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LOCAL ALGEBRAS AND A NEW VERSION OF YOUNG'S ORTHOGONAL FORM

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1. Introduction

An explicit construction of representations of symmetric groups which essentially differ from Specht modules was presented by A. Young in 1931 in one of the last papers of his renowned series. It is generally known as Young's orthogonal form (see [2]). Mathematicians almost did not pay attention to this construction in their further investigations and did not consider it to be essential for the representation theory of symmetric groups.

In very recent years new relationships of representation theory of Coxeter groups with mathematical physics (Young–Baxter equation) and topology (knot theory) have been discovered. This raised interest in explicit constructions of representations. On the other hand, models needed in applications should, to great extent, utilize the fact that the symmetric group is a Coxeter group and in particular, Coxeter generators and inductive structure of the group. Finally, it was an old problem of fundamental importance to clarify why Young diagrams play such a role in the general theory of the symmetric group S_n and to explain their a priori appearance in a description of representations (see the appendix to the Russian translation of [1]).

In this paper we describe a new method of construction of modules over the group algebras of Coxeter groups, and in particular, symmetric groups. Our method can also be applied to so-called local algebras which include Hecke algebras, braid algebras etc. The essential feature is a systematic application of inductive approach based on Coxeter generators and lattice theory. The method permits to derive Young's orthogonal form as a special case of a general construction which reveals the structure of representations of local algebras (local modules). The modules can be also considered as a discrete analogue of function spaces on flag manifolds in which one constructs

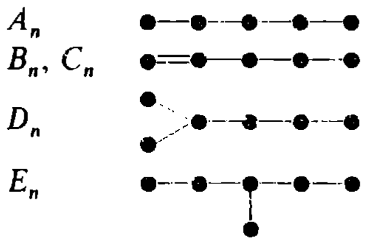
This paper is in final form and no version of it will be submitted for publication elsewhere.

representations of reductive groups. The reader can notice a relationship with the theory of Gelfand–Zeitlin bases, evolutionary models of quantum statistical physics and the theory of infinite symmetric group. More detailed presentation will appear elsewhere.

2. Local algebras

An algebra A_n (over a field k) with generators $\sigma_1, \ldots, \sigma_n$ is called *local* if $\sigma_i \sigma_{i+k} = \sigma_{i+k} \sigma_i$ for all k such that $|k| \geq k_0$ for a fixed k_0. In the sequel we always assume $k_0 = 2$. The same definition can also be introduced in the category of Lie algebras, groups etc. The word "local" is used here in a "physical" sense: distant generators commute, i.e., do not affect each other, whereas close (local) affect each other. An algebra is called *stationary* (*almost stationary*) if all relations among its generators $\sigma_i, \ldots, \sigma_{i+s}$ do not depend on i (do not depend on i starting with a fixed i_0).

Classical series of Coxeter groups A_n, B_n, C_n, D_n, E_n are examples of local, almost stationary algebras; A_n is stationary.



Other examples include braid groups and Hecke algebras. Relations for Coxeter generators of S_n (series A_n) are as follows:

(1) $$\sigma_i^2 = 1, \quad i = 1, \ldots, n-1,$$

(2) $$\sigma_i \sigma_{i+k} = \sigma_{i+k} \sigma_i, \quad k \geq 2,$$

(3) $$(\sigma_i \sigma_{i+1})^3 = 1, \quad i = 1, \ldots, n-2.$$

The following are relations for Hecke algebra:

(1) $$(\sigma_i - q)(\sigma_i + 1) = 0, \quad i = 1, ,,,, n-1.$$

(2) $$\sigma_i \sigma_{i+k} = \sigma_{i+k} \sigma_i, \quad k \geq 2,$$

(3) $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-2.$$

For $q = 1$ we get $k(S_n)$. The relations (2), (3) define braid group. Passage to other generators gives another presentation of Hecke algebras:

(1) $$\tau_i^2 = 1, \quad i = 1, \ldots, n-1,$$

(2) $$\tau_i \tau_{i+k} = \tau_{i+k} \tau_i, \quad k \geq 2,$$

(3) $$(\tau_i \tau_{i+1})^3 + a(\tau_i \tau_{i+1})^2 - a(\tau_i \tau_{i+1}) - 1 = 0,$$

where $a = ((q - q^{-1})/(q + q^{-1}))^2$, $i = 1, \ldots, n-2$.

In this presentation it is a special case of so-called algebras generated by reflections.

From new examples we mention the following generalization of PSL(2):

(1) $$(\sigma_i \sigma_{i+1})^3 = 1, \quad i = 1, \ldots, n-1,$$

(2) $$\sigma_i \sigma_{i+k} = \sigma_{i+k} \sigma_i, \quad k \geqslant 2,$$

(3) $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-1.$$

For $n = 2$ we get $\mathrm{PSL}(2, \mathbf{Z})$.

3. Local modules over local algebras

Let L be a graded poset (partially ordered set) with $\phi = \inf L$ and let $Y = Y(L)$ be the space of all paths of maximal length in L; for $A \in Y(L)$ we have $A = (\phi = A_0, A_1, \ldots, A_n)$, $A_i \in L$, $A_i \lessdot A_{i+1}$ (1), $i = 0, 1, \ldots, n-1$. Let k be a fixed field ($k = \mathbf{Q}, \mathbf{R}$ or $\mathbf{C}$) and let $V = k^Y$ be the vector space with basis $\{e_A\}_{A \in Y}$.

Let A_n be a local algebra. The space V is called a *local module* over A_n if the action of the generator $\sigma_i \in A_n$, $i = 1, \ldots, n$, has the form

(*) $$\sigma_i e_A = \sum_{A'} c_i^{A'} e_{A'}$$

where the summation ranges over all paths A' for which $A'_j = A_j$ for $j \neq i$ and $c_i^{A'}$ depends on A_{i-1}, A_i, A_{i+1} and A'_i only. In other terms σ_i sends the ith vertex of the path into a linear combination of paths which differ from A only in the ith place and with coefficients depending on the 2-interval of the path between A_{i-1} and A_{i+1} (2).

We say that an A_n-module V has a local form if there exists a poset L such that V is isomorphic to $k^{Y(L)}$ as an A_n-module.

Remark. If (*) is satisfied then automatically $\sigma_i \sigma_{i+k} = \sigma_{i+k} \sigma_i$ on V. Consequently, in order to define the structure of a local module we must check the relation between σ_i and σ_{i+1}, $i = 1, \ldots, n-1$, only.

THEOREM 1. *Every finite-dimensional module over a local algebra A_n has a local form.*

The proof is based on a special construction of the Gelfand–Zeitlin basis. The problem consists in describing a class of posets for which a local module over given local algebras A_n can be defined. We shall restrict ourselves to $A_n = k(S_n)$ in this paper.

(1) $x \lessdot y$ means that $x < z < y$ implies $z = x$ or $z = y$.

(2) An *interval* $[a, b]$ of the poset is the set $\{z: a < z < b\}$; a 1-*interval* is a pair (a, b) where $a \lessdot b$, a 2-interval is defined analogously.

4. Local modules for symmetric groups and flags of ideals in a poset

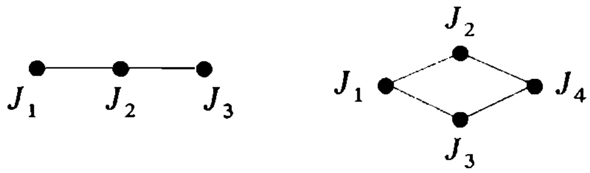
If $A_n = k(S_n)$ then any σ_i is an involution ($\sigma_i^2 = 1$) and we must check the relation $(\sigma_i\sigma_{i+1})^3 = 1$ only.

We shall consider a special class of posets, namely distributive lattices. If L is a distributive lattice then by Birkhoff's theorem there exists a poset J such that L is the lattice of its ideals ([3]). We write $L = L(J)$ in such a case. A path Λ in $L(J)$ is a flag of ideals and it is an element of $Y(L(J))$ if $\Lambda_0 = \phi, \Lambda_i = I_i, I_i = I_{i-1} \cup \{x_i\}, I_i$ an ideal in J, $i = 0, 1, \ldots, n$, $I_n = J$. Roughly speaking every element Λ of Y is determined by a sequence $\{x_i\}$ in J such that $\forall i \exists j < i \; x_j \lessdot x_i$.

LEMMA 1. *Any 2-interval in a distributive lattice has one of the two forms*:

(1) $[J_1, J_3] = \{J_1, J_2, J_3\}, \quad J_2 = J_1 \cup \{x\}, \quad J_3 = J_2 \cup \{y\}, \quad x < y.$

(2) $[J_1, J_4] = \{J_1, J_2, J_3, J_4\}, \quad J_2 = J_1 \cup \{x\}, \quad J_3 = J_1 \cup \{y\},$
$J_4 = J_1 \cup \{x, y\},$ *x and y are not comparable.*

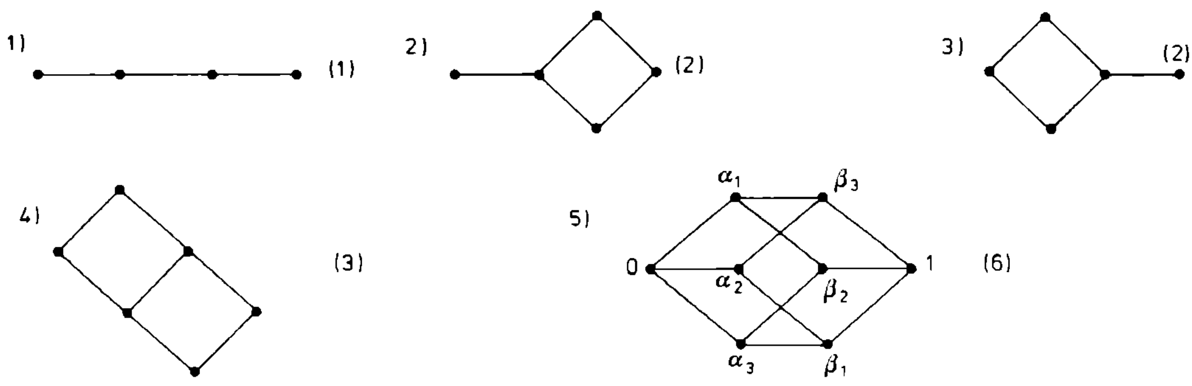


It is enough to define an action of generators σ_i on 1 and 2-dimensional subspaces which correspond to pairs of ideals of the first or second type mentioned in Lemma 1. An involution in k^1 is defined by $\varepsilon = \pm 1$ and in k^2 by a matrix of the form

$$\begin{pmatrix} \varrho & \sqrt{1-\varrho^2} \\ \sqrt{1-\varrho^2} & -\varrho \end{pmatrix}$$

where $\varrho \in k$ is such that $\sqrt{1-\varrho^2} \in k$. Therefore a representation of S_n in k^Y is determined if for every $i = 1, \ldots, n-1$, and every pair of ideals (I, I') such that $\#(I' \setminus I) = 2$, numbers $\varepsilon = \varepsilon_{(I_1, I'_1)}$ and $\varrho = \varrho_{(I_2, I'_2)}$ are given where (I_1, I'_1) is a pair of the first type and (I_2, I'_2) of the second type.

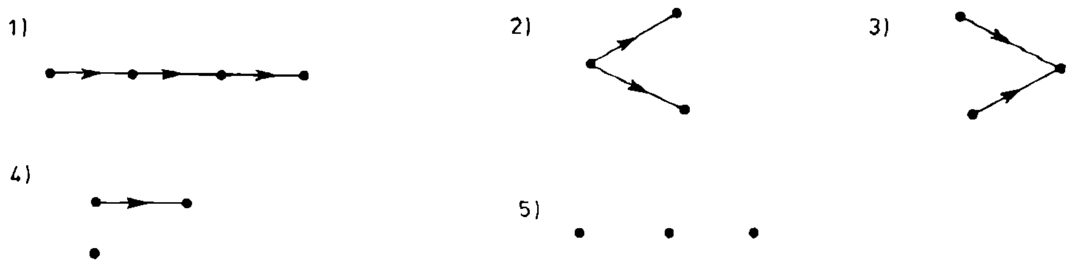
LEMMA 2. *Any 3-interval in a distributive lattice has one of the 5 forms*:



([3]) A set $I \subset J$ is an *ideal* in a poset J if $x \in I$ and $y \leqslant x$ imply $y \in I$.

In brackets we indicated the number of paths or the dimensions of the corresponding vector spaces.

The intervals listed in Lemma 2 correspond to the following posets:



Now let us assume that the involution σ_i acts on the vector space corresponding to the initial 2-intervals of a 3-interval (listed in Lemma 2) and σ_{i+1} acts on the vector space corresponding to the final 2-intervals of the same 3-interval.

THEOREM 2. *We have* $(\sigma_i \sigma_{i+1})^3 = 1$ *for the action defined above if and only if the following conditions (up to isomorphism) are satisfied in each of the cases* (1)–(5) *of Lemma* 2:

(1) $$\sigma_i = \sigma_{i+1} = \pm \mathrm{Id}$$

(2) $$\sigma_i = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{i+1} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \quad \text{or } \sigma_i = \sigma_{i+1} = \pm \mathrm{Id},$$

(3) $$\sigma_i = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad \sigma_{i+1} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or } \sigma_i = \sigma_{i+1} = \pm \mathrm{Id},$$

(4) $$\sigma_i = \begin{bmatrix} d^{-1} & \sqrt{1-d^{-2}} & 0 \\ \sqrt{1-d^{-2}} & -d^{-1} & 0 \\ 0 & 0 & \varepsilon \end{bmatrix},$$

$$\sigma_{i+1} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & (d+\varepsilon)^{-1} & \sqrt{1+(d+\varepsilon)^{-2}} \\ 0 & \sqrt{1-(d+\varepsilon)^{-2}} & -(d+\varepsilon)^{-1} \end{bmatrix}$$

where $\varepsilon = \pm 1$, $d \in \mathbf{R} \setminus \{0\}$.

(5) *We denote* $k^\sigma = V_1 \oplus V_2 \oplus V_3 = W_1 \oplus W_2 \oplus W_3$, $\dim V_j = \dim W_j = 2$, $j = 1, 2, 3$, *where* V_j *and* W_j *correspond to 2-intervals* $[0, \beta_j]$ *and* $[\alpha_j, 1]$, *respectively* (*see Lemma* 2); *then*

$$\sigma_i|_{V_j} = \sigma_{i+1}|_{W_j} = \begin{pmatrix} d_j^{-1} & \sqrt{1-d_j^{-2}} \\ \sqrt{1-d_j^{-2}} & -d_j^{-1} \end{pmatrix}, \quad j = 1, 2, 3$$

where $d_j \in \mathbf{R} \setminus \{0\}$, $d_1 + d_2 + d_3 = 0$.

COROLLARY. *The parameters which define the action of* σ_i *depend on* $J'' \setminus J' = \{x, y\}$, $x, y \in J$, *only* (*not on ideals* J', J'').

Now we can rewrite the conditions of Theorem 2 in terms of parameters $\varepsilon = \varepsilon(x, y)$ and $d = d(x, y)$. These functions are defined on the set of pairs $\{x, y\}$ where $x \lessdot y$ for the function ε and x, y are noncomparable for the function d.

Let $[J_1, J_2]$ be a 2-interval in $L(J)$, $J_1 \subset J_2 \subset J$. We shall denote by $\sigma|_{[J_1, J_2]}$ the action of σ on the 2 or 1-dimensional subspace which corresponds to $[J_1, J_2]$ (see the definition of local modules).

LEMMA 3. *The functions ε and d satisfy the following conditions*:

(1) *If* $[x, z] = \{x, y, z\}$, *then* $\varepsilon(x, y) = \varepsilon(y, z)$,

(2) $d(x, y) = d(y, x), \quad d(x_1, x_2)+d(x_2, x_3)+ \ldots +d(x_n, x_1) = 0,$

(3) *If* $z \lessdot x$, $z \lessdot y$ *and* $\varepsilon(z, x) = -\varepsilon(z, y)$ *then* $|d(x, y)| = 2$.

If $\varepsilon = \varepsilon(z, x) = \varepsilon(z, y)$, *then* $\sigma|_{[J_1, J_1 \cup \{x, y\}]} = \varepsilon \,\mathrm{Id}$.

(4) *If* $x \lessdot y$ *and* z *is incomparable with* x *and* y, *then*

$$d(z, y) = d(z, x)+\varepsilon(x, y).$$

The lemma follows from Theorem 2.

A triple (J, d, ε) where J is a finite poset and d, ε satisfy the conditions of Lemma 3 will be called *admissible*.

THEOREM 3. *Let (J, d, ε) be an admissible triple, $L(J)$ the distributive lattice of ideals of J, $\#(J) = n$, $Y = Y(L(J))$ the set of all maximal paths (flags) in $L(J)$ and $V = k^Y$. There exists the canonical $k(S_n)$-module structure on V which is determined by the action of Coxeter involutions σ_i, $i = 1, \ldots, n-1$, defined in the following way. Let $J' \subset J''$ be two ideals in J, $J''\backslash J' = \{x, y\}$, $\#(J') = i-1$, $\Lambda = (\phi, \Lambda_1, \ldots, \Lambda_{i-2}, J', J' \cup \{x\}, J'', \Lambda_{i+2}, \ldots, J) \in Y$ and e_Λ the basis element of V corresponding to Λ. Then*

(a) *for* $x \lessdot y$ *we have* $\sigma_i e_\Lambda = \varepsilon(x, y) e_\Lambda$,

(b) *for* x, y *noncomparable*, $\Lambda' = (\phi, \Lambda_1, \ldots, \Lambda_{i-2}, J', J' \cup \{y\}, J'', \Lambda_{i+2}, \ldots, J)$ *and* $W = ke_\Lambda + ke_{\Lambda'}$ *we have*

$$\sigma_{i|W} = \begin{pmatrix} \varrho & \sqrt{1-\varrho^2} \\ \sqrt{1-\varrho^2} & -\varrho \end{pmatrix}, \quad \varrho = d(x, y)^{-1}.$$

The theorem contains a rich information about $k(S_n)$-modules and the way of their construction.

EXAMPLE 1. Let J be a finite ideal (so-called Young diagram) in $Z_+ \oplus Z_+$ where $Z_+ = \{0, 1, \ldots\}$. Then $L(J)$ is the set of all Young tableaux of shape J. The functions ε and d are defined uniquely by Lemma 3 and conditions $\varepsilon((0, 0), (1, 0)) = 1 = -\varepsilon((0, 0), (0, 1))$, $d((1, 0), (0, 1)) = 2$. By Theorem 3 we obtain the irreducible representation of S_n corresponding to the diagram J in Young's orthogonal form.

EXAMPLE 2. Let J be a poset with k pairwise noncomparable points $\alpha_1, \alpha_2, \ldots, \alpha_k$; then $L(J)$ is the Boolean algebra with k atoms. It is enough to define real numbers $d(\alpha_i, \alpha_{i+1})$, $i = 1, \ldots, k-1$. The corresponding representation is the regular representation of S_k.

EXAMPLE 3. If $J = J^1 \setminus J^2$ where $J^1 \supset J^2$ are ideals in $Z_+ \oplus Z_+$, then J is a skew diagram and we obtain the obvious construction of the corresponding representation.

The same construction can be applied to representations of Hecke algebras, Coxeter groups etc.

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References

[1] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math. 682. Springer, Berlin 1978 (Russian translation, Mir, Moscow 1982).

[2] G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, Reading, Mass. 1981.

[3] A. M. Vershik and S. V. Kerov, *Characters and realizations of representations of the infinite-dimensional Hecke algebra and knot invariants*, Soviet Math. Dokl. 301 (1988), 777–780 (in Russian).