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Continuous mappings on continua II

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CONTENTS

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Introduction*

This paper is a continuation of a survey paper [78] by the first author in which he discussed some aspects of continuous mappings of continua. In this paper we survey some other aspects of mappings of continua. We have attempted to gather a large number of results scattered throughout the literature and to present them in a systematic way. We have provided proofs and strengthenings of many results in the literature. We have proved some new propositions that answer questions in the literature. We have also listed some problems from the literature as well as some new problems.

The paper begins with a study of the general notion of aposyndesis and its application to Jones' decompositions of homogeneous continua.

The greatest part of this paper is concerned with monotone decompositions of continua: information which we can obtain by studying decompositions of continua in general, irreducible continua, general homogeneous continua, homogeneous atriodic continua, homogeneous multicoherent continua and homogeneous continua which embed in 2-manifolds.

We extend Bellamy's result by showing the existence of weakly confluent mappings from hereditarily indecomposable continua onto an arbitrary continuum; we indicate how one can use a result of this type in studying for example the hyperspace of an arbitrary continuum.

By a modification of an unpublished proof of Bellamy that Waraszkiewicz's spirals do not have a common model we obtain short and not so technical proofs of Russo's results from [104]. The last part of this paper is exactly a review of known results and open questions concerning the existence of common models, universal continua, common image and incomparable families.

The reader may refer to [47] and [78] for terms that are not defined here.

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1. General notion of aposyndesis

All the spaces we consider are Hausdorff and compact. The collection of all closed nonempty sets of X we denote by 2^X and the collection of all subcontinua of X we denote by $C(X)$.

Let A be a subset of a space X. Then A^0, $\bar{A}$ and $\mathrm{bd}(A)$ denote the interior, the closure and the boundary of A in X, respectively. Sometimes we will wrote $\mathrm{cl}(A)$ instead of $\bar{A}$. If $\mathscr{B}$ is a collection of subsets of X, then $\mathscr{B}^*$ $= \bigcup \{B: B \in \mathscr{B}\}$ and $\mathscr{B}^0 = \bigcup \{B^0: B \in \mathscr{B}\}$.

Let T be a fixed collection of closed subsets of a space such that $X \in T$. If $A \subset X$, then we define

$$T(A) = \bigcap \{\bar{V}: V \text{ is open, } \bar{A} \subset V, \text{ there is a finite collection}$$
$$\mathscr{B} \subset T \text{ such that } \mathrm{bd}(V) \subset \mathscr{B}^0 \text{ and } \mathscr{B}^* \cap A = \emptyset\},$$

$$T^1(A) = T(A) \quad \text{and} \quad T^n(A) = T(T^{n-1}(A)) \quad \text{for} \quad n > 1.$$

If $A = T(A)$, then we say that A is *T-closed*. We say that X is *T-aposyndetic at A with respect to B* if there is $K \in T$ such that $A \subset K^0 \subset K \subset X \setminus B$. The space is called *T-aposyndetic* if it is T-aposyndetic at each point with respect to each other point. If T is fixed and $T = C(X)$, then we will drop the letter T and simply write aposyndetic for T-aposyndetic.

(1.1) $A \subset T(A)$ *and* $T(A)$ *is closed.*

(1.2)   *If* $A \subset B$, *then* $T(A) \subset T(B)$.

(1.3)   *If* $x \in T(A) \cap B^0$ *and* $B \in T$, *then* $B \cap A \neq \emptyset$.

(1.4)   *If* U *is open,* $T(A) \subset U$, *then there are an open set* V *and a finite collection* $\mathscr{B} \subset T$ *such that* $\mathrm{bd}(V) \subset \mathscr{B}^0$, $\mathscr{B}^* \cap A = \emptyset$ *and* $T(A) \subset \bar{V} \subset U$.

Let G be open and such that $T(A) \subset G \subset \bar{G} \subset U$. By the definition of $T(A)$ and compactness of X there are open sets V_i and finite families $\mathscr{B}_i \subset T$ such that $\bar{A} \subset V_i$, $\mathrm{bd}(V_i) \subset \mathscr{B}_i^0$, $\mathscr{B}_i^* \cap A = \emptyset$ and $T(A) \subset \bar{V}_i$ for $i = 1, \ldots, n$ and $\bar{V}_1 \cap \ldots \cap \bar{V}_n \subset G$. Since for each $B \in \mathscr{B}_i$ we have $B \cap A = \emptyset$, we infer $B^0 \cap T(A) = \emptyset$ by (1.3). Thus $T(A) \subset V_i$ for each $i = 1, \ldots, n$. Therefore, $T(A) \subset V_1 \cap \ldots \cap V_n \subset G$, thus $T(A) \subset V_1 \cap \ldots \cap V_n \subset \bar{G} \subset U$. Let $\mathscr{B} = \bigcup_{i=1}^{n} \mathscr{B}_i$. Since $\mathrm{bd}(V_1 \cap \ldots \cap V_n) \subset \mathrm{bd}(V_1) \cap \ldots \cap \mathrm{bd}(V_n) \subset \mathscr{B}^0$ and $\mathscr{B}^* \cap A = \emptyset$, the set V satisfies all required conditions.

(1.5)   *If* $T(A) = M \cup N$ *where M and N are separated and nonempty, then* $A \cap M \neq \emptyset$ *and* $T(A \cap M) \subset M$.

There are open sets U_1 and U_2 such that $M \subset U_1$, $N \subset U_2$ and $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. It follows from (1.4) that there are an open set V and a finite

collection $\mathscr{B} \subset T$ such that $\mathrm{bd}(V) \subset \mathscr{B}^0$, $\mathscr{B}^* \cap A = \emptyset$ and $T(A) \subset \bar{V}$ $\subset U_1 \cup U_2$. Since $\mathrm{bd}(V \cap U_i) \subset \mathrm{bd}(V)$ for $i = 1, 2$, we infer $\emptyset \neq A \cap M$ $\subset T(A \cap M) \subset U$; thus $T(A \cap M) \subset M$.

(1.6)　　*If U is open, $T(A) \subset U$, then there is an open set V such that $A \subset V$ and for each $B \subset V$ we have $T(B) \subset U$.*

Since $T(A) \subset U$, there are an open set G and a finite collection $\mathscr{B} \subset T$ such that $T(A) \subset G \subset \bar{G} \subset U$, $\mathrm{bd}(G) \subset \mathscr{B}^0$ and $\mathscr{B}^* \cap A = \emptyset$ by (1.4). Let V $= G \cap (X \setminus \mathscr{B}^*)$. Then $A \subset V$ and V is open. If $B \subset V$, then $\mathscr{B}^* \cap B = \emptyset$; thus $T(B) \subset \mathrm{cl}(G) \subset U$.

Condition (1.5) implies

(1.7)　　*if A is connected, then $T(A)$ is a continuum.*

Condition (1.6) implies

(1.8)　　*the mappings $A \to T^n(A)$ are upper semi-continuous.*

Condition (1.5) implies

(1.9)　　*if A is closed and B is a component of $T(A)$, then $B \cap A \neq \emptyset$ and $T(A \cap B) \subset B$.*

As in [9] we obtain

(1.10)　　*$A \to T(A)$ is additive (i.e. if $\mathscr{B} \subset 2^X$ and $\mathscr{B}^*$ is closed, then $T(\mathscr{B}^*)$ $= \bigcup \{T(B): B \in \mathscr{B}\}$) iff for each $A, B \in 2^X$ we have $T(A \cup B) = T(A) \cup T(B)$.*

Suppose $x \notin C = \bigcup \{T(B): B \in \mathscr{B}\}$. Then for each $B \in \mathscr{B}$ there is $F_B \in 2^X$ such that $B \subset F_B^0$ and $x \notin T(F_B)$ by (1.6). Since $\mathscr{B}^*$ is compact there is a finite subcollection $\mathscr{C} \subset \mathscr{B}$ such that $\mathscr{B}^* \subset \bigcup \{F_B: B \in \mathscr{C}\}$. Hence, by (1.2), hypothesis and induction $T(\mathscr{B}^*) \subset T(\bigcup \{F_B: B \in \mathscr{C}\}) = \bigcup \{T(F_B): B \in \mathscr{C}\}$. Since for all $B \in \mathscr{C}$, $x \notin T(F_B)$, it follows that $x \notin T(\mathscr{B}^*)$. Thus $T(\mathscr{B}^*) \subset C$. The converse inclusion is obvious by (1.2).

(1.11)　　*If X is T-aposyndetic, then $T(x) = \{x\}$ for each $x \in X$.*

The converse of (1.11) is not true in general.

(1.12)　　EXAMPLE. Let $X = [0, 1]$, $T = \{A \in C(X): A \subset X \setminus \{\frac{1}{2}\}$ or $A = X\}$. Then $T(x) = \{x\}$ for each $x \in X$ and X is not T-aposyndetic at $\frac{1}{2}$.

(1.13)　　*If X is T-aposyndetic and $A \to T(A)$ is additive, then each closed set is T-closed. Moreover, $T(A) = \bar{A}$ for each $A \subset X$.*

In fact, $T(x) = \{x\}$ for each $x \in X$ by (1.11). Therefore, if $A = \bar{A}$, then $T(A) = A$. In particular, $T(T(A)) = T(A)$ by (1.1). Hence, $A \to T(A)$ is the Kuratowski's closure operation which gives the same family of closed sets as the original closure operation on X. Thus $\bar{A} = T(A)$ for each $A \subset X$.

(1.14)   EXAMPLE. Let $A(x, y)$ denote the closed interval joining points x and y in the Euclidean plane R^2. Put

$$X_0 = A((0, 1), (1, 0)) \cup A((1, 0), (-1, 0)) \cup A((0, 1), (-1, 0)),$$
$$X_n = A((0, 1), ((1+1/n)^2, -1/n)) \cup A(((1+1/n)^2, -1/n), (-(1+1/n)^2, -1/n)) \cup$$
$$\cup A((0, 1), (-(1+1/n)^2, -1/n))$$

for $n = 1, 2, \ldots$ and

$$X = X_0 \cup \bigcup_{n=1}^{\infty} (X_n \cup A((1, -1/2n), (1, -1/(2n+1))) \cup$$
$$\cup A((-1, -1/(2n+1)), (-1, -1/(2n+2)))).$$

Let $T_1 = C(X)$ and $T_2 = \{K \in C(X): K \text{ is arcwise connected}\}$. It is easy to see that X is T_1-aposyndetic but X is not T_2-aposyndetic.

Jones' notion of aposyndesis has a quite big literature (see [36]).

2. Relation T for special families

If T is a family of subcontinua of a continuum X such that $X \in T \subset C(X)$, then we will consider the following properties:

(a$_0$) if $x \in B_1^0 \cap \ldots \cap B_n^0$, $\{B_1, \ldots, B_n\} \subset T$, then there is $B \in T$ such that $x \in B^0 \subset B \subset B_1 \cap \ldots \cap B_n$;

(a$_1$) if V is an open set in X, $\mathscr{B}$ is a finite subfamily of T such that $\mathrm{bd}(V) \subset \mathscr{B}^0$ and $\mathscr{B}^*$ is connected, then $V \cup \mathscr{B}^* \in T$;

(a$_2$) if V is an open set in X, $\mathscr{B}$ is a finite subfamily of T such that $\mathrm{bd}(V) \subset \mathscr{B}^0$ then each component of $V \cup \mathscr{B}^*$ belongs to T;

(a$_3$) if V is an open set in X, $K \in T$, $\mathscr{B}$ is a finite family of T such that $\mathrm{bd}(V) \subset \mathscr{B}^0$ then each component of $(K \cap V) \cup \mathscr{B}^*$ belongs to T;

(d) each element of T is closed domain in X (i.e. a connected set which is the closure of its interior);

(h) if h is a homeomorphism from X onto itself and $A \in T$, then $h(A) \in T$;

(u$_1$) if V is an open set in X, $A = \bar{A} \subset C_i \in T$ and $C_i \subset C_{i+1}^0 \subset X \setminus \bar{V}$ for $i = 1, 2, \ldots$, then there is $C \in T$ such that $\bigcup_{i=1}^{\infty} C_i \subset C \subset X \setminus V$;

(u$_2$) if V is an open set in X, $\mathscr{B} \subset T$, $A = \bar{A} \subset B^0$ for each $B \in \mathscr{B}$, $\mathscr{B}^* \subset X \setminus \bar{V}$ and $\mathscr{B}^*$ is open then there is $C \in T$ such that $\mathscr{B}^* \subset C \subset X \setminus V$;

(u_3) if $A = \bar{A} \subset B^0 \subset B \in T$, V is an open set in X and $B \subset X \setminus \bar{V}$, then there is $C(A, V) \in T$ such that $C(A, V) \subset X \setminus V$, $C(A, V) \cap \bar{V} \neq \emptyset$, and if $A \subset D^0 \subset D \in T$ and $D \subset X \setminus \bar{V}$, then $D \subset C(A, V)$.

It is clear that

(2.1) $(a_3) \Rightarrow (a_2) \Rightarrow (a_1)$,

(2.2) $(u_3) \Rightarrow (u_2) \Rightarrow (u_1)$,

(2.3)   *if X is an arbitrary continuum and $T = C(X)$, then T has properties* (a_3), (h) *and* (u_3),

(2.4)   *if X is an arbitrary continuum and T is the collection of all connected closed domains of X, then T has properties* (a_3), (d), (h) *and* (u_2).

Recall that a continuum X is a θ-continuum (θ_n-continuum) if the complement of every subcontinuum of X has a finite number of components (at most n components).

(2.5)   *If X is a θ-continuum and $Q_1, \ldots, Q_n \in C(X)$, then the set $X \setminus (Q_1 \cup \ldots \cup Q_n)$ has a finite number of components.*

In fact, we will prove this by induction (compare [29]). Let $Q_1, \ldots, Q_{n+1} \in C(X)$. We may assume that $Q_1, \ldots, Q_{n+1}$ are pairwise disjoint. Then there is a component C of $X \setminus (Q_1 \cup \ldots \cup Q_{n+1})$ such that the closure of C intersects two different Q_i; say $\bar{C} \cap Q_1 \neq \emptyset \neq \bar{C} \cap Q_2$. Then each component of $X \setminus (Q_1 \cup \ldots \cup Q_{n+1})$ different from C is a component of $X \setminus ((Q_1 \cup \bar{C} \cup Q_2) \cup Q_3 \cup \ldots \cup Q_{n+1})$ which completes the proof.

From (2.5) immediately follows

(2.6)   *if X is an arbitrary θ-continuum and $T = C(X)$ then T has property* (a_0).

3. Properties of T

(3.1)   *If T has property* (a_2), *then*

(i) $x \in T(A) \Leftrightarrow (x \in B^0 \subset B \in T \Rightarrow A \cap B \neq \emptyset)$.

In fact, suppose that $x \notin T(A)$. Then there are an open set V and a finite subfamily $\mathscr{B}$ of T such that $\bar{A} \subset V \subset \bar{V} \subset X \setminus \{x\}$, $\mathrm{bd}(V) \subset \mathscr{B}^0$ and $\mathscr{B}^* \cap A = \emptyset$. Since the set $X \setminus \bar{V}$ is open and $\mathrm{bd}(X \setminus \bar{V}) \subset \mathrm{bd}(V)$, the set $(X \setminus \bar{V}) \cup \mathscr{B}^*$ has a finite number of components and x belongs to the interior of one of them. Therefore, we obtain $\Leftarrow$ by (a_2). The converse also holds by (1.3).

(3.2)   *If T has property (a_3) and $T(A) = M \cup N$ where M and N are separated and nonempty, then $T(A \cap M) = M$.*

Suppose $p \in M \setminus T(A \cap M)$. Since M and N are separated, there are open sets U_1 and U_2 such that $M \subset U_1$, $N \subset U_2$, $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Since $T(A) \subset U_1 \cup U_2$, there are an open set V and a finite collection $\mathscr{B} \subset T$ such that $T(A) \subset V \subset \bar{V} \subset U_1 \cup U_2$, $\mathrm{bd}(V) \subset \mathscr{B}^0$ and $\mathscr{B}^* \subset X \setminus A$ by (1.4). Since $p \notin T(A \cap M)$, there is $B \in T$ such that $p \in B^0 \subset B \subset X \setminus (A \cap M)$ by (2.1) and (3.1). The property (a_3) implies that there is $L \in T$ such that $p \in L^0 \subset L \subset (B \cap U_1 \cap V) \cup \mathscr{B}^* \subset X \setminus A$; thus $p \notin T(A)$ by (3.1), a contradiction. The converse inclusion follows from (1.5).

As an immediate consequence of (1.8), (1.9) and (3.2) we obtain

(3.3)   *If T has property (a_3), A is closed and B is a component of $T(A)$, then $A \cap B \neq \emptyset$ and $T(A \cap B) = B$.*

Repeating the proof of Lemma 3 in [2] one can obtain that

(3.4)   *If T has property (a_2) and $A \to T(A)$ is continuous, then T is idempotent, i.e. $T(T(A)) = T(A)$.*

Theorem 3 in [2] can be reformulated as follows:

(3.5)   *If $T = C(X)$, X is T-aposyndetic, and $A \to T(A)$ is continuous, then $A \to T(A)$ is additive and X is locally connected.*

We have the following observation (compare (1.13)):

(3.6)   *If T has property (a_2), X is T-aposyndetic, and $A \to T(A)$ is additive, then $T(A) = \bar{A}$ for each $A \subset X$ and X is locally connected.*

Indeed, if $x \in V$ and V is open, then $T(X \setminus V) = X \setminus V$. Therefore, there is $B \in T \ (\subset C(X))$ such that $x \in B^0 \subset B \subset V$ by (3.1).

(3.7)   *If T has properties (a_0) and (a_2) then $A \to T(A)$ is additive.*

Let $A = \bar{A}$ and $x \in X \setminus A$. It suffices to show that if A has the property that $y \in A$ implies $T(y) \subset X \setminus \{x\}$, then $T(A) \subset X \setminus \{x\}$. For each $y \in A$ there is an open set V_y and $K \in T$ such that $y \in V_y$ and $x \in K^0 \subset K \subset X \setminus V_y$. Since A is compact, we can find a finite collection $\{K_1, ..., K_n\} \subset T$ such that $x \in K_1^0 \cap ... \cap K_n^0 \subset K_1 \cap ... \cap K_n \subset X \setminus A$. The property (a_0) implies that there is $K \in T$ such that $x \in K^0 \subset K \subset K_1 \cap ... \cap K_n$. From (3.1) we infer $x \notin T(A)$.

The following proposition has a very simple proof (see [2], p. 587):

(3.8)   *If $T_1 = C(X)$ and $T_2 = C(f(X))$ where f is continuous, then $T_2(A) \subset f T_1 f^{-1}(A)$ where $A \subset f(X)$.*

4. *T*-aposyndesis in homogeneous continua

A topological transformation group (G, X) is a topological group G together with a topological space X and a continuous map $(g, x) \to gx$ of $G \times X$ into X such that $(gh)x = g(hx)$ and if e is the identity of G, then $ex = x$ for all g, h in G and x in X.

(G, X) is *polonais* if G and X are polonais, i.e., they are separable and metrizable by a complete metric.

For each x in X let $G^x = \{gx: g \in G\}$ and $\pi_x: G \to X$ be such that $\pi_x(g) = gx$. The mapping π_x is of course continuous. It follows from Theorem (2.1) in [24] that we have

EFFROS' THEOREM. *Let (G, X) be a polonais transformation group and $x \in X$. Then $\pi_x: G \to \pi_x(G)$ is open if and only if each set G^y is of second category in itself for each $y \in X$ if and only if each set G^y is G_δ in X for each $y \in X$.*

A metric in a given metric space X we will denote by $d(\ ,\ )$ and if $A \subset X$ and $\varepsilon > 0$ then $B(A, \varepsilon)$ will denote the union of all balls with centre in A and radius $< \varepsilon$. H will always denote the group of all homeomorphisms from X onto itself.

Recall that a space X is homogeneous if for each two points x and y in X there is a homeomorphism h from X onto itself such that $h(x) = y$. It easily follows from Effros' Theorem (compare [107], [41], [55])

(4.1)   (*ε-push property*) *If X is a homogeneous metric compact space and $\varepsilon > 0$, then there is $\delta > 0$ such that if $d(x, y) < \delta$ then there exists an ε-homeomorphism h from X onto itself (i.e. $d(z, h(z)) < \varepsilon$ for all z in X) such that $h(x) = y$* (such number δ we will call an *Effros number for ε*).

A space X such that for each three different points x, y and z in X there is a homeomorphism h from X onto itself such that $h(x) = x$ and $h(y) = z$ will be called *r-homogeneous* (homogeneous by rotations).

We have

(4.2)   (*ε-push property* II) *If X is a r-homogeneous metric compact space, $x \in X$ and $\varepsilon > 0$, then there is $\delta > 0$ such that if $y, z \in X \setminus B(x, \varepsilon)$ and $d(y, z) < \delta$ then there is an ε-homeomorphism h from X onto itself such that $h(x) = x$ and $h(y) = z$.*

It suffices to consider $G = \{g \in H: g(x) = x\}$ and apply Effros' Theorem.

If X is a nonlocally connected metric continuum, then there is a point x and an open set U such that x does not belong to the interior of the closure of the component of U containing x. An easy application of (4.2) gives (compare [107] and see [17] for another types of homogeneity)

(4.3)   *If X is a r-homogeneous metric continuum then it is locally connected.*

We have

(4.4)   *If X is a metric homogeneous continuum, T has property* (h), $h \in H$ *and* $x \in X$, *then* $h(T(x)) = T(h(x))$ *and the mapping* $x \rightarrow T(x)$ *is continuous.*

In fact, if $\varepsilon > 0$ and δ is an Effros number for ε and if $d(x, y) < \delta$ then there is an ε-homeomorphism $h \in H$ such that $h(x) = y$. Therefore, $T(y) = h(T(x)) \subset B(T(x), \varepsilon)$ and $T(x) \subset B(h(T(x)), \varepsilon) = B(T(y), \varepsilon)$; thus dist$(T(x), T(y)) < \varepsilon$, i.e. $x \rightarrow T(x)$ is continuous.

(4.5)   *If X is a metric homogeneous continuum, T has properties* (h) *and* (u_2), *then T has property* (u_3) *and for each* $D \in T$ *and open nonempty set V such that* $\bar{V} \subset X \backslash D$ *and* $D^0 \neq \emptyset$ *the space X is T-aposyndetic at D with respect to* $\bar{V}$.

Proof. Let $A = \bar{A} \subset D^0 \subset D \in T$ and $D \subset X \backslash \bar{U}$ where U is open, let ε_0 = inf$\{d(x, y): x \in A, y \in X \backslash D^0\}$, ε_1 = inf$\{d(x, y): x \in D$ and $y \in \bar{U}\}$ and ε = $\frac{1}{2}$min$\{\varepsilon_0, \varepsilon_1\}$. Then $\varepsilon > 0$. Let δ be the Effros number for ε and put H_ε = $\{h \in H: d(z, h(z)) < \varepsilon\}$ for all z in X. Consider $W = \bigcup \{h(D): h \in H_\varepsilon\}$. Then

(4.5.1) $A \subset (h(D))^0$ for each $h \in H_\varepsilon$.

In fact, if $x \in X \backslash (h(D))^0 \cap A$, then there is $y \in X \backslash D$ such that $h(y) = x$ and $d(x, y) < \varepsilon < \varepsilon_0$, a contradiction with the choice of ε_0.

Moreover the choice of δ and ε implies

(4.5.2) $B(D, \delta) \subset W \subset B(D, \varepsilon) \subset X \backslash \bar{U}$.

Let $x \in W$. Then $x \in h_0(D)$ for some $h_0 \in H_\varepsilon$. Then, for each $z \in X$, $d(z, h_0(z)) < \varepsilon$. Since X is compact, there is $\eta > 0$ such that $d(z, h_0(z)) < \eta$ $< \varepsilon$ for each $z \in X$. Let ξ be the Effros number for $\varepsilon - \eta$ and H_x = $\{h \in H: d(z, h(z)) < \varepsilon - \eta$ for all $z \in X\}$ and put $W_x = \bigcup \{hh_0(A): h \in H_x\}$. If $h \in H_x$, then $hh_0 \in H_\varepsilon$; therefore, $W_x \subset W$. Since $B(h_0(A), \xi) \subset W_x$, we infer that $x \in W^0$; thus,

(4.5.3) W is open.

Conditions (4.5.1), (4.5.2), (4.5.3) and property (u_2) imply that

(4.5.4)   there is $C \in T$ such that $\bar{W} \subset C \subset X \backslash U$.

Therefore, if V is a fixed open set, $A = \bar{A} \subset X \backslash \bar{V}$, then

(4.5.5)   if $A \subset D^0 \subset D \in T$ and $D \subset X \backslash \bar{V}$, then there is $C \in T$ such that $D \subset C^0 \subset X \backslash \bar{V}$.

Put $\mathscr{B} = \{D \in T: A \subset D^0 \subset D \subset X \backslash \bar{V}\}$. Then $\mathscr{B}^* \subset X \backslash \bar{V}$, $\mathscr{B}^*$ is connected and $\mathscr{B}^*$ is open by (4.5.5). Therefore, there is $C(A, V) \in T$ such that $\overline{\mathscr{B}^*} \subset C(A, V) \subset X \backslash V$ by (u_2). The choice of $\mathscr{B}$ completes the proof of (u_3), i.e. (4.5) holds.

We will now prove, similarly,

(4.6)   *If X is a metric homogeneous continuum, T has properties* (h), (d), (a_1) *and* (u_1), *then T has property* (u_3) *and for each $D \in T$ and open nonempty set V such that $\bar{V} \subset X \setminus D$ the space X is T-aposyndetic at D with respect to $\bar{V}$.*

Proof. Let $A = \bar{A} \subset D^0 \subset D \subset X \setminus \bar{U}$, $\bar{V} \subset U$, $D \in T$, where U and V are open. Let $\varepsilon_0 = \inf \{d(x, y): x \in A, y \in X \setminus D^0\}$, $\varepsilon_1 = \inf \{d(x, y): x \in D$ and $y \in \bar{U}\}$ and $\varepsilon = \frac{1}{2} \min \{\varepsilon_0, \varepsilon_1\}$. Then $\varepsilon > 0$. Let δ be the Effros number for ε. Since T has property (d) for each $x \in D$ there is $y_x \in D^0$ such that $d(x, y_x) < \delta$. Let $h_x \in H$ be such that $h_x(y_x) = x$ and $d(h_x(z), z) < \varepsilon$ for all $z \in X$. Then $x \in (h_x(D))^0$. Since X is compact, we obtain that there are $h_1, \ldots, h_n \in H$ such that $D \subset (h_1(D))^0 \cup \ldots \cup (h_n(D))^0$ and $d(h_i(z), z) < \varepsilon$ for all $z \in X$ and $i = 1, \ldots, n$. Moreover, if $x \in X \setminus (h_i(D))^0 \cap A$, then there is $y \in X \setminus D$ such that $h_i(y) = x$ and $d(x, y) < \varepsilon < \varepsilon_0$, a contradiction with the choice of ε_0. Thus, $A \subset (h_i(D))^0$ for each $i = 1, \ldots, n$. Therefore, $h_1(D) \cup \ldots \cup h_n(D)$ is connected. Hence, $h_1(D) \cup \ldots \cup h_n(D) \in T$ by (a_1) and (h). The choice of ε implies that $h_1(D) \cup \ldots \cup h_n(D) \subset X \setminus \bar{U}$. This construction gives the following

(4.6.1)   if V is a fixed open set, $A = \bar{A} \subset X \setminus \bar{V}$, $A \subset D^0 \subset D \in T$ and $D \subset X \setminus \bar{V}$, then there is $C \in T$ such that $D \subset C^0 \subset X \setminus \bar{V}$.

Put $\mathscr{B} = \{D \in T: A \subset D^0 \subset D \subset X \setminus \bar{V}\}$. Then $\mathscr{B}^* \subset X \setminus \bar{V}$. It follows from (4.6.1) that $\mathscr{B}^* = \mathscr{B}^0$. Since X has a countable base there are $D_i \in \mathscr{B}$ such that $\mathscr{B}^* = D_1 \cup D_2 \cup \ldots$.

Observe now that the property (a_1) implies

(4.6.2)   if $C_i \in T$, $A \subset C_i^0 \subset C_i \subset X \setminus \bar{V}$ for $i = 1, 2$, then $A \subset (C_1 \cup C_2)^0$ $\subset C_1 \cup C_2 \subset X \setminus \bar{V}$ and $C_1 \cup C_2 \in T$.

By induction and (4.6.1) and (4.6.2) using D_i we can construct $B_i \in T$ such that $A \subset B_i \subset B_{i+1}^0 \subset X \setminus \bar{V}$ for $i = 1, 2, \ldots$ and $\mathscr{B}^* = B_1 \cup B_2 \cup \ldots$. Therefore, there is $C(A, V) \in T$ such that $\overline{\mathscr{B}^*} \subset C(A, V) \subset X \setminus V$ by (u_1). The choice of $\mathscr{B}$ implies that $C(A, V)$ has the properties required in (u_3).

5. Colocal connectedness and *T*-aposyndesis

The proof of the following is based on the proof of the main results in [65]:

(5.1)   (Krasinkiewicz, Minc) *Let X be a metric continuum and let T have properties* (a_1) *and* (u_3). *If A is a proper subset of X such that $A \in T$ and for each $D \in T$ such that $A \subset D \not\subseteq X$ there is $x \in X$ such that X is T-aposyndetic at D with respect to x, then there is a proper open subset G of X such that*

(i) $G = \bigcup\limits_{n=0}^{\infty} C_n$, $A \subset C_0$, $C_n \subset C_{n+1}^0$, $C_n \in T$ *for* $n = 1, 2, \ldots$,

(ii) bd(G) *is a continuum and* $\bar{G} = X$,

(iii) *if* $\emptyset \neq K^0 \subset K \in T$ *and* $K \cap \text{bd}(G) \neq \emptyset$, *then* $\text{bd}(G) \subset K$.

Moreover, if $T = C(X)$, $K \in T$ *and* $K \cap \text{bd}(G) \neq \emptyset \neq K \cap G$, *then* $\text{bd}(G) \subset K$.

Proof. Let $U_1, U_2, \ldots$ be a base for X and $\text{diam}\, U_n \underset{n}{\to} 0$. We will construct a sequence $W_0, W_1, \ldots$ of open sets in X and a sequence of continua $C_0, C_1, \ldots$ such that for $n \geq 1$ we have

$(5.1.1)_n$ $W_{n-1} \cap A = \emptyset$, $C(A, W_{n-1}) = C_{n-1} \subset C_n^0$, $C_{n-1} \in T$,

$(5.1.2)_n$ $\text{diam}\, W_n < 1/n$,

$(5.1.3)_n$ $C_n \cap U_n \neq \emptyset$,

$(5.1.4)_n$ if $x \in U_n$ and X is T-aposyndetic at C_{n-1} with respect to x, then $W_n \subset U_n$.

Let W_0 be an arbitrary open set such that X is T-aposyndetic at A with respect to $\bar{W}_0$ and $C_0 = C(A, W_0)$ by (u_3). Assume that sets $W_0, \ldots, W_{n-1}$ are constructed. Firstly, assume that there is $x \in U_n$ such that X is T-aposyndetic at C_{n-1} with respect to x. Then there is a continuum $B \in T$ such that $C_{n-1} \subset B^0 \subset B \subset X \setminus \{x\}$. In this case let W_n be an open neighbourhood of x satisfying $(5.1.2)_n$ and $\bar{W}_n \subset U_n \setminus B$ and $C_n = C(A, W_n)$, then also $(5.1.3)_n$ holds. In the opposite case there is $x \in X$ such that X is T-aposyndetic at C_{n-1} with respect to x. Let $B \in T$ be a continuum such that $C_{n-1} \subset B^0 \subset B \subset X \setminus \{x\}$. Let W_n be an open neighbourhood of x such that $\bar{W}_n \subset X \setminus B$ and $(5.1.2)_n$ holds. It suffices to prove $(5.1.3)_n$. We will show that $U_n \subset C_n = C(A, W_n)$. In fact, in the opposite case X is T-aposyndetic at C_{n-1} with respect to each point from $U_n \setminus C_n$ by the choice of C_n.

Put $G = \bigcup\limits_{n=0}^{\infty} C_n$. The set $X \setminus G = \text{bd}(G)$ is nonempty, closed and boundary. Suppose $X \setminus G = P \cup Q$ where P and Q are closed, nonempty and disjoint. Let U and V be open neighbourhoods of P and Q such that $\bar{U} \cap \bar{V} = \emptyset$. There is an integer n such that $(X \setminus U) \cap (X \setminus V) \subset C_n^0$. Let U_m be such that $m > n$, $\bar{U}_m \subset U \setminus C_n$ and $p \in U_m \cap P$. The property (a_1) implies that $(X \setminus \bar{U}) \cup C_n \in T$. Since X is T-aposyndetic at C_{m-1} with respect to p, we obtain $W_m \subset U_m$ by $(5.1.4)_m$. But $A \subset ((X \setminus \bar{U}) \cup C_n)^0$ and $(X \setminus \bar{U}) \cup C_n \subset X \setminus \bar{U}_m \subset X \setminus \bar{W}_m$, we obtain $(X \setminus \bar{U}) \cup C_n \subset C(A, W_m) = C_m$, a contradiction.

Now, let $\emptyset \neq K^0 \subset K \in T$ and $K \cap \text{bd}(G) \neq \emptyset$. Then there is an open set V such that $\bar{V} \subset K^0 \cap (X \setminus (\text{bd}(G)))$. Moreover, there is C_n such that $\bar{V} \subset C_n^0$. The property (a_1) implies that $K \cup C_n \in T$. Suppose $p \in \text{bd}(G) \setminus K$. Let U_m be such that $m > n$, $\bar{U}_m \subset X \setminus (K \cup C_n)$ and $p \in U_m$. Since X is T-aposyndetic at C_{m-1} with respect to p, we obtain $W_m \subset U_m$ by $(5.1.4)_m$. But

since $A \subset (K \cup C_n)^0$ and $K \cup C_n \subset X \setminus \bar{U}_m \subset X \setminus \bar{W}_m$,   we   obtain $K \cup C_n \subset C(A, W_m) = C_m$, a contradiction.

We say that a continuum X is *colocally T-connected* (*semi-locally T-connected*) at C if for each open set U containing C there is an open set V such that $C \subset V \subset U$ and $X \setminus V$ is connected and belongs to T ($X \setminus V$ has a finite number of components and each of them belongs to T).

(5.2)   If T has property (a_2), then $A = T(A)$ if and only if X is semi-locally T-connected at A.

(5.3)   If T has property (a_1) and $\mathscr{B} \subset T$ is such that if $A, B \in \mathscr{B}$, then there is $C \in \mathscr{B}$ such that $A \cup B \subset C^0$, then X is colocally T-connected at each component of $X \setminus \mathscr{B}^*$.

In fact, the set $X \setminus \mathscr{B}^*$ is closed. Let K be a component of $X \setminus \mathscr{B}^*$ and let U be an open neighbourhood of K in X. Then there is an open set V such that the set $V \cap (X \setminus \mathscr{B}^*)$ is closed-open in $X \setminus \mathscr{B}^*$ and $K \subset V \subset U$. Then $\mathrm{bd}(V) \subset \mathscr{B}^*$. Since $\mathrm{bd}(V)$ is compact there is a continuum $A \in \mathscr{B}$ such that $\mathrm{bd}(V) \subset A^0$. Since T has property (a_1), we infer $(X \setminus V) \cup A \in T$. Put $G = V \setminus A$. Then $K \subset G \subset U$, $X \setminus G$ is connected and belongs to T.

(5.4)   If T has property (a_1) and X is colocally T-connected at A_1 and at A_2, then X is colocally T-connected at $A_1 \cap A_2$ provided A_1 and A_2 are closed and $A_1 \cup A_2 \neq X$.

In fact, let U be an open neighbourhood of $A_1 \cap A_2$. Since A_1 and A_2 are closed, there are open sets U_1 and U_2 such that $A_1 \subset U_1$, $A_2 \subset U_2$ and $U_1 \cap U_2 \subset U$. Since X is colocally T-connected at A_1 and at A_2 there are open sets V_1 and V_2 such that $A_i \subset V_i \subset U_i$, $X \setminus V_i$ is connected and $X \setminus V_i \in T$ for $i = 1, 2$. We may assume that there is an open set G contained in $(X \setminus V_1)^0 \cap (X \setminus V_2)^0$, thus $(X \setminus V_1) \cup (X \setminus V_2) \in T$ by (a_1). Therefore, $A_1 \cap A_2 \subset V_1 \cap V_2 \subset U$ and $X \setminus (V_1 \cap V_2) \in T$; thus X is colocally T-connected at $A_1 \cap A_2$.

(5.5)   If T has property (a_1) and X is colocally T-connected at A, A is closed, then there is a continuum $K \subset A$ such that X is colocally T-connected at K and K is minimal with respect to this property.

This proposition easily follows as in (5.3).

6. Decompositions and terminal continua

Let T be a collection of closed subsets of a compact Hausdorff space X. We say that a continuous mapping f from X onto a Hausdorff space $f(X)$ is *T-admissible* provided $f(A)$ is a point for each $A \in T$. We have (see [94], compare [30])

(6.1)   *There is a unique function h which is T-admissible and such that for each T-admissible function f there is a continuous mapping g from the space $h(X)$ onto $f(X)$ such that $f = g \circ h$. Moreover, if $T \subset C(X)$, then h is monotone.*

In fact, let $F = \{f: f$ is T-admissible mapping from $X\}$. Take the product $Y = \prod \{f(X): f \in F\}$ and the diagonal mapping $h = \underset{f \in F}{\triangle} f: X \to Y$. Then h has the required properties. The decomposition of X into components of point inverses under h is also T-admissible provided $T \subset C(X)$; and hence h is monotone.

This minimal T-admissible mapping h will be called a *core T-admissible mapping of* X and the sets $h^{-1}(t)$ for $t \in h(X)$ will be called *T-layers*.

If $S = \{T(x): x \in X\}$, then S-admissible mappings (S-layers), we will call $T(x)$-*admissible mappings* (and $T(x)$-*layers*, respectively). We have

(6.2)   *A core* $T(x)$-*admissible mapping is always monotone.*

It is known (see [59] for a simple proof)

(6.3)   (Dyer) *Let X and Y be nondegenerate metric continua and let* $f: X \to Y$ *be a monotone open surjection. Then there exists a dense* G_δ-*subset A of Y having the following property: for each* $y \in A$, *for each continuum* $B \subset f^{-1}(y)$, *for each x from the interior of B in* $f^{-1}(y)$ *and for each neighbourhood U of B in X, there exists a continuum* $Z \subset X$ *containing B and a neighbourhood V of y in Y such that* $x \in Z^0$, $(f|Z)^{-1}(V) \subset U$ *and* $f|Z: Z \to Y$ *is a monotone surjection.*

(6.4)   *Let* $g: X \to Y$ *be a mapping between compact metric spaces. Then* $\{Y \in Y\}$: $g^{-1}: Y \to 2^X$ *is continuous at* $y\}$ *is a dense* G_δ-*subset of Y (see* [70]).

We reformulate Theorem 1 from [102] as follows

(6.5)   (Rogers) *Let g be a quotient mapping from a homogeneous metric continuum X onto a space Y (Y is not assumed to be Hausdorff) such that* $T = \{g^{-1}(y): y \in Y\} \subset C(X)$ *has property* (h). *Then g is open, each two elements of T are homeomorphic and homogeneous, and the space Y is a homogeneous continuum.* (In particular, Y is Hausdorff.)

The fact that Y is Hausdorff easily follows from Effros' theorem. The rest is a simple consequence of (h) and (6.4). In fact if $y, y' \in Y$ and $y \neq y'$, then let $\varepsilon = \frac{1}{3}\inf\{d(x, x'): x \in g^{-1}(y)$ and $x' \in g^{-1}(y')\}$ and let $H_\varepsilon = \{h \in H: d(h(z), z) < \varepsilon$ for all $z \in X\}$. Then the images under g of the sets $V = \{h(g^{-1}(y)): h \in H_\varepsilon\}$ and $U = \{h(g^{-1}(y')): h \in H_\varepsilon\}$ are open and disjoint.

By a *curve* we mean a 1-dimensional continuum. Recall that a mapping $f: X \to Y$ from a metric space X onto Y is *completely regular* if for given $\varepsilon > 0$ and $y \in Y$ there exists an open set V in Y containing y such that if $y' \in V$, then there is a homeomorphism h from $f^{-1}(y)$ to $f^{-1}(y')$ such that

$d(x, h(x)) < \varepsilon$ for $x \in f^{-1}(y)$. Each completely regular mapping is open. It is known (see [83], Th. 3.2, Corollary 3.3)

(6.6)   (Mason, Wilson) *If f is a completely regular monotone mapping of a metric curve onto a nondegenerate continuum Y, then $\dim Y = 1$ and for each $y \in Y$, $f^{-1}(y)$ is tree-like.*

(6.7)   (Rogers) *If we have the same assumptions and notation as in (6.5), then g is completely regular.*

Let y' belong to the open set $g(W)$ where W is open in X and of diameter less than an Effros number of ε. Let x' belong to $g^{-1}(y') \cap W$. Let $h: X \to X$ be an ε-homeomorphism such that $h(x) = x'$. It follows from (h) that h maps $g^{-1}(y)$ onto $g^{-1}(y')$. Therefore, g is completely regular.

Combining (6.6) and (6.7) we obtain (compare [100])

(6.8)   (Rogers) *If we assume in addition to the conditions in (6.5) that X is a curve, then we have that for each $y \in Y$ the set $g^{-1}(y)$ is tree-like and Y is a curve.*

A subcontinuum Q of X is called *terminal* if $K \in C(X)$ and $K \cap Q \neq \emptyset$ imply $K \subset Q$ or $Q \subset K$. A subcontinuum Q of a homogeneous metric continuum X will be called ε-*strongly terminal* if Q is terminal and $h(Q) = Q$ for each ε-homeomorphism h of X onto itself with $h(Q) \cap Q \neq \emptyset$.

(6.9)   *If Q is an ε-strongly terminal subcontinuum of a metric homogeneous continuum X, then Q is homogeneous.*

In fact, let δ be the Effros number for ε. Fix two points x and y in Q. Since Q is a continuum there is a finite sequence $x_0, \ldots, x_n$ of points of Q such that $x_0 = x$, $x_n = y$ and $d(x_{i-1}, x_i) < \delta$ for $i = 1, \ldots, n$. The choice of δ implies that there are ε-homeomorphisms h_i from X onto itself such that $h_i(x_{i-1}) = x_i$ for $i = 1, \ldots, n$. Since $h_i(Q) \cap Q \neq \emptyset$, we have $h_i(Q) = Q$. Take $h = h_n \circ \ldots \circ h_1$. Then $h(Q) = Q$ and $h(x) = y$. Therefore, Q is homogeneous.

(6.10)   *If Q is an ε-strongly terminal proper subcontinuum of a metric homogeneous continuum X, then there are a subcontinuum K of X containing Q and a monotone completely regular mapping φ from K onto a nondegenerate continuum Y such that for each $y \in Y$ the set $\varphi^{-1}(y)$ is homeomorphic to Q.*

Proof. Let $\delta > 0$ be the Effros number for $\varepsilon/3$ and let K be a component of $\overline{B(Q, \delta/2)}$ containing Q. Since X is a continuum we have that $Q \subset K \neq Q$. Let h and g be arbitrary $\varepsilon/3$-homeomorphisms of X onto itself such that $h(Q) \cap K \neq \emptyset$ and $h(Q) \cap g(Q) \neq \emptyset$. Then

(6.10.1) $h(Q) = g(Q) \subset K$.

Since $h(Q)$ is also terminal, we obtain $h(Q) \subset K$. Since $h^{-1}(g(Q)) \cap Q \neq \emptyset$ and $h^{-1}g$ is an ε-homeomorphism, we obtain $h^{-1}(g(Q)) = Q$; thus $g(Q) = hh^{-1}(g(Q)) = h(Q)$.

$\left(\begin{matrix} B \cup \\ W \end{matrix} \right)$

Therefore, the images of Q under $\varepsilon/3$-homeomorphisms form a decomposition of K. Let φ be the canonical mapping of this decomposition. We will now prove that

(6.10.2) φ *is continuous, monotone, open and completely regular.*

In fact, let h be an $\varepsilon/3$-homeomorphism of X onto itself such that $h(Q) \cap K \neq \emptyset$. Therefore there is a number $\eta > 0$ such that $d(z, h(z)) < \eta < \varepsilon/3$ for $z \in X$. Let $\xi > 0$ be the Effros number for $\omega < \varepsilon/3 - \eta$ and let $x \in h(Q)$ and $y \in B(x, \xi) \cap K$. Then there is an ω-homeomorphism g from X onto itself such that $g(x) = y$. Since gh is an $\varepsilon/3$-homeomorphism and $\mathrm{dist}(h(Q), gh(Q)) < \omega$, we obtain that φ is continuous and open. As in the proof of (6.7) we see that φ is completely regular. The proof is complete.

Combining (6.6) and (6.10) we infer

(6.11)   *If Q is an ε-strongly terminal proper subcontinuum of a metric homogeneous curve, then Q is tree-like.*

Combining (6.3) and (6.10) we infer

(6.12)   *If Q is an ε-strongly terminal proper subcontinuum of a metric homogeneous continuum, then Q is indecomposable.*

We have

(6.13)   *If Q is a proper terminal subcontinuum of a metric homogeneous continuum X, then Q is indecomposable.*

Proof. Suppose that Q is the union of two proper subcontinua A and B and let $a \in A \setminus B$ and $b \in B \setminus A$. Put $\varepsilon = \frac{1}{2} \min \{d(a, B), d(b, A)\}$. Let h be an ε-homeomorphism of X onto itself such that $h(Q) \cap Q \neq \emptyset$. To obtain a contradiction it suffices to show that $h(Q) = Q$ by (6.12).

We may assume that $h(A) \cap Q \neq \emptyset$. Since B is not contained in $h(A)$, we have $h(A) \subset Q$. Therefore $h(B)$ intersects Q and similarly we obtain $h(B) \subset Q$, thus $h(Q) \subset Q$. Since h^{-1} is an ε-homeomorphism and $h^{-1}(Q) \cap Q \neq \emptyset$, the same arguments give us the inclusion $h^{-1}(Q) \subset Q$, i.e. $Q \subset h(Q)$.

The above proof is a simplification of our proof given us by J. Krasinkiewicz.

It follows from Theorem (10.2.2) in [23] that

(6.14)   *If f is a mapping from a 1-dimensional continuum X onto a tree-like continuum Y such that sets $f^{-1}(y)$ are tree-like continua, then X is a tree-like continuum.*

7. Decompositions of homogeneous continua

The following proposition generalizes Jones's result about decomposition of homogeneous continua (see [52], [53]):

(7.1)   *If X is a homogeneous metric continuum, $T \subset C(X)$, there is*

$D \in T \setminus \{X\}$ such that $D^0 \neq \emptyset$, T has properties (a_1) and (h) and either T has properties (d) and (u_1) or T has property (u_2), then the core $T(x)$-admissible mapping φ has the following properties

(i) φ is continuous, open, and monotone,

(ii) for each $x \in X$ we have $T(x) = \varphi^{-1}\varphi(x)$ and X is colocally T-connected at $T(x)$,

(iii) $T(x)$-layers are homeomorphic, boundary sets in X which are homogeneous,

(iv) if $\emptyset \neq K^0 \subset K \in T$ and $K \cap T(x) \neq \emptyset$, then $T(x) \subset K$ for each $x \in X$,

(v) $\varphi(X)$ is a homogeneous metric continuum which is colocally S-connected at each point where $S = \{\varphi(A): A \in T\}$.

Moreover, if $T = C(X)$, then $T(x)$-layers are indecomposable terminal subcontinua of X and if X is a curve, $T = C(X)$, then $T(x)$-layers are tree-like.

Proof. According to (4.5) and (4.6) we can apply (5.1). Let G be as in (5.1) and $R = \mathrm{bd}(G)$. By (5.3) X is colocally T-connected at R. The properties of R imply that if $x \in R$, then $T(x) \subset R$. If $y \in R \setminus T(x)$ then there is $K \in T$ such that $K^0 \cap R \neq \emptyset$ and $K \subset X \setminus \{x\}$, because R is a continuum. Therefore (5.1) (iii) implies that $R \subset K$, a contradiction. Hence, for each $x \in R$, we have $R = T(x)$.

Now let $h \in H$ so that $h(R) \cap R \neq \emptyset$. Since X is colocally T-connected at R and $h(R)$ the property (iii) of (5.1) implies $h(R) = R$. The remaining arguments are clear by (1.8), (6.2), (6.4) (compare the proof of (6.5) and (iv)).

Recall that a mapping f from X onto Y is atomic if for each continuum $K \subset X$ we have $K = f^{-1}f(K)$ provided $f(K)$ is nondegenerate. It is known (see [25])

(7.2)   each atomic mapping is monotone.

Observe that by (7.1)

(7.3)   If X is a homogeneous metric decomposable continuum, $T = C(X)$, then the core $T(x)$-admissible mapping is atomic.

The following problem remains open

(7.4)   Does it follow that the atomic image of a homogeneous continuum is homogeneous?

We have

(7.5)   If X is a homogeneous metric decomposable continuum, $T = C(X)$, φ is the core $T(x)$-admissible mapping and f is a monotone mapping from X onto a Hausdorff space, then either $f^{-1}f(x) \subset \varphi^{-1}\varphi(x)$ or $\varphi^{-1}\varphi(x) \subset f^{-1}f(x)$ for each $x \in X$ (this follows from the fact that a mapping φ of continua is atomic if and only if each point inverse under φ is a terminal continuum in the domain).

It follows from (6.3)

(7.6)   *If F is an open atomic mapping from a metric continuum X, then there is a dense G_δ set A in $f(X)$ such that $f^{-1}(t)$ is indecomposable for each $t \in A$.*

From (4.5) and (7.1) we easily conclude

(7.7)   *If X is a homogeneous metric decomposable continuum, $T_1 = C(X)$ and $T_2 = \{A \in C(X): A$ is the closure of an open set which is continuum-wise connected\}, then $T_1(x) = T_2(x)$ for each $x \in X$.*

8. Indecomposable continua and colocal connectedness

Using Bellamy's mapping approach (see [7], [64], compare [80]) one can prove the following:

(8.1)   (Bellamy, Rogers) *Let C, K and X be Hausdorff continua such that $\emptyset \neq K \cap C \neq K$, $C, K \subset X$ and D be the indecomposable Knaster's continuum. Then either there is a subcontinuum Q of X such that $K \cap C$ is contained in the interior of $Q \cap K$ in K and $Q \cap K \neq K$ or there is a continuous mapping f from X onto D such that $f(K) = D$ and $f(C) = (0, 0, \ldots)$.*

Sketch of the proof. D is an inverse limit of arcs $I_n = [0, 1]$ with bonding maps $h_n: I_{n+1} \to I_n$ given by $h_n = h$ and $h(t) = 2t$ for $t \leqslant 1/2$ and $h(t) = 2 - 2t$ for $t \geqslant 1/2$. Suppose, that there is no continuum Q such that $K \cap C$ is contained in the interior of $Q \cap K$ in K and $Q \cap K \neq K$. Let U and V be open sets such that $\bar{U} \cap \bar{V} = \emptyset$, $C \subset U$ and $V \cap K \neq \emptyset$. Let f_1 be an arbitrary Urysohn function from X onto $[0, 1]$ such that $f_1(\bar{U}) = 0$ and $f_1(\bar{V}) = 1$. Assume, $f_1, \ldots, f_n$ are constructed in this way that $C \subset (f_m^{-1}(0))^0$, $K \cap (f_m^{-1}(1))^0 \neq \emptyset$ and $f_m = h \circ f_{m-1}$ for $m > 1$. It suffices to construct f_{n+1} having such properties. Let C^* be the component of $X \setminus (f_n^{-1}(1))^0$ containing C. Then by assumptions the set $(f_n^{-1}(0))^0 \cap K$ is not contained in C^*. Therefore there is a decomposition $M \cup N$ of $X \setminus (f_n^{-1}(1))^0$ such that the sets $M \cap K \cap (f_n^{-1}(0))^0$ and $N \cap K \cap (f_n^{-1}(0))^0$ are nonempty. Assume $C^* \subset M$ and define $f_{n+1}(x) = \frac{1}{2} f_n(x)$ for $x \in M \cup f_n^{-1}(1)$ and $f_{n+1}(x) = 1 - \frac{1}{2} f_n(x)$ for $x \in N$.

It is known (see [64])

(8.2)   (Krasinkiewicz, Minc) *A Hausdorff continuum X can be mapped onto an indecomposable continuum Y if and only if there are two closed and disjoint sets A and B in X such that each subcontinuum of X irreducible between them is indecomposable.*

Recall that a continuum is *δ-connected* if each pair of its points can be joined by a hereditarily decomposable continuum.

It follows from (8.1) and (8.2) that

(8.3)   If X is a δ-connected continuum and $X \neq K \in T = C(X)$, then X can not be mapped onto an indecomposable continuum and there is a point $x \in X$ such that X is T-aposyndetic at K with respect to x.

Propositions (8.3) and (5.1) imply (compare [65])

(8.4)   (Krasinkiewicz, Minc) If X is a δ-connected metric continuum and $X \neq K \in T = C(X)$ then there is a continuum $L \subset X \setminus K$ such that $L^0 = \emptyset$, X is colocally T-connected at L and L is terminal in X (thus if X is arcwise connected, then L is degenerate).

In the nonmetric case we have

(8.5)   If X is a δ-connected continuum and $X \neq K \in T = C(X)$, then there is a continuum $L \subset X \setminus K$ such that $L^0 = \emptyset$, X is colocally T-connected at L and L is minimal with respect to these properties. Moreover, the intersection of each two such continua is empty.

Proof. Applying (8.3) we can construct a sequence of proper subcontinua of X such that $K \subset C_1^0 \subset C_1 \subset C_2^0 \subset C_2 \subset \ldots$ Then X is colocally T-connected at each component of $X \setminus (C_1 \cup C_2 \cup \ldots)$ by (5.3). From (5.5), we infer that there is a continuum $L \subset X \setminus K$ such that X is colocally T-connected at L and L is minimal with respect to this property. If the interior of L is nonempty, then $\overline{X \setminus L}$ is a proper subcontinuum of X so we can apply (8.3) once more to obtain a sequence of proper subcontinua of X such that $\overline{X \setminus L} \subset D_1^0 \subset D_1 \subset D_2^0 \subset D_2 \subset \ldots$ Taking an arbitrary component of $X \setminus (D_1 \cup D_2 \cup \ldots)$ we obtain a contradiction by (5.3). Now, if we have two such continua L_1 and L_2, then $L_1 \cup L_2 \neq X$; thus X is colocally T-connected at $L_1 \cap L_2$ by (4). Moreover, X is colocally T-connected at each component of $L_1 \cap L_2$, a contradiction.
From (8.5) we infer   .

(8.6)   If X is a δ-connected homogeneous continuum and $T = C(X)$, then X can be covered by a collection of mutually disjoint boundary continua at which X is colocally T-connected.

Unfortunately, we de not know whether this decomposition is upper semi-continuous. We conclude only

(8.7) If X is a δ-connected r-homogeneous continuum and $T = C(X)$, then X is colocally T-connected at each point (because elements of the covering from (8.6) must be degenerate).

In the metric case we have much more (by (7.1))

(8.8)   (Rogers) If X is a δ-connected homogeneous metric continuum and $T = C(X)$, then X is colocally T-connected at each point.

9. Closed domains in homogeneous continua

Recall that a set U is *continuum-wise connected* if each two points of U can be joined in U by a continuum. Put

$$R(A) = \{B \in 2^X: \text{ there is an open set } U \text{ such that } A \subset U \subset \bar{U} = B$$
$$\text{and } U \text{ is continuum-wise connected}\}.$$

(9.1)   If $B \in R(A)$, then B is a closed connected domain.

Now, we will prove

(9.2)   If X is a homogeneous metric continuum and $\emptyset \neq A^0 \subset A \in C(X) \setminus \{X\}$, then $R(A) \setminus \{X\} \neq \emptyset$. Moreover, if $B \in R(A) \setminus \{X\}$, then there is a continuous mapping α from $[0, 1]$ into $C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(t') \in R(\alpha(t))$ for $0 \leqslant t < t' \leqslant 1$.

Proof. If $\varepsilon > 0$ and $\emptyset \neq A^0 \subset A \in C(X) \setminus \{X\}$, then we put

$$R(A, \varepsilon) = \text{cl}\left(\bigcup \{L \in C(X): A \subset L^0 \subset L \subset \overline{(B(A, \varepsilon))}^0\}\right).$$

It follows from (4.5) that

(9.2.1) $R(A, \varepsilon) \in R(A)$ and $R(A, \varepsilon) \subset \overline{B(A, \varepsilon)}$.

Now, let $B \in R(A) \setminus \{X\}$ and U be an open set such that $A \subset U \subset \bar{U} = B$ and U is continuum-wise connected.

Fix $\varepsilon > 0$ and assume $0 < \varepsilon < \text{dist}(A, B)$ where dist denotes the Hausdorff metric in the hyperspace 2^X. It suffices to show that there is a finite collection $B_1, B_2, \ldots, B_n$ such that $B_{i+1} \in R(B_i)$ for $i = 1, 2, \ldots, n-1$, $B_1 \in R(A)$ and $B \in R(B_n)$, $\text{dist}(A, B_1) < \varepsilon$, $\text{dist}(B_n, B) < \varepsilon$ and $\text{dist}(B_{i-1}, B_i) < \varepsilon$ for $i = 2, 3, \ldots, n$.

We may assume that $\inf \{d(x, y): x \in A, y \in X \setminus U\} = 2$. Let $A = B_0 = C_0$ and by induction we define

C_{k+1} is the component of $B(C_k, \varepsilon/2) \cap \left(X \setminus B(X \setminus U, 1/(k+1))\right)$ containing C_k and

$B_{k+1} = R\left(C_{k+1}, \min\left(\varepsilon/2, 1/(2(k+1)(k+2))\right)\right)$.

It follows from the construction that

(9.2.2) $C_k \subset B_k \subset C_{k+1} \subset U$ for each $k = 0, 1, 2, \ldots$
(9.2.3) $\text{dist}(C_k, C_{k+1}) \leqslant \varepsilon$ for $k = 0, 1, 2, \ldots$

According to (9.2.2), (9.2.3) we need only prove that there in n such that $\text{dist}(C_n, B) = \text{dist}(C_n, \bar{U}) \leqslant \varepsilon$. Take a finite set F such that $\text{dist}(F, \bar{U}) < \varepsilon/2$ and $F \subset U$. Since U is continuum-wise connected, there is a continuum Q such that $A \cup F \subset Q \subset U$. There is k such that $Q \cap B(X \setminus U, 1/k) = \emptyset$. From the properties of the hyperspace $C(X)$ we conclude that there are continua $A = K_1, K_2, \ldots, K_m = Q$ such that $K_i \subset K_{i+1}$ and $\text{dist}(K_i, K_{i+1}) < \varepsilon/2$ for $i = 1, \ldots, m-1$. Of course,

$$K_2 \subset B(A, \varepsilon/2) \cap (X \setminus B(X \setminus U, 1/k)) \subset B(C_k, \varepsilon/2) \cap (X \setminus B(X \setminus U, 1/k)).$$

Therefore, $K_2 \subset C_{k+1}$. Since

$$K_3 \subset B(K_2, \varepsilon/2) \cap (X \setminus B(X \setminus U, 1/k)) \subset B(C_{k+1}, \varepsilon/2) \cap (X \setminus B(X \setminus U, 1/(k+1))),$$

we obtain $K_3 \subset C_{k+2}$. Therefore, $K_m \subset C_{k+m-1}$ by an easy induction. The inclusion $F \subset Q = K_m$ gives that $\text{dist}(B_{k+m-1}, B) \leqslant \varepsilon$ by (9.2.2). In this way the proof is complete.

The same arguments show

(9.3)   If X is a metric continuum, $A \in C(X)$ and $B \in R(A) \setminus \{X\}$, then there is a continuous mapping α from $[0, 1]$ into $C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$, $\alpha(t) \subset \alpha(t') \neq \alpha(t)$ and $\alpha(t) \subset B^0$ for $0 \leqslant t < t' < 1$.

10. Irreducible continua

It is obvious that

(10.1)   If X is an irreducible continuum between a and b and $a \in C \in C(X)$, then $X \setminus C$ is connected. Moreover, X is a θ_2-continuum.

A *composant* of a point p in a continuum X is the set of all points which can be joined with p by a proper subcontinuum of X. It is clear

(10.2)   If X is a continuum, C is a composant of X, then $\bar{C} = X$.

Now, we have the following even in the nonmetric case (compare [70], p. 211):

(10.3)   If X is an irreducible continuum between a and b, C is the composant of a in X, then the set $\overline{X \setminus C}$ is connected. Moreover, if it is not a boundary set, then it is an indecomposable connected closed domain.

Proof. (Kuratowski). Suppose $\overline{X \setminus C} = M \cup N$, where M and N are closed and disjoint and with $b \in M$. There is an open set G such that $M \subset G \subset \bar{G} \subset X \setminus N$. Then $(\bar{G} \setminus G) \cap (M \cup N) = \emptyset$. Let K be the component of the point b in $\bar{G}$. Then $K \cap (\bar{G} \setminus G) \neq \emptyset$ and if $p \in K \cap (\bar{G} \setminus G)$, then $p \in C$. Therefore, there is a continuum P such that $a, p \in P \subset C$. Thus, $K \cup P$ is a continuum containing a and b. Hence, $K \cup P = X$. Therefore, $N \subset K \cup P \subset \bar{G} \cup C$. Thus $N \subset C$ and $X \setminus C \subset (X \setminus N) \cap (M \cup N) = M$; i.e. $\overline{X \setminus C} \subset M$ and $N = \emptyset$. This implies that the set $\overline{X \setminus C}$ is connected.

Let $Q = \overline{X \setminus \overline{X \setminus C}}$. Suppose that $\overline{X \setminus C}$ is not boundary. Then $Q \neq X$. Since $\overline{X \setminus C}$ is a continuum containing b, by (10.1), we obtain that Q is a continuum and if it is nonempty, then $a \in Q$ and $Q \subset C$. Therefore, $X \setminus C \subset X \setminus Q$ and $\overline{X \setminus C} \subset \overline{X \setminus Q} = X \setminus \overline{X \setminus \overline{X \setminus C}} \subset \overline{X \setminus C}$. Thus, $\overline{X \setminus C} = \overline{X \setminus Q}$, i.e. $\overline{X \setminus C}$ is a closed domain.

Suppose $\overline{X \setminus C} = M \cup N$, where M and N are proper subcontinua of $\overline{X \setminus C}$. If $N \setminus C = \emptyset$, then $X \setminus C \subset X \setminus N$ and $X \setminus C \subset \overline{X \setminus C} \setminus N \subset M$. Thus, $\overline{X \setminus C} \subset M$, a contradiction. Therefore, the sets $M \setminus C$ and $N \setminus C$ are nonempty.

If $Q = \emptyset$, then $\overline{X \setminus C} = X = M \cup N$. If $a \in M$, then $M = X$ since $M \not\subset C$, which completes the proof. Assume $Q \neq \emptyset$ and then $a \in Q$. Since $X = \overline{X \setminus \overline{X \setminus C}} \cup \overline{X \setminus C} = Q \cup M \cup N$, we may assume $Q \cap M \neq \emptyset$ by the connectedness of X. But then $Q \cup M = X$. Thus, $\overline{X \setminus Q} \subset M$. Since $\overline{X \setminus Q} = \overline{X \setminus C}$, we obtain $\overline{X \setminus C} = M$, i.e. $N = \emptyset$ and the proof is complete.

We say that a set $A \subset X$ is *a set of irreducibility* of X if there is a point $a \in X$ such that no proper subcontinuum of X contains $\{a\} \cup A$. It is known (see [69])

(10.4)   (Kuratowski) *A set A is a set of irreducibility of a metric continuum X if and only if there are no proper subcontinua P and R of X such that $X = P \cup R$ and $A \subset P \cap R$.*

As a corollary we obtain (see [105])

(10.5)   (Sorgenfrey) *Each nondegenerate unicoherent metric continuum X which is not a triod is irreducible between two points.*

S h o r t   p r o o f. Since each indecomposable metric continuum is irreducible, we may assume that $X = H \cup K$ where H and K are proper subcontinua of X. Since X is unicoherent, the set $H \cap K$ is a continuum. If $H \cap K$ is not a set of irreducibility of H, then $H = P \cup R$ and $H \cap K \subset P \cap R$ where P and R are proper subcontinua of H; $P \cap R = P \cap (R \cup K)$ is connected since X is unicoherent and $X = P \cup R \cup (P \cap R) \cup K$, but then $X = P \cup R \cup K$ is a triod, a contradiction. Therefore, we may assume that H is irreducible between a and $H \cap K$ and K is irreducible between b and $H \cap K$. Let W be a proper subcontinuum of X containing a and b. Then $(K \setminus H) \cup (H \setminus K) \subset W$. Let C be a component of $(W \cap K) \setminus H$ which contains b. Then $C \cup (H \cap K)$ is a continuum which contains b and $H \cap K$. Hence $C \cup (H \cap K) = K$. It follows that $W \cup H = W \cup K = W \cup (H \cap K) = X$ and $X = (K \cap W) \cup (H \cap W) \cup (K \cap H)$ is a triod, as above, a contradiction.

Recall that by an *n-od* we mean a continuum X containing a proper subcontinuum C such that $X \setminus C$ is a union of n separated nonvoid sets $U_1, \ldots, U_n$. The set C is called a *kernel* of X and the sets $C \cup U_i$ are the *links of X*. The next two propositions were first proved in [63].

(10.6)   (Krasinkiewicz, Minc) *If X is a continuum irreducible about a finite set $\{a_1, \ldots, a_n\}$ where $n > 2$ is either of the form $n = 2k - 1$ or $n = 2k$, then X can be represented as a union of k irreducible subcontinua.*

P r o o f (Krasinkiewicz, Minc). Let $n = 2k$, $k > 1$. Let M be a subcon-

tinuum of X irreducible about $\{a_3,\ldots,a_n\}$ and let N be a subcontinuum of X irreducible between a_1 and a_2. Suppose $M \cap N = \emptyset$ and $G = X \backslash (M \cup N)$. Each subcontinuum of X meeting both M and N contains G. Let M_1 be a subcontinuum of X irreducible about $\{a_2,\ldots,a_{n-1}\}$ and let N_1 be a subcontinuum of X irreducible between a_1 and a_n. Then $G \subset M_1 \cap N_1$. By the induction assumption (10.6) holds for n.

(10.7)   (Krasinkiewicz, Minc) *If X is a unicoherent metric continuum which cannot be represented as a union of k irreducible subcontinua, then X is a $(2k+1)$-od.*

Short proof. Suppose (10.7) fails. Let n be the maximal natural number such that X is an n-od but not an m-od for $m > n$. By (10.5) we have $3 \leqslant n < 2k+1$. Let X_0 be a subcontinuum of X such that $X \backslash X_0$ is a union of n separated nonvoid sets $X_1,\ldots,X_n$. By (10.4) X_0 is an irreducibility set of each link $X_i \cup X_0$. Therefore, each subcontinuum of X containing some $a_j \in X_j$ and meeting X_0 contains $\bar{X}_j$. Then X is irreducible about the set $\{a_1,\ldots,a_n\}$. Indeed, if Y is a proper subcontinuum of X irreducible about $\{a_1,\ldots,a_n\}$, then $X \backslash Y \subset X_0$ and $X_0 \backslash Y \neq \emptyset$. By the unicoherence of X the set $Y \cap X_0$ is connected. It follows that X is an $(n+1)$-od with kernel $Y \cap X_0$ and links X_0, $X_j \cup (Y \cap X_0)$ for $j \geqslant 1$, a contradiction. By (10.6) X is a union of k irreducible continua. With this contradiction the proof is complete.

11. Decompositions onto locally connected continua

Let us first observe that

(11.1)   EXAMPLE. There is a continuum which has no minimal monotone decomposition onto a locally connected continuum.

In fact, let

$$X_0 = \{(x, y, 0) \in R^3 : 0 \leqslant x, y \leqslant 1\} \quad \text{and} \quad X_n = \{(x, y, 1/n) : 0 \leqslant x, y \leqslant 1\}$$

and π_i be the projection from R^3 onto the ith axis where $i = 1, 2, 3$. Consider the space $X = \bigcup_{n=0}^{\infty} X_n$ and the equivalence relation $\sim$ on X defined by: if $x, y \in X$, then $x \sim y$ if and only if either $x = y$ or $\pi_1(x) = \pi_1(y)$, $\pi_2(x) = \pi_2(y)$ and $\{\pi_1(x), \pi_2(x)\} \cap \{0, 1\} \neq \emptyset$. Let $Y = X/\sim$ and φ be the canonical mapping from X onto Y. Put $g_i(t) = \pi_i \varphi^{-1}(t)$ for $t \in Y$ and $i = 1, 2$. Then Y is a continuum which is not locally connected; for $i = 1, 2$ g_i is a monotone mapping from Y onto $[0,1]$ and the monotone decomposition of Y, which refines decompositions $\{g_1^{-1}(t): t \in [0, 1]\}$ and $\{g_2^{-1}(t): t \in [0, 1]\}$, gives the identity mapping on Y.

One can observe that if we consider copies of the Sierpiński universal

plane curve instead of discs X_n, then our constructed example is even one-dimensional.

Now we have

(11.2)   *If X is a continuum, $T = C(X)$ has property (a_0), then the core $T(x)$-admissible mapping φ on X is a mapping which gives the minimal monotone decomposition onto a locally connected continuum and the $T(x)$-layers are T-closed.*

Proof. Firstly, from (3.7) we obtain

(11.2.1) $T\big(\varphi^{-1}(t)\big) = \varphi^{-1}(t)$ for each $t \in \varphi(X)$.

Now, if $T_1 = C\big(\varphi(X)\big)$, then we will prove that

(11.2.2) T_1 has property (a_0).

Indeed, let $t \in B_1^0 \cap \ldots \cap B_n^0$, $\{B_1, \ldots, B_n\} \subset T_1$. Since φ is monotone, we have $\{\varphi^{-1}(B_1), \ldots, \varphi^{-1}(B_n)\} \subset T$. Moreover, if $x \in \varphi^{-1}(t)$, then $x \in (\varphi^{-1}(B_1))^0 \cap \ldots \cap (\varphi^{-1}(B_n))^0$. Since T has property (a_0), there is $B_x \in T$ such that $x \in B_x^0 \subset B_x \subset \varphi^{-1}(B_1) \cap \ldots \cap \varphi^{-1}(B_n)$. By the compactness of $\varphi^{-1}(t)$, we find a finite collection $B_{x_1}, \ldots, B_{x_m}$ such that $B_{x_1}^0, \ldots, B_{x_m}^0$ cover $\varphi^{-1}(t)$. Then $t \in (\varphi(B_{x_1}) \cup \ldots \cup \varphi(B_{x_m}))^0 \subset \varphi(B_{x_1}) \cup \ldots \cup \varphi(B_{x_m}) \subset B_1 \cup \ldots \cup B_n$, and the set $\varphi(B_{x_1}) \cup \ldots \cup \varphi(B_{x_m})$ is a continuum, i.e. (11.2.2) holds.

(11.2.3) $\varphi(X)$ is T_1-aposyndetic.

In fact, if $t, t' \in \varphi(X)$ and $t \neq t'$, then using (11.2.1) and compactness of $\varphi^{-1}(t')$ we can find a finite collection $\{B_1, \ldots, B_n\} \subset T$ such that $\varphi^{-1}(t') \subset B_1^0 \cup \ldots \cup B_n^0 \subset B_1 \cup \ldots \cup B_n \subset X \setminus \varphi^{-1}(t)$ and $\varphi^{-1}(t') \cap B_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. Then $t' \in (\varphi(B_1) \cup \ldots \cup \varphi(B_n))^0 \subset \varphi(B_1) \cup \ldots \cup \varphi(B_n) \subset \varphi(X) \setminus \{t\}$ and $\varphi(B_1) \cup \ldots \cup \varphi(B_n)$ is a continuum, i.e., (11.2.3) holds.

Now, applying (3.6), (3.7) and conditions (11.2.2) and (11.2.3) we obtain

(11.2.4) $\varphi(X)$ is a locally connected continuum.

It remains to show that if ψ is a monotone continuous mapping from X onto a locally connected continuum Y, then ψ is $T(x)$-admissible. Take $x, x' \in X$ such that $\psi(x) \neq \psi(x')$. Since Y is locally connected, there is a continuum B such that $\psi(x') \in B^0 \subset B \subset Y \setminus \{\psi(x)\}$. Then $x' \in (\psi^{-1}(B))^0 \subset \psi^{-1}(B) \subset X \setminus \{x\}$ and $\psi^{-1}(B)$ is a continuum, because ψ is monotone. Therefore, $x' \notin T(x)$. Thus, $T(x) \subset \psi^{-1}\psi(x)$, i.e. ψ is $T(x)$-admissible.

From (11.2) we infer that

(11.3)   *If a continuum X is either hereditarily unicoherent or a θ-continuum and $T = C(X)$, then the core $T(x)$-admissible mapping φ on X is a minimal*

monotone mapping onto a locally connected continuum and the $T(x)$-layers are T-closed.

In particular (see [70], p. 216)

(11.4)   (Kuratowski) *If X is an irreducible continuum, then X has a minimal monotone decomposition into an arc.*

Now, we have

(11.5)   EXAMPLE. There is a continuum X such that if $T = C(X)$ and φ is the core $T(x)$-admissible mapping on X with $T_1 = C(\varphi(X))$, then $\varphi(X)$ is not T_1-aposyndetic, but the core $T^2(x)$-admissible mapping ξ maps X onto a locally connected continuum.

First, let F be the Cantor ternary set lying in $[0, 1]$, η be the Cantor step-function from F onto $[0, 1]$ and $\psi(t) = |t - \frac{1}{2}|$ for $t \in R$. Let $A(x, y)$ denote the line segment joining points x and y in the Euclidean space R^3. Put

$$X_1 = \bigcup \{A((t, -1, z), (\eta(t), 0, z)): t \in F, z \geqslant 0\},$$

$$X_2 = \bigcup \{A((x, -1/n, 1/n), (x, -1, 0)): x \geqslant 0, n = 1, 2, ...\},$$

$$X_3 = \bigcup \{A((0, 0, 0), (1/2, 1/n, 0)) \cup A((1, 0, 0), (1/2, 1/n, 0)): n = 1, 2, ...\}$$

and

$$X_0 = (X_1 \cap X_2) \cup X_3 \cup (X_1 \cap \{(x, y, 0): x, y \in R\}).$$

Now, define an equivalence relation on X_0 by:

$$(x, y, z) \sim (x', y', z') \Leftrightarrow (x, y, z) = (x', y', z')$$

or

$$\psi(x) = \psi(x') \quad \text{and} \quad y = y' = -1.$$

Put $X = X_0/\sim$. It is easily observed that X has all of the required properties. $\varphi(X)$ is obtained from X_3 by folding the limit arc $A((0, 0, 0), (1, 0, 0))$ in half. $\xi(X)$ is a *bouquet* of a null sequence of circles.

(11.6)   *If X is a θ-continuum, $\emptyset \neq K^0 \subset K \in C(X) = T$ and K is indecomposable, then K is a closed domain, K^0 is connected and if $x \in K$, then $K \subset T(x)$.*

In fact, the set $X \setminus K$ has a finite number of components; thus, the set $X \setminus (\overline{X \setminus K}) = K^0$ has a finite number of components by (2.5). Now, if C is a component of K^0, then $\overline{C}$ is a subcontinuum of K with nonempty interior; thus $\overline{C} = K$ because K is indecomposable. Therefore, K is a closed domain and $K^0 = C$.

Now, let $x \in K$ and $y \in K^0 \setminus \{x\}$. If $y \notin T(x)$, then there is $L \in C(X)$ such that $y \in L^0 \subset L \subset X \setminus \{x\}$. By property (a_0) (see (2.6)) there is $M \in C(X)$ such that $y \in M^0 \subset M \subset K \cap L$. But then M is a proper subcontinuum of K with nonempty interior, a contradiction. Hence, $K^0 \subset T(x)$. Thus, $K \subset T(x)$.

Propositions (11.3) and (11.6) imply

(11.7)   *If f is a monotone mapping from a θ-continuum X onto a locally connected continuum Y such that $(f^{-1}(y))^0 = \emptyset$ for each $y \in Y$, then every indecomposable continuum in X has an empty interior.*

It is clear that

(11.8)   *The monotone image of a θ-continuum is a θ-continuum.*

It is know that (see [29])

(11.9)   (Fitzgerald) *A locally connected metric θ-continuum is a finite graph.*

Now, we will prove (compare [32])

(11.10)   (Kuratowski, Gordh) *If X is an irreducible continuum between a and b such that each indecomposable subcontinuum of X has an empty interior, then $\{T^2(x): x \in X\}$ is a core $T(x)$-admissible decomposition of X with boundary $T(x)$-layers, where $T = C(X)$.*

Sketch of proof. Let C_a and C_b be composants of a and b in X, respectively. It follows from (10.3) that the sets $X \setminus C_a$ and $X \setminus C_b$ are boundary continua. If $c \in C_a \cap C_b$ and $X_{a,c}$ and $X_{b,c}$ are continua irreducible between a and c, b and c respectively, then $X = X_{a,c} \cup X_{b,c}$ and $X_{a,c}$ and $X_{b,c}$ are proper subcontinua of X such that $X_{a,c} \subset C_a$ and $X_{b,c} \subset C_b$. Continua $X_{a,c}$ and $X_{b,c}$ can be decomposed, therefore, there is a continuum K such that $c \in K^0 \subset K \subset C_a \cap C_b$. Hence $T(a) \subset X \setminus C_b$ and $T(b) \subset X \setminus C_a$. Thus $T(a) = X \setminus C_b$ and $T(b) = X \setminus C_b$. Moreover, if $x \in X \setminus C_b$, then $T(x) = X \setminus C_b$.

Now, if $C_{a,c}$ is the composant of a in $X_{a,c}$ and if $C_{b,c}$ is the composant of b in $X_{b,c}$, then the same arguments show that for each $x \in X \setminus (C_{a,c} \cup C_{b,c})$ we have $T^2(x) = X \setminus (C_{a,c} \cup C_{b,c})$; thus also $T^2(x) = T^2(y)$ or $T^2(x) \cap T^2(y) = \emptyset$ for $x, y \in X$.

The iterations of T also give decompositions for θ_n-continua. More precisely, we have (see [38] for the proof)

(11.11)   (Grace, Vought) *Let X be a compact, metric θ_n-continuum. Then X admits a monotone upper semi-continuous decomposition $\mathcal{D}$ such that the elements of $\mathcal{D}$ have void interior and the quotient space $X/\mathcal{D}$ is a finite graph if and only if $[T(H)]^0 = \emptyset$ for every subcontinuum H with void interior where $T = C(X)$. Furthermore, $\mathcal{D} = \{T^{n(n+1)}(x): x \in X\}$.*

The requirement that X contains no indecomposable subcontinuum with nonvoid interior in (11.11) is too weak (even for θ_1-continua, see [108] for an example and a discussion of the planar case). Grace in [37] has proved a version of (11.11) for θ-continua. It is clear that

(11.12)   *If X is irreducible about $\{a_1, \ldots, a_n\}$, then X is a θ_n-continuum.*

The bibliography on upper semi-continuous decompositions of irreducible continua is rather large begining with [50], [57], [58], [68] and [69]. A more complete study on irreducible continua is contained in [106]. Decompositions of continua irreducible about a finite set were studied in [103], [109] and [110] (for further remarks see Section 18 here).

12. Locally connected homogeneous continua

We have the following simple observation

(12.1)   *If X is a locally connected homogeneous metric continuum which is not a simple closed curve, then X contains simple closed curves of arbitrarily small diameters.*

In fact, if X does not contain simple closed curves of small diameter, then it is a local dendrite (see [70] for the definition and properties). Therefore, it can be decomposed into a finite union of dendrites having finite intersections. Since X fails to contain end-points, we obtain X is a finite graph. Since X is homogeneous and has only a finite set of remification points, we obtain that X has no remification point. Thus it is a simple closed curve.

It is known (see [1], [84])

(12.2)   (Anderson, Mazurkiewicz) *The simple closed curve and the Menger universal curve are the only homogeneous, locally connected metric curves.*

Recall that a continuum X is *Suslinian* if every collection of nondegenerate mutually disjoint subcontinua is countable.

(12.3)   *If X is a Suslinian homogeneous metric continuum, then X is a simple closed curve.*

Since X is Suslinian it must be a curve which is locally connected by Theorem 2 in [14]. According to (12.2) X is a simple closed curve.

Each manifold without boundary is homogeneous. The Hilbert cube is homogeneous with respect to countable closed sets (see [56]). This last proposition can be generalized (see [31]); namely: if P is the product of a countably infinite number of manifolds with boundary and U and V are countable dense subsets of P, then there is a homeomorphism h of P onto itself such that $h(U) = V$. Some other properties (and problems) concerning homogeneous manifolds one can find in [15].

It is an open question whether each arcwise connected homogeneous continuum is locally connected (K. Kuperberg). We know (see [10]) that an arcwise connected homogeneous metric continuum contains no arc-cut point or end-point (for locally connected homogeneous metric continua this is a simple observation).

13. Atriodic homogeneous continua

A continuum X is said to be *triodic* if it contains three continua such that the common part of all three of them is both a nonvacuous proper subcontinuum of each of them and the common part of every two of them. A continuum is *atriodic* if it contains no triodic continuum. It is clear that

(13.1)   *The intersection of any two subcontinua of an atriodic continuum has at most two components.*

(13.2)   *Suppose X is an atriodic continuum and K is a proper subcontinuum of X which is not unicoherent. Then K is terminal in X. In particular, $K^0 = \emptyset$.*

Proof. Let A and B be proper subcontinua of K such that $K = A \cup B$ and $A \cap B = P \cup Q$ where P and Q are disjoint continua (compare (13.1)). Let A' be a continuum in A irreducible between P and Q and let B' be a continuum in B irreducible between $P \cap A'$ and $Q \cap A'$. Then

(13.2.1) $P \cap A'$ and $Q \cap A'$ are continua and $A' \cup P \cup Q = A$ and $B' = B$.

In fact, the sets $P \cap A'$ and $Q \cap A'$ are continua by (13.1) because $A' \cap B = (P \cap A') \cup (Q \cap A')$. Now, let U and V be open neighbourhoods of P and Q such that $\bar{U} \cap \bar{V} = \emptyset$ and let E and F be components of $B \cap \bar{U}$ and $B \cap \bar{V}$ containing P and Q, respectively. If $A \setminus (A' \cup P \cup Q) \neq \emptyset$, then $A \cup E \cup F$ is a triod, a contradiction.

According to (13.2.1) we may assume that A and B are irreducible between each point from P and each point from Q. Let $K' \in C(X)$ such that $K' \setminus K \neq \emptyset \neq K' \cap K$. It suffices to prove $K \subset K'$. Just suppose $K \not\subset K'$. By (13.1) $K \cap K'$ has at most two components C_1 and C_2. We must consider three cases:

(a) $C_1 \cap B = \emptyset$. Let U_1, U_2 and U_3 be open neighbourhoods with disjoint closures of C_1, P and Q, respectively. If D_1, D_2 and D_3 denote the components of $\bar{U}_1 \cap K'$, $\bar{U}_2 \cap B$ and $\bar{U}_3 \cap B$, respectively, containing C_1, P and Q, respectively, then the set $(A \cup D_1) \cup (A \cup D_2) \cup (A \cup D_3)$ is a triod, a contradiction.

(b) $C_1 \cap P \neq \emptyset$ and $C_1 \cap Q = \emptyset$. Let U be an open neighbourhood of $C_1 \cup P$ such that $\bar{U} \subset X \setminus (C_2 \cup Q)$. If D_1, D_2 and D_3 denote the components

of $\bar{U} \cap K'$, $\bar{U} \cap (A \cup C_1)$, $\bar{U} \cap (B \cup C_1)$, respectively, and such that $C_1 \cup P \subset D_1 \cap D_2 \cap D_3$, then the set $D_1 \cup D_2 \cup D_3$ is a triod, a contradiction.

(c) $C_1 \cap P \neq \emptyset$ and $C_1 \cap Q \neq \emptyset$. Since A and B are irreducible, we may assume that $A \subset C_1$. The assumption and symmetry imply, by cases (a) and (b) that $C_2 = \emptyset$ and $C_1 \cap B$ has exactly two components D_1 and D_2 (containing P and Q, respectively). Let U_1 and U_2 be open neighbourhood of D_1 and D_2 with disjoint closures and let K_1 and K_2 be components of $B \cap \bar{U}_1$ and $B \cap \bar{U}_2$ containing D_1 and D_2 respectively. Then the set $(C_1 \cup K_1) \cup (C_1 \cup K_2) \cup K'$ is a triod, a contradiction.

(13.3)   *If Q is an indecomposable subcontinuum of an atriodic homogeneous metric continuum X, then Q is terminal in X.*

Suppose that $K \in C(X)$ and $K \setminus Q \neq \emptyset \neq Q \cap K$. The set $K \cap Q$ has at most two components C_1 and C_2. Let U be an open neighbourhood of C_1 such that $\bar{U} \cap C_2 = \emptyset$ and let K_0 be a component of $\bar{U} \cap K$. Then $K_0 \setminus Q \neq \emptyset \neq Q \setminus K_0$ and $K_0 \cap Q$ is a continuum contained in one composant of Q. Let $a \in K_0 \setminus Q$, $b \in Q \setminus K_0$ and $x_0 \in Q \cap K_0$. Let $\varepsilon = \frac{1}{2} \min \{d(a, Q), d(b, K_0)\}$ and let δ be an Effros number for ε. Take points $x_1, x_2 \in B(x_0, \delta)$ such that x_0, x_1, x_2 are in different composants of Q and let g and h be ε-homeomorphisms of X onto X such that $g(x_0) = x_1$ and $h(x_0) = x_2$. The choice of ε implies that $K_0 \cup h(K_0) \cup g(K_0) \subset Q \setminus \{b\}$ and $\{a, h(a), g(a)\} \subset X \setminus Q$. If the continua K_0, $h(K_0)$ and $g(K_0)$ are pairwise disjoint, then $Q \cup K_0 \cup h(K_0) \cup g(K_0)$ is a triod, a contradiction. If $K_0 \cup h(K_0) \cup g(K_0)$ is a continuum, then the intersection $(K_0 \cup h(K_0) \cup g(K_0)) \cap Q$ has at least three components, a contradiction by (13.1). Since Q has infinitely many composants and each of these is dense in Q this completes the proof.

We have (compare with the introduction in [73])

(13.4)   *If X is a homogeneous, atriodic nondegenerate metric continuum, then either X is one-dimensional or every nondegenerate subcontinuum of X is infinite dimensional.*

Proof. Suppose that there is a subcontinuum K of X such that $\dim K = 2$ and K is the union of a finite number of closed sets each of diameter less than 1. At least one of these closed sets has dimension 2 and at least one of its components has dimension 2. Hence, K has a 2-dimensional subcontinuum K_1 of diameter less than 1. Similarly, K_1 has a 2-dimensional subcontinuum K_2 of diameter less than 1/2, K_2 has a 2-dimensional subcontinuum K_3 of diameter less than 1/3,... Let $\{p\} = \bigcap_{n=1}^{\infty} K_n$.

Now, let Q be an arbitrary nondegenerate subcontinuum of X. Take three different points x_1, x_2, x_3 lying in Q. Since for each point $q \in X$ we have the same property as at the point p, we can find disjoint continua Q_i

such that $x_i \in Q_i$ and $\dim Q_i = 2$ for $i = 1, 2, 3$. Since X is atriodic, we obtain that one of continua Q_1, Q_2, Q_3 is contained in Q; thus $\dim Q = 2$.

Therefore, if X is finite dimensional, then it is a curve. Now, assume that X is infinite dimensional. Since each subcontinuum of X, by the above arguments, is either one-dimensional or infinite dimensional, as above, we can find a point $p \in X$ and a sequence of continua $C_1, C_2, \ldots$ such that $C_{i+1} \subset C_i$, $\dim C_i = \infty$ for $i = 1, 2, \ldots$ and $\bigcap\limits_{n=1}^{\infty} C_i = \{p\}$. Let Q be an arbitrary nondegenerate subcontinuum of X. Take three different points x_1, x_2, x_3 lying in Q. We can find disjoint continua Q_i such that $x_i \in Q_i$ and $\dim Q_i = \infty$ for $i = 1, 2, 3$. Since X is atriodic, we obtain that one of continua Q_1, Q_2, Q_3 is contained in Q. Thus, $\dim Q = \infty$.

The same arguments can be applied to show

(13.5) *If X is a homogeneous, nondegenerate, metric continuum which does not contain an n-od (in fact an ω-od), then either X is one-dimensional or every nondegenerate subcontinuum of X is infinite dimensional.*

We do not know the answers to the following questions (compare [73], Question 1): Does there exist a homogeneous nondegenerate metric continuum which is atriodic (resp. hereditarily unicoherent, hereditarily indecomposable) and which is not one-dimensional? [1]

We have the following simple observation (see [52], see also [35])

(13.6) (Jones) *If X is a hereditarily unicoherent, homogeneous metric continuum, then X is indecomposable.*

In fact, if X is decomposable and $T = C(X)$, then the image of X under the core $T(x)$-admissible mapping φ is a homogeneous hereditarily unicoherent metric nontrivial continuum, which is colocally T-connected at each point by Theorem (7.1) where $T_1 = C(\varphi(X))$. If $\varphi(X)$ were locally connected it would be a dendrite and hence a point. Therefore $\varphi(X)$ is not locally connected. Let $x \in \varphi(X)$ and U be an open neighbourhood of x in $\varphi(X)$ such that $x \notin C^0$, where C is the component of $\bar{U}$ containing x. There is an open neighbourhood V of x such that $V \subset U$ and $\varphi(X) \setminus V$ is a continuum. Let A be the unique continuum irreducible between x and $\varphi(X) \setminus V$ in C and let $y \in A \setminus \{x\}$. Since $\varphi(X)$ is hereditarily unicoherent, we infer that $x \in T_1(y)$, a contradiction because $\varphi(X)$ is colocally T_1-connected.

(13.7) *If f is a monotone open mapping from an atriodic continuum X, then f is atomic.*

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[1] One answer follows from the recent paper of J. T. Rogers, Jr. *Homogeneous, hereditarily indecomposable continua are tree-like*, Huston J. Math. 8 (1982), 421–428.

It suffices to show that if $K \in C(X)$, $y \in f(X)$ and $f^{-1}(y) \cap K \neq \emptyset$, then either $f^{-1}(y) \subset K$ or $K \subset f^{-1}(y)$. Suppose that $K \not\subset f^{-1}(y)$ and $f^{-1}(y) \not\subset K$. Then we can find $y_1, y_2 \in f(K)$ such that y, y_1, y_2 are different and $K \not\subset f^{-1}(y_1)$ and $K \not\subset f^{-1}(y_2)$ by the openness of f. The ·set $K \cup f^{-1}(y) \cup f^{-1}(y_1) \cup f^{-1}(y_2)$ is a triod, a contradiction.

(13.8)   *Every proper subcontinuum of an atriodic homogeneous metric continuum X is unicoherent.*

It follows from (13.2) that a proper subcontinuum Q of X which is not unicoherent is decomposable and terminal. This is a contradiction to (6.13).
It is known (see [89])

(13.9)   (Moore) *If, in the plane, G is an uncountable set of triodic continua, there exists an uncountable subset H of G such that every two continua of the set H have a point in common.*

14. Arcs and pseudoarcs in homogeneous continua

In [14], p. 228, R. H. Bing proved that each homogeneous circle-like (inverse limit of circles) metric continuum that contains an arc is a solenoid. This result was used by Hagopian in [44] to prove the following

(14.1)   *A homogeneous metric continuum with only arcs as proper nondegenerate subcontinua is a solenoid (i.e. it is homeomorphic to an inverse limit of circles with covering maps as the bonding maps).*

Now, we have (see [14] and [102])

(14.2)   *If A is an arc component of a homogeneous metric space X, B is an arc in X such that $B \cap \bar{A} \neq \emptyset$, then $B \subset \bar{A}$ and $\bar{A}$ is homogeneous.*

Let $b \in B \setminus \bar{A}$ and $a \in B \cap \bar{A}$. Let $\varepsilon = d(b, \bar{A})$ and δ be an Effros number for ε. Let p be a point of A such that $d(a, p) < \delta$ and let h be an ε-homeomorphism of X onto X such that $h(p) = a$. Since $h(A)$ is an arc component of X containing a, we ·infer $b \in h(A)$. Thus $d(b, \bar{A}) < \varepsilon$, a contradiction.
Let p and q be two points of an arc component of X. The sum of all arcs in X that have p as an endpoint and contain q is called a *ray starting at p and going by q*. We will say that X is *simple acyclic* (resp. *simple atriodic*) if X does not contain a simple closed curve (resp. a simple triod, i.e. a union of three arcs having only one end-point in common).

(14.3)   (Rogers) *If A is an arc component of a homogeneous metric continuum*

X which is simple acyclic and simple atriodic, and $a, b \in A$, *then the ray starting at a and going by b is dense in A.*

Proof (Rogers). Firstly observe that

(14.3.1)   No arc in $\bar{A}$ contains an open set of $\bar{A}$.

If some arc in $\bar{A}$ contains an open set of $\bar{A}$, then the homogeneity of $\bar{A}$ would imply that $\bar{A}$ is a 1-manifold, i.e. a simple closed curve.

(14.3.2)   *If p is a point of A, and if one of the rays R starting at p has the property that* $\bar{R} = \bar{A}$, *then the other ray starting at p also has this property.*

Suppose S is the other ray starting at p and q is a point of $A \setminus \bar{S}$. Let $\varepsilon = d(q, \bar{S})$ and δ be an Effros number for $\varepsilon/2$. Let $[p, r]$ be an arc in A containing q in its interior such that $d(p, r) < \delta$. Let h be an $\varepsilon/2$-homeomorphism such that $h(p) = r$. Since q is not a point of $h(\bar{S})$, it follows that $h(S)$ is the ray starting at r that does not contain q and that q is not a point of $\overline{h(S)}$. But $h(S) \supset R \setminus [p, r]$. Hence $[p, r]$ contains an open set of $\bar{A}$ containing q. This contradicts (14.3.1).

Now, suppose S is a ray starting at $p \in A$ and $q \in A \setminus \bar{S}$. Let $\varepsilon = d(q, \bar{S})$ and δ be an Effros number for $\varepsilon/2$. It follows from the proof of (14.3.2) that no arc $[p, r]$ of A containing q has the property that $d(p, r) < \delta$. Hence, if T is the ray starting at q and not containing S, then $d(\bar{S}, \bar{T}) \geqslant \delta$. Hence $[p, q]$ contains an open set of $\bar{A}$. We have the following generalization of Rogers' theorem (see [102]):

(14.4)   *If the atriodic, homogeneous, metric continuum X contains an arc, then X is a solenoid.*

Proof. Firstly, by (13.4) we obtain

(14.4.1) X is a curve.

Let A be an arc component of X. Then

(14.4.2)   Each proper subcontinuum K of $\bar{A}$ is an arc.

Let $p \in K$. Since, by (14.3), each of the rays starting at p is dense in $\bar{A}$, there is an arc $[q_1, q_2]$ containing p the endpoints of which are not in K. If r were a point of $K \setminus [q_1, q_2]$, then there would be an arc $[r, s]$ such that s is not a point of K. Then $K \cup [q_1, q_2] \cup [r, s]$ would contain a triod.

(14.4.2) and (14.1) imply that $\bar{A}$ is a solenoid. By (6.8) we conclude $\bar{A} = X$.

(14.5)   *If X is a homogeneous metric curve which is simple atriodic, then either X is a simple closed curve or X is simple acyclic.*

In fact, if X contains a simple closed curve A as a proper subset, then A is an arc component of X. According to (14.2), homeomorphic copies of A

give a decomposition of X. But by (6.8) we obtain that A is acyclic, a contradiction.

Now, we have

(14.6)   *If X is a homogeneous atriodic metric continuum which is T-aposyndetic where $T = C(X)$, then X is a simple closed curve.*

It suffices to show that X is locally connected by (14.4). According to (7.1) we have that X is colocally T-connected at each point. Let $y \in X$ and U be an open neighbourhood of y in X. Then there is an open neighbourhood V of y such that $V \subset U$ and $X \setminus V$ is a continuum. At most two components of U intersect V (otherwise we have a triod). Therefore the component of $\bar{U}$ containing y contains this point in its interior. Hence X is locally connected.

(14.7)   *Let X be a homogeneous metric and atriodic decomposable continuum. Then there is an open atomic mapping f from X onto a simple closed curve Y such that the sets $f^{-1}(y)$ are homeomorphic, homogeneous and indecomposable.*

Proof. According to (7.1) and (7.3) there is an open atomic mapping f from X such that sets $f^{-1}(y)$ are homeomorphic and homogeneous and $f(X)$ is a homogeneous metric continuum which is colocally connected at each point. It follows from (7.6) that the sets $f^{-1}(y)$ are indecomposable. Since f is monotone, we infer that $f(X)$ is atriodic. By (14.6) we finally obtain that $f(X)$ is a simple closed curve.

(14.8)   *Every metric homogeneous atriodic continuum containing a nondegenerate hereditarily decomposable continuum is a solenoid.*

Proof. Let Q be a proper hereditarily decomposable subcontinuum of a metric homogeneous atriodic continuum X. Since Q is unicoherent (by (13.8)) and it is not a triod, we infer, by (10.5) that Q is irreducible between some two points a and b. Proposition (11.10) implies that there is a continuous monotone mapping φ from Q onto $[0, 1]$ such that $\varphi^{-1}(t)$ is a boundary set in Q for each $t \in [0, 1]$.

Fix $t \in [0, 1] \setminus \{0, 1\}$ and let $W = \varphi^{-1}(t)$. Take two numbers t_0, t_1 such that $0 < t_0 < t < t_1 < 1$ and φ^{-1} is continuous at t_0 and t_1. Then the set $\varphi^{-1}[t_0, t_1]$ is a continuum which is irreducible between each point of $\varphi^{-1}(t_0)$ and each point of $\varphi^{-1}(t_1)$. Let $t_0 < t_0' < t < t_1' < t_1$. For $A, B \subset X$ put

$$\varepsilon(A, B) = \inf \{d(x, y) : x \in A, \ y \in B\}$$

and

$$\varepsilon = \tfrac{1}{2} \min \{\varepsilon(\varphi^{-1}[t_0, t_1], \ \varphi^{-1}(0) \cup$$
$$\cup \varphi^{-1}(1)), \ \varepsilon(\varphi^{-1}(t_0), \ \varphi^{-1}[t_0', 1]), \ \varepsilon(\varphi^{-1}(t_1), \ \varphi^{-1}[0, t_1'])\}.$$

Let δ be an Effros number for ε, $v \in \varphi^{-1}(t_0', t_1') \cap B(W, \delta)$ and let $\dot{w} \in W$ be

such that $d(w, v) < \delta$. There is an ε-homeomorphism h from X onto itself such that $h(w) = v$. The choice of ε and the atriodicity of X imply that $\varphi^{-1}(t_0', t_1') \subset h(\varphi^{-1}[t_0, t_1]) \subset Q \setminus (\varphi^{-1}(0) \cup \varphi^{-1}(1))$. Since W is a layer of $\varphi^{-1}[t_0', t_1']$, we have that $h(W)$ is a layer of $h(\varphi^{-1}[t_0, t_1])$ and the layer of Q containing v is a layer of $h(\varphi^{-1}[t_0, t_1])$. Therefore $h(W)$ is the layer of v in Q. Thus,

(14.8.1)   there are an $\varepsilon > 0$ and $s, r \in [0, 1]$ such that $0 < s < t < r < 1$ and images of W under ε-homeomorphisms give the decomposition of $\varphi^{-1}[s, r]$ onto the arc $[s, r]$.

Since $\varphi^{-1}[s, r]$ is atriodic, we infer that the sets $\varphi^{-1}(p)$ for $p \in [s, r]$ are indecomposable continua by (7.6). Therefore, the set $\varphi^{-1}[s, r]$ is an arc, because Q is hereditarily decomposable. Thus, X is a solenoid by (14.4) and the proof is complete.

One can observe that (14.8.1) is still valid if we assume only that Q is decomposable. Using this fact we can construct an indecomposable continuum in X which has a composant with continuous monotone decomposition onto a line. Unfortunately, we are not able to show anything about the other composants of this continuum. This approach seemed to us a good way to solve the following questions:

Let X be a homogeneous metric atriodic (one-dimensional) continuum which contains a decomposable subcontinuum. Does it follow that there is an open atomic mapping f from X onto some solenoid Y such that the sets $f^{-1}(y)$ are homeomorphic, homogeneous, hereditarily indecomposable (tree-like) continua? Is every indecomposable proper subcontinuum of X hereditarily indecomposable?

The same approach was used by Jones in [42], where he gave a sketch of a proof of the following:

(14.9)   (Jones, Hagopian) *Every planable homogeneous indecomposable continuum is hereditarily indecomposable.*

The complete proof of (14.9) is in [54], where Hagopian constructed just such an indecomposable continuum with a nice composant but he then used the notion of accessibility in the plane which is not available here.

The following solves Bing's question (see [14], p. 288, compare [45]):

(14.10)   (Hagopian, Rogers) *Each homogeneous circle-like continuum other than a solenoid contains a pseudoarc. Moreover, every circle-like homogeneous continuum is a pseudo-arc, a solenoid of pseudoarcs or a solenoid.*

It follows from (6.8) that a solenoid of solenoids cannot exist (Rogers' remark in [100]). Rogers in [99] has constructed an uncountable collection of homogeneous continua called solenoids of pseudoarcs (compare [45]).

It is known (see [101], compare [27], [28] and [97] and Hagopian in [43] gave an easy proof that the pseudocircle is not homogeneous)

(14.11)   (Rogers) *Each homogeneous indecomposable plane continuum is a tree-like continuum; in particular, the pseudocircle is not homogeneous.*

Moreover (see [93], Theorem 3.6)

(14.12)   (Oversteegen, Tymchatyn) *Each subcontinuum of a homogeneous indecomposable plane continuum has span zero* (see [72] for the definition).

It has been conjectured by Lelek that the class of continua with span zero coincides with the class of all chainable continua.

Further questions concering homogeneous continua can be found in [10], [73] and [75].

15. Homogeneous continua in compact 2-manifolds

Now we will follow [14]. Recall the following notations concerning the abutting of arcs in the plane R^2. Suppose ab, cd and ef are arcs in R^2 such that $ab \cap cd = \{c\}$ and $ab \cap ef = \{e\}$ are interior points of ab. Then cd and ef are said to abut on opposite sides of ab if there is a homeomorphism of R^2 onto itself that takes ab onto a horizontal segment and cd, ef onto vertical segments which lie except for their points of contact with ab on opposite sides of the line containing ab.

A sequence of arcs $A_1, A_2, \ldots$ is said to *converge homeomorphically to an arc* A_∞ if for each positive number ε there is an integer n such that if $n < i$, there is a homeomorphism of A_i onto A_∞ that moves no point more than ε.

Suppose ab, cd, ef are arcs such that cd and ef abut on ab from opposite sides. A sequence of arcs $A_1, A_2, \ldots$ converging homeomorphically to ab is said to *converge homeomorphically from the cd side of ab* if none of the arcs intersects ab and all but possibly a finite number of these arcs intersects cd. Two sequences of arcs converging homeomorphically to ab are said to *converge homeomorphically from opposite sides* if one of the sequence converges from the cd side of ab and the other from the ef side of ab. It is known (see [14], Theorem 3 and 4)

(15.1)   (Bing) *If W is an uncountable collection of mutually exclusive arcs in R^2, then there is an element w of W and two sequences of elements of W converging homeomorphically to w from opposite sides.*

(15.2)   (Bing) *Suppose $B, B_1, B_2, \ldots$ is a sequence of mutually exclusive arcs in R^2 such that $B_1, B_3, \ldots$ and $B_2, B_4, \ldots$ converge homeomorphically to B from*

opposite sides. If C is a continuum intersecting each B_i but neither end of B and h is a homeomorphims of $C \cup B \cup B_1 \cup B_2 \cup \ldots$ into R^2, then $h(B_1)$, $h(B_3)$,... and $h(B_2)$, $h(B_4)$,... converge homeomorphically to $h(B)$ from opposite sides.

The following proposition generalizes Theorem 1 from [14] and it gives the answer to Bing's question (see [14], p. 228).

(15.3)   (Bing) *If X is a 1-dimensional homogeneous continuum that contains arcs and lies on a compact 2-manifold M, then X is a simple closed curve.*

Proof (A modification of Bing's proof). Let A be an arbitrary arc component of X. Then $\bar{A}$ is a homogeneous continuum by (14.2). $\bar{A}$ is not the Menger universal curve U (because U is not locally planable (see [1])). We may assume that $\bar{A}$ is not a simple closed curve, because, otherwise, homeomorphic copies of $\bar{A}$ give a decomposition of X with layers which are not tree-like, which is impossible (compare (6.8)). Therefore, according to (12.2) we obtain that $\bar{A}$ is not locally connected. It follows from Theorem 2 in [14] that $\bar{A}$ contains an open set U with uncountably many components. Of course we may assume that $\bar{U}$ is planable. If $\bar{A}$ contained a simple triod, it would follow from the homogeneity of $\bar{A}$ that each component of U would contain a simple triod. This would violate the fact that the plane does not contain uncountably many exclusive triods (compare (13.9)). Therefore $\bar{A}$ is a simple atriodic and, by (14.5)

(15.3.1) $\bar{A}$ is a simple atriodic and simple acyclic.

Moreover,

(15.3.2) $\bar{A}$ has uncountably many arc components.

If $\bar{A}$ had only countably many arc components, it would follow from the Baire Category Theorem that one of the arcs in A contains an open subset of $\bar{A}$. The homogeneity of $\bar{A}$ would then imply that $\bar{A}$ is a 1-manifold, i.e. the simple closed curve, a contradiction.

(15.3.3)   If C is a nondegenerate subcontinuum of $\bar{A}$ that is not an arc, then C intersects uncountably many arc components of $\bar{A}$.

This is true by (15.3.2) if $C = \bar{A}$ so we suppose that C is a proper subcontinuum of $\bar{A}$. Let p be a point of $\bar{A} \setminus C$ and B be the arc component of $\bar{A}$ containing p. Since each ray is dense in $\bar{A}$ (by (14.3)), there is a sequence of points p_1, p_{-1}, p_2, p_{-2},... of $B \setminus C$ such that B is the union of the arcs $p_i p_{i+1}$ and no two of the $p_i p_{i+1}$'s intersect except possibly at an end-point of each. If one considers the intersections of these arcs $p_i p_{i+1}$ with C, one finds that $B \cap C$ is the sum of a countable collection of mutually exclusive closed sets. Since no continuum is the sum of a countably infinite number of mutually exclusive closed point sets, C intersects uncountably many arc components of $\bar{A}$.

(15.3.4)   Each nondegenerate proper subcontinuum of $\bar{A}$ is an arc.

Suppose C is a nondegenerate proper subcontinuum of $\bar{A}$ that is not an arc. Let ab be an arc which is a component of the intersection $C \cap E$ where E is an arc component of $\bar{A}$ (compare (15.3.3)). Since $\bar{A} \subset M$ and M is a 2-manifold, there is an open disk D containing ab in its interior. Take a neighbourhood V of ab such that $\bar{V} \subset D$ and let C' be a component of $\bar{V} \cap C$ containing ab. Then C' is a nondegenerate proper subcontinuum of $\bar{A}$ that is not an arc and that has a neighbourhood V which is planable. It follows from (15.3.3) and the fact that each ray is dense in $\bar{A}$ that V contains an uncountable collection $\mathscr{C}$ of mutually exclusive arcs each of which intersects C' but no one of which has an end on C'. It follows from (15.1) that there is one of these arcs B that has two sequences $B_1, B_3, \ldots$ and $B_2, B_4, \ldots$ of the arcs from $\mathscr{C}$ converging homeomorphically to B from opposite sides. Considering only small motions of $C' \cup B \cup B_1 \cup B_2 \cup \ldots$ we are still in the plane. Applying (15.2) we infer that there is an $\varepsilon > 0$ such that under no ε-homeomorphism h of $C' \cup B \cup B_1 \cup B_2 \cup \ldots$ into V is the image of any interior point of B accessible from the complement of $h(C' \cup B \cup B_1 \cup \cup B_2 \cup \ldots)$. But such images form an open set in V by Effros' theorem, a contradiction.

Proposition (14.1) and condition (15.3.4) imply that $\bar{A}$ is a solenoid, which is not a simple closed curve. This is impossible, because no solenoid which is not a simple closed curve can be embedded into 2-manifold (see [60], [61], compare [86]).

The proof of (15.3) is complete.

Now, we will generalize Jones' result from [51].

(15.4)   (Jones) *If X is a 1-dimensional homogeneous continuum that lies on a compact 2-manifold M and that is colocally T-connected where $T = C(X)$, then X is a simple closed curve.*

Proof. According to (15.3) we may assume that X is not locally connected. It follows from Theorem 2 in [14] that X contains an open set U with uncountably many components. We may assume that $\bar{U}$ is planable, $X \setminus U$ is a continuum and $\bar{U}$ has uncountably many components each of which meets U. There is a component C of $\bar{U}$ which meets U and which is atriodic. Take a point c belonging to $C \cap U$ and let V be an open neighbourhood of c such that $X \setminus V$ is a continuum and $\bar{V} \subset U$. Denote the component of $\bar{V}$ containing c by K. Then $K \subset C$ and the set $K \cap (X \setminus V)$ has at most two components C_1 and C_2 (otherwise C contains a triod; in fact, if C_1, C_2, and C_3 are three different components of $K \cap (X \setminus V)$ and V_1, V_2 and V_3 are open neighbourhoods of C_1, C_2 and C_3, respectively, such that $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$ are pairwise disjoint and $\bar{V}_1 \cup \bar{V}_2 \cup \bar{V}_3 \subset U$; then the components of $\bar{V}_i \cap C$ containing C_i for $i = 1, 2, 3$ together with K form a triod). If either C_1

or C_2 is not a set of irreducibility of K, then using (10.4) in an easy way we construct a triod in C; thus in each case K is an irreducible continuum. Moreover, since X is colocally T-connected at each point of $K \setminus (C_1 \cup C_2)$ we infer that K is decomposable. We can show that each indecomposable subcontinuum of K is a boundary set in K. In fact, if Q is an indecomposable subcontinuum of K with a nonempty interior in K, then Q is a proper subcontinuum of K and $\overline{K \setminus Q}$ is a proper subset of K. It follows from (10.1) that $\overline{K \setminus Q}$ is a union of at most two subcontinua, each of which intersects Q and fails to contain K, but K is terminal (the proof is similar to that one of (13.3)). Now, if W is a layer of K, then using the colocal connectedness of X, we easily check that W is degenerate. Therefore, X contains an arc. Thus, by (15.3), X is a simple closed curve, a contradiction.

It follows from (15.4), (14.9), (14.11) and (14.12) that the class of homogeneous plane continua may be classified as follows: (a) a simple closed curve, (b) a tree-like hereditarily indecomposable continuum with span zero, (c) a continuum which has a nice decomposition onto a continuum of type (a) with layers of type (b). The pseudoarc represents a continuum of type (b) (see [11], [88]) and the so-called circle of pseudoarcs is of type (c) (see [16]). Moreover, each homogeneous nondegenerate chainable continuum is a pseudoarc (see [13]).

It is known (see [116], Theorem 11, p. 1159)

(15.5)   (Wisner) *If a proper subcontinuum X of a 2-manifold M is decomposable and homogeneous, then there is an atomic open mapping f from X onto a simple closed curve Y such that the sets $f^{-1}(y)$ are homeomorphic, homogeneous, indecomposable tree-like continua.*

Proof. According to (7.1) there is a monotone open mapping f from X such that f is atomic (compare (7.3)), the sets $f^{-1}(y)$ are homeomorphic and homogeneous continua and $f(X)$ is a homogeneous metric continuum which is colocally T-connected at each point where $T = C(f(X))$. It follows from (7.6) that the sets $f^{-1}(y)$ are indecomposable. It is obvious that X is a curve (if not, then it is a 2-manifold; thus, $X = M$, a contradiction). Therefore, the sets $f^{-1}(y)$ are tree-like continua by (6.6). Theorem 1 in [95] implies that the collection $\{f^{-1}(y): y = f(X)\}$ together with individual points of $M \setminus X$ forms an upper semi-continuous decomposition of M such that the decomposition space is homeomorphic to M. By (15.4) we infer $f(X)$ is a simple closed curve. The proof is complete.

The following question (this is a modification of a known question concerning plane continua) remains open: Let X be an indecomposable homogeneous subcontinuum of a metric 2-manifold (of a plane). Does it follow that X is a pseudoarc? (has span zero?) For partial results in the plane compare (14.12).

16. Multicoherence and homogeneity

A continuum X is called ω-*coherent* ($\aleph_0$-*coherent*) provided for each finite collection $\mathscr{B}$ of subcontinua of X with nonempty interiors and such that $\mathscr{B}^* = X$ the set $\bigcap \{B: B \in \mathscr{B}\}$ has a finite (countable) number of components. It is clear that

(16.1)   *Every ω-coherent continuum is $\aleph_0$-coherent and every hereditarily finitely multicoherent continuum is ω-coherent* (a continuum X is *hereditarily finitely multicoherent* if the intersection of every two subcontinua of X has finitely many components). ·

Now we will prove

(16.2)   *If X is a $\aleph_0$-coherent, homogeneous, metric and T-aposyndetic continuum where $T = C(X)$, then X is a simple closed curve.*

Proof. It follows from (7.1) that X is colocally T-connected continuum. Firstly, we will prove that

(16.2.1) X contains subcontinua with nonempty interiors of arbitrarily small diameter.

Indeed, let U be an open set with diameter less than ε. If U has only one point in the boundary then $\bar{U}$ is a continuum of diameter less than 2ε. Therefore, we may assume that $\operatorname{bd}(U)$ is nondegenerate. Let a, b be two different points from $\operatorname{bd}(U)$ and $0 < \delta < \frac{1}{3}d(a, b)$ and $B(u, 3\delta) \subset U$ for some $u \in U$. For each point $x \in \operatorname{bd}(U)$ let V_x be an open neighbourhood of x of diameter less than δ and such that $X \setminus V_x$ is a continuum with nonempty interior. Since $\operatorname{bd}(U)$ is compact, we can choose a finite collection $\mathscr{V}$ from $\{V_x: x \in \operatorname{bd}(U)\}$ which covers $\operatorname{bd}(U)$. Let $\mathscr{B} = \{X \setminus V: V \in \mathscr{V}\}$. Then $\mathscr{B}$ is a finite family of subcontinua of X with nonempty interiors. Moreover, $\mathscr{B}^* = X$, because if $x \in V \in \mathscr{V}$, then there is $V' \in \mathscr{V}$ such that $V' \cap V = \emptyset$ by the choice of δ; and thus $x \in X \setminus V'$. Since X is $\aleph_0$-coherent we obtain that the set $\bigcap \{B: B \in \mathscr{B}\}$ has a countable number of components. The set $B(u, \delta)$ is contained in $\bigcap \{B: B \in \mathscr{B}\} = X \setminus \mathscr{V}^* \subset X \setminus \operatorname{bd}(U)$; therefore, there is a component C of $\bigcap \{B: B \in \mathscr{B}\}$ contained in U and which has a nonempty interior in X by the Baire category theorem, i.e. (16.2.1) holds.

Now, since X is homogeneous and compact, from (16.2.1) we conclude that each point in X is a limit of continua with nonempty interiors. Using Effros' theorem we obtain

(16.2.2) X is locally connected.

We have

(16.2.3) X contains no nondegenerate indecomposable continuum.

Suppose, on the contrary, that Q is a nondegenerate indecomposable

subcontinuum of X. Let a and b be two different points of Q. According to (16.2.2) there are disjoint continua Q_a and Q_b containing a and b in their interiors. Since X is colocally T-connected at a, there is· an open neighbourhood V such that $a \in V \subset Q_a^0$ and $X \setminus V$ is a continuum with a nonempty interior. Since $X = (X \setminus V) \cup (Q \cup Q_a) \cup (Q \cup Q_b)$ and X is $\aleph_0$-coherent, the set $(X \setminus V) \cap (Q \cup Q_a) \cap (Q \cup Q_b) = (X \setminus V) \cap Q$ has a countable number of components, a contradiction, because Q is indecomposable and, hence, the complement of the open set $V \cap Q$ in Q has uncountably many components.

From (16.2.3) we infer that X is a curve (compare for example [12]). Therefore, by (16.2.2) and (12.2) X is either a simple closed curve or the Menger universal curve. Since the universal curve contains indecomposable continua, we finally obtain that X is a simple closed curve by (16.2.3).

The following proposition generalizes Theorem 12 in [116], p. 1160 (compare (16.1))

(16.3)   (Wisner) *If X is an $\aleph_0$-coherent, homogeneous, metric and decomposable continuum, then there is an open atomic mapping f from X onto a simple closed curve Y such that the sets $f^{-1}(y)$ are homeomorphic, homogeneous indecomposable continua.*

Moreover, if X is a curve, then each proper subcontinuum of X is a tree-like continuum.

This is a simple consequence of (6.6), (6.14), (7.1), (7.3), (7.6) and (16.2).

17. Types of aposyndesis

Recall that a continuum X is *mutually aposyndetic (aposyndetic, semi-aposyndetic)* if for each two points a and b of X there are subcontinua H and K of X containing a and b, respectively, in their interiors and $H \cap K = \emptyset$ ($H \subset X \setminus \{b\}$ and $K \subset X \setminus \{a\}$; either $H \subset X \setminus \{b\}$ or $K \subset X \setminus \{a\}$, respectively). It is obvious that

(17.1)   *Every mutually aposyndetic continuum is aposyndetic and every aposyndetic continuum is semi-aposyndetic.*

The following proposition solves Problem 47 from University of Houston Mathematics Problem Book (UHMPB).

(17.2)   *If f is an atomic mapping from a semi-aposyndetic continuum onto a nondegenerate continuum Y, then f is a homeomorphism.*

Proof. Let y be an arbitrary point in Y. Since f is atomic, we infer that the set $f^{-1}(y)$ is a terminal subcontinuum of X (compare (7.2)) and, hence

$f^{-1}(y)^0 = \emptyset$. If $a, b \in f^{-1}(y)$, then there exists a continuum K such that $a \in K^0 \subset K \subset X \setminus \{b\}$ (or $b \in K^0 \subset K \subset X \setminus \{a\}$ and then arguments are similar), because X is semi-aposyndetic. Since $f^{-1}(y)$ is terminal, we conclude $K \subset f^{-1}(y)$; thus, $a \in (f^{-1}(y))^0$, a contradiction. Therefore, the set $f^{-1}(y)$ has only one point, i.e. f is a homeomorphism.

Recall that a continuum X is *semi-locally connected* if for each point $x \in X$ and each open neighbourhood U of x there is an open set V such that $x \in V \subset U$ and $X \setminus V$ has a finite number of components. It follows immediately from the definitions

(17.3)   *A continuum X is aposyndetic if and only if it is semi-locally connected.*

The following example solves Problem 48 from UHMPB.

(17.4)   *There is a mutually aposyndetic unicoherent continuum Y which is not locally connected (and which is colocally connected and contractible with respect to the circle).*

We use the notation from Example (11.1). It is obvious that Y is mutually aposyndetic and colocally connected. According to Theorem 2 in [70], p. 437 it suffices to show that if f is an arbitrary mapping from Y onto the unit circle S, then $f \sim 1$ (i.e. f is homotopic to a constant mapping). Since $\varphi(x_0)$ is contractible, we conclude that $f|\varphi(x_0) \sim 1$. Theorem 9 in [70], p. 408 implies that there is an open set G such that $\varphi(x_0) \subset G$ and $f|G \sim 1$. By the construction of Y we may assume that $Y \setminus G$ has finitely many components with disjoint closures and that these components are discs with connected (homeomorphic to S) boundaries. By induction and Theorem 5 in [70], p. 435 we consequently obtain $f \sim 1$.

We do not know whether such an example can be one-dimensional, i.e. whether each unicoherent, mutually aposyndetic one-dimensional continuum is locally connected. The question of Jones whether a metric homogeneous one-dimensional aposyndetic continuum is locally connected is still unsolved even in the following special case: Is it true that a metric, homogeneous, colocally connected unicoherent (one-dimensional) continuum is locally connected? (the Cartesian product of a circle and a pseudoarc is an example of a metric homogeneous aposyndetic continuum which is not locally connected).

It is clear that

(17.5)   *If C is a connected subset of a continuum X such that for each $x \in C$ there is a subcontinuum K of X with $x \in K^0 \subset K \subset C$, then C is continuum-wise connected.*

A continuum X is said to *have property A* if for each point $x \in X$ and for each continuum K of $X \setminus \{x\}$ there is a continuum in $X \setminus K$ which contains x in its interior. It follows immediately from the definitions that

(17.6)   *Every continuum which has property A is mutually aposyndetic.*

Now, we have

(17.7)   *If a continuum X has property A, $x \in X$ and $X \setminus \{x\}$ is connected, then X is connected im kleinen at x.*

Indeed, let V be an open neighbourhood of x. According to (17.6) there is a finite collection $K_1, K_2, \ldots, K_n$ of subcontinua of X such that $X \setminus V \subset K_1 \cup K_2 \cup \ldots \cup K_n \subset X \setminus \{x\}$. The set $X \setminus \{x\}$ is continuum-wise connected by (17.5); thus there is a continuum K such that $K_1 \cup K_2 \cup \ldots \cup K_n \subset K \subset X \setminus \{x\}$. Since X has property A there is a continuum H such that $x \in H^0 \subset H \subset X \setminus K$. But then $H \subset V$, i.e. X is connected im kleinen at x.

The following proposition gives a positive solution to Problem 21 in UHMPB.

(17.8)   *A continuum X is locally connected if and only if X has property A.*

In fact, let x be a point of X such that $X \setminus \{x\}$ is not connected. According to (17.7) it suffices to show that X is connected im kleinen at x. Let C be a component of $X \setminus \{x\}$. Then $C \cup \{x\}$ is a continuum which has property A. To see this let $y \in C$ and let K be a continuum with $y \in K \subset X \setminus \{x\}$. Since X has property A there is a continuum H with $x \in H^0 \subset H \subset X \setminus K$. But then $H \cap (C \cup \{x\})$ is a continuum in $C \cup \{x\}$ with x in its interior relative to $C \cup \{x\}$ and not intersecting K. The other half of the proof that $C \cup \{x\}$ has property A is trivial. Now, let V be an open neighbourhood of x. Since each component of $X \setminus \{x\}$ is an open set in X only finitely many components of $X \setminus \{x\}$, say $C_1, \ldots, C_n$ intersect $X \setminus V$. Since the continua $X_i = C_i \cup \{x\}$ have property A and $X_i \cup \{x\}$ is connected, we obtain that the X_i are connected im kleinen at x, by (17.7). Therefore, for $i = 1, \ldots, n$ there is a continuum K_i in $(C_i \cup \{x\}) \cap V$ containing x in its interior relative to X_i. Put

$$K = \{x\} \cup \bigcup_{i=1}^{n} K_i \cup \mathrm{cl}(\bigcup \{C : C \text{ is a component of } X \setminus \{x\} \text{ and } C \subset V\}).$$

Then K is a continuum containing x in its interior and contained in $\bar{V}$; thus X is connected im kleinen. The proof of (17.8) is complete.

18. More about decompositions

(18.1)   *If $f: X \to Y$ is a monotone surjection from a Hausdorff arc X (i.e. continuum with exactly two nonseparating points) onto a Hausdorff space Y, then Y is an arc.*

Proof. Let a and b be the points which do not separate X and let $y \in Y \setminus \{f(a), f(b)\}$. The set $f^{-1}(y)$ is a continuum; thus, $X \setminus f^{-1}(y) = U \cup V$, where U and V are open and connected. Therefore, the sets $f(U)$ and $f(V)$ are connected and $f(U) \cup \{y\}$ and $f(V) \cup \{y\}$ are closed. Moreover, $f(U) \cap f(V) = \emptyset$ (if $z \in f(U) \cap f(V)$, then $f^{-1}(z)$ is not connected), $Y \setminus \{y\}$ $= f(U) \cup f(V)$ and $f(U)$ and $f(V)$ are separated.

(18.2)   (Whyburn) *If $f \colon X \to Y$ is a continuous surjection from an arcwise connected Hausdorff continuum X onto a Hausdorff space Y, then Y is arcwise connected.*

Proof. Assume that X is a Hausdorff arc with end-points a and b. Then we can linearly order X. Let $c, d \in Y$ be such that $a \in f^{-1}(c)$ and $b \in f^{-1}(d)$. Let A be a minimal closed subset of X with respect to the property that $a, b \in A$ and if uv is a component of $X \setminus A$ with endpoints u and v then $f(u) = f(v)$. Define a function g from X into Y by $f(t) = g(t)$ for $t \in A$ and for each component uv of $X \setminus V$, we put $g(t) = f(u)$ where $u < t < v$. It is easy to verify that g is monotone. Therefore $g(X)$ is an arc in Y joining c and d by (18.1).

It is known (see [18], [21]) that

(18.3)   (Charatonik) *If X is an arbitrary continuum and $T = \{C \in C(X) \colon C$ is a layer of an irreducible continuum in $X\}$, and φ is the core T-admissible mapping from X, then $\varphi(X)$ is hereditarily arcwise connected.*

Proof. Let A be an irreducible continuum between a and b in $\varphi(X)$, where $a \neq b$. Let $x \in \varphi^{-1}(a)$, $y \in \varphi^{-1}(b)$ and let $B \subset \varphi^{-1}(A)$ be a continuum irreducible between x and y. Then the layer of B which (in the sense of Kuratowski) contains x is different from the layer of B which contains y; thus there exists a mapping ψ from B onto $I = [0, 1]$ such that the sets $\psi^{-1}(t)$ are layers of B for $t \in I$. It is easy to verify that $f = \varphi \circ \psi^{-1} \colon I \to \varphi(X)$ is well-defined and continuous. Since $a, b \in f(I) \subset A$ and $f(I)$ is arcwise connected, we infer that A is an arc (compare (18.2)).

The decomposition described in (18.3) may be trivial even when the continuum X is nondegenerate and hereditarily decomposable (see [19], [77]). Every hereditarily arcwise connected continuum is a decomposition space of a continuous decomposition described in (18.3) (see [58], [82], Corollary 2). The Charatonik question (see [20], Problem 4) whether for a given dendroid Y there exists a λ-dendroid X which has a continuous decomposition of the type described in (18.3) onto Y, still remains open. Decompositions of the type from (18.3) for smooth continua are studied in [33], [34] and [79].

We do not know the answer to the following question: let Y be an arbitrary continuum. Do there exists a continuum X and an atomic (and

open) mapping f from X onto Y such that for each $y \in Y$ the set $f^{-1}(y)$ is nondegenerate?

(18.4)   *If X is an arbitrary continuum, $T = \{C \in C(X) \colon C$ is indecomposable$\}$ and φ is the core T-admissible mapping from X, then $\varphi(X)$ is hereditarily decomposable.*

Proof. Let B be an indecomposable subcontinuum of $\varphi(X)$ and let $A \subset \varphi^{-1}(B)$ be a minimal subcontinuum of X which is mapped under φ onto B. Then A is an indecomposable continuum; thus, $B = \varphi(A)$ is degenerate.

(18.5)   *If X is an arbitrary continuum, $T = \{C \in C(X) \colon C$ is discoherent$\}$ and φ is the core T-admissible mapping from X, then $\varphi(X)$ is hereditarily unicoherent.*

Proof. Let B be a discoherent decomposable subcontinuum of $\varphi(X)$ (see [70] for the definition of discoherence; if a continuum is not hereditarily unicoherent, then it contains a discoherent decomposable subcontinuum). Then there are two points a, b belonging to B and two continua B_1, B_2 irreducible between a and b such that $B = B_1 \cup B_2$ and a and b are contained in different components of $B_1 \cap B_2$. If A_1 and A_2 are minimal subcontinua of X which are mapped under φ onto B_1 and B_2, respectively, then the continuum $\varphi^{-1}(a) \cup \varphi^{-1}(b) \cup A_1 \cup A_2$ contains a discoherent continuum which is mapped onto B under φ. Thus, B is degenerate.

(18.6)   *If X is an arbitrary continuum, $T = \{C \in C(X) \colon C$ is a convergence subcontinuum of $X\}$ and φ is the core T-admissible mapping from X, then $\varphi(X)$ is hereditarily locally connected.*

It is easy to construct continua which do not have a minimal monotone decomposition onto a hereditarily arcwise connected continuum (hereditarily decomposable continuum, hereditarily unicoherent continuum, hereditarily locally connected continuum; compare Example (11.1)). If we assume additionally that X is hereditarily unicoherent then in each case such a minimal decomposition exists. If in (18.5) we assume that X is locally connected, then the decomposition described there is a minimal decomposition of X onto a continuum which does not contain cyclic elements in the sense of Whyburn.

19. Mappings from hereditarily indecomposable continua

Firstly we shall prove the following:

(19.1)   (Hurewicz) *There exists a monotone mapping from some metric curve onto the Hilbert cube.*

Proof. Let $Q = \prod_{i=1}^{\infty} I_i$, $I_i = [0, 1]$ and C_i be the Cantor ternary sets lying in I_i for $i = 1, 2,\ldots$. Let $U = \{(x_1, x_2,\ldots) \in Q : x_i \in C_i$ for each i except at most one$\}$. Then

(19.1.1) $\qquad\qquad\qquad\qquad U$ is one-dimensional.

Let $A_i = \prod_{j=1}^{\infty} A_j^i$ where $A_j^i = C_j$ for $i \neq j$ and $A_j^i = I_i$ for $i = j$. The sets A_i are compact and one-dimensional and $U = \bigcup_{i=1}^{\infty} A_i$. Thus, $\dim U = 1$.

(19.1.2) $\qquad\qquad\qquad\qquad U$ is compact.

If $x^n = (x_1^n, x_2^n,\ldots) \to (x_1, x_2,\ldots)$ and $x^n \in U$ and $x_{i_0}^{n_k} \in I_{i_0} \setminus C_{i_0}$ for infinitely many $n_1, n_2,\ldots$, then $x_i^{n_k} \in C_i$ for $i \neq i_0$. Therefore, $x_i \in C_i$ for each $i \neq i_0$.

(19.1.3) $\quad$ If $x = (x_1, x_2,\ldots) \in U$, then there is a connected set containing x and $C = \prod_{i=1}^{\infty} C_i$ and contained in U.

We may assume that $x_i \in C_i$ for $i > 1$. Put $B_1(x) = \{(t_1, t_2,\ldots): 0 \leqslant t_1 = 1, t_i = x_i$ for $i > 1\}$. Then $x \in B_1(x)$ and $B_1(x)$ is connected. Take $B_n(x) = \{(t_1, t_2,\ldots): t_i \in C_i$ for $i < n$, $t_i = x_i$ for $i > n$, $t_n \in [0, 1]\}$. The sets $B^n(x) = B_1(x) \cup \ldots \cup B_n(x)$ are connected. Thus, $\overline{B(x)}$ is connected where $B(x) = \bigcup_{n=1}^{\infty} B^n(x)$. Then $\{x\} \cup C \subset \overline{B(x)}$.

From (19.1.3) we obtain

(19.1.4) $\qquad\qquad\qquad\qquad U$ is connected.

Now, let h_i be the Cantor step function from C_i onto I_i. Define $f: U \to Q$ by $f(x_1, x_2,\ldots) = (h_1(x_1), h_2(x_2),\ldots)$. Since $C \subset U$ and $f(C) = Q$ we infer that f is onto. It remains to prove that F is monotone.

For a fixed $x = (x_1, x_2,\ldots) \in U$ let $a_i = b_i = x_i$ if $x_i \in C_i$ and it is not an end-point of a conjugate interval to C_i and a_i, b_i are the endpoints of such an interval containing x_i in the opposite case. We assume $x_i \in C_i$ for $i > 1$. Put $D = \prod_{i=1}^{\infty} \{a_i, b_i\}$, $D_1 = \{(t_1, t_2,\ldots): a_1 \leqslant t_1 \leqslant b_1, t_i = x_i$ for $i > 1\}$. Then $x \in D_1$, D_1 is connected and $f(D_1) = f(x)$. Take $D_n = \{(t_1, t_2,\ldots): t_i \in \{a_i, b_i\}$ for $i < n$, $t_i = x_i$ for $i > n$ and $t_n \in [a_n, b_n]\}$. As above $D_0 = D_1 \cup D_2 \cup \ldots$ is connected and $D \subset \bar{D}_0$ and $f(D_0) = f(x)$. Thus f is monotone, and the proof is complete.

Proposition (19.1) having, as we see, a quite simple proof will be applied in the next propositions. Wilson, applying more complicated techniques, obtained in [115] (compare [67]) a very nice general result.

(19.2) $\quad$ (Anderson, Wilson) *Every locally connected metric continuum X is an image of the Menger universal curve U under a monotone and open mapping f such that the sets $f^{-1}(x)$ are homeomorphic to U.*

Recall that a mapping f from a continuum X onto Y is *weakly confluent* if for each subcontinuum K in Y there exists a continuum C in X such that $f(C) = K$. The following proposition generalizes a result from [8]:

(19.3)   *Every metric continuum X is a weakly confluent image of some hereditarily indecomposable metric curve.*

Proof. From Theorem 3.5 in [40] we infer that the continuum X can be embedded in a compactification Y of the half line such that each continuous mapping from any continuum onto Y is weakly confluent. There exists an hereditarily indecomposable metric continuum M with dim $M = 3$ (see [12]) and there exists a weakly confluent mapping f from M onto the cube I^3 (see [46], [91]). By Proposition (19.1) there exists a curve $K \subset I^3$ and a monotone mapping g from K onto the Hilbert cube Q containing Y. Let N be a subcontinuum of M such that $g^{-1}(Y) = f(N)$. Then $h = g \circ f|N$ is weakly confluent. Let $Y_n = X \cup [n, \infty)$ and N_n continua in N such that $h(N_n) = Y_n$. We may suppose $N \supset N_1 \supset N_2 \supset \ldots$ Put $N_0 = \operatorname{Lim} N_n$. Then $h(N_0) = X$ and since $h|N_n$ is weakly confluent for each $n = 1, 2, \ldots$ we conclude that $h|N_0$ is weakly confluent. In this way the proof is complete, because every hereditarily indecomposable metric continuum is an image under a monotone open map of a hereditarily indecomposable metric curve (see [74]).

We do not know whether every Hausdorff continuum is an image of a hereditarily indecomposable continuum. ·

(19.4)   (Kelly, Mazurkiewicz) *If X is a metric continuum, then $C(X)$ is an image of the Cantor fan ($=$ the cone over the Cantor set).*

Short proof. By (19.3), we may assume that X is hereditarily indecomposable. Let μ be a Whitney map on $C(X)$ (see [90]) with $\mu(X) = 1$ and f be a mapping from the Cantor set C onto X. The mapping $g : C \times I \to C(X)$ is defined by $g(c, t) = (f(c), \mu^{-1}(t) \cap A(c))$ where $A(c)$ denotes all subcontinua of X containing c. Then $g(C \times \{1\}) = \{X\}$.

20. Common model

We shall say that a metric continuum K is a *common model* for the class $\mathcal{K}$ of metric continua provided each member of $\mathcal{K}$ is a continuous image of K. It is well-known from the Hahn–Mazurkiewicz theorem that an arc is a common model for locally connected metric continua. W. Kuperberg (see [66]) has shown that the cone over the Cantor set is a common model for uniformly pathwise connected continua. Mioduszewski [87], Lelek [71] and

Fearnley [26] have shown that the pseudoarc is a common model for arc-like continua, and Rogers [98] has shown that there is a pseudosolenoid which is a common model for circle-like continua.

On the other hand, in 1934 Waraszkiewicz [112] constructed a collection of plane continua which has no common model. Based on this result, Bellamy [5] has shown that there is no common model for indecomposable continua. Krasinkiewicz and Minc [62] proved that there is no common hereditarily decomposable model for planar fans. (An easy modification of their proof gives that there is no hereditarily decomposable common model for hereditarily decomposable arc-like continua.) Ingram [48], [49] has constructed an uncountable collection of planar atriodic indecomposable (hereditarily indecomposable) tree-like continua without a common model. In 1979, R. L. Russo [104] showed that there is no model for: planar tree-like continua, arcwise connected continua, planar indecomposable continua. He also proved that if $\mathscr{P}$ is a collection of polyhedra such that $\mathscr{P}$-like continua have a common model, then $\mathscr{P} = \{arc\}$ or $\mathscr{P} = \{circle\}$ or $\mathscr{P} = \{arc, circle\}$. Russo's proofs are based on Waraszkiewicz's very technical and long proof from [112]. Several years ago D. P. Bellamy gave a nice short proof of Waraszkiewicz's results from [112]. Unfortunately, Bellamy never published his proof. Now, we will obtain some of Russo's results by a modification of Bellamy's original proof.

Let a, b, c be complex numbers which are cubic roots of 1. Put $T = \{rz: z \in \{a, b, c\}$ and $r \in [0, 1]\}$. We define a mapping h from the set of real numbers R onto T as follows: $h(2k+1) = 0$, $h(6k) = a$, $h(6k+2) = b$, $h(6k+4) = c$ and h is linear on the interval $[k, k+1]$ for each integer k. Let $I = [0, 3]$ and $\pi: I \times R \to I \times T$ be a mapping given by $\pi(t, x) = (t, h(x))$. Denote the projection from $I \times R$ onto R by α, the projection from $I \times T$ onto T by β and the projection from $I \times R$ onto I by δ.

Observe that

(20.1)   *If $x, y \in R \times I$ and $|\alpha(x) - \alpha(y)| = 3$, then $\varrho(\beta\pi(x), \beta\pi(y)) = 1$,*

where ϱ is the arclength metric on T.

Let σ be the metric on $I \times T$ defined by $\sigma(x, y) = \max\{\varrho(\beta(x), \beta(y)), |\delta(x) - \delta(y)|\}$ for $x, y \in I \times T$, and let d denote the sup metric on $(I \times T)^X$ which is induced by σ for any compact Hausdorff space X. Consider the family $\mathscr{S} = \{\langle k_i \rangle_{i=0}^{\infty}: k_i$ is an integer which can be divided by 6 and $6^i < k_i \leqslant 6^{i+1}\}$. Observe that

(20.2)   *If $\langle k_i \rangle_{i=0}^{\infty} \in \mathscr{S}$, then $k_0 = 6$.*

Let $\langle a_i \rangle_{i=0}^{\infty}$ and $\langle b_i \rangle_{i=0}^{\infty}$ be fixed sequences in I such that $a_0 < 1$, $a_{n+1} < b_n < a_n$ for each $n = 0, 1, 2,...$ and $\lim a_n = \lim b_n = 0$.

Now for $K = \langle k_i \rangle_{i=0}^{\infty} \in \mathscr{S}$ define $F_K: (0, 3] \to R$ by $F_K(t) = 0$ if $1 \leqslant t \leqslant 3$, $F_K(a_i) = k_i$, $F_K(b_i) = -k_i$, F_K is linear on each $[a_{i+1}, b_i]$, $[b_i, a_i]$

and $[a_0, 1]$. Then define $\hat{F}_K : (0, 3] \to I \times R$ by $\hat{F}_K(t) = (t, F_K(t))$. The image of $\hat{F}_K$ is the graph of F_K. Next, define the Waraszkiewicz style spiral W_K (approximating the triod T) for each $K \in \mathscr{S}$ by

$$W_K = (\{0\} \times T) \cup (\pi \circ \hat{F}_K(0, 3]) \subset I \times T.$$

Next, suppose $K = \langle k_i \rangle_{i=0}^{\infty}$, $L = \langle l_i \rangle_{i=0}^{\infty} \in \mathscr{S}$ and $K \neq L$, and $h_K : X \to W_K \subset I \times T$ and $h_L : X \to W_L \subset I \times T$ are continuous and onto where X is some fixed Hausdorff continuum. Then

(20.3) $d(h_K, h_L) \geqslant 1.$

Suppose that $d(h_K, h_L) < 1$. Let n be the smallest integer such that $k_n \neq l_n$ and suppose $l_n < k_n$. Then $k_n - l_n \geqslant 6$.

If Q is a continuum in X such that one of the sets $h_K(Q)$ and $h_L(Q)$ intersects $\pi \circ \hat{F}_K[2, 3] = \pi \circ \hat{F}_L[2, 3]$ and both are arcs then we denote the mapping $\pi^{-1}|h_M(Q)$ by H_M^Q and γ_M^Q is defined by $\gamma_M^Q(x) = \alpha H_M^Q h_M(x)$ for $x \in Q$ and $M \in \{K, L\}$. The mapping H_M^Q is a homeomorphism. Moreover,

(20.3.1) $|\gamma_K^Q(x) - \gamma_L^Q(x)| < 3$ for $x \in Q.$

Indeed, suppose that $|\gamma_K^Q(x_0) - \gamma_L^Q(x_0)| \geqslant 3$ for some $x_0 \in Q$. Since Q is connected and $|\gamma_K^Q(p) - \gamma_L^Q(p)| = 0$ for some $p \in Q$, we infer that there is $x_1 \in Q$ such that $|\gamma_K^Q(x_1) - \gamma_L^Q(x_1)| = 3$. Then, by (20.1), $\varrho(\beta h_K(x_1), \beta h_L(x_1)) = 1$. Therefore, $\sigma(h_K(x_1), h_L(x_1)) \geqslant 1$ and, thus, $d(h_K, h_L) \geqslant 1$, a contradiction.

Choose $p \in X$ such that $h_K(p) = (3, a)$ and let J denote the component of $h_K^{-1}([a_n, 3] \times T)$ containing p. Then $h_K(J) = \pi \circ \hat{F}_K([a_n, 3])$ which is an arc. Moreover, $h_L(J)$ is also an arc. If $\pi \circ \hat{F}_L(2)$, $\pi \circ \hat{F}_L(a_{n+2}) \in h_L(J)$ there is a continuum J' contained in J which is mapped by h_L onto $\pi \circ \hat{F}_L([a_{n+2}, 2])$. We know also that $h_K(J') \subset \pi \circ \hat{F}_K([a_n, 3])$. Then, by (20.3.1), we have

$$|\gamma_K^{J'}(x) - \gamma_L^{J'}(x)| < 3   \text{for}   x \in J'.$$

Observe that the maximum value of $F_K(t)$ for $t \in [a_n, 3]$ is k_n. Therefore, $\gamma_K^{J'}(x) \leqslant k_n$ for $x \in J'$. Let $w \in J'$ be such that $h_L(w) = \pi \circ \hat{F}_L(a_{n+2})$. Then $\gamma_L^{J'}(w) = l_{n+2}$. Thus, $\gamma_L^{J'}(w) > 6^{n+2}$. But $\gamma_K^{J'}(w) \leqslant k_n \leqslant 6^{n+1}$. Therefore, $|\gamma_K^{J'}(w) - \gamma_L^{J'}(w)| \geqslant 6^{n+2} - 6^{n+1} > 3$, a contradiction, i.e. $h_L(J)$ is an arc. Hence, by (20.3.1), we obtain

$$|\gamma_K^J(x) - \gamma_L^J(x)| < 3   \text{for}   x \in J.$$

Now, by Janiszewski's boundary-bumping theorem there is $q \in J$ such that $h_K(q) = \pi \circ \hat{F}_K(a_n)$ and, hence, $H_K^J h_K(q) = (a_n, k_n)$.

The minimum value of $\gamma_K^J(x)$ for $x \in J$ is thus $F_K(b_{n-1}) = -k_{n-1}$ ($n \geqslant 2$ by (20.2)). Hence, the minimum value of $\gamma_L^J(x)$ for $x \in J$ is not smaller than $-k_{n-1} - 3$, and since $k_{n-1} \leqslant 6^n$ we have $\gamma_L^J(x) \geqslant -6^n - 3$. Since $-l_n < -6^n$, we obtain that the projection of the set $\gamma_L^J(J)$ onto the interval I does not

contain the value b_n. Since $\delta\gamma_L^J(p) \geqslant 1 > b_n$ and J is connected it easily follows that $H_L^J h_L(J) \subseteq \hat{F}_L[b_n, 3]$.

Finally, observe that the maximum value of $F_L(t)$ for $t \in [b_n, 3]$ is l_n and $l_n \leqslant k_n - 6$. However, $\gamma_K^J(q) = k_n$ and $\gamma_L^J(q) \leqslant l_n$. Thus, $|\gamma_K^J(q) - \gamma_L^J(q)| \geqslant 6$, a contradiction. The proof of (20.3) is complete.

Any set $\{h_K\}_{K \in \mathcal{S}}$ of maps h_K of X onto W_K would be an uncountable discrete subset of $(I \times T)^X$ by (20.3). Therefore,

(20.4)   *The collection $\{W_K : K \in \mathcal{S}\}$ has no common model.*

It is easily seen that every W_K is a planar λ-dendroid; so

(20.5)   *There is no common model for planar λ-dendroids.*

Note, also (as in [104]) that for every $K \in \mathcal{S}$, $W_K \times I$ is aposyndetic. Further, $W_K \times I$ can be mapped onto W_K by the projection. Thus,

(20.6)   *There is no common model for aposyndetic continua.*

It still is an open question, whether there exists a model for planar aposyndetic continua.

(20.7)   *There is an uncountable collection of indecomposable tree-like plane continua which has no common model.*

We can construct as in [104] for every $K \in \mathcal{S}$ an indecomposable tree-like continuum $I(W_K)$ in the plane such that W_K is a retract of $I(W_K)$. Conditions (19.3) and (20.4) imply

(20.8)   *There is no common model for hereditarily indecomposable curves.*

Since each W_K is an image of some Waraszkiewicz spiral we have an alternative proof that

(20.9)   *Waraszkiewicz spirals do not have a common model.*

It is still an open question whether there are common models for planar arcwise connected continua, fans and dendroids (see [104], compare [62]), but we have

(20.10)   *Arcwise connected continua do not have a common model.*

In fact, consider the planes P_n given by $x = nz$ for $n = 1, 2, \ldots$ and P_0 is the plane $z = 0$. For $n = 0, 1, 2, \ldots$ let $\hat{F}_K^n = (\hat{F}_K((0, 3]) \times R) \cap P_n$. Let L_0 be the semi-circle $(x - 1.5)^2 + z^2 = 1.5^2$, $y = 0$, $z \geqslant 0$; let L_1 be the semi-circle $(x - 3)^2 + (z - 1.5)^2 = 1.5^2$, $y = 0$, $x \geqslant 3$ and for $i \geqslant 2$ let L_i be the line segment with end-points $(a_i, k_i, a_i/(i-1))$ and $(a_i, k_i, a_i/i)$. Let $W_K' = \{0\} \times$ $\times R \times \{0\} \cup \bigcup\limits_{n=0}^{\infty} (\hat{F}_K^n \cup L_n)$ and define an equivalence relation $\sim$ on W_K' by x

$\sim y$ in W_K' if and only if $x = y$ or $x = (0, x', 0)$, $y = (0, y', 0)$ and $h(x')$ $= h(y')$. Let $W_K^* = W_K'/\sim$. Then the collection $(W_K^*)_{K\in\mathscr{S}}$ has no common model by a proof similar to that of (20.3).

21. Final remarks

(a) *Common image.* We shall say that a continuum K is a *common image* for the class $\mathscr{K}$ of continua provided each member of $\mathscr{K}$ can be mapped onto K. We know that: each metric locally connected continuum is a common image for all nondegenerate continua (this is a trivial consequence of the Hahn–Mazurkiewicz theorem and Urysohn's lemma); the cone over the harmonic sequence is a common image for all nonlocally connected continua (see [8]), the cone over the Cantor set is a common image for continua containing an open set with uncountably many components (see [3]), the Knaster's indecomposable continuum is a common image for indecomposable continua (see [7]; compare (8.1) here); the pseudoarc is a common image for hereditarily indecomposable continua (see [6]).

In the metric case we know much more: if f is a continuous mapping from a subcontinuum Q of a hereditarily unicoherent metric continuum X onto the cone over the harmonic sequence (the cone over the Cantor set, Knaster's indecomposable continuum) Y, then there is an extension f^* of f from X onto Y; and if f is a continuous mapping from a subcontinuum Q of a hereditarily indecomposable metric continuum X onto the pseudoarc Y, then there is an extension f^* of f to X (these results were first announced by D. P. Bellamy; they are proved in [80] and [81]).

There is a λ-dendroid, which is not arcwise connected and which can not be mapped onto a $\sin\frac{1}{x}$-curve (see [63]). It is unknown whether the $\sin\frac{1}{x}$-curve is a common image for planar λ-dendroids, which are not arcwise connected.

(b) *Universal continua.* Let $\mathscr{K}$ be a class of continua. An element $X \in \mathscr{K}$ is said to be *universal* for $\mathscr{K}$ provided each member of $\mathscr{K}$ can be imbedded in X.

The following classes of continua have universal continua: plane curves (see [70], p. 275), n-dimensional metric continua, arc-like continua, tree-like continua (see [85] where it is proved that $\mathscr{P}$-like continua have a universal continuum provided $\mathscr{P}$ is an amalgamble class of polyhedra), dendrites (see [113]), completely regular curves (see [47]), smooth dendroids (see [39]).

The following classes of continua do not have universal objects: regular curves (see [92]), (plane) fans, (plane) dendroids, (plane) λ-dendroids, (plane) hereditarily decomposable continua, (plane) Suslinian continua, (plane)

rational continua (see [62]), hereditarily decomposable arc-like continua (an easy modification of the proofs in [62]; it suffices to construct a suitable family of hereditarily decomposable arc-like continua instead of the family of fans which is considered in [62]. Much more is proved in [62]; namely that such a universal object for the constructed family of fans can not be hereditarily decomposable); smooth planar dendroids (see [76]).

It is still an open question whether hereditarily indecomposable continua have a universal continuum (H. Cook).

(c) *Incomparable families.* A family $\mathscr{K}$ of continua is called *incomparable* if no member of $\mathscr{K}$ is a continuous image of any other member of $\mathscr{K}$.

The first uncountable incomparable family of continua was constructed by Waraszkiewicz in [111]. There exists an uncountable incomparable collection of circularly chainable (planar circularly chainable, chainable) continua (see [4], [22], [96]).

We do not know whether there exists an incomparable uncountable collection of arcwise connected continua (fans, dendroids) (B. Knaster).

Added in the proof. Recently in the paper "Atriodic homogeneous continua" Ch. Hagopian has proved that every atriodic homogeneous metric continuum that is not a solenoid and has a decomposable subcontinuum admits a continuous decomposition to a solenoid. It solves some questions from Sections 13 and 14 here.

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