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Classification of generalized affine symmetric spaces of dimension $n \leqslant 4$

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Introduction

In [10] O. Kowalski introduced the concept of a tangentially regular s-manifold and considered the problem of the existence and uniqueness of a connection invariant under all symmetries s_x on those manifolds. Further, in [11], O. Kowalski gave a definition of a generalized affine symmetric space.

In the present paper we shall classify simply connected generalized affine symmetric spaces and present a complete list of these spaces of dimensions $n \leqslant 4$.

Throughout the present paper we shall make use of the concepts and theorems published by the author in [17].

We exclude from our classification spaces which are symmetric in the sense of Cartan, that is, we assume the torsion tensor $T \neq 0$.

At the same time, we shall limit ourselves to the primitive generalized affine symmetric spaces, i.e. those which are not products of generalized affine symmetric spaces [17].

For our purpose, it is indispensable to represent generalized affine symmetric spaces by certain tangentially regular s-manifolds first. However, it is not necessary to give a list of all such s-manifolds of a given dimension to obtain a complete list of generalized affine symmetric spaces of that dimension. Namely, we are going to show that the finding of certain privileged s-manifolds is quite sufficient.

§ 1. Tangentially regular s-structure

Following O. Kowalski [10], we define a tangentially regular s-structure on a smooth manifold M as a family $\{s_x\}$, $x \in M$, of diffeomorphisms satisfying the following axioms:

(1.1) $s_x(x) = x$;

(1.2) the tangent map $(s_x)_{*x}\colon T_x(M) \to T_x(M)$ has no fixed vectors except the null vector;

(1.3) $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$;

(1.4) the map $(x, y) \mapsto s_x(y)$ is smooth.

The diffeomorphisms s_x, $x \in M$, are called generalized symmetries of M. The pair $(M, \{s_x\})$ is called a tangentially regular s-manifold (or shortly, an s-manifold).

DEFINITION 1. An s-structure $\{s_x\}$ on M is said to be of *order* k ($k > 1$ being an integer) if $(s_x)^k = \mathrm{id}$ for all $x \in M$, and if k is the least integer with this property. If such an integer k does not exist, then $\{s_x\}$ is said to be of *infinite order*.

DEFINITION 2. An automorphism of $(M, \{s_x\})$ onto itself is a *diffeomorphism* Φ: $M \to M$ such that $\Phi \circ s_x = s_{\Phi(x)} \circ \Phi$ for each $x \in M$.

Let us remark that all symmetries s_x of M are automorphisms.

In [10] the following basic theorem was proved:

THEOREM A. *Let $(M, \{s_x\})$ be a connected s-manifold. Denote by S a tensor field of type $(1, 1)$ given by $S_x = (s_x)_{*x}$ for all $x \in M$. Then*

1) *There is a unique connection $\tilde{\nabla}$ on M (called the canonical connection) such that $\tilde{\nabla}$ is invariant under all s_x and $\tilde{\nabla} S = 0$. $\tilde{\nabla}$ is complete and has parallel curvature and parallel torsion.*

2) *The group* $\mathrm{Aut}(M)$ *of all automorphisms of $(M, \{s_x\})$ is a transitive Lie transformation group which is a closed subgroup of the full affine tranformation group $A(M)$ with respect to $\tilde{\nabla}$. The automorphism of $(M, \{s_x\})$ are exactly those affine transformations which leave the tensor field S invariant.*

3) *Let G denote the component of unity of* $\mathrm{Aut}(M)$, *let o be a fixed point of M, and let G_0 be the corresponding isotropy subgroup. Then the homogeneous space G/G_0 is reductive in a canonical way and, under the standard identification $G/G_0 \cong M$, the connection $\tilde{\nabla}$ coincides with the canonical connection of the second kind of G/G_0.*

DEFINITION 3. Two s-manifolds $(M, \{s_x\})$, $(M', \{s'_y\})$ are called *isomorphic* if there is a diffeomorphism Φ: $M \to M'$ (called an isomorphism) such that $\Phi \circ s_x = s'_{\Phi(x)} \circ \Phi$ for each $x \in M$.

They are said to be *locally isomorphic* if, for every two points $p \in M$, $p' \in M'$, there is a diffeomorphism Φ of a neighbourhood $U \ni p$ onto a neighbourhood $U' \ni p'$ (called a local isomorphism) with the following property: For each $x \in U$ there is neighbourhood $V_x \subset U \cap s_x^{-1}(U)$ such that $\Phi \circ s_x = s'_{\Phi(x)} \circ \Phi$ holds on V_x.

In [17] we proved the following

THEOREM B. *Let $(M, \{s_x\})$, $(M', \{s'_y\})$ be two s-manifolds with the canonical connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ respectively, and let $U \subset M$, $U' \subset M'$ be open sets. Then a diffeomorphism Φ: $U \to U'$ is a local isomorphism of $(M, \{s_x\})$ into $(M', \{s'_y\})$ if and only if Φ is a local affine map of $(M, \tilde{\nabla})$ into $(M', \tilde{\nabla}')$ such that $\Phi\ (S_{|U}) = S'_{|U'}$.*

DEFINITION 4. A connected affine manifold $(M, \tilde{\nabla})$ is called a *generalized affine symmetric space* (shortly: *g.a.s. space*) if M admits at least one tangentially regular s-structure $\{s_x\}$ such that $\tilde{\nabla}$ is its canonical connection. (An s-structure with this property is said to be admissible).

DEFINITION 5. A generalized affine symmetric space $(M, \tilde{\nabla})$ is said to be of *order* k if it admits an s-structure of order k and does not admit any s-structure of order $l < k$.

$(M, \tilde{\nabla})$ is said to be of *infinite order* if it admits only s-structures of infinite order.

DEFINITION 6. An infinitesimal s-manifold is a *collection* (V, S_0, T_0, R_0)(1) where V is a real vector space and S_0, T_0, R_0 are tensors of types (1,1), (1,2), (1,3), respectively, such that the following conditions are satisfied:

(i) Both S_0 and $I_0 - S_0$ are non-singular linear transformations of V.

(ii) For every $X, Y \in V$ the endomorphism $R_0(X, Y)$ acting as a derivation on the tensor algebra $\mathcal{T}(V)$ satisfies $R_0(X, Y) S_0 = 0$, $R_0(X, Y) T_0 = 0$, $R_0(X, Y) R_0 = 0$.

(iii) The tensors T_0 and R_0 are invariant by S_0:

$$T_0(S_0 X, S_0 Y) = S_0\big(T_0(X, Y)\big),$$

$$R_0(S_0 X, S_0 Y) S_0 Z = S_0\big(R_0(X, Y) Z\big).$$

(iv) $\quad T_0(X, Y) = -T_0(Y, X), \quad R_0(X, Y) = -R_0(Y, X).$

(v) The first Bianchi identity holds:

$$\mathfrak{S}\big(R_0(X, Y)Z - T_0(T_0(X, Y), Z)\big) = 0.$$

(vi) The second Bianchi identity holds:

$$\mathfrak{S}\big(R_0(T_0(X, Y), Z)\big) = 0.$$

Here $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y, Z.

DEFINITION 7. Two infinitesimal s-manifolds (V_i, S_i, T_i, R_i), $i = 1, 2$, will be said to be *isomorphic* if there is a linear isomorphism $f\colon V_1 \to V_2$ of vector spaces such that $f(S_1) = S_2$, $f(T_1) = T_2$, $f(R_1) = R_2$.

We have proved in [17] that for each point $x \in M$ of a tangentially regular s-manifold $(M, \{s_x\})$ the collection (M_x, S_x, T_x, R_x) is an infinitesimal s-manifold and all these infinitesimal s-manifolds are mutually isomorphic (in a natural sense).

DEFINITION 8. The infinitesimal type of an s-manifold $(M, \{s_x\})$ is the *isomorphism class of infinitesimal s-manifolds* (M_x, S_x, T_x, R_x), $x \in M$.

We have proved in [17], Theorem 5, that a simply connected tangentially regular s-manifold $(M, \{s_x\})$ is locally uniquely determined by its infinitesimal

(1) In [17] the infinitesimal s-manifold was denoted by $(V, S_0, \tilde{T}_0, \tilde{R}_0)$ to emphasize the relation of $\tilde{T}, \tilde{R}$ to the connection $\tilde{\nabla}$; now the sign "$\sim$" will be omitted.

type. We have also described the construction of a simply connected tangentially regular s-manifold $(M, \{s_x\})$ having a given infinitesimal s-manifold (V, S_0, T_0, R_0) as its infinitesimal type.

Let us now sum up briefly the construction that we have mentioned. Let (V, S_0, T_0, R_0) be an infinitesimal s-manifold. Let $\mathfrak{h}$ be the Lie algebra of all endomorphisms A of V which, as derivations of the tensor algebra $\mathscr{T}(V)$, satisfy $A(S_0) = 0$, $A(T_0) = 0$, $A(R_0) = 0$. Particularly, $R_0(X, Y) \in \mathfrak{h}$ for every $X, Y \in V$. Following a construction of K. Nomizu [16], we define a Lie algebra $\mathfrak{g}$ to be the direct sum $V + \mathfrak{h}$ with the multiplication given by

$$[X, Y] = (-T_0(X, Y), -R_0(X, Y)),$$
$$[A, X] = AX, \quad [X, A] = -AX,$$
$$[A, B] = A \circ B - B \circ A, \qquad A, B \in \mathfrak{h}, \quad X, Y \in V.$$

Conditions (v) and (vi) in Definition 6 imply the Jacobi identities. Let G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$, and let H be the connected Lie subgroup corresponding to the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Then H is a closed subgroup of G and $M = G/H$ is a reductive homogeneous space with respect to the decomposition $\mathfrak{g} = V + \mathfrak{h}$. Denote by $\tilde{\nabla}$ the canonical connection of the second kind of G/H [16]. Let us first identify $\mathfrak{g} = V + \mathfrak{h}$ with the tangent space G_e and then V with the tangent space $(G/H)_0$ at the origin of G/H via the projection $\pi \colon G \to G/H$.

Starting from S_0, T_0, R_0, we can construct (in a unique way) tensor fields S, T, R on G/H which are G-invariant and also parallel with respect to $\tilde{\nabla}$. Then there is a family $\{s_x\}$ of affine transformations of $(G/H, \tilde{\nabla})$ uniquely determined by S. Here $(G/H, \{s_x\})$ is an s-manifold for which $\tilde{\nabla}$ is the canonical connection. Also, we can show that R and T are the curvature tensor field and the torsion tensor field of $\tilde{\nabla}$, respectively. Hence we deduce that our s-manifold has the prescribed infinitesimal type. The uniqueness follows from a corollary of Theorem 1 [17].

We have proved ([17], Theorem 6) that our construction has the same result if we replace the algebra $\mathfrak{h}$ by its subalgebra $\mathfrak{h}^\circ \subset \mathfrak{h}$ supposing only that $R_0(X, Y) \in \mathfrak{h}^\circ$ for any $X, Y \in V$.

Remark. If we take the Lie subalgebra $\mathfrak{h}^\circ \subset \mathfrak{h}$ generated by all curvature transformations $R_0(X, Y)$, $X, Y \in V$, then the group G' is locally isomorphic to the transvection group of $(M, \tilde{\nabla})$ [10].

Let $(M, \{s_x\})$ be a simply connected s-manifold. Then for each $x \in M$ the transformation S_x has the same eigenvalues $\Theta_1, \ldots, \Theta_n$, which will thus be called the eigenvalues of S.

Definition 9. Let $(M, \{s_x\})$ be an s-manifold and (V, S_0, T_0, R_0) its infinitesimal type. The symmetries s_x are called *semi-simple* if S_0 is completely reducible on the complexification V^c of V.

The eigenvalues of the s-structure $\{s_x\}$ are defined as the eigenvalues of S_0.

In [17] we proved the following

THEOREM C. *Let M be a simply connected manifold. Each admissible s-structure $\{s_x\}$ on the space $(M, \tilde{\nabla})$ can be replaced by an admissible s-structure $\{s'_x\}$ having the same eigenvalues (including multiplicity) and such that the symmetries s'_x are semi-simple.*

DEFINITION 10. Let $\mathscr{A}^n$ be the set of all n-tuples (θ_i) of complex numbers such that $\theta_i \neq 0, 1$ for $i = 1, \dots, n$; there is a permutation ϱ of the indices $1, \dots, n$ such that $\varrho^2 =$ identity and $\theta_{\varrho(i)} = \bar{\theta}_i$ for $i = 1, \dots, n$. The elements $(\theta_i) \in \mathscr{A}^n$ will be called *systems of eigenvalues*.

It is obvious that a family of all eigenvalues of an n-dimensional s-structure belongs to $\mathscr{A}^n$. Such a family will be called a system of eigenvalues of the s-structure. Thus, a system of eigenvalues of an s-structure is uniquely determined up to a permutation.

DEFINITION 11. The relations $\theta_i \cdot \theta_j = \theta_k$, $i \neq j$; $\theta_r \cdot \theta_s = 1$; $\theta_r = \bar{\theta}_s$ satisfied by the numbers $\theta_1, \dots, \theta_n$, $(\theta_1, \dots, \theta_n) \in \mathscr{A}^n$ are called *characteristic relations*. We shall denote by $\Sigma(\theta_i)$ the set of all characteristic relations satisfied by an element (θ_i) of $\mathscr{A}^n$.

DEFINITION 12. We write $(\theta_i) \prec (\theta'_i)$ if and only if $\Sigma(\theta_i) \subseteq \Sigma(\theta'_i)$ after a possible re-numeration of the numbers θ'_i. Further, write $(\theta_i) \sim (\theta'_i)$ if and only if $(\theta_i) \prec (\theta'_i)$ and $(\theta'_i) \prec (\theta_i)$.

In [17] we proved the following

THEOREM D. *Let M be a simply connected n-dimensional manifold. If the space $(M, \tilde{\nabla})$ admits an s-structure $\{s_x\}$ with a system of eigenvalues (θ_i), and if $(\theta'_i) \succ (\theta_i)$ in $\mathscr{A}^n$, then $(M, \tilde{\nabla})$ admits an s-structure $\{s'_x\}$ with the system of eigenvalues (θ'_i).*

DEFINITION 13. A system of eigenvalues $(\theta_i) \in \mathscr{A}^n$ is called *maximal* if for any $(\theta'_i) \in \mathscr{A}^n$ the relation $(\theta'_i) \succ (\theta_i)$ implies $(\theta'_i) \sim (\theta_i)$.

On account of Theorem D each simply connected generalized affine symmetric space $(M, \tilde{\nabla})$ admits the s-structure $\{s_x\}$ with a maximal system of eigenvalues. In particular, if $(M, \tilde{\nabla})$ is of finite order, than it admits the s-structure of finite order with a maximal system of eigenvalues.

Finally, let us mention a fact which is very useful for applications. Let (V, S_0, T_0, R_0) be an infinitesimal s-manifold, $(\theta_1, \dots, \theta_n)$ the system of eigenvalues of S_0, and $\Sigma(\theta_i)$ the set of characteristic relations of (θ_i).

DEFINITION 14. By a *reduced system of characteristic relations for the system of eigenvalues* (θ_i) we mean the set $\Sigma^r(\theta_i) \subset \Sigma(\theta_i)$ which arises by deleting from $\Sigma(\theta_i)$ all characteristic relations of the from $\theta_i \cdot \theta_j = \theta_k$ where $T(U, V) = 0$ for all eigenvectors U corresponding to θ_i and all eigenvectors V corresponding to θ_j.

Let (V, S, T_0, R_0) be an infinitesimal s-manifold and (θ'_i) a system of eigenvalues. Further, let $\Sigma^r(\theta_i)$ be a reduced system of characteristic relations of (V, S, T_0, R_0) and $\Sigma(\theta'_i)$ the system of characteristic relations of (θ'_i). If $\Sigma^r(\theta_i) \subset \Sigma(\theta'_i)$, then there exists an infinitesimal s-manifold (V, S'_0, T_0, R_0) such that $(\theta'_1, \ldots, \theta'_n)$ are eigenvalues of S'_0. In other words, each generalized affine symmetric space obtained from the tensors T_0 and R_0 can also be obtained from the system of eigenvalues $(\theta'_1, \ldots, \theta'_n)$.

§ 2. The description of a method of classification

According to the previous section, it is possible to represent generalized affine symmetric spaces by s-manifolds with semi-simple symmetries and with maximal systems of eigenvalues. Thus we can begin by determining all maximal systems of eigenvalues for the given dimension n. Next we shall consider the n-dimensional vector space V and its complexification V^c. Further, for a fixed maximal system $(\theta_1, \ldots, \theta_n)$ there will be constructed all non-isomorphic infinitesimal s-manifolds (V, S_0, T_0, R_0), with the S_0 of eigenvalues $(\theta_1, \ldots, \theta_n)$. Let σ be a permutation of the natural numbers $\{1, \ldots, n\}$ such that $\sigma^2 = \mathrm{id}$, $\theta_{\sigma(i)} = \bar{\theta}_i$, and let $\{U_1, \ldots, U_n\}$ be a basis of V^c such that $U_{\sigma(i)} = \bar{U}_i$. We define $S_0\colon V^c \to V^c$ by the relation $S_0 U_i = \theta_i U_i$, $i = 1, \ldots, n$. Next we put

$$T_0(U_i, U_j) = \sum_k t^k_{ij} U_k \quad \text{for} \quad i, j = 1, 2, \ldots, n,$$

where t^k_{ij} are arbitrary complex numbers. We consider the relations $T_0(X, Y) = -T_0(Y, X)$, $S_0(T_0) = T_0$, and then obtain the dependence of T_0 upon a number of variables. By a change of the basis $\{U_1, \ldots, U_n\}$, where $U_1, \ldots, U_n$ remain to be complex eigenvectors of S_0, we minimize the number of these variables, reducing them to some "canonical" forms.

Let us consider a certain canonical type (S_0, T_0). Now we compute the Lie algebra $\mathfrak{k}$ of all endomorphisms A of the space V such that $A(S_0) = 0$, $A(T_0) = 0$. Next we put $R_0(U_i, U_j) U_k = \sum r^l_{ijk} U_l$, where r^l_{ijk} – are complex variables. First, we express the relation ${}^tR_0(U_i, U_j) \in \mathfrak{k}$ for all U_i, U_j, and next all the remaining conditions (ii)-(vi) of the definition of an infinitesimal s-manifold. In general, R_0 depends upon a number of variables. If it is possible, we minimize the number of these parameters by a change of the basis $\{U_1, \ldots, U_n\}$ (leaving the form of T_0 and S_0 invariant).

Knowing the form of T_0 and R_0, we determine a subalgebra $\mathfrak{h}^\circ \subset \mathfrak{k}$ generated by all transformations $R_0(X, Y)$; $X, Y \in V$. In particular, we may assume $\mathfrak{h}^\circ = (0)$ whenever $R_0 = 0$.

Let us have an infinitesimal s-manifold (V, S_0, T_0, R_0) in the canonical form (with a maximal system of eigenvalues). Making use of the construction of K. Nomizu, we determine the algebra $\mathfrak{g} = V+\mathfrak{h}$. The transformation $S_0: V \to V$ defines an automorphism $S_+ = S_0 + \mathrm{id}_{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$. Finally, S_+ determines an automorphism σ of the corresponding simply connected Lie group G such that $(G^\sigma)^0 \subset H \subset G^\sigma$. (Here H is the closed connected subgroup of G with the Lie algebra $\mathfrak{h}$.)

We shall show how the triplet (G, H, σ) determines directly the s-manifold with the infinitesimal type (V, S_0, T_0, R_0) and also the corresponding generalized affine symmetric space.

First of all, the automorphism $\sigma: G \to G$ admits a transformation s_0: $G/H \to G/H$ given by the relation $s_0 \circ \pi = \pi \circ \sigma$, when $\pi: G \to G/H$. Then the regular s-structure $\{s_x\}$ on G/H is given by the relation $s_x = g \circ s_0 \circ g^{-1}$ for any $x \in G/H$ and $g \in \pi^{-1}(x)$. Particularly, s_0 is the symmetry at the origin.

On the other hand, the induced automorphism $\sigma_*: \mathfrak{g} \to \mathfrak{g}$ completely determines a decomposition $\mathfrak{g} = V+\mathfrak{h}$. Here V is the unique subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$ and invariant with respect to σ_*. Hence we have described G/H as a reductive homogeneous space and we can define the canonical connection.

For these reasons, we shall represent all spaces of our classification list in the form (G, H, σ). Here it is not essential whether G is simply connected or not.

In our calculations, we shall usually determine the group G and the automorphism σ in the following way: we represent the algebra $\mathfrak{g}$ faithfully as an algebra $\mathfrak{g}^*$ of infinitesimal transformations of a Cartesian space R^k and we obtain the group G as a connected group of transformations of this space corresponding to the infinitesimal transformations. Further, we seek a diffeomorphism φ of the space R^k for which the tangent transformation φ_* induces the automorphism S_+ of the algebra $\mathfrak{g}^*$, (here $\mathfrak{g}^*$ is identified with $\mathfrak{g}$). Then the automorphism σ is given by the formula $\sigma: g \mapsto \varphi \circ g \circ \varphi^{-1}$ for $g \in G$, which follows from [8] Chapter I.

Let us remark that the group G obtained by our geometrical method is not always simply connected, particularly if we are looking for the simplest matrix representation.

Remark. We shall make a simplification in our notation, namely we shall write (V, S, T, R) instead of (V, S_0, T_0, R_0) if there is no risk of confusion.

§ 3. Two-dimensional symmetric spaces

LEMMA 1. *Let $(M, \{s_x\})$ be a tangentially regular s-manifold and $(\theta_1, \ldots, \theta_n)$ the eigenvalues of tensor S. If $\theta_i \cdot \theta_j \neq \theta_k$ for $i, j, k = 1, \ldots, n$, $i \neq j$, then the space $(M, \tilde{\nabla})$ is locally symmetric.*

Proof. Let $p \in M$. In the complexification $(M_p)^c$ of the tangent space M_p every vector X is a linear combination of eigenvectors corresponding to the eigenvalues $\theta_1, \ldots, \theta_n$.

If U_i, U_j are the eigenvectors corresponding to the eigenvalues θ_i, θ_j, $i \neq j$, then

$$S(T(U_i, U_j)) = T(SU_i, SU_j) = \theta_i \theta_j T(U_i, U_j), \tag{3.1}$$

where tensors T and S denote linear extensions to the complex space $(M_p)^c$.

The condition $\theta_i \theta_j \neq \theta_k$ implies $T(U_i, U_j) = 0$, that is, $T = 0$. Thus the space $(M, \tilde{\nabla})$ is locally symmetric.

LEMMA 2. *Let $(M, \{s_x\})$ be a tangentially regular s-manifold and $(\theta_1, \ldots, \theta_n)$ the eigenvalues of tensor S. If $\theta_i \cdot \theta_j \neq 1$ for $i, j = 1, \ldots, n$, $i \neq j$, then the curvature tensor of $(M, \tilde{\nabla})$ is zero.*

Proof. By the definition of an infinitesimal s-manifold we have for arbitrary $X, Y, Z \in V^c$:

$$S(R(X, Y)Z) = R(SX, SY)SZ \tag{3.2}$$

and

$$R(X, Y)(S) = 0. \tag{3.3}$$

Conditions (3.2) and (3.3) imply the following relations:

$$R(SX, SY) = R(X, Y). \tag{3.4}$$

Let $U_1, \ldots, U_n$ be the eigenvectors in the complexification $(M_p)^c$ of M_p corresponding to the eigenvalues $\theta_1, \ldots, \theta_n$. Then for $i \neq j$

$$R(SU_i, SU_j) = R(\theta_i U_i, \theta_j U_j) = \theta_i \theta_j R(U_i, U_j),$$

and hence for $\theta_i \theta_j \neq 1$ we obtain $R(U_i, U_j) = 0$. Thus for $\theta_i \theta_j \neq 1$, $i, j = 1, \ldots, n$, we have $R = 0$.

THEOREM 3.1. *All generalized symmetric affine spaces of two dimensions are locally symmetric.*

Proof. Let θ_1, θ_2 be the eigenvalues of S_0 in a two-dimensional tangentially regular s-manifold $(M, \{s_x\})$. Since all eigenvalues of S_0 are different from 1, their product cannot be an eigenvalue in any case. Hence, by Lemma 1, $T = 0$ and $(M, \tilde{\nabla})$, is a locally symmetric space.

§ 4. Three-dimensional generalized affine symmetric spaces

THEOREM 4.1. *The only maximal systems of eigenvalues for dimension $n = 3$ (for which at least one relation $\theta_i \cdot \theta_j = \theta_k$, $i \neq j$ is satisfied) are the following:*

$$(i, -i, -1), \quad (\alpha, -\alpha, -1), \quad (\alpha^2, \alpha, 1/\alpha), \quad (\alpha^2, \alpha, \alpha)$$

where α is a real number, $\alpha \neq 0, \pm 1$.

Proof. Every system of eigenvalues for dimension $n = 3$ takes one of the forms

$$(4.1) \qquad (\theta, \bar{\theta}, \gamma) \quad \text{or} \quad (\alpha, \beta, \gamma),$$

where θ is a complex number, and α, β, γ are real numbers different from 0 and 1.

In the first case there may hold the following characteristic relations:

$$\theta\bar{\theta} = \gamma, \quad \theta\gamma = \bar{\theta}, \quad \theta\bar{\theta} = 1.$$

The first and the second relations as well as the first and the third cannot hold simultaneously. The second and the third relations imply

$$\theta = i, \quad \gamma = -1.$$

Hence in the first case we obtain two maximal systems of eigenvalues,

$$(4.2) \qquad (i, -i, -1) \quad \text{and} \quad (\theta, \bar{\theta}, \theta\bar{\theta}),$$

where θ is a complex number.

In the second case, where all three numbers α, β, γ are real, we have the following possible characteristic relations:

$$\begin{aligned} \alpha\beta &= \gamma, & \alpha\gamma &= \beta, & \beta\gamma &= \alpha, \\ \alpha\beta &= 1, & \alpha\gamma &= 1, & \beta\gamma &= 1, \\ \alpha &= \beta, & \alpha &= \gamma, & \beta &= \gamma. \end{aligned}$$

The maximal systems of eigenvalues in this case have the form

$$(4.3) \qquad (\alpha, -\alpha, -1), \quad (\alpha^2, \alpha, 1/\alpha) \quad \text{and} \quad (\alpha^2, \alpha, \alpha),$$

where α is a real number different from $0, \pm 1$.

It is easy to see that the system $(\theta, \bar{\theta}, \theta\bar{\theta})$ is contained in the system $(\alpha, \alpha, \alpha^2)$. Finally, we have obtained four maximal systems. Q.E.D.

We shall now deal with the individual maximal systems and determine the corresponding generalized affine symmetric spaces.

1. The maximal system $(i, -i, -1)$. Let $S\colon V \to V$ be a linear transformation with eigenvalues $(i, -i, -1)$ and $T \neq 0$ the tensor of type (1,2) antisymmetric and invariant under S, i.e. $T(X, Y) = -T(Y, X)$ and $S(T) = T$. Let us denote by the same letters the linear extensions of S and T to the complexification $V^c = V \otimes_R C$ of V. Let $U \in V^c$ be a complex eigenvector such that $SU = iU$ and $W \in V$ a real eigenvector such that $SW = -W$. Then $S\bar{U} = -i\bar{U}$. The condition $S(T) = T$ means that $T(SX, SY) = S(T(X, Y))$ for any $X, Y \in V^c$.

Thus we have

$$\begin{aligned} S(T(U, \bar{U})) &= T(SU, S\bar{U}) = -i^2 T(U, \bar{U}) \Rightarrow T(U, \bar{U}) = 0, \\ S(T(U, W)) &= T(SU, SW) = -iT(U, W) \Rightarrow T(U, W) = \tau\bar{U}, \\ S(T(\bar{U}, W)) &= T(S\bar{U}, SW) = iT(\bar{U}, W) \Rightarrow T(\bar{U}, W) = \bar{\tau}U, \end{aligned}$$

where $\tau \neq 0$ is a complex variable. Assuming $\tau = \varrho e^{2i\varphi}$, $\varrho > 0$, and putting $U' = e^{-i\varphi} U$, $W' = (1/\varrho) W$, we have $T(U', W') = \bar{U}'$, $\bar{T}(\bar{U}', W') = U'$. We now conclude that for a pair of tensors (S, T) in V there is a basis $(U, \bar{U}, W)$ for V^c $(W \in V)$ such that

$$\begin{gathered} SU = iU, \quad S\bar{U} = -i\bar{U}, \quad SW = -W, \\ T(U, \bar{U}) = 0, \quad T(U, W) = \bar{U}, \quad T(\bar{U}, W) = U. \end{gathered} \tag{4.4}$$

Thus we have obtained the canonical form of an admissible pair (S, T), $T \neq 0$.

Let $\mathfrak{k}$ denote the Lie algebra of all real endomorphisms $A\colon V^c \to V^c$ which, as derivations, satisfy $A(S) = A(T) = 0$. The relation $A(S) = 0$ means $A \circ S = S \circ A$ and hence

$$AU = \sigma U, \quad A\bar{U} = \bar{\sigma}\bar{U}, \quad AW = \varrho W \tag{4.5}$$

(ϱ — real number).

The relation $A(T) = 0$ means that $A(T(X, Y)) = T(AX, Y) + T(X, AY)$ for any $X, Y \in V^c$. So we have

$$\begin{aligned} A(T(U, W)) &= (\sigma + \varrho)\ T(U, W), \\ A(T(\bar{U}, W)) &= (\bar{\sigma} + \varrho)\ T(\bar{U}, W), \end{aligned}$$

and hence

$$\bar{\sigma} = \sigma + \varrho, \quad \sigma = \bar{\sigma} + \varrho. \tag{4.6}$$

It follows from (4.6) that $\varrho = 0$ and σ is a real number. Thus

$$AU = \sigma U, \quad A\bar{U} = \sigma\bar{U} \quad \text{and} \quad AW = 0.$$

Consequently, algebra $\mathfrak{k}$ is one-dimensional and generated by endomorphism B satisfying the conditions

$$BU = U, \quad B\bar{U} = \bar{U}, \quad BW = 0. \tag{4.7}$$

Let (V, S, T, R) be an infinitesimal s-manifold with the tensors S, T satisfying (4.4). For any $X, Y \in V$ $R(X, Y) \in \mathfrak{k}$ and for any $Z, Z' \in V^c$ we have $R(Z, Z') \in \mathfrak{k} \otimes_R C$. It follows from $R(SX, SY) = R(X, Y)$ that

$$\begin{aligned} R(SU, SW) &= -iR(U, W) &&\Rightarrow R(U, W) = 0, \\ R(S\bar{U}, SW) &= iR(\bar{U}, W) &&\Rightarrow R(\bar{U}, W) = 0, \\ R(SU, S\bar{U}) &= R(U, \bar{U}) &&\Rightarrow R(U, \bar{U}) = \tau B, \end{aligned} \tag{4.8}$$

where τ is a complex number.

Now we shall make use of $R(U, \bar{U})(R) = 0$. We have

$$R(\tau BU, \bar{U}) + R(U, \tau B\bar{U}) = 0.$$

On account of (4.7), $2\tau R(U, \bar{U}) = 0$, that is,

$$R(U, \bar{U}) = 0. \tag{4.9}$$

From (4.8) and (4.9) we obtain $R = 0$.

The "Nomizu algebra" $\mathfrak{g}$ is given by the formula $[X, Y] = -T(X, Y)$ for any $X, Y \in V$, since by $R = 0$ we may assume $\mathfrak{h} = (0)$. Putting $U = X + iY$, $Z = W$, $X, Y, Z \in V$, we have

$$SX = -Y, \quad SY = X, \quad SZ = -Z. \tag{4.10}$$

The multiplication in the Lie algebra $\mathfrak{g}$ is given by

$$[X, Y] = 0, \quad [X, Z] = -X, \quad [Y, Z] = Y. \tag{4.11}$$

We may give the representation of this algebra by infinitesimal affine transformation of the plane $R^2(x, y)$, namely

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.12}$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} e^{-c} & 0 & a \\ 0 & e^{c} & b \\ 0 & 0 & 1 \end{Vmatrix}. \tag{4.13}$$

The underlying manifold of the group G is the Cartesian space $R^3(a, b, c)$. Thus G is the required simply connected Lie group with the Lie algebra $\mathfrak{g}$. Diffeomorphism φ of the plane R^2 inducing the automorphism S_0 of algebra $\mathfrak{g}$ is given by the transformation

$$x' = y, \quad y' = -x. \tag{4.14}$$

The automorphism σ of the group G determining symmetry on R^3 is given by the formulas

$$a' = -b, \quad b' = a, \quad c' = -c. \tag{4.15}$$

Since $\sigma^4 = \mathrm{id}$, the order of the s-structure is four.

2. The maximal system $(\alpha, -\alpha, -1)$. Let $S\colon V \to V$ be a linear transformation with eigenvalues $(\alpha, -\alpha, -1)$, α real number different from $0, \pm 1$, and $T \neq 0$ a tensor of type $(1, 2)$ antisymmetric and invariant under S. Let X, Y, Z be eigenvectors of S,

$$SX = \alpha X, \quad SY = -\alpha X, \quad SZ = -Z. \tag{4.16}$$

By the condition $S(T) = T$ we obtain

$$T(X, Y) = 0, \quad T(X, Z) = aY, \quad T(Y, Z) = bX, \tag{4.17}$$

where a, b are real numbers, $a^2+b^2 > 0$. Let us consider the following cases:

1) $a \cdot b \neq 0$; then, assuming $X = aX'$, $Y = \sqrt{|ab|}\, Y'$, $Z = \sqrt{|ab|}\, Z'$ we have

(4.18) $$T(X', Y') = 0, \quad T(X', Z') = Y', \quad T(Y', Z') = \operatorname{sgn}(ab) X'.$$

2) $ab = 0$. For instance, let $a = 0$ and $b \neq 0$.

Then, assuming $X = (1/b)X'$, $Y = Y'$, $Z = Z'$, we have

(4.19) $$T(X', Y') = 0, \quad T(X', Z') = 0, \quad T(Y', Z') = X'.$$

Thus we have obtained three types of canonical forms of tensor $T \neq 0$.

(4.20) $$\begin{aligned} &T(X, Y) = 0, \quad T(X, Z) = Y, \quad T(Y, Z) = X, \\ &T(X, Y) = 0, \quad T(X, Z) = Y, \quad T(Y, Z) = -X, \\ &T(X, Y) = 0, \quad T(X, Z) = 0, \quad T(Y, Z) = X. \end{aligned}$$

Now let $\mathfrak{k}$ denote the Lie algebra of all real endomorphisms $A: V \to V$ which, as derivations, satisfy $A(S) = A(T) = 0$. Let (V, S, T, R) be an infinitesimal s-manifold with the tensors S, T satisfying (4.16) and (4.20). Then $R(X, Y) \in \mathfrak{k}$ for any $X, Y \in V$. By Lemma 2 we obtain $R = 0$. In this case the infinitesimal s-manifold is of the form $(V, S, T, 0)$. So, as $\mathfrak{h} \subset \mathfrak{k}$, we may assume $\mathfrak{h}^{\circ} = (0)$. We have just obtained three Lie algebras with multiplications defined as follows:

(4.21) $$\begin{aligned} &\mathfrak{g}_1\colon [X, Y] = 0, \quad [X, Z] = -Y, \quad [Y, Z] = -X, \\ &\mathfrak{g}_2\colon [X, Y] = 0, \quad [X, Z] = -Y, \quad [Y, Z] = X, \\ &\mathfrak{g}_3\colon [X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = -X. \end{aligned}$$

We shall prove that these algebras are different from one another. Algebra $\mathfrak{g}_3$ cannot be isomorphic with algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. For, two-dimensional subalgebras $[\mathfrak{g}_1, \mathfrak{g}_1]$ and $[\mathfrak{g}_2, \mathfrak{g}_2]$ of the form

$$[\mathfrak{g}_i, \mathfrak{g}_i] = \{a[X, Y]+b[X, Z]+c[Y, Z]\} \quad (i = 1, 2, 3),$$

where a, b and c are real numbers, are non-isomorphic to the $[\mathfrak{g}_3, \mathfrak{g}_3]$ which is a one-dimensional subalgebra.

Algebra $\mathfrak{g}_1$ is not isomorphic with $\mathfrak{g}_2$.

To prove this consider the mapping

$$L_V\colon [\mathfrak{g}, \mathfrak{g}] \to [\mathfrak{g}, \mathfrak{g}]$$

defined by $L_V(U) = [U, V]$ where

$$V \in \mathfrak{g} - \{[\mathfrak{g}, \mathfrak{g}]\}, \quad U \in [\mathfrak{g}, \mathfrak{g}].$$

In the case of algebra $\mathfrak{g}_1$ we obtain

$$L_V(X) = -cY, \quad L_V(Y) = -cX, \quad c \neq 0.$$

The eigenvalues of this transformation are real.

In the case of algebra $\mathfrak{g}_2$ we obtain

$$L_V(X) = -cY, \quad L_V(Y) = cX, \quad c \neq 0.$$

The eigenvalues of this transformation are imaginary.

Algebra $\mathfrak{g}_1$ is isomorphic with algebra $\mathfrak{g}$ obtained for the maximal system $(i, -i, -1)$. If X', Y', Z' denote the basis vectors of algebra $\mathfrak{g}$, the isomorphism is given by the assignment

$$X \mapsto X'-Y', \quad Y \mapsto X'+Y', \quad Z \mapsto Z'.$$

There remain two non-isomorphic Lie algebras $\mathfrak{g}_2$ and $\mathfrak{g}_3$. The representations of these algebras can be given by the respective infinitesimal transformations of Cartesian spaces. The representation of the algebra $\mathfrak{g}_2$ may be given by an infinitesimal transformation of the space $R^3(x, y, z)$

$$X = \frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial x}, \quad Z = -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+\frac{\partial}{\partial z}. \tag{4.22}$$

The corresponding Lie group G is the group of all matrices of the from

$$\begin{Vmatrix} \cos c & -\sin c & 0 & a \\ \sin c & \cos c & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{4.23}$$

The underlying manifold G is the Cartesian space $R^3(a, b, c)$. G is the required simply connected Lie group with the Lie algebra $\mathfrak{g}_2$. The diffeomorphism φ of R^3 inducing the automorphism S_0 of $\mathfrak{g}_2$ is given by the transformation

$$x' = -\frac{1}{\alpha}x, \quad y' = \frac{1}{\alpha}y, \quad z' = -z. \tag{4.24}$$

The automorphism σ of the group G determining symmetry on R^3 is given by formulas

$$a' = -\frac{1}{\alpha}a, \quad b' = \frac{1}{\alpha}b, \quad c' = -c. \tag{4.25}$$

This symmetry is of infinite order.

R e m a r k. The algebra $\mathfrak{g}_2$ can also be given by a group of transformations of R^2; however, this group is not simply connected. We may give the representation of this algebra by infinitesimal affine transformations of the plane $R^2(x, y)$, namely

$$X = \frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial x}, \quad Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

The corresponding Lie group G is the group of matrices of the form

$$\begin{Vmatrix} \cos c & -\sin c & a \\ \sin c & \cos c & b \\ 0 & 0 & 1 \end{Vmatrix}.$$

For algebra $\mathfrak{g}_3$ we have

$$X = \frac{\partial}{\partial x}, \quad Y = y\frac{\partial}{\partial x}, \quad Z = \frac{\partial}{\partial y}. \tag{4.26}$$

The corresponding Lie group is the group of all matrices of the form

$$\begin{Vmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{Vmatrix}. \tag{4.27}$$

Here the underlying manifold G is the Cartesian space $R^3(a, b, c)$. The symmetry S_0 is of infinite order and is given at the initial point $(0, 0, 0)$ of R^3 by the rule

$$a' = -\frac{1}{\alpha}a, \quad b' = -b, \quad c' = \frac{1}{\alpha}c. \tag{4.28}$$

3. The maximal system $(\alpha^2, \alpha, \alpha)$. Let $S\colon V \to V$, X, Y, Z be the eigenvectors of S, that is,

$$SX = \alpha^2 X, \quad SY = \alpha Y, \quad SZ = \alpha Z. \tag{4.29}$$

By the condition $S(T) = T$ we obtain

$$T(X, Y) = 0, \quad T(X, Z) = 0, \quad T(Y, Z) = aX, \tag{4.30}$$

where a is a real number $\neq 0$.

By a suitable change of the basis we may have $a = 1$. From Lemma 2 $R = 0$. Therefore the infinitesimal s-manifold is of the form $(V, S, T, 0)$, where S and T satisfy (4.29) and (4.30). Here $\mathfrak{h} \subset \mathfrak{k}$, $R = 0$ and consequently $\mathfrak{h}^\circ = (0)$. The Lie algebra $\mathfrak{g}$ is defined by multiplication:

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = -X. \tag{4.31}$$

It is immediately seen that this algebra is isomorphic to algebra $\mathfrak{g}_3$ obtained in the case of the maximal system $(\alpha, -\alpha, -1)$.

4. The maximal system $(\alpha^2, \alpha, 1/\alpha)$. As in the previous cases, let $S\colon V \to V$ be the linear transformation with eigenvalues $(\alpha^2, \alpha, 1/\alpha)$. Let X, Y, Z be the eigenvectors of S, i.e.

$$SX = \alpha^2 X, \quad SY = \alpha Y, \quad SZ = (1/\alpha)Z. \tag{4.32}$$

The condition $S(T) = T$ implies

$$T(X, Y) = 0, \quad T(X, Z) = aY, \quad T(Y, Z) = 0, \tag{4.33}$$

where a is real number not equal to 0.

By a change of the basis we may have $a = 1$. Now we shall find the Lie algebra $\mathfrak{k}$. By the condition $A(S) = 0$ we have

$$AX = \tau X, \quad AY = \varrho Y, \quad AZ = \sigma Z, \tag{4.34}$$

where τ, ϱ and σ are real numbers. $A(T) = 0$ implies $\sigma = \varrho - \tau$. Consequently, the Lie algebra $\mathfrak{k}$ is two-dimensional. Let $\{A_1, A_2\}$ be a basis of $\mathfrak{k}$, where

$$\begin{aligned} A_1 X &= X, & A_1 Y &= 0, & A_1 Z &= -Z, \\ A_2 X &= 0, & A_2 Y &= Y, & A_2 Z &= Z. \end{aligned} \tag{4.35}$$

From $R(SX, SY) = R(X, Y)$ for all $X, Y \in V$ we have

$$\begin{gathered} R(X, Y) = 0, \quad R(X, Z) = 0, \\ R(Y, Z) = \tau A_1 + \varrho A_2. \end{gathered} \tag{4.36}$$

The first Bianchi identity gives

$$\mathfrak{S}(R(X, Y)Z) = \mathfrak{S}\big(T(T(X, Y), Z)\big),$$

that is,

$$R(Y, Z)X = (\tau A_1 + \varrho A_2)X = 0,$$

and hence $\tau = 0$. Thus $R(Y, Z)$ is determined by endomorphism A_2. Now we shall make use of the condition $R(Y, Z)(R) = 0$, namely $R(\varrho A_2 Y, Z) + R(Y, \varrho A_2 Z) = 0$, and hence $R(Y, Z) = 0$.

The infinitesimal s-manifold is of the form $(V, S, T, 0)$. The Lie algebra $\mathfrak{g}$ is defined by multiplication:

$$[X, Y] = 0, \quad [X, Z] = -Y, \quad [Y, Z] = 0. \tag{4.37}$$

This algebra is isomorphic to the algebra $\mathfrak{g}_3$ obtained in the case of the maximal system $(\alpha, -\alpha, -1)$.

§ 5. Four-dimensional generalized affine symmetric spaces

THEOREM 5.1. *The only maximal systems of eigenvalues for dimension* $n = 4$ *are the following*:

(1) $\left(\alpha, \frac{1}{\alpha^2}, \frac{1}{\alpha}, \alpha^2\right)$, α — *real number* $\neq 0, \pm 1$,

(2) $\left(\frac{1}{\alpha^2}, \alpha, \frac{1}{\alpha}, \alpha\right)$,

(3) $\left(\alpha^2, \frac{1}{\alpha}, \alpha, \alpha\right)$,

(4) $(-1, \alpha, -\alpha, -1)$,

(5) $\left(\alpha^2, \frac{1}{\alpha}, \alpha, \alpha^3\right)$,

(6) $\left(-1, \alpha, -\alpha, \frac{1}{\alpha}\right)$,

(7) $\left(\frac{1}{\alpha^2}, \alpha^3, \alpha, \frac{1}{\alpha}\right)$,

(8) $\left(\alpha, -\frac{1}{\alpha}, -1, -1\right)$,

(9) $\left(\alpha, \frac{1}{\alpha^2}, \frac{1}{\alpha}, \frac{1}{\alpha^2}\right)$,

(10) $(\alpha, \alpha, \alpha^2, \alpha)$,

(11) $(-1, \alpha, -\alpha, -\alpha)$,

(12) $(-1, \alpha, -\alpha, -\alpha^2)$,

(13) $(\alpha, \alpha, \alpha^2, \alpha^3)$,

(14) $(\alpha, \alpha, \alpha^2, \alpha^2)$,

(15) $(-1, -1, -1, -1)$,

(16) $(i, -i, -1, -1)$,

(17) $(\theta, \theta^2, \theta, \theta^2)$, $\theta = e^{2\pi i/3}$,

(18) $(\theta, \theta^2, \theta^3, \theta^4)$, $\theta = e^{2\pi i/5}$.

Proof. Every system of eigenvalues for $n = 4$ takes one of the following forms:

A) $(\alpha, \beta, \gamma, \delta)$,

B) $(\theta, \bar{\theta}, \gamma, \delta)$,

C) $(\theta, \bar{\theta}, \tau, \bar{\tau})$,

where $\alpha, \beta, \gamma, \delta$ are real and θ, τ are complex numbers. Each one of them is different from 0 and 1.

Ad A) When all the eigenvalues are real, there are two possible cases: the product of any two eigenvalues is different from any third one — then we have the maximal system $(-1, -1, -1, -1)$ satisfying all the remaining characteristic relations; there are three eigenvalues such that the product of two of them is equal to the third one then each system is of the form $(\alpha, \beta, \alpha\beta, \delta)$. In the second case, from the other possible characteristic relations we obtain the maximal systems (1)-(14).

Ad B) In the case where two eigenvalues are real and the other two are complex we have the following possible characteristic relations:

$$\theta \cdot \bar{\theta} = \gamma, \quad \theta\bar{\theta} = \delta, \quad \gamma \cdot \theta = \bar{\theta}, \quad \delta \cdot \theta = \bar{\theta},$$
$$\theta \cdot \bar{\theta} = 1, \quad \gamma\delta = 1, \quad \gamma = \delta.$$

If $\gamma = \delta$, then we have the following possible systems of relations: $\theta\bar{\theta} = 1$, $\gamma^2 = 1$, $\gamma \cdot \theta = \bar{\theta}$ or $\theta \cdot \bar{\theta} = \gamma$. Hence we obtain two maximal systems, $(i, -i, -1, -1)$ and $(\theta, \bar{\theta}, \theta\bar{\theta}, \theta\bar{\theta})$, where $|\theta| \neq 1$. If $\gamma \neq \delta$, then we obtain the maximal system $(\theta, \bar{\theta}, \theta\bar{\theta}, 1/\theta\bar{\theta})$, where $|\theta| \neq 1$. It can easily be seen that the system $(\theta, \bar{\theta}, \theta\bar{\theta}, \theta\bar{\theta})$ is contained in the real system $(\alpha, \alpha, \alpha^2, \alpha^2)$, and the system $(\theta, \bar{\theta}, \theta\bar{\theta}, 1/\theta\bar{\theta})$ is contained in $(i, -i, -1, -1)$. Thus in this case the essential system is $(i, -i, -1, -1)$.

Ad C) Now we have the following possible characteristic relations:

$$\theta\tau = \bar{\theta}, \quad \theta\tau = \bar{\tau}, \quad \theta\bar{\tau} = \tau, \quad \theta\bar{\tau} = \bar{\theta},$$
$$\theta\bar{\theta} = 1, \quad \theta\tau = 1, \quad \theta\bar{\tau} = 1,$$
$$\theta = \tau, \quad \bar{\theta} = \tau,$$

If $\theta \neq \tau$ and $\theta \neq \bar{\tau}$, then the possible relations are

$$\theta\tau = \bar{\theta}, \quad \theta\bar{\tau} = \tau \quad \text{or} \quad \theta\tau = \bar{\tau}, \quad \theta\bar{\tau} = \bar{\theta},$$

and

$$\theta\bar{\theta} = 1 \quad \text{or} \quad \theta\tau = 1 \quad \text{or} \quad \theta\bar{\tau} = 1.$$

Finally we have two possibilities:

$$\theta\tau = \bar{\theta}, \quad \theta\bar{\tau} = \tau, \quad \theta\bar{\theta} = 1$$

or

$$\theta\tau = \bar{\tau}, \quad \theta\bar{\tau} = \bar{\theta}, \quad \theta\bar{\theta} = 1.$$

In both cases we obtain the system $(\theta, \theta^2, \theta^3, \theta^4)$, where $\theta = e^{2\pi i/}$. If $\tau = \theta$, then each of the relations $\theta\tau = \bar{\theta}$, $\theta\tau = \tau$ leads to a contradiction. There remain possible system of relations:

$$\theta\bar{\tau} = \bar{\theta}, \quad \theta\tau = \bar{\tau}, \quad \theta\bar{\theta} = 1$$

or

$$\theta\tau = \bar{\theta}, \quad \theta\tau = \bar{\tau}, \quad \theta\tau = 1.$$

The first system implies $\tau^3 = 1$, the second $\theta^3 = 1$. Thus in both cases we obtain the maximal system $(\theta, \theta^2, \theta, \theta^2)$ where $\theta = e^{2\pi i/3}$. Also when $\tau = \bar{\theta}$ we obtain the same system. Q.E.D.

We shall now consider the individual maximal systems one by one and determine the corresponding generalized affine symmetric spaces.

1. The maximal system $(\alpha^2, \alpha, 1/\alpha, 1/\alpha^2)$. Let V be a 4-dimensional vector space, and $S\colon V \to V$ a real linear transformation with eigenvalues $(\alpha^2, \alpha, 1/\alpha, 1/\alpha^2)$, $\alpha \neq 0, \pm 1$. Further, let T be a tensor of type (1,2) antisymmetric and invariant under S.

Let $\{X_1, X_2, X_3, X_4\}$ be eigenvectors of transformation S, i.e.

$$SX_1 = \alpha^2 X_1, \quad SX_2 = \alpha X_2, \quad SX_3 = \frac{1}{\alpha} X_3, \quad SX_4 = \frac{1}{\alpha^2} X_4.$$

By the condition $S(T) = T$ we have

$$\begin{aligned} &T(X_1, X_2) = 0, \quad T(X_1, X_3) = aX_2, \quad T(X_1, X_4) = 0,\\ &T(X_2, X_3) = 0, \quad T(X_2, X_4) = bX_3, \quad T(X_3, X_4) = 0. \end{aligned}$$

where a, b are real numbers, $a^2+b^2 > 0$.

Let us consider the following cases:

A) $a \cdot b \neq 0$; then assuming $X_1 = aX'_1$, $X_4 = bX'_4$ we obtain $T(X'_1, X_3) = X_2$, $T(X_2, X'_4) = X_3$.

Thus we conclude that in this case for the pair (S, T) on V there exists a basis (denoted also by $\{X_1, X_2, X_3, X_4\}$) such that:

$$\begin{aligned} &SX_1 = \alpha^2 X_1, \quad SX_2 = \alpha X_2, \quad SX_3 = \frac{1}{\alpha} X_3, \quad SX_4 = \frac{1}{\alpha^2} X_4,\\ &T(X_1, X_2) = 0, \quad T(X_1, X_3) = X_2, \quad T(X_1, X_4) = 0, \qquad (5.1)\\ &T(X_2, X_3) = 0, \quad T(X_2, X_4) = X_3, \quad T(X_3, X_4) = 0. \end{aligned}$$

Let $\mathfrak{k}$ denote the Lie algebra of all real endomorphisms $A\colon V \to V$ which, as derivations, satisfy $A(S) = A(T) = 0$. The relation $A(S) = 0$ means that $A \circ S = S \circ A$ and hence

$$AX_1 = pX_1, \quad AX_2 = qX_2, \quad AX_3 = rX_3, \quad AX_4 = sX_4,$$

where p, q, r, s are real numbers. On account of $A(T) = 0$ we obtain

$$\begin{aligned} &A(T(X_1, X_3)) = T(AX_1, X_3) + T(X_1, AX_3) \Rightarrow q = p+r,\\ &A(T(X_2, X_4)) = T(AX_2, X_4) + T(X_2, AX_4) \Rightarrow r = q+s. \end{aligned}$$

So algebra $\mathfrak{k}$ is two-dimensional. Let $\{A_1, A_2\}$ be the basis of this algebra, and

$$\begin{aligned} &A_1 X_1 = X_1, \quad A_1 X_2 = 0, \quad A_1 X_3 = -X_3, \quad A_1 X_4 = -X_4,\\ &A_2 X_1 = 0, \quad A_2 X_2 = X_2, \quad A_2 X_3 = X_3, \quad A_2 X_4 = 0. \end{aligned}$$

Let (V, S, T, R) be an infinitesimal s-manifold with the tensors S and T satisfying (5.1). Then for any $X, Y \in V$ we have $R(X, Y) \in \mathfrak{k}$. The conditions $S(R) = R$ and $R(X, Y)(S) = 0$ imply $R(SX, SY) = R(X, Y)$. Thus we have

$$R(X_1, X_2) = R(X_1, X_3) = R(X_2, X_4) = R(X_3, X_4) = 0,$$
$$R(X_1, X_4) = \tau_1 A_1 + \tau_2 A_2,$$
$$R(X_2, X_3) = \varrho_1 A_1 + \varrho_2 A_2,$$

where τ_i, ϱ_i are real numbers.

Further, the first Bianchi identity $\mathfrak{S}(R(X, Y)Z) = \mathfrak{S}(T(T(X, Y), Z))$ must hold in V and particularly we have

$$R(X_2, X_3)X_1 = 0 \Rightarrow \varrho_1 = 0,$$
$$R(X_1, X_4)X_2 = X_2 \Rightarrow \tau_2 = 1,$$
$$R(X_1, X_4)X_3 = -X_3 \Rightarrow \tau_1 = 2.$$

Hence we obtain

$$R(X_1, X_4) = 2A_1 + A_2,$$
$$R(X_2, X_3) = \varrho_2 A_2.$$

For $A \in \mathfrak{h}$, $(AR)(X, Y, Z) = 0$ for any X, $Y, Z \in V$, that is,

$$A \circ R(X, Y) - R(X, Y) \circ A = R(AX, Y) + R(X, AY).$$

Since $R(X, Y) \in \mathfrak{h}$, we have $R(X, Y)(R) = 0$. In the case of $R(X_2, X_3) = \varrho_2 A_2$ we have

$$R(\varrho_2 A_2 X_2, X_3) + R(X_2, \varrho_2 A_2 X_3) = 0,$$

and hence

$$R(X_2, X_3) = 0.$$

The algebra $\mathfrak{h}^\circ$ generated by the curvature endomorphism $R(X, Y)$ is one-dimensional and is generated by an endomorphism of the form $A = R(X_1, X_4) = 2A_1 + A_2$

$$AX_1 = 2X_1, \quad AX_2 = X_2, \quad AX_3 = -X_3, \quad AX_4 = -2X_4.$$

In this case the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ formed by means of the "Nomizu construction" is of the form:

(5.2)

	X_1	X_2	X_3	X_4	A
X_1	0	0	$-X_2$	$-A$	$-2X_1$
X_2	0	0	0	$-X_3$	$-X_2$
X_3	X_2	0	0	0	X_3
X_4	A	X_3	0	0	$2X_4$
A	$2X_1$	X_2	$-X_3$	$-2X_4$	0

We may now give the representation of this algebra by infinitesimal affine transformations of the plane $R^2(x, y)$, namely

$$X_1 = y\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = -x\frac{\partial}{\partial y},$$

$$A = -x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

The corresponding Lie group G is the group of all matrices of the form [9].

$$\begin{Vmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{Vmatrix}, \quad \text{where } ad-bc = 1.$$

The subgroup H is the group of matrices of the form

$$\begin{Vmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

G/H is a reductive homogeneous space. The diffeomorphism φ of the plane $R^2(x, y)$ inducing the automorphism S of algebra $\mathfrak{g}$ is given by the transformation

$$x' = \frac{1}{\alpha}x, \quad y' = \alpha y.$$

The automorphism σ of the group G is given by

$$a' = a, \quad b' = \frac{1}{\alpha^2}b, \quad c' = \alpha^2 c,$$

$$d' = d, \quad e' = \frac{1}{\alpha}e, \quad f' = \alpha f.$$

This transformation is an automorphism preserving the invariant subgroup H.

B) $a \neq 0$, $b = 0$. In this case we have

$$T(X_1, X_2) = T(X_1, X_4) = T(X_2, X_3) = T(X_2, X_4) = T(X_3, X_4) = 0,$$
$$T(X_1, X_3) = X_2.$$

The algebra $\mathfrak{k}$ is three-dimensional. Let $\{A_1, A_2, A_3\}$ be the basis of this algebra, and

$$\begin{aligned} &A_1 X_1 = X_2, && A_1 X_2 = 0, && A_1 X_3 = -X_3, && A_1 X_4 = 0, \\ &A_2 X_1 = 0, && A_2 X_2 = X_2, && A_2 X_3 = X_3, && A_2 X_4 = 0, \\ &A_3 X_1 = 0, && A_3 X_2 = 0, && A_3 X_3 = 0, && A_3 X_4 = X_4. \end{aligned}$$

Then

$$R(X_1, X_2) = R(X_1, X_3) = R(X_2, X_3) = R(X_3, X_4) = 0,$$

$$R(X_1, X_4) = \sum_{i=1}^{3} \tau_i A_i, \quad R(X_2, X_3) = \sum_{i=1}^{3} \varrho_i A_i.$$

On account of the first Bianchi identity, for the triples of basis vectors X_1, X_2, X_3; X_1, X_2, X_4; X_2, X_3, X_4 and X_1, X_3, X_4 we have

$$R(X_2, X_3)X_1 = 0, \; R(X_1, X_4)X_2 = 0, \; R(X_2, X_3)X_4 = 0, \; R(X_1, X_4)X_3 = 0,$$

respectively. Hence

$$\varrho_1 = 0, \quad \tau_2 = 0, \quad \varrho_3 = 0, \quad \tau_1 = 0.$$

Thus we obtain

$$R(X_1, X_4) = \tau_3 A_3; \quad R(X_2, X_3) = \varrho_2 A_2.$$

Further, by the condition $R(X, Y)(R) = 0$ we have

$$R(\tau_3 A_3 X_1, X_4) + R(X_1, \tau_3 A_3 X_4) = 0 \Rightarrow R(X_1, X_4) = 0,$$
$$R(\varrho_2 A_2 X_2, X_3) + R(X_2, \varrho_2 A_2 X_3) = 0 \Rightarrow R(X_2, X_3) = 0,$$

and hence $R = 0$.

In this case the infinitesimal s-manifold is of the form $(V, S, T, 0)$, where S and T satisfy conditions (5.3). Here we may assume $\mathfrak{h}^\circ = 0$. The "Nomizu algebra" $\mathfrak{g}$ is given by $[X, Y] = -T(X, Y)$ for all $X, Y \in V$. Thus multiplication in algebra $\mathfrak{g}$ is defined as follows:

(5.4)

	X_1	X_2	X_3	X_4
X_1	0	0	$-X_2$	0
X_2	0	0	0	0
X_3	X_2	0	0	0
X_4	0	0	0	0

We have obtained a decomposable algebra, i.e. a reducible infinitesimal s-manifold. This leads to a decomposable generalized affine symmetric space $(M, \tilde{\nabla}) = (M_1, \tilde{\nabla}_1) \times (M_2, \tilde{\nabla}_2)$ [17].

C) $a = 0$, $b \neq 0$. In this case we have $R = 0$ and the algebra $\mathfrak{g}$ is decomposable.

2. The maximal system $(1/\alpha^2, 1/\alpha, \alpha, \alpha)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = T(X_2, X_3) = T(X_2, X_4) = T(X_3, X_4) = 0,$$
$$T(X_1, X_3) = aX_2, \quad T(X_1, X_4) = bX_2, \tag{5.5}$$

where a, b are real numbers and $a^2+b^2>0$. Let $a\neq 0$; then assuming $X'_1 = X_1$, $X'_2 = X_2$, $X'_3 = (1/a)X_3$, $X'_4 = bX_3 - aX_4$, we obtain $T(X'_1, X'_3) = X'_2$, $T(X'_1, X'_4) = 0$. For algebra $\mathfrak{k}$ we obtain: $A\in\mathfrak{k}$, that is,

$$AX_1 = kX_1, \quad AX_2 = lX_2, \quad AX_3 = pX_3+qX_4, \quad AX_4 = rX_3+sX_4,$$

where k, l, p, q, r, s are real numbers. By condition $A(T) = 0$ we have

$$A(T(X_1, X_3)) = T(AX_1, X_3)+T(X_1, AX_3),$$

whence

$$lX_2 = kX_2+pX_2 \Rightarrow p = l-k,$$

$$A(T(X_1, X_4)) = T(AX_1, X_4)+T(X_1, AX_4),$$

and thus

$$0 = rX_2 \Rightarrow r = 0.$$

So algebra $\mathfrak{k}$ is four-dimensional with the basis $\{A_1, A_2, A_3, A_4\}$ where

$$\begin{array}{llll} A_1X_1 = X_1, & A_1X_2 = 0, & A_1X_3 = -X_3, & A_1X_4 = 0, \\ A_2X_1 = 0, & A_2X_2 = X_2, & A_2X_3 = X_3, & A_2X_4 = 0, \\ A_3X_1 = 0, & A_3X_2 = 0, & A_3X_3 = X_4, & A_3X_4 = 0, \\ A_4X_1 = 0, & A_4X_2 = 0, & A_4X_3 = 0, & A_4X_4 = X_4. \end{array}$$

Let (V, S, T, R) be the infinitesimal s-manifold with tensors S, T satisfying (5.5) $(a = 1, b = 0)$. Then $R(X, Y)\in\mathfrak{k}$ and from $R(SX, SY) = R(X, Y)$ we obtain

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_3, X_4) = 0,$$

$$R(X_2, X_3) = \sum_{i=1}^{4} \tau_i A_i, \quad R(X_2, X_4) = \sum_{i=1}^{4} \varrho_i A_i.$$

The second Bianchi identity $\mathfrak{S}\big(R(T(X, Y), Z)\big) = 0$ must hold and particularly we get

$$R(T(X_1, X_3), X_4)+R(T(X_3, X_4), X_1)+R(T(X_4, X_1), X_3) = 0,$$

that is,

$$R(X_2, X_4) = 0.$$

By the first Bianchi identity we have

$$\mathfrak{S}(R(X_1, X_2)X_3) = \mathfrak{S}\big(T(T(X_1, X_2), X_3)\big),$$

whence

$$R(X_2, X_3)X_1 = 0 \Rightarrow \tau_1 = 0.$$

$$\mathfrak{S}(R(X_2, X_3)X_4) = \mathfrak{S}\big(T(T(X_2, X_3), X_4)\big)$$

and thus

$$R(X_2, X_3)X_4 = 0 \Rightarrow \tau_4 = 0.$$

Hence

$$R(X_2, X_3) = \tau_2 A_2 + \tau_3 A_3.$$

From $R(X, Y)(R) = 0$ we obtain

$$R\big((\tau_2 A_2+\tau_3 A_3)X_2, X_3\big)+R\big(X_2, (\tau_2 A_2+\tau_3 A_3)X_3\big) = 0,$$

$$2\tau_2 R(X_2, X_3) = 0.$$

Hence

$$R(X_2, X_3) = \tau A_3, \quad \tau \text{ being an arbitrary real number.}$$

For $\tau = 0$ we obtain a decomposable algebra. When $\tau \neq 0$, we take a new basis $X_i = \sqrt{|\tau_i|}\, X_i'$, $i = 2, 3, 4$. Then

$$T(X_1, X_3') = X_2', \quad R(X_2', X_3') = \operatorname{sgn}\tau \cdot A_3,$$

$$A_3 X_1 = 0, \quad A_3 X_2' = 0, \quad A_3 X_3' = X_4', \quad A_3 X_4' = 0.$$

So algebra $\mathfrak{h}^\circ$ is one-dimensional and is generated by endomorphism A_3. The Lie algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is given by

(5.6)

	X_1	X_2	X_3	X_4	A
X_1	0	0	$-X_2$	0	0
X_2	0	0	$-kA$	0	0
X_3	X_2	kA	0	0	$-X_4$
X_4	0	0	0	0	0
A	0	0	X_4	0	0

for $k = \pm 1$. These algebras are isomorphic. If $\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{A}$ denote the basis vector of the algebras for $k = -1$, the isomorphism is given by the assignment

$$X_1 \mapsto \bar{X}_1, \quad X_2 \mapsto \bar{X}_2, \quad X_3 \mapsto \bar{X}_3, \quad X_4 \mapsto \bar{X}_4, \quad A \mapsto -\bar{A}.$$

Now we may give the representation of this algebra by proper infinitesimal transformations of $R^4(x, y, u, v)$, namely

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = -x\frac{\partial}{\partial y}+v\frac{\partial}{\partial u}-y\frac{\partial}{\partial v},$$

$$X_4 = \frac{\partial}{\partial u}, \quad A = \frac{\partial}{\partial v}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} 1 & t & -\frac{1}{2}t^2 & \frac{1}{6}t^3 & a \\ 0 & 1 & -t & \frac{1}{2}t^2 & b \\ 0 & 0 & 1 & -t & c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The subgroup H is the group of all matrices of the form

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

G/H is a reductive homogeneous space. A symmetry s_0 is determined by the transformation

$$t' = \frac{1}{\alpha}t, \quad a' = \frac{1}{\alpha}a, \quad b' = b, \quad c' = \alpha c, \quad d' = \alpha^2 d.$$

3. The maximal system $(\alpha^2, 1/\alpha, \alpha, \alpha)$. Let S: $V \to V$ be the linear transformation with the eigenvalues $(\alpha^2, 1/\alpha, \alpha, \alpha)$, $\alpha \neq 0, \pm 1$ and let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_3) = T(X_1, X_4) = T(X_2, X_3) = T(X_2, X_4) = 0,$$
$$T(X_1, X_2) = aX_3 + bX_4, \quad T(X_3, X_4) = cX_1,$$

where a, b, c are real numbers and $a^2+b^2+c^2 > 0$.

Let us consider the following cases:

A) $a^2+b^2 > 0$, $c \neq 0$. If $a \neq 0$, then assuming

$$X_1' = -cX_1, \quad X_2' = -\frac{1}{c}X_2, \quad X_3' = aX_3+bX_4, \quad X_4' = -\frac{1}{a}X_4,$$

we obtain

$$T(X_1', X_2') = X_3', \quad T(X_3', X_4') = X_1'.$$

Thus we conclude that in this case for the pair (S, T) on V there exists a basis (denoted also by $\{X_1, X_2, X_3, X_4\}$) such that

$$SX_1 = \alpha^2 X_1, \quad SX_2 = \frac{1}{\alpha}X_2, \quad SX_3 = \alpha X_3, \quad SX_4 = \alpha X_4,$$

(5.7) $$T(X_1, X_3) = T(X_1, X_4) = T(X_2, X_3) = T(X_2, X_4) = 0,$$
$$T(X_1, X_2) = X_3, \quad T(X_3, X_4) = X_1.$$

Let $\mathfrak{k}$ denote the Lie algebra of all real endomorphisms $A\colon V \to V$ acting as derivations satisfying $A(S) = 0$. Then

$$AX_1 = kX_1, \quad AX_2 = lX_2, \quad AX_3 = pX_3 + qX_4, \quad AX_4 = rX_3 + sX_4.$$

By the condition $A(T) = 0$ we have

$$A\big(T(X_1, X_2)\big) = T(AX_1, X_2) + T(X_1, AX_2),$$

whence

$$pX_3 + qX_4 = (k+l)X_3 \Rightarrow q = 0, \quad p = k+l,$$

$$A\big(T(X_3, X_4)\big) = T(AX_3, X_4) + T(X_3, AX_4),$$

and thus

$$kX_1 = (p+s)X_1 \Rightarrow s = k-p.$$

So algebra $\mathfrak{k}$ is three-dimensional. Let $\{A_1, A_2, A_3\}$ be the basis of this algebra, and

$$\begin{aligned} &A_1X_1 = X_1, \quad &&A_1X_2 = 0, \quad &&A_1X_3 = X_3, \quad &&A_1X_4 = 0, \\ &A_2X_1 = 0, &&A_2X_2 = X_2, &&A_2X_3 = X_3, &&A_2X_4 = -X_4, \\ &A_3X_1 = 0, &&A_3X_2 = 0, &&A_3X_3 = 0, &&A_3X_4 = X_3. \end{aligned} \tag{5.8}$$

Let (V, S, T, R) be an infinitesimal s-manifold with the tensors S and T satisfying (5.7). Then $R(X, Y) \in \mathfrak{k}$ and from $R(SX, SY) = R(X, Y)$ we obtain

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_3, X_4) = 0,$$

$$R(X_2, X_3) = \sum_{i=1}^{3} \tau_i A_i, \quad R(X_2, X_4) = \sum_{i=1}^{3} \varrho_i A_i.$$

The first Bianchi identity must hold in V, and particulary for the triples of basis vectors X_1, X_2, X_3; X_1, X_2, X_4 and X_2, X_3, X_4 we have

$$R(X_2, X_3)X_1 = 0, \quad R(X_2, X_4)X_1 = 0,$$

and

$$R(X_2, X_3)X_4 - R(X_2, X_4)X_3 = X_3,$$

respectively. Hence

$$\tau_1 = 0, \quad \varrho_1 = 0, \quad \tau_2 = 0, \quad \tau_3 = 2 + \varrho_2.$$

Thus we obtain

$$R(X_2, X_3) = (2+\varrho_2)A_3,$$
$$R(X_2, X_4) = A_1 + \varrho_2 A_2 + \varrho_3 A_3.$$

By the condition $A(R) = 0$ we have $\varrho_2 = -2$. Hence

$$\begin{aligned} &R(X_2, X_3) = 0, \\ &R(X_2, X_4) = A_1 - 2A_2 + \varrho_3 A_3. \end{aligned} \tag{5.9}$$

On account of (5.8) and (5.9) we obtain

$$R(X_2, X_4)X_1 = X_1, \quad R(X_2, X_4)X_2 = -2X_2, \quad R(X_2, X_4)X_3 = -X_3,$$
$$R(X_2, X_4)X_4 = 2X_4 + \varrho_3 X_3.$$

We define a new basis by taking

$$X_4' = X_4 + \tfrac{1}{3}\varrho_3 X_3.$$

Then

$$R(X_2, X_4')X_4' = R(X_2, X_4)(X_4 + \tfrac{1}{3}\varrho_3 X_3) = 2X_4'.$$

Hence we conclude that in this case there is a basis (denoted also by X_1, X_2, X_3, X_4) (1) such that

$$T(X_1, X_2) = X_3, \quad T(X_3, X_4) = X_1, \quad R(X_2, X_4) = A,$$
$$AX_1 = X_1, \quad AX_2 = -2X_2, \quad AX_3 = -X_3, \quad AX_4 = 2X_4.$$

So the algebra $\mathfrak{h}^\circ$ generated by the curvature endomorphism $R(X, Y)$ is one-dimensional and is generated by endomorphism A. Multiplication in the algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined as follows:

(5.10)

	X_1	X_2	X_3	X_4	A
X_1	0	$-X_3$	0	0	$-X_1$
X_2	X_3	0	0	$-A$	$2X_2$
X_3	0	0	0	$-X_1$	X_3
X_4	0	A	X_1	0	$-2X_4$
A	X_1	$-2X_2$	$-X_3$	$2X_4$	0

This algebra is isomorphic with the one obtained in the case of the maximal system $(\alpha^2, \alpha, 1/\alpha, 1/\alpha^2)$. Let us denote the basis vectors of algebra (5.2) by $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{A}\}$; the isomorphism is given by the assignment

$$\bar{X}_1 \mapsto -X_2, \quad \bar{X}_2 \mapsto X_3, \quad \bar{X}_3 \mapsto X_1, \quad \bar{X}_4 \mapsto X_4, \quad \bar{A} \mapsto -A.$$

B) $a^2 + b^2 > 0$, $c = 0$. Let $a \neq 0$; then there exists a basis $\{X_1, X_2, X_3, X_4\}$ in V such that

$$SX_1 = \alpha^2 X_1, \quad SX_2 = \frac{1}{\alpha} X_2, \quad SX_3 = \alpha X_3, \quad SX_4 = \alpha X_4,$$

(5.11)
$$\begin{aligned} T(X_1, X_3) &= T(X_1, X_4) = T(X_2, X_3) = T(X_2, X_4) \\ &= T(X_3, X_4) = 0, \\ T(X_1, X_2) &= X_3. \end{aligned}$$

(1) From now on we shall not change the notation of the basis.

From the conditions $A(S) = A(T) = 0$ it follows that in the present case the algebra $\mathfrak{k}$ is four-dimensional with a basis $\{A_1, A_2, A_3, A_4\}$ such that

$$\begin{array}{llll} A_1 X_1 = X_1, & A_1 X_2 = 0, & A_1 X_3 = X_3, & A_1 X_4 = 0, \\ A_2 X_1 = 0, & A_2 X_2 = X_2, & A_2 X_3 = X_3, & A_2 X_4 = 0, \\ A_3 X_1 = 0, & A_3 X_2 = 0, & A_3 X_3 = 0, & A_3 X_4 = X_3, \\ A_4 X_1 = 0, & A_4 X_2 = 0, & A_4 X_3 = 0, & A_4 X_4 = X_4. \end{array}$$

Further, from $R(SX, SY) = R(X, Y)$ we obtain

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_3, X_4) = 0,$$

$$R(X_2, X_3) = \sum_{i=1}^{4} \tau_i A_i, \quad R(X_2, X_4) = \sum_{i=1}^{4} \varrho_i A_i.$$

By the first Bianchi identity we have

$$R(X_2, X_3) X_1 = 0 \Rightarrow \tau_1 = 0,$$
$$R(X_2, X_4) X_1 = 0 \Rightarrow \varrho_1 = 0,$$
$$R(X_2, X_3) X_4 - R(X_2, X_4) X_3 = 0 \Rightarrow \tau_4 = 0, \ \tau_3 = \varrho_2.$$

Hence

$$R(X_2, X_3) = \tau_2 A_2 + \varrho_2 A_3, \quad R(X_2, X_4) = \sum_{i=2}^{4} \varrho_i A_i.$$

From the condition $A(R) = 0$ we obtain $\tau_2 = \varrho_2 = \varrho_4 = 0$. Finally we have

$$R(X_2, X_3) = 0, \quad R(X_2, X_4) = \varrho_3 A_3.$$

For $\varrho_3 = 0$ we obtain a decomposable algebra.

For $\varrho_3 \neq 0$ we define a new basis

$$X_i = \sqrt{|\varrho_3|}\, X_i', \quad i = 2, 3, 4.$$

Then

$$T(X_1, X_2') = X_3', \quad R(X_2', X_4') = \operatorname{sgn} \varrho_3 \cdot A_3,$$

$$A_3 X_1 = 0, \quad A_3 X_2' = 0, \quad A_3 X_3' = 0, \quad A_3 X_4' = X_3'.$$

The algebra $\mathfrak{h}^\circ$ is one-dimensional and is generated by endomorphism A_3. Multiplication in the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined as follows:

(5.12)

	X_1	X_2	X_3	X_4	A
X_1	0	$-X_3$	0	0	0
X_2	X_3	0	0	$-kA$	0
X_3	0	0	0	0	0
X_4	0	kA	0	0	$-X_3$
A	0	0	0	X_3	0

for $k = \pm 1$. These algebras are isomorphic. If $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{A}\}$ denote the basis vectors of the algebra for $k = -1$, the isomorphism is given by the assignment $X_1 \mapsto -\bar{X}_1$, $X_2 \mapsto -\bar{X}_2$, $X_3 \mapsto \bar{X}_3$, $X_4 \mapsto \bar{X}_4$, $A \mapsto \bar{A}$.

Now we may give the representation of this algebra by proper infinitesimal transformations of $R^3(x, y, z)$, namely

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X = -\frac{\partial}{\partial y},$$

$$X_4 = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \quad A = z\frac{\partial}{\partial y}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} 1 & b & e & -c \\ 0 & 1 & d & a+\frac{1}{2}d^2 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The subgroup H is the group of all matrices of the form

$$\begin{Vmatrix} 1 & 0 & e & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

G/H is a reductive homogeneous space. The following transformation may be given as an example of symmetry s_0:

$$a' = \frac{1}{\alpha^2}a, \quad b' = \alpha b, \quad c' = \frac{1}{\alpha}c, \quad d' = \frac{1}{\alpha}d, \quad e' = e.$$

C) $a = b = 0$, $c \neq 0$. In this case for the pair (S, T) there is a basis in V such that

$$SX_1 = \alpha^2 X_1, \quad SX_2 = \frac{1}{\alpha}X_2, \quad SX_3 = \alpha X_3, \quad SX_4 = \alpha X_4,$$

$$\begin{aligned} &T(X_1, X_2) = T(X_1, X_3) = T(X_1, X_4) = T(X_2, X_3) \\ &\qquad = T(X_3, X_4) = 0, \\ &T(X_3, X_4) = X_1. \end{aligned} \tag{5.13}$$

From the conditions $A(S) = A(T) = 0$ it follows that algebra $\mathfrak{k}$ is five-dimensional. Denoting its basis by $\{A_1, A_2, A_3, A_4, A_5\}$ we have

$$A_1X_1 = X_1, \quad A_1X_2 = 0, \quad A_1X_3 = 0, \quad A_1X_4 = X_4,$$
$$A_2X_1 = 0, \quad A_2X_2 = X_2, \quad A_2X_3 = 0, \quad A_2X_4 = 0,$$

$$A_3X_1 = 0, \quad A_3X_2 = 0, \quad A_3X_3 = X_3, \quad A_3X_4 = -X_4,$$
$$A_4X_1 = 0, \quad A_4X_2 = 0, \quad A_4X_3 = X_4, \quad A_4X_4 = 0,$$
$$A_5X_1 = 0, \quad A_5X_2 = 0, \quad A_5X_3 = 0, \quad A_5X_4 = X_3.$$

From the condition $R(SX, SY) = R(X, Y)$ we obtain

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_3, X_4) = 0,$$
$$R(X_2, X_3) = \sum_{i=1}^{5} \tau_i A_i, \quad R(X_2, X_4) = \sum_{i=1}^{5} \varrho_i A_i.$$

By the first Bianchi identity we have

$$R(X_2, X_3)X_1 = 0 \Rightarrow \tau_1 = 0,$$
$$R(X_2, X_4)X_1 = 0 \Rightarrow \varrho_1 = 0,$$
$$R(X_2, X_3)X_4 - R(X_2, X_4)X_3 = 0 \Rightarrow \varrho_3 = \tau_5, \ \varrho_4 = -\tau_3.$$

So we obtain

$$R(X_2, X_3) = \sum_{i=2}^{5} \tau_i A_i,$$
$$R(X_2, X_4) = \varrho_2 A_2 + \tau_5 A_3 - \tau_3 A_4 + \varrho_5 A_5.$$

From $R(X, Y)(R) = 0$ we obtain the following relations:

$$\begin{aligned} \varrho_2\tau_2 - \varrho_2\tau_3 + \tau_2\tau_5 &= 0,\\ \tau_2\tau_5 + \tau_3\tau_5 + \varrho_5\tau_4 &= 0,\\ \tau_3^2 + \tau_4\tau_5 + \tau_2\tau_3 &= 0,\\ 3\tau_5^2 + \tau_2\varrho_5 - 3\tau_3\varrho_5 &= 0, \end{aligned} \tag{5.14}$$

$$\begin{aligned} \varrho_2\tau_2 - \varrho_2\tau_3 + \tau_2\tau_5 &= 0,\\ \tau_3\tau_5 - \varrho_2\tau_3 + \varrho_5\tau_4 &= 0,\\ 3\tau_3^2 + \varrho_2\tau_4 + 3\tau_4\tau_5 &= 0,\\ \tau_5^2 - \varrho_2\tau_5 - \varrho_5\tau_3 &= 0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} \tau_2^2 + \tau_2\tau_3 + \varrho_2\tau_4 &= 0,\\ \tau_2\tau_3 + \tau_3^2 + \tau_4\tau_5 &= 0,\\ \tau_2\tau_4 &= 0,\\ \tau_2\tau_5 + \tau_3\tau_5 + \varrho_5\tau_4 &= 0, \end{aligned} \tag{5.16}$$

$$\begin{aligned} \varrho_2^2 - \varrho_2\tau_5 + \varrho_5\tau_2 &= 0,\\ \varrho_2\tau_5 - \tau_5^2 + \varrho_5\tau_3 &= 0,\\ -\varrho_2\tau_3 + \tau_3\tau_5 + \varrho_5\tau_4 &= 0,\\ \varrho_2\varrho_5 &= 0. \end{aligned} \tag{5.17}$$

Let us observe first that $\tau_2 = \varrho_2 = 0$. In fact, if $\tau_2 \neq 0$, then the third equation of (5.16), the third equation of (5.16) and the first equation of (5.16) lead to a contradiction. Similarly, if $\varrho_2 \neq 0$, then the fourth equation of (5.17), the fourth equation of (5.14) and the first equation of (5.17) give a contradiction. Hence

$$R(X_2, X_3) = \sum_{i=3}^{5} \tau_i A_i, \qquad (5.18)$$
$$R(X_2, X_4) = \tau_5 A_3 - \tau_3 A_4 + \varrho_5 A_5.$$

By the condition

$$R(R(X_2, X_3)X_2, X_3) + R(X_2, R(X_2, X_3)X_3) = 0$$

we have

$$\tau_3 R(X_2, X_3) + \tau_4 R(X_2, X_4) = 0. \qquad (5.19)$$

If $\tau_3 = \tau_4 = 0$, then from the fourth equation of (5.14) we obtain $\tau_5 = 0$. Thus

$$R(X_2, X_3) = 0, \quad R(X_2, X_4) = \varrho_5 A_5. \qquad (5.20)$$

If $(\tau_3)^2 + (\tau_4)^2 \neq 0$, then (5.19) may be writen in the form

$$R(X_2, \tau_3 X_3 + \tau_4 X_4') = 0.$$

This means that in subspace $V^{(\alpha)}$ there is a vector $X_3' = \tau_3 X_3 + \tau_4 X_4$ for which

$$R(X_2, X_3') = 0. \qquad (5.21)$$

We choose X_4' as an arbitrary non-zero vector such that X_3' and X_4' form the basis in the subspace $V^{(\alpha)}$. Next we choose endomorphisms $A'_i \in \mathfrak{k}$, $i = 1, 2, 3, 4, 5$, such that conditions (5.13), (5.18) and (5.21) are satisfied. So we have

$$R(X_2, X_3') = \sum_{i=3}^{5} \tau_i' A_i' = 0,$$
$$R(X_2, X_4') = \tau_5' A_3' - \tau_3' A_4' + \varrho_5' A_5'.$$

Hence

$$\tau_3' = \tau_4' = \tau_5' = 0,$$

that is,

$$R(X_2, X_4') = \varrho_5' A_5'. \qquad (5.22)$$

From (5.20) and (5.22) we obtain

$$R(X_2, X_3) = 0, \quad R(X_2, X_4) = kA_5, \quad k = 0, 1.$$

For $k = 0$ we obtain the decomposable algebra $\mathfrak{g} = V$.

For $k = 1$ we obtain the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$, where $\mathfrak{h}^\circ$ is a one-dimensional algebra generated by endomorphism A_5.

Multiplication in the algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined as follows:

(5.23)

	X_1	X_2	X_3	X_4	A
X_1	0	0	0	0	0
X_2	0	0	0	$-A$	0
X_3	0	0	0	$-X_1$	0
X_4	0	A	X_1	0	$-X_3$
A	0	0	0	X_3	0

We shall give the representation of this algebra by proper infinitesimal transformations of $R^4(x, y, u, v)$, namely

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u},$$

$$X_4 = -u\frac{\partial}{\partial x} + v\frac{\partial}{\partial u} - y\frac{\partial}{\partial v}, \quad A = \frac{\partial}{\partial v}.$$

The corresponding Lie group G is the group of all matrices of the form:

$$\begin{Vmatrix} 1 & -t & -\frac{1}{2}t^2 & \frac{1}{6}t^3 & a \\ 0 & 1 & t & -\frac{1}{2}t^2 & b \\ 0 & 0 & 1 & -t & c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The subgroup H is the group of all matrices of the form

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

G/H is a reductive homogeneous space. A symmetry s_0 is determined by the transformation

$$t' = \frac{1}{\alpha}t, \quad a' = \frac{1}{\alpha^2}a, \quad b' = \frac{1}{\alpha}b, \quad c' = c, \quad d' = \alpha d.$$

4. The maximal system $(\alpha, -\alpha, -1, -1)$. Let $S\colon V \to V$ be the linear transformation with eigenvalues $(\alpha, -\alpha, -1, -1)$, $\alpha \neq 0, \pm 1$ and let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of S. We have

$$(5.24)\qquad \begin{aligned} &SX_1 = \alpha X_1, \quad SX_2 = -\alpha X_2, \quad SX_3 = -X_3, \quad SX_4 = -X_4,\\ &T(X_1, X_2) = 0, \quad T(X_1, X_3) = aX_2, \quad T(X_1, X_4) = bX_2,\\ &T(X_2, X_3) = cX_1, \quad T(X_2, X_4) = dX_1, \quad T(X_3, X_4) = 0, \end{aligned}$$

where a, b, c, d are real numbers, $a^2+b^2+c^2+d^2 > 0$.

For the algebra $\mathfrak{k}$ we have

$$\begin{aligned} AX_1 &= kX_1,\\ AX_2 &= lX_2,\\ AX_3 &= pX_3+qX_4,\\ AX_4 &= rX_3+sX_4. \end{aligned}$$

Let us denote by W the eigenspace for the eigenvalue -1. Conditions (5.24) may be written in the form

$$\begin{gathered} SX_1 = \alpha X_1, \quad SX_2 = -\alpha X_2, \quad SZ = -Z, \quad Z \in W,\\ T(X_1, X_2) = T(X_3, X_4) = 0,\\ T(X_1, Z) = f(Z)\cdot X_2, \quad T(X_2, Z) = g(Z)\cdot X_1, \end{gathered}$$

where $f(Z)$ and $g(Z)$ are linear forms defined in W, not all equal to zero.

Let us consider the following two cases:

A) $\operatorname{Ker} f \oplus \operatorname{Ker} g = W$,

B) $\dim(\operatorname{Ker} f + \operatorname{Ker} g) = 1$.

Ad A. We choose $X_3 \in \operatorname{Ker} g$, $X_4 \in \operatorname{Ker} f$, $X_4 \neq \lambda X_3$. Then the tensor T must satisfy the following conditions:

$$(5.25)\qquad \begin{gathered} T(X_1, X_2) = T(X_1, X_4) = T(X_2, X_3) = T(X_3, X_4) = 0,\\ T(X_1, X_3) = aX_2, \quad T(X_2, X_4) = dX_1, \end{gathered}$$

$a^2+d^2 > 0$.

A.I. $ad \neq 0$; then by a suitable change of the basis we obtain $a = d = 1$. From $A(T) = 0$ we obtain

$$\begin{aligned} A(T(X_1, X_3)) &= T(AX_1, X_3)+T(X_1, AX_3) \Rightarrow l = k+p,\\ A(T(X_1, X_4)) &= T(AX_1, X_4)+T(X_1, AX_4) \Rightarrow r = 0,\\ A(T(X_2, X_3)) &= T(AX_2, X_3)+T(X_2, AX_3) \Rightarrow q = 0,\\ A(T(X_2, X_4)) &= T(AX_2, X_4)+T(X_2, AX_4) \Rightarrow k = l+s. \end{aligned}$$

So we have

$$AX_1 = kX_1, \quad AX_2 = lX_2, \quad AX_3 = (l-k)X_3, \quad AX_4 = (k-l)X_4.$$

The algebra $\mathfrak{k}$ is two-dimensional. Let $\{A_1, A_2\}$ be the basis of this algebra, and

$$A_1X_1 = X_1, \quad A_1X_2 = 0, \quad A_1X_3 = -X_3, \quad A_1X_4 = X_4,$$
$$A_2X_1 = 0, \quad A_2X_2 = X_2, \quad A_2X_3 = X_3, \quad A_2X_4 = -X_4.$$

Let (V, S, T, R) be an infinitesimal s-manifold with the tensors S and T satisfying (5.25), $(a = d = 1)$. Then $R(X, Y) \in \mathfrak{k}$ and from $R(SX, SY) = R(X, Y)$ we obtain

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_3) = R(X_2, X_4) = 0,$$
$$R(X_3, X_4) = \tau A_1 + \varrho A_2.$$

By the first Bianchi identity we have

$$R(X_3, X_4)X_1 = X_1 \Rightarrow \tau = 1$$

$$R(X_3, X_4)X_2 = -X_2 \Rightarrow \varrho = -1.$$

Hence we obtain

$$R(X_3, X_4) = A_1 - A_2 = A.$$

The algebra $\mathfrak{h}^\circ$ is one-dimensional and is generated by endomorphism A, where

$$AX_1 = X_1, \quad AX_2 = -X_2, \quad AX_3 = -2X_3, \quad AX_4 = 2X_4.$$

Multiplication in the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined as follows:

(5.26)

	X_1	X_2	X_3	X_4	A
X_1	0	0	$-X_2$	0	$-X_1$
X_2	0	0	0	$-X_1$	X_2
X_3	X_2	0	0	$-A$	$2X_3$
X_4	0	X_1	A	0	$-2X_4$
A	X_1	$-X_2$	$-2X_3$	$2X_4$	0

This algebra is isomorphic with the one obtained in the case of the maximal system $(\alpha^2, \alpha, 1/\alpha, 1/\alpha^2)$.

Let as denote the basis vectors of algebra (5.2) by $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{A}\}$; the isomorphism is given by the assignment:

$$\bar{X}_1 \mapsto -X_3, \quad \bar{X}_2 \mapsto X_2, \quad \bar{X}_3 \mapsto X_1, \quad \bar{X}_4 \mapsto X_4, \quad \bar{A} \mapsto -A.$$

A.II. $a \cdot d = 0$, $a^2+d^2 > 0$. Let $a = 0$, $d \neq 0$; in this case we have $T(X_1, X_2) = T(X_1, X_3) = T(X_1, X_4) = T(X_2, X_3) = T(X_3, X_4) = 0$, $T(X_2, X_4) = X_1$.

The algebra $\mathfrak{k}$ is four-dimensional with a basis $\{A_1, A_2, A_3, A_4\}$ where

$$\begin{array}{llll} A_1 X_1 = X_1, & A_1 X_2 = 0, & A_1 X_3 = 0, & A_1 X_4 = X_4, \\ A_2 X_1 = 0, & A_2 X_2 = X_2, & A_2 X_3 = 0, & A_2 X_4 = -X_4, \\ A_3 X_1 = 0, & A_3 X_2 = 0, & A_3 X_3 = X_3, & A_3 X_4 = 0, \\ A_4 X_1 = 0, & A_4 X_2 = 0, & A_4 X_3 = 0, & A_4 X_4 = X_3. \end{array}$$

Further, from $R(X, Y) \in \mathfrak{k}$, $R(SX, SY) = R(X, Y)$ we obtain

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_3) = R(X_2, X_4) = 0,$$
$$R(X_3, X_4) = \sum_{i=1}^{4} \tau_i A_i.$$

By the first Bianchi identity we have

$$R(X_3, X_4) X_1 = 0 \Rightarrow \tau_1 = 0,$$
$$R(X_3, X_4) X_2 = 0 \Rightarrow \tau_2 = 0.$$

By the condition $R(X_3, X_4)(R) = 0$ we obtain $\tau_3 = 0$. Hence

$$R(X_3, X_4) = \tau_4 A_4.$$

So algebra $\mathfrak{h}^\circ$ is one-dimensional and is generated by endomorphism A_4

For $\tau_4 = 0$ we obtain a decomposable algebra $\mathfrak{g} = V$.

When $\tau_4 \neq 0$, we take a new basis $X_i = \sqrt{|\tau_4|}\, X_i'$, $i = 1, 3, 4$; then

$$T(X_2, X_4') = X_1', \quad R(X_3', X_4') = \operatorname{sgn} \tau_4 \cdot A_4,$$
$$A_4 X_1' = 0, \quad A_4 X_2 = 0, \quad A_4 X_3' = 0, \quad A_4 X_4' = X_3'.$$

The Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is given by

(5.27)

	X_1	X_2	X_3	X_4	A
X_1	0	0	0	0	0
X_2	0	0	0	$-X_1$	0
X_3	0	0	0	$-kA$	0
X_4	0	X_1	kA	0	$-X_3$
A	0	0	0	X_3	0

for $k = \pm 1$. Now we may give the representations of these algebras by proper infinitesimal transformations of $R^4(x, y, u, v)$.

For $k = 1$ we have

$$X_1 = -\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u},$$

$$X_4 = y\frac{\partial}{\partial x} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \quad A = \frac{\partial}{\partial v}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} 1 & t & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & \cos t & \sin t & c \\ 0 & 0 & -\sin t & \cos t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The subgroup H is the group of all matrices of the form

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}, \tag{5.28}$$

G/H is a reductive homogeneous space. The following transformation may be given as an example of symmetry s_0:

$$t' = -t, \quad a' = \frac{1}{\alpha}a, \quad b' = -\frac{1}{\alpha}b, \quad c' = -c, \quad d' = d. \tag{5.29}$$

For $k = -1$ we have:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u},$$

$$X_4 = y\frac{\partial}{\partial x} + v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \quad A = \frac{\partial}{\partial v}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} 1 & t & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & \cos ht & \sin ht & c \\ 0 & 0 & \sin ht & \cos ht & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The subgroup H is the group of all matrices of the form (5.28). Here transformation (5.29) is an example of symmetry s_0.

A.III. $a \neq 0$, $d = 0$ resolves itsels into A.II.

Ad B. $\dim(\operatorname{Ker} f + \operatorname{Ker} g) = 1$. We choose $X_3 \in \operatorname{Ker} f$, $X_4 \neq \lambda X_3$. Then we have for T

$$T(X_1, X_2) = T(X_1, X_3) = T(X_2, X_3) = T(X_3, X_4) = 0,$$
$$T(X_1, X_4) = bX_2, \quad T(X_2, X_4) = dX_1, \tag{5.30}$$
$$b^2 + d^2 > 0.$$

B.I. If $bd \neq 0$, then assuming $X_1 = bX'_1$, $X_2 = \sqrt{|bd|}\, X'_2$, $X_3 = X'_3$, $X_4 = \sqrt{|bd|}\, X'_4$ we obtain $T(X'_1, X'_4) = X'_2$, $T(X'_2, X'_4) = \pm X'_1$. From the conditions $A(S) = A(T) = 0$ it follows that in the present case the algebra $\mathfrak{k}$ is three-dimensional with a basis $\{A_1, A_2, A_3\}$ such that

$$\begin{array}{llll} A_1 X_1 = X_1, & A_1 X_2 = X_2, & A_1 X_3 = 0, & A_1 X_4 = 0, \\ A_2 X_1 = 0, & A_2 X_2 = 0, & A_2 X_3 = X_3, & A_2 X_4 = 0, \\ A_3 X_1 = 0, & A_3 X_2 = 0, & A_3 X_3 = 0, & A_3 X_4 = X_3. \end{array}$$

Hence

$$R(X_3, X_4) = \tau A_1 + \varrho A_2 + \varkappa A_3.$$

By the first Bianchi identity we have

$$R(X_3, X_4) X_1 = 0 \Rightarrow \tau = 0.$$

By the condition $R(X_3, X_4)(R) = 0$ we obtain $\varrho = 0$. Hence

$$R(X_3, X_4) = \varkappa A_3.$$

The algebra $\mathfrak{h}^\circ$ is one-dimensional and is generated by endomorphism A_3.

For $\varkappa = 0$ we obtain a decomposable algebra $\mathfrak{g} = V$.

When $\varkappa \neq 0$, the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is given by

(5.31)

	X_1	X_2	X_3	X_4	A
X_1	0	0	0	$-X_2$	0
X_2	0	0	0	kX_1	0
X_3	0	0	0	$-\varkappa A$	0
X_4	X_2	$-kX_1$	$\varkappa A$	0	$-X_3$
A	0	0	0	X_3	0

for $k = \pm 1$. Now we may give the representations of these algebras by proper infinitesimal transformations of $R^4(x, y, u, v)$.

For $k = 1$ we have

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u},$$

$$X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - \varkappa u\frac{\partial}{\partial v} + v\frac{\partial}{\partial u}, \quad A = \frac{\partial}{\partial v}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\left\|\begin{matrix} \cos t & \sin t & 0 & 0 & a \\ -\sin t & \cos t & 0 & 0 & b \\ 0 & 0 & \cos\sqrt{\varkappa}\,t & \frac{1}{\sqrt{\varkappa}}\sin\sqrt{\varkappa}\,t & c \\ 0 & 0 & -\sqrt{\varkappa}\sin\sqrt{\varkappa}\,t & \cos\sqrt{\varkappa}\,t & d \\ 0 & 0 & 0 & 0 & 1 \end{matrix}\right\|,$$

for $\varkappa > 0$, and the group of all matrices of the form:

$$\left\|\begin{matrix} \cos t & \sin t & 0 & 0 & a \\ -\sin t & \cos t & 0 & 0 & b \\ 0 & 0 & \cos h\sqrt{-\varkappa}\,t & \frac{1}{\sqrt{-\varkappa}}\sin h\sqrt{-\varkappa}\,t & c \\ 0 & 0 & \sqrt{-\varkappa}\sin h\sqrt{-\varkappa}\,t & \cos h\sqrt{-\varkappa}\,t & d \\ 0 & 0 & 0 & 0 & 1 \end{matrix}\right\|,$$

for $\varkappa < 0$.

The subgroup H is the group of all matrices of the form:

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right\|.$$

G/H is a reductive homogeneous space. The following transformation may by given as an example of symmetry s_0:

$$t' = -t, \quad a' = \frac{1}{\alpha}a, \quad b' = -\frac{1}{\alpha}b, \quad c' = -c, \quad d' = d.$$

For $k = -1$ the algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is isomorphic with the one obtained in the case of the maximal system $(-i, -i, -1, -1)$. The isomorphism is given by the assignment

$$X \mapsto X_1+X_2, \quad Y \mapsto -X_1+X_2, \quad V_1 \mapsto X_4, \quad V_2 \mapsto X_3, \quad A \mapsto A.$$

B.II. $b \cdot d = 0$, $b^2+d^2 > 0$. Let $b \neq 0$, $d = 0$; in this case we have

$$T(X_1, X_2) = T(X_1, X_3) = T(X_2, X_3) = T(X_2, X_4) = T(X_3, X_4) = 0,$$
$$T(X_1, X_4) = X_2.$$

The algebra $\mathfrak{k}$ is four-dimensional with a basis $\{A_1, A_2, A_3, A_4\}$ where

$$\begin{array}{llll} A_1X_1 = X_1, & A_1X_2 = 0, & A_1X_3 = 0, & A_1X_4 = -X_4, \\ A_2X_1 = 0, & A_2X_2 = X_2, & A_2X_3 = 0, & A_2X_4 = X_4, \\ A_3X_1 = 0, & A_3X_2 = 0, & A_3X_3 = X_3, & A_3X_4 = 0, \\ A_4X_1 = 0, & A_4X_2 = 0, & A_4X_3 = 0, & A_4X_4 = X_3. \end{array}$$

Hence

$$R(X_3, X_4) = \sum_{i=1}^{4} \tau_i A_i.$$

By the first Bianchi identity we have

$$R(X_3, X_4)X_1 = 0 \Rightarrow \tau_1 = 0,$$
$$R(X_3, X_4)X_2 = 0 \Rightarrow \tau_2 = 0.$$

By the condition $R(X_3, X_4)(R) = 0$ we obtain $\tau_3 = 0$. Hence

$$R(X_3, X_4) = \tau_4 A_4.$$

For $\tau_4 = 0$ we obtain a decomposable algebra $\mathfrak{g}$.

When $\tau_4 \neq 0$, we take a new basis $X_i = \sqrt{|\tau_4|}\, X_i'$, $i = 2, 3, 4$. Then

$$T(X_1, X_4') = X_2', \quad R(X_3', X_4') = \pm A_4,$$
$$A_4X_1 = 0, \quad A_4X_2' = 0, \quad A_4X_3' = 0, \quad A_4X_4' = X_3'.$$

Multiplication in the algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is defined as follows:

(5.32)

	X_1	X_2	X_3	X_4	A
X_1	0	0	0	$-X_2$	0
X_2	0	0	0	0	0
X_3	0	0	0	$-kA$	0
X_4	X_2	0	kA	0	$-X_3$
A	0	0	0	X_3	0

for $k = \pm 1$. This algebra is isomorphic with the algebra (5.27). Let us denote the basis vectors of algebra (5.27) by $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{A}\}$; the isomorphism is given by the assignment

$$X_1 \mapsto \bar{X}_2, \quad X_2 \mapsto \bar{X}_1, \quad X_3 \mapsto \bar{X}_3, \quad X_4 \mapsto \bar{X}_4, \quad A \mapsto \bar{A}.$$

5. The maximal system $(\alpha^2, 1/\alpha, \alpha, \alpha^3)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = aX_3, \quad T(X_1, X_3) = bX_4, \quad T(X_1, X_4) = 0,$$
$$T(X_2, X_3) = 0, \quad T(X_2, X_4) = cX_1, \quad T(X_3, X_4) = 0,$$

$a^2+b^2+c^2 > 0$.

A) Let $a \cdot b \cdot c \neq 0$; then, assuming

$$X_1 = abcX'_1, \quad X_2 = X'_2, \quad X_3 = bcX'_3, \quad X_4 = abc^2 X'_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_1, X'_3) = X'_4, \quad T(X'_2, X'_4) = X'_1.$$

Let $\mathfrak{k}$ denote the Lie algebra of all real endomorphisms $A\colon V \to V$ which, as derivations, satisfy $A(S) = A(T) = 0$. By the condition $A(S) = 0$ we have

$$AX_1 = pX_1, \quad AX_2 = qX_2, \quad AX_3 = rX_3, \quad AX_4 = sX_4.$$

On account of $A(T) = 0$ we obtain

$$p = 2r, \quad q = -r, \quad s = 3r.$$

So algebra $\mathfrak{k}$ is one-dimensional and is generated by endomorphism B

$$BX_1 = 2X_1, \quad BX_2 = -X_2, \quad BX_3 = X_3, \quad BX_4 = 3X_4.$$

Let (V, S, T, R) be an infinitesimal s-manifold. By the condition $R(SX, SY) = R(X, Y)$ we have

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_4) = R(X_3, X_4) = 0,$$
$$R(X_2, X_3) = \tau B.$$

By the first Bianchi identity we have

$$\mathfrak{S}(R(X_1, X_2)X_3) = \mathfrak{S}\big(T(T(X_1, X_2), X_3)\big)$$
$$R(X_2, X_3)X_1 = T(X_2, X_4),$$
$$2\tau X_1 = X_1 \Rightarrow \tau = \tfrac{1}{2}$$

and

$$\mathfrak{S}(R(X_2, X_3)X_2) = \mathfrak{S}\big(T(T(X_2, X_3), X_2)\big)$$
$$R(X_2, X_3)X_2 = 0,$$
$$-\tau X_2 = 0 \Rightarrow \tau = 0;$$

so the case $abc \neq 0$ leads to a contradiction.

B) Let $ab \neq 0, c = 0$; then, assuming

$$X_1 = ab\bar{X}_1, \quad X_2 = \bar{X}_2, \quad X_3 = b\bar{X}_3, \quad X_4 = ab\bar{X}_4,$$

we obtain

$$T(\bar{X}_1, \bar{X}_2) = \bar{X}_3, \quad T(\bar{X}_1, \bar{X}_3) = \bar{X}_4.$$

In the present case the algebra $\mathfrak{k}$ is two-dimensional with a basis $\{A_1, A_2\}$ such that

$$\begin{aligned} &A_1\bar{X}_1 = \bar{X}_1, \quad &&A_1\bar{X}_2 = -\bar{X}_2, \quad &&A_1\bar{X}_3 = 0, \quad &&A_1\bar{X}_4 = \bar{X}_4, \\ &A_2\bar{X}_1 = 0, \quad &&A_2\bar{X}_2 = \bar{X}_2, \quad &&A_2\bar{X}_3 = \bar{X}_3, \quad &&A_2\bar{X}_4 = \bar{X}_4. \end{aligned}$$

Hence

$$R(\bar{X}_2, \bar{X}_3) = \tau A_1 + \varrho A_2.$$

By the first Bianchi identity we have

$$\begin{aligned} R(\bar{X}_2, \bar{X}_3)\bar{X}_1 = 0 &\Rightarrow \tau = 0, \\ R(\bar{X}_2, \bar{X}_3)\bar{X}_4 = 0 &\Rightarrow \varrho = 0. \end{aligned}$$

In this case the infinitesimal s-manifold is of the form $(V, S, T, 0)$. Here we may assume $\mathfrak{h}^\circ = (0)$. The "Nomizu algebra" $\mathfrak{g}$ is given by

$$[X, Y] = -T(X, Y) \quad \text{for all } X, Y \in V.$$

Thus multiplication in the algebra $\mathfrak{g}$ is defined as follows:

(5.33)

	$\bar{X}_1$	$\bar{X}_2$	$\bar{X}_3$	$\bar{X}_4$
$\bar{X}_1$	0	$-\bar{X}_3$	$-\bar{X}_4$	0
$\bar{X}_2$	$\bar{X}_3$	0	0	0
$\bar{X}_3$	$\bar{X}_4$	0	0	0
$\bar{X}_4$	0	0	0	0

Now we may give the representation of this algebra by infinitesimal affine transformations of $R^3(x, y, z)$, namely

$$\bar{X}_1 = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \quad \bar{X}_2 = \frac{\partial}{\partial x}, \quad \bar{X}_3 = \frac{\partial}{\partial y}, \quad \bar{X}_4 = \frac{\partial}{\partial z}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} 1 & t & \frac{1}{2}t^2 & a \\ 0 & 1 & t & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The underlying manifold of the group G is the Cartesian space $R^4(t, a, b, c)$. Thus G is the required simply connected Lie group with the Lie algebra $\mathfrak{g}$. The elements $\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4 \in \mathfrak{g}$ can be represented by the left-invariant vector fields

$$\bar{X}_1 = -\frac{\partial}{\partial t}, \quad \bar{X}_2 = \frac{\partial}{\partial a} + t\frac{\partial}{\partial b} + \frac{1}{2}t^2\frac{\partial}{\partial c},$$

$$\bar{X}_3 = \frac{\partial}{\partial b} + t\frac{\partial}{\partial c}, \quad \bar{X}_4 = \frac{\partial}{\partial c}.$$

A possible symmetry s_0 is the transformation

$$t' = \frac{1}{\alpha^2}t, \quad a' = \frac{1}{\alpha^3}a, \quad b' = \frac{1}{\alpha}b, \quad c' = \alpha c,$$

and it can easily be verified that this is the automorphism of G.

C) Let $a \cdot c \neq 0$, $b = 0$; then, assuming

$$X_1 = acX'_1, \quad X_2 = X'_2, \quad X_3 = cX'_3, \quad X_4 = ac^2 X'_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_2, X'_4) = X'_1.$$

In the present case the algebra $\mathfrak{k}$ is two-dimensional with a basis $\{A_1, A_2\}$ such that

$$A_1X_1 = X_1, \quad A_1X_2 = 0, \quad A_1X_3 = X_3, \quad A_1X_4 = X_4,$$
$$A_2X_1 = 0, \quad A_2X_2 = X_2, \quad A_2X_3 = X_3, \quad A_2X_4 = -X_4.$$

Hence

$$R(X_2, X_3) = \tau A_1 + \varrho A_2.$$

By the first Bianchi identity we have

$$R(X_2, X_3)X_1 = 0 \Rightarrow \tau = 0,$$
$$R(X_2, X_3)X_4 = 0 \Rightarrow \varrho = 0.$$

In this case the infinitesimal s-manifold is of the form $(V, S, T, 0)$. Here we may assume $\mathfrak{h}^\circ = (0)$. Multiplication in the Lie algebra $\mathfrak{g}$ is defined as follows:

(5.34)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	0	0
X_2	X_3	0	0	$-X_1$
X_3	0	0	0	0
X_4	0	X_1	0	0

This algebra is isomorphic with the algebra (5.33). The isomorphism is given by the assignment

$$X_1 \mapsto \bar{X}_3, \quad X_2 \mapsto \bar{X}_1, \quad X_3 \mapsto -\bar{X}_4, \quad X_4 \mapsto \bar{X}_2.$$

D) Let $bc \neq 0$, $a = 0$; then, assuming

$$X_1 = bcX'_1, \quad X_2 = X'_2, \quad X_3 = bcX'_3, \quad X_4 = bc^2 X'_4,$$

we obtain

$$T(X'_1, X'_3) = X'_4, \quad T(X'_2, X'_4) = X'_1.$$

In the present case the algebra $\mathfrak{k}$ is two-dimensional with a basis $\{A_1, A_2\}$ such that

$$A_1 X_1 = X_1, \quad A_1 X_2 = 0, \quad A_1 X_3 = 0, \quad A_1 X_4 = X_4,$$
$$A_2 X_1 = 0, \quad A_2 X_2 = X_2, \quad A_2 X_3 = -X_3, \quad A_2 X_4 = -X_4,$$

Hence

$$R(X_2, X_3) = \tau A_1 + \varrho A_2.$$

By the first Bianchi identity we have

$$R(X_2, X_3) X_1 = X_1 \Rightarrow \tau = 1,$$
$$R(X_2, X_3) X_4 = -X_4 \Rightarrow \varrho = 2.$$

Thus we obtain

$$R(X_2, X_3) = A_1 + 2A_2 = A.$$

$$AX_1 = X_1, \quad AX_2 = 2X_2, \quad AX_3 = -2X_3, \quad AX_4 = -X_4.$$

Multiplication in the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined as follows:

(5.35)

	X_1	X_2	X_3	X_4	A
X_1	0	0	$-X_4$	0	$-X_1$
X_2	0	0	$-A$	$-X_1$	$-2X_2$
X_3	X_4	A	0	0	$2X_3$
X_4	0	X_1	0	0	X_4
A	X_1	$2X_2$	$-2X_3$	$-X_4$	0

This algebra is isomorphic with the algebra (5.2). Let us denote basis vectors of algebra (5.2) by $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{A}\}$; the isomorphism is given by the assignment

$$X_1 \mapsto \bar{X}_2, \quad X_2 \mapsto \bar{X}_1, \quad X_3 \mapsto \bar{X}_4, \quad X_4 \mapsto \bar{X}_3, \quad A \mapsto \bar{A}.$$

In the cases

E) $a \neq 0, \quad b = c = 0,$

F) $b \neq 0, \quad a = c = 0,$

G) $c \neq 0, \quad a = b = 0,$

we always have $R = 0$. The algebras $\mathfrak{g} = V \dotplus (0)$ are decomposable.

6. The maximal system $(-1, \alpha, -\alpha, 1/\alpha)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = aX_3, \quad T(X_1, X_3) = bX_2, \quad T(X_1, X_4) = 0,$$
$$T(X_2, X_3) = 0, \quad T(X_2, X_4) = 0, \quad T(X_3, X_4) = cX_1,$$

$a^2 + b^2 + c^2 > 0$.

A) Let $abc \neq 0$; then, assuming

$$X_1 = \sqrt{|ab|}\, X'_1, \quad X_2 = aX'_2, \quad X_3 = \sqrt{|ab|}\, X'_3, \quad X_4 = cX'_4,$$

we obtain

(5.36) $\quad T(X'_1, X'_2) = X'_3, \quad T(X'_1, X'_3) = \operatorname{sgn}(ab)\, X'_2, \quad T(X'_3, X'_4) = X'_1.$

Let (V, S, T, R) be an infinitesimal s-manifold. By the condition $R(SX, SY) = R(X, Y)$ we have

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_3) = R(X_3, X_4) = 0.$$

By the second Bianchi identity we have

$$\mathfrak{S}\big(R(T(X_1, X_3), X_4)\big) = 0 \Rightarrow R(X_2, X_4) = 0.$$

Further, the first Bianchi identity must hold in V and particularly we have

$$\mathfrak{S}(R(X_1, X_2)X_4) = \mathfrak{S}\big(T(T(X_1, X_2), X_4)\big);$$

hence $0 = T(X_3, X_4)$, and this contradicts (5.36).

B) Let $a \cdot b \neq 0$, $c = 0$; then, assuming

$$X_1 = \sqrt{|ab|}\, X'_1, \quad X_2 = aX'_2, \quad X_3 = \sqrt{|ab|}\, X'_3, \quad X_4 = X'_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_1, X'_3) = \operatorname{sgn}(ab)\, X'_2.$$

By the second Bianchi identity we have

$$\mathfrak{S}\big(R(T(X_1, X_3), X_4)\big) = 0 \Rightarrow R(X_2, X_4) = 0.$$

Here we may assume $\mathfrak{h}^\circ = (0)$. Multiplication in the algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is defined as follows:

(5.37)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	$-kX_2$	0
X_2	X_3	0	0	0
X_3	kX_2	0	0	0
X_4	0	0	0	0

for $k = \pm 1$. We have obtained a decomposable algebra.

C) Let $a \cdot c \neq 0$, $b = 0$; then, assuming

$$X_1 = X'_1, \quad X_2 = aX'_2, \quad X_3 = X'_3, \quad X_4 = cX'_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_3, X'_4) = X'_1.$$

Let $\mathfrak{k}$ denote the Lie algebra of all real endomorphisms $A\colon V \to V$ which, as derivations, satisfy $A(S) = A(T) = 0$. By the condition $A(S) = 0$ we have

$$AX_1 = pX_1, \quad AX_2 = qX_2, \quad AX_3 = rX_3, \quad AX_4 = sX_4.$$

On account of $A(T) = 0$ we obtain

$$r = p+q, \quad s = -q.$$

So algebra $\mathfrak{k}$ is two-dimensional with a basis $\{A_1, A_2\}$ such that

$$\begin{aligned} &A_1X_1 = X_1, \quad A_1X_2 = 0, \quad A_1X_3 = X_3, \quad A_1X_4 = 0,\\ &A_2X_1 = 0, \quad A_2X_2 = X_2, \quad A_2X_3 = X_3, \quad A_2X_4 = -X_4. \end{aligned}$$

By the condition $R(SX, SY) = R(X, Y)$ we have

$$\begin{aligned} &R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_3) = R(X_3, X_4) = 0,\\ &R(X_2, X_4) = \tau A_1 + \varrho A_2. \end{aligned}$$

By the first Bianchi identity we have

$$\begin{aligned} R(X_2, X_4)X_1 &= X_1 \Rightarrow \tau = 1,\\ R(X_2, X_4)X_3 &= -X_3 \Rightarrow \varrho = -2. \end{aligned}$$

Thus we obtain

$$R(X_2, X_4) = A_1 - 2A_2.$$

The algebra $\mathfrak{h}^\circ$ is one-dimensional and is generated by endomorphism $A = A_1 - 2A_2$. Multiplication in the "Nomizu algebra" $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined as follows:

(5.38)

	X_1	X_2	X_3	X_4	A
X_1	0	$-X_3$	0	0	$-X_1$
X_2	X_3	0	0	$-A$	$2X_2$
X_3	0	0	0	$-X_1$	X_3
X_4	0	A	X_1	0	$-2X_4$
A	X_1	$-2X_2$	$-X_3$	$2X_4$	0

This algebra is isomorphic with the algebra (5.2). Let us denote the basis vectors of algebra (5.2) by $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{A}\}$; the isomorphism is given by the assignment

$$X_1 \mapsto \bar{X}_2, \quad X_2 \mapsto \bar{X}_4, \quad X_3 \mapsto \bar{X}_3, \quad X_4 \mapsto -\bar{X}_1, \quad A \mapsto \bar{A}.$$

D) Let $b \cdot c \neq 0$, $a = 0$; then, assuming

$$X_1 = X_1', \quad X_2 = \frac{1}{b} X_2', \quad X_3 = X_3', \quad X_4 = cX_4',$$

we obtain

$$T(X_1', X_3') = X_2', \quad T(X_3', X_4') = X_1'.$$

In the present case the algebra $\mathfrak{k}$ is two-dimensional with a basis $\{A_1, A_2\}$ such that

$$A_1X_1 = X_1, \quad A_1X_2 = 0, \quad A_1X_3 = -X_3, \quad A_1X_4 = 2X_4,$$
$$A_2X_1 = 0, \quad A_2X_2 = X_2, \quad A_2X_3 = X_3, \quad A_2X_4 = -X_4.$$

Hence

$$R(X_2, X_4) = \tau A_1 + \varrho A_2.$$

By the first Bianchi identity we have

$$R(X_2, X_4)X_1 = 0 \Rightarrow \tau = 0,$$
$$R(X_2, X_4)X_3 = 0 \Rightarrow \varrho = 0.$$

Hence

$$R(X_2, X_4) = 0.$$

Here we may assume $\mathfrak{h}^\circ = (0)$. Multiplication in the algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is defined as follows:

(5.39)

	X_1	X_2	X_3	X_4
X_1	0	0	$-X_2$	0
X_2	0	0	0	0
X_3	X_2	0	0	$-X_1$
X_4	0	0	X_1	0

This algebra is isomorphic with the algebra (5.33). The isomorphism is given by the assignment:

$$X_1 \mapsto -\bar{X}_3, \quad X_2 \mapsto \bar{X}_4, \quad X_3 \mapsto \bar{X}_1, \quad X_4 \mapsto -\bar{X}_2.$$

In the cases

E) $a = 0, \quad b = c = 0,$

F) $b \neq 0, \quad a = c = 0,$

G) $c \neq 0, \quad a = b = 0,$

we always have $R = 0$. The algebras $\mathfrak{g} = V+(0)$ are decomposable.

7. The maximal system $(1/\alpha^2, \alpha^3, \alpha, 1/\alpha)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = aX_3, \quad T(X_1, X_3) = bX_4, \quad T(X_1, X_4) = 0,$$
$$T(X_2, X_3) = 0, \quad T(X_2, X_4) = 0, \quad T(X_3, X_4) = 0;$$

$a^2+b^2 > 0$.

A) Let $a \cdot b \neq 0$; then, assuming

$$X_1 = abX'_1, \quad X_2 = X'_2, \quad X_3 = bX'_3, \quad X_4 = abX'_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_1, X'_3) = X'_4.$$

By the condition $R(SX, SY) = R(X, Y)$ we have

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_3) = R(X_2, X_4) = 0.$$

By the second Bianchi identity we have

$$\mathfrak{S}\big(R(T(X_1, X_2), X_4)\big) = 0 \Rightarrow R(X_3, X_4) = 0.$$

Here we may assume $\mathfrak{h}^\circ = (0)$. Multiplication in the algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is defined as follows:

(5.40)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	$-X_4$	0
X_2	X_3	0	0	0
X_3	X_4	0	0	0
X_4	0	0	0	0

This algebra is isomorphic with algebra (5.33).

In the cases

B) $a \neq 0,\ b = 0$,

C) $a = 0,\ b \neq 0$

we have $R = 0$. The algebras $\mathfrak{g} = V+(0)$ are decomposable.

8. The maximal system $(\alpha, -1/\alpha, -1, -1)$. By the condition $S(T) = T$ we have

$$T(X_1, X_3) = T(X_1, X_4) = T(X_2, X_3) = T(X_2, X_4) = T(X_3, X_4) = 0,$$
$$T(X_1, X_2) = aX_3+bX_4, \quad a^2+b^2 > 0.$$

By a proper change of $\{X_1, X_2, X_3, X_4\}$ we obtain $T(X_1, X_2) = X_3$, or $T(X_1, X_2) = X_4$.

By the condition $R(SX, SY) = R(X, Y)$ we have

$$R(X_1, X_2) = R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_3) = R(X_2, X_4) = 0.$$

By the second Bianchi identity we have $R(X_3, X_4) = 0$. Hence $R = 0$ and $\mathfrak{h}^\circ = (0)$. The algebra $\mathfrak{g} = V+(0)$ is decomposable.

9. The maximal system $(\alpha, 1/\alpha, 1/\alpha^2, 1/\alpha^2)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformations S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = T(X_2, X_3) = T(X_2, X_4) = T(X_3, X_4) = 0,$$
$$T(X_1, X_3) = aX_2, \quad T(X_1, X_4) = bX_2, \quad a^2+b^2 > 0.$$

Let $a \neq 0$; then, assuming

$$X_1' = X_1, \quad X_2' = X_2, \quad X_3' = \frac{1}{a} X_3, \quad X_4' = bX_3-aX_4,$$

we obtain

$$T(X_1', X_3') = X_2', \quad T(X_1', X_4') = 0.$$

For the algebra $\mathfrak{k}$ we have

$$AX_1 = kX_1, \quad AX_3 = pX_3+qX_4,$$

$$AX_2 = lX_2, \quad AX_4 = rX_3+sX_4.$$

From $A(T) = 0$ we obtain

$$p = l-k, \quad r = 0.$$

So algebra $\mathfrak{k}$ is four-dimensional with a basis $\{A_1, A_2, A_3, A_4\}$ where

$$\begin{array}{llll} A_1 X_1 = X_1, & A_1 X_2 = 0, & A_1 X_3 = -X_3, & A_1 X_4 = 0, \\ A_2 X_1 = 0, & A_2 X_2 = X_2, & A_2 X_3 = X_3, & A_2 X_4 = 0, \\ A_3 X_1 = 0, & A_3 X_2 = 0, & A_3 X_3 = X_4, & A_3 X_4 = 0, \\ A_4 X_1 = 0, & A_4 X_2 = 0, & A_4 X_3 = 0, & A_4 X_4 = X_4. \end{array}$$

Further, from $R(X, Y)\in \mathfrak{k}$, $R(SX, SY) = R(X, Y)$ we obtain

$$R(X_1, X_3) = R(X_1, X_4) = R(X_2, X_3) = R(X_2, X_4) = R(X_3, X_4) = 0,$$

$$R(X_1, X_2) = \sum_{i=1}^{4} \tau_i A_i.$$

By the first Bianchi identity we have

$$R(X_1, X_2)X_3 = 0 \Rightarrow \tau_2 = \tau_1, \ \tau_3 = 0,$$

$$R(X_1, X_2)X_4 = 0 \Rightarrow \tau_4 = 0.$$

Hence

$$R(X_1, X_2) = \tau_1 A_1+\tau_1 A_2.$$

From $R(X_1, X_2)(R) = 0$ we obtain $\tau_1 = 0$. Hence $R = 0$. The "Nomizu algebra" $\mathfrak{g} = V+(0)$ is decomposable.

10. The maximal system $(\alpha^2, \alpha, \alpha, \alpha)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = T(X_1, X_3) = T(X_1, X_4) = 0,$$

$$T(X_2, X_3) = aX_1, \quad T(X_2, X_4) = bX_1, \quad T(X_3, X_4) = cX_1,$$

$a^2+b^2+c^2 > 0$.

Let $a \neq 0$; then, assuming

$$X'_1 = X_1, \quad X'_2 = \frac{1}{a} X_2, \quad X'_3 = X_3, \quad X'_4 = cX_2-bX_3+aX_4,$$

we obtain

$$T(X'_2, X'_3) = X'_1, \quad T(X'_2, X'_4) = 0, \quad T(X'_3, X'_4) = 0.$$

Let $\mathfrak{h}$ denote the Lie algebra of all real endomorphisms $A: V \to V$ which, as derivations, satisfy $A(S) = A(T) = A(R) = 0$. Particularly, we have $R(X, Y) \in \mathfrak{h}$. By the condition $R(SX, SY) = R(X, Y)$ we have $R(X, Y) = 0$. The algebra $\mathfrak{g} = V + (0)$ is decomposable.

11. The maximal system $(-1, \alpha, -\alpha, -\alpha)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = aX_3 + bX_4, \quad T(X_1, X_3) = cX_2,$$
$$T(X_1, X_4) = dX_2, \quad T(X_2, X_3) = T(X_2, X_4) = T(X_3, X_4) = 0,$$

$a^2 + b^2 + c^2 + d^2 > 0$.

Let us consider the following cases:

A.I) $a \neq 0$, $c \cdot d \neq 0$, $ac + bd \neq 0$; then, assuming

$$X_1' = |ac+bd|^{1/2} X_1, \quad X_3' = aX_3 + bX_4,$$
$$X_2' = |ac+bd|^{1/2} X_2, \quad X_4' = |ac+bd| \cdot d^{-1} X_4,$$

we obtain

$$T(X_1', X_2') = X_3', \quad T(X_1', X_3') = \operatorname{sgn}(ac+bd) X_2', \quad T(X_1', X_4') = X_2'.$$

By the condition $R(SX, SY) = R(X, Y)$ we have $R(X, Y) = 0$. Multiplication in the algebra $\mathfrak{g} = V$ is defined as follows:

(5.41)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	kX_2	$-X_2$
X_2	X_3	0	0	0
X_3	$-kX_3$	0	0	0
X_4	X_2	0	0	0

for $k = \pm 1$. Substituing $X_4' = X_4 - kX_3$, we can verify that these algebras are decomposable.

A.II) Let $a \neq 0$, $c \cdot d \neq 0$, $ac + bd = 0$; then, assuming

$$X_1' = X_1, \quad X_2' = X_2, \quad X_3' = aX_3 - \frac{ac}{d} X_4, \quad X_4' = \frac{1}{d} X_4,$$

we obtain

$$T(X_1', X_2') = X_3', \quad T(X_1', X_3') = 0, \quad T(X_1', X_4') = X_2'.$$

The Lie algebra $\mathfrak{g} = V+(0)$ is given by

(5.42)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	0	$-X_2$
X_2	X_3	0	0	0
X_3	0	0	0	0
X_4	X_2	0	0	0

This algebra is isomorphic with algebra (5.33). The isomorphism is given by the assignment

$$X_1 \mapsto \bar{X}_1, \quad X_2 \mapsto \bar{X}_3, \quad X_3 \mapsto \bar{X}_4, \quad X_4 \mapsto \bar{X}_2.$$

In all remaining cases we find that these algebras are decomposable.

12. The maximal system $(-1, \alpha, -\alpha, -\alpha^2)$. Let $\{X_1, X_2, X_3, X_4\}$ be the eigenvectors of transformation S. By the condition $S(T) = T$ we have

$$T(X_1, X_2) = aX_3, \quad T(X_1, X_3) = bX_2, \quad T(X_1, X_4) = 0,$$
$$T(X_2, X_3) = cX_4, \quad T(X_2, X_4) = 0, \quad T(X_3, X_4) = 0,$$

$a^2+b^2+c^2 > 0$.

Let us consider the following cases:

A) $a \cdot b \cdot c \neq 0$; then, assuming

$$X'_1 = |ab|^{-1/2} X_1, \quad X'_3 = \operatorname{sgn} a \cdot X_3,$$
$$X'_2 = \left|\frac{b}{a}\right|^{1/2} X_2, \quad X'_4 = \operatorname{sgn}(ac) \cdot \left|\frac{b}{a}\right|^{1/2} X_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_1, X'_3) = \operatorname{sgn}(ab) X'_2, \quad T(X'_2, X'_3) = X'_4.$$

In virtue of Lemma 2, $R = 0$. The Lie algebra $\mathfrak{g} = V+(0)$ is given by

(5.43)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	$-kX_2$	0
X_2	X_3	0	$-X_4$	0
X_3	kX_2	X_4	0	0
X_4	0	0	0	0

for $k = \pm 1$. Now we may give the representations of these algebras by proper infinitesimal transformations of $R^3(x, y, z)$.

For $k = 1$ we have

$$X_1 = y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y}+x\frac{\partial}{\partial z},$$

$$X_3 = \frac{\partial}{\partial x}-y\frac{\partial}{\partial z}, \quad X_4 = 2\frac{\partial}{\partial z}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} \cos ht & \sin ht & 0 & a \\ \sin ht & \cos ht & 0 & b \\ b\cos ht - a\sin ht & b\sin ht - a\cos ht & 1 & c \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$t' = -t, \quad a' = \frac{1}{\alpha}a, \quad b' = -\frac{1}{\alpha}b, \quad c' = -\frac{1}{\alpha^2}c.$$

For $k = -1$ we have

$$X_1 = y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y}+x\frac{\partial}{\partial z},$$

$$X_3 = \frac{\partial}{\partial x}-y\frac{\partial}{\partial z}, \quad X_4 = 2\frac{\partial}{\partial z}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\begin{Vmatrix} \cos t & \sin t & 0 & a \\ -\sin t & \cos t & 0 & b \\ b\cos t + a\sin t & b\sin t - a\cos t & 1 & c \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$t' = -t, \quad a' = \frac{1}{\alpha}a, \quad b' = -\frac{1}{\alpha}b, \quad c' = -\frac{1}{\alpha^2}c.$$

B) Let $a = 0$, $b \cdot c \neq 0$; then, assuming

$$X'_1 = |b|^{-1/2}X_1, \quad X'_2 = \operatorname{sgn} b \cdot |b|^{1/2}X_2,$$
$$X'_3 = X_3, \quad X'_4 = \operatorname{sgn} b \cdot |b|^{1/2} \cdot cX_4,$$

we obtain

$$T(X'_1, X'_3) = X'_2, \quad T(X'_2, X'_3) = X'_4.$$

The Lie algebra $\mathfrak{g} = V + (0)$ is given by

(5.44)

	X_1	X_2	X_3	X_4
X_1	0	0	$-X_2$	0
X_2	0	0	$-X_4$	0
X_3	X_2	X_4	0	0
X_4	0	0	0	0

This algebra is isomorphic with algebra (5.33). The isomorphism is given by the assignment

$$X_1 \mapsto \bar{X}_2, \quad X_2 \mapsto -\bar{X}_3, \quad X_3 \mapsto \bar{X}_1, \quad X_4 \mapsto \bar{X}_4.$$

C) Let $b = 0$, $ac \neq 0$; then, assuming

$$X_1' = |a|^{-1/2} X_1, \quad X_3' = \operatorname{sgn} a \cdot X_3,$$
$$X_2' = |a|^{-1/2} X_2, \quad X_4' = \operatorname{sgn} a \cdot |a|^{-1/2} \cdot cX_4,$$

we obtain

$$T(X_1', X_2') = X_3', \quad T(X_2', X_3') = X_4'.$$

The Lie algebra $\mathfrak{g} = V + (0)$ is given by

(5.45)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	0	0
X_2	X_3	0	$-X_4$	0
X_3	0	X_4	0	0
X_4	0	0	0	0

This algebra is isomorphic with algebra (5.33). The isomorphism is given by the assignment

$$X_1 \mapsto -\bar{X}_2, \quad X_2 \mapsto \bar{X}_1, \quad X_3 \mapsto \bar{X}_3, \quad X_4 \mapsto \bar{X}_4.$$

In the cases

D) $c = 0$, $a \cdot b \neq 0$,

E) $a \neq 0$; $b = c = 0$,

F) $b \neq 0$, $a = b = 0$,

G) $c \neq 0$, $a = b = 0$

we find that the algebras are decomposable.

13. The maximal system $(\alpha, \alpha, \alpha^2, \alpha^3)$. For this system we have

$$T(X_1, X_2) = aX_3, \quad T(X_1, X_3) = bX_4, \quad T(X_1, X_4) = 0,$$
$$T(X_2, X_3) = cX_4, \quad T(X_2, X_4) = 0, \quad T(X_3, X_4) = 0,$$

$a^2+b^2+c^2 > 0$.

Let us consider the following cases:

A) $abc \neq 0$; then, taking a new basis

$$X_1' = \frac{1}{b} X_1 + \frac{bc-1}{c} X_2, \quad X_2' = bX_2, \quad X_3' = aX_3, \quad X_4' = abcX_4,$$

we obtain

$$T(X_1', X_2') = X_3', \quad T(X_1', X_3') = X_4', \quad T(X_2', X_3') = X_4'.$$

In virtue of Lemma 2, $R = 0$. The Lie algebra $\mathfrak{g} = V + (0)$ is given by

(5.46)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	$-X_4$	0
X_2	X_3	0	$-X_4$	0
X_3	X_4	X_4	0	0
X_4	0	0	0	0

Taking a new basis

$$X_1' = \tfrac{1}{2}(X_1+X_2), \quad X_2' = X_2 - X_1, \quad X_3' = X_3, \quad X_4' = X_4,$$

we see that the algebra is isomorphic with (5.33).

B) $a = 0$, $bc \neq 0$; then, assuming

$$X_1' = \frac{1}{b} X_1 + \frac{bc-1}{c} X_2, \quad X_2' = bX_2, \quad X_3' = X_3, \quad X_4' = bcX_4,$$

we obtain

$$T(X_1', X_3') = X_4', \quad T(X_2', X_3') = X_4'.$$

The Lie algebra $\mathfrak{g}$ is given by

(5.47)

	X_1	X_2	X_3	X_4
X_1	0	0	$-X_4$	0
X_2	0	0	$-X_4$	0
X_3	X_4	X_4	0	0
X_4	0	0	0	0

Taking a new basis

$$X'_1 = X_1+X_2, \quad X'_2 = X_1-X_2, \quad X'_3 = X_3, \quad X'_4 = X_4,$$

we see that the algebra is decomposable.

C) $ab \neq 0$, $c = 0$; then, assuming

$$X'_1 = \frac{1}{b} X_1, \quad X'_2 = bX_2, \quad X'_3 = aX_3, \quad X'_4 = aX_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_1, X'_3) = X'_4.$$

The algebra $\mathfrak{g}$ obtained in this case is isomorphic with (5.33).

D) $ac \neq 0$, $b = 0$; then, assuming

$$X'_1 = \frac{1}{c} X_1, \quad X'_2 = cX_2, \quad X'_3 = aX_3, \quad X'_4 = ac^2 X_4,$$

we obtain

$$T(X'_1, X'_2) = X'_3, \quad T(X'_2, X'_3) = X'_4.$$

Multiplication in the algebra $\mathfrak{g} = V$ is defined as follows:

(5.48)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	0	0
X_2	X_3	0	$-X_4$	0
X_3	0	X_4	0	0
X_4	0	0	0	0

This algebra is isomorphic with algebra (5.33). The isomorphism is given by the assignment

$$X_1 \mapsto -\bar{X}_2, \quad X_2 \mapsto \bar{X}_1, \quad X_3 \mapsto \bar{X}_3, \quad X_4 \mapsto \bar{X}_4.$$

In the cases

E) $a \neq 0, \quad b = c = 0,$

F) $b \neq 0, \quad a = c = 0,$

G) $c \neq 0, \quad a = b = 0,$

we find that the algebras are decomposable.

14. The maximal system $(\alpha, \alpha, \alpha^2, \alpha^2)$. For this system, we have

$$T(X_1, X_2) = aX_3+bX_4, \quad a^2+b^2 > 0,$$
$$T(X_1, X_3) = T(X_1, X_4) = T(X_2, X_3) = T(X_2, X_4) = T(X_3, X_4) = 0.$$

Let $a \neq 0$; then, assuming

$$X'_1 = X_1, \quad X'_2 = X_2, \quad X'_3 = aX_3+bX_4, \quad X'_4 = \frac{1}{a}X_4,$$

we obtain $T(X'_1, X'_2) = X_3$. In virtue of Lemma 2, $R = 0$. The Lie algebra $\mathfrak{g} = V+(0)$ is given by

(5.49)

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	0	0
X_2	X_3	0	0	0
X_3	0	0	0	0
X_4	0	0	0	0

We have obtained a decomposable algebra.

15. The maximal system $(-1, -1, -1, -1)$. For this system $T(X, Y) = 0$, and the spaces obtained are ordinary symmetrical ones.

16. The maximal system $(i, -i, -1, -1)$. Let V be a 4-dimensional vector space, V^c its complexification and S: $V^c \to V^c$ a real linear transformation with the eigenvalues given above. Further, let T be the tensor on V satisfying the usual conditions, and denote by the same letter its linear extension to V^c. We can find a basis $\{U, \bar{U}, V_1, V_2\}$ of eigenvectors of S such that

$$SU = iU, \quad S\bar{U} = -i\bar{U}, \quad SV_1 = -V_1, \quad SV_2 = -V_2.$$

By the condition $S(T) = T$ we have

$$T(U, \bar{U}) = T(V_1, V_2) = 0, \quad T(U, V_1) = \alpha\bar{U},$$
$$T(\bar{U}, V_1) = \bar{\alpha}U, \quad T(U, V_2) = \beta\bar{U}, \quad T(\bar{U}, V_2) = \bar{\beta}U.$$

Here α, β are complex variables, $\alpha\bar{\alpha}+\beta\bar{\beta} > 0$. Let $\alpha = \varrho e^{i\varphi}$, $\varrho \neq 0$; then, assuming $U' = e^{-i\varphi/2}U$, $V'_1 = V_1/\varrho$, we obtain

$$T(U', V'_1) = \bar{U}', \quad T(U', V_2) = e^{-i\varphi}\beta\bar{U}', \quad T(\bar{U}', V_2) = e^{i\varphi}\bar{\beta}U'.$$

Now applying the previous notation we have

$$T(U, \bar{U}) = T(V_1, V_2) = 0, \quad T(U, V_1) = \bar{U}, \quad T(\bar{U}, V_1) = U,$$
$$T(U, V_2) = \gamma\bar{U}, \quad T(\bar{U}, V_2) = \bar{\gamma}U.$$

Let us consider the following cases:

I. γ – a real number,

II. γ – a complex number $\neq 0$.

Ad I). We define a new basis, taking

$$V_2' = V_2 - \gamma V_1; \quad \text{then} \quad T(U, V_2') = T(\bar{U}, V_2') = 0.$$

Hence we conclude that in this case there is a basis (we denote it by $\{U, \bar{U}, V_1, V_2\}$ again) such that:

$$\begin{gathered} T(U, \bar{U}) = T(V_1, V_2) = T(U, V_2) = T(\bar{U}, V_2) = 0, \\ T(U, V_1) = \bar{U}, \quad T(\bar{U}, V_1) = U. \end{gathered} \tag{5.50}$$

Let $\mathfrak{k}$ denote the Lie algebra of all real endomorphisms $A\colon V^c \to V^c$, which, as derivations, satisfy $A(S) = A(T) = 0$.

$A(S) = 0$ means that $AU = \lambda U$, $A\bar{U} = \bar{\lambda}\bar{U}$, $AV_1 = pV_1 + qV_2$, $AV_2 = rV_1 + sV_2$; p, q, r, s are real numbers.

$A(T) = 0$ implies particularly $A(T(U, V_i)) = T(AU, V_i) + T(U, AV_i)$, $i = 1, 2$, whence $\bar{\lambda} - \lambda = p$, $r = 0$. Hence $p = 0$, $\lambda = \bar{\lambda}$. So algebra $\mathfrak{k}$ is three-dimensional with a basis $\{A_1, A_2, A_3\}$ such that

$$\begin{array}{llll} A_1 U = U, & A_1 \bar{U} = \bar{U}, & A_1 V_1 = 0, & A_1 V_2 = 0, \\ A_2 U = 0, & A_2 \bar{U} = 0, & A_2 V_1 = V_2, & A_2 V_2 = 0, \\ A_3 U = 0, & A_3 \bar{U} = 0, & A_3 V_1 = 0, & A_3 V_2 = V_2. \end{array}$$

Let (V, S, T, R) be the infinitesimal s-manifold with tensor S, T satisfying (5.50). Then $R(X, Y) \in \mathfrak{k}^c$ and from $R(SX, SY) = R(X, Y)$ we obtain

$$R(U, V_k) = R(\bar{U}, V_k) = 0, \quad k = 1, 2,$$

$$R(U, \bar{U}) = i \sum_{j=1}^{3} \varrho_j A_j, \quad R(V_1, V_2) = \sum_{j=1}^{3} \omega_j A_j,$$

ϱ_j, ω_j being real numbers.

By the second Bianchi identity we have

$$\begin{gathered} R(U, U) V_1 = 0 \Rightarrow \varrho_2 = 0, \\ R(U, \bar{U}) V_2 = 0 \Rightarrow \varrho_3 = 0, \\ R(V_1, V_2) U = 0 \Rightarrow \omega_1 = 0. \end{gathered}$$

Hence

$$\begin{gathered} R(U, \bar{U}) = i\varrho_1 A_1 = B, \\ R(V_1, V_2) = \omega_2 A_2 + \omega_3 A_3 = C. \end{gathered}$$

From the condition $R(X, Y)(R) = 0$ we obtain

$$\begin{gathered} R(BU, \bar{U}) + R(U, B\bar{U}) = 0 \Rightarrow R(U, \bar{U}) = 0, \\ R(CV_1, V_2) + R(V_1, CV_2) = 0 \Rightarrow \omega_3 = 0. \end{gathered}$$

Finally we have $R(V_1, V_2) = \varkappa A_2$, $\varkappa$ being real number.

Let us consider the basis $\{X, Y, V_1, V_2\}$ of V given by $U = X+iY$. Then

$$SX = -Y, \quad SY = X, \quad SV_1 = -V_1, \quad SV_2 = -V_2,$$

$$T(X, Y) = 0, \quad T(X, V_1) = X, \quad T(X, V_2) = 0,$$
$$T(Y, V_1) = -Y, \; T(Y, V_2) = 0, \quad T(V_1, V_2) = 0,$$

$$R(X, Y) = 0, \quad R(X, V_k) = R(Y, V_k) = 0, \quad k = 1, 2,$$

$$R(V_1, V_2) = \varkappa A, \quad \varkappa - \text{real number},$$

$$AX = 0, \quad AY = 0, \quad AV_1 = V_2, \quad AV_2 = 0.$$

Multiplication in the Lie algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is defined by

(5.51)

	X	Y	V_1	V_2	A
X	0	0	$-X$	0	0
Y	0	0	Y	0	0
V_1	X	$-Y$	0	$-\varkappa A$	$-V_2$
V_2	0	0	$\varkappa A$	0	0
A	0	0	V_2	0	0

For $\varkappa = 0$ we obtain a decomposable algebra.

For $\varkappa \neq 0$ we may give the representation of this algebra by proper infinitesimal transformations of the $R^4(x, y, u, v)$, namely

$$X = \frac{\partial}{\partial x}+\frac{\partial}{\partial y}, \quad Y = -\frac{\partial}{\partial x}+\frac{\partial}{\partial y},$$

$$V_1 = -y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+\varkappa u\frac{\partial}{\partial v}+v\frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial u},$$

$$A = \frac{\partial}{\partial v}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\left\|\begin{matrix} \cos ht & -\sin ht & 0 & 0 & a \\ -\sin ht & \cos ht & 0 & 0 & b \\ 0 & 0 & \cos h\sqrt{\varkappa}\,t & \frac{1}{\sqrt{\varkappa}}\sin h\sqrt{\varkappa}\,t & c \\ 0 & 0 & \sqrt{\varkappa}\sin h\sqrt{\varkappa}\,t & \cos\sqrt{\varkappa}\,t & d \\ 0 & 0 & 0 & 0 & 1 \end{matrix}\right\|$$

for $\varkappa > 0$, and the group of all matrices of the form

$$\begin{Vmatrix} \cos ht & -\sin ht & 0 & 0 & a \\ -\sin ht & \cos ht & 0 & 0 & b \\ 0 & 0 & \cos\sqrt{-\varkappa}\,t & \frac{1}{\sqrt{-\varkappa}}\sin\sqrt{-\varkappa}\,t & c \\ 0 & 0 & -\sqrt{-\varkappa}\sin\sqrt{-\varkappa}\,t & \cos\sqrt{-\varkappa}\,t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

for $\varkappa < 0$. The subgroup H is the group of all matrices of the form

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

A symmetry s_0 is determined by the transformation

$$t' = -t, \quad a' = -b, \quad b' = a, \quad c' = -c, \quad d' = d.$$

This symmetry is of order four.

Ad II. Let $\gamma = \gamma_1 + i\gamma_2, \gamma_2 \neq 0$. Then, assuming

$$V_2' = \frac{1}{\gamma_2} V_2 - \frac{\gamma_1}{\gamma_2} V_1,$$

we obtain

$$T(U, V_2') = i\bar{U}, \quad T(\bar{U}, V_2') = -iU.$$

Now, applying the previous notation, we have

$$T(U, \bar{U}) = T(V_1, V_2) = 0, \quad T(U, V_1) = \bar{U},$$
$$T(\bar{U}, V_1) = U, \quad T(U, V_2) = i\bar{U}, \quad T(\bar{U}, V_2) = -iU.$$

For the algebra $\mathfrak{k}$ we obtain

$$A(T(U, V_1)) = T(AU, V_1) + T(U, AV_1)$$
$$\bar{\lambda} - \lambda = p + qi,$$
$$A(T(U, V_2)) = T(AU, V_2) + T(U, AV_2)$$
$$i(\bar{\lambda} - \lambda) = r + si.$$

Let $\lambda = \lambda_1 + \lambda_2 i$; then we have

$$p = 0, \quad s = 0, \quad r = -q = 2\lambda_2.$$

So algebra $\mathfrak{k}$ is two-dimensional with a basis $\{A_1, A_2\}$ where

$$A_1 U = U, \quad A_1 \bar{U} = \bar{U}, \quad A_1 V_1 = 0, \quad A_1 V_2 = 0,$$
$$A_2 U = iU, \quad A_2 \bar{U} = -i\bar{U}, \quad A_2 V_1 = -2V_2, \quad A_2 V_2 = 2V_1.$$

Further,

$$R(U, V_k) = R(\bar{U}, V_k) = 0, \quad k = 1, 2,$$
$$R(U, \bar{U}) = i\varrho_1 A_1 + i\varrho_2 A_2,$$
$$R(V_1, V_2) = \omega_1 A_1 + \omega_2 A_2.$$

By the second Bianchi identity we have

$$R(U, \bar{U}) V_1 = 0 \Rightarrow \varrho_2 = 0,$$
$$R(V_1, V_2) U = -2iU \Rightarrow \omega_1 = 0, \quad \omega_2 = -2.$$

From the condition $R(X, Y)(R) = 0$ we obtain $R(U, \bar{U}) = 0$. Let us consider the basis $\{X, Y, V_1, V_2\}$ of V given by $U = \frac{1}{2}(X + iY)$. Then we have

$$SX = -Y, \quad SY = X, \quad SV_1 = -V_1, \quad SV_2 = -V_2,$$
$$T(X, Y) = 0, \quad T(X, V_1) = X, \quad T(X, V_2) = Y,$$
$$T(Y, V_1) = -Y, \quad T(Y, V_2) = X, \quad T(V_1, V_2) = 0,$$
$$R(X, Y) = 0, \quad R(X, V_k) = R(Y, V_k) = 0, \quad k = 1, 2,$$
$$R(V_1, V_2) = -2A,$$
$$AX = -Y, \quad AY = X, \quad AV_1 = -2V_2, \quad AV_2 = 2V_1.$$

Multiplication in the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined by

(5.52)

	X	Y	V_1	V_2	A
X	0	0	$-X$	$-Y$	Y
Y	0	0	Y	$-X$	$-X$
V_1	X	$-Y$	0	$2A$	$2V_2$
V_2	Y	X	$-2A$	0	$-2V_1$
A	$-Y$	X	$-2V_2$	$2V_1$	0

This algebra is isomorphic with (5.56); the isomorphism is given by the assignment

$$X \mapsto X_1, \quad Y \mapsto Y_1, \quad V_1 \mapsto X_2, \quad V_2 \mapsto -Y_2, \quad A \mapsto A.$$

17. The maximal system $(\theta, \bar{\theta} = \theta^2, \theta, \bar{\theta} = \theta^2)$, $\theta = e^{2\pi i/3}$. Let us denote the basis of V^c by $\{U_1, U_2, \bar{U}_1, \bar{U}_2\}$ where

$$SU_1 = \theta U_1, \quad SU_2 = \theta U_2, \quad S\bar{U}_1 = \bar{\theta}\bar{U}_1, \quad S\bar{U}_2 = \bar{\theta}\bar{U}_2.$$

By the condition $S(T) = T$ we have

$$T(U_1, U_2) = \alpha\bar{U}_1 + \beta\bar{U}_2,$$
$$T(\bar{U}_1, \bar{U}_2) = \bar{\alpha}U_1 + \bar{\beta}U_2,$$
$$T(U_i, \bar{U}_j) = 0, \quad i, j = 1, 2,$$

If $T \neq 0$, then $\varrho^2 = \alpha\bar{\alpha} + \beta\bar{\beta} > 0$. Let us replace U_1, U_2 by the new eigenvectors

$$U'_1 = \frac{1}{\varrho}(\bar{\alpha}U_1 + \bar{\beta}U_2),$$
$$U'_2 = \frac{1}{\varrho^2}(-\beta U_1 + \alpha U_2)$$

and then write again U_1, U_2 instead of U'_1, U'_2. We get the following set of relations:

$$\text{(5.53)} \quad \begin{aligned} &SU_1 = \theta U_1, \quad SU_2 = \theta U_2, \quad S\bar{U}_1 = \bar{\theta}\bar{U}_1, \quad S\bar{U}_2 = \bar{\theta}\bar{U}_2, \\ &T(U_1, U_2) = \bar{U}_1, \quad T(\bar{U}_1, \bar{U}_2) = U_1, \quad T(U_i, \bar{U}_j) = 0. \end{aligned}$$

Let $\mathfrak{k}$ be the Lie algebra of all real endomorphisms A of V^c which, as derivations, annihilate S and T; then

$$AU_1 = aU_1 + bU_2,$$
$$AU_2 = cU_1 + dU_2;$$

here a, b, c, d are complex variables.

$A(T) = 0$ implies particularly

$$A(T(U_1, U_2)) = T(AU_1, U_2) + T(U_1, AU_2) \Rightarrow \bar{a}\bar{U}_1 + \bar{b}\bar{U}_2 = (a+d)\bar{U}_1.$$

Hence

$$a + d = \bar{a}, \quad \bar{b} = 0.$$

Let

$$a = \alpha + \beta i, \quad c = \gamma + \delta i, \quad d = -2\beta i;$$

then

$$AU_1 = (\alpha + \beta i)U_1,$$
$$AU_2 = (\gamma + \delta i)U_1 - 2\beta i U_2.$$

Thus the algebra $\mathfrak{k}$ is four-dimensional with a basis $\{A_1, A_2, A_3, A_4\}$ where

$$\text{(5.54)} \quad \begin{aligned} &A_1 U_1 = U_1, \quad A_1\bar{U}_1 = \bar{U}_1, \quad A_1 U_2 = 0, \quad A_1\bar{U}_2 = 0, \\ &A_2 U_1 = iU_1, \quad A_2\bar{U}_1 = -i\bar{U}_1, \quad A_2 U_2 = -2iU_2, \quad A_2\bar{U}_2 = 2i\bar{U}_2, \\ &A_3 U_1 = 0, \quad A_3\bar{U}_1 = 0, \quad A_3 U_2 = U_1, \quad A_3\bar{U}_2 = \bar{U}_1, \\ &A_4 U_1 = 0, \quad A_4\bar{U}_1 = 0, \quad A_4 U_2 = iU_1, \quad A_4\bar{U}_2 = -i\bar{U}_1. \end{aligned}$$

Let (V, S, T, R) be an infinitesimal s-manifold with tensors S, T satisfying (5.53). Then $R(X, Y) \in \mathfrak{k}^c$ and from $R(SX, SY) = R(X, Y)$ we obtain

$$R(U_1, U_2) = R(\bar{U}_1, \bar{U}_2) = 0,$$
$$R(U_1, \bar{U}_1) = i \sum_{k=1}^{4} \tau_k A_k,$$
$$R(U_1, \bar{U}_2) = \sum_{k=1}^{4} (\varrho_k + i\sigma_k) A_k,$$
$$R(U_2, \bar{U}_1) = \sum_{k=1}^{4} (-\varrho_k + i\sigma_k) A_k,$$
$$R(U_2, \bar{U}_2) = i \sum_{k=1}^{4} \omega_k A_k.$$

By the first Bianchi identity we have

$$R(U_2, \bar{U}_1) U_1 - R(U_1, \bar{U}_1) U_2 = 0,$$

hence

$$\varrho_1 + \sigma_2 - \tau_4 = 0, \quad \varrho_2 - \sigma_1 + \tau_3 = 0, \quad \tau_2 = 0,$$

and

$$R(U_2, \bar{U}_2) U_1 - R(U_1, \bar{U}_2) U_2 = U_1,$$

hence

$$\varrho_2 = 0, \quad \sigma_2 = 0, \quad -\omega_2 - \varrho_3 + \sigma_4 - 1 = 0, \quad \omega_1 - \sigma_3 - \varrho_4 = 0.$$

Therefore

$$\begin{aligned} & R(U_1, \bar{U}_1) = i(\tau_1 A_1 + \tau_3 A_3 + \tau_4 A_4), \\ (5.55)\quad & R(U_1, \bar{U}_2) = (\tau_4 + i\tau_3) A_1 + (\varrho_3 + i\sigma_3) A_3 + (\varrho_4 + i\sigma_4) A_4, \\ & R(U_2, \bar{U}_2) = i[(\sigma_3 + \varrho_4) A_1 + (\sigma_4 - \varrho_3 - 1) A_2 + \omega_3 A_3 + \omega_4 A_4. \end{aligned}$$

From the condition $R(X, Y)(R) = 0$ we obtain

$$\tau_1 = \tau_3 = \tau_4 = 0,$$
$$(\varrho_3 + \sigma_4)\omega_2 = (\varrho_4 - \sigma_3)\omega_2 = 0,$$
$$3\sigma_4 \omega_2 - 3\sigma_3 \omega_1 + 2(\varrho_4 \sigma_3 - \varrho_3 \sigma_4) = 0,$$
$$3\varrho_3 \omega_2 + 3\varrho_4 \omega_1 - 2(\varrho_4 \sigma_3 - \varrho_3 \sigma_4) = 0,$$
$$\varrho_3 \omega_1 - \varrho_4 \omega_2 = 0,$$
$$\sigma_4 \omega_1 + \sigma_3 \omega_2 = 0,$$
$$\omega_3 \sigma_3 + \omega_4 \varrho_3 = 0,$$
$$\omega_3 \sigma_4 + \omega_4 \varrho_4 = 0.$$

By these relations we obtain from (5.55) the following three possibilities:

A) $R(U_1, \bar{U}_1) = 0,\ R(U_1, \bar{U}_2) = 0,\ R(U_2, \bar{U}_2) = -iA_2,$

B) $R(U_1, \bar{U}_1) = 0,\ R(U_1, \bar{U}_2) = -A_3,\ R(U_2, \bar{U}_2) = ikA_3,\ k = 0, 1.$

C) $R(U_1, \bar{U}_1) = 0,\ R(U_1, \bar{U}_2) = -\frac{3}{8}A_3+\frac{3}{8}iA_4,\ R(U_2, \bar{U}_2) = -\frac{1}{4}iA_2.$

Ad A). Let us consider the basis $\{X_1, Y_1, X_2, Y_2\}$ of V given by

$$U_1 = \frac{1}{\sqrt{2}}(X_1+iY_1), \quad U_2 = \frac{1}{2}(X_2+iY_2).$$

Then

$$\begin{aligned}
&T(X_1, X_2) = X_1, && T(X_1, Y_1) = 0, && T(X_1, Y_2) = -Y_1,\\
&T(X_2, Y_1) = Y_1, && T(X_2, Y_2) = 0, && T(Y_1, Y_2) = -X_1,\\
&R(X_1, X_2) = 0, && R(X_1, Y_1) = 0, && R(X_1, Y_2) = 0,\\
&R(X_2, Y_1) = 0, && R(X_2, Y_2) = 2A, && R(Y_1, Y_2) = 0,
\end{aligned}$$

$$AX_1 = -Y_1, \quad AX_2 = 2Y_2, \quad AY_1 = X_1, \quad AY_2 = -2X_2.$$

Multiplication in the Lie algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is defined by

(5.56)

	X_1	Y_1	X_2	Y_2	A
X_1	0	0	$-X_1$	Y_1	Y_1
Y_1	0	0	Y_1	X_1	$-X_1$
X_2	X_1	$-Y_1$	0	$-2A$	$-2Y_2$
Y_2	$-Y_1$	$-X_1$	$2A$	0	$2X_2$
A	$-Y_1$	X_1	$2Y_2$	$-2X_2$	0

Now we may give the representation of this algebra by proper infinitesimal transformations of the plane $R^2(x, y)$, namely

$$X_1 = \frac{\partial}{\partial x}, \quad Y_1 = \frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x},$$

$$Y_2 = x\frac{\partial}{\partial y}+y\frac{\partial}{\partial x}, \quad A = x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}.$$

The corresponding Lie group G is the group of all matrices of the form [12]

$$\left\|\begin{matrix} a & b & e\\ c & d & f\\ 0 & 0 & 1\end{matrix}\right\|, \quad ad-bc = 1.$$

The subgroup H is the group of all matrices of the form

$$\begin{Vmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

$M = G/H$ is a reductive homogeneous space.

A symmetry s_0 is determined by the transformation

$$\begin{aligned} a' &= \tfrac{1}{4}a+\tfrac{1}{4}\sqrt{3}b+\tfrac{1}{4}\sqrt{3}c+\tfrac{3}{4}d, \\ b' &= -\tfrac{1}{4}\sqrt{3}a+\tfrac{1}{4}b-\tfrac{3}{4}c+\tfrac{1}{4}\sqrt{3}d, \\ c' &= -\tfrac{1}{4}\sqrt{3}a-\tfrac{3}{4}b+\tfrac{1}{4}c+\tfrac{1}{4}\sqrt{3}d, \\ d' &= \tfrac{3}{4}a-\tfrac{1}{4}\sqrt{3}b+\tfrac{1}{4}\sqrt{3}c+\tfrac{1}{4}d, \\ e' &= -\tfrac{1}{2}e-\tfrac{1}{2}\sqrt{3}f, \\ f' &= \tfrac{1}{2}\sqrt{3}e-\tfrac{1}{2}f. \end{aligned}$$

This symmetry is of order three.

Ad B) Assuming $U_1 = \frac{1}{2}(X_1+iY_1)$, $U_2 = \frac{1}{2}(X_2+iY_2)$, we obtain

$$\begin{array}{lll} T(X_1, X_2) = X_1, & T(X_1, Y_1) = 0, & T(X_1, Y_2) = -Y_1, \\ T(X_2, Y_1) = Y_1, & T(X_2, Y_2) = 0, & T(Y_1, Y_2) = -X_1, \\ R(X_1, X_2) = -2A, & R(X_1, Y_1) = 0, & R(X_1, Y_2) = 0, \\ R(X_2, Y_1) = 0, & R(X_2, Y_2) = -2kA, & R(Y_1, Y_2) = -2A, \end{array}$$

$$AX_1 = 0, \quad AX_2 = X_1, \quad AY_1 = 0, \quad AY_2 = Y_1.$$

Multiplication in the Lie algebra $\mathfrak{g} = V+\mathfrak{h}^\circ$ is defined by

(5.57)

	X_1	Y_1	X_2	Y_2	A
X_1	0	0	$-X_1+2A$	Y_1	0
Y_1	0	0	Y_1	X_1+2A	0
X_2	X_1-2A	$-Y_1$	0	0	$-X_1$
Y_2	$-Y_1$	$-X_1-2A$	0	0	$-Y_1$
A	0	0	X_1	Y_1	0

for $k = 0$, and

(5.58)

	$\tilde{X}_1$	$\tilde{Y}_1$	$\tilde{X}_2$	$\tilde{Y}_2$	$\tilde{A}$
$\tilde{X}_1$	0	0	$-\tilde{X}_1+2\tilde{A}$	$\tilde{Y}_1$	0
$\tilde{Y}_1$	0	0	$\tilde{Y}_1$	$\tilde{X}_1+2\tilde{A}$	0
$\tilde{X}_2$	$\tilde{X}_1-2\tilde{A}$	$-\tilde{Y}_1$	0	$2\tilde{A}$	$-\tilde{X}_1$
$\tilde{Y}_2$	$-\tilde{Y}_1$	$-\tilde{X}_1-2\tilde{A}$	$-2\tilde{A}$	0	$-\tilde{Y}_1$
$\tilde{A}$	0	0	$\tilde{X}_1$	$\tilde{Y}_1$	0

for $k = 1$. These algebras are isomorphic. The isomorphism is given by the assignment

$$\tilde{X}_1 \mapsto X_1, \quad Y_1 \mapsto \tilde{Y}_1, \quad \tilde{X}_2 \mapsto X_2+\tfrac{1}{2}Y_1, \quad \tilde{Y}_2 \mapsto Y_2-\tfrac{1}{2}X_1, \quad \tilde{A} \mapsto A.$$

Now we may give the representation of algebra (5.57) by infinitesimal transformations of $R^3(x, y, z)$, namely

$$X_1 = -\frac{\partial}{\partial x}-4\frac{\partial}{\partial z}, \qquad Y_1 = -\frac{\partial}{\partial y},$$

$$X_2 = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-2z\frac{\partial}{\partial z}, \quad Y_2 = x\frac{\partial}{\partial y}+3y\frac{\partial}{\partial x},$$

$$A = -\frac{\partial}{\partial x}+2\frac{\partial}{\partial z}.$$

The corresponding Lie group G is the group of all matrices of the form

$$\left\|\begin{matrix} e^s\cosh t & e^s\sinh t & 0 & a \\ e^s\sinh t & e^s\cosh t & 0 & b \\ 0 & 0 & e^{-2s} & c \\ 0 & 0 & 0 & 1 \end{matrix}\right\|.$$

The subgroup H is the group of all matrices of the form

$$\left\|\begin{matrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2a \\ 0 & 0 & 0 & 1 \end{matrix}\right\|.$$

A symmetry s_0 is determined by the transformation

$$a' = \tfrac{1}{2}a-\tfrac{1}{2}b-\tfrac{1}{4}c,$$

$$b' = \tfrac{1}{2}a-\tfrac{1}{2}b+\tfrac{1}{4}c,$$

$$c' = -2a-2b,$$

$$u' = v,$$

$$v' = -(u+v),$$

where $u = s+t$, $v = s-t$.

Ad C) In this case the algebra $\mathfrak{h}^\circ$ is three-dimensional with a basis $\{A, B, C\}$ such that

$$AU_1 = iU_1, \quad BU_1 = 0, \quad CU_1 = 0,$$
$$AU_2 = -2iU_2, \quad BU_2 = U_1, \quad CU_2 = iU_1.$$

Assuming

$$U_1 = \tfrac{1}{2}(X_1 + iY_1), \quad U_2 = \tfrac{1}{2}(X_2 + iY_2),$$

we obtain

$$T(X_1, X_2) = 2X_1, \quad T(X_1, Y_1) = 0, \quad T(X_1, Y_2) = -2Y_1,$$
$$T(X_2, X_1) = 2Y_1, \quad T(X_2, Y_2) = 0, \quad T(Y_1, Y_2) = -2X_1,$$
$$R(X_1, X_2) = -3B, \quad R(X_1, Y_1) = 0, \quad R(X_1, Y_2) = -3C,$$
$$R(X_2, X_1) = -3C, \quad R(X_2, Y_2) = 2A, \quad R(Y_1, Y_2) = -3B,$$

$$AX_1 = -Y_1, \quad AX_2 = 2Y_2, \quad AY_1 = X_1, \quad AY_2 = -2X_2,$$
$$BX_1 = 0, \quad BX_2 = X_1, \quad BY_1 = 0, \quad BY_2 = Y_1,$$
$$CX_1 = 0, \quad CX_2 = -Y_1, \quad CY_1 = 0, \quad CY_2 = X_1.$$

Multiplication in the Lie algebra $\mathfrak{g} = V + \mathfrak{h}^\circ$ is defined by

(5.59)

	X_1	Y_1	X_2	Y_2	A	B	C
X_1	0	0	$-2X_1+3B$	$2Y_1+3C$	Y_1	0	0
Y_1	0	0	$2Y_1-3C$	$2X_1+3B$	$-X_1$	0	0
X_2	$2X_1-3B$	$-2Y_1+3C$	0	$-2A$	$-2Y_2$	$-X_1$	Y_1
Y_2	$-2Y_1-3C$	$-2X_1-3B$	$2A$	0	$2X_2$	$-Y_1$	$-X_1$
A	$-Y_1$	X_1	$2Y_2$	$-2X_2$	0	$3C$	$-3B$
B	0	0	X_1	Y_1	$-3C$	0	0
C	0	0	$-Y_1$	X_1	$3B$	0	0

Now we may give the representation of this algebra by proper a infinitesimal transformation of $R(x, y, u, v)$

$$X_1 = \frac{\partial}{\partial x}, \quad Y_1 = \frac{\partial}{\partial y},$$

$$X_2 = (-2x+u)\frac{\partial}{\partial x} + (2y-v)\frac{\partial}{\partial y} + 3x\frac{\partial}{\partial u} - 3y\frac{\partial}{\partial v},$$

$$Y_2 = (2y+v)\frac{\partial}{\partial x}+(2x+u)\frac{\partial}{\partial y}+3y\frac{\partial}{\partial u}+3x\frac{\partial}{\partial v},$$

$$A = -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+3v\frac{\partial}{\partial u}+3u\frac{\partial}{\partial v},$$

$$B = \frac{\partial}{\partial u}, \quad C = \frac{\partial}{\partial v}.$$

18. The maximal system $(\theta, \bar{\theta} = \theta^4, \varepsilon = \theta^3, \bar{\tau} = \theta^2)$, $\theta = e^{2\pi i/5}$. Let (V, S, T, R), $T \neq 0$, be an infinitesimal s-manifold with the above system of eigenvelues. Let U_1, U_2 be the eigenvectors corresponding to the eigenvalues θ and τ. Then

$$SU_1 = \theta U_1, \quad SU_2 = \theta^3 U_2, \quad S\bar{U}_1 = \theta^4 \bar{U}_1, \quad S\bar{U}_2 = \theta^2 \bar{U}_2.$$

By the condition $S(T) = T$ we have

$$T(U_1, U_2) = \alpha\bar{U}_1, \quad T(U_1, \bar{U}_2) = \beta U_2,$$
$$T(\bar{U}_1, \bar{U}_2) = \bar{\alpha} U_1, \quad T(\bar{U}_1, U_2) = \bar{\beta}\bar{U}_2,$$

where α, β are complex numbers, not both equal to zero. If $\alpha \neq 0$, $\beta = 0$, then the reduced set of characteristic relations is $\theta\tau = \bar{\theta}$, $\theta\bar{\theta} = 1$. Completing this set by a new characteristic relation $\theta = \tau$, we obtain the maximal system (17).

Similarly, if $\alpha = 0$, $\beta \neq 0$, the reduced set of characteristic relations is $\theta\bar{\tau} = \tau$, $\theta\bar{\theta} = 1$, which is contained in the maximal system (17).

Let us now suppose that $\alpha\beta \neq 0$ and calculate the Lie algebra $\mathfrak{h}$ of all real endomorphisms A of V^c such that $A(S) = A(T) = 0$.

$A(S) = 0$ means that

$$AU_1 = \lambda U_1, \quad A\bar{U}_1 = \bar{\lambda}\bar{U}_1, \quad AU_2 = \mu U_2, \quad A\bar{U}_2 = \bar{\mu}\bar{U}_2.$$

$A(T) = 0$ implies

$$\alpha(\bar{\lambda}-\lambda-\mu) = 0, \quad \beta(\mu-\lambda-\bar{\mu}) = 0.$$

Hence $\lambda = \mu = 0$ and $\mathfrak{h} = (0)$.

This means that $R = 0$ and the first Bianchi identity $\mathfrak{S}\big(T(T(X, Y), Z)\big) = 0$ holds. Particularly

$$\mathfrak{S}\big(T(T(U_1, U_2), \bar{U}_1)\big) = \beta\bar{\beta} U_2,$$
$$\mathfrak{S}\big(T(T(U_1, U_2), \bar{U}_2)\big) = \alpha\bar{\alpha} U_1,$$

and hence $\alpha = \beta = 0$, a contradiction. We cannot obtain new infinitesimal s-manifolds in this way.

Classification list

Dimension $n = 3$. There is only one type (I) of generalized affine symmetric spaces of order 4 and two types (II and III) of infinite order. All these spaces are primitive and are represented by the following matrix groups and their automorphisms;

I
$$G_I = \begin{Vmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$x' = -y, \quad y' = x, \quad z' = -z.$$

II
$$G_{II} = \begin{Vmatrix} \cos z & -\sin z & x \\ \sin z & \cos z & y \\ 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$x' = -\alpha x, \quad y' = \alpha y, \quad z' = -z, \quad \alpha \neq 0, \pm 1.$$

III
$$G_{III} = \begin{Vmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$x' = -\alpha x, \quad y' = -y, \quad z' = \alpha z, \quad \alpha \neq 0, \pm 1.$$

Dimension $n = 4$. There are exactly three types (I, II and III below) of generalized affine symmetric spaces of order 3, two types (IV and V) of order 4 and 11 types (VI–XVI) of infinite order.

All these spaces are primitive and they are represented by the following regular homogeneous s-manifolds (G, H, σ):

A. Generalized affine symmetric spaces of order 3.

I. $M_1 = G_1/H_1$, where

$$G_1 = \begin{Vmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{Vmatrix}, \quad ad - bc = 1.$$

$$H_1 = \begin{Vmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

An automorphism σ is determined by the transformation

$$\begin{aligned} a' &= \tfrac{1}{4}a+\tfrac{1}{4}\sqrt{3}\,b+\tfrac{1}{4}\sqrt{3}\,c+\tfrac{3}{4}d,\\ b' &= -\tfrac{1}{4}\sqrt{3}\,a+\tfrac{1}{4}b-\tfrac{3}{4}c+\tfrac{1}{4}\sqrt{3}\,d,\\ c' &= -\tfrac{1}{4}\sqrt{3}\,a+\tfrac{3}{4}b+\tfrac{1}{4}c+\tfrac{1}{4}\sqrt{3}\,d,\\ d' &= \tfrac{3}{4}a-\tfrac{1}{4}\sqrt{3}\,b-\tfrac{1}{4}\sqrt{3}\,c+\tfrac{1}{4}d,\\ e' &= -\tfrac{1}{2}e-\tfrac{1}{2}\sqrt{3}\,f,\\ f' &= \tfrac{1}{2}\sqrt{3}\,e-\tfrac{1}{2}f. \end{aligned}$$

II. $M_2 = G_2/H_2$,

$$G_2 = \begin{Vmatrix} e^s\cosh t & e^s\sinh t & 0 & a\\ e^s\sinh t & e^s\cosh t & 0 & b\\ 0 & 0 & e^{-2s} & c\\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

$$H_2 = \begin{Vmatrix} 1 & 0 & 0 & -a\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2a\\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

An automorphism σ is determined by the transformation

$$\begin{aligned} a' &= \tfrac{1}{2}a-\tfrac{1}{2}b-\tfrac{1}{4}c,\\ b' &= \tfrac{1}{2}a-\tfrac{1}{2}b+\tfrac{1}{4}c,\\ c' &= -2a-2b,\\ u' &= v,\\ v' &= -u+v, \end{aligned}$$

where $u' = s+t$, $v = s-t$.

III. $M_3 = G_3/H_3$, where the corresponding local regular s-triplet $(\mathfrak{g}_3, \mathfrak{h}_3, \nu)$ is defined as follows: $\mathfrak{g}_3$ is isomorphic to a 7-dimensional Lie algebra of infinitesimal transformations of the space $R^4(x, y, u, v)$ with the basis:

$$X_1 = \frac{\partial}{\partial x},\quad X_2 = \frac{\partial}{\partial y},$$

$$X_3 = (-2x+u)\frac{\partial}{\partial x}+(2y-v)\frac{\partial}{\partial y}+3x\frac{\partial}{\partial u}-3y\frac{\partial}{\partial v},$$

$$X_4 = (2y+v)\frac{\partial}{\partial x}+(2x+u)\frac{\partial}{\partial y}+3y\frac{\partial}{\partial u}+3x\frac{\partial}{\partial v},$$

$$A = -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+3v\frac{\partial}{\partial u}-3u\frac{\partial}{\partial v},$$

$$B = \frac{\partial}{\partial u},\quad C = \frac{\partial}{\partial v}.$$

$\mathfrak{h}_3$ corresponds to the 3-dimensional subalgebra (A, B, C) and ν is an automorphism of order 3 of $\mathfrak{g}_3$ induced by the following transformation of R^4

$$x' = \tfrac{1}{2}\sqrt{3}x + \tfrac{1}{2}y, \quad y' = -\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{3}y, \quad u' = u, \quad v' = v.$$

B. Generalized affine symmetric spaces of order 4.

IV. $M_4 = G_4/H_4$,

$$G_4 = \begin{Vmatrix} \cosh t & -\sinh t & 0 & 0 & a \\ -\sinh t & \cosh t & 0 & 0 & b \\ 0 & 0 & \cosh \varkappa t & (1/\varkappa)\sinh \varkappa t & c \\ 0 & 0 & \varkappa \sinh \varkappa t & \cosh \varkappa t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

for $\varkappa > 0$,

$$H_4 = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

An automorphism σ is determined by the transformation

$$t' = -t, \quad a' = -b, \quad b' = a, \quad c' = -c, \quad d' = d.$$

V. $M_5 = G_5/H_4$,

$$G_5 = \begin{Vmatrix} \cosh t & -\sinh t & 0 & 0 & a \\ -\sinh t & \cosh t & 0 & 0 & b \\ 0 & 0 & \cos \varkappa t & (1/\varkappa)\sin \varkappa t & c \\ 0 & 0 & -\varkappa \sin \varkappa t & \cos \varkappa t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

for $\varkappa > 0$. An automorphisms σ is determined by the transformation

$$t' = -t, \quad a' = -b, \quad b' = a, \quad c' = -c', \quad d' = d.$$

C. Generalized affine symmetric spaces of infinite order.

VI. $M_6 = G_6$,

$$G_6 = \begin{Vmatrix} 1 & t & \tfrac{1}{2}t^2 & a \\ 0 & 1 & t & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$a' = \alpha^3 a, \quad b' = \alpha b, \quad c' = \frac{1}{\alpha} c, \quad t' = \alpha^2 t, \quad \alpha \neq 0, \pm 1.$$

VII. $M_7 = G_7$,

$$G_7 = \begin{Vmatrix} \cosh t & \sinh t & 0 & a \\ \sinh t & \cosh t & 0 & b \\ b\cosh t - a\sinh t & b\sinh t - a\cosh t & 1 & c \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$t' = -t, \quad a' = \alpha a, \quad b' = -\alpha b, \quad c' = -\alpha^2 c, \quad \alpha \neq 0, \ \pm 1.$$

VIII. $M_8 = G_8$,

$$G_8 = \begin{Vmatrix} \cos t & \sin t & 0 & a \\ -\sin t & \cos t & 0 & b \\ b\cos t + a\sin t & b\sin t - a\cos t & 1 & c \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

A possible symmetry s_0 is the transformation

$$t' = -t, \quad a' = \alpha a, \quad b' = -\alpha b, \quad c' = -\alpha^2 c, \quad \alpha \neq 0, \ \pm 1.$$

IX. $M_9 = G_9/H_9$,

$$G_9 = \begin{Vmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{Vmatrix}, \quad ad - bc = 1,$$

$$H_9 = \begin{Vmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

An automorphism σ is determined by the transformation

$$a' = a, \quad b' = \frac{1}{\alpha^2} b, \quad c' = \alpha^2 c, \quad d' = d, \quad e' = \frac{1}{\alpha} e, \quad f' = \alpha f, \quad \alpha \neq 0, \ \pm 1.$$

X. $M_{10} = G_{10}/H_{10}$,

$$G_{10} = \begin{Vmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

$$H_{10} = \begin{Vmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

An automorphism σ is determined by the transformation

$$a' = \frac{1}{\alpha} a, \quad b' = b, \quad c' = \alpha c, \quad d' = \alpha d, \quad e' = \alpha^2 e, \quad \alpha \neq 0, \ \pm 1.$$

XI. $M_{11} = G_{11}/H_{11}$,

$$G_{11} = \begin{Vmatrix} 1 & t & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & \cos t & \sin t & c \\ 0 & 0 & -\sin t & \cos t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

$$H_{11} = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

An automorphism σ is determined by the transformation

$$t' = -t, \quad a' = \alpha a, \quad b' = -\alpha b, \quad c' = -c, \quad d' = d, \quad \alpha \neq 0, \ \pm 1.$$

XII. $M_{12} = G_{12}/H_{11}$,

$$G_{12} = \begin{Vmatrix} 1 & t & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & \cosh t & \sinh t & c \\ 0 & 0 & \sinh t & \cosh t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

An automorphism σ is determined by the transformation

$$t' = -t, \quad a' = \alpha a, \quad b' = -\alpha b, \quad c' = -c, \quad d' = d, \quad \alpha \neq 0, \ \pm 1.$$

XIII. $M_{13} = G_{13}/H_{11}$,

$$G_{13} = \begin{Vmatrix} \cos t & \sin t & 0 & 0 & a \\ -\sin t & \cos t & 0 & 0 & b \\ 0 & 0 & \cos \varkappa t & 1/\varkappa \sin \varkappa t & c \\ 0 & 0 & -\varkappa \sin \varkappa t & \cos \varkappa t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

for $\varkappa > 0$. An automorphism σ is determined by the transformation

$$t' = -t, \quad a' = \alpha a, \quad b' = -\alpha b, \quad c' = -c, \quad d' = d, \quad \alpha \neq 0, \ \pm 1.$$

XIV. $M_{14} = G_{14}/H_{11}$,

$$G_{14} = \begin{Vmatrix} \cos t & \sin t & 0 & 0 & a \\ -\sin t & \cos t & 0 & 0 & b \\ 0 & 0 & \cosh \varkappa t & (1/\varkappa) \sinh \varkappa t & c \\ 0 & 0 & \varkappa \sinh \varkappa t & \cosh \varkappa t & d \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

for $\varkappa > 0$. An automorphism σ is determined by the transformation

$$t' = -t, \quad a' = \alpha a, \quad b' = -\alpha b, \quad c' = -c, \quad d' = d, \quad \alpha \neq 0, \pm 1.$$

XV. $M_{15} = G_{15}/H_{15}$,

$$G_{15} = \left\| \begin{matrix} 1 & t & -\frac{1}{2}t^2 & \frac{1}{6}t^3 & a \\ 0 & 1 & -t & \frac{1}{2}t^2 & b \\ 0 & 0 & 1 & -t & c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|,$$

$$H_{15} = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|.$$

An automorphism σ is determined by the transformation

$$t' = \frac{1}{\alpha} t, \quad a' = \frac{1}{\alpha} a, \quad b' = b, \quad c' = \alpha c, \quad d' = \alpha^2 d, \quad \alpha \neq 0, \pm 1.$$

XVI. $M_{16} = G_{16}/H_{16}$,

$$G_{16} = \left\| \begin{matrix} 1 & -t & -\frac{1}{2}t^2 & \frac{1}{6}t^3 & a \\ 0 & 1 & t & -\frac{1}{2}t^2 & b \\ 0 & 0 & 1 & -t & c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|,$$

$$H_{16} = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|.$$

An automorphism σ is determined by the transformation

$$t' = \alpha t, \quad a' = \alpha^2 a, \quad b' = \alpha b, \quad c' = c, \quad d' = \frac{1}{\alpha} d, \quad \alpha \neq 0, \pm 1.$$

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