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STRUCTURAL THEOREMS FOR ULTRADISTRIBUTIONS

STEVAN PILIPOVIĆ

Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia
E-mail: pilipovi%unsim@yubgef51.bitnet

Abstract. A characterization of bounded sets and a boundary value representations in ${\mathcal{D}'}^*_{L^s}$ and ${\mathcal{S}'}^*$ are given. A decomposition of an ultradistribution, with appropriate assumptions on the corresponding singular spectra, is easily proved.

Introduction. In [9] Komatsu has given the analysis of Sato's hyperfunctions ([18]) as boundary values of a harmonic functions. This idea has been also used by several other mathematicians. Hörmander's monumental monography [6] contains in Chapter 9 such an approach to the hyperfunction theory; see also Matsuzawa's approach [11].

While in [6] the sheaf of distributions was considered, Komatsu [9] have studied sheafs $\mathcal{C}^*$ and $\mathcal{C}_*$, the microlocalization of spaces of ultradistributions and ultradifferentiable functions, which correspond to Gevrey classes $* = (M_p)$ or $* = \{M_p\}$. Komatsu's approach is simple and deep. Recall $\mathcal{C}^*$ and $\mathcal{C}_*$ are constructed by exact sequences

$$0 \to \mathcal{A} \to \mathcal{D}'^* \to \pi_*\mathcal{C}^* \to 0, \quad 0 \to \mathcal{A} \to \mathcal{E}^* \to \pi_*\mathcal{C}_* \to 0,$$

where $\mathcal{A}$ is the space of real analytic functions and $\pi : S^*M \to M$ is the canonical projection, $S^*M = (T^*M \setminus M)/\mathbf{R}_+$ and M is a real analytic manifold. The analysis of various ultradistributional subsheafs of the sheaf of microfunctions $\mathcal{C}$ is given in [4]; for the theory of ultradistributions spaces we refer to [8], [1], [4] and [10].

On the other hand, in [2], [3], [7], [12], [14], [15] we have studied subspaces of ultradistribution spaces as well as the convolution and the integral transformations in them. The most interesting subspaces are ${\mathcal{D}'}^*_{L^t}$ and ${\mathcal{S}'}^*$. Such investigations may be considered as a study of the structure of $\mathcal{C}^*$.

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The investigations of the singular spectrum of an $f \in \mathcal{D}'^*$ based on the theory of $\mathcal{S}'^*$ spaces are given in [16] (see also [17]).

In this paper we will present structural characterizations of bounded sets and boundary value (hyperfunction) representations in $\mathcal{D}'^*_{L^t}$ and $\mathcal{S}'^*$. We will give both improvements of results from [3] and [15] and some new ones. Also we will point out some problems which are not solved yet.

Notation and definitions. By M_p, $p \in \mathbf{N}$, we denote a sequence of positive numbers, where $M_0 = 1$. We define additionally $M_p^* = M_p/p!$, $m_p = M_p/M_{p-1}$, $m_p^* = M_p^*/M_{p-1}^*$, $p \in \mathbf{N}$. The following conditions will be considered.

$(M.1)^*$ $M_p^* \le M_{p-1}^* M_{p+1}^*$, $p \in \mathbf{N} \Rightarrow (M.1)$ $M_p^2 \le M_{p-1}M_{p+1}$, $p \in \mathbf{N}$.

$(M.2)$ There exist $A > 0$ and $H > 0$ such that

$$M_p \le AH^p M_q M_{p-q}, \quad p \in \mathbf{N}_0,\ 0 \le q \le p\ (\mathbf{N}_0 = \mathbf{N} \cup \{0\}),$$

which implies

$(M.2)'$ There exist $A > 0$ and $H > 0$ such that

$$M_{p+1} \le AH^p M_p, \quad p \in \mathbf{N}_0.$$

$(M.3)$ There exists $A > 0$ such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le ApM_p/M_{p+1}, \quad p \in \mathbf{N}_0,$$

which implies

$$(M.3)' \qquad \sum_{p=0}^{\infty} M_{p-1}/M_p < \infty.$$

Condition $(M.1)$ will always be assumed.

The associated growth functions are defined by

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \ln \frac{\rho^p}{M_p}, \quad M^*(\rho) = \sup_{p \in \mathbf{N}_0} \ln \frac{\rho^p p!}{M_p}, \quad \rho > 0.$$

An operator formaly of the form $\sum_{\alpha \in \mathbf{N}_0^n} a_\alpha \partial^\alpha$ is said to be of (M_p)-*class* (resp. of $\{M_p\}$-*class*) if for some $H > 0$ and $C > 0$ (resp. for every $H > 0$ there is $C > 0$)

$$|a_\alpha| \le CH^{|\alpha|}/M_{|\alpha|}, \quad \alpha \in \mathbf{N}_0^n.$$

We will use the common symbol $*$ for (M_p) and $\{M_p\}$.

Recall that $\mathcal{D}_{K,h}^{M_p}$, $K \subset\subset \mathbf{R}^n$, $h > 0$, is the space of smooth functions on $\mathbf{R}^n$ supported by K such that

$$\|\phi\|_{K,h} = \sup_{x \in K, \alpha \in \mathbf{N}_0^n} \frac{|\phi^{(\alpha)}(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty.$$

For the definitions of the spaces $\mathcal{D}_K^*$, $\mathcal{D}^*$ and $\mathcal{E}^*$ we refer to [8].

The spaces $\mathcal{D}'^*_{L^t}$. Let $s \in [1,\infty]$. Recall that

$$\mathcal{D}^{M_p,h}_{L^s} = \Big\{\phi \in C^\infty(\mathbf{R}^n) : \sup_{\alpha\in\mathbf{N}_0^n} \frac{\|\phi^{(\alpha)}\|_{L^s}}{h^{|\alpha|}M_{|\alpha|}} < \infty\Big\}, \qquad h > 0,$$

$$\mathcal{D}^{(M_p)}_{L^s} = \operatorname*{proj\,lim}_{h\to 0} \mathcal{D}^{M_p,h}_{L^s}, \qquad \mathcal{D}^{\{M_p\}}_{L^s} = \operatorname*{ind\,lim}_{h\to\infty} \mathcal{D}^{M_p,h}_{L^s}.$$

and $\dot{\mathcal{B}}^*$ is the completion of $\mathcal{D}^*$ in $\mathcal{D}^*_{L^\infty}$.

The strong dual of $\mathcal{D}^*_{L^s}$ for $s \in [1,\infty)$ is $\mathcal{D}'^*_{L^t}$, where $t = s/(s-1)$, and $(\dot{\mathcal{B}}^*)' = \mathcal{D}'^*_{L^1}$.

The next theorem is proved in [15]. We recall its formulation with the improvements concerning the use of conditions on M_p.

Theorem 1 (see [15]). (i) *A set $B \subset \mathcal{D}'^{(M_p)}_{L^t}$, $t \in (1,\infty]$, is bounded if and only if every $f \in B$ can be represented in the form*

$$f = \sum_{\alpha\in\mathbf{N}_0^n} D^\alpha f_\alpha, \qquad f_\alpha \in L^t,\ \alpha \in \mathbf{N}_0^n, \tag{1}$$

and there exist $d > 0$ and $C > 0$ independent of $f \in B$ such that

$$\sum_{\alpha\in\mathbf{N}_0^n} d^{|\alpha|} M_{|\alpha|} \|f_\alpha\|_{L^t} < C.$$

(ii) *Assume* $(M.3)'$. *A set $B \subset \mathcal{D}'^{(M_p)}_{L^1}$ is bounded if and only if every $f \in B$ is of the form*

$$f = \sum_{\alpha\in\mathbf{N}_0^n} D^\alpha f_\alpha, \qquad f_\alpha \in \mathcal{M}^1 = \mathcal{C}_0', \qquad \alpha \in \mathbf{N}_0^n,$$

and there exist $d > 0$ and $C > 0$ such that

$$\sum_{\alpha\in\mathbf{N}_0^n} d^{|\alpha|} M_{|\alpha|} \|f_\alpha\|_{\mathcal{M}^1} < C,$$

where $\mathcal{C}_0$ is the space of continuous functions which tend to 0 as $|x| \to \infty$ with the supremum norm and $\|\ \|_{\mathcal{M}^1}$ is the dual norm.

(iii) *$f \in \mathcal{D}'^{\{M_p\}}_{L^t}$, $t \in (1,\infty)$, if and only if it is of the form* (1) *and for every $d > 0$ there is $C_d > 0$ such that*

$$\sum_{\alpha\in\mathbf{N}_0^n} d^{|\alpha|} M_{|\alpha|} \|f_\alpha\|_{L^t} < C_d.$$

(iv) *Assume* $(M.3)'$. *Then $f \in \mathcal{D}'^{\{M_p\}}_{L^1}$ if and only if f is of the form* (2) *and for every $d > 0$ there is $C_d > 0$ such that*

$$\sum_{\alpha\in\mathbf{N}_0^n} d^{|\alpha|} M_{|\alpha|} \|f_\alpha\|_{\mathcal{M}^1}.$$

Problem 1. Characterize bounded sets in $\mathcal{D}'^{\{M_p\}}_{L^t}$, $t \in [1,\infty]$.

Let $\Re$ be the set of all strictly increasing sequences (r_p) which tend to ∞ with the partial ordering $(r_p) \le (s_p)$ if $r_p \le s_p$, $p \in \mathbf{N}$. Let $s \in [1,\infty]$. Then

$$\widetilde{\mathcal{D}}^{\{M_p\}}_{L^s,(r_p)} = \left\{\phi \in C^\infty(\mathbf{R}^n) : \sup \frac{\|\phi^{(\alpha)}\|_{L^s}}{(\prod_{i=1}^{|\alpha|} r_i)M_{|\alpha|}} < \infty\right\},$$

$$\widetilde{\mathcal{D}}^{\{M_p\}}_{L^s} = \operatorname{proj}\lim_{(r_p)} \widetilde{\mathcal{D}}^{\{M_p\}}_{L^s,(r_p)}.$$

Under assumption $(M.3)'$ we define $\widetilde{\dot{\mathcal{B}}}^{\{M_p\}}$ as the completion of $\mathcal{D}^{\{M_p\}}$ in $\widetilde{\mathcal{D}}^{\{M_p\}}_{L^\infty,(r_p)}$.

Theorem 2 (see [15]). (i) *$\widetilde{\mathcal{D}}^{\{M_p\}}_{L^s} = \mathcal{D}^{\{M_p\}}_{L^s}$, $s \in [1,\infty)$, and the inclusion mapping $\mathcal{D}^{\{M_p\}}_{L^s} \to \widetilde{\mathcal{D}}^{\{M_p\}}_{L^s}$, is continuous.*

(ii) *Under $(M.3)'$, $\widetilde{\dot{\mathcal{B}}}^{\{M_p\}} = \dot{\mathcal{B}}^{\{M_p\}}$ and the inclusion mapping $\dot{\mathcal{B}}^{\{M_p\}} \to \widetilde{\dot{\mathcal{B}}}^{\{M_p\}}$ is continuous.*

Problem 2. Are the spaces $\widetilde{\mathcal{D}}^{\{M_p\}}_{L^s}$, $s \in [1,\infty)$ and $\widetilde{\dot{\mathcal{B}}}^{\{M_p\}}$ quasibarrelled?

Let $r > 0$ (resp. $(r_p) \in \Re$). We shall use the following operators of class (M_p) (resp. $\{M_p\}$) : $P_r(D) = (1-\Delta^2)^{n'} \prod_{p=1}^\infty (1-\Delta^2 r^{-2} m_p^{-2})$, $P_{r_p}(D) = (1-\Delta^2)^{n'} \prod_{p=1}^\infty (1-\Delta^2 r_p^{-2} m_p^{-2})$, where $n' > n$ and Δ is the Laplacian (cf. [8]).

Theorem 3. *Assume $(M.2)$ and $(M.3)$. Let $A \subset \mathcal{D}'$.*

(i) (*see* [15]) *A is a bounded* (*resp. equicontinuous*) *subset of $\mathcal{D}'^{(M_p)}_{L^t}$* (*resp. $\widetilde{\mathcal{D}}'^{\{M_p\}}_{L^t}$*), *$t \in [1,\infty]$ if and only if there exist $r > 0$* (*resp. $(r_p) \in \Re$*) *and a bounded subset A_1 of L^t such that*

$$A \ni f = P_r(D) f_1 + f_2, \quad f_1, f_2 \in A_1$$
$$(\textit{resp. } A \ni f = P_{r_p}(D) f_1 + f_2,\ f_1, f_2 \in A_1).$$

(ii) *Let $t \in [1,\infty]$. Then A is a bounded subset of $\mathcal{D}'^{\{M_p\}}_{L^t}$ if and only if there exist $(r_p) \in \Re$ and a bounded subset A_1 of L^t such that*

$$A \ni f = P_{r_p}(D) f_1 + f_2, \quad f_1, f_2 \in A_1.$$

Proof. (ii) Let Ω be a bounded open set in $\mathbf{R}^n$, $\Omega \ni 0$, $K = \overline{\Omega}$. Let $\phi \in \mathcal{D}^{M_p}_{K,h}$. First, we show that, for every $f \in A$, $f * \phi$ is a continuous linear functional on $\mathcal{D}^{\{M_p\}}$ endowed with the topology of L^s.

The spaces $\mathcal{D}^{\{M_p\}}_{L_s}$, $s \in [1,\infty)$ and $\dot{\mathcal{B}}^{\{M_p\}}$ are barrelled, which implies that A is equicontinuous. This implies that for every $h > 0$ there exists $C > 0$ such that, for every $\psi \in \mathcal{D}^{\{M_p\}}$,

$$\begin{aligned} |\langle f * \check{\phi}, \psi\rangle| = |\langle f, \phi * \psi\rangle| &\le C\|\phi * \psi\|_{L^s,h} \\ &\le C\|\phi\|_{K,h}\|\psi\|_{L^s} \le C_1\|\psi\|_{L^s}. \end{aligned} \tag{3}$$

Since $\mathcal{D}^{\{M_p\}}$ is dense in L^s (in the completion of $\mathcal{D}^{\{M_p\}}$ in L^∞, if $s = \infty$), it follows that $\{f * \phi : f \in A\}$ is a set of continuous functions bounded in L^t.

Moreover, (3) implies that for every $h>0$ there exists $C>0$ such that

$$\left|\left\langle f*\frac{\check{\psi}}{\|\psi\|_{L^s}},\phi\right\rangle\right|\le C\|\phi\|_{K,h}$$

for every $f\in A$ and $\psi\in\mathcal{D}^{\{M_p\}}$. It follows that $\{f*\check{\psi}/\|\psi\|_{L^s}:f\in A\}$ is a bounded set in $\mathcal{D}'^{\{M_p\}}_K$ and thus equicontinuous. This implies that, for some $(r_p)\in\Re$ and some $C>0$,

$$\left|\left\langle f*\frac{\check{\psi}}{\|\psi\|_{L^s}},\phi\right\rangle\right|\le C\|\phi\|_{K(r_p)},\qquad \phi\in\mathcal{D}^{\{M_p\}}_K,\ f\in A,\ \psi\in\mathcal{D}^{\{M_p\}},$$

where

$$\|\phi\|_{K,(r_p)}=\sup_{\substack{x\in K\\ \alpha\in\mathbf{N}_0^n}}\frac{|\phi^{(\alpha)}(x)|}{(\prod_{i=1}^{(\alpha)}r_i)M_{|\alpha|}}.$$

Recall that the family of norms $\|\ \|_{K,(r_p)}$, $(r_p)\in\Re$, determines the topology of $\mathcal{D}^{\{M_p\}}_K$ (see [8]).

This implies

$$|\langle f*\check{\psi},\phi\rangle|\le C\|\psi\|_{L^s}\|\phi\|_{K,(r_p)},\qquad \phi\in\mathcal{D}^{\{M_p\}}_K,\ \psi\in\mathcal{D}^{\{M_p\}},\ f\in A,$$

and

$$\|f*\check{\phi}\|_{L^t}\le C\|\phi\|_{K,(r_p)}$$

Let B_1 be the unit ball in L^s and B be a bounded subset of $\mathcal{D}^{\{M_p\}}_K$. Then, for every $\psi\in B_1\cap\mathcal{D}^{\{M_p\}}$ and $\phi\in B$,

$$\langle f*\check{\psi},\check{\phi}\rangle=|\langle f*\phi,\psi\rangle|\le\|f*\phi\|_{L^t}\|\psi\|_{L^s}\le\|f*\phi\|_{L^t}<C<\infty.$$

This implies that the set $\{f*\psi:f\in A,\psi\in B_1\cap\mathcal{D}^{\{M_p\}}\}$ is equicontinuous in $\mathcal{D}^{\{M_p\}}_K$. So there exists a neighbourhood of zero

$$V_{r_p}(\epsilon)=\{\theta\in\mathcal{D}^{\{M_p\}}_K:\|\theta\|_{K,r_p}<\epsilon\},\qquad \epsilon>0,$$

in $\mathcal{D}^{\{M_p\}}_K$ such that

$$\theta\in V_{r_p}\Rightarrow|\langle f*\check{\psi},\check{\theta}\rangle|=|\langle f*\theta,\psi\rangle|\le 1,\qquad \psi\in B_1\cap\mathcal{D}^{\{M_p\}}.$$

Define $\mathcal{D}^{\{M_p\}}_{\Omega(r_p)}=\operatorname{ind\,lim}_{K\subset\subset\Omega}\mathcal{D}^{\{M_p\}}_{K,(r_p)}$. This implies that for every $\theta\in\mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$ there is $C>0$ such that

$$|\langle f*\check{\psi},\check{\theta}\rangle|=|\langle f*\theta,\psi\rangle|\le C,\qquad \psi\in B_1\cap\mathcal{D}^{\{M_p\}},\ f\in A,$$

and thus

$$|\langle f*\theta,\psi\rangle|\le C\|\psi\|_{L^s},\qquad \psi\in\mathcal{D}^{\{M_p\}},\ f\in A.$$

This proves that, for every $\theta\in\mathcal{D}^{\{M_p\}}_{\Omega,r_p}$, $\{f*\theta:f\in A\}$ is a bounded set in L^t.

By the parametrix method (see [9] or [15]) we have

$$f = P_{r_p}(u * f) + \psi * f, \qquad u \in \mathcal{D}^{\{M_p\}}_{\Omega, r_p},\ \psi \in \mathcal{D}^{\{M_p\}}(\Omega),$$

and the assertion is proved.

PROBLEM 3. Characterize bounded sets in $\widetilde{\mathcal{D}}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$.

The next theorem is an improvement of results from [3]. Namely, in [3] instead of cones, octants were considered and for $t = 1$ and $t = \infty$ the assertion was proved with the assumption $(M.3)$. The proof of this theorem is omitted, since it is very similar to that of Theorem 8, which is given in the next section with all the details.

THEOREM 4. *Suppose that* Γ *is an open convex cone in* $\mathbf{R}^n$, $s \in [1, \infty]$ *and* M_p *satisfies* $(M.1)^*$ *and* $(M.2)$. *Suppose that* $F(x + \sqrt{-1}y)$ *is holomorphic in* $T^\Gamma = \mathbf{R}^n + \sqrt{-1}\Gamma$ *and there are* $L > 0$ *and* $C > 0$ (*resp. for every* $L > 0$ *there is* $C > 0$) *such that*

$$\|F(\cdot + \sqrt{-1}y)\|_{L^t} \le Ce^{M^*(L/|y|)}, \qquad |y| < \gamma,\ y \in \Gamma.$$

Then

$$\lim_{\Gamma \ni y \to 0} F(\cdot + \sqrt{-1}y) \overset{\mathcal{D}'^*_{L^t}}{=} F(\cdot + \sqrt{-1}0).$$

The converse assertion is given in the following form:

THEOREM 5 (see [3]). *Assume* $(M.1)^*$, $(M.2)$ *and* $(M.3)'$ *and let* $f \in \mathcal{D}^*_{L^t}$, $t \in (1, \infty)$. *Then there exists a holomorphic function* F *in*

$$\{z \in (\mathbf{C} \setminus \mathbf{R})^n : |y_i| < \gamma,\ i = 1, \dots, n\}$$

such that

$$\|F(x + \sqrt{-1}y)\|_{L^t} \le C \exp\Big(\sum_{i=1}^n M_i^*(k/|y_i|)\Big), \qquad 0 \ne |y_i| < \gamma,\ i = 1, \dots, n,$$

for some $k > 0$ *and some* $C > 0$ (*resp. for every* $k > 0$ *there is* $C > 0$) *and*

$$f(x) = \lim_{\epsilon \to 0} \sum_e \Big(\prod_{i=1}^n e_i\Big) F(x_1 + e_1 \epsilon \sqrt{-1}, \dots, x_n + e_n \epsilon \sqrt{-1}) \qquad \text{in } \mathcal{D}'^*_{L^t}$$

where $e = (e_1, \dots, e_n)$, $e_i = 1$ *or* -1, $i = 1, \dots, n$.

PROBLEM 4. Are the assertions of Theorem 5 true for $t = 1$ and $t = \infty$?

Tempered ultradistributions. We now recall from [12] the definitions and basic structural properties of tempered ultradistribution spaces. Assume $(M.3)'$ to hold.

The space of smooth functions φ on $\mathbf{R}^n$ such that, for all $m > 0$,

$$\sigma_{m,2}(\varphi) = \Bigg(\sum_{\alpha, \beta \in \mathbf{N}_0^n} \int_{\mathbf{R}^n} \left| \frac{m^{|\alpha+\beta|}}{M_{|\alpha|} M_{|\beta|}} (q(x))^{|\beta|} \varphi^{(\alpha)}(x) \right|^2 dx \Bigg)^{1/2} < \infty,$$

where $q(x)=(1+|x|^2)^{1/2}$, equipped with the topology induced by the norm $\sigma_{m,2}$, is denoted by $\mathcal{S}_2^{M_p,m}$. The strong duals of

$$\mathcal{S}^{(M_p)}=\operatorname*{proj\,lim}_{m\to\infty}\mathcal{S}_2^{M_p,m}\quad\text{and}\quad\mathcal{S}^{\{M_p\}}=\operatorname*{ind\,lim}_{m\to\infty}\mathcal{S}_2^{M_p,m}$$

are called the *spaces of tempered ultradistributions of Beurling* and *Roumieau type*. If we assume $(M.2)'$ to hold, then for every fixed $p\in[1,\infty]$, the family of norms $\{\sigma_{m,2}:m>0\}$ is equivalent to the family of norms $\{\sigma_{m,p}:m>0\}$ where instead of L^2 norm we put L^p norms.

The Fourier transformation is an isomorphism of $\mathcal{S}^*$ onto $\mathcal{S}^*$.

If $(M.2)$ holds, then $\mathcal{D}^*\hookrightarrow\mathcal{S}^*\hookrightarrow\mathcal{E}^*$ and $\mathcal{S}^*\hookrightarrow\mathcal{S}$, where "$A\hookrightarrow B$" means that A is dense in B and the inclusion mapping is continuous.

$\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ are (FS)- and (LS)-spaces, respectively. If $(M.2)$ holds, they are (FN)- and (LN)-spaces, respectively.

Note that $\mathcal{S}^{\{M_p\}}=\operatorname{proj\,lim}_{r_i,s_j\in\Re}\mathcal{S}^{M_p}_{r_i,s_j}$ where $\mathcal{S}^{M_p}_{r_i,s_j}$ is the space of functions φ from C^∞ such that

$$\gamma_{r_i,s_j}(\varphi)=\sup\left\{\frac{\|q(x)^{|\beta|}\partial^\alpha\varphi\|_{L^2}}{M_{|\alpha|}r(\alpha)M_{|\beta|}s(\beta)}:\alpha,\beta\in\mathbf{N}_0^n\right\}<\infty,$$

where $r(\alpha)=\prod_{i=1}^{|\alpha|}r_i$, $s(\beta)=\prod_{j=1}^{|\beta|}s_j$.

The following two theorems give the characterization of bounded sets in $\mathcal{S}'^*$.

THEOREM 6 (see [12]). *A set $B\subset\mathcal{S}'^{(M_p)}$ (resp. $\mathcal{S}'^{\{M_p\}}$) is bounded if and only if every $f\in B$ can be represented in the form*

$$\sum_{\alpha,\beta\in\mathbf{N}_0^n}^{\infty}D^\alpha(q^{|\beta|}f_{\alpha,\beta}),$$

where $f_{\alpha,\beta}\in L^2$ and for some $d>0$ (resp. every $d>0$) there exists $D>0$ independent of $f\in B$ such that

$$\sum_{\alpha,\beta\in\mathbf{N}_0^n}^{\infty}d^{\alpha+\beta}M_{|\alpha|}M_{|\beta|}\|f_{\alpha,\beta}\|_{L^2}<D.$$

THEOREM 7 (see [15]). *Assume $(M2)$ and $(M3)$. Let $A\subset\mathcal{D}'^{(M_p)}$ (resp. $A\subset\mathcal{D}'^{\{M_p\}}$). Then A is a bounded subset of $S'^{(M_p)}$ (resp. A is a bounded subset of $S'^{\{M_p\}}$) if and only if $f\in A$ is of the form $f=P(D)F$, $F\in A_1$, where P is an operator of class (M_p) (resp. of class $\{M_p\}$) and A_1 is a set of continuous functions on $\mathbf{R}^n$ such that for some $k>0$ and some $C>0$ (resp. for every $k>0$ there is $C>0$)*

$$|F(x)|\le C\exp M(k|x|)),\qquad F\in A_1,\ x\in\mathbf{R}^n.$$

Remark. Let Ω be an open set in $\mathbf{R}^n$ and $\mathcal{S}^*(\Omega)$ be the space defined by the families of norms with Ω instead of $\mathbf{R}^n$. Then we obtain the corresponding space $\mathcal{S}'^*(\Omega)$ and the presheaf $\Omega\to\mathcal{S}'^*(\Omega)$, $\Omega\subset\mathbf{R}^n$. The associated sheaf is $\Omega\to\mathcal{D}'^*(\Omega)$, $\Omega\subset\mathbf{R}^n$.

Until the end of the paper we shall assume that conditions $(M.1)^*$, $(M.2)$ and $(M.3)'$ are satisfied by the sequence M_p.

The proof of the next theorem is based on Hörmander's proof of the corresponding assertions in $\mathcal{D}'$ ([6, Theorem 3.1.15]) and on the almost analytic extension (see [13]).

THEOREM 8. *Let* Γ *be an open convex cone in* $\mathbf{R}^n$ *and* F *be an analytic function in*

$$Z=\{z\in \mathbf{C}^n : \operatorname{Im} z\in\Gamma,\ |\operatorname{Im} z|<\gamma\}$$

for some $\gamma>0$. *Moreover, assume that, in* (M_p)*-case, there exist constants* $a>0$, $b>0$, $C>0$, *and, in* $\{M_p\}$*-case, for arbitrary* $a>0$ *and* $b>0$ *there is a constant* $C=C_{a,b}$ *such that*

$$|F(x+\sqrt{-1}y)|\le Ce^{M(a|x|)+M^*(b/|y|)},\qquad x+\sqrt{-1}y\in Z.$$

Then

$$F(x+\sqrt{-1}y)\overset{\mathcal{S}'^*}{\to} F(x+\sqrt{-1}0),\qquad y\to 0,\ y\in\Gamma.$$

Proof. Let $h>0$ and $\varphi\in C^\infty$ such that

$$\sup_{\alpha\in\mathbf{N}_0,\,x\in\mathbf{R}^n}\frac{h^{|\alpha|}}{M_{|\alpha|}}|\phi^{(\alpha)}(x)|e^{M(h|x|)}<\infty. \tag{4}$$

Note that the family of norms defined by (4), $h>0$, is equivalent to $\sigma_{m,2}$, $m>0$ ([12]). As in [13], let κ be a non-negative function which belongs to $\mathcal{D}^*(\mathbf{R})$ such that $\operatorname{supp}\kappa\subset[-2,2]$ and $\kappa|_{[-1,1]}=1$. Moreover, let

$$\kappa_p(t)=\kappa(4tm_p^*/h),\qquad t,h>0,\ p\in\mathbf{N}_0,$$

where $m_p^*=m_p/p$.

The almost analytic extension of φ is defined by

$$\Phi(z)=\sum_{p\in\mathbf{N}_0^n}\frac{\varphi^{(p)}(x)}{p!}(\sqrt{-1}y)^p\kappa_p(y),\qquad z=x+\sqrt{-1}y\in\mathbf{C}^n,$$

where

$$p!=p_1!\dots p_n!,\qquad (\sqrt{-1}y)^p=(\sqrt{-1}y_1)^{p_1}\dots(\sqrt{-1}y_n)^{p_n},$$
$$\kappa_p(y)=\kappa_{p_1}(y_1)\dots\kappa_{p_n}(y_n)$$

for $p=(p_1,\dots,p_n)$ and $y=(y_1,\dots,y_n)$. The function Φ is smooth in $\mathbf{R}^{2n}$.

Fix $Y=(Y_1,\dots,Y_n)\in\Gamma$, $Y\neq 0$. We assume that $Y_i\neq 0$ for $i=1,\dots,n$ (if $Y_i=0$, for some $i=1,\dots,n$, the proof is the same). Applying the following inequalities:

$$\frac14 h/m_{p_i}^*\le Y_it_i\le\frac12 h/m_{p_i}^*,\qquad i=1,\dots,n,\ p\in\mathbf{N}_0,$$
$$M_p^*/m_p^{*p-1}\le M_{p-1}^*/m_{p-1}^{*p-2}\le\dots\le M_2^*,\qquad p\in\mathbf{N}_0,$$

$$\max\left\{\kappa(t), \left|\frac{d}{dt}\kappa(t)\right|\right\} \le C, \qquad t \in \mathbf{R},$$

one can show that are constants m, C_h, D_h, $H_h > 0$ such that

$$\exp[M(h|x|) + M^*(1/H_h|t|)]\left|\frac{\partial}{\partial \bar{z}_i}\Phi(x+\sqrt{-1}tY)\right| \le C_h\sigma_{m,2}(\varphi) \tag{5}$$

and

$$\exp[M(h|x|)]|\Phi(x+\sqrt{-1}tY)| \le D_h\sigma_{m,2}(\varphi) \tag{6}$$

for all $x \in \mathbf{R}^n$, $t \in (0,1]$ and $i = 1,\ldots,n$.

Put $Z_Y = \{z = x+\sqrt{-1}tY : x \in \mathbf{R}^n, 0 \le t \le 1\}$. Suppose that ψ is an analytic function and θ, $\frac{\partial}{\partial \bar{z}_i}\theta$, $i = 1,\ldots,n$, are continuous functions in a neighbourhood of Z_Y. Moreover, assume that $\psi\theta$ is integrable in this neighbourhood and that for every fixed $m \in \{1,\ldots,n\}$,

$$|(\psi\theta)(\ldots, a_{i_1} + \sqrt{-1}tY_{i_1}, \ldots, a_{i_m} + \sqrt{-1}tY_{i_m}, \ldots)| \to 0 \quad \text{as } a \to \infty$$
$$\text{uniformly in } t \in [0,1], \text{ and } x_j \in \mathbf{R},\ j \ne i_1,\ldots,i_m,$$

where $i_1,\ldots,i_m \in \{1,\ldots,n\}$ and $|a_{i_j}| = a > 0$.

Under the above assumptions we have

$$\int_{\mathbf{R}^n} \theta(x_1,\ldots,x_n)\psi(x_1,\ldots,x_n)\,dx_1,\ldots,dx_n$$
$$= \int_{\mathbf{R}^n} \theta(x+\sqrt{-1}Y)\psi(x+\sqrt{-1}Y))\,dx$$
$$+ 2\sqrt{-1}\sum_{i=1}^n Y_i \int_0^1 \int_{\mathbf{R}^n} \frac{\partial\theta}{\partial \bar{z}_i}(x+\sqrt{-1}tY)\psi(x+\sqrt{-1}tY)\,dt\,dx.$$

The proof of this assertion follows from Stokes' formula on

$$Z_{Y_a} = \{z = x+\sqrt{-1}tY : -a \le x_i \le a,\ i = 1,\ldots,n,\ 0 \le t \le 1\}$$

by letting $a \to \infty$. There exists a constant $C > 0$ such that

$$t \le C|y+tY|, \qquad y \in \Gamma,\ t > 0$$

(see [6, p. 66]). Fix $y \in \Gamma$. Then the functions

$$\psi(z) = F(z+\sqrt{-1}y), \quad \theta(z) = \Phi(z), \quad z \in Z_Y,$$

satisfy the above assumptions. This follows from the assumptions of the theorem and inequalities (5) and (6). By letting $y \to 0$, $y \in \Gamma$, we have

$$\langle F(x+\sqrt{-1}0), \varphi(x)\rangle = \int_{\mathbf{R}^n} F(x+\sqrt{-1}Y)\Phi(x+\sqrt{-1}Y)\,dx$$
$$+2\sqrt{-1}\sum_{i=1}^n Y_i \int_0^1 \int_{\mathbf{R}^n} \frac{\partial}{\partial \bar{z}_i}\Phi(x+\sqrt{-1}tY)F(x+\sqrt{-1}Yt)\,dt\,dx$$

for every $\varphi \in S^*$.

Moreover, for some $C > 0$ and $h > 0$ (resp. for every $h > 0$ there is $C > 0$) we have

$$|\langle F(x+\sqrt{-1}0), \varphi(x)\rangle| \le C\sigma_{h,2}(\varphi), \qquad \varphi \in \mathcal{S}^*. \ \blacksquare$$

As in [6], put

$$I(\xi) = \int\limits_{|\omega|=1} e^{-\langle \omega,\xi\rangle}\, d\omega, \qquad \xi \in \mathbf{R}^n,$$

$$K(z) = (2\pi)^{-n} \int \frac{e^{\sqrt{-1}\langle z,\xi\rangle}}{I(\xi)}\, d\xi, \qquad z \in D\mathbf{R}^n = \{z \in \mathbf{C}^n : |\operatorname{Im} z| < 1\}.$$

We call K *Hörmander's kernel*. The boundary value representation of an element from $\mathcal{S}'^*$ is given in the following

THEOREM 9 (see [16]). *Let $u \in \mathcal{S}'^*$ and define*

$$U(z) = (u * K)(z) = \langle u(t), K(x - t + \sqrt{-1}y)\rangle,$$

for $z = x + \sqrt{-1}y \in D\mathbf{R}^n$. Then U is analytic in $D\mathbf{R}^n$ and,

- *in (M_p)-case: there are some constants $a, b, C > 0$,*
- *in $\{M_p\}$-case: for arbitrary $a, b > 0$, there exists a constant $C = C_{a,b} > 0$,*

such that

$$(7) \qquad |U(z)| \le C \exp[M(a|x|) + M^*(b/(1-|y|))], \qquad z = x + \sqrt{-1}y \in D\mathbf{R}^n.$$

and

$$(8) \qquad \langle u, \phi\rangle = \int\limits_{S^{n-1}} \langle U(\cdot + \sqrt{-1}\omega), \phi\rangle d\omega, \qquad \phi \in \mathcal{S}^*.$$

Conversely, if U satisfies (7) *then* (8) *defines an ultradistribution from $\mathcal{S}'^*$.*

Microlocal analysis. Microlocal analysis given in [16] via Hörmander's and Poisson's kernels is based on Theorems 8 and 9. In this section we will recall only two assertions from [16] and prove a theorem on the so-called weak suppleness of the microlocal support of an ultradistribution. Recall that, for an ultradistribution $f \in \mathcal{D}'^*$, the point $(x, \omega) \in S^*\Omega = \Omega \times S^{n-1}$ is not in SS_*f (resp. not in SS^*f) if and only if there exist a neighbourhood $U \subset \Omega$ of x and a conic neighbourhood Γ of ω of the form $\Gamma = \{\xi \ne 0 : |\xi/|\xi| - \omega| < \eta\}$ such that for every $\phi \in \mathcal{D}^*(U)$ the following holds:

- in (M_p)-case: for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that $|\widehat{\phi f}(\xi)| \le C_\epsilon e^{-M(\epsilon|\xi|)}$, $\xi \in \Gamma$ (resp. there are $k > 0$, and $C > 0$ such that $|\widehat{\phi f}(\xi)| \le Ce^{M(k|\xi|)}, \xi \in \Gamma$),
- in $\{M_p\}$-case: there exist $k > 0$ and $C > 0$ such that $|\widehat{\phi f}(\xi)| \le Ce^{-M(k|\xi|)}$, $\xi \in \Gamma$ (resp. for every $\epsilon > 0$, there is $C_\epsilon > 0$ such that $|\widehat{\phi f}(\xi)| \le C_\epsilon e^{M(k|\xi|)}, \xi \in \Gamma$).

Note that $SS_{\{M_p\}}$ is equivalent to the Hörmander notion of the wave front set WF_L (see [6]).

Let U be an open set in $\mathbf{C}^n$. Then a function F is said to be in $O^*|_{D\mathbf{R}^n}(U)$ (resp. $O^*|_{D\mathbf{R}^n}(U)$) if F is holomorphic in $D\mathbf{R}^n \cap U$ and, for every compact set $K \subset\subset U$,

- in (M_p)-case: there are $C > 0$ and $k > 0$,
- in $\{M_p\}$-case: for every $k > 0$ there is $C > 0$,

such that

$$|F(z)| \le C \exp\left[M^*\left(\frac{k}{1-|y|}\right)\right], \qquad z = x + \sqrt{-1}y \in K \cap D\mathbf{R}^n$$

(resp. for every compact set $K \subset\subset U$,

- in (M_p)-case: for every ultradifferential operator $P(D)$ of class (M_p),
- in $\{M_p\}$-case: for every ultradifferential operator $P(D)$ of class $\{M_p\}$,

$$P(D)F(z) \text{ is bounded in } K \cap D\mathbf{R}^n.)$$

With the notation of Theorem 9 we have

THEOREM 10 (see [16]). *Let $u \in \mathcal{S}'^*$. Then*

(a) *$q = (x, \omega) \notin SS^*u$ if and only if U is $\mathcal{O}^*$ in a neighboourhood of $x - \sqrt{-1}\omega$.*
(b) *$q \notin SS_*u$ if and only if U is $\mathcal{O}_*$ in a neighbourhood if $x - \sqrt{-1}\omega$.*
(c) *$q \notin SSu$ if and only if U is analytic at $x - \sqrt{-1}\omega$ (i.e. in a neighbourhood of this point).*

THEOREM 11 (see [16]). *Let G_j be closed subsets of S^{n-1} such that $\bigcup_{j=1}^r G_j = S^{n-1}$. Any $u \in \mathcal{S}'^*(\mathbf{R}^n)$ can be written as $u = \sum_{j=1}^r u_j$, $u_j \in \mathcal{S}'^*(\mathbf{R}^n)$ with*

(a) *$SS^*u_j \subset SS^*u \cap \mathbf{R}^n \times G_j$, $j = 1, \ldots, r$;*
(b) *$SS_*u_j \subset SS_*u \cap \mathbf{R}^n \times G_j$, $j = 1, \ldots, r$;*
(c) *$SSu_j \subset SSu \cap \mathbf{R}^n \times G_j$, $j = 1, \ldots, r$.*

If $u \in \mathcal{E}'^*$ then u_j, $j = 1, \ldots, r$, have compact supports as well.

It is proved in [5] that the sheafs $\mathcal{C}^*$ and $\mathcal{C}_*$ are supple, which means if $u \in \mathcal{C}^*$ (resp. $u \in \mathcal{C}_*$) and $SSu = Z_1 \cup Z_2$, where Z_i, $i = 1, 2$ are closed sets, then there are $u_i \in \mathcal{C}^*$ (resp. $\mathcal{C}_*$), $i = 1, 2$, such that $u = u_1 + u_2$ and $SSu_i \subset Z_i$, $i = 1, 2$. (The same holds with $u \in \mathcal{C}^*$ and $SS_*u = Z_1 \cup Z_2$.)

In the next assertion we will simply prove the suppleness if Z_i, $i = 1, 2$, are of special form.

Let $u \in \mathcal{D}'^*$. Put $\Sigma_* = \pi_2 SS_*u$, $\operatorname{sing\,supp}_* u = \pi_1 SS_*u$.

THEOREM 12. (a) *Assume $\Sigma_* = \Sigma_1 \cup \Sigma_2$ where Σ_1, Σ_2 are closed sets. Then there exist $u_1, u_2 \in \mathcal{D}'^*$ such that*

$$u = u_1 + u_2, \quad \pi_2 SS_*u_i \subset \Sigma_i, \quad i = 1, 2.$$

(b) *Assume* $\operatorname{sing\,supp}_* u = S_1 \cup S_2$, *where S_1, S_2 are closed sets. Then there exist $v_1, v_2 \in \mathcal{D}'^*$ such that*

$$u = v_1 + v_2, \quad \operatorname{sing\,supp}\, v_{i*} \subset S_i, \quad i = 1, 2.$$

P r o o f. (a) By using a partition of unity we may assume that $u \in \mathcal{E}'^*$. Put

$$\widetilde{\Sigma}_1 = \{\xi \in S^{n-1} : d(\xi, \Sigma_1) \le d(\xi, \Sigma_2)\},$$
$$\widetilde{\Sigma}_2 = \{\xi \in S^{n-1} : d(\xi, \Sigma_1) \ge d(\xi, \Sigma_2)\},$$
$$\Sigma_3 = \overline{\widetilde{\Sigma}_1 \setminus \Sigma_1}, \quad \Sigma_4 = \overline{\widetilde{\Sigma}_2 \setminus \Sigma_2}.$$

By Theorem 11,

$$u = \widetilde{u}_1 + \widetilde{u}_2 + \widetilde{u}_3 + \widetilde{u}_4, \quad SS_*\widetilde{u}_i \subset SS_*u \cap \mathbf{R}^n \times \Sigma_i, \quad i = 1, 2, 3, 4.$$

Hence $u = u_1 + u_2$, where $u_1 = \widetilde{u}_1 + \widetilde{u}_3$, $u_2 = \widetilde{u}_2 + \widetilde{u}_4$, is the decomposition with the asserted properties.

(b) By [6, Corollary 1.4.11], the exists $\phi \in C_0^\infty(\mathbf{R}^n \setminus (S_1 \cap S_2))$ such that

$$\phi(x) = \begin{cases} 0, & x \in S_1 \setminus (S_1 \cap S_2), \\ 1, & x \in S_2 \setminus (S_1 \cap S_2), \end{cases}$$

and for some $C > 0$ and $h > 0$,

$$|\phi^{(k)}(x)| \le \frac{Ch^k}{d(x)^k} \widetilde{M}_k, \quad k \in \mathbf{N}_0^n,$$

where $d(x) = \max\{d(x, S_1),\ d(x, S_2)\}$ and $\widetilde{M}_k \prec M_k$ (see [8]).

By [8] we have $u = P(D)v$, where $P = P_r$ or $P = P_{r_p}$ (see Theorem 3) and v is a compactly supported continuous function such that sing $\text{supp}_* u =$ sing $\text{supp}_* v$. Put $v_2 = P(D)\phi v$, $v_1 = P(D)((1 - \phi)v)$. Since

$$\text{sing supp}_* v_i \subset \text{ sing supp}_* u \cap S_i, \quad i = 1, 2,$$

and $u = v_1 + v_2$, the assertion is proved.

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