{x_1}^{j-1}(A_1^{-1} F) + \partial_{x_1}^{j-1}(A_1^{-1} E \partial_t u)$$

$$- \partial_{x_1}^{j-1} \Big(A_1^{-1} \sum_{i>1} A_i \partial_{x_1} u \Big) - \partial_{x_1}^{j-1}(A_1^{-1} Cu).$$

Hence using (4.1.37) and (4.1.38) in (4.1.40) we get

(4.1.41) $\|\partial_{x_1}^j u\|_{n-1-j, J \times \partial \Omega}$

$$\leq p_6(h)(\varepsilon \|F\|_{n-1, J \times \Omega} + \varepsilon^{-1} \|u\|_{n-1, J \times \Omega} + \|u\|_{n-1, J \times \partial \Omega}).$$

Putting (4.1.41) into (4.1.34) and multiplying both sides by $e^{-\eta t}$ we obtain

(4.1.42) $e^{-\eta t} \|u(T)\|_{n-1, \Omega}$

$$\leq p_7(h)e^{-\eta t}(\varepsilon\|F\|_{n-1,J\times\Omega} + \varepsilon^{-1}\|u\|_{n-1,J\times\Omega} + \|u\|_{n-1,J\times\partial\Omega})$$

$$\leq p_8(h)(\varepsilon\|F\|_{n-1,\eta,J\times\Omega} + \varepsilon^{-1}\|u\|_{n-1,\eta,J\times\Omega} + \|u\|_{n-1,\eta,J\times\partial\Omega}).$$

Taking next $\varepsilon = 2^{-1/2}$ and using (3.5.42) with $s = n-1$ we obtain (4.1.31). ∎

Consider now the problem

(4.1.43) $$\qquad\qquad L^{(*)}w = \Phi \qquad \text{in } J \times \bar{\Omega},$$

(4.1.44) $$\qquad\qquad L_B^{(*)}w = \gamma \qquad \text{on } J \times \partial\Omega,$$

(4.1.45) $$\qquad\qquad w(T,\cdot) = \varphi(\cdot) \qquad \text{in } \Omega,$$

where $L^{(*)}$ and $L_B^{(*)}$ are the formally adjoint operators to L and L_B given by (3.4.6) and (3.4.11), and $J = (-\infty, T]$, $T > 0$.

LEMMA 4.1.3. *Let assumptions* (3.1.1), (3.1.3), (3.1.4), (3.1.7), (3.1.14), (3.1.20), (3.4.13), (3.5.38), (3.5.39), (4.1.30) *be satisfied. Moreover, assume*

(4.1.46) $$\qquad \Phi \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega}), \qquad \Phi = 0 \text{ for } t > T,$$

(4.1.47) $$\qquad \gamma \in C_{(0)}^\infty(\mathbf{R} \times \partial\Omega), \qquad \gamma = 0 \text{ for } t > T,$$

(4.1.48) $$\qquad \varphi \in C_0^\infty(\Omega).$$

Then there exists a unique solution $w \in C_{(0)}^\infty(J \times \bar{\Omega})$ *of problem* (4.1.43)–(4.1.45) *fulfilling the estimate*

(4.1.49) $$\quad \eta^{1/2}\|w\|_{(1-n)^0,-\eta,J\times\Omega} + \|w\|_{(1-n)^0,-\eta,J\times\partial\Omega}$$

$$\leq \tilde{f}(\varepsilon_1,\varepsilon_2,\varepsilon_3)p(\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f})(\eta^{-1/2}\|\Phi\|_{(1-n)^0,-\eta,J\times\Omega}$$

$$+ \|\gamma\|_{(1-n)^0,-\eta,J\times\partial\Omega} + \|\varphi\|_{(1-n)^0,-\eta,\Omega}),$$

where p *is a polynomial;* $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ *and* $\bar{f}$ *are given by* (3.1.13) *with* $k > \max(m/2 + 5/2, (m+1)/2 + n - 1)$; $\tilde{f}$ *is a continuous function;* ε_1, ε_2, ε_3 *are the constants from Theorem* 3.2.1; *and* η *satisfies* (3.5.43).

P r o o f. By Theorems 3.4.1, 3.5.2 and 4.1.1 there exists a unique solution $w \in C_{(0)}^\infty(J \times \bar{\Omega})$ of problem (4.1.43)–(4.1.45). It suffices to prove (4.1.49).

Let u be the smooth solution of (3.1)–(3.2). Then integrating by parts and taking into account

(4.1.50) $$\qquad\qquad w = \begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \qquad \text{on } J \times \partial\Omega$$

(which follows from the form (3.4.11) of the operator $L_B^{(*)}$) we obtain

(4.1.51) $$(F,w)_{0,J\times\Omega} = (-u(T),\varphi)_{0,\Omega} + (u,\Phi)_{0,J\times\Omega}$$

$$+ (B_2 u^I,\gamma)_{0,J\times\partial\Omega} + (A_1^{II}u^{II},\gamma)_{0,J\times\partial\Omega} + \left(A_1 u, \begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix}\right)_{0,J\times\partial\Omega}.$$

Assume that $g = 0$. Then $u \in P = \ker L_B$ in $J \times \partial\Omega$ and

$$\begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix} \in P^{(*)} = (A_1[P])^\perp.$$

Hence the last component in (4.1.51) vanishes and we get the estimate

$$(4.1.52) \qquad |(F, w)_{0, J \times \Omega}| \leq \|u(T)\|_{n-1, \eta, \Omega} \|\varphi\|_{(1-n)^0, -\eta, \Omega}$$
$$+ \|u\|_{n-1, \eta, J \times \Omega} \|\Phi\|_{(1-n)^0, -\eta, J \times \Omega}$$
$$+ p_1(a)\|u\|_{n-1, \eta, J \times \partial\Omega} \|\gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega}.$$

Using (3.5.46) with $s = n - 1$ and (4.1.31) we get

$$|(F, w)_{0, J \times \Omega}| \leq p_2(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}) \widetilde{f}_1(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$
$$\times \eta^{-1/2} \|F\|_{n-1, \eta, J \times \Omega} (\|\varphi\|_{(1-n)^0, -\eta, \Omega}$$
$$+ \eta^{-1/2} \|\Phi\|_{(1-n)^0, -\eta, J \times \Omega} + \|\gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega}),$$

for η satisfying (3.5.43).

Since $F \in C^\infty_{(0)}(J \times \overline{\Omega})$ is arbitrary, from the definition of the norm $\|\cdot\|_{(1-n)^0, -\eta, J \times \Omega}$ we have

$$(4.1.53) \qquad \eta^{1/2} \|v\|_{(1-n)^0, -\eta, J \times \Omega} = \eta^{1/2} \sup_{F \in C^\infty_{(0)}(J \times \overline{\Omega})} \frac{|(F, w)_{0, J \times \Omega}|}{\|F\|_{n-1, \eta, J \times \Omega}}$$
$$\leq p_3(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}) \widetilde{f}_2(\varepsilon_1, \varepsilon_2, \varepsilon_3)(\|\varphi\|_{(1-n)^0, -\eta, \Omega}$$
$$+ \eta^{-1/2} \|\Phi\|_{(1-n)^0, -\eta, J \times \Omega} + \|\gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega}).$$

Assume next that $F = 0$ and $g = L_B A_1^{-1} \begin{bmatrix} \psi \\ 0 \end{bmatrix}$, where $\psi \in C^\infty_0(\mathbf{R} \times \Omega)$. Since

$$\begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix} \in (A_1[P])^\perp \quad \text{and} \quad L_B \left(u - A_1^{-1} \begin{bmatrix} \psi \\ 0 \end{bmatrix} \right) = 0,$$

we have

$$\left(A_1 \left(u - A_1^{-1} \begin{bmatrix} \psi \\ 0 \end{bmatrix} \right), \begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix} \right) = 0.$$

Hence

$$\left(A_1 u, \begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix} \right)_{0, J \times \Omega} = (\psi, w^I)_{0, J \times \Omega}.$$

Therefore (4.1.51) takes the form

$$(4.1.54) \qquad (\psi, w^I)_{0, J \times \partial\Omega} = (u(T), \varphi)_{0, \Omega} - (u, \Phi)_{0, J \times \Omega}$$
$$- (B_2 u^I, \gamma)_{0, J \times \partial\Omega} - (A_1^{II} u^{II}, \gamma)_{0, J \times \partial\Omega}.$$

Estimating $(\psi, w^I)_{0, J \times \partial \Omega}$ similarly to $(F, w)_{0, J \times \Omega}$ above we get

$$|(\psi, w^I)_{0, J \times \partial \Omega}| \leq \|u(T)\|_{n-1, \eta, \Omega} \|\varphi\|_{(1-n)^0, -\eta, \Omega}$$
$$+ \|u\|_{n-1, \eta, J \times \Omega} \|\Phi\|_{(1-n)^0, -\eta, J \times \Omega}$$
$$+ p_4(a) \|u\|_{n-1, \eta, J \times \partial \Omega} \|\gamma\|_{(1-n)^0, -\eta, J \times \partial \Omega} .$$

Using the same argument as before and taking into account that $w^{II} = S^{(*)} w^I + \gamma$, we obtain

(4.1.55) $\|w\|_{(1-n)^0, -\eta, J \times \partial \Omega} \leq p_5(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}) \tilde{f}_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$$\times (\|\varphi\|_{(1-n)^0, -\eta, \Omega} + \eta^{-1/2} \|\Phi\|_{(1-n)^0, -\eta, J \times \Omega} + \|\gamma\|_{(1-n)^0, -\eta, J \times \partial \Omega}) .$$

Combining (4.1.53) and (4.1.55) we get (4.1.49). ∎

LEMMA 4.1.4. *Let the assumption of Lemma 4.1.3 be satisfied. Then the solution* $w \in C_{(0)}^\infty(J \times \bar{\Omega})$ *of problem* (4.1.43)–(4.1.45) *satisfies the estimate*

(4.1.56) $\forall 0 \leq r \leq n - 1$

$\eta^{1/2} \|w\|_{r, -\eta, J \times \Omega} + \|w\|_{r, -\eta, J \times \partial \Omega}$

$$\leq p_r(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k, 0, \infty, K}) \tilde{f}_r(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times (\eta^{-1/2} \|\Phi\|_{r, -\eta, J \times \Omega} + \|\gamma\|_{r, -\eta, J \times \partial \Omega} + \|\varphi\|_{r, -\eta, \Omega}) ,$$

where p_r *are polynomials;* $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, $\bar{f}$ *are given by* (3.3.13) *with* $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$; $\tilde{f}_r$ *are continuous functions;* ε_1, ε_2, ε_3 *are the constants from Theorem 3.2.1;* η *satisfies an inequality of type* (3.5.43); *and* K *is the compact set from Theorem 3.3.1.*

P r o o f. We first prove by induction that for all $0 \leq s \leq n - 1$,

(4.1.57) $\eta^{1/2} \|w\|_{(s+1-n)^0, -\eta, J \times \Omega} + \|w\|_{(s+1-n)^0, -\eta, J \times \partial \Omega}$

$$\leq p_s(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k, 0, \infty, K}) \tilde{f}^s(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times (\|\varphi\|_{(s+1-n)^0, -\eta, \Omega} + \eta^{-1/2} \|\Phi\|_{(s+1-n)^0, -\eta, J \times \Omega}$$
$$+ \|\gamma\|_{(s+1-n)^0, -\eta, J \times \partial \Omega}) .$$

For $s = 0$ this was proved in Lemma 4.1.3. Take $s \geq 1$ and notice that $\partial_{t, x'}^\sigma w$ (where $1 \leq |\sigma| \leq n - 1$) satisfies

(4.1.58) $L^{(*)} \partial_{t, x'}^\sigma w = \partial_{t, x'}^\sigma L^{(*)} w + (L^{(*)} \partial_{t, x'}^\sigma w - \partial_{t, x'}^\sigma L^{(*)} w)$ in $J \times \Omega$,

(4.1.59) $L_B^{(*)} \partial_{t, x'}^\sigma w = \partial_{t, x'}^\sigma L_B^{(*)} w + (L_B^{(*)} \partial_{t, x'}^\sigma w - \partial_{t, x'}^\sigma L_B^{(*)} w)$ on $J \times \partial \Omega$,

(4.1.60) $\partial_{t, x'}^\sigma w|_{t=T} = h_\sigma(\partial_x^\beta \tilde{L}^{(*)}, \partial_x^\beta \varphi)$ in Ω,

where h_σ is a linear function in the derivatives $\partial_x^\beta \varphi$ ($|\beta| \leq |\sigma|$), with coefficients being products of E^{n-1} and the coordinates of the vector $\partial_x^\beta \tilde{L}^{(*)}$,

where

$$\widetilde{L}^{(*)} = (A_j^{(*)} \ (j=1,\dots,m), G_j \ (j=1,\dots,m), H, C^*),$$

$$G_j(z,x_1) = \partial_{x_j} \widetilde{A}_j^*(z,x_1,\widetilde{v}(z,x_1)) + \sum_{i=1}^{n} \partial_{v_i} \widetilde{A}_j^*(z,x_1,\widetilde{v}(z,x_1))\partial_{x_j}\widetilde{v}_i(z,x_1),$$

$$H(z,x_1) = \partial_t \widetilde{E}^*(z,x_1,\widetilde{v}(z,x_1)) + \sum_{i=1}^{n} \partial_{v_i} \widetilde{E}^*(z,x_1,\widetilde{v}(z,x_1))\partial_t\widetilde{v}_i(z,x_1), \quad \cdot$$

with $z = (t,x')$ (we have used here (3.4.7)–(3.4.8)).

As in Lemma 4.1.3, let u be the smooth solution of problem (3.1)–(3.2). Then integrating by parts and taking into account (4.1.50) we get

(4.1.61) $\quad (F, \partial_{t,x'}^\sigma w)_{0,J\times\Omega} = -(u(T), h_\sigma(\partial_x^\beta \widetilde{L}^{(*)}, \partial_x^\beta\varphi))_{0,\Omega}$

$$+ (u, \partial_{t,x'}^\sigma \Phi)_{0,J\times\Omega} + (u, L^{(*)}\partial_{t,x'}^\sigma w - \partial_{t,x'}^\sigma L^{(*)}w)_{0,J\times\Omega}$$

$$+ (B_2 u^I, \partial_{t,x'}^\sigma \gamma)_{0,J\times\partial\Omega} + (A_1^{II}u^{II}, \partial_{t,x'}^\sigma\gamma)_{0,J\times\partial\Omega}$$

$$+ (B_2 u^I, L_B^{(*)}\partial_{t,x'}^\sigma w - \partial_{t,x'}^\sigma L_B^{(*)}w)_{0,J\times\partial\Omega}$$

$$+ (A_1^{II}u^{II}, L_B^{(*)}\partial_{t,x'}^\sigma w - \partial_{t,x'}^\sigma L_B^{(*)}w)_{0,J\times\partial\Omega}$$

$$+ \left(A_1 u, \begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix}\right)_{0,J\times\partial\Omega} \quad \cdot$$

Let $|\sigma| = 1$. Then

(4.1.62) $\quad (F, \partial_{t,x'}^\sigma w)_{0,J\times\Omega} = -(u(T), h_\sigma(\partial_x^\beta \widetilde{L}^{(*)}, \partial_x^\beta\varphi))_{0,\Omega}$

$$+ (u, \partial_{t,x'}^\sigma \Phi)_{0,J\times\Omega} + (u\partial_{t,x'}^\sigma E^*, w)_{0,J\times\Omega}$$

$$- \left(u \sum_{j=1}^{m} \partial_{t,x'}^\sigma A_j^*, w_{x_j}\right)_{0,J\times\Omega} - \left(u\left(\sum_{j=1}^{m} G_j - H - C^*\right), w\right)_{0,J\times\Omega}$$

$$+ (B_2 u^I, \partial_{t,x'}^\sigma \gamma)_{0,J\times\partial\Omega} + (A_1^{II}u^{II}, \partial_{t,x'}^\sigma\gamma)_{0,J\times\partial\Omega}$$

$$+ (B_2 u^I \partial_{t,x'}^\sigma L_B^{(*)}, w)_{0,J\times\partial\Omega} + (A_1^{II}u^{II}\partial_{t,x'}^\sigma L_B^{(*)}, w)_{0,J\times\partial\Omega}$$

$$+ \left(A_1 u, \begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix}\right)_{0,J\times\partial\Omega} \quad \cdot$$

Assume that $g = 0$. Then the last term in (4.1.62) vanishes and we obtain

(4.1.63) $\quad |(F, \partial_{t,x'}^\sigma w)_{0,J\times\Omega}| \le \widetilde{p}^{\,1}(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\widetilde{f}, |\widetilde{v}|_{k,0,\infty,K})$

$$\times \left(\|u(T)\|_{n-1,\eta,\Omega} \sum_{|\beta|\le 1} \|\partial_x^\beta\varphi\|_{(1-n)^0,-\eta,\Omega}\right.$$

$$+ \|u\|_{n-1,\eta,J\times\Omega} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma\Phi\|_{(1-n)^0,-\eta,J\times\Omega}$$

$$+ \|u\|_{n-1,\eta,J\times\partial\Omega} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\Omega}$$

$$+ \|u\|_{n-1,\eta,J\times\partial\Omega} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0,-\eta,J\times\partial\Omega}$$

$$+ \|u\|_{n-1,\eta,J\times\partial\Omega} \|w\|_{(1-n)^0,-\eta,J\times\partial\Omega} \Big) .$$

Using now (3.5.46) with $s = n - 1$ and (4.1.31) and summing up over all σ with $|\sigma| \le 1$ we obtain for sufficiently large η (satisfying (3.5.43)) the estimate

$$(4.1.64) \quad \eta^{1/2} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\Omega} \le \widetilde{p}^2(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})$$

$$\times \widetilde{f}_1(\varepsilon_1,\varepsilon_2,\varepsilon_3)\Big(\sum_{|\beta|\le 1} \|\partial_x^\beta \varphi\|_{(1-n)^0,-\eta,\Omega}$$

$$+ \eta^{-1/2} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0,-\eta,J\times\Omega} + \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0,-\eta,J\times\partial\Omega} \Big) .$$

Assuming now $F = 0$ and $g = L_B^{(*)} A_1^{-1} \begin{bmatrix} \psi \\ 0 \end{bmatrix}$ (where $\psi \in C_0^\infty(\mathbf{R}\times\Omega)$), in the same way we get

$$(4.1.65) \quad \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\partial\Omega} \le \widetilde{p}^3(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})$$

$$\times \widetilde{f}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3)\Big(\sum_{|\beta|\le 1} \|\partial_x^\beta \varphi\|_{(1-n)^0,-\eta,\Omega}$$

$$+ \eta^{-1/2} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0,-\eta,J\times\Omega} + \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\Omega}$$

$$+ \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0,-\eta,J\times\partial\Omega} \Big) .$$

Adding (4.1.64) and (4.1.65) and taking η satisfying (3.5.43) we obtain

$$(4.1.66) \quad \eta^{1/2} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\Omega} + \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\partial\Omega}$$

$$\le \widetilde{p}^4(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\widetilde{f}_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)$$

$$\times \Big(\sum_{|\beta|\le 1} \|\partial_x^\beta \varphi\|_{(1-n)^0,-\eta,\Omega} + \eta^{-1/2} \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0,-\eta,J\times\Omega}$$

$$+ \sum_{|\sigma|\le 1} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0,-\eta,J\times\partial\Omega} \Big) .$$

In order to estimate $\partial_{x_1} w$ in $J \times \Omega$ we shall use the identity

$$(4.1.67) \quad \partial_{x_1} w = (A_1^*)^{-1} \Big[L^{(*)} w + E^* \partial_t w - \sum_{j=2}^{m} A_j^* \partial_{x_j} w$$

$$- \Big(\sum_{j=1}^{m} G_j - H - C^* \Big) w \Big]$$

and the inequality

$$(4.1.68) \qquad \|\Phi\|_{(1-n)^0, -\eta, J \times \Omega} \le C\eta^{-1} \|\partial_t \Phi\|_{(1-n)^0, -\eta, J \times \Omega}$$

if $\Phi(T) = 0$ (see [21, p. 279]).

Thus we find that $\eta^{1/2} \|\partial_{x_1} v\|_{(1-n)^0, -\eta, J \times \Omega}$ is also estimated by the right-hand side of (4.1.65). Hence

$$(4.1.69) \quad \eta^{1/2} \sum_{|\nu| \le 1} \|\partial_{t,x}^\nu w\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{|\sigma| \le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \partial\Omega}$$

$$\le \widetilde{p}^5(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}^1(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times \Big(\sum_{|\beta| \le 1} \|\partial_x^\beta \varphi\|_{(1-n)^0, -\eta, \Omega} + \eta^{-1/2} \sum_{|\sigma| \le 1} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0, -\eta, J \times \Omega}$$

$$+ \sum_{|\sigma| \le 1} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega} \Big).$$

Notice further that applying integration by parts, for every $\psi \in C_{(0)}^\infty(J \times \overline{\Omega})$ we obtain

$$|(\partial_{t,x'}^\sigma \Phi, \psi)_{0, J \times \Omega}| = |(\Phi, \partial_{t,x'}^\sigma \psi)_{0, J \times \Omega}| \le C \|\Phi\|_{(2-n)^0, -\eta, J \times \Omega} \|\psi\|_{n-1, \eta, J \times \Omega}$$

for a constant $C > 0$ and for all σ with $|\sigma| = 1$. Therefore

$$(4.1.70) \quad \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0, -\eta, J \times \Omega} = \sup_{\psi \in C_{(0)}^\infty(J \times \overline{\Omega})} \frac{|(\partial_{t,x'}^\sigma \Phi, \psi)_{0, J \times \Omega}|}{\|\psi\|_{n-1, \eta, J \times \Omega}}$$

$$\le c_1 \|\Phi\|_{(2-n)^0, -\eta, J \times \Omega}$$

for a constant $c_1 > 0$. In the same way we can prove

$$(4.1.71) \qquad \|\partial_x^\beta \varphi\|_{(1-n)^0, -\eta, \Omega} \le c_2 \|\varphi\|_{(2-n)^0, -\eta, \Omega},$$

$$(4.1.72) \qquad \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega} \le c_3 \|\gamma\|_{(2-n)^0, -\eta, J \times \partial\Omega}$$

for some constants $c_2, c_3 > 0$.

Using (4.1.70)–(4.1.72) in (4.1.69) we obtain

$$(4.1.73) \quad \eta^{1/2} \sum_{|\nu| \le 1} \|\partial_{t,x}^\nu w\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{|\sigma| \le 1} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \partial\Omega}$$

$$\le p^1(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}^1(\varepsilon_1, \varepsilon_2, \varepsilon_3)(\|\varphi\|_{(2-n)^0, -\eta, \Omega}$$

$$+ \eta^{-1/2}\|\Phi\|_{(2-n)^0,-\eta,J\times\Omega} + \|\gamma\|_{(2-n)^0,-\eta,J\times\partial\Omega}\big)\,.$$

Assume now that for $j = 0, 1, \ldots, r-1$ (where $r \le n-1$) we have proved the estimate

$$(4.1.74) \quad \eta^{1/2} \sum_{|\nu|\le j} \|\partial_{t,x}^\nu w\|_{(1-n)^0,-\eta,J\times\Omega} + \sum_{|\sigma|\le j} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\partial\Omega}$$

$$\le \widetilde{p}^j(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\widetilde{f}^j(\varepsilon_1,\varepsilon_2,\varepsilon_3)$$

$$\times \Big(\sum_{|\nu|\le j} \|\partial_x^\nu \varphi\|_{(1-n)^0,-\eta,\Omega} + \eta^{-1/2} \sum_{|\sigma|\le j} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0,-\eta,J\times\Omega}$$

$$+ \sum_{|\sigma|\le j} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0,-\eta,J\times\partial\Omega} \Big)\,,$$

where the $\widetilde{p}^j$ are polynomials.

We shall prove that the above estimate is true for $j = r$. To do this consider problem $(4.1.58)$–$(4.1.60)$ with $|\sigma| = r$ and let u be the smooth solution of problem (3.1)–(3.2). Using $(4.1.61)$ we get

$$(4.1.75) \quad (F, \partial_{t,x'}^\sigma w)_{0,J\times\Omega} = -(u(T), h_\sigma(\partial_x^\beta \widetilde{L}^{(*)}, \partial_x^\beta \varphi))_{0,\Omega}$$

$$+ (u, \partial_{t,x'}^\sigma \Phi)_{0,J\times\Omega} + \sum_{\mu<\sigma} \binom{\sigma}{\mu}(u\partial_{t,x'}^{\sigma-\mu}E^*, \partial_{t,x'}^\mu(\partial_t w))_{0,J\times\Omega}$$

$$- \sum_{\mu<\sigma} \binom{\sigma}{\mu}\Big(u\sum_{j=1}^m \partial_{t,x'}^{\sigma-\mu}A_j^*, \partial_{t,x'}^\mu(\partial_{x_j}w)\Big)_{0,J\times\Omega}$$

$$+ \sum_{\mu<\sigma} \binom{\sigma}{\mu}\Big(u\partial_{t,x'}^{\sigma-\mu}\Big(\sum_{j=1}^m G_j - H - C^*\Big), \partial_{t,x'}^\mu w\Big)_{0,J\times\Omega}$$

$$+ (B_2 u^I, \partial_{t,x'}^\sigma \gamma)_{0,J\times\partial\Omega} + (A_1^{II} u^{II}, \partial_{t,x'}^\sigma \gamma)_{0,J\times\partial\Omega}$$

$$- \sum_{\mu<\sigma} \binom{\sigma}{\mu}(B_2 u^I \partial_{t,x'}^{\sigma-\mu}L_B^{(*)}, \partial_{t,x'}^{\sigma-\mu}w)_{0,J\times\partial\Omega}$$

$$- \sum_{\mu<\sigma} \binom{\sigma}{\mu}(A_1^{II} u^{II} \partial_{t,x'}^{\sigma-\mu}L_B^{(*)}, \partial_{t,x'}^\mu w)_{0,J\times\partial\Omega}$$

$$+ \Big(A_1 u, \begin{bmatrix} w^I \\ S^{(*)}w^I \end{bmatrix}\Big)_{0,J\times\partial\Omega}\,.$$

Assume that $g = 0$. Then using $(4.1.74)$ with $j = r-1$, $(3.5.46)$ with $s = n-1$ and $(4.1.31)$, and summing up over all σ with $|\sigma| \le r$ we obtain from $(4.1.75)$ the inequality

$$(4.1.76) \quad \eta^{1/2} \sum_{|\sigma|\le r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0,-\eta,J\times\Omega}$$

$$\leq p_1(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}_1(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times \left(\sum_{|\beta| \leq r} \|\partial_x^\beta \varphi\|_{(1-n)^0, -\eta, \Omega} + \eta^{-1/2} \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0, -\eta, J \times \Omega} \right.$$

$$\left. + \eta^{-1/2} \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega} \right),$$

where p_1 is a polynomial and $\widetilde{f}_1$ is a continuous function.

Assuming now that $F = 0$ and $g = L_B^* A_1^{-1} \begin{bmatrix} \psi \\ 0 \end{bmatrix}$ (where $\psi \in C_0^\infty(\mathbf{R} \times \Omega)$) we get

$$(4.1.77) \quad \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \partial\Omega}$$

$$\leq p_2(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}_2(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times \left(\sum_{|\beta| \leq r} \|\partial_x^\beta \varphi\|_{(1-n)^0, -\eta, \Omega} + \eta^{-1/2} \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0, -\eta, J \times \Omega} \right.$$

$$\left. + \eta^{-1/2} \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega} \right),$$

Adding (4.1.76) and (4.1.77) for η satisfying (3.5.43) we obtain

$$(4.1.78) \quad \eta^{1/2} \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \partial\Omega}$$

$$\leq p_3(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times \left(\sum_{|\beta| \leq r} \|\partial_x^\beta \varphi\|_{(1-n)^0, -\eta, \Omega} + \eta^{-1/2} \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \Phi\|_{(1-n)^0, -\eta, J \times \Omega} \right.$$

$$\left. + \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega} \right).$$

Notice that

$$\sum_{|\nu| \leq r} \|\partial_{t,x}^\nu w\|_{(1-n)^0, -\eta, J \times \Omega} = \sum_{p=0}^r \|\partial_{t,x}^p w\|_{(1-n)^0, -\eta, J \times \Omega}$$

$$= \sum_{p=0}^r \|\partial_{t,x'}^p w\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{\substack{0 \leq q \leq r-1 \\ q + q_1 \leq r}} \|\partial_{t,x'}^q \partial_{x_1}^{q_1} w\|_{(1-n)^0, -\eta, J \times \Omega}$$

(where $\partial_{t,x}^p w$, $\partial_{t,x'}^p w$ and $\partial_{t,x'}^q \partial_{x_1}^{q_1} w$ denote the vectors composed of the derivatives of w of orders p and $q + q_1$ respectively).

Applying now the formula (see [28])

$$(4.1.79) \quad \partial_{x_1}^{q_1} w = \sum_{i=0}^{q_1-1} \partial_{t,x}^i f_i(A_1^{*-1}, \widetilde{L}^{(*)}) \partial_{t,x}^{q_1-i-1} \varPhi$$

$$+ \sum_{j=0}^{q_1} \partial_{t,x}^j g_j(A_1^{*-1}, \widetilde{L}^{(*)}) \partial_{t,x'}^{q_1-j} w + \sum_{j=0}^{q_1-1} \partial_{t,x}^j h_j(A_1^{*-1}, \widetilde{L}^{(*)}) \partial_{t,x'}^{q_1-j-1} w$$

(where f_i, g_j and h_j are polynomials with respect to their arguments) we obtain

$$(4.1.80) \quad \|\partial_{t,x'}^q \partial_{x_1}^{q_1} w\|_{(1-n)^0, -\eta, J \times \Omega}$$

$$\leq c \Big(\sum_{i=0}^{q_1-1} \sum_{\bar{q}=0}^q \|\partial_{t,x'}^{q-\bar{q}} \partial_{t,x}^i f_i(A^{*-1}, \widetilde{L}^{(*)}) \partial_{t,x}^{\bar{q}+q_1-i-1} \varPhi\|_{(1-n)^0, -\eta, J \times \Omega}$$

$$+ \sum_{j=0}^{q_1} \sum_{\bar{q}=0}^q \|\partial_{t,x'}^{q-\bar{q}} \partial_{t,x}^j g_j(A^{*-1}, \widetilde{L}^{(*)}) \partial_{t,x'}^{\bar{q}+q_1-j} w\|_{(1-n)^0, -\eta, J \times \Omega}$$

$$+ \sum_{j=0}^{q_1-1} \sum_{\bar{q}=0}^q \|\partial_{t,x'}^{q-\bar{q}} \partial_{t,x}^j h_j(A^{*-1}, \widetilde{L}^{(*)}) \partial_{t,x'}^{\bar{q}+q_1-j-1} w\|_{(1-n)^0, -\eta, J \times \Omega} \Big)$$

$$\leq \widetilde{p}_4(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}_4(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times \Big(\sum_{|\sigma| \leq r-1} \|\partial_{t,x'}^\sigma \varPhi\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \Omega} \Big) .$$

Using now (4.1.68) for $\partial_{t,x'}^\sigma \varPhi$ and (4.1.78), (4.1.80) we get

$$(4.1.81) \quad \eta^{1/2} \sum_{|\nu| \leq r} \|\partial_{t,x}^\nu w\|_{(1-n)^0, -\eta, J \times \Omega} + \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma w\|_{(1-n)^0, -\eta, J \times \partial\Omega}$$

$$\leq \widetilde{p}_5(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}_5(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times \Big(\sum_{|\beta| \leq r} \|\partial_x^\beta \varPhi\|_{(1-n)^0, -\eta, \Omega} + \eta^{-1/2} \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \varPhi\|_{(1-n)^0, -\eta, J \times \Omega}$$

$$+ \sum_{|\sigma| \leq r} \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega} \Big) .$$

Thus we have proved (4.1.74) for $0 \leq j \leq n - 1$. Using in (4.1.81) the inequalities

$$(4.1.82) \qquad \|\partial_{t,x'}^\sigma \varPhi\|_{(1-n)^0, -\eta, J \times \Omega} \leq C_1 \|\varPhi\|_{(j+1-n)^0, -\eta, J \times \Omega} ,$$

$$(4.1.83) \qquad \|\partial_x^\beta \varphi\|_{(1-n)^0, -\eta, \Omega} \leq C_2 \|\varphi\|_{(j+1-n)^0, -\eta, \Omega} ,$$

$$(4.1.84) \qquad \|\partial_{t,x'}^\sigma \gamma\|_{(1-n)^0, -\eta, J \times \partial\Omega} \leq C_3 \|\gamma\|_{(j+1-n)^0, -\eta, J \times \partial\Omega}$$

$(0 \le |\sigma| = j \le n - 1)$ and Lemma 2.1.4 we obtain

$$(4.1.85) \quad \eta^{1/2}\|w\|_{(j+1-n)^0,-\eta,J\times\Omega} + \|w\|_{(j+1-n)^0,-\eta,J\times\partial\Omega}$$

$$\le p^j(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\widetilde{f}^j(\varepsilon_1,\varepsilon_2,\varepsilon_3)$$

$$\times (\|\varphi\|_{(j+1-n)^0,-\eta,\Omega} + \eta^{-1/2}\|\Phi\|_{(j+1-n)^0,-\eta,J\times\Omega}$$

$$+ \|\gamma\|_{(j+1-n)^0,-\eta,J\times\partial\Omega}),$$

where $0 \le j \le n - 1$.

For $j = n - 1$ we have the estimate (4.1.56) with $r = 0$. For $0 < r \le n - 1$ we obtain that estimate similarly to (4.1.85), i.e. considering problem (4.1.58)–(4.1.60) and using (4.1.56) with $r = 0$. ∎

Remark 4.1.2. Since problem (4.1.4)–(4.1.6) is formally adjoint to (4.1.43)–(4.1.45), the smooth solution u of the former with $F \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$, $F = 0$ for $t < 0$, $g \in C_0^\infty(\mathbf{R} \times \partial\Omega)$, $g = 0$ for $t < 0$ and $f \in C_0^\infty(\Omega)$ satisfies the estimate (4.1.56) with some polynomials p_r, continuous functions $\widetilde{f}_r$ and the constants ε_1^*, ε_2^*, ε_3^* from Theorem 3.4.1. In particular,

$$(4.1.86) \quad \eta^{1/2}\|u\|_{0,\eta,[0,\infty)\times\Omega} + \|u\|_{0,\eta,[0,\infty)\times\partial\Omega}$$

$$\le p_0(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\widetilde{f}_0(\varepsilon_1^*,\varepsilon_2^*,\varepsilon_3^*)$$

$$\times (\|f\|_{0,\eta,\Omega} + \eta^{-1/2}\|F\|_{0,\eta,[0,\infty)\times\Omega} + \|g\|_{0,\eta,[0,\infty)\times\partial\Omega}).$$

Using the same argument as in Remark 3.5.1 we find that the restriction of u to $[0,T]$ satisfies

$$(4.1.87) \quad \eta^{1/2}\|u\|_{0,\eta,[0,T]\times\Omega} + \|u\|_{0,\eta,[0,T]\times\partial\Omega}$$

$$\le p_0(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\widetilde{f}_0(\varepsilon_1^*,\varepsilon_2^*,\varepsilon_3^*)$$

$$\times (\|f\|_{0,\eta,\Omega} + \eta^{-1/2}\|F\|_{0,\eta,[0,T]\times\Omega} + \|g\|_{0,\eta,[0,T]\times\partial\Omega})$$

(in (4.1.86) and (4.1.87), η fulfils the inequality from Theorem 3.4.1).

Let now w (as before) be the solution of problem (4.1.43)–(4.1.45). We shall find an estimate for $w(t,\cdot)$ for $t \le T$. To do this consider once again an operator $P(t,x,\partial_t,\partial_x)$ given by (4.1.14) and satisfying assumptions (4.1.15)–(4.1.16).

LEMMA 4.1.5. *There exists a constant $c > 0$ independent of $|P|_0$ (see (4.1.18)) such that for all T with $t \le T$ and for $\varphi \in H^n([t,T] \times \Omega)$ with $\varphi = 0$ in a neighbourhood of $\{(t,x) : t = T, x_1 = 0\}$ and for sufficiently small $\varepsilon > 0$,*

$$(4.1.88) \quad \|\varphi(t)\|_{n-1,\Omega} \le c\Big(\|\varphi(T)\|_{n-1,\Omega} + \varepsilon\|P\varphi\|_{n-1,[t,T]\times\Omega}$$

$$+\varepsilon^{-1}\|\varphi\|_{n-1,[t,T]\times\Omega} + \sum_{j=0}^{n-1}\|\partial_{x_1}^j\varphi\|_{n-1-j,[t,T]\times\partial\Omega}\Big).$$

P r o o f. The proof is analogous to that of Lemma 4.1.1. Integrating (4.1.20) by parts over $J\times\Omega$ we get

$$(4.1.89)\qquad \int_\Omega Q^0(\varphi(t)) \leq c\Big(\|\varphi(T)\|_{n-1,\Omega}^2 + \varepsilon^2\|P\varphi\|_{0,J\times\Omega}$$

$$+\varepsilon^{-2}\|\varphi\|_{n-1,J\times\Omega}^2 + \sum_{j=0}^{n-1}\|\partial_{x_1}^j\varphi\|_{n-1-j,J\times\partial\Omega}^2\Big)$$

for small $\varepsilon > 0$.

If $n = 1$, then (4.1.22) finishes the proof of the lemma. For $n > 1$, (4.1.88) follows from (4.1.89), (4.1.24) and the following two inequalities:

(4.1.90) $\forall\varphi \in H^{n-1}([t,T]\times\Omega) : \varphi = 0$ in a neighbourhood of $\{(t,x) : t = T, x_1 = 0\}$ and $\forall 0 \leq j \leq n - 2$

$$\|\partial_{x_1}^j\varphi(t)\|_{n-1-j-1/2,\Omega} \leq c\|\partial_{x_1}^j\varphi\|_{n-1-j,[t,T]\times\partial\Omega} ,$$

(4.1.91) $\|\varphi(t)\|_{n-2,\Omega} \leq \sqrt{2}\|\varphi\|_{n-1,[t,T]\times\Omega} + \|\varphi(T)\|_{n-2,\Omega}.$

To prove (4.1.90), set

$$\widetilde{\varphi} = \begin{cases} \partial_{x_1}^j\varphi & \text{for } t \leq T, \ x_1 = 0 , \\ 0 & \text{for } t \geq T, \ x_1 = 0 . \end{cases}$$

Then (4.1.25) with J replaced by $[t,\infty)$, applied to $\widetilde{\varphi}$, yields (4.1.90).

In order to obtain (4.1.91) integrate the identity

$$\frac{d}{dt}\|\varphi(t)\|_{n-2,\Omega}^2 = \sum_{|\alpha|\leq n-2}(\partial_x^\alpha\varphi, \partial_t(\partial_x^\alpha\varphi))_{0,\Omega} + \sum_{|\alpha|\leq n-2}(\partial_t(\partial_x^\alpha\varphi), \partial_x^\alpha\varphi)_{0,\Omega}$$

from T to t. ∎

LEMMA 4.1.6. *Let assumptions* (3.1.1), (3.1.3), (3.1.4), (3.1.7), (3.1.14), (3.1.20), (3.4.13), (3.5.38), (3.5.39), (4.1.30) *and* (4.1.48) *be satisfied. Moreover, assume*

(4.1.92) $\Phi = 0, \quad \gamma = 0 .$

Then for the solution w of problem (4.1.43)–(4.1.45) *we have*

$$(4.1.93)\quad \forall t \leq T \quad \|w(t)\|_{n-1,-\eta,\Omega} \leq p(\bar{a},\bar{b},\bar{c},\bar{d},\bar{e},\bar{f},|v|_{k,0,\infty,K})$$
$$\times \widetilde{f}(\varepsilon_1,\varepsilon_2,\varepsilon_3)\|w(T)\|_{n-1,-\eta,\Omega} ,$$

where p is a polynomial; $\widetilde{f}$ is a continuous function; $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ are given by (3.3.13) *with $k > \max(m/2 + 5/2, (m + 1)/2 + n)$; η satisfies an inequality of type* (3.5.43); *K is the compact set from Theorem 3.3.1.*

P r o o f. Consider the operator $L^{(*)}$ as a matrix of differential operators and let $L^{(*)co}$ denote the matrix of differential operators whose entries are the corresponding cofactors of the matrix $L^{(*)}$.

Let $w \in C_{(0)}^{\infty}(\mathbf{R} \times \bar{\Omega})$ be the solution of problem (4.1.43)–(4.1.45). Then each component w_j of w satisfies

$$(4.1.94) \quad (\det L^{(*)})w_j = (L^{(*)co}\Phi)_j$$
$$+((\text{operator of order } (n-1))w)_j \quad (j = 1, \ldots, n).$$

The operator $\det L^{(*)}$ fulfils assumptions (4.1.15)–(4.1.16). Hence applying Lemma 4.1.5 to w_j $(j = 1, \ldots, n)$ and summing up the resulting inequalities over j we get

$$(4.1.95) \quad \|w(t)\|_{n-1,\Omega} \le p_1(h^{(*)})\Big(\varepsilon\|w(T)\|_{n-1,\Omega} + \varepsilon\|\Phi\|_{n-1,[t,T]\times\Omega}$$
$$+\varepsilon^{-1}\|w\|_{n-1,[t,T]\times\Omega} + \sum_{j=0}^{n-1}\|\partial_{x_1}^j w\|_{n-1-j,[t,T]\times\partial\Omega}\Big),$$

where $h^{(*)} = \|\widetilde{L}^{(*)}\|_{C_B^{n-1}(\mathbf{R}\times\Omega)}$.

The identity (4.1.67) yields the estimate

$$(4.1.96) \quad \|\partial_{x_1}w\|_{n-2,[t,T]\times\partial\Omega}$$
$$\le p_2(h^{(*)})(\|\Phi\|_{n-2,[t,T]\times\partial\Omega} + \|w\|_{n-1,[t,T]\times\partial\Omega}).$$

Using the Young inequality and the trace theorem we obtain

$$(4.1.97) \quad \|\Phi\|_{n-2,[t,T]\times\Omega} = \|L^{(*)}w\|_{n-2,[t,T]\times\Omega} \le p_3(h^{(*)})\|w\|_{n-1,[t,T]\times\Omega}.$$

From (4.1.96)–(4.1.97) we get

$$(4.1.98) \quad \|\partial_{x_1}w\|_{n-2,[t,T]\times\Omega} \le p_4(h^{(*)})(\varepsilon\|\Phi\|_{n-1,[t,T]\times\Omega}$$
$$+ \varepsilon^{-1}\|w\|_{n-1,[t,T]\times\Omega} + \|w\|_{n-1,[t,T]\times\partial\Omega}).$$

In the same way we can prove

$$(4.1.99) \quad \|\partial_{x_1}^j w\|_{n-1-j,[t,T]\times\partial\Omega} \le p_5(h^{(*)})(\varepsilon\|\Phi\|_{n-1,[t,T]\times\Omega}$$
$$+ \varepsilon^{-1}\|w\|_{n-1,[t,T]\times\Omega} + \|w\|_{n-1,[t,T]\times\partial\Omega}) \quad (j = 1, \ldots, n).$$

Using now (4.1.92) and (4.1.99) in (4.1.95) we obtain

$$(4.1.100) \quad \|w(t)\|_{n-1,\Omega} \le p_6(h^{(*)})(\varepsilon\|w(T)\|_{n-1,\Omega}$$
$$+ \varepsilon^{-1}\|w\|_{n-1,[t,T]\times\Omega} + \|w\|_{n-1,[t,T]\times\partial\Omega}).$$

Multiplying both sides by $e^{\eta t}$, taking $\varepsilon = \eta^{-1/2}$ and applying (4.1.56) with $r = n - 1$ we obtain (4.1.93). ∎

LEMMA 4.1.7. *Let assumptions* (3.1.1), (3.1.3), (3.1.4), (3.1.7), (3.1.14), (3.1.20), (3.4.13), (3.5.38), (3.5.39), (4.1.30) *and* (4.1.7)–(4.1.9) *be satisfied.*

Then the solution $u \in C^\infty_{(0)}([0,T] \times \bar{\Omega})$ of problem (4.1.1)–(4.1.3) *fulfils the estimate*

(4.1.101) $\|u(T)\|_{0,\eta,\Omega} \leq p(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k,0,\infty,K}) \widetilde{f}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$$\times (\|f\|_{0,\eta,\Omega} + \eta^{-1/2}\|F\|_{0,\eta,[0,T]\times\Omega} + \|g\|_{0,\eta,[0,T]\times\partial\Omega}),$$

where p is a polynomial with coefficients independent of F, g, f, T and η; $\widetilde{f}$ is a continuous function; $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ are given by (3.3.13) *with $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$; ε_1, ε_2, ε_3 are the constants from Theorem 3.2.1; and η satisfies an inequality of type* (3.5.43).

Proof. Let $u \in C^\infty_{(0)}([0,T] \times \bar{\Omega})$ be the solution of problem (4.1.1)–(4.1.3) and let $w \in C^\infty_{(0)}(J \times \bar{\Omega})$ be the solution of (4.1.43)–(4.1.45) (the assumptions of Lemma 4.1.6 are satisfied). Then

$$(u(T), w(T))_{0,\Omega} = (u(0), w(0))_{0,\Omega} - (F, w)_{0,[0,T]\times\Omega} + (A_1 u, w)_{0,[0,T]\times\partial\Omega}.$$

Since $u^I - Su^{II} = g$ for $x_1 = 0$, we have

$$u = \begin{bmatrix} Su^{II} \\ u^{II} \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix} \quad \text{for } x_1 = 0.$$

Hence $u \equiv \begin{bmatrix} g \\ 0 \end{bmatrix} \pmod{P}$ for $x_1 = 0$. Therefore taking into account that $\gamma = 0$ (see assumption (4.1.92)) we have

$$(A_1 u, w)_{0,[0,T]\times\partial\Omega} = \left(A_1 \begin{bmatrix} u^{II} \\ Su^{II} \end{bmatrix}, w\right)_{0,[0,T]\times\partial\Omega} + (A_1^I g, w^I)_{0,[0,T]\times\partial\Omega}$$

$$+ (B_2 g, w^{II})_{0,[0,T]\times\partial\Omega} = (A_1^I g, w^I)_{0,[0,T]\times\partial\Omega} + (B_2 g, w^{II})_{0,[0,T]\times\partial\Omega}.$$

By both above identities we have

$$(u(T), w(T))_{0,\Omega} = (u(0), w(0))_{0,\Omega} - (F, w)_{0,[0,T]\times\Omega}$$

$$+ (A_1^I g, w^I)_{0,[0,T]\times\partial\Omega} + (B_2 g, w^{II})_{0,[0,T]\times\partial\Omega}.$$

Using now (4.1.93) and (4.1.56) with $r = n - 1$ we get

(4.1.102) $|(u(T), \varphi)_{0,\Omega}| \leq p_1(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k,0,\infty,K})$

$$\times \widetilde{f_1}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \|\varphi\|_{n-1,-\eta,\Omega}(\|f\|_{(1-n)^o,\eta,\Omega}$$

$$+ \eta^{-1/2}\|F\|_{(1-n)^o,\eta,[0,T]\times\Omega} + \|g\|_{(1-n)^o,\eta,[0,T]\times\partial\Omega}).$$

From (4.1.102) and since $w(T) = \varphi \in C^\infty_0(\Omega)$ is arbitrary we have

$$\|u(T)\|_{(1-n)^o,\eta,\Omega} \leq p_2(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, |v|_{k,0,\infty,K}) \widetilde{f_1}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times (\|f\|_{(1-n)^o,\eta,\Omega} + \eta^{-1/2}\|F\|_{(1-n)^o,\eta,[0,T]\times\Omega} + \|g\|_{(1-n)^o,\eta,[0,T]\times\partial\Omega}).$$

Notice now that $\partial^\sigma_{t,x'} u$ (where $1 \leq \sigma \leq n - 1$) satisfies

(4.1.103) $L\partial^\sigma_{t,x'} u = \partial^\sigma_{t,x'} Lu + (L\partial^\sigma_{t,x'} u - \partial^\sigma_{t,x'} Lu)$ in $[0,T] \times \bar{\Omega}$,

(4.1.104) $L_B \partial^\sigma_{t,x'} u = \partial^\sigma_{t,x'} L_B u + (L_B \partial^\sigma_{t,x'} u - \partial^\sigma_{t,x'} L_B u)$ on $[0,T] \times \partial\Omega$,

(4.1.105) $\partial^\sigma_{t,x'} u|_{t=0} = \widetilde{h}_\sigma(\partial^\beta_x \widetilde{L}, \partial^\beta_x f)$ in Ω,

where $\widetilde{h}_\sigma$ is a linear function in the derivatives of $\partial^\beta_x \varphi$ ($|\beta| \leq |\sigma|$) with coefficients being products of A_1^{-1} and of components of $\partial^\beta \widetilde{L}$. Using the same argument as in the proof of Lemma 4.1.4 (applying (4.1.103)–(4.1.105)) we can show by induction that

$$\sum_{|\nu| \leq r} \|\partial^\nu_{t,x} u(T)\|_{(1-n)^o, \eta, \Omega} \leq \widetilde{p}_r(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}_r(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times (\|f\|_{(r+1-n)^o, \eta, \Omega} + \eta^{-1/2} \|F\|_{(r+1-n)^o, \eta, [0,T] \times \Omega} + \|g\|_{(r+1-n)^o, \eta, [0,T] \times \partial\Omega})$$

for $0 \leq r \leq n - 1$. Hence, using [9, Theorem 2.5.4] we obtain

$$\|u(T)\|_{(r+1-n)^o, \eta, \Omega} \leq p_r(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \widetilde{f}^r(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\times (\|f\|_{(r+1-n)^o, \eta, \Omega} + \eta^{-1/2} \|F\|_{(r+1-n)^o, \eta, [0,T] \times \Omega}$$

$$+ \|g\|_{(r+1-n)^o, \eta, [0,T] \times \partial\Omega})$$

for $0 \leq r \leq n - 1$. In particular, for $r = n - 1$ we have the assertion of the lemma. ∎

DEFINITION 4.1.1 (see [21]).   A function $u \in L^2([0,T] \times \Omega)$ is called a *strong solution* of problem (4.1.1)–(4.1.3) if there exist a sequence $u_\mu \in C^\infty_{(0)}([0,T] \times \overline{\Omega})$ such that $L_B u_\mu = g$ on $[0,T] \times \partial\Omega$ and a function $u_0 \in L^2([0,T] \times \partial\Omega)$ satisfying, as $\mu \to \infty$,

(4.1.106) $\|u_\mu - u\|_{0,[0,T] \times \Omega} \to 0$,

(4.1.107) $\|u_\mu - u_0\|_{0,[0,T] \times \partial\Omega} \to 0$,

(4.1.108) $\|L u_\mu - F\|_{0,[0,T] \times \Omega} \to 0$,

(4.1.109) $\|u_\mu(0, \cdot) - f(\cdot)\|_{0,\Omega} \to 0$,

(4.1.110) $L_B u_0 = g$ on $[0,T] \times \partial\Omega$.

THEOREM 4.1.2. *Let assumptions* (3.1.1), (3.1.3), (3.1.4), (3.1.7), (3.1.14), (3.1.20), (3.4.13), (3.5.38), (3.5.39) *and* (4.1.30) *be satisfied. Furthermore, assume that*

(4.1.111) $F \in H^s([0,T] \times \Omega)$, $g \in H^s([0,T] \times \partial\Omega)$, $f \in \mathring{H}^s(\Omega)$,

where $0 \leq s \leq k$, $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$, *and*

(4.1.112) $\forall 0 \leq j \leq s - 1$ $\partial^j_t F|_{t=0} = \partial^j_t g|_{t=0} = 0$.

Then there exists a unique strong solution $u \in \mathring{H}^s_0([0,T] \times \Omega) \cap H^s([0,T] \times \partial\Omega)$ *of problem* (4.1.1)–(4.1.3) *satisfying for* $0 \leq t \leq T$ *the estimate*

(4.1.113) $|u(t)|^2_{s,0,\Omega} e^{-2\eta t} + \eta \|u\|^2_{s,\eta,\Omega'} + \|u\|^2_{s,\eta,\partial\Omega'}$

$$\leq \widetilde{f}_s((\varepsilon_i)_{i=1}^3, (\varepsilon_i^*)_{i=1}^3)[p_s(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \|F\|_{s,\eta,\Omega^t}^2$$

$$+ q_s(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \|g\|_{s,\eta,\partial\Omega^t}^2$$

$$+ r_s(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K}) \|f\|_{s,\Omega}^2],$$

where $\Omega^t = [0,t] \times \Omega$, $\partial\Omega^t = [0,t] \times \partial\Omega$; $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{e}$ and $\overline{f}$ are given by (3.3.13); p_s, q_s and r_s are polynomials for $0 \leq s < k$, while $p_k = \widehat{p}_k + p_{k-1} e^{2\eta T_2^*}$, $q_k = \widehat{q}_k + q_{k-1} e^{2\eta T_2^*}$ and $r_k = \widehat{r}_k$ ($\widehat{p}_k, \widehat{q}_k, \widehat{r}_k$ are polynomials, $T_2^* > 0$ is such that the function S is constant outside a compact set $[T_1^*, T_2^*] \times K_x \subset \mathbf{R} \times \overline{\Omega}$); $(\varepsilon_i)_{i=1}^3$ and $(\varepsilon_i^*)_{i=1}^3$ are the constants from Theorems 3.2.1 and 3.4.1 respectively; $\widetilde{f}_s$ are continuous functions; K is the compact set from Theorem 3.3.1; and η satisfies the inequality

$$(4.1.114) \qquad \eta > \widetilde{f}((\varepsilon_i)_{i=1}^3, (\varepsilon_i^*)_{i=1}^3) p(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K})$$

(p is a polynomial, $\widetilde{f}$ is a continuous function).

P r o o f. Take sequences (F_σ), (g_σ) and (f_σ) such that $F_\sigma \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$ and $F_\sigma = 0$ for $t < 0$, $g_\sigma \in C_{(0)}^\infty(\mathbf{R} \times \partial\Omega)$ and $g_\sigma = 0$ for $t < 0$, $f_\sigma \in C_0^\infty(\Omega)$ and, as $\sigma \to 0$

$$(4.1.115) \qquad\qquad F_\sigma \to F \quad \text{in } H^s(\Omega^T),$$

$$(4.1.116) \qquad\qquad g_\sigma \to g \quad \text{in } H^s(\partial\Omega^T),$$

$$(4.1.117) \qquad\qquad f_\sigma \to f \quad \text{in } \overset{\circ}{H}{}^s(\Omega).$$

By Theorem 4.1.1, Remark 4.1.2 and Lemma 4.1.7 there exists a unique solution $u_\sigma \in C_{(0)}^\infty([0,T] \times \overline{\Omega})$ of problem (4.1.1)-(4.1.3) with F, g, f replaced by F_σ, g_σ, f_σ respectively, satisfying for $0 \leq t \leq T$ the inequality

$$(4.1.118) \quad \|u_\sigma(t)\|_{0,\eta,\Omega}^2 + \eta \|u_\sigma\|_{0,\eta,\Omega^t}^2 + \|u_\sigma\|_{0,\eta,\partial\Omega^t}^2$$

$$\leq \widetilde{f}_0((\varepsilon_i)_{i=1}^3, (\varepsilon_i^*)_{i=1}^3) p_0(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K})$$

$$\times (\|F_\sigma\|_{0,\eta,\Omega^t}^2 + \|g_\sigma\|_{0,\eta,\partial\Omega^t}^2 + \|f_\sigma\|_{0,\Omega}^2).$$

Repeating now the proof of [28, Lemma 3.2] we get (4.1.113) with u, F, g and f replaced by u_σ, F_σ, g_σ and f_σ. Therefore by (4.1.115)-(4.1.118) and inequality (4.1.113) written for the differences $u_\sigma - u_{\tilde\sigma}$, $F_\sigma - F_{\tilde\sigma}$, $g_\sigma - g_{\tilde\sigma}$ and $f_\sigma - f_{\tilde\sigma}$ there exists $u \in \Pi_0^s(\Omega^T) \cap H^s(\partial\Omega^T)$ such that $u_\sigma \to u$ in $\Pi_0^s(\Omega^T) \cap H^s(\partial\Omega^T)$. Then u is a strong solution of problem (4.1.1)-(4.1.3), satisfying (4.1.113). ∎

THEOREM 4.1.3. Let assumptions (3.1.1), (3.1.3), (3.1.4), (3.1.6a)-(3.1.6b) with $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$, (3.1.7), (3.1.14), (3.1.20), (3.4.13), (4.1.30) and (4.1.111), (4.1.112) with $s = k$ be fulfilled. Moreover, assume

$$(4.1.119) \quad v \in \Pi_0^k(\Omega^T) \cap H^k(\partial\Omega^T),$$

(4.1.120)   *there exist sequences of $n \times n$-matrix-valued functions $E_\mu, A_{j\mu}$*
$(j = 1, \ldots, m), C_\mu, S_\mu \in C^\infty(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$ such that:

(a) *E_μ, $A_{j\mu}$, C_μ, S_μ satisfy for any μ conditions (3.1.1), (3.1.3),*
(3.1.4), (3.1.7), (3.1.14), (3.1.20), (3.4.13), (4.1.30) and $E_\mu, A_{j\mu}$,
$C_\mu, S_\mu \to E, A_j, C, S$ in $C_B^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$ as $\mu \to 0$;

(b) *for any $z_0 = (t_0, x_0, v_0, \eta_0, \zeta_0) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n \times [0, \infty) \times \mathbf{R}^m$ with*
$|\eta_0| + |\zeta_0| \neq 0$ the function

$$m_{z_0}(\mu) = \begin{cases} M(z_0) & for \ \mu = 0, \\ M_\mu(z_0) & for \ 0 < \mu \le 1 \end{cases}$$

(where $M(z_0)$ is given by (3.1.21) and $M_\mu(z_0)$ by the same for-
mula with the subscript μ added where appropriate) is continu-
ous in an interval $[0, \mu_0]$ (where $\mu_0 \le 1$) and the multiplicities
of its eigenvalues in $[0, \mu_0]$ are constant.

Then there exists a unique solution $u \in \Pi_0^k(\Omega^T) \cap H^k(\partial\Omega^T)$ of problem
(4.1.1)–(4.1.3) satisfying for $0 \le s \le k$ the estimate

(4.1.121) $|u(t)|_{s,0,\Omega}^2 e^{-2\eta t} + \eta\|u\|_{s,\eta,\Omega^\iota}^2 + \|u\|_{s,\eta,\partial\Omega^\iota}^2$

$$\le \widetilde{p}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T})\|F\|_{s,\eta,\Omega^\iota}^2$$

$$+ \widetilde{q}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T})\|g\|_{s,\eta,\partial\Omega^\iota}^2$$

$$+ \widetilde{r}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T})\|f\|_{s,\Omega}^2 \,,$$

where $\widetilde{p}_s$ and $\widetilde{q}_s$ are polynomials for $0 \le s < k$, and $\widetilde{p}_k = \widehat{\widetilde{p}}_k + \widetilde{p}_{k-1}e^{2\eta T_2^*}$,
$\widetilde{q}_k = \widehat{\widetilde{q}}_k + \widetilde{q}_{k-1}e^{2\eta T_2^*}$ ($\widehat{\widetilde{p}}$ and $\widehat{\widetilde{q}}$ are polynomials; $T_2^* > 0$ is such that the
function S is constant outside a compact set $[T_1^*, T_2^*] \times K_x \subset \mathbf{R} \times \bar{\Omega}$)); $\widetilde{r}_s$
are polynomials for $0 \le s \le k$; and η satisfies

(4.1.122) $\eta > \widetilde{p}(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T})$

($\widetilde{p}$ is a polynomial).

P r o o f.   We first prove the theorem under the assumption that $v \in C_{(0)}^\infty(\mathbf{R} \times \bar{\Omega})$. To do this take sequences $E_\mu, A_{j\mu}$ $(j = 1, \ldots, m), C_\mu, S_\mu$ satisfying assumption (4.1.120).

In view of Theorem 4.1.2, for any $\mu < \mu_0$ (where $\mu_0 \le 1$ is a constant) the problem

(4.1.123) $E_\mu(v)u_{\mu t} - \sum_{j=1}^m A_{j\mu}(v)u_{\mu x_j} - C_\mu(v)u_\mu = F$ in Ω^T,

(4.1.124) $[I, -S_\mu(v)]u_\mu = g$ on $\partial\Omega^T$,

(4.1.125) $u_\mu(0, \cdot) = f(\cdot)$ in Ω

has a unique solution $u_\mu \in \Pi_0^k(\Omega^T) \cap H^k(\partial\Omega^T)$, fulfilling for $0 \le t \le T$ and $0 \le s \le k$ the estimate

$$(4.1.126) \quad |u_\mu(t)|_{s,0,\Omega}^2 e^{-2\eta t} + \eta\|u_\mu\|_{s,\eta,\Omega'}^2 + \|u_\mu\|_{s,\eta,\partial\Omega'}^2$$
$$\le \widetilde{f}_s((\varepsilon_{i\mu})_{i=1}^3, (\varepsilon_{i\mu}^*)_{i=1}^3)[p_s(\overline{a}_\mu, \overline{b}_\mu, \overline{c}, \overline{d}, \overline{e}_\mu, \overline{f}_\mu, |v|_{k,0,\infty,K})\|F\|_{s,\eta,\Omega'}^2$$
$$+ q_s(\overline{a}_\mu, \overline{b}_\mu, \overline{c}, \overline{d}, \overline{e}_\mu, \overline{f}_\mu, |v|_{k,0,\infty,K})\|g\|_{s,\eta,\partial\Omega'}^2$$
$$+ r_s(\overline{a}_\mu, \overline{b}_\mu, \overline{c}, \overline{d}, \overline{e}_\mu, \overline{f}_\mu, |v|_{k,0,\infty,K})\|f\|_{s,\Omega}^2],$$

where

$$(4.1.127) \quad \eta > \widetilde{f}((\varepsilon_{i\mu})_{i=1}^3, (\varepsilon_{i\mu}^*)_{i=1}^3) p(\overline{a}_\mu, \overline{b}_\mu, \overline{c}, \overline{d}, \overline{e}_\mu, \overline{f}_\mu, |v|_{k,0,\infty,K}),$$

$\overline{a}_\mu = |\widetilde{L}_\mu''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}$, $\overline{b}_\mu = \|\widetilde{L}_\mu\|_{C_B^k(\mathbf{R} \times \bar\Omega \times \mathbf{R}^m)} + \|S_\mu\|_{C_B^k(\mathbf{R} \times \bar\Omega \times \mathbf{R}^m)}$, $\overline{e}_\mu = |S_\mu''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}$, $\overline{f}_\mu = \|S_\mu''\|_{k,\eta,\mathbf{R}^m \times \{x_1=0\}}$, and $\widetilde{L}_\mu''$ and $\widetilde{L}_\mu$ are given by (3.3.11) and (3.3.12) with E'', A_j'', C'' and E, A_j, C replaced by the respective operators with subscript μ.

Since $v \in C_{(0)}^\infty(\mathbf{R} \times \bar\Omega)$, in view of (4.1.120)(a) we have

$$(4.1.128) \quad \overline{a}_\mu, \overline{b}_\mu, \overline{e}_\mu, \overline{f}_\mu \to \overline{a}, \overline{b}, \overline{e}, \overline{f} \quad \text{as } \mu \to 0,$$

while by the proof of Theorem 3.2.1

$$(4.1.129) \quad (\varepsilon_{i\mu})_{i=1}^3, (\varepsilon_{i\mu}^*)_{i=1}^3 \to (\varepsilon_i)_{i=1}^3, (\varepsilon_i^*)_{i=1}^3 \quad \text{as } \mu \to 0.$$

By [1, Theorem 2.8]

$$(4.1.130) \quad \exists \widetilde{c} > 0 \; \forall 1 \le p < \infty \; \forall \mu < \mu_0 \; \forall 0 \le t \le T \; \forall 0 \le s \le k$$
$$|u_\mu|_{s,0,p,\Omega'}^2 \le \widetilde{c}|u_\mu|_{s,0,\infty,\Omega'}^2.$$

Therefore using (4.1.128)–(4.1.130) in (4.1.126) and (4.1.127) we obtain

$$(4.1.131) \quad c_0|u_\mu(t)|_{s,0,p,\Omega}^2 e^{-2\eta t} + \eta\|u_\mu\|_{s,\eta,\Omega'}^2 + \|u_\mu\|_{s,\eta,\partial\Omega'}^2$$
$$\le p_s^*(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K})\|F\|_{s,\eta,\Omega'}^2$$
$$+ q_s^*(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K})\|g\|_{s,\eta,\partial\Omega'}^2$$
$$+ r_s^*(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K})\|f\|_{s,\Omega}^2,$$

(where $0 \le s \le k$, $0 \le t \le T$, $1 \le p < \infty$; p_s^*, q_s^*, r_s^* are functions of the same type as p_s, q_s, r_s; $c_0 = 1/\widetilde{c}_0$) and

$$(4.1.132) \quad \eta > p^*(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, |v|_{k,0,\infty,K})$$

(where p^* is a polynomial).

Hence we can choose a subsequence of (u_μ), still denoted by (u_μ), such that

$$\forall 1 \le p < \infty \; \forall 0 \le s \le k \quad u_\mu \to u \text{ weakly in } \Pi_{0,p}(\Omega^T) \cap H^s(\Omega^T)$$

and $u_\mu \to u$ strongly in $C^1([0,T] \times \widehat{K}_x)$ as $\mu \to 0$ for any compact set $\widehat{K}_x \subset \overline{\Omega}$. Therefore (4.1.120)(a) implies that u satisfies (4.1.1)–(4.1.3). Moreover, by (4.1.131) and [1, Theorem 2.8] u satisfies

$$(4.1.133) \quad c_0|u(t)|^2_{s,0,\Omega}\, e^{-2\eta t} + \eta\|u\|^2_{s,\eta,\Omega^t} + \|u\|^2_{s,\eta,\partial\Omega^t}$$
$$\leq p^*_s(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\|F\|^2_{s,\eta,\Omega^t}$$
$$+ q^*_s(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\|g\|^2_{s,\eta,\partial\Omega^t}$$
$$+ r^*_s(\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},|v|_{k,0,\infty,K})\|f\|^2_{s,\Omega}\,,$$

where $0 \leq t \leq T$, $0 \leq s \leq k$.

(4.1.133) implies additionally the uniqueness of u.

Assume now that v satisfies (4.1.119). Hence $v \in \Pi^k_0(\Omega^T) \cap H^{k+1/2}(\Omega^T)$. Consider the following cases:

$1°$ $[0,T] \subset [T^*_1, T^*_2]$;

$2°$ $[T^*_1, T^*_2] \subset [0,T]$;

$3°$ $[T^*_1, T^*_2] \cap [0,T] = [0, T^*_2] \vee [T^*_1, T^*_2] \cap [0,T] = [T^*_1, T]$;

$4°$ $[0,T] \cap [T^*_1, T^*_2] = \emptyset$.

Consider first case $1°$. Define $v^* = v|_{[0,T] \times K_x}$ and using Lemma 2.1.3 extend v^* to a function $\widetilde{v}^* \in \Pi^k_0([T^*_1, T^*_2] \times K_x) \cap H^{k+1/2}([T^*_1, T^*_2] \times K_x)$ which satisfies

$$(4.1.134) \quad |\widetilde{v}^*|_{k,0,\infty,[T^*_1,T^*_2]\times K_x} + \|\widetilde{v}^*\|_{k+1/2,[T^*_1,T^*_2]\times K_x}$$
$$\leq c_1(|v^*|_{k,0,\infty,[0,T]\times K_x} + \|v^*\|_{k+1/2,[0,T]\times K_x})$$
$$\leq c_2(|v|_{k,0,\infty,\Omega^T} + \|v\|_{k+1/2,\Omega^T})$$
$$\leq c_3(|v|_{k,0,\infty,\Omega^T} + \|v\|_{k,\partial\Omega^T})\,,$$

where the constants $c_1, c_2, c_3 > 0$ are independent of v. Applying again Lemma 2.1.3 extend $\widetilde{v}^*$ to a function $E_{\mathbf{R}\times\mathbf{R}^m}\widetilde{v}^* \in \Pi^k_0(\mathbf{R} \times \mathbf{R}^m) \cap H^{k+1/2}(\mathbf{R} \times \mathbf{R}^m)$ satisfying

$$(4.1.135) \quad |E_{\mathbf{R}\times\mathbf{R}^m}v^*|_{k,0,\infty,\mathbf{R}\times\Omega} + \|E_{\mathbf{R}\times\mathbf{R}^m}\widetilde{v}^*\|_{k+1/2,\mathbf{R}\times\Omega}$$
$$\leq c_4(|v^*|_{k,0,\infty,[T^*_1,T^*_2]\times K_x} + \|v^*\|_{k+1/2,[T^*_1,T^*_2]\times K_x})\,,$$

where $E_{\mathbf{R}\times\mathbf{R}^m}$ is the extension operator from the proof of Lemma 2.1.3.

Since $\forall\, 1 \leq p < \infty$, $\widetilde{v}^* \in \Pi^k_{0,p}([T^*_1, T^*_2] \times K_x) \cap H^{k+1/2}([T^*_1, T^*_2] \times K_x)$, the proof of Lemma 2.1.3 implies that $\forall\, 1 \leq p < \infty$, $E_{\mathbf{R}\times\mathbf{R}^m}\widetilde{v}^* \in \Pi^k_{0,p}(\mathbf{R} \times \mathbf{R}^m) \cap H^{k+1/2}(\mathbf{R} \times \mathbf{R}^m)$.

Take now a nonnegative real function J with the properties:

(a) $J \in C^\infty_0(\mathbf{R} \times \mathbf{R}^m)$,

(b) $J(t,x) = 0$ if $\sqrt{t^2 + |x|^2} \geq 1$,

(c) $\displaystyle\int_{\mathbf{R}\times\mathbf{R}^m} J(t,x)\,dt\,dx = 1,$

and a function w defined almost everywhere in $\mathbf{R}\times\mathbf{R}^m$. Define operators P_i $(i = 1,2,\ldots)$ by

(4.1.136) $$P_i w = J_{1/i} * (\psi_i w),$$

where $J_{1/i}(t,x) = (1/i)^{-m-i}J(it,ix)$, $\psi_i(t,x) = \psi(\sqrt{t^2 + |x|^2 - i})$, $\psi \in C^\infty(\mathbf{R})$ and

$$\psi(s) = \begin{cases} 1 & \text{if } s \le 0, \\ 0 & \text{if } s \ge 1. \end{cases}$$

Set

(4.1.137) $$v_i = P_i E_{\mathbf{R}\times\mathbf{R}^m}\widetilde{v}^* \quad \text{for } i = 1,2,\ldots$$

By [3, Lemma 5.2] and [1, Lemma 2.18, the proofs of Lemma 3.15, Theorems 7.38 and 7.40], we have $v_i \in C_0^\infty(\mathbf{R}\times\mathbf{R}^m)$ and

(4.1.138) $$v_i|_{[T_1^*,T_2^*]\times K_\varepsilon} \to \widetilde{v}^*$$

$$\text{in } \Pi_{0,p}^k([T_1^*,T_2^*]\times K_x)\cap H^{k+1/2}([T_1^*,T_2^*]\times K_x)$$

for any $1 \le p < \infty$ as $i \to \infty$. Moreover,

(4.1.139) $$|v_i|_{k,0,\infty,[T_1^*,T_2^*]\times K_\varepsilon} + \|v_i\|_{k+1/2,[T_1^*,T_2^*]\times K_\varepsilon}$$
$$\le c_5(|\widetilde{v}^*|_{k,0,\infty,[T_1^*,T_2^*]\times K_\varepsilon} + \|\widetilde{v}^*\|_{k+1/2,[T_1^*,T_2^*]\times K_\varepsilon}).$$

In fact, (4.1.135)–(4.1.137) and the proof of [1, Theorem 7.38] and [3, Lemma 5.2] yield

$$|v_i|_{k,0,p,[T_1^*,T_2^*]\times K_\varepsilon} + \|v_i\|_{k+1/2,[T_1^*,T_2^*]\times K_\varepsilon}$$
$$\le c_6(|P_i E_{\mathbf{R}\times\mathbf{R}^m}\widetilde{v}^*|_{k,0,\infty,[T_1^*,T_2^*]\times K_\varepsilon} + \|P_i E_{\mathbf{R}\times\mathbf{R}^m}\widetilde{v}^*\|_{k+1/2,[T_1^*,T_2^*]\times K_\varepsilon})$$
$$\le c_7(|E_{\mathbf{R}\times\mathbf{R}^m}\widetilde{v}^*|_{k,0,\infty,\mathbf{R}\times\mathbf{R}^m} + \|E_{\mathbf{R}\times\mathbf{R}^m}v^*\|_{k+1/2,\mathbf{R}\times\mathbf{R}^m})$$
$$\le c_8(|\widetilde{v}^*|_{k,0,\infty,[T_1^*,T_2^*]\times K_\varepsilon} + \|\widetilde{v}^*\|_{k+1/2,[T_1^*,T_2^*]\times K_\varepsilon})$$

for any $1 \le p < \infty$, which implies (4.1.139) by [1, Theorem 2.8].

Now, (4.1.134) and (4.1.139) imply

(4.1.140) $$|v_i|_{k,0,\infty,[T_1^*,T_2^*]\times K_\varepsilon} + \|v_i\|_{k+1/2,[T_1^*,T_2^*]\times K_\varepsilon}$$
$$\le c(|v|_{k,0,\infty,\Omega^T} + \|v\|_{k,\partial\Omega^T}),$$

where $c > 0$ is a constant.

Consider now the problem

(4.1.141) $$E(v_i)u_{it} - \sum_{j=1}^m A_j(v_i)u_{ix_j} - C(v_i)u_i = F \quad \text{in } \Omega^T,$$

(4.1.142) $$[I, -S(v_i)]u_i = g \quad \text{on } \partial\Omega^T,$$

(4.1.143) $u_i(0, \cdot) = f(\cdot)$ in Ω .

By the considerations from the first part of the proof there exists a unique
solution $u_i \in \Pi_0^k(\Omega^T) \cap H^k(\partial \Omega^T)$ of problem (4.1.141)–(4.1.143) and it
satisfies for $\eta > p^*(\overline{a}_i, \overline{b}, \overline{c}_i, \overline{d}, \overline{e}_i, \overline{f}_i, |v_i|_{k,0,\infty,K})$ the estimate (4.1.133) with
$\overline{a}, \overline{c}, \overline{e}$ and $\overline{f}$ replaced by

$$\overline{a}_i = |\widetilde{L}_i''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)} \, ,$$

$$\overline{c}_i = \sum_{|\alpha| \leq 2} \sup_{\zeta = (\eta,\zeta):|\zeta| \leq 1} |\partial_\zeta^\alpha \Psi_{1i}''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)}$$

$$+ \sum_{|\alpha| \leq 2}^{\zeta} \sup_{\zeta = (\eta,\zeta):|\zeta| = 1} |\partial_\zeta^\alpha \Psi_{2i}|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)} \, ,$$

$$\overline{e}_i = |S_i''|_{k,0,\infty,\mathbf{R}^m \times [0,\infty)} \, , \qquad \overline{f}_i = \|S_i''\|_{k,\eta,\mathbf{R}^m \times [0,\infty)} \, ,$$

where $\widetilde{L}_i'' = (E_i'', A_{ji}'' (j = 1, \ldots, m), C_i'')$, $G_i(t,x) = G_i(t,x,v_i(t,x))$ for
$G_i = E_i$, A_{ji} $(j = 1, \ldots, m)$, C_i, S_i; thus $G_i = E_i, A_{ji}, C_i, S_i$. Therefore
Lemma 2.1.2, assumption (3.1.7) and (4.1.140), when applied in (4.1.133),
give the estimate

(4.1.144) $c_0 |u_i|_{s,0,p,\Omega}^2 e^{-2\eta t} + \eta \|u_i\|_{s,\eta,\Omega^t}^2 + \|u_i\|_{s,\eta,\partial \Omega^t}^2$

$$\leq \widetilde{p}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial \Omega^T}) \|F\|_{s,\eta,\Omega^t}^2$$

$$+ \widetilde{q}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial \Omega^T}) \|g\|_{s,\eta,\partial \Omega^t}^2$$

$$+ \widetilde{r}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial \Omega^T}) \|f\|_{s,\Omega}^2$$

for any $1 \leq p < \infty$, $0 \leq t \leq T$ and $0 \leq s \leq k$. Hence we can choose a
subsequence of (u_i), still denoted by (u_i), such that

$$\forall 0 \leq s \leq k \; \forall 1 \leq p < \infty \quad u_i \to u \text{ weakly in } \Pi_0^k(\Omega^T) \cap H^s(\partial \Omega^T)$$

and $u_i \to u$ strongly in $C^1([0,T] \times \widehat{K}_x)$ for any compact set $\widehat{K}_x \subset \overline{\Omega}$, where u
satisfies (4.1.1)–(4.1.3) and the estimate (4.1.121) holds for u with η fulfilling
(4.1.122). Furthermore, u is unique.

In case 2° instead of (4.1.134) we have

(4.1.145) $|v^*|_{k,0,\infty,[T_1^-,T_2^-] \times K_x} + \|v^*\|_{k+1/2,[T_1^-,T_2^-] \times K_x}$

$$\leq c(|v|_{k,0,\infty,\Omega^T} + \|v\|_{k,\partial \Omega^T})$$

for some constant $c > 0$. In this case v_i is given by

(4.1.146) $v_i = P_i E_{\mathbf{R} \times \mathbf{R}^m} v^*$, $i = 1, 2, \ldots$

In case 3° we extend the functions $v^* = v|_{[0,T_2^*]}$ and $v^* = v|_{[T_1^*,T]}$ respec-
tively to $[T_1^*, T_2^*]$ and we repeat the same argument as in case 1°.

In case 4°, (4.1.1)–(4.1.3) is a problem with constant coefficients, thus
the conclusion is also satisfied. ∎

Remark 4.1.3. If $E, A_j, C, S \in C^\infty(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$, then assumption (4.1.120) is automatically satisfied. Unfortunately, at present the author is not able to present any other example.

Remark 4.1.4. Using the same argument as in Remark 3.5.3 we can replace, in Theorem 4.1.3 and 4.1.4, assumption (3.1.4) by (3.5.47).

4.2. A mixed problem with zero initial condition. Consider now problem (4.1.1)–(4.1.3) with $f = 0$. In this case the assumptions of Theorems 4.1.2–4.1.3 can be weakened. Instead of Theorem 4.1.2 we obtain

THEOREM 4.2.1. *Let assumptions* (3.1.1)–(3.1.5), (3.1.7), (3.1.14), (3.1.20), (3.1.24), (3.4.13), (3.5.38), (3.5.39) *be satisfied. Moreover, assume*

$$(4.2.1) \qquad F \in H^s(\Omega^T), \quad g \in H^s(\partial\Omega^T), \quad f = 0,$$

where $0 \le s \le k$, $k > m/2 + 5/2$;

$$(4.2.2) \qquad \forall 0 \le j \le s-1 \quad \partial_t^j F|_{t=0} = \partial_t^j g|_{t=0} = 0.$$

Then there exists a unique strong solution $u \in H^s(\partial\Omega^T)$ *of problem* (4.1.1)–(4.1.3), *satisfying for* $0 \le t \le T$ *and* $0 \le s \le k$ *the estimate*

$$(4.2.3) \quad \eta\|u\|_{s,\eta,\Omega^\cdot}^2 + \|u\|_{s,\eta,\partial\Omega^\cdot}^2$$
$$\le \tilde{f}_s((\varepsilon_i)_{i=1}^3, (\varepsilon_i^*)_{i=1}^3)(p_s(\bar{a}', \bar{b}, \bar{c}', \bar{d}, \bar{e}', \bar{f}', \|v\|_{k,K})\|F\|_{s,\eta,\Omega^\cdot}^2$$
$$+ q_s(\bar{a}', \bar{b}, \bar{c}', \bar{d}, \bar{e}', \bar{f}', \|v\|_{k,K})\|g\|_{s,\eta,\partial\Omega^\cdot}^2),$$

where $\bar{b}$ *and* $\bar{d}$ *are given by* (3.3.13);

$$\bar{a}' = \|\tilde{L}''\|_{k,\mathbf{R}^m \times [0,\infty)} = \|\tilde{\mathcal{L}}''\|_{k,\mathbf{R} \times \Omega}$$

$(\tilde{L}''$ *and* $\tilde{\mathcal{L}}''$ *are given by* (3.3.12)),

$$\bar{c}' = \sum_{|\alpha|\le 2} \sup_{\zeta=(\eta,\zeta):|\zeta|\le 1} \|\partial_\zeta^\alpha \Psi_1''\|_{k,\mathbf{R}^m \times [0,\infty)}$$
$$+ \sum_{|\alpha|\le 2} \sup_{\zeta=(\eta,\zeta):|\zeta|\le 1} \|\partial_\zeta^\alpha \Psi_2''\|_{k,\mathbf{R}^m \times [0,\infty)},$$

$$\bar{e}' = \|S''\|_{k,\mathbf{R}^m \times [0,\infty)}, \quad \bar{f}' = \|S''\|_{k,\eta,\mathbf{R}^m};$$

p_s *and* q_s *are polynomials for* $0 \le s < k$, *and* $p_k = \hat{p}_k + p_{k-1}e^{2\eta T_2^*}$ *and* $q_k = \hat{q}_k + q_{k-1}e^{2\eta T_2^*}$ ($\hat{p}_k$ *and* $\hat{q}_k$ *are polynomials,* $T_2^* > 0$ *is such that* S'' *vanishes outside a compact set* $K = [T_1^*, T_2^*] \times K_x \subset \mathbf{R} \times \bar{\Omega}$); $(\varepsilon_i)_{i=1}^3$ *and* $(\varepsilon_i^*)_{i=1}^3$ *are the constants from Theorems 3.2.1 and 3.4.1;* $\tilde{f}_s$ *are continuous functions; and* η *satisfies*

$$(4.2.4) \qquad \eta > \tilde{f}((\varepsilon_i)_{i=1}^3, (\varepsilon_i^*)_{i=1}^3)p(\bar{a}', \bar{b}, \bar{c}', \bar{d}, \bar{e}', \bar{f}', \|v\|_{k,K})$$

(p *is a polynomial,* $\tilde{f}$ *is a continuous function*).

P r o o f. Assume first that $F \in C^\infty_{(0)}(\mathbf{R} \times \bar{\Omega})$, $F = 0$ for $t < 0$, and $g \in C^\infty_0(\mathbf{R} \times \partial\Omega)$, $g = 0$ for $t < 0$. Then by Theorem 3.5.2 there exists a unique solution $u \in C^\infty_{(0)}(\mathbf{R} \times \Omega)$ of the boundary value problem (3.1)–(3.2) such that $u = 0$ for $t < 0$. Thus $u|_{[0,T] \times \Pi}$ is the solution of (4.1.1)–(4.1.3) and in view of Remarks 3.3.1 and 3.5.2 it satisfies the estimate (4.2.3) with $s = 0$.

Using a similar argument to the proof of [28, Lemma 3.2], we get (4.2.3) for $0 \leq s \leq k$.

Assume now that F and g satisfy (4.2.1)–(4.2.2) and take sequences (F_σ) and (g_σ), $0 < \sigma < 1$, such that $F_\sigma \in C^\infty_{(0)}(\mathbf{R} \times \Omega)$ and $F_\sigma = 0$ for $t < 0$, $g_\sigma \in C^\infty_0(\mathbf{R} \times \partial\Omega)$ and $g_\sigma = 0$ for $t < 0$ and, as $\sigma \to 0$,

(4.2.5) $F_\sigma \to F$ in $H^s(\Omega^T)$, $g_\sigma \to g$ in $H^s(\partial\Omega^T)$.

Applying the same argument as in the proof of Theorem 4.1.2 we obtain the assertion. ∎

The theorem given below corresponds to Theorem 4.1.3.

T H E O R E M 4.2.2. *Let assumptions* (3.1.1), (3.1.3), (3.1.4), (3.1.6a)–(3.1.6b) *with* $k > m/2 + 5/2$, (3.1.7), (3.1.14), (3.1.20), (3.4.13), (4.1.30), (4.1.120) *and* (4.2.1) *and* (4.2.2) *with* $s = k$ *be satisfied. Moreover, assume*

(4.2.6) $v \in H^k(\partial\Omega^T)$.

Then there exists a unique solution $u \in H^k(\partial\Omega^T)$ *of problem* (4.1.1)–(4.1.3), *satisfying for* $0 \leq t \leq T$ *and* $0 \leq s \leq k$ *the estimate*

(4.2.7) $\eta\|u\|^2_{s,\eta,\Omega'} + \|u\|^2_{s,\eta,\partial\Omega'} \leq \widetilde{p}_s(\|v\|_{k,\partial\Omega^T})\|F\|^2_{s,\eta,\Omega'}$

$+ \widetilde{q}_s(\|v\|_{k,\partial\Omega^T})\|g\|^2_{s,\eta,\partial\Omega'}$,

where $\widetilde{p}_s$ *and* $\widetilde{q}_s$ *are polynomials for* $0 \leq s < k$, *and* $\widetilde{p}_k = \widehat{\widetilde{p}}_k + \widetilde{p}_{k-1}e^{2\eta T^*_2}$, $\widetilde{q}_k = \widehat{\widetilde{q}}_k + \widetilde{q}_{k-1}e^{2\eta T^*_2}$ $(\widehat{\widetilde{p}}_k$ *and* $\widehat{\widetilde{q}}_k$ *are polynomials;* T^*_2 *is such as in Theorem* 4.2.1); *and* η *satisfies*

(4.2.8) $\eta > \widetilde{p}(\|v\|_{k,\partial\Omega^T})$

$(\widetilde{p}$ *is a polynomial*).

P r o o f. The proof is analogous to that of Theorem 4.1.3. ∎

5. A mixed problem for a system of linear hyperbolic equations of the first order in a bounded domain

In this chapter we consider the mixed problem (4.1.1)–(4.1.3) in a bounded domain $\Omega \subset \mathbf{R}^m$ with boundary $\partial\Omega$ of class C^∞.

Let us assume the following conditions:

(5.1)   all functions given in (4.1.1)–(4.1.3) are real;

(5.2)   system (4.1.1) is strictly hyperbolic, i.e. $\forall \omega = (\omega_1, \ldots, \omega_m) \in \mathbf{R}^m \setminus \{0\}$ and $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$ the equation $\det(\lambda E(t, x, v) - \sum_{j=1}^{m} A_j(t, x, v)\omega_j) = 0$ has n distinct real roots;

(5.3) $\det E(t, x, v) \neq 0$, $\forall (t, x, v) \in \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n$;

(5.4) $\det A_\nu(t, x, v) \neq 0$, $\forall (t, x, v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n$, where $A_\nu = -\sum_{j=1}^{m} A_j\nu_j$, $\nu = (\nu_1, \ldots, \nu_m)$ is the unit outward vector normal to the boundary $\partial\Omega$;

(5.5) $E, A_j (j = 1, \ldots, m), C, S \in C_B^k(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$, where $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$;

(5.6) $\forall (t, x, v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n$, $E^{-1}(t, x, v)A_\nu(t, x, v)$ has l negative and $n - l$ positive eigenvalues;

(5.7) $\det(B_{2\nu}(t, x, \alpha(t, x))S(t, x, \alpha(t, x)) + A_\nu^{II}(t, x, \alpha(t, x)) \neq 0$, $\forall (t, x) \in \mathbf{R} \times \partial\Omega$, where $\alpha \in C^0(\mathbf{R} \times \bar{\Omega})$ is an arbitrary n-vector-valued function and

$$A_\nu = \begin{bmatrix} A_\nu^I & B_{1\nu} \\ B_{2\nu} & A_\nu^{II} \end{bmatrix}$$

(where the entries are matrix-valued functions of appropriate dimensions).

Consider now the function

(5.8) $$M_\nu(t, x, v, s, i\omega') = A_\nu^{-1}(t, x, v)\Big[E(t, x, v)s - i\sum_{j=2}^{m} A_j(t, x, v)\omega_j\Big],$$

where $s = \eta + i\xi$, $\eta > 0$, $\xi \in \mathbf{R}$, $\omega' = (\omega_2, \ldots, \omega_m)$, $(t, x, v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n$.

In the same way as in Lemma 3.1.1 we can prove that $\forall (t, x, v) \in \mathbf{R} \times \partial\Omega \times \mathbf{R}^n$ and $\forall (s, \omega') \in \mathbf{C} \times \mathbf{R}^{m-1} : \mathrm{Re}\, s > 0$, $M_\nu(t, x, v, s, i\omega')$ has exactly l eigenvalues κ with $\mathrm{Re}\,\kappa < 0$.

Let vector-valued functions $\varphi_1(t, x, v, s, i\omega'), \ldots, \varphi_l(t, x, v, s, i\omega')$ be an orthonormal basis of the l-dimensional linear space spanned by the generalized eigenvectors corresponding to the eigenvalues of M_ν with negative real parts. Define

(5.9) $$N(t, x, \alpha, v, s, \omega') = [I, -S(t, x, \alpha(t, x))]$$
$$\cdot [\varphi_1(t, x, v, s, \omega'), \ldots, \varphi_l(t, x, v, s, \omega')],$$

where $\alpha \in C^0(\mathbf{R} \times \bar{\Omega})$ is arbitrary. Assume

(5.10) $\det N(t, x, \alpha, v, s, \omega')$ is a continuous function for $\mathrm{Re}\, s \geq 0$ and

$$|\det N(t, x, \alpha, v, s, \omega')| > \varepsilon^*$$

$\forall (t, x, v) \in \mathbf{R} \times \partial \Omega \times \mathbf{R}^n \ \forall (s, \omega') \in \mathbf{C} \times \mathbf{R}^{m-1} : \operatorname{Re} s > 0, |s| + |\omega| \neq$
0, where $\varepsilon^* > 0$ is independent of t, x, v and α.

Assume now that the function v occurring in the coefficients of problem
(4.1.1)–(4.1.3) belongs to $\Pi_0^k(\Omega^T) \cap H^k(\partial \Omega^T)$. By Lemma 2.1.3 there exists
an extension $v^* \in \Pi_0^k(\mathbf{R} \times \Omega) \cap H^{k+1/2}(\mathbf{R} \times \Omega)$ of v to $\mathbf{R} \times \Omega$ such that $v^* = 0$
outside a compact set $\widehat{K}_t \subset \mathbf{R}$ and

(5.11) $|v^*|_{k,0,\infty,\mathbf{R} \times \Omega} + \|v^*\|_{k+1/2,\mathbf{R} \times \Omega} \leq c(|v|_{k,0,\infty,\Omega^T} + \|v\|_{k,\partial \Omega^T})$,

where $c > 0$ is a constant.

Consider the boundary value problem

(5.12) $E(v^*) u_t - \sum_{j=1}^m A_j(v^*) u_{x_j} - C(v^*) u = F$ in $\mathbf{R} \times \Omega$,

(5.13) $[I, -S(v^*)] u = g$ on $\mathbf{R} \times \partial \Omega$.

THEOREM 5.1. *Let assumptions* (5.1)–(5.7) *and* (5.10) *be satisfied. If*
$u \in C_{(0)}^\infty(\mathbf{R} \times \overline{\Omega})$ *is a solution of* (5.12)–(5.13), *then for* $0 \leq s \leq k$,

(5.14) $\eta \|u\|_{s,\eta,\mathbf{R} \times \Omega}^2 + \|u\|_{s,\eta,\mathbf{R} \times \partial \Omega}^2 \leq p_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial \Omega^T}, \overline{b})$

$\times \eta^{-1} \|F\|_{s,\eta,\mathbf{R} \times \Omega}^2 + q_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial \Omega^T}, \overline{b}) \|g\|_{s,\eta,\mathbf{R} \times \partial \Omega}^2$,

where $\overline{b}$ *is given by* (3.3.13); p_s *and* q_s *are polynomials for* $0 \leq s < k$, *and*
$p_k = \widehat{p}_k + p_{k-1} e^{2\eta T^*}$ *and* $q_k = \widehat{q}_k + q_{k-1} e^{2\eta T^*}$ ($\widehat{p}_k$ *and* $\widehat{q}_k$ *are polynomials,*
$T^* > 0$ *is a certain constant); and* η *satisfies*

(5.15) $\eta > p(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial \Omega^T}, \overline{b})$

(p *is a polynomial*).

Proof. Cover $\overline{\Omega}$ by two families $\{K_{R_q^1}\}$ and $\{K_{R_q^2}\}$ ($q = 1, \ldots, \mu + \nu$)
of balls with centres $x^{0q} \in \overline{\Omega}$ such that $\forall\, 1 \leq q \leq \mu + \nu$, $K_{R_q^1} \Subset K_{R_q^2}$, and
$\partial \Omega$ is covered by the balls $\{K_{R_q^1}\}$ and $\{K_{R_q^2}\}$ ($q = 1, \ldots, \mu$) with radii R_q^1
and R_q^2 so small that in $\mathbf{R} \times (K_{R_q^i} \cap \Omega) \times \mathbf{R}^n$ ($i = 1, 2$) assumptions (5.4),
(5.6) and (5.7) are satisfied, while the remaining balls of both families do
not intersect $\partial \Omega$. Assume moreover that there exists $N_0 > \mu$ such that the
intersection of any $N_0 + 1 \leq \mu$ of the balls covering $\partial \Omega$ is empty.

Set next $\Omega_q^i = K_{R_q^i} \cap \Omega$, $\Gamma_q^i = K_{R_q^i} \cap \partial \Omega$, $i = 1, 2$, $q = 1, \ldots, \mu$. Associate
with every point $x^{0q} \in \partial \Omega$ the coordinates $(z_i)_{i=1}^m$ given by (2.1.2), and next
introduce the coordinates $(y_i)_{i=1}^m$ by means of (2.1.3). We have

(5.16) $x = \Psi_q(y)$,

(5.17) $y = \Psi_q^{-1}(x)$,

where $x = (x_i)_{i=1}^m$, $y = (y_i)_{i=1}^m$, Ψ_q and Ψ_q^{-1} are of class C_B^∞. The function
Ψ_q^{-1} maps Ω_q^i ($i = 1, 2$) onto $\Sigma_{R_q^i} = \{y : [y_1 + h_q(y')]^2 + |y'|^2 < R_q^i, y_1 > 0\}$,

and Γ_q^i onto $\sigma_{R_q^i} = \{y : [y_1 + h_q(y')]^2 + |y'|^2 < R_q^i, y_1 = 0\}$.

Now assume additionally that $\forall\, 1 \le q \le \mu$, R_q^1 is so small in comparison with R_q^2 that there exists $P_q = \{x \in \mathbf{R}^m : 0 < x_1 < b_1, a_i < x_i < b_i, i = 2, \ldots, m\}$ such that $\Sigma_{R_q^1} \subset P_q \subset \Sigma_{R_q^2}$.

Define further

(5.18) $\widehat{G}(t, y, v) = G(t, x, v)|_{x = \Psi_q(y)} = G(t, \Psi_q(y), v)$

(where $G = E, A_j\, (j = 1, \ldots, m), C, S$) and

(5.19) $E^0 = \widehat{E}, \quad A_i^0 = \widehat{A}_j \widehat{y}_{i,j}\ (i = 1, \ldots, m), \quad C^0 = \widehat{C}, \quad S^0 = \widehat{S},$

where

$$\widehat{y}_{i,j} = \left.\frac{\partial y_i}{\partial x_j}\right|_{x = \Psi_q(y)}.$$

From the form of Ψ_q^{-1} it follows that $A_1^0 = -\widehat{A}_{\widehat{\nu}} = \widehat{A}_\nu$ and $A_i^0 = \widehat{A}_j c_{ij}$ for $i = 2, \ldots, m$, where $\widehat{\nu}(y) = \nu(\Psi_q(y))$.

Now, with the covering $\{K_{R_q^i}\}_{q=1}^{\mu+\nu}$ of $\overline{\Omega}$ associate a partition of unity $\{\psi_q\}_{q=1}^{\mu+\nu}$ satisfying the properties:

    (a) $\psi_q \in C_0^\infty(\mathbf{R}^m)$, supp $\psi_q \subset K_{R_q^1}$, $\forall\, 1 \le q \le \mu + \nu$;

    (b) $0 \le \psi_q \le 1$, $\forall\, 1 \le q \le \mu + \nu$;

    (c) $\sum_{q=1}^{\mu+\nu} \psi_q = 1$, $\forall\, x \in \overline{\Omega}$.

Set

(5.20) $\sigma_q(t, x) = \sigma(t, x)\psi_q(x)$,   where $\sigma = u, F, g$.

Let $[T_1', T_2']$ be an interval outside which $u \in C_0^\infty(\mathbf{R} \times \overline{\Omega})$ vanishes. Then $\sigma_q = 0$ outside $[T_1', T_2'] \times (\Omega_q^1 \cup \Gamma_q^1)$.

Since u satisfies (5.12)–(5.13), $u_q|_{[T_1,T_2] \times (\Omega_q^2 \cup \Gamma_q^2)}$ (where $[T_1', T_2'] \subset (T_1, T_2)$) is for any $1 \le q \le \mu$ the solution of the problem

(5.21) $Lu_q = F_q + [L, \psi_q]u$ in $[T_1, T_2] \times \Omega_q^2$,

(5.22) $L_B u_q = g_q$ on $[T_1, T_2] \times \Gamma_q^2$,

where

$$L = E(v^*)\partial_t - \sum_{j=1}^m A_j(v^*)\partial_{x_j} - C(v^*),$$

$$L_B = [I, -S(v^*)],$$

$$[L, \psi_q]u = L(\psi_q u) - \psi_q Lu.$$

Applying the coordinates given by (5.17) we replace (5.21)–(5.22) by

(5.23) $\widehat{L}\widehat{u}_q = \widehat{F}_q + [\widehat{L}, \widehat{\psi}_q]\widehat{u}$ in $[T_1, T_2] \times P_q$,

(5.24) $\widehat{L}_B \widehat{u}_q = \widehat{g}_q$ on $[T_1, T_2] \times p_q$,

where

$$p_q = \{x \in \mathbf{R}^m : x_1 = 0, a_i < x_i < b_i, i = 2, \ldots, m\},$$

$$\widehat{L} = \widehat{E}(\widehat{v}^*)\partial_t - \sum_{i,j=1}^m \widehat{A}_j(\widehat{v}^*)\partial_{x_j} y_i \partial_{y_i} - \widehat{C}(\widehat{v}^*)$$

$$= E^0(\widehat{v}^*)\partial_t - \sum_{i=1}^m A_i^0(\widehat{v}^*)\partial_{y_i} - C^0(\widehat{v}^*),$$

$$\widehat{L}_B = [I, -\widehat{S}(\widehat{v}^*)] = [I, -S^0(\widehat{v}^*)],$$

$$\widehat{h}(t,y) = h|_{[T_1,T_2] \times (\Omega_q^2 \cup \Gamma_q^2)}(t, \Psi_q(y))$$

for $h = v^*, F_q, g_q, \psi_q, u_q, u$ and

$$\widehat{G}(\widehat{v}^*) = \widehat{G}(t, y, \widehat{v}^*(t, y))$$

for G being the restriction to $[T_1, T_2] \times (\Omega_q^2 \cup \Gamma_q^2) \times \widehat{K}_v$ of E, A_i, C, S, and $\widehat{G}$ on the right-hand side given by (5.18); $\widehat{K}_v$ is a cube such that $v^*(t, x) \in \widehat{K}_v$, $\forall (t, x) \in [T_1, T_2] \times (\Omega_q^2 \cup \Gamma_q^2)$.

Take constant matrices $E^{0'}, A_i^{0'} (i = 1, \ldots, m), C^{0'}, S^{0'}$ satisfying (5.1)-(5.4), (5.6), (5.7), (5.10) and consider the differences

$$E^{0''} = E^0 - E^{0'}, \quad A_i^{0''} = A_i^0 - A_i^{0'}, \quad C^{0''} = C^0 - C^{0'}, \quad S^{0''} = S^0 - S^{0'}.$$

Applying Remark 2.1.1 extend $E^{0''}, A_i^{0''}, C^{0''}$ and $S^{0''}$ defined on $[T_1, T_2] \times (P_q \cup p_q) \cup \widehat{K}_v$ to functions $\widetilde{E}^{0''}, \widetilde{A}_i^{0''}, \widetilde{C}^{0''}, \widetilde{S}^{0''} \in C^k(\mathbf{R} \times \{y : y_1 \geq 0\} \times \mathbf{R}^n)$ such that

(5.25)
$$\|\widetilde{E}^{0''}\|_{\widetilde{X}} \leq c\|E^{0''}\|_X, \quad \|\widetilde{A}_i^{0''}\|_{\widetilde{X}} \leq c\|A_i^{0''}\|_X,$$

$$\|\widetilde{C}^{0''}\|_{\widetilde{X}} \leq c\|C^{0''}\|_X, \quad \|\widetilde{S}^{0''}\|_{\widetilde{X}} \leq c\|S^{0''}\|_X,$$

$$X = C_B^k([T_1, T_2] \times (P_q \cup p_q) \times \widehat{K}_v),$$

$$\widetilde{X} = C_B^k(\mathbf{R} \times \{y : y_1 \geq 0\} \times \mathbf{R}^n)$$

(where $c > 0$ is a constant) and $\widetilde{E}^{0''} = \widetilde{A}_i^{0''} = \widetilde{C}^{0''} = \widetilde{S}^{0''} = 0$ outside a compact set $K_{t,y,v} \subset \mathbf{R} \times \{y : y_1 \geq 0\} \times \mathbf{R}^n$. Then the functions

(5.26) $\quad \widetilde{E}^0 = E^{0'} + \widetilde{E}^{0''}, \quad \widetilde{A}_i^0 = A_i^{0'} + \widetilde{A}_i^{0''}, \quad \widetilde{C}^0 = C^{0'} + \widetilde{C}^{0''}, \quad \widetilde{S}^0 = S^{0'} + \widetilde{S}^{0''}$

are C^k extensions of E^0, A_i^0, C^0 and S^0 to $\mathbf{R} \times \{y : y_1 \geq 0\} \times \mathbf{R}^n$ such that $\widetilde{E}^0 = E^{0'}, \widetilde{A}_i^0 = A_i^{0'}, \widetilde{C}^0 = C^{0'}, \widetilde{S}^0 = S^{0'}$ outside $K_{t,y,v}$.

Applying Lemma 2.1.3 extend next $\widehat{u}(t, y) = u|_{[T_1,T_2] \times (\Omega_q^2 \cup \Gamma_q^2)}(t, \Psi_q(y)) \in C_{(0)}^\infty([T_1, T_2] \times (\Sigma_{R^2} \cup \sigma_{R_q^2}))$ to a function $\widetilde{u} \in C_{(0)}^\infty(\mathbf{R} \times \{y : y_1 \geq 0\})$, and $\widehat{v}^* \in \Pi_0^k([T_1, T_2] \times P_q) \cap H^{k+1/2}([T_1, T_2] \times P_q)$ to a function $\widetilde{v}^*$ such that

(5.27) $\quad |\widetilde{v}^*|_{k,0,\infty,\mathbf{R} \times \{y : y_1 > 0\}} + \|\widetilde{v}^*\|_{k+1/2, \mathbf{R} \times \{y : y_1 > 0\}}$

$$\leq c(|\widehat{v}^*|_{k,0,\infty,[T_1,T_2] \times P_q} + \|\widehat{v}^*\|_{k+1/2,[T_1,T_2] \times P_q})$$

and $\widetilde{v}^* = 0$ outside a compact set $\widehat{K}_{t,y} \subset \mathbf{R} \times \{y : y_1 \geq 0\}$.

Since $\widehat{u}_q \in C_{(0)}^\infty([T_1, T_2] \times (\Sigma_{R_q^2} \cup \sigma_{R_q^2}))$ and $\partial_{t,y}^j \widehat{u}_q = \partial_{t,y}^j \widehat{F}_q = \partial_{t,y}^j \widehat{g}_q = 0$ for $t \in (T_1, T_2) \setminus [T_1', T_2']$, $y \in \Sigma_{R_q^2} \setminus \Sigma_{R_q^1}$ and $j \in \mathbf{N} \cup \{0\}$, the zero extensions $\widetilde{\widehat{u}}_q \in C_{(0)}^\infty(\mathbf{R} \times \{y : y_1 \geq 0\})$, $\widetilde{\widehat{F}}_q$ and $\widetilde{\widehat{g}}_q$ of $\widehat{u}_q$, $\widehat{F}_q$ and $\widehat{g}_q$ satisfy

$$(5.28) \qquad \widetilde{\widehat{L}}\widetilde{\widehat{u}}_q = \widetilde{\widehat{F}}_q + [\widetilde{\widehat{L}}, \widetilde{\widehat{\psi}}_q]\widetilde{\widehat{u}} \qquad \text{in } \mathbf{R} \times \{y : y_1 \geq 0\},$$

$$(5.29) \qquad \widetilde{\widehat{L}}_B \widetilde{\widehat{u}}_q = \widetilde{\widehat{g}}_q \qquad \text{on } \mathbf{R} \times \{y : y_1 = 0\},$$

where

$$\widetilde{\widehat{L}} = \widetilde{E}^0(\widetilde{\widehat{v}}^*)\partial_t - \sum_{j=1}^m \widetilde{A}_j^0(\widetilde{\widehat{v}}^*)\partial_{y_i} - \widetilde{C}^0(\widetilde{\widehat{v}}^*),$$

$$\widetilde{\widehat{L}}_B = [I, -\widetilde{S}^0(\widetilde{\widehat{v}}^*)],$$

$\widetilde{\widehat{\psi}}_q$ is the zero extension of $\widehat{\psi}_q$ to $\mathbf{R} \times \{y : y_1 \geq 0\}$. From assumptions (5.2)–(5.4), (5.6), (5.7) and (5.10) it follows that the functions $\widetilde{E}^0(t, y, v)$, $\widetilde{A}_i^0(t, y, v)$, $\widetilde{C}^0(t, y, v)$ and $\widetilde{S}^0(t, y, v)$ satisfy (4.1.30), (3.1.3), (3.1.4), (3.1.14), (3.4.13) for $(t, y, v) \in [T_1, T_2] \times (P_q \cup p_q) \times \widehat{K}_v$ and (3.1.20) for $(t, y', v) \in [T_1, T_2] \times p_q \times \widehat{K}_v$. Hence repeating the proof of Theorem 3.2.1 we find that there exists a matrix-valued function $R = R(t, y, v, \eta, \zeta)$ satisfying assertions $1°$–$3°$ of that theorem, $4°$ for $(t, y, v, \eta, \zeta) \in [T_1, T_2] \times (P_q \cup p_q) \times \widehat{K}_v \times [\eta_1, \infty) \times \mathbf{R}^m$ with M replaced by

$$\widetilde{M}^0(t, y, v, \eta, \zeta) = \widetilde{A}_1^{0-1}(t, y, v)\Big((\eta + i\xi)\widetilde{E}^0(t, y, v) - i\sum_{j=2}^m \widetilde{A}_j^0(t, y, v)\omega_j\Big),$$

and $5°$ for $(t, y', v, \eta, \zeta) \in [T_1, T_2] \times p_q \times \widehat{K}_v \times [\eta_1, \infty) \times \mathbf{R}^m$. By Theorem 3.2.1, R has the form (3.2.61) with M' and M'' replaced by

$$M^{0\prime}(\eta, \zeta) = A_1^{0\prime-1}\Big((\eta + i\xi)E^{0\prime} - i\sum_{j=2}^m A_j^{0\prime}\omega_j\Big),$$

$$\widetilde{M}^{0\prime\prime}(t, y, v, \eta, \zeta) = \widetilde{M}^0(t, y, v, \eta, \zeta) - M^{0\prime}(\eta, \zeta).$$

Set now $z = (t, y')$ and take $G = \widetilde{E}^0, \widetilde{A}_j^0, \widetilde{C}^0, \widetilde{S}^0$ in (3.3.2), and $J = \widetilde{M}^0, R, \Psi_1, \Psi_2$ in (3.3.3). Moreover, in (3.3.4) we replace E, A_j, C, S and M by $\widetilde{E}^0$, $\widetilde{A}_j^0$, etc., and E', A_j', C', S', M', E'', A_j'', C'', S'' and M'' by $E^{0\prime}$, $A_j^{0\prime}$, etc.

Additionally, define

$$(5.30) \qquad \widetilde{L}^{0\prime\prime} = (\widetilde{E}^{0\prime\prime}, (\widetilde{A}_j^{0\prime\prime})_{j=1}^m, \widetilde{C}^{0\prime\prime}), \quad \widetilde{L}^0 = (\widetilde{E}^0, (\widetilde{A}_j^0)_{j=1}^m, \widetilde{C}^0).$$

$\widetilde{L}^{0\prime\prime}$, $\widetilde{S}^{0\prime\prime}$, $\widetilde{M}^{0\prime\prime}$ and R'' vanish outside a compact set $K_{t,y} \subset \mathbf{R} \times \{y : y_1 \geq 0\}$.

Repeating the argument from the proof of Theorem 3.3.1 and taking into account that $\widehat{\widetilde{u}}_q = 0$ outside $[T_1, T_2] \times (P_q \cup p_q)$ we obtain the estimate

$$\eta \|\widehat{\widetilde{u}}_q\|^2_{0,\eta,\mathbf{R}\times\{y:y_1>0\}} + \|\widehat{\widetilde{u}}_q\|^2_{0,\eta,\mathbf{R}\times\{y:y_1=0\}}$$

$$\leq \widetilde{f}_0(\varepsilon^0_{q1}, \varepsilon^0_{q2}, \varepsilon^0_{q3})[\overline{p}_0(\overline{a}^0, \overline{b}^0, \overline{c}, \overline{d}, \overline{e}^0)\eta^{-1}(\|\widehat{\widetilde{F}}_q\|^2_{0,\eta,\mathbf{R}\times\{y:y_1>0\}}$$

$$+ \|[\widetilde{L}, \widehat{\widetilde{\psi}}_q]\widehat{\widetilde{u}}\|^2_{0,\eta,\mathbf{R}\times\{y:y_1>0\}}) + \overline{q}_0(\overline{a}^0, \overline{b}^0, \overline{c}, \overline{d}, \overline{e}^0)\|\widehat{\widetilde{g}}_q\|^2_{0,\eta,\mathbf{R}\times\{y:y_1=0\}}],$$

where

$$\overline{a}^0 = |\widetilde{L}^{0\prime\prime}|_{k,0,\infty,\mathbf{R}\times\{y:y_1>0\}},$$

$$\overline{b}^0 = \|\widetilde{L}^0\|_{C^k_B(\mathbf{R}\times\{y:y_1\geq 0\}\times\mathbf{R}^n)} + \|\widetilde{S}^0\|_{C^k_B(\mathbf{R}\times\{y:y_1\geq 0\}\times\mathbf{R}^n)}$$

$$(\widetilde{L}^0 = (\widetilde{E}^0, (\widetilde{A}^0_i)^m_{i=1}, \widetilde{C}^0)),$$

$$\overline{e}^0 = |\widetilde{S}^{0\prime\prime}|_{k,0,\infty,\mathbf{R}\times\{y:y_1\geq 0\}};$$

$\overline{c}$ and $\overline{d}$ are given by (3.3.13); $\overline{p}_0$ and $\overline{q}_0$ are polynomials; ε^0_{q1}, ε^0_{q2} and ε^0_{q3} are the constants from Theorem 3.2.1(4°, 5°); and $\widetilde{f}_0$ is a continuous function.

Repeating the proof of [28, Lemma 3.2] for η satisfying an inequality analogous to (3.5.43) and for $0 \leq s \leq k$ we get

$$\eta \|\widehat{\widetilde{u}}_q\|^2_{s,\eta,\mathbf{R}\times\{y:y_1>0\}} + \|\widehat{\widetilde{u}}_q\|^2_{s,\eta,\mathbf{R}\times\{y:y_1=0\}}$$

$$\leq \widetilde{f}_s(\varepsilon^0_{q1}, \varepsilon^0_{q2}, \varepsilon^0_{q3})[\overline{p}_s(\overline{a}^0, \overline{b}^0, \overline{c}, \overline{d}, \overline{e}^0, \overline{f}^0)\eta^{-1}(\|\widehat{\widetilde{F}}_q\|^2_{s,\eta,\mathbf{R}\times\{y:y_1>0\}}$$

$$+ \|[\widetilde{L}, \widehat{\widetilde{\psi}}_q]\widehat{\widetilde{u}}\|^2_{s,\eta,\mathbf{R}\times\{y:y_1>0\}}) + \overline{q}_s(\overline{a}^0, \overline{b}^0, \overline{c}, \overline{d}, \overline{e}^0, \overline{f}^0)\|\widehat{\widetilde{g}}_q\|^2_{s,\eta,\mathbf{R}\times\{y:y_1=0\}}],$$

where

$$\overline{f}^0 = \|\widetilde{S}^{0\prime\prime}\|_{k,\eta,\mathbf{R}\times\{y:y_1=0\}},$$

and $\overline{p}_s$, $\overline{q}_s$ have the same form as p_s, q_s in Theorem 3.5.2.

Using now (5.11), (5.27), (5.25) and [1, Theorem 3.35] we obtain

$$(5.31) \quad \eta \|u_q\|^2_{s,\eta,\mathbf{R}\times\Omega} + \|u_q\|^2_{s,\eta,\mathbf{R}\times\partial\Omega} \leq \widetilde{f}_s(\varepsilon^0_{q1}, \varepsilon^0_{q2}, \varepsilon^0_{q3})$$

$$\times [\overline{\overline{p}}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T}, \overline{b})\eta^{-1}(\|F_q\|^2_{s,\eta,\mathbf{R}\times\Omega} + \|[L,\psi_q]u_q\|^2_{s,\eta,\mathbf{R}\times\Omega})$$

$$+ \overline{\overline{q}}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T}, \overline{b})\|g_q\|^2_{s,\eta,\mathbf{R}\times\partial\Omega}],$$

where $1 \leq q \leq \mu$ ($\overline{\overline{p}}_s$ and $\overline{\overline{q}}_s$ have analogous forms to p_s and q_s from Theorem 3.5.2).

The analogous estimate can be derived for $\mu + 1 \leq q \leq \mu + \nu$. Then $K_{R^i_q} \cap \Omega = \emptyset$ ($i = 1, 2$) and instead of (5.22)–(5.23) we consider the Cauchy problem using the results of Section 3.6.

Now, since $u = \sum_{q=1}^{\mu+\nu} u_q$, (5.31) gives

(5.32)
$$\eta\|u\|_{s,\eta,\mathbb{R}\times\Omega}^2 + \|u\|_{s,\eta,\mathbb{R}\times\partial\Omega}^2$$

$$\leq \widetilde{p}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T}, \overline{b})\eta^{-1}$$

$$\times \left(\|F\|_{s,\eta,\mathbb{R}\times\Omega}^2 + \sum_{q=1}^{\mu+\nu} \|[L,\psi_q]u\|_{s,\eta,\mathbb{R}\times\Omega}^2 \right)$$

$$+ \widetilde{q}_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T}, \overline{b})\|g\|_{s,\eta,\mathbb{R}\times\partial\Omega}^2 ,$$

where $\widetilde{p}_s$ and $\widetilde{q}_s$ depend on $\sum_{q=1}^{\mu+\nu} \widetilde{f}_s(\varepsilon_{q1}^0, \varepsilon_{q2}^0, \varepsilon_{q3}^0)$.

We now prove that for any $1 \leq q \leq \mu + \nu$,

(5.33) $\forall 0 \leq s \leq k$ $\|[L,\psi_q]u\|_{s,\eta,\mathbb{R}\times\Omega} \leq p_s^1(|v|_{k,0,\infty,\Omega^T}, \overline{b})\|u\|_{s,\eta,\mathbb{R}\times\Omega}$,

where the p_s^1 are polynomials.

Consider first the case when $s = 0$. Applying Lemma 2.1.2 and (5.11) we have

$$\|[L,\psi_q]u\|_{0,\eta,\mathbb{R}\times\Omega} = \left\| \left(E\partial_t\psi_q - \sum_{j=1}^m A_j\partial_{x_j}\psi_q \right)u \right\|_{0,\eta,\mathbb{R}\times\Omega}$$

$$\leq p_0^*(|v|_{k,0,\infty,\Omega^T}, \overline{b})\|u\|_{0,\eta,\mathbb{R}\times\Omega} .$$

Assume now that (5.33) holds with $s = i - 1$, where $0 < i \leq k$. Then

$$\partial_{t,x}^i([L,\psi_q]u) = \partial_{t,x}^i\left[\left(E\partial_t\psi_q - \sum_{j=1}^m A_j\partial_{x_j}\psi_q \right)u \right]$$

$$= \sum_{\mu=0}^i \binom{i}{\mu} \partial_{t,x}^{i-\mu}E\partial_{t,x}^\mu(\partial_t\psi_q u)$$

$$- \sum_{j=1}^m \sum_{\mu=0}^i \binom{i}{\mu} \partial_{t,x}^{i-\mu}A_j\partial_{t,x}^\mu(\partial_{x_j}\psi_q u) .$$

Hence

$$\|\partial_{t,x}^i([L,\psi_q]u)\|_{0,\eta,\mathbb{R}\times\Omega}^2$$

$$\leq c\Big[\sum_{\mu=0}^i \int_{\mathbb{R}\times\Omega} |\partial_{t,x}^{i-\mu}E''|^2 |\partial_{t,x}^\mu(\partial_t\psi_q u)|^2 e^{-2\eta t}\, dx\, dt$$

$$+ \sum_{j=1}^m \sum_{\mu=0}^i \int_{\mathbb{R}\times\Omega} |\partial_{t,x}^{i-\mu}A_j''|^2 |\partial_{t,x}^\mu(\partial_{x_j}\psi_q u)|^2 e^{-2\eta t}\, dx\, dt\Big]$$

$$\leq c\Big[\sum_{\mu=0}^i \int_{\mathbb{R}} \left(\|\partial_{t,x}^{i-\mu}E''\|_{0,2p,\Omega}^2 \|\partial_{t,x}^\mu(\partial_t\psi_q u)\|_{0,2p',\Omega}^2 \right.$$

$$+ \sum_{j=1}^{m} \|\partial_{t,x}^{i-\mu} A_j''\|_{0,2p,\Omega}^2 \|\partial_{t,x}^{\mu}(\partial_{x_j} \psi_q u)\|_{0,2p',\Omega}^2 \Big) e^{-2\eta t} \, dt \Big],$$

where $1/p + 1/p' = 1$.

Take p and p' such that

$$\frac{1}{2p'} \geq \frac{m - 2(i - \mu)}{2m} \quad \text{and} \quad \frac{1}{2p} \geq \frac{m - 2(k - i + \mu)}{2m},$$

$\mu = 0, \ldots, i, 0 \leq i \leq k$. (This can be done since $\frac{m-2(i-\mu)}{2m} + \frac{m-2(k-i+\mu)}{2m} < 1$.)
By the Sobolev lemma and assumption (5.5) we have

$$\|\partial_{t,x}^{i-\mu} E''\|_{0,2p,\Omega}^2 \leq c \|\partial_{t,x}^{i-\mu} E''\|_{k-i+\mu,\Omega}^2 \leq c |E''|_{k,0,\infty,\mathbf{R}\times\Omega}^2,$$

$$\|\partial_{t,x}^{i-\mu} A_j''\|_{0,2p,\Omega}^2 \leq c \|\partial_{t,x}^{i-\mu} A_j''\|_{k-i+\mu,\Omega}^2 \leq c |A_j''|_{k,0,\infty,\mathbf{R}\times\Omega}^2,$$

$$\|\partial_{t,x}^{\mu}(\partial_t \psi_q u)\|_{0,2p',\Omega}^2 \leq c \|\partial_{t,x}^{\mu}(\partial_t \psi_q u)\|_{i-\mu,\Omega}^2,$$

$$\|\partial_{t,x}^{\mu}(\partial_{x_j} \psi_q u)\|_{0,2p',\Omega}^2 \leq c \|\partial_{t,x}^{\mu}(\partial_{x_j} \psi_q u)\|_{i-\mu,\Omega}^2.$$

Using the above inequalities we get

$$\|\partial_{t,x}^{i}([L, \psi_q] u)\|_{0,\eta,\mathbf{R}\times\Omega}^2 \leq p_i^*(|v|_{k,0,\infty,\Omega^T}, \overline{b}) \|u\|_{i,\eta,\mathbf{R}\times\Omega}.$$

Together with the inductive assumption, this yields (5.33) with $s = i$.

Using now (5.33) in (5.32) and assuming that η satisfies (5.15) we obtain (5.14), where p_s and q_s depend on $\sum_{q=1}^{\mu+\nu} \tilde{f}_s(\varepsilon_{q1}^0, \varepsilon_{q2}^0, \varepsilon_{q3}^0)$. ■

THEOREM 5.2. *Let assumptions* (5.1)–(5.4), (5.6), (5.7) *and* (5.10) *be satisfied. Moreover, assume*

(5.34) $v \in C_{(0)}^{\infty}(\mathbf{R} \times \overline{\Omega})$,

(5.35) $E, A_j \, (j = 1, \ldots, m), C, S \in C_B^{\infty}(\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n)$,

(5.36) $F \in L_\eta^2(\mathbf{R} \times \Omega), \quad g \in L_\eta^2(\mathbf{R} \times \partial\Omega)$.

Then problem (5.12)–(5.13) *has a unique strong solution* $u \in L_\eta^2(\mathbf{R} \times \Omega)$ *(see Definition 3.5.1) for which the estimate* (5.14) *holds with* $s = 0$.

Proof. The proof is the same as that of Theorem 3.5.1 ([5, Theorem 1] is also true for a bounded domain). ■

THEOREM 5.3. *Let assumptions* (5.1)–(5.4), (5.6), (5.7), (5.10), (5.34), (5.35) *be fulfilled. Furthermore; assume that*

(5.37) $F \in C_{(0)}^{\infty}(\mathbf{R} \times \Omega) \quad and \quad F = 0 \; for \; t < 0$,

            $g \in C_{(0)}^{\infty}(\mathbf{R} \times \partial\Omega) \quad and \quad g = 0 \; for \; t < 0$.

Then problem (5.12)–(5.13) *has a unique solution* $u \in C_{(0)}^{\infty}(\mathbf{R} \times \overline{\Omega})$ *for which the estimate* (5.14) *is true.*

Proof. The proof is the same as that of Theorem 3.5.2 ([24, Theorems 1 and 3] are also true for a bounded domain). ∎

Remark 5.1. Let $F \in C^\infty_{(0)}(\mathbf{R} \times \Omega)$, $g \in C^\infty_{(0)}(\mathbf{R} \times \partial\Omega)$ and let $J \subset \mathbf{R}$ be an interval. Using the same argument as in Remarks 3.5.1 and 3.5.2 we can prove that for η satisfying (5.15) the estimates (3.5.37) and (3.5.45) hold.

Now we shall consider problem (4.1.1)–(4.1.3).

THEOREM 5.4. *Let assumptions* (5.1)–(5.4), (5.6), (5.7), (5.10), (5.34), (5.35) *be fulfilled. Moreover, let*

$$(5.38) \qquad F \in H^s(\Omega^T), \quad g \in H^s(\partial\Omega^T), \quad f \in \dot{H}^s(\Omega),$$

where $0 \le s \le k$, $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$,

$$(5.39) \qquad \forall 0 \le j \le s - 1 \quad \partial_t^j F|_{t=0} = \partial_t^j g|_{t=0} = 0.$$

Then there exists a unique strong solution (in the sense of Definition 4.1.1) $u \in \Pi_0^s(\Omega^T) \cap H^s(\partial\Omega^T)$ *of* (4.1.1)–(4.1.3) *satisfying for* $0 \le t \le T$ *the estimate*

$$(5.40) \qquad |u(t)|^2_{s,0,\Omega}\, e^{-2\eta t} + \eta\|u\|^2_{s,\eta,\Omega^t} + \|u\|^2_{s,\eta,\partial\Omega^t}$$
$$\le p_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T}, \overline{b})\|F\|^2_{s,\eta,\Omega^t}$$
$$+ q_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T}, \overline{b})\|g\|^2_{s,\eta,\partial\Omega^t}$$
$$+ r_s(|v|_{k,0,\infty,\Omega^T}, \|v\|_{k,\partial\Omega^T}, \overline{b})\|f\|^2_{s,\eta,\Omega},$$

where η *satisfies an inequality of type* (5.15).

Proof. Assume first that $F \in C^\infty_{(0)}(\mathbf{R} \times \overline{\Omega})$ and $F = 0$ for $t < 0$, $g \in C^\infty_{(0)}(\mathbf{R} \times \partial\Omega)$ and $g = 0$ for $t < 0$ and $f \in C^\infty_0(\Omega)$. Then, as in Theorem 4.1.1, we can prove that there exists a unique solution $u \in C^\infty_{(0)}([0,T] \times \overline{\Omega})$ of (4.1.1)–(4.1.3).

We now prove that u satisfies (5.40). To do this take the same coverings $\{K_{R_q^1}\}$ and $\{K_{R_q^2}\}$ $(q = 1,\ldots,\mu+\nu)$ of $\overline{\Omega}$ as in Theorem 5.1 and the same partition of unity $\{\psi_q\}$. Since u is the solution of (4.1.1)–(4.1.3), u_q is the solution of the problem

$$(5.41) \qquad Lu_q = F_q + [L,\psi_q]u \quad \text{in } \Omega_q^T = [0,T] \times \Omega_q^2,$$

$$(5.42) \qquad L_B u_q = g_q \qquad \text{on } \Gamma_q^T = [0,T] \times \Gamma_q^2,$$

$$(5.43) \qquad u_q(0,\cdot) = f_q(\cdot) \qquad \text{in } \Omega_q^1,$$

where

$$L = E(v)\partial_t - \sum_{j=1}^m A_j(v)\partial_{x_j} - C(v), \quad L_B = [I, -S(v)].$$

and where we use notation (5.20). As in Theorem 5.1, applying the coordinates given by (5.17) we replace (5.41)–(5.43) by the problem

(5.44) $\widehat{L}\widehat{u}_q = \widehat{F}_q + [\widehat{L}, \widehat{\psi}_q]\widehat{u}$ in $[0,T] \times P_q$,

(5.45) $\widehat{L}_B\widehat{u}_q = \widehat{g}_q$ on $[0,T] \times p_q$,

(5.46) $\widehat{u}_q(0, \cdot) = \widehat{f}_q(\cdot)$ in P_q ,

where

$$\widehat{L} = \widehat{E}(\widehat{v})\partial_t - \sum_{i,j=1}^{m} \widehat{A}_j(\widehat{v})\partial_{x_j}v_i\partial_{y_i} - \widehat{C}(\widehat{v})$$

$$= E^0(\widehat{v})\partial_t - \sum_{i=1}^{m} A_i^0(\widehat{v})\partial_{y_i} - C^0(\widehat{v}) ,$$

$$\widehat{L}_B = [I, -\widehat{S}(\widehat{v})] = [I, -S^0(\widehat{v})] ,$$

$$\widehat{h}(t,y) = h|_{[0,T]\times(\Omega_q^2\cup\Gamma_q^2)}(t, \Psi_q(y))$$

for $h = v, F_q, g_q, \psi_q, f_q, u_q, u$,

$$\widehat{G}(\widehat{v}) = \widehat{G}(t, y, \widehat{v}(t,y))$$

for G being the restriction to $\mathbf{R} \times (\Omega_q^2 \cup \Gamma_q^2) \times \mathbf{R}^n$ of E, A_i, C, S, and $\widehat{G}$ on the right-hand side given by (5.18).

Let now $\widetilde{E}^0$, $\widetilde{A}_i^0$, $\widetilde{C}^0$ and $\widetilde{S}^0$ be C^k extensions of E^0, A_i^0, C^0 and S^0 from $[0,T] \times (P_q \cup p_q) \times \widehat{K}_v$ ($\widehat{K}_v$ is a cube such that $v(t,x) \in \widehat{K}_v$, $\forall (t,x) \in [0,T] \times (\Omega_q^2 \cup \Gamma_q^2)$) to $\mathbf{R} \times \{y : y_1 \geq 0\} \times \mathbf{R}^n$ such that $\widetilde{E}^0(t,y,v) = E^{0\prime}$, $\widetilde{A}_i^0(t,y,v) = A_i^{0\prime}$, $\widetilde{C}^0(t,y,v) = C^{0\prime}$, $\widetilde{S}^0(t,y,v) = S^{0\prime}$ outside a compact set $K_{t,y,v} \subset \mathbf{R} \times \{y : y_1 \geq 0\} \times \mathbf{R}^n$ ($E^{0\prime}$, $A_i^{0\prime}$, $C^{0\prime}$ and $S^{0\prime}$ are constant matrices satisfying (5.1)–(5.4), (5.6), (5.7), (5.10)) and

(5.47)
$$\|\widetilde{E}^{0\prime\prime}\|_{\widetilde{X}} \leq c\|E^{0\prime\prime}\|_X , \quad \|\widetilde{A}_i^{0\prime\prime}\|_{\widetilde{X}} \leq c\|A_i^{0\prime\prime}\|_X ,$$
$$\|\widetilde{C}^{0\prime\prime}\|_{\widetilde{X}} \leq c\|C^{0\prime\prime}\|_X , \quad \|\widetilde{S}^{0\prime\prime}\|_{\widetilde{X}} \leq c\|S^{0\prime\prime}\|_X ,$$
$$X = C_B^k([0,T] \times (P_q \cup p_q) \times \widehat{K}_v) ,$$
$$\widetilde{X} = C_B^k(\mathbf{R} \times \{y : y_1 \geq 0\} \times \mathbf{R}^n) ,$$

where $\widetilde{E}^{0\prime\prime} = \widetilde{E}^0 - E^{0\prime}$, $\widetilde{A}_i^{0\prime\prime} = \widetilde{A}_i^0 - A_i^{0\prime}$, $\widetilde{C}^{0\prime\prime} = \widetilde{C}^0 - C^{0\prime}$, $\widetilde{S}^{0\prime\prime} = \widetilde{S}^0 - S^{0\prime}$, and $E^{0\prime\prime}$, $A_i^{0\prime\prime}$, $C^{0\prime\prime}$ and $S^{0\prime\prime}$, are given by (5.26).

Next, extend $\widehat{u} \in C_{(0)}^{\infty}([0,T] \times (\Sigma_{R_q^2} \cup \sigma_{R_q^2}))$ to $\widetilde{u} \in C_{(0)}^{\infty}([0,T] \times \{y : y_1 \geq 0\})$, and $\widehat{v} \in C_{(0)}^{\infty}(\mathbf{R} \times (\Sigma_{R_q^2} \cup \sigma_{R_q^2}))$ from $[0,T] \times (P_q \cup p_q)$ to a function $\widetilde{v} \in \Pi_0^k([0,T] \times \{y : y_1 > 0\}) \cap H^k([0,T] \times \{y : y_1 > 0\})$ satisfying

(5.48) $|\widetilde{v}|_{k,0,\infty,[0,T]\times\{y:y_1>0\}} + \|\widetilde{v}\|_{k,[0,T]\times\{y:y_1=0\}}$
$$\leq c(|\widehat{v}|_{k,0,\infty,[0,T]\times P_q} + \|\widehat{v}\|_{k,[0,T]\times p_q}) .$$

Let further $\widetilde{\widehat{u}}_q \in C^\infty_{(0)}([0,T] \times \{y : y_1 \geq 0\})$, $\widetilde{\widehat{F}}_q$ and $\widetilde{\widehat{g}}_q$ be extensions of $\widehat{u}_q$, $\widehat{F}_q$ and $\widehat{g}_q$ such that $\widehat{F}_q = \widehat{f}_q = 0$ outside $\Sigma_{R^1_q}$ and $\widehat{g}_q = 0$ outside $\sigma_{R^1_q}$. Then (5.44)–(5.46) yield

$$(5.49) \qquad \widetilde{L}\widetilde{\widehat{u}}_q = \widetilde{\widehat{F}}_q + [\widetilde{L}, \widetilde{\widehat{\psi}}_q]\widetilde{\widehat{u}} \quad \text{in } [0,T] \times \{y \in \mathbf{R}^m : y_1 > 0\}\,,$$

$$(5.50) \qquad \widetilde{L}_B\widetilde{\widehat{u}}_q = \widetilde{\widehat{g}}_q \qquad\qquad \text{on } [0,T] \times \{y \in \mathbf{R}^m : y_1 = 0\}\,,$$

$$(5.51) \qquad \widetilde{\widehat{u}}_q(0,\cdot) = \widetilde{\widehat{f}}_q(\cdot) \qquad \text{in } \{y \in \mathbf{R}^m : y_1 > 0\}\,,$$

where

$$\widetilde{L} = \widetilde{E}^0(\widetilde{v})\partial_t - \sum_{i=1}^m \widetilde{A}^0_i(\widetilde{v})\partial_{y_i} - \widetilde{C}^0(\widetilde{v})\,,$$

$$\widetilde{L}_B = [I, -\widetilde{S}^0(\widetilde{v})]\,.$$

Repeating the argument from Lemmas 4.1.2–4.1.7 and Theorem 4.1.2 and applying Theorem 5.1 we obtain for $0 \leq t \leq T$ the estimate

$$(5.52) \quad |\widetilde{\widehat{u}}_q(t)|^2_{s,0,\{y:y_1>0\}}e^{-2\eta t} + \eta\|\widetilde{\widehat{u}}_q\|^2_{s,\eta,[0,t]\times\{y:y_1>0\}} + \|\widetilde{\widehat{u}}_q\|^2_{s,\eta,[0,t]\times\{y:y_1=0\}}$$

$$\leq \widetilde{p}_s(|\widetilde{\widehat{v}}|_{k,0,\infty,[0,T]\times\{y:y_1>0\}}, \|\widetilde{\widehat{v}}\|_{k,[0,T]\times\{y:y_1=0\}})$$

$$\times (\|\widetilde{\widehat{F}}_q\|^2_{s,\eta,[0,t]\times\{y:y_1>0\}} + \|[\widetilde{L}, \widetilde{\widehat{\psi}}_q]\widetilde{\widehat{u}}\|^2_{s,\eta,[0,t]\times\{y:y_1>0\}})$$

$$+ \widetilde{q}_s(|\widetilde{\widehat{v}}|_{k,0,\infty,[0,T]\times\{y:y_1>0\}}, \|\widetilde{\widehat{v}}\|_{k,[0,T]\times\{y:y_1=0\}})\|\widetilde{\widehat{g}}_q\|^2_{s,\eta,[0,t]\times\{y:y_1=0\}}$$

$$+ \widetilde{r}_s(|\widetilde{\widehat{v}}|_{k,0,\infty,[0,T]\times\{y:y_1>0\}}, \|\widetilde{\widehat{v}}\|_{k,[0,T]\times\{y:y_1=0\}})\|\widetilde{\widehat{f}}_q\|^2_{s,\{y:y_1>0\}}\,,$$

where $\widetilde{p}_s$, $\widetilde{q}_s$ and $\widetilde{r}_s$ depend on $\widetilde{f}_s((\varepsilon^0_{qi})^3_{i=1})$, $(\varepsilon^{0*}_{qi})^3_{i=1})$ $((\varepsilon^0_{qi})^3_{i=1}$ and $(\varepsilon^{0*}_{qi})^3_{i=1}$ are the constants from Theorems 3.2.1 and 3.4.1 respectively, $\widetilde{f}_s$ are continuous functions). Using the same argument as in the proof of Theorem 5.1 and using (5.48) we get (5.40) with p_s, q_s and r_s depending on $\sum_{q=1}^{\mu+\nu} \widetilde{f}_s((\varepsilon^0_{qi})^3_{i=1}$, $(\varepsilon^{0*}_{qi})^3_{i=1})$.

Take next $F \in H^s(\Omega^T)$, $g \in H^s(\partial\Omega^T)$ and $f \in \mathring{H}^s(\Omega)$. Repeating the argument from the proof of Theorem 4.1.2 we get the assertion with p_s, q_s and r_s and the polynomial p from (5.15) depending on the above sum. ∎

THEOREM 5.5. *Let assumptions* (5.1)–(5.4), (5.5) *with* $k > \max(m/2 + 5/2, (m+1)/2 + 2n - 1)$, (5.6), (5.7), (5.10) *and* (5.38), (5.39) *with* $s = k$ *be fulfilled. Moreover, assume that*

$$(5.53) \qquad v \in \Pi^k_0(\Omega^T) \cap H^k(\partial\Omega^T);$$

(5.54) *there exist sequences of* $n \times n$-*matrix-valued functions* E_μ, $A_{j\mu}$ $(j = 1,\ldots,m)$, C_μ, $S_\mu \in C^\infty(\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n)$ *such that:*

(a) E_μ, $A_{j\mu}$ $(j = 1, \ldots, m)$, C_μ, S_μ *satisfy for any μ conditions* (5.1)–
(5.7), (5.10) *and* $E_\mu, A_{j\mu}, C_\mu, S_\mu \to E, A_j, C, S$ *in* $C_B^k(\mathbf{R} \times \overline{\Omega} \times \mathbf{R}^n)$
as $\mu \to 0$;

(b) *there exists $\varepsilon > 0$ such that for each* $z_0 = (t_0, x_0, v_0, \eta_0, \zeta_0) \in$
$\mathbf{R} \times \overline{\Omega}_\varepsilon \times \mathbf{R}^n \times [0, \infty) \times \mathbf{R}^n$ *with* $|\eta_0| + |\zeta_0| \neq 0$ *(where $\Omega_\varepsilon =$*
$\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$) the function

$$m_{z_0}(\mu) = \begin{cases} M_\nu(z_0) & \text{for } \mu = 0, \\ M_{\nu\mu}(z_0) & \text{for } 0 < \mu \leq 1, \end{cases}$$

where

$$M_{\nu\mu}(z_0) = A_{\nu\mu}^{-1}(t_0, x_0, v_0)\Big((\eta_0 + i\xi_0)E_\mu(t_0, x_0, v_0)$$

$$- i\sum_{j=2}^m A_{j\mu}(t_0, x_0, v_0)\omega_{0j}\Big),$$

*is continuous in an interval $[0, \mu_0]$ $(\mu_0 \leq 1)$ and the multiplicities
of its eigenvalues are constant in $[0, \mu_0]$.*

*Then there exists a unique solution $u \in \Pi_0^k(\Omega^T) \cap H^k(\partial\Omega^T)$ of (4.1.1)–
(4.1.3) satisfying the estimate (5.40) for $0 \leq t \leq T$ and $0 \leq s \leq k$, with η
fulfilling (5.15).*

P r o o f. The proof is analogous to that of Theorem 4.1.3. Notice only
that we can choose extensions $\widetilde{E}_\mu^0$, $\widetilde{A}_{i\mu}^0$, $\widetilde{C}_\mu^0$ and $\widetilde{S}_\mu^0$ of E_μ^0, $A_{i\mu}^0$, C_μ^0 and
S_μ^0 in such a way that $\sum_q \widetilde{f}_s((\varepsilon_{qi\mu}^0)_{i=1}^3, (\varepsilon_{qi\mu}^{0*})_{i=1}^3)$ can be estimated by a
constant independent of μ. These extensions can be chosen for example
in the form $\widetilde{G}_\mu^0 = G_\mu^0\phi_1 + G^{0\prime}\phi_2$, where $G = E, A_i, C, S$; $\phi_1 + \phi_2 = 1$ in
$[0, T] \times \Sigma_{R_q^2} \times \widehat{K}_v$, ϕ_1 and ϕ_2 are of class C^∞, $\mathrm{supp}\,\phi_1 \subset (T_1, T_2) \times \Sigma_{R_q^2} \times U_v$,
$\mathrm{supp}\,\phi_2 \subset (\mathbf{R} \setminus [0, T]) \times (\{y : y_1 \geq 0\} \setminus \Sigma_{R_q^1}) \times (\mathbf{R}^n \setminus \widehat{K}_v)$ $((T_1, T_2)$ is an interval
such that $[0, T] \subset (T_1, T_2)$, and U_v is an open set such that $\widehat{K}_v \subset U_v$ and
$\mathrm{dist}(\widehat{K}_v, \partial U_v) = \delta$ for some constant $\delta > 0$). ∎

R e m a r k 5.2. If $f = 0$, then in Theorems 5.1, 5.4 and 5.5 it suffices to
assume that $k > m/2 + 5/2$, and in Theorem 5.5 we can replace (5.53) by
$v \in H^k(\partial\Omega^T)$. Furthermore, in Theorem 5.5 we can replace (5.2) by weaker
assumptions, namely by (3.1.2) and by the assumption that the matrices

$$A(t, x, v, \omega) = iE^{-1}(t, x, v)\sum_{j=1}^m A_j(t, x, v)\omega_j,$$

$$M_\nu(t, x, v, \eta, \zeta) = A_\nu^{-1}(t, x, v)\Big((\eta + i\xi)E(t, x, v)$$

$$- i\sum_{j=2}^m A_j(t, x, v)\omega_j\Big)$$

satisfy (3.1.5) and (3.1.24) respectively and that the approximating matrices from Theorem 5.5 (i.e. $A_\mu(t,x,v,\omega) = i\sum_{j=1}^{m} A_{j\mu}(t,x,v)\omega_j$ and $M_{\nu\mu}(t,x,v,\eta,\zeta)$) also fulfil (3.1.5) and (3.1.24). The neighbourhood θ_{δ,z_0} from (3.1.24) is here defined as $\theta_{\delta,z_0} = \{z \in \mathbf{R} \times \bar{\Omega}_\epsilon \times \mathbf{R}^n \times [0,\infty) \times \mathbf{R}^m : |z - z_0| < \delta\}$.

R e m a r k 5.3. If $E, A_j, C, S \in C_B^\infty(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^n)$, then assumption (5.54) is automatically satisfied. Again, we know no other example.

6. A mixed problem for a nonlinear system of hyperbolic equations of the first order

Let $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$ or let $\Omega \subset \mathbf{R}^m$ be a bounded domain with C^∞ boundary. Consider the nonlinear problem

$$(6.1) \qquad E(u)u_t - \sum_{j=1}^{m} A_j(u)u_{x_j} - C(u)u = F \quad \text{in } \Omega^T = [0,T] \times \Omega,$$

$$(6.2) \qquad\qquad [I, -S(u)]u = g \qquad\qquad \text{on } \partial\Omega^T = [0,T] \times \partial\Omega,$$

$$(6.3) \qquad\qquad u(0,\cdot) = f(\cdot) \qquad\qquad \text{in } \Omega,$$

where $T > 0$; $G(u) = G(t,x,u)$, $G = E, A_j \ (j = 1,\ldots,m), C, S$.

THEOREM 6.1.

$1°$ Let $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$. Assume that conditions (3.1.1), (3.1.3), (3.1.4), (3.1.6a)–(3.1.6b) with $k > \max(m/2+5/2, (m+1)/2+2n-1)$, (3.1.7), (3.1.14), (3.1.20), (3.4.13), (4.1.30) and (4.1.120) are satisfied. Then there exists $0 < t^* \le T$ such that for all $F \in H^k(\Omega^t)$ and $g \in H^k(\partial\Omega^t)$ (where $0 < t \le t^*$) satisfying $\forall 0 \le j \le k-1$, $\partial_t^j F|_{t=0} = \partial_t^j g|_{t=0} = 0$, and for all $f \in \mathring{H}^{k+1}(\Omega)$ with $\|f\|_{k,\Omega}$ sufficiently small, there exists a unique solution $u \in \Pi_0^k(\Omega^t) \cap H^k(\partial\Omega^t)$ of problem (6.1)–(6.3).

$2°$ Let Ω be a bounded domain in $\mathbf{R}^m$ with C^∞ boundary. Assume that conditions (5.1)–(5.4), (5.5), (5.53) with $k > \max(m/2 + 5/2, (m + 1)/2 + 2n - 1)$, (5.6), (5.7), (5.10), (5.38)–(5.39) with $s = k$ and (5.54) are satisfied. Then there exists $0 < t^* \le T$ such that for all $F \in H^k(\Omega^T)$ and $g \in H^k(\partial\Omega^t)$ (where $0 < t \le t^*$) satisfying $\forall 0 \le j \le k-1$, $\partial_t^j F|_{t=0} = \partial_t^j g|_{t=0} = 0$, and for all $f \in \mathring{H}^{k+1}(\Omega)$ with $\|f\|_{k,\Omega}$ sufficiently small, there exists a unique solution $u \in \Pi_0^k(\Omega^t) \cap H^k(\partial\Omega^t)$ of problem (6.1)–(6.3).

$3°$ Let assumptions of either $1°$ or $2°$ be fulfilled. Moreover, if $\forall x \in \Omega$, $f(x) \in Q$ (where Q is an open subset of $\mathbf{R}^m$) and t^* is sufficiently small, then $\forall(\tau,x) \in \Omega^t$, $u(\tau,x) \in Q$.

P r o o f. The proofs of $1°$ and $2°$ are analogous. As in [28], consider the

problem

$$(6.4) \qquad L(u_{p-1})u_p \equiv E(t,x,u_{p-1})u_{pt} - \sum_{j=1}^{m} A_j(t,x,u_{p-1})u_{px_j}$$

$$- C(t,x,u_{p-1})u_p = F \qquad \text{in } \Omega^t,$$

$$(6.5) \qquad L_B(u_{p-1})u_p \equiv [I, -S(t,x,u_{p-1})]u_p = g \quad \text{on } \partial\Omega^t,$$

$$(6.6) \qquad u_p|_{t=0} = f \qquad \text{in } \Omega,$$

where $p \in \mathbb{N} \cup \{0\}$, $u_0 \equiv f$.

Introduce the function

$$(6.7) \qquad v_p = u_p - f,$$

which satisfies the problem

$$(6.8) \qquad E(t,x,u_{p-1})v_{pt} - \sum_{j=1}^{m} A_j(t,x,u_{p-1})v_{px_j} - C(t,x,u_{p-1})v_p$$

$$= F - \sum_{j=1}^{m} A_j(t,x,u_{p-1})f_{x_j} - C(t,x,u_{p-1})f \quad \text{in } \Omega^t,$$

$$(6.9) \qquad [I, -S(t,x,u_{p-1})]v_p = g \qquad \text{on } \partial\Omega^t,$$

$$(6.10) \qquad v_p|_{t=0} = 0 \qquad \text{in } \Omega.$$

By Theorem 5.5 there exists a unique solution $v_p \in \Pi_0^k(\Omega^t) \cap H^k(\partial\Omega^t)$ of problem (6.8)–(6.10) satisfying

$$(6.11) \quad |||v_p|||_{k,t}^2 \le (e^{2\eta t} + 1)\Big[p_s(|u_{p-1}|_{k,0,\infty,\Omega^t}, e^{2\eta t}\|u_{p-1}\|_{k,\eta,\partial\Omega^t}, \overline{b})$$

$$\times \Big(\|F\|_{k,\eta,\Omega^t}^2 + \sum_{j=1}^{m} \|A_j f_{x_j}\|_{k,\eta,\Omega^t}^2 + \|Cf\|_{k,\eta,\Omega^t}^2\Big)$$

$$+ q_s(|u_{p-1}|_{k,0,\infty,\Omega^t}, e^{2\eta t}\|u_{p-1}\|_{k,\eta,\partial\Omega^t}, \overline{b})\|g\|_{k,\eta,\partial\Omega^t}^2\Big],$$

where $|||\cdot|||_{k,t}^2 = |\cdot|_{k,0,\infty,\Omega^t}^2 + \|\cdot\|_{k,\eta,\Omega^t}^2 + \|\cdot\|_{k,\eta,\partial\Omega^t}^2$, and η satisfies

$$(6.12) \qquad \eta > q(|u_{p-1}|_{k,0,\infty,\Omega^t}, e^{2\eta t}\|u_{p-1}\|_{k,\eta,\partial\Omega^t}, \overline{b}),$$

where q is a polynomial. From (6.7) we have

$$(6.13) \qquad \|u_p\| \le \|v_p\| + \|f\| \quad \forall p,$$

where $\|\cdot\| = |\cdot|_{k,0,\infty,\Omega^t}, \|\cdot\|_{k,\eta,\Omega^t}, \|\cdot\|_{k,\eta,\partial\Omega^t}$. We can choose η sufficiently large, t^* and ε sufficiently small ($\varepsilon \to 0$ as $t^* \to 0$) and f with $\|f\|_{k,\Omega}$ so small that

$$(6.14) \qquad \forall t \in [0, t^*] \; \forall p \in \mathbb{N} \cup \{0\} \quad |||v_p|||_{k,t} \le \varepsilon.$$

In fact, the above inequality is satisfied for any $\eta, t^*, \varepsilon > 0$ if $p = 0$. Take now any $p \in \mathbb{N}$. Then we can choose η sufficiently large and t^*, ε, $\|f\|_{k,\Omega}$ so small that

$$
(6.15) \quad (e^{2\eta t} + 1)\Big[p_s(\varepsilon + \|f\|_{k,\Omega}, \varepsilon e^{2\eta t}, \bar{b})
$$

$$
\times \Big(\|F\|_{k,\eta,\Omega^t}^2 + \sum_{j=1}^{m} \|A_j f_{x_j}\|_{k,\eta,\Omega^t}^2 + \|Cf\|_{k,\eta,\Omega^t}^2 \Big)
$$

$$
+ q_s(\varepsilon + \|f\|_{k,\Omega}, \varepsilon e^{2\eta t}, \bar{b}) \|g\|_{k,\eta,\partial\Omega^t}^2 \Big] \leq \varepsilon^2 \quad \forall t \in [0, t^*],
$$

$$
(6.16) \quad \eta > q(\varepsilon + \|f\|_{k,\Omega}, \varepsilon e^{2\eta t}, \bar{b}).
$$

From (6.13), (6.15), (6.16) and the fact that $f|_{\partial\Omega} = 0$ it follows that if $|||v_{p-1}|||_{k,t} \leq \varepsilon$, $\forall t \in [0, t^*]$, then (6.11) holds. Using next (6.13) we get $|||v_p|||_{k,t} \leq \varepsilon$, $\forall t \in [0, t^*]$. Therefore (6.14) is true.

We now prove the convergence of (u_p). To do this consider $U_p = u_p - u_{p-1}$, which is the solution of the problem

$$
(6.17) \quad
\begin{aligned}
L(u_{p-1})U_p &= -[L(u_{p-1}) - L(u_{p-2})]u_{p-1}, \\
L_B(u_{p-1})U_p &= -[L_B(u_{p-1}) - L_B(u_{p-2})]u_{p-1}, \\
U_p|_{t=0} &= 0 \quad \text{for } p \in \mathbb{N}, \\
U_0 &= 0.
\end{aligned}
$$

Applying the mean value theorem we have

$$
(6.18) \quad
\begin{aligned}
\widetilde{L}(u_{p-1}) - \widetilde{L}(u_{p-2}) &= \widetilde{L}'(\widetilde{u}_{p-1})(u_{p-1} - u_{p-2}), \\
L_B(u_{p-1}) - L_B(u_{p-2}) &= L_B'(\widetilde{u}_{p-1})(u_{p-1} - u_{p-2}),
\end{aligned}
$$

where $\widetilde{L}(v) = (E(v), (A_j(v))_{j=1}^m, C(v))$ and $\widetilde{u}_{p-1} \in (u_{p-1}, u_{p-2})$. From (6.17) and (6.18) and Theorem 5.5 we obtain

$$
(6.19) \quad |||U_p|||_{k-1,t}^2 \leq (e^{2\eta t} + 1)\widetilde{p}(|u_{p-1}|_{k,0,\infty,\Omega^t}, e^{2\eta t}\|u_{p-1}\|_{k,\eta,\partial\Omega^t},
$$

$$
|\widetilde{u}_{p-1}|_{k,0,\infty,\Omega^t}, e^{2\eta t}\|\widetilde{u}_{p-1}\|_{k,\eta,\partial\Omega^t}, \bar{b})|||u_{p-1}|||_{k,t}^2 |||U_{p-1}|||_{k-1,t}^2,
$$

where $\widetilde{p}$ is a polynomial and η satisfies (6.16).

Take t^* and ε so small that

$$
(e^{2\eta t} + 1)\widetilde{p}(|u_{p-1}|_{k,0,\infty,\Omega^t}, e^{2\eta t}\|u_{p-1}\|_{k,\eta,\partial\Omega^t},
$$

$$
|\widetilde{u}_{p-1}|_{k,0,\infty,\Omega^t}, e^{2\eta t}\|\widetilde{u}_{p-1}\|_{k,\eta,\partial\Omega^t}, \bar{b})|||u_{p-1}|||_{k,t}^2 < 1.
$$

Consequently, $U_p \to 0$ in $\Pi_0^{k-1}(\Omega^t) \cap H^{k-1}(\partial\Omega^t)$. Hence by (6.14) and (6.7) there exists $u \in \Pi_0^k(\Omega^t) \cap H^k(\partial\Omega^t)$ such that $u_p \to u$ in $\Pi_0^{k-1}(\Omega^t) \cap H^{k-1}(\partial\Omega^t)$ and u satisfies problem (6.1)–(6.3).

We now prove that u is unique. Suppose that u_1 and u_2 are two solutions

of (6.1)–(6.3). Then $U = u_1 - u_2$ satisfies

$$L(u_2)U = -[L(u_1) - L(u_2)]u_2 = -L'(\tilde{u})u_1 U\,,$$

(6.20) $\qquad L_B(u_2)U = -[L_B(u_1) - L_B(u_2)]u_2 = -L'_B(\tilde{u})u_1 U\,,$

$$U|_{t=0} = 0\,,$$

where $\tilde{u} = \theta u_1 + (1 - \theta)u_2$, $0 < \theta < 1$. From Theorem 5.5 we obtain the estimate

(6.21) $\quad |U(t)|^2_{k-1,0,\Omega}\, e^{-2\eta t} + \eta\|U\|^2_{k-1,\eta,\Omega^t} + \|U\|^2_{k-1,\eta,\partial\Omega^t}$

$$\leq p_0(|u_2|_{k,0,\infty,\Omega^t}, \|u_2\|_{k,\partial\Omega^t}, \overline{b})\|L'(\tilde{u})u_1 U\|^2_{k-1,\eta,\Omega^t}$$

$$+ q_0(|u_2|_{k,0,\infty,\Omega^t}, \|u_2\|_{k,\partial\Omega^t}, \overline{b})\|L'_B(\tilde{u})u_1 U\|^2_{k-1,\eta,\partial\Omega^t}$$

for η, t, ε and $\|f\|_{k,\Omega}$ satisfying (6.16).

The first part of the proof shows that

(6.22) $\qquad \|u_i\| \leq \varepsilon + \|f\|$, where $\|\cdot\| = |\cdot|_{k,0,\infty,\Omega^t}, \|\cdot\|_{k,\eta,\Omega^t}$,

(6.23) $\qquad \|u_i\|_{k,\eta,\partial\Omega^t} \leq \varepsilon$ for $i = 1, 2$.

Using (6.22) and (6.23) in (6.21) we get

(6.24) $\qquad |U(t)|^2_{k-1,0,\Omega}\, e^{-2\eta t} + \eta\|U\|^2_{k-1,\eta,\Omega^t} + \|U\|^2_{k-1,\eta,\partial\Omega^t}$

$\leq \widetilde{p}_0(|u_2|_{k,0,\infty,\Omega^t}, \|u_2\|_{k,\partial\Omega^t}, |\tilde{u}|_{k,0,\infty,\Omega^t}, \|\tilde{u}\|_{k,\partial\Omega^t}, \overline{b})\|u_1\|^2_{k,\eta,\Omega^t}\|U\|^2_{k-1,\eta,\Omega^t}$

$+ \widetilde{q}_0(|u_2|_{k,0,\infty,\Omega^t}, \|u_2\|_{k,\partial\Omega^t}, |\tilde{u}|_{k,0,\infty,\Omega^t}, \|\tilde{u}\|_{k,\partial\Omega^t}, \overline{b})$

$$\times \|u_1\|^2_{k,\eta,\partial\Omega^t}\|U\|^2_{k-1,\eta,\partial\Omega^t}$$

$\leq (\varepsilon + \|f\|_{k,\Omega})^2 \widetilde{p}_0(\varepsilon + \|f\|_{k,\Omega}, \varepsilon e^{2\eta t}, 2(\varepsilon + \|f\|_{k,\Omega}), 2\varepsilon e^{2\eta t}, \overline{b})\|U\|^2_{k-1,\eta,\Omega^t}$

$+ \widetilde{q}_0(\varepsilon + \|f\|_{k,\Omega}, \varepsilon e^{2\eta t}, 2(\varepsilon + \|f\|_{k,\Omega}), 2\varepsilon e^{2\eta t}, \overline{b})\|U\|^2_{k-1,\eta,\partial\Omega^t}\,,$

where $\widetilde{p}_0$ and $\widetilde{q}_0$ are polynomials.

Take now t^*, ε and $\|f\|_{k,\Omega}$ so small that

$$(\varepsilon + \|f\|_{k,\Omega})\widetilde{p}_0(\varepsilon + \|f\|_{k,\Omega}, \varepsilon e^{2\eta t}, 2(\varepsilon + \|f\|_{k,\Omega}), 2\varepsilon e^{2\eta t}, \overline{b}) < \eta/2\,,$$

$$\varepsilon\widetilde{q}_0(\varepsilon + \|f\|_{k,\Omega}, \varepsilon e^{2\eta t}, 2(\varepsilon + \|f\|_{k,\Omega}), 2\varepsilon e^{2\eta t}, \overline{b}) < 1/2\,.$$

Then (6.24) turns into

$$|U(t)|^2_{k-1,0,\Omega}\, e^{-2\eta t} + \tfrac{1}{2}\eta\|U\|^2_{k-1,\eta,\Omega^t} + \tfrac{1}{2}\|U\|^2_{k-1,\eta,\partial\Omega^t} \leq 0\,.$$

This implies the uniqueness of solution of (6.1)–(6.3).

$3°$ follows from (6.7) and (6.14). ∎

Remark 6.1. Assume that $f = 0$. Then the assertion of Theorem 6.1 with $u \in H^k(\partial\Omega^t)$ holds under weaker assumptions. In the case when $\Omega = \{x \in \mathbf{R}^m : x_1 > 0\}$, the assumptions of Theorem 4.2.2 suffice. In the case when Ω is a bounded domain with C^∞ boundary it suffices to take the assumptions of Remark 5.2.

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