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Conjugate norms in $\mathbb{C}^n$ and related geometrical problems

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Abstract

We consider $\mathbb{C}^n$ as a normed space equipped with a complex norm F and we investigate some geometrical problems related with the notion of a conjugate norm F^*. A crucial role in our considerations is played by the classical Shmul'yan theorem on exposed points in dual spaces. Many applications of this theorem are given for different problems including characterization of linear (biholomorphic) equivalence for a class of balls in $\mathbb{C}^n$, calculation of the group of linear automorphisms (Section 4) and for problems related to the complex method of interpolation (Sections 5–7). The main result is an effective formula for interpolating norms for the couple $(\mathbb{R}^n \check{\otimes} \mathbb{C}, \mathbb{R}^n \hat{\otimes} \mathbb{C})$ (Section 5) and, more generally, for the couple $(H \check{\otimes} \mathbb{C}, H \hat{\otimes} \mathbb{C})$, where H is a real Hilbert space. In Section 3 we present connections of conjugate norms with problems of pluripotential theory and approximation theory. Here a special role is played by a class of complex norms that are natural complexifications of norms in $\mathbb{R}^n$. In Section 2 we consider some properties of such norms, in particular we prove an essential generalization of a result by Hahn and Pflug.

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To Anna and Natalia

0. Introduction

(a) Let $X = (\mathbb{R}^n, f)$ be a normed space with unit ball $E = \{x \in X : f(x) \le 1\}$. Then E is a convex symmetric body and its polar $E^* := \{x \in \mathbb{R}^n : x \cdot y \le 1 \ \forall y \in E\}$ is again a convex symmetric body (and $(E^*)^* = E$) which is the unit ball for the norm $f^*(x) := \sup_{y\in E} |x \cdot y|$. We call this norm *conjugate* or *dual* (to f) because $(\mathbb{R}^n, f^*)$ is isometrically isomorphic to the usual dual X^*. Since $f(x) = \sup_{y\in E^*} |x\cdot y|$ for $x \in \mathbb{R}^n$, we can easily define a norm $F(z) = \mathcal{F}(f, z)$ in $\mathbb{C}^n$ such that $F(x) = f(x)$ on $\mathbb{R}^n$ (here $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$). Namely, if we put

$$\mathcal{F}(f, z) = \sup_{y\in E^*} |z \cdot y|$$

we obtain a complex norm that extends the norm f. We call this norm the *natural complexification* of f. If F is a norm in $\mathbb{C}^n$ then we define its complex conjugate by

$$F^*(z) := \sup\{|z \cdot w| : F(w) \le 1\}.$$

This definition coincides with the previous one for $\mathbb{R}^{2n}$: the dual norm of F in $\mathbb{C}^n$ is equal to its dual in $\mathbb{R}^{2n}$. In this paper we consider some geometrical problems related to norms and dual norms in $\mathbb{C}^n$. We restrict our interest mainly to a class of natural complexifications of norms in $\mathbb{R}^n$. Let us present two interesting examples where this notion plays an important role.

The classical Markov inequality (an important tool in approximation theory) says that for any polynomial P,

$$|P'(x)| \le (\deg P)^2 \|P\|_{[-1,1]}, \quad x \in [-1, 1].$$

If $\|P\|_{[-1,1]} \le 1$ and P has real resp. complex coefficients, this is equivalent to saying that $\frac{1}{(\deg P)^2} P'(x) \in [-1, 1]$ resp. $\frac{1}{(\deg P)^2} P'(x) \in \overline{\mathbb{D}}$, $\mathbb{D}$ is the unit disk in $\mathbb{C}$. A far-reaching generalization for higher dimensions is the following (see also Section 3). Let E be a convex symmetric body in $\mathbb{R}^n$, so E is the unit ball in $(\mathbb{R}^n, f)$. Let P be a polynomial of n variables with real or complex coefficients such that $\deg P \ge 1$ and $\|P\|_E \le 1$. Then for all $x \in E$,

$$\frac{1}{(\deg P)^2} \operatorname{grad} P(x) \in E^* \quad \text{if } P \in \mathbb{R}[x_1, \ldots, x_n]$$

and

$$\frac{1}{(\deg P)^2} \operatorname{grad} P(x) \in \{z \in \mathbb{C}^n : \mathcal{F}(f^*, z) \le 1\} \quad \text{if } P \in \mathbb{C}[x_1, \ldots, x_n].$$

There is an interesting completion of the above fact. It is well known that a convex function f is differentiable for almost all $x \in \mathbb{R}^n$. Let G be the set of all gradients of f

at points of its differentiability. Then E^* is determined by G, namely

$$E^* = \overline{\mathrm{conv}(G)}.$$

This follows by combining two classical results: the Straszewicz and Shmul'yan theorems (see the second part of introduction and Section 1). Thus we get a relation between the gradients of polynomials on a ball E and the gradients of the norm f for which E is the unit ball. The equality $E^* = \overline{\mathrm{conv}(G)}$ is very useful if we want to calculate the dual norm f^*. Usually, it is not difficult to show $f^*(x) \le \phi(x)$, where ϕ is a norm. Much more difficult is to check that we have equality. However, we have a tool to do this: it is sufficient to check that $\phi(\operatorname{grad} f(x)) = 1$ at points of differentiability of f. We shall repeat this many times in the paper, in particular it will play a crucial role in Sections 5 and 6.

A modern approach to polynomial inequalities is based on pluripotential methods which use some extremal functions. One of the most important extremal functions is the *Siciak extremal function* Φ_E associated with compact subsets of $\mathbb{C}^n$:

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p} : p \in \mathbb{C}[z],\ \deg p \ge 1,\ \|p\|_E \le 1\}, \quad z \in \mathbb{C}^n.$$

If E is the unit ball for a norm F in $\mathbb{C}^n$ we have $\Phi_E(z) = \max(1, F(z))$. If E is the unit ball for a norm f in $\mathbb{R}^n$ the situation is much more complicated. In that case one can prove the following interesting formula:

$$\Phi_E(z) = h(\mathcal{F}(\phi, (z, i)))$$

where $h(t) = t + \sqrt{t^2 - 1}$ for $t \ge 1$ and ϕ is a norm in $\mathbb{R}^{n+1}$, $\phi(x, t) = (f(x)^2 + t^2)^{1/2}$. The reader will find in Section 3 another interesting connection between pluripotential theory and geometry of normed spaces $(\mathbb{R}^n, f)$ and their complexifications.

(b) Let now (X, f) be a Banach space over $\mathbb{R}$. Assume the norm f is Gateaux resp. Fréchet differentiable at $x_0 \in X$, i.e. there exists $d_{x_0} f \in X^*$ such that

$$d_{x_0} f(y) = \lim_{t \to 0} \frac{f(x_0 + ty) - f(x_0)}{t},$$

resp.

$$f(x_0 + h) - f(x_0) = d_{x_0} f(h) + r(h) f(h), \quad r(h) \to 0 \quad \text{as } f(h) \to 0.$$

Let $E = \{x \in X : f(x) \le 1\}$ and let

$$E^* = \{x^* \in X^* : f^*(x^*) := \sup_{x \in E} |x^*(x)| \le 1\}.$$

One can check that $d_{x_0} f$ *is an extreme point of* E^*. Indeed, we have the following basic inequality:

$$d_{x_0} f(x - x_0) \le f(x) - f(x_0)$$

for all $x \in X$. This implies $|d_{x_0} f(x)| \le d_{x_0} f(x_0) + f(x) - f(x_0)$. Putting $x = x_0,\ x = 0$ and $x = 2x_0$ we get

$$d_{x_0} f(x_0) = f(x_0) \quad \text{and} \quad |d_{x_0} f(x)| \le f(x) \quad \text{for all } x \in X,$$

which gives $f^*(d_{x_0} f) = 1$. Now, observe that

If $\ell \in E^*$ *and* $\ell(x_0) = f(x_0)$, *then* $\ell = d_{x_0} f$.

This is a consequence of $\ell(x-x_0)\leq f(x)-f(x_0)$, which is equivalent to $\ell(th)\leq f(x_0+th)-f(x_0)$, which implies $\ell = d_{x_0}f$. Now, suppose $d_{x_0}f = (1-\alpha)\ell_1+\alpha\ell_2$, where $0<\alpha<1$ and $\ell_1,\ell_2\in E^*$. Taking $x_1 = x_0/f(x_0)$ we have $\ell_1(x_1),\ell_2(x_1)\in[-1,1]$, $d_{x_0}f(x_1)=1$, which implies $\ell_1(x_1)=\ell_2(x_1)=1$. By the earlier observation, $\ell_1=\ell_2=d_{x_0}f$. So, $d_{x_0}f$ is an extreme point of E^*. What is more, if f is Gateaux differentiable at x_0, then

$$\ell\in E^*,\ \ell\neq d_{x_0}f \Rightarrow \ell(x_0)<f(x_0)=d_{x_0}f(x_0).$$

In other words, if $Y_0=\{x^*\in X^*: x^*(x_0)=f(x_0)\}$, then $Y_0\cap E^* = d_{x_0}f$. We then say that $d_{x_0}f$ is a w^*-*exposed* point of E^* and is w^*-exposed by x_0 (see e.g. [R-S]). If f is Fréchet differentiable at x_0, then we easily check that for each $\varepsilon>0$ there exists $\delta>0$ such that

$$E^*\cap\{x^*\in X^*: x^*(x_0)>f(x_0)-\delta\}\subset\{x^*\in X^*: f^*(x^*-d_{x_0}f)\leq\varepsilon\}$$

and then we say that $d_{x_0}f$ is a w^*-*strongly exposed* point of E^* and it is w^*-strongly exposed by x_0 (see also [R-S]). Recapitulating, the Gateaux differential of a norm at some point is a w^*-exposed point of the dual ball and the Fréchet differential (of a norm at some point) is a w^*-strongly exposed point of the dual ball. It is the easy part of the remarkable Shmul'yan result [SH], who discovered that the reverse statement is also true: a w^*-exposed (resp. w^*-strongly exposed) point x^* of the dual ball which is w^*-exposed (w^*-strongly exposed) by x_0 is exactly $d_{x_0}f$. In particular, the norm f is (Gateaux or Fréchet) differentiable at x_0. It is interesting that this nontrivial result is less known, even to many specialists in functional analysis. Better known are some global versions of this theorem, which are also called the (global) Shmul'yan theorem (see e.g. [DIE], [CI]) and play an important role in the geometry of Banach spaces: relations between differentiable properties of a norm in a Banach space and geometrical properties of the unit ball in the dual space. For example, if (X,f) is a reflexive Banach space then we have

PROPOSITION 0.1 (see [CI]). *The following conditions are equivalent*:

(1) X *is smooth* (*the norm* f *is Gateaux differentiable*).

(2) X^* *is strictly convex* (*the unit ball has the strictly convex boundary*).

PROPOSITION 0.2 (see [CI]). *The following conditions are equivalent*:

(1) X *is uniformly smooth* (f *is uniformly Fréchet differentiable on the unit sphere*).

(2) X^* *is uniformly convex* (*for each* $\varepsilon>0$ *there exists* $\delta>0$ *such that if* $f^*(x^*)=f^*(y^*)=1$ *and* $f^*(x^*-y^*)\geq\varepsilon$, *then* $f^*(x^*+y^*)\leq 2(1-\delta)$).

The last two conditions can be expressed in terms of the *modulus of smoothness* or *convexity* (see e.g. [DI]):

$$\varrho_f(\tau) := \tfrac{1}{2}\sup_{f(x)=f(y)=1}(f(x+\tau y)+f(x-\tau y)-2),\quad \lim_{\varrho\to 0}\frac{\varrho_f(\tau)}{\tau}=0;$$
$$\Delta_{f^*}(\varepsilon) := \tfrac{1}{2}\inf\{2-f^*(x^*+y^*): f^*(x^*)=f^*(y^*)=1,\ f^*(x^*-y^*)\geq\varepsilon\},$$
$$\Delta_{f^*}(\varepsilon)>0\quad\text{for } \varepsilon\in(0,2].$$

The precise relation between these two notions is given by the Lindenstrauss duality formula (see [DI])

$$\varrho_f(\tau) = \sup_{0<\varepsilon\le 2} [\tau\varepsilon/2 - \Delta_{f^*}(\varepsilon)], \quad \tau > 0.$$

Ruess and Stegall found very interesting applications of the local Shmul'yan theorem to characterization of some kind of extreme points in duals of operator spaces (see [R-S]; in this paper we have found information on the local Shmul'yan theorem; we do not know of any other paper, except Shmul'yan's original paper, where this result is proved).

(c) We restrict our attention mainly to the standard finite-dimensional case of Banach spaces: $\mathbb{R}^n$ and $\mathbb{C}^n$. It seems that the local Shmul'yan theorem is more interesting in that case and has many applications to different problems. The aim of our paper is to show how to use the Shmul'yan theorem in some situations where the structure of exposed points plays a crucial role.

The dual space to $(\mathbb{R}^n, f)$ is, by the Riesz theorem, isometrically isomorphic to $\mathbb{R}^n$ endowed with some norm, which we call the *conjugate* or *dual* norm. Analogously we define the conjugate norm of a complex norm in $\mathbb{C}^n$ regarded as a real norm in $\mathbb{R}^{2n}$. In Sections 1 and 2 we present basic properties of conjugate norms. If we know the dual norm, which is possible in many specific cases, applying the Shmul'yan theorem we can find exposed points in the original space $(\mathbb{R}^n, f)$. On the other hand, the Shmul'yan theorem is necessary for calculating the dual norm (see Corollary 1.4); more precisely, for an estimate from below of the conjugate norm. The Shmul'yan theorem is also useful in the case where we know the exposed points in $(\mathbb{R}^n, f)$ and we want to find the norm f: we calculate the conjugate norm f^* and next we can obtain a form of f, which is equal to f^{**}. Such a situation occurs, for example, if $\exp f = \{\ell x_0 : \ell \in G\}$, where $x_0 \in \mathbb{R}^n \setminus \{0\}$ is a fixed point and G is the group of linear isometries of some space $(\mathbb{R}^n, g)$. Note that G is isomorphic (see e.g. Proposition 1.10) to a subgroup of the group $O(n)$ of orthogonal automorphisms of $\mathbb{R}^n$ (this can be shown with the help of the notion of dual norm) but it seems that it is still an open problem which subgroups of $O(n)$ are the groups of linear isometries of some space $(\mathbb{R}^n, g)$. (In the complex case the situation is analogous.) In Sections 1 (real case) and 2 (complex case) we give some basic properties of the group of linear isometries.

Especially interesting for us is the case of complex finite-dimensional Banach spaces $\mathbb{C}^n$ equipped with a complex norm. Let Ω be a convex bounded domain in $\mathbb{C}^n$. If $n = 1$, then, by the Riemann theorem, Ω is biholomorphically equivalent to the unit disc $\mathbb{D}$. In particular, Ω has a noncompact (transitive) group of biholomorphic automorphisms and all convex domains on the plane $\mathbb{C}$ are biholomorphically equivalent. It is well known that this is not true for $n > 1$ (see, e.g., [KR]). It is rather difficult to find a convex bounded domain with noncompact automorphism group or to show that a given Ω has a compact automorphism group. Recently Bedford and Pinchuk [B-P] have classified (up to biholomorphic equivalence) all convex bounded domains which possess a noncompact automorphism group under the assumption that the boundary is smooth of finite type. All the canonical domains in [B-P] are circular, and therefore are open balls for some norms in $\mathbb{C}^n$. If we have two open balls in $\mathbb{C}^n$, then the problem of their biholomorphic equivalence is reduced, by the Braun–Kaup–Upmeier theorem (cf. [B-K]), to the question when such domains are linearly equivalent.

Since exposed points are invariant with respect to linear maps, it may be very helpful to know exposed points. In Section 4 we give examples related to a *natural* complexification of a given norm f in $\mathbb{R}^n$ (see Propositions 4.9 and 4.14). In particular, we find a form of linear automorphisms of some convex circular domains (see Propositions 4.6 and 4.12). We also give examples of domains with nonsmooth boundary (for which the Bedford–Pinchuk result does not work) which have a compact group of automorphisms (see Theorem 4.20). Here the crucial role is also played by the fact that we can find exposed points of the boundary of a given domain. However, the method we adopt for our purpose, based on Kim's scaling technique, is very laborious and therefore we consider only the case $n = 2$.

The main goal of Section 2 is to investigate the natural complexification $\mathcal{F}(f, z)$ of a norm f in $\mathbb{R}^n$ and its dual. The norm $\mathcal{F}(f, z)$ is the complex norm in $\mathbb{C}^n$ that extends the norm f and is minimal in some classes of norms which have a similar property. The problem which motivated our investigations is the characterization of such classes. In particular, we generalize an earlier result of Hahn and Pflug (cf. [H-P]), who considered the case of the Euclidean norm in $\mathbb{R}^n$. They found an explicit formula for $\mathcal{F}(|\cdot|, z)$, which, in fact, had been discovered by A. Turowicz twenty years earlier (see [D], where its generalization to the case of real Hilbert spaces is given). This is the reason why we call this norm the *Turowicz norm* T_n. Better known is its conjugate norm, which is equal to the Lie norm L_n. The unit ball with respect to the Lie norm is an example of Cartan's homogeneous domain, whence it has a noncompact group of automorphisms. Recently K. T. Kim [KI2] has shown that the unit ball with respect to the Turowicz norm has a compact automorphism group. This was the second source of our motivations. The importance of the norms $\mathcal{F}(f, z)$ is shown in Section 3. These norms are, for example, related to Markov's inequality for a convex symmetric body E in $\mathbb{R}^n$ (see Proposition 3.6) and with a complex foliation associated with the generalized Green function of the complement E in $\mathbb{C}^n$ (see Proposition 3.1). There are connections between the complex equilibrium measure λ_E asociated with a ball E and with the dual norms of the norm Λ_x defined by $\mathcal{F}(f, \cdot)$: $\Lambda_x(v) = \inf\{t > 0 : \ \mathcal{F}(f, v + itx) \le t\}$. These norms play a crucial role for some polynomial inequalities. There is also an application of the equilibrium norm to the problem of estimating the volume product $\mathrm{vol}(E) \cdot \mathrm{vol}(E^*)$.

In Section 5 we consider some problems relating to the complex method of interpolation applied to conjugate norms in $\mathbb{C}^n$. We give a few applications where the interpolation of $\mathbb{C}^n$ equipped with two conjugate norms as a couple of Banach spaces is essential (see e.g. Proposition 5.4 and the proof of Proposition 2.7(5)). The main result of this section is Theorem 5.10, where we calculate interpolating norms for the couple $(\mathbb{C}^n, L_n)$ and $(\mathbb{C}^n, T_n)$. Equivalently, we interpolate two different tensor products: projective $\mathbb{R}^n \hat{\otimes}_{\mathbb{R}} \mathbb{C}$ and injective $\mathbb{R}^n \check{\otimes}_{\mathbb{R}} \mathbb{C}$. We extend this result in Section 7 to the case of the complexification of a real Hilbert space. The interpolating norms satisfy Clarkson's type inequalities—such inequalities are related to the notion of a uniformly convex space and by duality (cf. Proposition 0.2) also to the notion of a uniformly smooth space and appropriate moduli of convexity or smoothness. We observe a more general fact: if we interpolate $(\mathbb{C}^n, F)$ and $(\mathbb{C}^n, F^*)$, then interpolating norms satisfy Clarkson's type inequalities (see Theorem 5.18) and one can obtain nontrivial estimates of moduli of convexity and smoothness (see Corollary 5.22). As an application of the complex method of interpolation we construct a norm F in $\mathbb{C}^2$ such that $(\mathbb{C}^2, F)$ is isometrically isomorphic to its dual but is not isometrically isomorphic to the Hilbert space $\mathbb{C}^2$.

The main result of Section 6 is Theorem 6.10, where we interpolate the injective and projective tensor products of $\mathbb{C}^n$ and $\mathbb{C}^k$. This result is also sharp. Let us note here that if we consider interpolation of the same kind tensor products of complex Banach spaces, then a more general result of Kouba [KO1], [KO2] can be applied. Generally, it is rather difficult to calculate the interpolating norms explicitly. In our paper we show how to apply the Shmul'yan theorem to verify that some norm is the interpolating norm for a given couple of spaces.

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1. Conjugate norms in $\mathbb{R}^n$

Let E be a convex symmetric body in $\mathbb{R}^n$ (i.e. E is convex, compact with nonempty interior and $E = -E$). Let E^* denote its polar

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \ \forall y \in E\}.$$

Here "$\cdot$" denotes the canonical inner product in $\mathbb{R}^n$. Define

$$f(x) := \sup\{|x \cdot y| : y \in E^*\} \quad \text{for } x \in \mathbb{R}^n$$

and

$$f^*(x) := \sup\{|x \cdot y| : y \in E\} \quad \text{for } x \in \mathbb{R}^n.$$

Then both functions f and f^* are norms in $\mathbb{R}^n$ and E, E^* are the unit balls for f and f^*, respectively. The following assignment gives a canonical isomorphism between the space $\mathbb{R}^n$ and its dual $(\mathbb{R}^n)^* := \mathcal{L}(\mathbb{R}^n, \mathbb{R})$:

$$\ell : \mathbb{R}^n \ni x \mapsto \{y \mapsto x \cdot y\} \in (\mathbb{R}^n)^*.$$

Since $f^* \circ \ell^{-1}$ is a norm in $(\mathbb{R}^n)^*$ induced by the norm f, we can briefly say that f^* is the dual norm in $\mathbb{R}^n$ and E^* is the dual ball in $\mathbb{R}^n$. In other words, ℓ is a linear isometry between $(\mathbb{R}^n, f^*)$ and $(\mathbb{R}^n, f)^*$.

From the definition of the dual norm one gets

PROPOSITION 1.1. *Let $f,\ f_k,\ g$ be norms in $\mathbb{R}^n$. Then*

(1) $f^{**} = f$.

(2) $f \le g$ *iff* $g^* \le f^*$.

(3) *If $f_k \to f$ uniformly on S^{n-1}, then $f_k^* \to f^*$ uniformly on S^{n-1}.*

(4) *If l is a linear automorphism of $\mathbb{R}^n$ and l^* denotes its Euclidean adjoint operator, then $(f \circ l)^* = f^* \circ (l^*)^{-1}$.*

(5) *The unit balls* $\{f \le 1\}$ *and* $\{g \le 1\}$ *are linearly homeomorphic iff* $g = f \circ l$, *where* l *is a linear automorphism of* $\mathbb{R}^n$.

(6) *If* l *is a linear mapping of* $\{f \le 1\}$ *onto* $\{g \le 1\}$, *then* l^* *is a linear homeomorphism between* $\{g^* \le 1\}$ *and* $\{f^* \le 1\}$.

(7) (Generalized Hölder inequality) $|x \cdot y| \le f(x) f^*(y)$ *for* $x, y \in \mathbb{R}^n$.

Note that the inequality in (7) is always sharp. Of course, it is useful if we can calculate the dual norm f^*. We give many examples where it is possible (mainly in $\mathbb{R}^{2n}$ and $\mathbb{C}^n$).

If E is a convex body in $\mathbb{R}^n$, then we denote by $\operatorname{extr}(E)$ and $\exp(E)$ the sets of all extreme points of E and of all points of strict convexity (which are also called *exposed* points), respectively. Recall that $x_0 \in E$ is an *extreme point* iff $E \setminus \{x_0\}$ is convex, and x_0 is an *exposed point* iff there exists a support hyperplane H of E at the point x_0 such that $H \cap E = \{x_0\}$.

The importance of extremal points in the theory of convex sets is well known. Less known is the analogous result on exposed points. It is interesting that Straszewicz's theorem [ST] was proved earlier than the famous Krein–Milman theorem [K-M]. Both results for convex bodies are presented in the proposition below.

Proposition 1.2 (see e.g. [L]). *If* E *is a convex body in* $\mathbb{R}^n$, *then*

(1) (Krein–Milman's theorem) $E = \operatorname{conv}\{\overline{\operatorname{extr}(E)}\}$.

(2) (Straszewicz's theorem) $E = \operatorname{conv}\{\overline{\exp(E)}\}$.

(3) $\exp(E) \subset \operatorname{extr}(E)$, $\overline{\exp(E)} = \overline{\operatorname{extr}(E)}$. *If* $n = 2$, *then* $\operatorname{extr}(E)$ *is compact.*

It is easily seen that extreme and exposed points are invariants of linear homeomorphisms of convex bodies.

Since a norm f is a convex function on $\mathbb{R}^n$, f is Fréchet differentiable at a point x_0 iff it is Gateaux differentiable at x_0. Let

$$\mathcal{D}(f) := \{x \in \mathbb{R}^n : f \text{ is differentiable at } x\}.$$

It is well known that $\mathbb{R}^n \setminus \mathcal{D}(f)$ has zero Lebesgue measure. Now we can formulate a very important property of the dual ball. If f is a norm in $\mathbb{R}^n$ and E denotes its unit closed ball, then we define

$$\exp f := \exp(E).$$

Proposition 1.3 (Shmul'yan's theorem in $\mathbb{R}^n$). *If* f *is a norm in* $\mathbb{R}^n$, *then*

$$\exp f^* = \{\operatorname{grad} f(x) : x \in \mathcal{D}(f)\}.$$

The proof can be found in Shmul'yan's original paper [SH], where its general version for Banach spaces is given. A slightly more general result can also be found in [RO]. We present here another simple proof of the local Shmul'yan theorem in $\mathbb{R}^n$, which is sufficient for our goals. We shall need the infinite-dimensional case of the Shmul'yan theorem only in the last section.

Proof of Proposition 1.3. Assume $x_0 \in \mathcal{D}(f)$. Without loss of generality we may assume $f(x_0) = 1$. Let $y_0 = \operatorname{grad} f(x_0)$. If $y \in E^*$, then, by 1.1(7),

$$y \cdot x_0 \le f^*(y) f(x_0) = 1.$$

Suppose $y_1 \in E^*,\ y_1 \cdot x_0 = 1$. Let $v \in \mathbb{R}^n,\ t > 0$. Then $(x_0 + tv) \cdot y_1 \le f(x_0 + tv)$ or, equivalently,

$$v \cdot y_1 \le \frac{f(x_0 + tv) - f(x_0)}{t},$$

which implies

$$v \cdot y_1 \le d_{x_0} f(v) = v \cdot y_0, \quad v \in \mathbb{R}^n.$$

But the last inequality holds if and only if $v \cdot y_1 = v \cdot y_0$. Therefore, $y_1 = y_0$. This shows that $H = \{y \in \mathbb{R}^n : y \cdot x_0 = 1\}$ is a support hyperplane at y_0 such that $E^* \cap H = \{y_0\}$. This means that $y_0 \in \exp(E^*)$.

Conversely, suppose $y_0 \in \exp(E^*)$. Then there exists an $x_0 \in \mathbb{R}^n$ such that $y_0 \cdot x_0 = 1$, $y \cdot x_0 < 1$ for $y \in E^* \setminus \{y_0\}$, and consequently $f(x_0) = 1$.

We show that f is differentiable at x_0 and $\operatorname{grad} f(x_0) = y_0$. It is enough to prove that the limit $\lim_{t\to 0} \frac{f(x_0+tv)-f(x_0)}{t}$ exists for n linearly independent vectors v. Put $v_1 = x_0$ and let $v_2, \ldots, v_n$ be an orthogonal basis of $\{y_0\}^\perp$. Then $v_1, \ldots, v_n$ are linearly independent. We have

$$\frac{f(x_0 + tv_1) - f(x_0)}{t} = 1$$

and for fixed $j \ge 2,\ t > 0$,

$$0 = (\pm v_j) \cdot y_0 \le \frac{f(x_0 \pm tv_j) - f(x_0)}{t}.$$

It is well known (see e.g. [KU]) that $\lim_{t\to 0+} \frac{f(x_0+tv)-f(x_0)}{t}$ exists for all $v \in \mathbb{R}^n$. Let $t_n \to 0$. There exist $y_n \in E^*$ such that $f(x_0 + t_n v_j) = (x_0 + t_n v_j) \cdot y_n$, and thus

$$\frac{f(x_0 + t_n v_j) - f(x_0)}{t_n} \le v_j \cdot y_n, \quad \lim_{n\to\infty} x_0 \cdot y_n = 1.$$

Since E^* is compact, we may choose $y_1 \in E^*$ and a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y_1$. This implies $x_0 \cdot y_1 = 1$ and therefore $y_1 = y_0$. For this reason, $v_j \cdot y_{n_k} \to 0$ and we obtain

$$\lim_{t\to 0+} \frac{f(x_0 + tv_j) - f(x_0)}{t} = \lim_{k\to\infty} \frac{f(x_0 + t_{n_k} v_j) - f(x_0)}{t_{n_k}} = 0.$$

Similarly,

$$\lim_{t\to 0+} \frac{f(x_0 - tv_j) - f(x_0)}{t} = 0.$$

This means that

$$\lim_{t\to 0} \frac{f(x_0 + tv_j) - f(x_0)}{t} = 0 = v_j \cdot y_0$$

for $j = 2, \ldots, n$. So, f is differentiable at x_0, and since $v_j \cdot \operatorname{grad} f(x_0) = v_j \cdot y_0$, we obtain $y_0 = \operatorname{grad} f(x_0)$. This completes the proof of the proposition.

We shall often make use of

COROLLARY 1.4. *Let f, g be norms in $\mathbb{R}^n$ such that $g(\operatorname{grad} f(x)) \le 1$ for $x \in \mathcal{D}(f)$. Then $f^* \ge g$.*

PROOF. By the Shmul'yan theorem, $g(y) \le 1$ for $y \in \exp f^*$. Hence, by 1.2(2), $g(y) \le 1$ for $f^*(y) \le 1$, which implies $f^* \ge g$.

EXAMPLE 1.5. Let $f(x) = |x_1| + |x|$, $x = (x_1, x_2) \in \mathbb{R}^2$, where $|x|$ for $x \in \mathbb{R}^n$ (or $x \in \mathbb{C}^n$) denotes the standard Euclidean norm. Then $\mathcal{D}(f) = \{x \in \mathbb{R}^2 : x_1 \neq 0\}$ and, by Proposition 1.3,

$$\exp f^* = \{(x_1/|x| + \mathrm{sign}(x_1), x_2/|x|) : x_1 \neq 0\}.$$

Since this set is not compact, we have a proper inclusion

$$\exp(E^*) \subset \mathrm{extr}(E^*).$$

REMARK 1.6. If f is a norm in $\mathbb{R}^n$ and $x \in \mathcal{D}(f)$, then

$$\mathrm{grad}\, f(x) \cdot x = f(x), \quad f^*(\mathrm{grad}\, f(x)) = 1.$$

The second equality follows from Proposition 1.3 while the first is a consequence of homogeneity of f. These two conditions determine $\mathrm{grad}\, f(x)$: if $x_0 \in \mathcal{D}(f)$ and $y_0 \cdot x_0 = f(x_0)$, $f^*(y_0) = 1$, then $y_0 = \mathrm{grad}\, f(x_0)$.

REMARK 1.7 (see [B3, Prop. 1.8]). If f is an arbitrary norm in $\mathbb{R}^n$, then there exists an increasing sequence of norms f_k such that $f_k \nearrow f$ and $f_k \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$.

PROOF. Let $E = \{f(x) \leq 1\}$. We have

$$f(x) = \sup\{|x \cdot w| : w \in E^*\} = \lim_{k\to\infty} \left(\frac{1}{\mathrm{vol}(E^*)} \int_{E^*} (x \cdot w)^{2k}\, dw \right)^{1/(2k)}$$

and we may put

$$f_k(x) = \left(\frac{1}{\mathrm{vol}(E^*)} \int_{E^*} (x \cdot w)^{2k}\, dw \right)^{1/(2k)}.$$

Denote by $\Gamma(f)$ the group of linear automorphisms (linear isometries) of the Banach space $(\mathbb{R}^n, f)$. The basic properties of this group are gathered in

PROPOSITION 1.8. (1) *The groups $\Gamma(f)$ and $\Gamma(f^*)$ are isomorphic.*
(2) *If $\ell \in \Gamma(f)$ then $|\det \ell| = 1$.*
(3) *If $f_p(x) = (\int_{E^*} |x \cdot w|^p\, dw)^{1/p}$, then $\Gamma(f) \subset \Gamma(f_p)$ for all $1 \leq p \leq \infty$.*
(4) *$\Gamma(f)$ is isomorphic to a subgroup of the orthogonal group $O(n)$.*

PROOF. (1) is a straightforward corollary to 1.1(6). Now, let $\ell \in \Gamma(f)$. Since $\mathrm{vol}(E) = \mathrm{vol}(\ell(E)) = |\det \ell|\, \mathrm{vol}(E)$, we have $|\det \ell| = 1$. We also have

$$f_p(\ell x) = \left(\int_{E^*} |x \cdot \ell^* w|^p |\det \ell^*|\, dw \right)^{1/p} = \left(\int_{\ell^*(E^*)} |x \cdot w|^p\, dw \right)^{1/p} = f_p(x),$$

which implies $\ell \in \Gamma(f_p)$. In particular, $\ell \in \Gamma(f_2)$. Observe that

$$f_2(x) = |\mathcal{I}^{1/2} x|,$$

where $\mathcal{I} = [a_{ij}] = [\int_{E^*} w_i w_j\, dw]$ is the matrix of inertia of E^* (see [M-P]; the reader can also find in [M-P] that the unit ball for the norm f_2 is the *Binet fundamental ellipsoid* of E^*). The group $\Gamma(|\cdot| \circ \mathcal{I}^{1/2})$ is equal to $\mathcal{I}^{-1/2} O(n) \mathcal{I}^{1/2}$. Hence, if $\ell \in \Gamma(f)$, then we get $\mathcal{I}^{1/2} \ell \mathcal{I}^{-1/2} = B$ with some $B \in O(n)$. The last statement is obtained by defining the map

$$\phi : \Gamma(f) \ni \ell \mapsto \mathcal{I}^{1/2} \ell \mathcal{I}^{-1/2} \in O(n).$$

Corollary 1.9. *If* $\ell \in \Gamma(f)$ *and* $\ell\mathcal{I} = \mathcal{I}\ell$ *(*$\mathcal{I}$ *is the matrix of inertia of* E^**), then* $\ell \in O(n)$*. In particular,* $\Gamma(f) \subset O(n)$ *if and only if*

$$\ell\mathcal{I} = \mathcal{I}\ell, \quad \ell \in \Gamma(f).$$

If $\Gamma(\|\cdot\|_1) \subset \Gamma(f)$*, then* $\Gamma(f) \subset O(n)$*. (Here* $\|x\|_1 = |x_1| + \ldots + |x_n|$*.)*

As another application of the norm f_2 we easily obtain a new version of the classical Mazur characterization of Euclidean norms.

Corollary 1.10 ([MA], see [A]). *If* $\Gamma(f)$ *acts transitively on* $S_E = \{f(x) = 1\}$*, then for any* $x_0 \in \mathbb{R}^n \setminus \{0\}$*,*

$$f(x) = \frac{f(x_0)}{|\mathcal{I}^{1/2}x_0|}|\mathcal{I}^{1/2}x|.$$

Remark 1.11. Let B_f be the unit ball in the space $\mathcal{L}(\mathbb{R}^n)$ of linear operators in $(\mathbb{R}^n, f)$ with the usual operator norm:

$$B_f = \{L \in \mathcal{L}(\mathbb{R}^n) : \|L\| \le 1\}, \quad \|L\| = \sup_{f(x)=1} f(Lx).$$

The following fact is well known (see e.g. [L-P] or [A]):

$$\Gamma(f) \subset \operatorname{extr}(B_f).$$

This inclusion is, in most cases, proper. For example, if $\mathcal{D}(f) = \mathbb{R}^n \setminus \{0\}$ or $\mathcal{D}(f^*) = \mathbb{R}^n \setminus \{0\}$, then (see [A]) the equality $\Gamma(f) = \operatorname{extr}(B_f)$ implies that f is the Euclidean norm in $\mathbb{R}^n$. In Section 6 we shall show, applying the Shmul'yan theorem, the classical result $\Gamma(f) = \exp(B_f)$ for the Euclidean norm in $\mathbb{R}^n$. Note also the following Lindenstrauss–Perles result (see [L-P]): If $f(y) = 1$ and $x \in \operatorname{extr}(E)$, then there exists $L \in \operatorname{extr}(B_f)$ such that $y = Lx$. This implies the following connection between the norm f and the operator norm $\|\cdot\|$:

$$f(x)\varrho(E) = \sup\{|Lx| : \|L\| \le 1\},$$

where $\varrho(E) = \frac{1}{2}\operatorname{diam}(E)$ is the radius of E. In particular, if two norms f and g give the same operator norm, then they are proportional. Note also the formula

$$\varrho(E)\varrho(E^*) = \sup\{\|L\|_2 : \|L\| \le 1\},$$

where $\|\cdot\|_2$ is the operator norm induced by the canonical Euclidean norm.

Now we show an interesting application of the notion of the matrix of inertia. We denote by $\Gamma'(f)$ the set of all $L' \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ which commute with every $L \in \Gamma(f)$. We say E *has enough symmetries* if $\Gamma'(f)$ consists only of the scalar multiples of the identity operator I_n.

Proposition 1.12. *Let* $(\mathbb{R}^n, f)$ *be a space with enough symmetries. Assume* $\mathcal{J} \in \mathrm{GL}(\mathbb{R}^n)$ *is a selfadjoint operator, positive definite such that* $\mathcal{J}\Gamma(f)\mathcal{J}^{-1} \subset O(n)$*. Then there exists a constant* $c > 0$ *such that* $\mathcal{J} = c\mathcal{I}_{E^*}^{1/2}$*, where* $\mathcal{I}_{E^*}$ *is represented by the matrix of inertia of* E^**. (If additionally* $\Gamma(f) \subset O(n)$*, then* $\mathcal{I}_{E^*} = cI_n$*.) Moreover,*

$$\mathcal{I}_E \cdot \mathcal{I}_{E^*} = \omega(E)I_n, \quad \textit{where} \quad \omega(E) = \frac{1}{n}\int_{E\times E^*} (x\cdot y)^2\,dx\,dy.$$

If we drop the assumption on enough symmetries, then the formula may be false.

Proof. For all $L \in \Gamma(f)$ we have the following equalities:

$$\mathcal{J} L \mathcal{J}^{-2} L^* \mathcal{J} = I_n, \quad \mathcal{I}_{E^*}^{1/2} L \mathcal{I}_{E^*}^{-1} L^* \mathcal{I}_{E^*}^{1/2} = I_n.$$

This directly implies that

$$\mathcal{J}^2 L \mathcal{J}^{-2} = \mathcal{I}_{E^*} L \mathcal{I}_{E^*}^{-1} \quad \text{and thus} \quad \mathcal{I}_{E^*}^{-1} \mathcal{J}^2 L = L \mathcal{I}_{E^*}^{-1} \mathcal{J}^2.$$

Hence, there exists a constant $c > 0$ such that $\mathcal{J}^2 = c^2 \mathcal{I}_{E^*}$ and so $\mathcal{J} = c\mathcal{I}_{E^*}^{1/2}$. We have $\mathcal{I}_E^{1/2} \Gamma(f^*) \mathcal{I}_E^{-1/2} \subset O(n)$. This gives

$$\mathcal{I}_E^{-1/2} \Gamma(f) \mathcal{I}_E^{1/2} \subset O(n).$$

Therefore, there exists a positive constant ω such that $\mathcal{I}_E \cdot \mathcal{I}_{E^*} = \omega I_n$. Now, it is easy to calculate the value of ω.

We now show that the assumption on enough symmetries is essential. Define

$$f(x) = |x_1| + |x_2| + |x_1 + x_2/2|, \quad E = \{x \in \mathbb{R}^2 : f(x) \le 1\}.$$

Then

$$E = \operatorname{conv}\{\pm(0, 2/3), \pm(1/2, 0), \pm(1/3, -2/3)\},$$
$$E^* = \operatorname{conv}\{\pm(0, 3/2), \pm(2, 3/2), \pm(2, -1/2)\},$$
$$\mathcal{I}_E = 3^{-5} \begin{bmatrix} 23/2 & -6 \\ -6 & 24 \end{bmatrix}, \quad \mathcal{I}_{E^*} = \begin{bmatrix} 6 & 3 \\ 3 & 37/6 \end{bmatrix}.$$

It is easily seen that $\mathcal{I}_E \cdot \mathcal{I}_{E^*} \notin \mathbb{R}_+ I_2$. Here $\Gamma(f) = \{\pm I_2\}$.

One of the fundamental results in the local theory of Banach spaces is the famous John theorem.

Proposition 1.13 ([J]; see e.g. [PI2], [T]). *If E is a convex symmetric body in $\mathbb{R}^n$, then there exists a unique ellipsoid $D_E^{\max}$ of maximal volume* (John's ellipsoid) *contained in E.*

Applying this result and Propositions 1.8 and 1.12 we show the following description of John's ellipsoids. This fact seems to be unknown to specialists in the local theory of Banach spaces.

Theorem 1.14. *Let E be the closed unit ball in the space $(\mathbb{R}^n, f)$ with enough symmetries. Let $\mathcal{I}_E \in \mathrm{GL}(\mathbb{R}^n)$ be represented by the matrix of inertia of E. Then*

$$D_E^{\max} = \varrho(\mathcal{I}_E^{1/2}(E^*))^{-1} \mathcal{I}_E^{1/2}(\mathbb{B}_n),$$
$$\operatorname{vol}(D_E^{\max}) = \varrho(\mathcal{I}_E^{1/2}(E^*))^{-n} (\det \mathcal{I}_E)^{1/2} \operatorname{vol}(\mathbb{B}_n).$$

If additionally $\Gamma(f) \subset O(n)$, then

$$D_E^{\max} = \varrho(E^*)^{-1} \mathbb{B}_n, \quad \operatorname{vol}(D_E^{\max}) = \varrho(E^*)^{-n} \operatorname{vol}(\mathbb{B}_n).$$

Proof. Let $\|x\| = |\mathcal{J}x|$ be the norm such that $D_E^{\max} = \{x \in \mathbb{R}^n : \|x\| \le 1\}$. Since the John ellipsoid is unique, $D_E^{\max}$ is invariant with respect to the group $\Gamma(f)$, i.e., $\Gamma(f) \subset \Gamma(\|\cdot\|)$. Therefore $\mathcal{J}\Gamma(f)\mathcal{J}^{-1} \subset O(n)$ and, by Proposition 1.12, $\mathcal{J} = c\mathcal{I}_E^{-1/2}$. Now, we easily calculate the constant c, which is the infimum of all α such that $f(x) \le \alpha|\mathcal{I}_E^{-1/2} x|$

for $x \in \mathbb{R}^n$. Hence,

$$c = \sup_{x\neq 0} f\left(\mathcal{I}_E^{1/2}\frac{\mathcal{I}_E^{-1/2}x}{|\mathcal{I}_E^{-1/2}x|}\right) = \sup_{|x|=1} f(\mathcal{I}_E^{1/2}x) = \varrho(\mathcal{I}_E^{1/2}(E^*)).$$

This completes the proof.

To end this section we recall the definition of some class of norms in $\mathbb{R}^2$. These norms will be used in Section 2.

Let $\Phi : \mathbb{R} \to [0,\infty)$ be an *Orlicz function* (i.e. $\Phi(0) = 0$, $\Phi(t) \to \infty$ as $t \to \infty$, Φ is even, convex) such that $\Phi(1) = 1$. Define

$$I_\Phi(x) = \Phi(x_1) + \Phi(x_2), \quad x \in \mathbb{R}^2,$$

and two standard norms in the Orlicz space $\mathbb{R}^2$:

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \le 1\} \quad \text{(Luxemburg's norm)}$$
$$\|x\|_\Phi^o = \sup\{|x\cdot y| : I_{\Phi^*}(y) \le 1\} \quad \text{(Orlicz's norm)},$$

where Φ^* is the Young conjugate of Φ:

$$\Phi^*(u) = \sup_{v>0}\{|u|v - \Phi(v)\}, \quad u \in \mathbb{R}.$$

It is easily seen that $B_\Phi = \{x \in \mathbb{R}^2 : \Phi(x_1)+\Phi(x_2) \le 1\}$ is the unit ball for the Luxemburg norm and $\|x\|_\Phi^* = \|x\|_{\Phi^*}^o$.

We also recall the following Amemiya formulas for the Orlicz and Luxemburg norms (see e.g. [R-R]):

(1) $$\|x\|_\Phi^o = \inf_{k>0}\frac{1}{k}(1 + I_\Phi(kx)),$$

(2) $$\|x\|_\Phi = \inf_{k>0}\frac{1}{k}\max(1, I_\Phi(kx)).$$

2. Conjugate norms in $\mathbb{C}^n$

Now we treat $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. This means that

$$\mathbb{R}^n \simeq \{(x_1, 0, x_2, 0, \ldots, x_n, 0) : x_j \in \mathbb{R}\} \subset \mathbb{C}^n.$$

If $z = x + iy$, then we put $\Re z := x$, $\Im z := y$ and $(x, y) := (x_1, y_1, \ldots, x_n, y_n)$. Each norm in $\mathbb{C}^n$ is a norm in $\mathbb{R}^{2n}$. For this reason we can define the conjugate norm of F in $\mathbb{C}^n$ as the conjugate norm of F in $\mathbb{R}^{2n}$.

Definition 2.1. Let F be a norm in $\mathbb{C}^n$. We set

$$F_{\mathbb{R}}^*(z) := \sup\{|x_1u_1 + y_1v_1 + \ldots + x_nu_n + y_nv_n| : F(u,v) \le 1\}$$

and call it the *conjugate norm* of F in $\mathbb{C}^n$.

We also define the conjugate norm of F over $\mathbb{C}$ as follows:

Definition 2.2.

$$F_{\mathbb{C}}^*(z) := \sup\{|z\cdot w| : F(w) \le 1\},$$

where $z\cdot w = z_1\overline{w}_1 + \ldots + z_n\overline{w}_n$ is the canonical scalar product in the Hilbert space $\mathbb{C}^n$.

Proposition 2.3. *If F is a norm in $\mathbb{C}^n$, then $F^*_{\mathbb{R}} = F^*_{\mathbb{C}}$.*

Proof. Let $z = x + iy$ and $w = u + iv$. We have $z \cdot w = x \cdot u + y \cdot v + i(y \cdot u - x \cdot v)$. Observe that

$$\begin{aligned}|z \cdot w| &= \sup_\theta |\cos\theta(x \cdot u + y \cdot v) + \sin\theta(y \cdot u - x \cdot v)| \\ &= \sup_\theta |x \cdot (\cos\theta u - \sin\theta v) + y \cdot (\sin\theta u + \cos\theta v)| \le F^*_{\mathbb{R}}(z),\end{aligned}$$

since

$$(\cos\theta u - \sin\theta v) + i(\sin\theta u + \cos\theta v) =: Z = (\cos\theta + i\sin\theta)(u + iv)$$

and $F(z) \le 1$ if $F(w) \le 1$. On the other hand,

$$F^*_{\mathbb{R}}(z) = \sup\{|\Re(z \cdot w)| : F(w) \le 1\} \le F^*_{\mathbb{C}}(z).$$

This completes the proof.

Definition 2.4. If F is a norm in $\mathbb{C}^n$, then we set $F^* := F^*_{\mathbb{R}} = F^*_{\mathbb{C}}$.

It is obvious that F^* is a norm in $\mathbb{C}^n$. $(\mathbb{R}^{2n}, F^*)$ is isometrically isomorphic to $(\mathbb{R}^{2n}, F)^*$ but, in general, $(\mathbb{C}^n, F^*)$ is not isometrically isomorphic to $(\mathbb{C}^n, F)^*$. (Note that the mapping $(\mathbb{C}^n, F)^* \ni \ell \mapsto (\ell(e_1), \ldots, \ell(e_n)) \in (\mathbb{C}^n, F^*)$ is not an isometry if F is not symmetric with respect to $\mathbb{R}^n$, i.e. $F \not\equiv \overline{F}$, where $\overline{F}(z) := F(\overline{z})$: $(\mathbb{C}^n, F)^* \simeq (\mathbb{C}^n, F^*)$ iff $F = \overline{F}$.)

An analogue of Proposition 1.1 is the following

Proposition 2.5. *Let F, F_k, G be norms in $\mathbb{C}^n$. Then*

(1) $\overline{F}^*(z) = F^*(\overline{z})$ *and* $F^{**} = F$.

(2) $F \le G$ *iff* $G^* \le F^*$.

(3) *If $F_k \to F$ uniformly on S^{2n-1}, then $F^*_k \to F^*$ uniformly on S^{2n-1}.*

(4) *If L is a linear automorphism of $\mathbb{C}^n$ and L^* denotes its Hermitian adjoint operator, then $(F \circ L)^* = F^* \circ (L^*)^{-1}$.*

(5) *The unit balls $\{F \le 1\}$ and $\{G \le 1\}$ are $\mathbb{C}$-linearly homeomorphic iff $G = F \circ L$, where L is a linear automorphism of $\mathbb{C}^n$.*

(6) *If L is a $\mathbb{C}$-linear mapping of $\{F \le 1\}$ onto $\{G \le 1\}$, then L^* is a linear isometric homeomorphism of $\{G^* \le 1\}$ and $\{F^* \le 1\}$.*

(7) (Generalized Hölder's inequality) $|z \cdot w| \le F(z)F^*(w)$ *for* $z, w \in \mathbb{C}^n$.

Remark 2.6. One can also easily check the equalities

$$\Re(\operatorname{grad} F(z) \cdot z) = F(z), \quad F^*(\operatorname{grad} F(z)) = 1$$

for $z \in \mathcal{D}(F)$. Here differentiability is understood in $\mathbb{R}^{2n}$ and

$$\operatorname{grad} F(z) = \left(\frac{\partial}{\partial x_1}F(z), \frac{\partial}{\partial y_1}F(z), \ldots \frac{\partial}{\partial x_n}F(z), \frac{\partial}{\partial y_n}F(z)\right) = 2\left(\frac{\partial}{\partial \overline{z}_1}F(z), \ldots, \frac{\partial}{\partial \overline{z}_n}F(z)\right).$$

Denote by $\Gamma(F)$ the group of all complex linear isometries of the Banach space $(\mathbb{C}^n, F)$. The basic properties of this group are analogous to those for $\mathbb{R}^n$ and are gathered below.

Proposition 2.7. *Let F be a norm in $\mathbb{C}^n$. Let E denote its unit ball and E^* the dual ball. Then*

(1) $\Gamma(F) \simeq \Gamma(F^*)$.

(2) *If* $\ell \in \Gamma(F)$, *then* $|\det \ell| = 1$.

(3) $\Gamma(F)$ *is isomorphic to a subgroup of the unitary group* $O(n)$. *This embedding is given by*

$$\Gamma(F) \ni \ell \mapsto \mathcal{I}^{1/2} \ell \mathcal{I}^{-1/2} \in U(n),$$

where $\mathcal{I} \in \mathrm{GL}(\mathbb{C}^n)$ *is represented* (*in the canonical basis*) *by*

$$\mathcal{I} = \Big[\int_{E^*} z_i \overline{z}_j \, dz \Big].$$

(4) *If* $\ell \in \Gamma(F)$ *and* $\mathcal{I}\ell = \ell\mathcal{I}$, *then* $\ell \in U(n)$. *In particular*, $\Gamma(F) \subset U(n)$ *iff* $\mathcal{I}\ell = \ell\mathcal{I}$ *for all* $L \in \Gamma(F)$.

(5) *If* $\ell, \ell^* \in \Gamma(F)$, *then* $\ell \in U(n)$. *In particular*, $\Gamma(F) \subset U(n)$ *iff* $\ell^* \in \Gamma(F)$ *for all* $\ell \in \Gamma(F)$.

(6) *If* $\ell \in \mathrm{GL}(\mathbb{C}^n)$, *then* $\ell \in \Gamma(F)$ *iff* $\|\ell\|_F = 1$ *and* $|\det L| = 1$, *i.e.*

$$\Gamma(F) = \{\ell \in \mathrm{extr}(B_F) : |\det \ell| = 1\}.$$

Proof. The proof of (1)–(4) goes along the same lines as in the real case. A short proof of (5) will be given in Section 5. The last property is a simple consequence of a much more general Carathéodory–Cartan–Kaup–Wu theorem for holomorphic mappings (see e.g. [KR]).

We are interested in the following question:

When does the equality $F^*(z) = F(Lz)$ *hold for all* $z \in \mathbb{C}^n$, *where* L *is a linear automorphism of* $\mathbb{C}^n$? *Is it true that* $F(z) = |Kz|$ *for some* $K \in \mathrm{GL}(\mathbb{C}^n)$?

If L is fixed, then the answer depends on the properties of L.

Proposition 2.8. *Let F be a norm in $\mathbb{C}^n$ such that $F^*(z) = F(Lz)$ for some $L \in \mathrm{GL}(\mathbb{C}^n)$. Then*

(1) *If* $L = L^*$ *and* $L \geq 0$, *then* $F(z) = |L^{-1/2}z|$.

(2) *If L is a unitary mapping, then* $F(z) = F(L^2 z)$. *If additionally* $I_n \in \overline{\{L^{2k+1}\}}_{k\in\mathbb{Z}}$, *then* $F(z) = |z|$.

Proof. Consider the basic case where $L = I_n$. Then, by the generalized Hölder inequality, we have $|z|^2 = |z \cdot z| \leq F(z)F^*(z) = F(z)^2$, which gives $|z| \leq F(z)$ and, by 2.5(2), $F(z) = F^*(z) \leq |z|$. Now let L be a selfadjoint, positive definite operator. Then $L^{1/2}$ is also a selfadjoint operator. Put $G(z) = F((L^{1/2})^* z) = F(L^{1/2}z)$. We now have

$$G^*(z) = F^*(L^{-1/2}z) = F(L^{1/2}z) = G(z).$$

This implies $G(z) = |z|$, and therefore $F(z) = |L^{-1/2}z|$.

Now assume L is a unitary operator. Observe that if $F^*(z) = F(Lz)$ with some $L \in \mathrm{GL}(\mathbb{C}^n)$, then

$$F(z) = F(L(L^*)^{-1}z).$$

In particular, if L is unitary, then $F(z) = F(L^2 z)$ and thus $F^*(z) = F(L^{2k+1}z)$, for any $k \in \mathbb{Z}$. If $I_n \in \overline{\{L^{2k+1}\}_{k\in\mathbb{Z}}}$, then $F^*(z) = F(z)$, which implies $F(z) = |z|$. The proof is complete.

Remark 2.9. If $n \geq 4$ is even, $n = 2k$ and F is an arbitrary norm in $\mathbb{C}^n$, then the norm G defined by

$$G(z_1, \ldots, z_n) = (F(z_1, \ldots, z_k)^2 + F^*(z_{k+1}, \ldots, z_n)^2)^{1/2}$$

satisfies $G^*(z) = G(Lz)$ with $L(z) = (z_{k+1}, \ldots, z_n, z_1, \ldots, z_k)$ and the norm

$$G_1(z, z_{n+1}) = (G(z)^2 + |z_{n+1}|^2)^{1/2}$$

satisfies

$$G_1^*(z, z_{n+1}) = G_1(L_1(z, z_{n+1}))$$

with $L_1(z, z_{n+1}) = (Lz, z_{n+1})$. So, if $n \geq 4$, there exist a number of norms g in $\mathbb{C}^n$ which satisfy $G^*(z) = G(Lz)$ (for some L) and are not of the form $|Kz|$. The same is true for $n = 2, 3$ but the construction of an example for $n = 2$ is slightly more delicate. It will be given in Section 5 basing on some interpolation techniques.

Now we define a *natural* complexification of a given norm in $\mathbb{R}^n$.

Definition 2.10. Let f be a norm in $\mathbb{R}^n$ and f^* its conjugate norm. Define

$$\mathcal{F}(f, z) = \mathcal{F}(z) := \sup\{|z \cdot \omega| : f^*(\omega) \leq 1\}.$$

It can be easily seen that $\mathcal{F}$ is a norm in $\mathbb{C}^n$ such that $\mathcal{F}_{|\mathbb{R}^n} = f$ and

$$\begin{aligned}\mathcal{F}(f, z) &= \max_\theta f(\cos\theta x - \sin\theta y)\\ &= \max_\theta \max(f(\cos\theta x - \sin\theta y), f(\sin\theta x + \cos\theta y))\\ &= \max_\theta \max(f(\Re(e^{i\theta} z)), f(\Im(e^{i\theta} z))), \quad z = x + iy \in \mathbb{C}^n.\end{aligned}$$

Less obvious is the fact that $\mathcal{F}^*$ extends the norm f^*:

Proposition 2.11. $\mathcal{F}^*_{|\mathbb{R}^n} = f^*$.

Proof. We have

$$\mathcal{F}^*(z) = \sup\{|z \cdot w| : \mathcal{F}(w) \leq 1\} = \sup\{|z \cdot w| : |w \cdot \omega| \leq 1\ \forall f^*(\omega) \leq 1\}.$$

Hence,

$$\begin{aligned}\mathcal{F}^*(x) &\geq \sup\{|x \cdot y| : y \in \mathbb{R}^n,\ |y \cdot \omega| \leq 1\ \forall f^*(\omega) \leq 1\}\\ &= \sup\{|x \cdot y| : f(y) \leq 1\} = f^*(x).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{F}^*(x) &= \sup\{|(x + i0) \cdot w| : \mathcal{F}(w) \leq 1\} = \sup\{|x \cdot u| : \mathcal{F}(u, v) \leq 1\}\\ &= \sup\{|x \cdot u| : |(u + iv) \cdot \omega| \leq 1\ \forall f^*(\omega) \leq 1\}\\ &\leq \sup\{|x \cdot u| : |u \cdot \omega| \leq 1\ \forall f^*(\omega) \leq 1\} = \sup\{|x \cdot u| : f(u) \leq 1\} = f^*(x).\end{aligned}$$

Corollary 2.12. *If $f(x) = |x|$, then $\mathcal{F}(x) = \mathcal{F}^*(x) = |x|$ for all $x \in \mathbb{R}^n$.*

We also have

Proposition 2.13. *Let f, g be norms in $\mathbb{R}^n$ with $f \le g$. Then*

$$\mathcal{F}(f,\cdot) \le \mathcal{F}(g,\cdot), \qquad \mathcal{F}^*(f^*,\cdot) \le \mathcal{F}^*(g^*,\cdot), \tag{1}$$

$$\mathcal{F}(f,\cdot) \le \mathcal{F}^*(f^*,\cdot), \qquad \mathcal{F}(f^*,\cdot) \le \mathcal{F}^*(f,\cdot). \tag{2}$$

Proof. (1) The first inequality in (1) immediately follows from the definition. We deduce the second from the first by Propositions 1.1(2) and 2.5(2).

(2) We have

$$\begin{aligned}\mathcal{F}^*(f^*,z) &= \sup\{|z\cdot w| : \mathcal{F}(f^*,w) \le 1\}\\ &\ge \sup\{|z\cdot w| : w\in\mathbb{R}^n,\ \mathcal{F}(f^*,w)\le 1\} = \sup\{|z\cdot\omega| : f^*(\omega)\le 1\} = \mathcal{F}(f,z).\end{aligned}$$

The second inequality is a consequence of the first.

The next result, which gives another point of view on some known facts, plays a crucial role in the investigations of the norms $\mathcal{F}(f,\cdot)$.

Proposition 2.14 (see e.g. [BO], [B-S] or [P1]). *If f is a norm in $\mathbb{R}^n$, then*

$$\mathcal{F}^*(f^*,z) = \|z\|_{c(f)},$$

where $\|z\|_{c(f)}$ is the crossnorm for f,

$$\|z\|_{c(f)} := \inf\Big\{\sum |\alpha_j| f(\omega_j) : z = \sum \alpha_j\omega_j,\ \omega_j\in\mathbb{R}^n,\ \alpha_j\in\mathbb{C}\Big\}.$$

It follows from the definition of the crossnorm that it is the largest norm in $\mathbb{C}^n$ which extends f. Geometrically, this means that the open ball $\{\|z\|_{c(f)} < 1\}$ is the smallest convex circular domain in $\mathbb{C}^n$ whose intersection with $\mathbb{R}^n$ is $\{f(x) < 1\}$. The fact that $\mathbb{C}^n$ endowed with the norm $\mathcal{F}^*(f^*,\cdot)$ may be interpreted as the *projective* tensor product $(\mathbb{R}^n, f)\ \hat{\otimes}_{\mathbb{R}}\ (\mathbb{R}^2, |\cdot|)$ has also other important consequences. We consider them later. Note also that $(\mathbb{C}^n, \mathcal{F}(f,\cdot))$ may be interpreted as the so-called *injective* tensor product $(\mathbb{R}^n, f)\ \check{\otimes}_{\mathbb{R}}\ (\mathbb{R}^2, |\cdot|)$ (see e.g. [P1]). The following inequality holds for every norm f:

$$\|z\|_{c(f)} \le \min_\theta (f(\cos\theta x - \sin\theta y) + f(\sin\theta x + \cos\theta y)),$$

where equality holds (for all $z\in\mathbb{C}^n$) if f is the Euclidean norm.

Now we investigate some extremal properties of the norms $\mathcal{F}(f,\cdot)$ and their duals. We need

Proposition 2.15. *Let f be a norm in $\mathbb{R}^n$. Then*

(1) *The functions* $\max(f(x), f(y))$, $f(x)+f(y)$ *and* $(f(x)^2+f(y)^2)^{1/2}$ *of* (x,y) *are norms in* $\mathbb{R}^{2n}$ *and*

$$(\max(f(x),f(y)))^* = f^*(x)+f^*(y), \quad (f(x)+f(y))^* = \max(f^*(x),f^*(y)),$$
$$[(f(x)^2+f(y)^2)^{1/2}]^* = [f^*(x)^2+f^*(y)^2]^{1/2},$$

where * is taken in the sense of $\mathbb{R}^{2n}$.

(2) *We have*

$$\begin{aligned}\max(f(x),f(y)) &\le \max_\theta \max(f(\Re(e^{i\theta}z)), f(\Im(e^{i\theta}z))) = \mathcal{F}(f,z)\\ &\le (f(x)^2+f(y)^2)^{1/2} \le \mathcal{F}^*(f^*,z)\\ &\le \min_\theta (f(\Re(e^{i\theta}z)) + f(\Im(e^{i\theta}z))) \le f(x)+f(y).\end{aligned}$$

Proof. The first statement of (1) is obvious. For the second observe that

$$(\max(f(x),f(y)))^* = \sup\{|x\cdot u + y\cdot v| : f(u), f(v) \le 1\} \le f^*(x)+f^*(y).$$

By Shmul'yan's theorem, we have

$$\exp(\max(f(x),f(y))^*) = \{\operatorname{grad} f(x) + i0 : x\in\mathcal{D}(f)\}\cup\{0+i\operatorname{grad} f(x) : x\in\mathcal{D}(f)\}.$$

This implies, by 1.4,

$$(\max(f(x),f(y)))^* \ge f^*(x)+f^*(y).$$

The second equality in (1) immediately follows from the first applied to f^*. If we apply Shmul'yan's and Straszewicz's theorems to the norm $(f(x)^2+f(y)^2)^{1/2}$, then by 1.4 we obtain

$$[(f(x)^2+f(y)^2)^{1/2}]^* \ge [f^*(x)^2+f^*(y)^2]^{1/2}.$$

Now, by the generalized Hölder inequality, we have

$$\begin{aligned}[(f(x)^2+f(y)^2)^{1/2}]^* &= \sup\{|x\cdot u+y\cdot v| : f(u)^2+f(v)^2\le 1\}\\ &= \sup_{f(v)\le 1}\{|x\cdot u|+|y\cdot v| : f(u)\le (1-f(v)^2)^{1/2}\}\\ &\le \sup_{f(v)\le 1}\{f^*(x)(1-f(v)^2)^{1/2}+f^*(y)f(v)\} = [f^*(x)^2+f^*(y)^2]^{1/2}.\end{aligned}$$

This completes the proof of (1). The first two inequalities of (2) easily follow from the definition of $\mathcal{F}(f,z)$. Applying these to f^* and using (1) gives the last two inequalities of (2).

One can also prove the following generalization of Proposition 2.15(1).

Proposition 2.16. *Let $\Phi:\mathbb{R}\to[0,\infty)$ be an Orlicz function such that $\Phi(1)=1$. Then $F(x,y)=\|(f(x),f(y))\|_\Phi$ is a norm in $\mathbb{R}^{2n}$ and*

$$F^*(x,y) = \|(f^*(x),f^*(y))\|^o_{\Phi^*}.$$

In particular, if $p\ge 1$, then

$$[(f(x)^p+f(y)^p)^{1/p}]^* = [f^*(x)^q+f^*(y)^q]^{1/q},$$

where $1/p+1/q=1$.

Proof. It is obvious that F is a norm in $\mathbb{R}^{2n}$. We have

$$\begin{aligned}F^*(x,y) &= \sup\{|x\cdot u+y\cdot v| : \Phi(f(u))+\Phi(f(v))\le 1\}\\ &\le \sup\{f^*(x)f(u)+f^*(y)f(v) : \Phi(f(u))+\Phi(f(v))\le 1\} = \|(f^*(x),f^*(y))\|^o_{\Phi^*}.\end{aligned}$$

Assume that $\|\cdot\|_\Phi$ and f are differentiable on $\mathbb{R}^2\setminus\{0\}$ and $\mathbb{R}^n\setminus\{0\}$, respectively (we refer to [G-H] for criteria of differentiability of the Luxemburg norm). Then F is differentiable on $\mathbb{R}^{2n}\setminus\{0\}$ with

$$\operatorname{grad} F(x,y) = D_1\|\cdot\|_\Phi(f(x),f(y))\operatorname{grad} f(x) + iD_2\|\cdot\|_\Phi(f(x),f(y))\operatorname{grad} f(y).$$

By 1.4 we get

$$F^*(x,y)\ge \|(f^*(x),f^*(y))\|^o_{\Phi^*}.$$

If $\|\cdot\|_\Phi$ is differentiable and f is an arbitrary norm, then we obtain the same inequality by approximating f by smooth norms f_k (see Remark 1.7). Finally, if $\|\cdot\|_\Phi$ is not smooth, we can approximate $\|\cdot\|_{\Phi}$ by $\|\cdot\|_{\Phi_k}$ which are differentiable, $\Phi_k(1)=1$ and $\|\cdot\|^o_{\Phi_k^*}\to\|\cdot\|^o_{\Phi^*}$ uniformly on S^1. This completes the proof.

REMARK 2.17. We also note the following generalization of Proposition 2.15(2):

$$[(f(x_1)^p+\ldots+f(x_k)^p)^{1/p}]^*=[f^*(x_1)^q+\ldots+f^*(x_k)^q]^{1/q}$$

for all $(x_1,\ldots,x_k)\in(\mathbb{R}^n)^k$. The proof is analogous to that of Proposition 2.15.

LEMMA 2.18 (Generalized Hahn and Pflug lemma [H-P]). *Let Φ be an Orlicz function such that $\Phi(1)=1$ and Φ is differentiable at* 0. *Let f be a norm in $\mathbb{R}^n$ and let N be a norm in $\mathbb{C}^n$ such that $N_{|\mathbb{R}^n}=f$ and either $N(z)\le\|(f(x),f(y))\|_{\Phi}$ or $N(z)\le\|(f(x),f(y))\|^o_{\Phi^*}$ for $z\in\mathbb{C}^n$. Then*

$$N(z)\ge\max(f(x),f(y)).$$

In particular, this is true if $N(z)\le(f(x)^p+f(y)^p)^{1/p}$ for some fixed $p>1$.

PROOF. We apply the method from [H-P], where this lemma was implicitly formulated and proved in the case where $f(x)=|x|$ and $\Phi(t)=t^2$.

If $x=0$ or $y=0$, then the statement trivially holds. If $x\ne0$ and $y\ne0$, then we define $\phi(\xi,\eta):=N(\xi x/f(x)+i\eta y/f(y))$ for $(\xi,\eta)\in\mathbb{R}^2$. It is clear that ϕ is a norm in $\mathbb{R}^2$. We have, by our first assumption, $\phi(\xi,\eta)\le\|(\xi,\eta)\|_{\Phi}$ and $\phi(\xi,\eta)=\|(\xi,\eta)\|_{\Phi}=1$ for $(\xi,\eta)=(\pm1,0),(0,\pm1)$. This implies $\{\|(\xi,\eta)\|_{\Phi}\le1\}\subset\{\phi(\xi,\eta)\le1\}$. But the Luxemburg norm $\|\cdot\|_{\Phi}$ is differentiable at points $(\xi,\eta)=(\varepsilon,0),(0,\varepsilon),\ \varepsilon=\pm1$, with tangent lines given by $\xi=\varepsilon$ or $\eta=\varepsilon$, respectively (this is a consequence of the second Amemiya formula). It is easily seen that such tangent lines must necessarily support the ball $\{\phi(\xi,\eta)\le1\}$ at the boundary points $(\varepsilon,0)$ and $(0,\varepsilon)$. This implies $\{\phi(\xi,\eta)\le1\}\subset\{\max(|\xi|,|\eta|)\le1\}$ or equivalently $\phi(\xi,\eta)\ge\max(|\xi|,|\eta|)$. Putting $\xi=f(x),\ \eta=f(y)$ completes the proof of this case.

Now observe that the assumption $\Phi(1)=1$ implies the strict convexity of points $\pm e_1=(\pm1,0)$, $\pm e_2=(0,\pm1)$ for the norm $\|(\xi,\eta)\|_{\Phi}$. Since $\|\pm e_i\|^o_{\Phi^*}=1$ and $(\pm e_i)\cdot(\pm e_i)=\|\pm e_i\|_{\Phi}$, the points $\pm e_i$ are points of differentiability for $\|(\xi,\eta)\|^o_{\Phi^*}$. For this reason the next steps of the proof of this case are the same as of the previous one.

We have the following characterization of minimality of the norms $\mathcal{F}(f,z)$.

THEOREM 2.19. *Let f be a norm in $\mathbb{R}^n$ and let N be a norm in $\mathbb{C}^n$ such that $N(x)=f(x)$ for $x\in\mathbb{R}^n$. Then the following conditions are equivalent.*

(1) *$\mathcal{F}(f,z)\le N(z)$ for all $z\in\mathbb{C}^n$.*
(2) *$\max(f(x),f(y))\le N(z)$ for all $z\in\mathbb{C}^n$.*
(3) *$N^*(z)\le f^*(x)+f^*(y)$ for all $z\in\mathbb{C}^n$.*
(4) *$N^*(\operatorname{grad} f(x))\le1$ for all $x\in\mathcal{D}(f)$.*
(5) *$N^*(x)=f^*(x)$ for all $x\in\mathbb{R}^n$.*
(6) *$N^*(z)\le\mathcal{F}^*(f,z)$ for all $z\in\mathbb{C}^n$.*

PROOF. The implications (1)⇒(2)⇒(3) follow easily from Propositions 2.15 and 2.5. (3) implies (4), by Proposition 1.3. Assume now (4). By Proposition 1.4 we have $N^*(x)\le f^*(x)$ for $x\in\mathbb{R}^n$. On the other hand,

$$N^*(x) = \sup\{|x\cdot u + 0\cdot v| : N(u+iv) \le 1\}$$
$$\ge \sup\{|x\cdot u| : N(u) \le 1\} = \sup\{|x\cdot u| : f(u)\le 1\} = f^*(x).$$

(5)⇒(6) is obvious and (6)⇒(1) is derived from Proposition 2.5. This completes the proof.

Corollary 2.20. *If f, g are two norms in $\mathbb{R}^n$, then*

(1) $\mathcal{F}(f+g,\cdot) \le \mathcal{F}(f,\cdot) + \mathcal{F}(g,\cdot)$.
(2) $\mathcal{F}(\max(f,g),\cdot) = \max(\mathcal{F}(f,\cdot),\mathcal{F}(g,\cdot))$.

Proof. We have $\mathcal{F}(f,x)+\mathcal{F}(g,x) = f(x)+g(x)$ and

$$\mathcal{F}(f,z)+\mathcal{F}(g,z) \ge \max(f(x),f(y)) + \max(g(x),g(y)) \ge \max(f(x)+g(x), f(y)+g(y)),$$

which gives (1). Similarly, $\max(\mathcal{F}(f,x),\mathcal{F}(g,x)) = \max(f(x),g(x))$ and

$$\max(\mathcal{F}(f,z),\mathcal{F}(g,z)) \ge \max(\max(f(x),g(x)),(f(y),g(y))),$$

which gives $\mathcal{F}(\max(f,g),\cdot) \le \max(\mathcal{F}(f,\cdot),\mathcal{F}(g,\cdot))$. The opposite inequality is a consequence of Theorem 2.12(1).

From the point of view of the theory of tensor products, Theorem 2.18 gives equivalent conditions for N to be a so-called *reasonable* norm for $(\mathbb{R}^n, f)\otimes_{\mathbb{R}}\mathbb{C}$ (cf. [L-C]). If a norm N satisfies these conditions, we briefly say that N is reasonable for f. In particular, $\mathcal{F}(f,\cdot)$ is the smallest reasonable norm for f.

As an application, we obtain the following extension of a result from [H-P].

Corollary 2.21. *Let f be a norm in $\mathbb{R}^n$ and let N be a norm in $\mathbb{C}^n$ such that $N(x) = f(x)$ for $x\in\mathbb{R}^n$ and either $N(z)\le \|(f(x),f(y))\|_{\Phi}$ or $N(z)\le \|(f(x),f(y))\|^{o}_{\Phi^*}$ with some Orlicz function Φ which satisfies the assumptions of Lemma* 2.18. *Then N is reasonable for f.*

Corollary 2.22. *Let f be a norm in $\mathbb{C}^n$. If there exists a smallest norm N_0 in $\mathbb{C}^n$ that extends the norm f, then necessarily $N_0 = \mathcal{F}(f,\cdot)$.*

Proof. We have $N_0(z)\le \mathcal{F}(f,z)\le (f(x)^2+f(y)^2)^{1/2}$ for each z. Thus, by Corollary 2.20, $N_0(z)\ge \mathcal{F}(f,z)$ and we get equality.

Remark 2.23. The following natural question arises: when does the smallest norm N_0 extending a norm f exist? We shall return to this question in a moment.

Now consider the special case $f(x) = |x|$. The crucial role in the investigation of complexifications of the Euclidean norm is played by the lemma below. Part (1) is a special case of the Drużkowski result (see [D]) for real Hilbert spaces. We refer to this part as to the *Drużkowski lemma*. The second part has been observed by the author (see also Section 6).

Lemma 2.24. *If $z\in\mathbb{C}^n$ is an arbitrary vector then*

(1) (Drużkowski [D]) *There exist $q\in\mathbb{R}$ and $z' = x'+iy'\in\mathbb{C}^n$ such that $z = e^{iq}z'$, $x'\cdot y' = 0$.*

(2) *Moreover,*

$$\min(|x'|,|y'|) = \lambda_1(z) = \frac{1}{\sqrt{2}}(|z|^2 - |z^2|)^{1/2},$$

$$\max(|x'|, |y'|) = \lambda_2(z) = \frac{1}{\sqrt{2}}(|z|^2 + |z^2|)^{1/2},$$

where $z^2 = z_1^2 + \ldots + z_n^2$.

As an application we get

Corollary 2.25. *For all* $z \in \mathbb{C}^n$, *we have*

$$\mathcal{F}(|\cdot|, z) = \frac{1}{\sqrt{2}}(|z|^2 + |z^2|)^{1/2},$$
$$\mathcal{F}^*(|\cdot|, z) = \frac{1}{\sqrt{2}}(|z|^2 - |z^2|)^{1/2} + \frac{1}{\sqrt{2}}(|z|^2 + |z^2|)^{1/2}.$$

The second function is called the *Lie norm* (see e.g. [D]) and will be denoted by $L_n(z)$. We denote the first one by $T_n(z)$ in honour of Professor A. Turowicz who was the first to obtain this formula (see [D]).

Proof. By Lemma 2.24, it is enough to prove the above formulas for $z = x + iy$ with $x \cdot y = 0$. Then $T_n(z) = \max(|x|, |y|)$ and $L_n(z) = |x| + |y|$. We may also assume that x and y are linearly independent. Let $\{v_1 = x/|x|,\ v_2 = y/|y|,\ v_3, \ldots, v_n\}$ be an orthonormal basis in $\mathbb{R}^n$. For $u \in S^{n-1},\ u = \alpha_1 v_1 + \ldots + \alpha_n v_n$, we have

$$\begin{aligned}\mathcal{F}(|\cdot|, x) &= \sup\{((x\cdot u)^2 + (y \cdot u)^2)^{1/2} : u \in S^{n-1}\}\\ &\le \sup\{(\alpha_1^2|x|^2 + \alpha_2^2|y|^2)^{1/2} : \alpha_1^2 + \alpha_2^2 \le 1\} = \max(|x|, |y|).\end{aligned}$$

Hence, by 2.14(2), $\mathcal{F}(|\cdot|, z) = T_n(z)$. By 2.14(2), we also have $\mathcal{F}^*(|\cdot|, z) \le |x| + |y| = L_n(z)$. Let $u = \frac{1}{\sqrt{2}}(v_1 - v_2)$, $v = \frac{1}{\sqrt{2}}(v_1 + v_2)$. Then $T_n(u + iv) = 1$ and $|(x+iy)\cdot(u+iv)| = |x| + |y|$, which gives the inequality $\mathcal{F}^*(|\cdot|, z) \ge L_n(z)$ and so equality holds. This completes the proof.

It follows from Corollary 2.25 that T_n and L_n are conjugate (see [D]). This seems to be unknown to authors who have recently investigated the norm T_n (see [H-P], [KI1]). Applying Shmul'yan's theorem we can easily derive:

Proposition 2.26. *We have the following formulas*:

$$\operatorname{grad} T_n(z) = \frac{1}{2T_n(z)}\left\{z + \frac{z^2}{|z^2|}\overline{z}\right\} \quad \textit{for } z^2 \ne 0,$$
$$\operatorname{grad} L_n(z) = \frac{1}{L_n(z)}\left\{z + \frac{|z|^2 z - z^2\overline{z}}{(|z|^4 - |z^2|^2)^{1/2}}\right\} \quad \textit{for } |z^2| < |z|^2,$$
$$\exp L_n = \{e^{i\theta}\omega : \omega \in S^{n-1},\ \theta \in \mathbb{R}\},$$
$$\exp T_n = \{x + iy : x, y \in S^{n-1},\ x\cdot y = 0\},$$
$$S_{T_n} = \{T_n(z) = 1\} = \{e^{i\theta}(tx + iy) : \theta \in \mathbb{R},\ t \in [-1, 1],\ x, y \in S^{n-1},\ x \cdot y = 0\}.$$

Now we show that, for some norms f in $\mathbb{R}^n$, the smallest norm N_0 in $\mathbb{C}^n$ that extends f does not exist. First observe that if N is reasonable for f, then the same is true for $\overline{N}$. In particular, if N is an arbitrary norm in $\mathbb{C}^n$ such that $N = f$ on $\mathbb{R}^n$, then the norm $\frac{1}{2}(N + \overline{N})$ is reasonable for f. So, if there exists a norm N such that $N = f$ on $\mathbb{R}^n$ and $N(z_0) < \mathcal{F}(f, z_0)$, then necessarily N is not symmetric. If $n = 2$, then such a norm was constructed (by R. Zeinstra) in [H-P] for the Euclidean norm. In this case (see [H-P]) we have

$$T_2(z_1, z_2) = \tfrac{1}{2}|z_1 - iz_2| + \tfrac{1}{2}|z_1 + iz_2|.$$

Consider the equation

$$(*) \qquad N(z) + N(\overline{z}) = 2T_2(z), \quad z \in \mathbb{C}^2,$$

where N is a norm in $\mathbb{C}^2$. Zeinstra's result is contained in

PROPOSITION 2.27. *If N is a norm in $\mathbb{C}^2$ that satisfies $(*)$, then there exists $\alpha \in (-1, 1)$ such that*

$$N(z) = N(\alpha, z) = \frac{1-\alpha}{2}|z_1 - iz_2| + \frac{1+\alpha}{2}|z_1 + iz_2|.$$

If $\alpha \neq 0$, then

$$N(\alpha, (1, \operatorname{sign}(\alpha)i)) = 1 - |\alpha| < 1 = T_2(1, \pm i).$$

Moreover, $\sup_{\alpha\in(-1,1)} N(\alpha, z) = L_2(z)$.

PROOF. We only prove the first part. Define $\alpha := 1 - N(1, i)$. Then $|\alpha| < 1$ and $N(1, -i) = 1 + \alpha$. Since

$$(z_1, z_2) = \frac{z_1 - iz_2}{2}(1, i) + \frac{z_1 + iz_2}{2}(1, -i),$$

we get

$$N(z) \le \frac{1-\alpha}{2}|z_1 - iz_2| + \frac{1+\alpha}{2}|z_1 + iz_2| = N(\alpha, z).$$

Similarly, $N(\overline{z}) \le N(-\alpha, z)$. Since $N(\alpha, z) + N(-\alpha, z) = 2T(z)$, we must have $N(z) = N(\alpha, z)$. This completes the proof.

Now observe that $N(\alpha, z) = T_2(z + i\alpha(z_2, -z_1))$. This permits us to generalize the counterexample contained in Proposition 2.27 to higher dimensions. If $n \ge 3$, we define

$$N_\alpha(z) = T_n(z + i\alpha(z_2, -z_1, 0, \dots, 0)).$$

PROPOSITION 2.28. *The norm N_α is reasonable for the Euclidean norm in $\mathbb{R}^n$ for no* $0 < |\alpha| < 1$.

PROOF. It is clear that N_α is a norm in $\mathbb{C}^n$. Since $T_n(x + iy) = \max(|x|, |y|)$ for $x \cdot y = 0$, we get $N_\alpha(x) = |x|$ for $x \in \mathbb{R}^n$. We also have, by Proposition 2.5(4),

$$N_\alpha^*(z) = L_n\left(\frac{z_1 - i\alpha z_2}{1-\alpha^2}, \frac{z_2 + i\alpha z_1}{1-\alpha^2}, z_3, \dots, z_n\right).$$

In particular,

$$N_\alpha^*(1, 0, 0, \dots) = \frac{1}{1-\alpha^2} L_n(1, \alpha i, 0, \dots) = \frac{1}{1-|\alpha|} > 1$$

if $0 < |\alpha| < 1$. Therefore, the norm N_α does not satisfy the condition (5) of Theorem 2.19. This means that N_α is not reasonable for the Euclidean norm.

EXAMPLE 2.29. By using the norms N_α, $0 < \alpha < 1$, we easily check that the norms $N_\alpha(z) + \max(|z_1|, \dots, |z_n|)$ and $|z_1| + N_\alpha(z)$ are not reasonable for the norms $\max(|x_1|, \dots, |x_n|) + |x|$ and $|x_1| + |x|$, respectively.

If $f(x) = |x|$ then, by Lemma 2.24, we can restate Corollary 2.21 as follows.

Proposition 2.30. *Let N be a norm in $\mathbb{C}^n$ such that $N(x) = |x|$ on $\mathbb{R}^n$. Let Φ be an Orlicz function such that $\Phi(1) = 1$ and Φ is differentiable at 0. If*

$$N(x+iy) \le \|(|x|,|y|)\|_{\Phi} \quad \text{for } x\cdot y = 0$$

or

$$N(x+iy) \le \|(|x|,|y|)\|^{o}_{\Phi^*} \quad \text{for } x\cdot y = 0,$$

then N is reasonable for the Euclidean norm.

Now choose an Orlicz function Φ that satisfies the above assumptions. Define two sets of reasonable norms:

$$\mathcal{A}_{\Phi} = \{N : N \text{ is a norm in } \mathbb{C}^n,\ N(x) = |x| \text{ on } \mathbb{R}^n \text{ and}$$
$$N(x+iy) \le \|(|x|,|y|)\|_{\Phi} \text{ for } x\cdot y = 0\},$$
$$\mathcal{B}_{\Phi} = \{N : N \text{ is a norm in } \mathbb{C}^n,\ N(x) = |x| \text{ on } \mathbb{R}^n \text{ and}$$
$$N(x+iy) \le \|(|x|,|y|)\|^{o}_{\Phi^*} \text{ for } x\cdot y = 0\}.$$

Denote by $N_{\min}$, $M_{\min}$ the minimal norms in $\mathcal{A}_{\Phi}$ and $\mathcal{B}_{\Phi}$ and, analogously, let $N_{\max}$, $M_{\max}$ be the maximal norms in A_{Φ} and $\mathcal{B}_{\Phi}$, respectively. (Observe that such maximal norms exist.) Then, by Proposition 2.30, $N_{\min} = M_{\min} = T_n$ for any Φ. If $\Phi(t) = t^2$ then $N_{\max}(z) = M_{\max}(z) = |z|$. A nontrivial generalization of this fact is

Theorem 2.31. *Let $\Phi : \mathbb{R} \to [0,\infty)$ be an Orlicz function such that $\Phi(1) = 1$, $\Phi^{-1}(0) = \{0\}$ and $\Phi(\sqrt{t})$ is a convex function. Let $\lambda_1(z)$, $\lambda_2(z)$ be the functions of Lemma* 2.24. *Then*

$$N_{\max}(z) = \|(\lambda_1(z),\lambda_2(z))\|_{\Phi}, \quad M_{\max} = N^*_{\max}(z) = \|(\lambda_1(z),\lambda_2(z))\|^{o}_{\Phi^*}.$$

In particular, if $\Phi(t) = |t|^p$, $p \ge 2$, then

$$N_{\max}(z) = (\lambda_1(z)^p + \lambda_2(z)^p)^{1/p}, \quad M_{\max}(z) = (\lambda_1(z)^q + \lambda_2(z)^q)^{1/q}, \quad 1/p + 1/q = 1.$$

Proof. Set $\Lambda_{\Phi}(z) = \|(\lambda_1(z),\lambda_2(z))\|_{\Phi}$. To prove that $N_{\max} = \Lambda_{\Phi}$ it is sufficient to show that Λ_{Φ} is a norm in $\mathbb{C}^n$. This function is homogeneous and vanishes only at the origin. So, the point is to prove its subadditivity. To do this we show that

$$(*) \qquad \Lambda_{\Phi}(z) \ge \|(|x|,|y|)\|_{\Phi} \quad \text{for all } z \in \mathbb{C}^n.$$

It is known (see e.g. [KU] or [M-O]) that if $\phi : [0,\infty) \to \mathbb{R}$ is a convex function, continuous at 0, $a, b, c \ge 0$, $c \ge a+b$, then

$$\phi(a) + \phi(b+c) \ge \phi(a+b) + \phi(c).$$

Applying this inequality to the function $\phi(t) = \Phi(\sqrt{t})$ and

$$a = k^2\frac{|z|^2 - |z^2|}{2}, \quad b = k^2\frac{|z|^2 - |x^2 - y^2|}{2}, \quad c = k^2\max(|x|^2,|y|^2), \quad k > 0,$$

we obtain

$$\Phi(k\lambda_1(z)) + \Phi(k\lambda_2(z)) \ge \Phi(k|x|) + \Phi(k|y|),$$

which gives, by the first Amemiya formula, the inequality $(*)$ (with equality if $x\cdot y = 0$). Now observe that the function $\|(|x|,|y|)\|_{\Phi}$ is a norm in $\mathbb{R}^{2n}$. Fix $z = x+iy$, $w = u+iv \in$

$\mathbb{C}^n$. By Lemma 2.24, there exists $\theta \in \mathbb{R}$ such that $e^{i\theta}(z+w) = x' + iy'$ and $x' \cdot y' = 0$. Since

$$e^{i\theta}(z+w) = \cos\theta(x+u) - \sin\theta(y+v) + i[\cos\theta(y+v) + \sin\theta(x+u)],$$

we obtain

$$\begin{aligned}\Lambda_\Phi(z+w) &= \Lambda_\Phi(e^{i\theta}(z+w))\\ &= \|(|\cos\theta(x+u) - \sin\theta(y+v)|, |\cos\theta(y+v) + \sin\theta(x+u)|)\|_\Phi\\ &\le \|(|\cos\theta x - \sin\theta y|, |\cos\theta y + \sin\theta x|)\|_\Phi\\ &\quad + \|(|\cos\theta u - \sin\theta v|, |\cos\theta v + \sin\theta u|)\|_\Phi\\ &\le \Lambda_\Phi(e^{i\theta} z) + \Lambda_\Phi(e^{i\theta} w) = \Lambda_\Phi(z) + \Lambda_\Phi(w).\end{aligned}$$

Now we show that $\Lambda_\Phi^* = \|(\lambda_1(z), \lambda_2(z))\|^o_{\Phi^*}$. It is enough to consider the case where $x, y \ne 0$, $x \cdot y = 0$. By 2.15 we have

$$\Lambda_\Phi^*(z) \le \|(|x|, |y|)\|^o_{\Phi^*}.$$

Observe that the space $(\mathbb{R}^2, \|\cdot\|_\Phi)$ is strictly convex. Assume that $\|x\|_\Phi = \|y\|_\Phi = \|(x+y)/2\|_\Phi = 1$. Then $\Phi(x_1) + \Phi(x_2) = 1$, $\Phi(y_1) + \Phi(y_2) = 1$ and $\Phi((x_1+y_1)/2) + \Phi((x_2+y_2)/2) = 1$. The condition that $\Phi(\sqrt{t})$ is a convex function implies (see e.g. [R-R]) that Φ satisfies Clarkson's inequality

$$\Phi\Big(\frac{t+\tau}{2}\Big) + \Phi\Big(\frac{t-\tau}{2}\Big) \le \frac{1}{2}(\Phi(t) + \Phi(\tau)).$$

From this we get $\Phi\big(\frac{x_1-y_1}{2}\big) + \Phi\big(\frac{x_2-y_2}{2}\big) \le 0$ and thus $x_1 = y_1$, $x_2 = y_2$. As a corollary, we obtain the differentiability of $\|\cdot\|^o_{\Phi^*}$ on $\mathbb{R}^2 \setminus \{0\}$ and the norm $F(x,y) = \|(|x|,|y|)\|^o_{\Phi^*}$ is also differentiable if $x, y \ne 0$. Let $u + iv = \operatorname{grad} F(x,y)$. We have

$$u = D_1\|\cdot\|^o_{\Phi^*}(|x|,|y|)\frac{x}{|x|}, \quad v = D_2\|\cdot\|^o_{\Phi^*}(|x|,|y|)\frac{y}{|y|},$$

$u \cdot v = 0$, $\|(|u|,|v|)\|_\Phi = 1 = \Lambda_\Phi(u+iv)$ and $x \cdot u + y \cdot v = \|(|x|,|y|)\|^o_{\Phi^*}$, whence

$$\Lambda_\Phi^*(x+iy) = \|(|x|,|y|)\|^o_{\Phi^*} = \|(\lambda_1(z), \lambda_2(z))\|^o_{\Phi^*}.$$

This completes the proof.

Corollary 2.32. *The function $\Lambda_p(z) = (\lambda_1(z)^p + \lambda_2(z)^p)^{1/p}$ (for $p \ge 1$) is a norm in $\mathbb{C}^n$ such that*

(1) $\Lambda_p(z) = (|x|^p + |y|^p)^{1/p}$ *if* $x \cdot y = 0$;
(2) $\Lambda_p(z) \ge (|x|^p + |y|^p)^{1/p}$, $z \in \mathbb{C}^n$ *for* $p \ge 2$;
(3) $\Lambda_p(z) \le (|x|^p + |y|^p)^{1/p}$, $z \in \mathbb{C}^n$ *for* $1 \le p \le 2$;
(4) $\Lambda_p^* = \Lambda_q$, $1/p + 1/q = 1$.

We complete Theorem 2.31 by the following

Theorem 2.33. *Let N be a norm in $\mathbb{C}^n$ such that $N(x) = |x|$ on $\mathbb{R}^n$. Suppose $O(n) \subset \Gamma(N)$ and let $\eta_0 = 1/N(e_1 + ie_2)$. If $\eta_0 = 1$, then $N(z) = T_n(z)$, and if $\eta_0 = 1,2$, then $N(z) = L_n(z)$. If $1/2 < \eta_0 < 1$ then one can find an Orlicz function $\Phi : \mathbb{R} \to [0,\infty)$ with $\Phi(1) = 1$ such that*

$$N(z) = \|(\lambda_1(z), \lambda_2(z))\|_\Phi.$$

Proof. If $x, y \in S^{n-1}$ with $x \cdot y = 0$ are fixed, then, since $O(n) \subset \Gamma(N)$,

$$N(\eta x + i\xi y) = N(\eta e_1 + i\xi e_2) \quad \text{for } (\eta, \xi) \in \mathbb{R}^2.$$

The function $f(\eta, \xi) = N(\eta e_1 + i\xi e_2)$ is a norm in $\mathbb{R}^2$ and $\Gamma(f) \supset \Gamma(|\cdot|_1)$. For this reason we may apply Grząślewicz's characterization of the unit ball in an Orlicz space $\mathbb{R}^2$ equipped with the Luxemburg norm (see [G]). It follows from Grząślewicz's paper that $f(\eta, \xi) = |\eta| + |\xi|$ if $\eta_0 = 1/2$ and $f(\eta, \xi) = \max(|\eta|, |\xi|)$ for $\eta_0 = 1$. If $1/2 < \eta_0 < 1$, then $f(\eta, \xi) = \|(\eta, \xi)\|_\Phi$, where Φ may be chosen as follows. For $t \in [0, 1]$, let

$$g(t) = \sup\{\tau > 0 : f(t, \tau) = 1\}$$

and define

$$\Phi(t) = \begin{cases} (1 - g(t))/(2(1 - \eta_0)) & \text{for } 0 \le t \le \eta_0, \\ 1/2 + (t - \eta_0)/(2(1 - \eta_0)) & \text{for } t > \eta_0. \end{cases}$$

Then $f(\eta, \xi) = \|(\eta, \xi)\|_\Phi$ and therefore $N(z) = \|(\lambda_1(z), \lambda_2(z))\|_\Phi$, which completes the proof.

We end this section by giving a few examples where it is possible to calculate the norm $\mathcal{F}(f, \cdot)$ or $\mathcal{F}^*(f^*, \cdot)$ explicitly (in $\mathbb{C}^n$ or on a subset of $\mathbb{C}^n$).

Example 2.34. Let $f(x) = \max(|x_1|, \ldots, |x_n|) = \|x\|_\infty$. We have

$$\begin{aligned} f^*(x) &= |x_1| + \ldots + |x_n| = \|x\|_1, \\ \mathcal{F}(f, z) &= \max(|z_1|, \ldots, |z_n|) = \|z\|_\infty, \\ \mathcal{F}(f^*, z) &= \sup\{|z_1\omega_1 + \ldots + z_n\omega_n| : \omega_j = \pm 1\}, \\ \mathcal{F}^*(f, z) &= |z_1| + \ldots + |z_n| = \|z\|_1. \end{aligned}$$

If $n = 2$, then

$$\mathcal{F}^*(f^*, z) = \tfrac{1}{2}|z_1 - z_2| + \tfrac{1}{2}|z_1 + z_2|.$$

If $n > 2$, by Shmul'yan's theorem, we get

$$\exp \mathcal{F}^*(f^*, \cdot) = \{e^{i\theta}\omega : \theta \in \mathbb{R},\ \omega \in \mathbb{R}^n,\ |\omega_j| = 1\}.$$

Moreover,

$$\exp \mathcal{F}(f, \cdot) = S^1 \times \ldots \times S^1, \quad \exp \mathcal{F}^*(f, \cdot) = \{e^{i\theta} e_j : \theta \in \mathbb{R},\ j = 1, \ldots, n\},$$

where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis in $\mathbb{R}^n$. Note also that for any $z \in \mathbb{C}^n$,

$$\|z\|_\infty \le T_n(z) \le |z| \le L_n(z) \le \|z\|_1.$$

Indeed, for $x \in \mathbb{R}^n$ we have $\max_{1 \le j \le n} |x_j| \le |x| \le |x_1| + \ldots + |x_n|$. Hence by Proposition 2.13 we obtain the required inequalities.

Example 2.35. Let $n = 2$ and $f_p(x) = \|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$. An easy calculation gives

$$\mathcal{F}(f_p, (x_1, ix_2)) = \begin{cases} \max(|x_1|, |x_2|), & 2 \le p \le \infty, \\ (|x_1|^{p'} + |x_2|^{p'})^{1/p'}, & 1 \le p < 2, \end{cases}$$

where $p' = 2p/(2 - p)$, $(x_1, x_2) \in \mathbb{R}^2$. Let now $N_p(z) := \mathcal{F}(f_p, (z_1, iz_2))$, $z \in \mathbb{C}^2$. If $p \ge 2$, then N_p is a reasonable norm for f_∞. In particular, $N_p^*(x) = |x_1| + |x_2|$ for $p \ge 2$ and $x \in \mathbb{R}^2$. On the other hand, $N_p^*(z) = \mathcal{F}^*(f_q^*, (z_1, iz_2))$. This implies $\mathcal{F}^*(f_p^*, (x_1, ix_2)) =$

$|x_1| + |x_2|$, $x \in \mathbb{R}^2$, for $1 \le p \le 2$. Thus, we obtain nontrivial examples of reasonable norms for the norms f_∞ and f_1, respectively.

We now give some application of our formulas. Define, for a given norm f in $\mathbb{R}^n$,

$$C(f) := \inf\{C > 0 : \mathcal{F}^*(f^*, z) \le C\mathcal{F}(f, z)\ \forall z \in \mathbb{C}^n\}.$$

It follows from Proposition 2.15 that $1 \le C(f) \le 2$. Moreover, by Proposition 2.5, $C(f) = C(f^*)$ and $C(f \circ l) = C(f)$ for any $l \in \mathrm{GL}(\mathbb{R}^n)$. Note also that

$$d((\mathbb{R}^n, f) \hat{\otimes} \mathbb{C}, (\mathbb{R}^n, f) \check{\otimes} \mathbb{C}) \le C(f),$$

where $d(\cdot,\cdot)$ denotes the Banach–Mazur distance between two Banach spaces (see e.g. [T]). Moreover,

$$\mathcal{F}(f, (1, i)) \cdot \mathcal{F}(f^*, (1, i)) = d((\mathbb{R}^2, f), (\mathbb{R}^2, |\cdot|))$$

if $\Gamma(f) \subset O(2)$ has enough symmetries. It is clear that $C(f) \ge \mathcal{F}^*(f^*, (1, i))/\mathcal{F}(f, (1, i))$ for $n = 2$. Hence, we obtain

Proposition 2.35. *Let f be a norm in $\mathbb{R}^2$. Assume that there exists $l \in \mathrm{GL}(\mathbb{R}^2)$ such that $f_\infty \le f \circ l \le f_2$ or $f_2 \le f \circ l \le f_1$. Then $C(f) \ge C(f_1)$. If $1 \le p \le \infty$, then*

$$C(f_p) \ge 2^{1-|1/p-1/2|},$$

with equality for $p = 1, 2, \infty$.

We conjecture that equality holds for any $1 \le p \le \infty$ and that for any norm f, we have

$$C(f) = 2d((\mathbb{R}^2, f), (\mathbb{R}^2, |\cdot|))^{-1}.$$

3. Extremal properties of norms $\mathcal{F}(f, \cdot)$ in pluripotential theory

In this section we present some extremal properties of the norms $\mathcal{F}(f, \cdot)$, which were obtained by the author in [B1–B4]. As was mentioned earlier, those results were partially a motivation for this paper.

In the constructive theory of functions as well as in pluripotential theory, an important role is played by some extremal functions. Especially important is the *Siciak extremal function Φ_E* (which is also called the *polynomial extremal function*) associated with a given compact subset E of $\mathbb{C}^n$,

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p} : p \in \mathbb{C}[z],\ \deg p \ge 1,\ \|p\|_E \le 1\}$$

for $z \in \mathbb{C}^n$ (see [SI1]). By the Zakharyuta–Siciak theorem (see [SI2]),

$$\log \Phi_E = V_E,$$

where V_E is the generalized Green function associated with E,

$$V_E(z) = \sup\{u(z) : u \in \mathcal{L}_n,\ u \le 0 \text{ on } E\},$$

where $\mathcal{L}_n$ is the Lelong class of all functions $u \in \mathcal{PSH}(\mathbb{C}^n)$ of logarithmic growth: $u(z) \le \mathrm{const} + \log^+ |z|$. For the background of pluripotential theory see e.g. [KL]. If $E \subset \mathbb{R}^n$

then we may replace complex polynomials in the definition of Φ_E by polynomials with real coefficients and moreover (see [B2])

$$\Phi_E(z) = \sup\{|h(p(z))|^{1/\deg p} : p \in \mathbb{R}[z],\ \deg p \geq 1,\ \|p\|_E \leq 1\},$$

where $h(\zeta) = \zeta + \sqrt{\zeta^2 - 1}$ for $\zeta \in \mathbb{C} \setminus [-1, 1]$ is the inverse of the Joukowski function $g(\zeta) = \frac{1}{2}(\zeta + 1/\zeta),\ |\zeta| > 1$, and

$$|h(\zeta)| = h\big(\tfrac{1}{2}|\zeta - 1| + \tfrac{1}{2}|\zeta + 1|\big), \quad \zeta \in \mathbb{C},$$

where $h(t) = t + \sqrt{t^2 - 1},\ t \geq 1$, with the usual arithmetic square root.

In the special case of a convex symmetric body in $\mathbb{R}^n$ we have (see [LU1], [B1])

$$\Phi_E(z) = \sup\big\{h\big(\tfrac{1}{2}|z \cdot w - 1| + \tfrac{1}{2}|z \cdot w + 1|\big) : w \in \exp(E^*)\big\}. \tag{3.1}$$

Moreover, we have

PROPOSITION 3.1 ([B2]). *Let f be a norm in $\mathbb{R}^n$ and let E be its unit ball. The mapping*

$$\phi : (\mathbb{C} \setminus \{|\zeta| \leq 1\}) \times \{\mathcal{F}(f, c) = 1\} \ni (\zeta, c) \mapsto \tfrac{1}{2}(\zeta c + \zeta^{-1}\overline{c}) \in \mathbb{C}^n$$

is a surjection onto $\mathbb{C}^n \setminus E$ and $V_E(\phi(\zeta, c)) = \log|\zeta|$.

One can also prove

PROPOSITION 3.2 ([B7]). *Let f be a norm in $\mathbb{R}^n$ and let E be its unit ball. Define a norm π in $\mathbb{R}^{n+1}$ by*

$$\pi(x, t) = (f(x)^2 + t^2)^{1/2}, \quad (x, t) \in \mathbb{R}^{n+1}.$$

Then for all $z \in \mathbb{C}^n$,

$$\Phi_E(z) = h(\mathcal{F}(\pi, (z, i))).$$

As an application we get a short proof of an interesting formula:

COROLLARY 3.3 ([LU2], [B1]). *If $E = \mathbb{B}_n$ then*

$$\Phi_{\mathbb{B}_n}(z) = (h(|z|^2 + |z^2 - 1|))^{1/2}.$$

PROOF. In the case $f(z) = |z|$ we have

$$\pi(x, t) = (x^2 + t^2)^{1/2} = |(x, t)|$$

and thus

$$\mathcal{F}(\pi(z, \zeta)) = \frac{1}{\sqrt{2}}(|z|^2 + |\zeta|^2 + |z^2 + \zeta^2|)^{1/2}.$$

Applying this formula and Proposition 3.2 together with the simple fact that $h(2t^2 - 1) = h(t)^2$, $t \geq 1$, we obtain the required formula.

Part (1) of the following proposition was first proved by Siciak (see [SI3]). The second part improves and generalizes a result from [A-B].

PROPOSITION 3.4. *Let f be a norm in $\mathbb{R}^n$ and let N be a norm in $\mathbb{C}^n$ that is reasonable for f. Put $E = \{f(x) \leq 1\}$ and $B = \{N(z) \leq 1\}$.*

(1) *If $p \in \mathbb{C}[x]$, then*

$$\|p\|_B \leq h(\sqrt{2})^{\deg p}\|p\|_E.$$

(2) *If* $p \in \mathbb{R}[x]$, $\deg p = k$, *then*

$$\|p\|_B \le T_k(\sqrt{2})\|p\|_E,$$

where $T_k(\zeta)$ *is the Chebyshev polynomial* $T_k(t) = \cos(k \arccos t)$, $t \in [-1,1]$.

PROOF. (1) Since N is a reasonable norm for f, we have $B \subset \{\mathcal{F}(f,z) \le 1\}$. It follows from the definition of the Siciak extremal function that

$$\|p\|_B \le (\sup_{z\in B} \Phi_E(z))^{\deg p}\|p\|_E \le (\sup\{\Phi_E(z) : \mathcal{F}(f,z) \le 1\})^{\deg p}\|p\|_E.$$

Moreover, if $\mathcal{F}(f,z) \le 1$ we have

$$\Phi_E(z) = h\big(\big(\sup\big\{\tfrac{1}{2}(|z\cdot w|^2+1) + \tfrac{1}{2}|(z\cdot w)^2 - 1| : w \in E^*\big\}\big)^{1/2}\big) \le h(\sqrt{2}),$$

which completes the proof of this case.

(2) Now suppose $p \in \mathbb{R}[x]$ and $\deg p = k \ge 1$ and take $q(z) = \frac{1}{\|p\|_E}p(z)$. Then $u(z) = \frac{1}{k}\log|h(q(z))| \in \mathcal{L}_n$ and $u_{|E} \le 0$ on E. Hence

$$|h(q(z))| \le \Phi_E(z)^k \le h(\sqrt{2})^k = h(g(h(\sqrt{2})^k)) = h(T_k(\sqrt{2}))$$

for $z \in B$. On the other hand,

$$h(\max(1,|q(z)|)) \le h\big(\tfrac{1}{2}|q(z)-1| + \tfrac{1}{2}|q(z)+1|\big) = |h(q(z))| \le h(T_k(\sqrt{2})),$$

and the proof is complete.

REMARK 3.5. For a norm f in $\mathbb{R}^n$, we define

$$\sigma_k(f) := \sup\{\|p\|_B/\|p\|_E : p \in \mathbb{C}[x],\ \deg p \le k\}$$
$$\sigma_k := \sup\{\sigma_k(f) : f \text{ is a norm in } \mathbb{R}^n\},$$

for $k = 1,2,\ldots$, where $E = \{f(x) \le 1\}$ and $B = \{\mathcal{F}(f,z) \le 1\}$. Then $\sigma_k(f) \le \sigma_k \le h(\sqrt{2})^k$. Moreover, for $k=1$,

$$C(f) \le \sigma_1(f) \le \sqrt{1 + C(f)^2},$$

whence $2 \le \sigma_1 \le \sqrt{5}$.

In the proof of the next proposition we give an application of Propositions 2.26 and 3.1. We shall need the definition of the *homogeneous polynomial extremal function* associated with a given compact subset E of $\mathbb{C}^n$ (see [SI2]). Let $\mathcal{H}_k$ be the set of all homogeneous polynomials in $\mathbb{C}^n$ of degree $k \ge 1$. Define

$$\Psi_k(z,E) := \sup\{|Q(z)|^{1/k} : Q \in \mathcal{H}_k,\ \|Q\|_E \le 1\}$$

for $z \in \mathbb{C}^n$, and

$$\Psi_E(z) = \sup\{\Psi_k(z,E) : k \in \mathbb{N}\}.$$

Let $E = \{x \in \mathbb{R}^n : f(x) \le 1\}$ be the unit ball for a norm f. Then we always have

$$\|z\|_{c(f)} \le \Psi_k(z,E) \le \Psi_E(z)$$

with equalities for $z = \zeta x$, $\zeta \in \mathbb{C}, x \in \mathbb{R}^n$. Indeed, let $\mathcal{E}$ be the unit ball for the crossnorm $\|\cdot\|_{c(f)}$. Then $\|z\|_{c(f)} = \sup\{|z\cdot w| : w \in \mathcal{E}^*\}$. If $Q(z) = z\cdot w$ and $\|Q\|_E \le 1$, this is equivalent to $\mathcal{F}(f^*,w) \le 1$, which gives $w \in \mathcal{E}^*$ and the required inequality follows.

Proposition 3.6 ([D], see also [SI2]). *We have*

$$\Psi_k(z,\mathbb{B}_n) = L_n(z) \quad \text{for } z\in\mathbb{C}^n, \quad \Psi_{\mathbb{B}_n} = L_n.$$

Proof. It suffices to consider the case where $z=\frac{1}{2}(\zeta c+\zeta^{-1}\overline{c})$ with $|\zeta|\ge 1,\ c=a+ib\in\mathbb{C}^n,\ |a|=1,\ 0<|b|\le 1,\ a\cdot b=0$. We denote by g the Joukowski function $\frac{1}{2}(\zeta+\zeta^{-1})$ and by $\widehat{g}$ the function $\frac{1}{2}(\zeta-\zeta^{-1})$. Set

$$f(\zeta)=\tfrac{1}{2}(\zeta c+\zeta^{-1}\overline{c}) \quad\text{and}\quad \phi(\zeta)=g(\zeta)+\widehat{g}(\zeta)|b|$$

for $|\zeta|\ge 1$. If $|\zeta|=1$, then $|f(\zeta)|=|\phi(\zeta)|$. Take $Q\in\mathcal{H}_k$ with $\|Q\|_{\mathbb{B}_n}\le 1$. Then the function

$$u(\zeta)=\frac{1}{k}\log|Q(f(\zeta))|-\log|\phi(\zeta)|, \quad |\zeta|\ge 1,$$

is subharmonic. Since u is bounded above and $u\le 0$ on S^1, by the maximum principle for subharmonic functions we get $u\le 0$ for $|\zeta|\ge 1$. Now, for $z=g(\zeta)a+i\widehat{g}(\zeta)b$, we easily check that

$$|\phi(\zeta)|=\{|z|^2+\widehat{g}(|\zeta|^2)|b|^2\}^{1/2} \quad\text{and}\quad \widehat{g}(|\zeta|^2)^2|b|^2=|z|^4-|z^2|^2.$$

This gives $|Q(z)|^{1/k}\le L_n(z)$ and the proof is complete.

Remark 3.7. A shorter (but less elementary) proof of the proposition can be deduced from the fact that the Shilov boundary of the Lie ball is equal to $\exp L_n$ (see [H]). However, our proof gives a little more. Namely, it follows from our argument that both $\log\Phi_{\mathbb{B}_n}$ and $\log\Psi_{\mathbb{B}_n}$ are harmonic on each leaf $\phi(\zeta)=\frac{1}{2}(\zeta c+\zeta^{-1}\overline{c})$ with $T_n(c)=1$ and $|\zeta|>1$. In general, if $\log\|z\|_{c(f)}$ is harmonic on $\{\phi_c(\zeta):|\zeta|>1\}$, where $\phi_c(\zeta)=\frac{1}{2}(\zeta c+\zeta^{-1}\overline{c})$ with $\mathcal{F}(f,c)=1$, then $\Psi_E(z)=\|z\|_{c(f)}$. One can show that this does not hold in the case of $\mathbb{R}^2$ with $f(x)=\|x\|_\infty$ (see [B6]).

Put

$$\gamma(z,\zeta)=\mathcal{F}(\pi,(z,\zeta))+(\mathcal{F}(\pi,(z,\zeta))^2-|\zeta|^2)^{1/2}, \quad (z,\zeta)\in\mathbb{C}^{n+1},$$

where $\pi(x,t)=(f(x)^2+t^2)^{1/2}$. Observe that $\gamma(z,i)=\Phi_E(z)$. Since the homogeneous extremal function is homogeneous, by Proposition 3.2 we can obtain the following upper bound for Ψ_E.

Proposition 3.8. *If f is a norm in $\mathbb{R}^n$ with unit ball E then for all $z\in\mathbb{C}^n$,*

$$\Psi_E(z)\le\inf_{\zeta\in\mathbb{C}}\gamma(z,\zeta). \tag{3.2}$$

Hence we obtain the inequalities

$$\|z\|_{c(f)}\le\Psi_E(z)\le 2\mathcal{F}(f,z), \quad z\in\mathbb{C}^n.$$

For $f(x)=|x|$ we have $\gamma(z,\pm i\sqrt{z^2})=\|z\|_{c(f)}$, whence, by (3.2), we get another proof of the equality $\Psi_{\mathbb{B}_n}(z)=L_n(z)$.

Let E be the unit ball for a norm f in $\mathbb{R}^n$. For a fixed $x\in\operatorname{int}(E)$ we define a norm Λ_x in $\mathbb{R}^n$ by

$$\Lambda_x(v)=\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}V_E(x+i\varepsilon v)=\sup_{w\in\exp(E^*)}\frac{|v\cdot w|}{(1-(x\cdot w)^2)^{1/2}}, \quad v\in\mathbb{R}^n.$$

The norm Λ_x can also be defined by

$$\Lambda_x(v) = \inf\{t > 0 : \mathcal{F}(f, v + itx) \le t\}$$

and thus it is also given by the condition $\mathcal{F}(f, v + i\Lambda_x(v)x) = \Lambda_x(v)$. In particular, if $f(x) = |x|$ then

$$\Lambda_x(v) = \left(|v|^2 + \frac{(x \cdot v)^2}{1 - x^2}\right)^{1/2}, \quad v \in \mathbb{R}^n.$$

For every norm we have the inequalities

$$f(v) \le \Lambda_x(v) \le f(v)(1 - f(x)^2)^{-1/2}, \quad v \in \mathbb{R}^n.$$

To present the next results, we need the following definition.

Let E be a compact subset of $\mathbb{R}^n$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then we denote by $G_{E,x}^{(\alpha)} = G_x^{(\alpha)}$ the following set of gradients of polynomials at a fixed point $x \in E$:

$$G_x^{(\alpha)} := \left\{\frac{1}{(\deg p)^\alpha} \operatorname{grad} p(x) : p \in \mathbb{K}[x_1, \dots, x_n],\ \deg p \ge 1,\ \|p\|_E \le 1\right\}.$$

Proposition 3.9 (Generalized Markov theorem, [B3]). *Let f be a norm in $\mathbb{R}^n$ and let $E = \{f(x) \le 1\}$. Then for every $x \in E$,*

$$G_x^{(2)} = \begin{cases} f^*(y) \le 1\} & \text{if } \mathbb{K} = \mathbb{R}, \\ \{\mathcal{F}(f^*, z) \le 1\} & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

In the real case we also have, for $x \in \operatorname{int}(E)$ and $1 \le \alpha < 2$,

$$E^* \subset G_x^{(\alpha)} \subset (1 - f(x)^2)^{-(2-\alpha)/2} E^*.$$

For $\alpha = 1$ we have more precisely

$$G_x^{(1)} = \{v : \Lambda_x(v) \le 1\} =: E(x). \tag{3.3}$$

One can also prove (see [B7]) the following generalization of the classical van der Corput–Schaake inequality (it is the case $m = n = 1$ of the proposition below).

Proposition 3.10 ([B7]). *Let $Q : (\mathbb{R}^n, f_1) \to (\mathbb{R}^m, f_2)$ be a polynomial mapping. If $f_1(x) < 1$ and $f_2(Q(x)) < 1$ then for all $v \in \mathbb{R}^n$,*

$$\Lambda_{Q(x)}(d_x Q(v)) \le (\deg Q)\Lambda_x(v)\|Q\|,$$

where $\|Q\| = \sup\{f_2(Q(x)) : f_1(x) \le 1\}$.

Remark 3.11. It has also been shown in [B3–B4] that

$$\lambda_{E|_{\operatorname{int}(E)}} = n! \operatorname{vol}(G_x^{(1)})\, dx = n! \operatorname{vol}(E^*(x))\, dx = \lambda(x)\, dx,$$

where λ_E is the complex equilibrium measure $\lambda_E := (dd^c V_E)^n$, where $(dd^c\cdot)^n$ is the complex Monge–Ampère operator. (Here E is the unit ball with respect to a norm f and $G_x^{(1)}$ corresponds to the case where $\mathbb{K} = \mathbb{R}$.) We also have

$$\lambda(x) = \begin{cases} n! \operatorname{vol}(\operatorname{conv}(S_x(E^*))) & \text{(see [BT])}, \\ n! \operatorname{vol}(\operatorname{conv}(S_x(\exp(E^*)))) & \text{(see [B3])}, \end{cases}$$

for $x \in \operatorname{int}(E)$, where $S_x(y) = (1 - (x \cdot y)^2)^{-1/2} y$. Since $\lambda_E(E) = \lambda_E(\mathbb{C}^n) = (2\pi)^n$ (see e.g. [KL]), for every convex body E in $\mathbb{R}^n$ we get

$$\operatorname{vol}(E)\operatorname{vol}(E^*) \le \frac{1}{n!}(2\pi)^n \le \operatorname{const}\sqrt{n}\operatorname{vol}(\mathbb{B}_n)^2.$$

This implies that there exist constants c_n such that

$$\kappa(E) := (\operatorname{vol}(E)\operatorname{vol}(E^*)/\operatorname{vol}(\mathbb{B}_n)^2)^{1/n} \le c_n$$

and $c_n \to 1$ as $n \to \infty$. The above result is a slightly weaker version of the known Santaló inequality (see e.g. [P2])

$$\kappa(E) \le 1.$$

To end this section we prove an interesting property of the equilibrium measure.

Theorem 3.12. *Let f_1 and f_2 be norms in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Let E_j be the unit ball for f_j. Then*

$$\lambda_{E_1\times E_2} = \lambda_{E_1} \otimes \lambda_{E_2}.$$

Proof. It is known ([SI1]) that

$$V_{E_1\times E_2}(z_1, z_2) = \max(V_{E_1}(z_1), V_{E_2}(z_2)), \quad (z_1, z_2) \in \mathbb{C}^{n_1+n_2},$$

which gives

$$\Lambda_{(x_1,x_2)}(v_1, v_2) = \max(\Lambda_{x_1}(v_1), \Lambda_{x_2}(v_2)).$$

Applying the method from the proof of Proposition 2.15 one can check that

$$\Lambda^*_{(x_1,x_2)}(v_1, v_2) = \Lambda^*_{x_1}(v_1) + \Lambda^*_{x_2}(v_2), \quad (v_1, v_2) \in \mathbb{R}^{n_1+n_2}.$$

Lemma 3.13. *For all $x_1 \in \operatorname{int}(E_1)$ and $x_2 \in \operatorname{int}(E_2)$,*

$$\operatorname{vol}((E_1 \times E_2)^*(x_1, x_2)) = \frac{n_1!n_2!}{(n_1+n_2)!}\operatorname{vol}(E_1^*(x_1))\operatorname{vol}(E_2^*(x_2)).$$

Proof. We have, by Fubini's theorem,

$$\begin{aligned}\operatorname{vol}((E_1 \times E_2)^*(x_1, x_2)) &= \int\limits_{E_1^*(x_1)} \operatorname{vol}(\{\Lambda^*_{x_2}(v_2) \le 1 - \Lambda^*_{x_1}(v_1)\})\, dv_1 \\ &= \operatorname{vol}(E_2^*(x_2)) \int\limits_{E_1^*(x_1)} (1 - \Lambda^*_{x_1}(v_1))^{n_2}\, dv_1.\end{aligned}$$

To finish the proof of the lemma we need the following claim.

Claim 3.14. *Let f be a norm in $\mathbb{R}^n$ and let E be the unit ball for f. Let $\phi \in L^1[0,1]$. Then*

$$\int\limits_E \phi(f(x))\, dx = n\operatorname{vol}(E)\int\limits_0^1 t^{n-1}\phi(t)\, dt.$$

In particular, for all $m > -1$,

$$\int\limits_E (1 - f(x))^m\, dx = nB(n, m+1)\operatorname{vol}(E),$$

where $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt$ is the Euler beta function.

PROOF. It suffices to consider the case where f is smooth. Let

$$\omega(\theta_1,\ldots,\theta_{n-1})=(\cos\theta_1\ldots\cos\theta_{n-1},\cos\theta_1\ldots\cos\theta_{n-2}\sin\theta_{n-1},\ldots,\sin\theta_1)$$

for $\theta_j\in[0,2\pi]$ and let

$$G(r,\theta)=r\frac{\omega(\theta)}{f(\omega(\theta))}=r\chi(\theta).$$

Then $|\mathrm{Jac}\,G(r,\theta)|=r^{n-1}|\mathrm{Jac}\,\chi(\theta)|$ and we can write

$$\begin{aligned}\int_E\phi(f(x))\,dx&=\int_0^1\int_{[0,2\pi]^{n-1}}r^{n-1}\phi(r)|\mathrm{Jac}\,\chi(\theta)|\,d\theta\,dr\\&=n\int_0^1t^{n-1}\phi(t)\,dt\int_0^1r^{n-1}\,dr\int_{[0,2\pi]^{n-1}}|\mathrm{Jac}\,\chi(\theta)|\,d\theta=n\int_0^1t^{n-1}\phi(t)\,dt\,\mathrm{vol}(E).\end{aligned}$$

Now let

$$f_{j,k}(x_j)=\bigg(\frac{1}{\mathrm{vol}(E_j^*)}\int_{E_j^*}(x_j\cdot w_j)^{2k}\,dw_j\bigg)^{1/(2k)},\quad x_j\in\mathbb{R}^{n_j},\ j=1,2,$$

and let $E_{j,k}=\{f_{j,k}(x_j)\le1\}$. Then $E_{j,k+1}\subset E_{j,k}$, $V_{E_{j,k}}\nearrow V_{E_j}$, $V_{E_{1,k}\times E_{2,k}}\nearrow V_{E_1\times E_2}$ and, by the continuity properties of the Monge–Ampère operator, $\lambda_{E_{j,k}}\to\lambda_{E_j}$, $\lambda_{E_{1,k}\times E_{2,k}}\to\lambda_{E_1\times E_2}$ and $\lambda_{E_{1,k}}\otimes\lambda_{E_{2,k}}\to\lambda_{E_1}\otimes\lambda_{E_2}$ in the weak sense. Hence it suffices to check that $\lambda_{E_{1,k}\times E_{2,k}}=\lambda_{E_{1,k}}\otimes\lambda_{E_{2,k}}$. The measures $\lambda_{E_{1,k}},\lambda_{E_{2,k}}$, $\lambda_{E_{1,k}\times E_{2,k}}$ and $\lambda_{E_{1,k}}\otimes\lambda_{E_{2,k}}$ have no mass on the boundary of $E_{1,k}$, $E_{2,k}$ and $E_{1,k}\times E_{2,k}$, respectively, and therefore

$$\begin{aligned}\lambda_{E_{j,k}}&=n_j!\,\mathrm{vol}(E_{j,k}^*(x_j))\,dx_j,\\\lambda_{E_{1,k}\times E_{2,k}}&=(n_1+n_2)!\,\mathrm{vol}((E_{1,k}\times E_{2,k})^*(x_1,x_2))\,dx_1\,dx_2.\end{aligned}$$

By Lemma 3.13 we have

$$\begin{aligned}&(n_1+n_2)!\,\mathrm{vol}((E_{1,k}\times E_{2,k})^*(x_1,x_2))\,dx_1\,dx_2\\&\qquad=n_1!\,\mathrm{vol}(E_{1,k}^*(x_1))\,dx_1\otimes n_2!\,\mathrm{vol}(E_{2,k}^*(x_2))\,dx_2,\end{aligned}$$

which gives the desired equality $\lambda_{E_{1,k}\times E_{2,k}}=\lambda_{E_{1,k}}\otimes\lambda_{E_{2,k}}$ and the proof is complete.

4. Biholomorphic inequivalence of some convex circular domains

A domain $\Omega\subset\mathbb{C}^n$ is called *circular* if it is invariant with respect to the rotation group: $e^{i\theta}\Omega\subset\Omega$ for each $\theta\in\mathbb{R}$. In particular, every open ball with respect to a norm F in $\mathbb{C}^n$ is circular. Let us recall that two domains $\Omega_1,\Omega_2\subset\mathbb{C}^n$ are *biholomorphically equivalent* ($\Omega_1\overset{\text{bih.}}{\sim}\Omega_2$) iff there exists a biholomorphic mapping $\phi:\Omega_1\to\Omega_2$. For circular domains Ω_1, Ω_2, the problem of biholomorphic equivalence reduces to a linear problem via the following remarkable result proved by Braun, Kaup and Upmeier:

PROPOSITION 4.1 ([B-K]). *Let Ω_1,Ω_2 be two bounded circular domains in $\mathbb{C}^n$. Then $\Omega_1\overset{\text{bih.}}{\sim}\Omega_2$ iff there exists $L\in\mathrm{GL}(\mathbb{C}^n)$ such that $L(\Omega_1)=\Omega_2$.*

It follows from the properties of linear mappings and convex sets that in the case where both circular domains are convex this problem reduces to a simpler one: when does there exist a linear homeomorphism between exposed points of two given bounded convex circular domains? However, if we know nothing about the geometry of exposed points of the closure of both domains the problem is still difficult. In this section we solve this problem for domains $\{\mathcal{F}(f,z)<1\}$ and $\{\mathcal{F}(g,z)<1\}$, where f and g are two norms in $\mathbb{R}^n$. As we have seen in Section 3 such domains are important for applications in pluripotential theory and in polynomial approximation. We also saw in Section 2 that the set $\exp\mathcal{F}(f,\cdot)$ does not have a simple description and its geometry is, in general, unknown. But the situation is different for the norm $\mathcal{F}^*(f^*,\cdot)$ or, equivalently, for the crossnorm $\|\cdot\|_{c(f)}$. By the remarkable Ruess and Stegall [R-S] result concerning exposed points of the tensor product of Banach spaces or by Heinrich's result [HE], we have

Proposition 4.2 (Heinrich–Ruess–Stegall theorem in $\mathbb{C}^n$). *Let f be a norm in $\mathbb{R}^n$. Then*

$$\exp\mathcal{F}^*(f^*,\cdot)=\exp\|\cdot\|_{c(f)}=\{e^{i\theta}\omega:\omega\in\exp f,\ \theta\in\mathbb{R}\}.$$

Remark 4.3. Note that some special cases of the above result were obtained in Section 2 with the aid of Shmul'yan's theorem. That theorem is, in fact, a basic tool in Ruess and Stegall's argument.

Corollary 4.4. *If f is a norm in $\mathbb{R}^n$ $(n>1)$, then the circular domains $\{\mathcal{F}(f,z)<1\}$ and $\{\mathcal{F}^*(f,z)<1\}$ are not biholomorphically equivalent to the unit Euclidean ball in $\mathbb{C}^n$.*

Proof. It follows from Proposition 4.2 that $\{\mathcal{F}^*(f,z)<1\}$ has boundary points which are not exposed, while the unit Euclidean ball in $\mathbb{C}^n$ has strictly convex boundary. Since $|z|^*=|z|$, Proposition 2.5 implies that $\{\mathcal{F}(f,z)<1\}$ is not biholomorphically equivalent to the unit Euclidean ball $\mathbb{B}_n$.

In our next considerations a vital role is played by Proposition 4.2 and by the following elementary lemma.

Lemma 4.5. *Let A,B be symmetric subsets of $\mathbb{R}^n$ and let A contain n linearly independent vectors. Assume that there exists $L\in\mathrm{GL}(\mathbb{C}^n)$ such that*

$$L(\{e^{i\theta}a:a\in A,\ \theta\in\mathbb{R}\})=\{e^{i\theta}b:b\in B,\ \theta\in\mathbb{R}\}.$$

Then there exists $\widetilde{L}\in\mathrm{GL}(\mathbb{R}^n)$ such that $\widetilde{L}(A)=B$.

Proof. Let $\mathcal{A}=\{a_1,\ldots,a_n\}\subset A$ be a set of linearly independent vectors and let

$$L(a_j)=e^{i\theta_j}b_j,\quad j=1,\ldots,n.$$

Then $\{b_1,\ldots,b_n\}$ is also a system of linearly independent vectors. Introduce an equivalence relation $\sim$ in $\mathcal{A}$ as follows:

$$a_k\sim a_l\ \Leftrightarrow\ e^{i\theta_k}=\pm e^{i\theta_l}.$$

Fix $a\in A$, $a=\sum_j\alpha_ja_j$. It is easy to check that if $\alpha_j\alpha_k\neq0$, then $a_j\sim a_k$. Let $\mathcal{A}=\mathcal{A}_1\cup\ldots\cup\mathcal{A}_k$ be the decomposition of $\mathcal{A}$ into $\sim$ equivalence classes and let $\mathcal{A}_l=[a_{j_l}]_\sim$, $l=1,\ldots,k$. Define

$$\widetilde{L}(a):=e^{-i\theta_{j_l}}L(a)\quad\text{if } a\in\mathcal{A}_l,\ l=1,\ldots,k$$

and extend $\widetilde{L}$ on the whole space $\mathbb{R}^n$. Then $\widetilde{L}$ has the required property. This completes the proof.

Let $\mathrm{Aut}(\Omega)$ denote the group of biholomorphic automorphisms of a domain $\Omega \subset \mathbb{C}^n$ and let $\mathrm{Aut}_{z_0}(\Omega) = \{\phi \in \mathrm{Aut}(\Omega) : \phi(z_0) = z_0\}$ (see e.g. [J-P]). If Ω is a circular domain and $0 \in \mathrm{int}(\Omega)$, then by the classical Cartan theorem (see e.g. [KR]) $\mathrm{Aut}_0(\Omega) \subset \mathrm{GL}(\mathbb{C}^n)$. If f is a norm in $\mathbb{R}^n$, then put

$$\mathcal{B}_f = \{z \in \mathbb{C}^n : \mathcal{F}(f, z) < 1\}, \quad \mathcal{B}_f^* = \{z \in \mathbb{C}^n : \mathcal{F}^*(f, z) < 1\}.$$

Applying Lemma 4.5 we derive

Proposition 4.6. (1) *For any norm f in $\mathbb{R}^n$,*

$$S^1 \cdot \Gamma(f) \subset \mathrm{Aut}_0(\mathcal{B}_{f^*}^*), \quad S^1 \cdot \Gamma(f) \subset \mathrm{Aut}_0(\mathcal{B}_f).$$

(2) *If there exist $a_1, \ldots, a_{n+1} \in \exp f$ such that $a_1, \ldots, a_n$ are linearly independent and $a_{n+1} = \sum_{j=1}^n \alpha_j a_j$ with $\alpha_j \neq 0,\ j = 1, \ldots, n$, then $S^1 \cdot \Gamma(f) = \mathrm{Aut}_0(\mathcal{B}_{f^*}^*)$.*

(3) *If a similar condition holds for* $\exp f^*$, *then $S^1 \cdot \Gamma(f) = \mathrm{Aut}_0(\mathcal{B}_f)$.*

Proof. (1) Let $\ell \in \Gamma(f)$. For $z \in \exp \|\cdot\|_{c(f)}$, we have $\|e^{i\theta}\ell z\|_{c(f)} = 1$, which gives $\|e^{i\theta}\ell\|_{c(f)} = 1$. Since $|\det(e^{i\theta}\ell)| = |\det \ell| = 1$, by 2.7(6) we have $e^{i\theta}\ell \in \mathrm{Aut}_0(\mathcal{B}_{f^*}^*)$, and hence $S^1 \cdot \Gamma(f) \subset \mathrm{Aut}_0(\mathcal{B}_{f^*}^*)$. By this inclusion applied to the norm f^* we get $S^1 \cdot \Gamma(f^*) \subset \mathrm{Aut}_0(\mathcal{B}_f^*)$ and therefore $S^1 \cdot \Gamma(f) \subset \mathrm{Aut}_0(\mathcal{B}_f)$.

(2) Let $L \in \mathrm{Aut}_0(\mathcal{B}_{f^*}^*)$. It follows from the proof of Lemma 4.5 that $a_j \sim a_k$ for all $1 \le j, k \le n$. Define $\ell \in \mathrm{GL}(\mathbb{R}^n)$ by

$$\ell(a_j) = e^{-i\theta_1} L(a_j), \quad j = 1, \ldots, n.$$

Then $\ell = e^{-i\theta_1} L \in \mathrm{Aut}_0(\mathcal{B}_{f^*}^*)$, which gives $f(\ell x) = \|\ell x\|_{c(f)} = \|x\|_{c(f)} = f(x)$, i.e. $\ell \in \Gamma(f)$.

(3) With our assumptions we have, by (2), $S^1 \cdot \Gamma(f^*) = \mathrm{Aut}_0(\mathcal{B}_f^*)$, which implies $S^1 \cdot \Gamma(f) = \mathrm{Aut}_0(\mathcal{B}_f)$, and the proof is complete.

In the special case of $f(x) = |x|$, our argument yields a new simple proof of the following known property (see [KI2], [J-P]):

Corollary 4.7. *If $f(x) = |x|$ then $\mathrm{Aut}_0(\mathbb{T}_n) = \mathrm{Aut}_0(\mathbb{L}_n) = S^1 \cdot O(n)$, where $\mathbb{T}_n = \{T_n(z) < 1\}$, $\mathbb{L}_n = \{L(z) < 1\}$.*

Remark 4.8. The example of the norm $f(x) = |x_1| + \ldots + |x_n|$ shows that our assumption on $\exp f$ in 4.6(2) is essential. Note also that, as in the complex case, if $\ell, \ell^* \in \Gamma(f)$, then $\ell \in O(n)$. Indeed, we have $\ell \in \mathrm{Aut}_0(\mathcal{B}_f)$ and $\ell \in \mathrm{Aut}_0(\mathcal{B}_f^*)$, which implies, by 2.7(5), $\ell \in U(n)$ and therefore $\ell \in O(n)$.

Now we can formulate the main result of this section related to the problem of biholomorphic inequivalence of the balls $\mathcal{B}_f$.

Theorem 4.9. *Let f and g be norms in $\mathbb{R}^n$. The following conditions are equivalent*:

(1) $\mathcal{B}_{f^*}^* \overset{\text{bih.}}{\sim} \mathcal{B}_{g^*}^*$;

(2) $\mathcal{B}_f \overset{\text{bih.}}{\sim} \mathcal{B}_g$;

(3) $\mathcal{B}_f^* \overset{\text{bih.}}{\sim} \mathcal{B}_g^*$;

(4) $\mathcal{B}_{f^*} \overset{\text{bih.}}{\sim} \mathcal{B}_{g^*}$;
(5) $\{f(x) < 1\}$ *and* $\{g(x) < 1\}$ *are linearly homeomorphic*;
(6) $\{f^*(x) < 1\}$ *and* $\{g^*(x) < 1\}$ *are linearly homeomorphic*;
(7) $\exp f$ *and* $\exp g$ *are linearly homeomorphic*;
(8) $\exp f^*$ *and* $\exp g^*$ *are linearly homeomorphic.*

Proof. Observe that by Propositions 1.1, 1.2 and well-known properties of linear mappings

$$(7)\Leftrightarrow(5)\Leftrightarrow(6)\Leftrightarrow(8).$$

Now we prove the basic implication (1)⇒(7). We may assume that there exists a linear automorphism L of $\mathbb{C}^n$ such that

$$L(\mathcal{B}^*_{f^*}) = \mathcal{B}^*_{g^*}.$$

Then, by Proposition 4.2 and by Lemma 4.5, there exists a linear automorphism $\widetilde{L}$ of $\mathbb{R}^n$ such that

$$\widetilde{L}(\exp f) = \exp g,$$

and condition (7) is satisfied. Conversely, assume that condition (7) holds. Then (5) holds as well. Thus, by Proposition 1.1, $g = f \circ \ell$, where ℓ is a linear automorphism of $\mathbb{R}^n$. Let L be the natural extension of ℓ onto the whole $\mathbb{C}^n$. Then

$$\begin{aligned}\mathcal{F}(g,z) &= \mathcal{F}(f\circ\ell, z) = \sup\{|z\cdot\omega| : f(\ell\omega) \le 1\}\\ &= \sup\{|(L^*)^{-1}z\cdot\ell\omega| : f(\ell\omega)\le 1\} = \sup\{|(L^*)^{-1}z\cdot\omega| : f(\omega)\le 1\}\\ &= (\mathcal{F}(f,\cdot))\circ(L^*)^{-1}(z).\end{aligned}$$

Hence $\mathcal{F}(f,\cdot) = \mathcal{F}(g,\cdot)\circ L^*$ and (2) is satisfied. By Proposition 2.5, (2)⇒(3) and, applying similar arguments to f^* and g^*, we get (3)⇒(8) and also (8)⇒(6)⇒(4). Finally, by Proposition 2.5, (4)⇒(1). The proof is complete.

Corollary 4.10. *For $n = 2$, the domain $\mathcal{B}_f$ is biholomorphically equivalent to the unit polydisc if and only if $\#\exp f = 4$.*

Remark 4.11. A natural question arises: are circular convex domains $\mathcal{B}_f$ and $\mathcal{B}^*_{f^*}$ biholomorphically equivalent? This problem is, in general, difficult. One of the difficulties is that in most cases $\mathrm{Aut}_0(\mathcal{B}_f) = \mathrm{Aut}_0(\mathcal{B}^*_{f^*})$.

Let f be a fixed norm in $\mathbb{R}^n$, $n > 1$. Define $G_\alpha(z) := (1-\alpha)\mathcal{F}(f,z) + \alpha\mathcal{F}^*(f^*,z)$ for $\alpha\in[0,1]$. Then G_α is a norm in $\mathbb{C}^n$ that extends f. We conjecture that the interpolating domains $\{G_\alpha < 1\}$ and $\{G_\beta < 1\}$ are not biholomorphically equivalent if $\alpha \ne \beta$. We prove this conjecture for f being the Euclidean norm.

Fix $\alpha\in[0,1]$ and define $N_\alpha := (1-\alpha)T_n + \alpha L_n$, $\Omega_\alpha := \{N_\alpha < 1\}$.

Proposition 4.12. *Fix $n \ge 2$. Then for all $\alpha \in (0,1)$,*

$$\mathrm{Aut}_0(\Omega_\alpha) = S^1\cdot O(n) = \mathrm{Aut}_0(\Omega^*_\alpha).$$

Proof. It is enough to prove the second equality. We prove it for $n = 2$ and $n = 3$. For higher dimensions the proof is similar. Since $\Gamma(T_n) = \Gamma(L_n) = S^1\cdot O(n)$, we have the inclusions

$$S^1 \cdot O(n) \subset \mathrm{Aut}_0(\Omega_\alpha), \quad S^1 \cdot O(n) \subset \mathrm{Aut}_0(\Omega_\alpha^*).$$

Next, observe that $\mathrm{Aut}_0(\Omega_\alpha), \mathrm{Aut}_0(\Omega_\alpha^*) \subset U(n)$. Since $O(n) \supset \Gamma(\|\cdot\|_1)$, we get

$$\Big[\int_{E^*} z_i \overline{z}_j \, dz\Big] = \mathrm{const} \cdot I_n,$$

where $E = \overline{\Omega}_\alpha$ or $E = \overline{\Omega_\alpha^*}$, and thus, by Proposition 2.5, we obtain the required inclusions. By Shmul'yan's theorem one can check that

$$\exp N_\alpha^* = \{e^{i\theta}(x + i\alpha y) : \theta \in \mathbb{R},\ x, y \in S^{n-1},\ x \cdot y = 0\}.$$

This implies that $S^1 \cdot O(n)$ acts transitively on $\exp N_\alpha^*$.

Now assume $n = 2$ and take $\ell \in \Gamma(N_\alpha^*)$. We now show that there exists $\ell' \in S^1 \cdot O(2)$ such that $\ell' \circ \ell = I_n$. Choose $\ell' \in S^1 \cdot O(2)$ such that $\ell_1 = \ell' \circ \ell$ has the form

$$\ell_1(1, i\alpha) = \ell_1(e_1 + i\alpha e_2) = (1, i\alpha),$$
$$\ell_1(i\alpha, 1) = \ell_1(e_2 + i\alpha e_1) = e^{i\theta}((\cos\phi, \sin\phi) \pm i\alpha(\sin\phi, -\cos\phi)).$$

Since $(1, i\alpha) \cdot (i\alpha, 1) = 0$, we get $\cos\phi = 0$ and

$$\ell_1(i\alpha, 1) = e^{i\theta} \sin\phi (i\alpha, 1).$$

If we take $\sin\phi = 1$ then

$$\ell_1^*(1, i\alpha) = (1, i\alpha), \quad \ell_1^*(i\alpha, 1) = e^{-i\theta}(i\alpha, 1).$$

By the assumption $N_\alpha \circ \ell_1^* = N_\alpha$. In particular, $N_\alpha(\ell_1^*(\cos\phi, \sin\phi)) = 1$ for each $\phi \in \mathbb{R}$. We have (see Section 2)

$$N_\alpha(z) = \frac{1}{\sqrt{2}}(|z|^2 + |z^2|)^{1/2} + \alpha\frac{1}{\sqrt{2}}(|z|^2 - |z^2|)^{1/2}.$$

Now we calculate

$$\ell_1^*(\cos\phi, \sin\phi) = \frac{1}{d}[(\cos\phi - i\alpha\sin\phi)(1, i\alpha) + (\sin\phi - i\alpha\cos\phi)e^{-i\theta}(i\alpha, 1)],$$

where $d = 1 + \alpha^2$, whence

$$|\ell_1^*(\cos\phi, \sin\phi)|^2 = 1$$

and

$$|\ell_1^*(\cos\phi, \sin\phi)^2| = \frac{1}{d^2}|4\alpha^2 + (1-\alpha^2)^2\cos\theta$$
$$+ i[2\alpha(1-\alpha^2)\sin 2\phi(1-\cos\theta) + (1-\alpha^2)^2\cos 2\phi \sin\theta]|.$$

Observe that the function $f(t) := (1+t)^{1/2} + \alpha(1-t)^{1/2}$ is injective for $0 \le t \le \lambda$ and for $\lambda < t \le 1$, where $\lambda = (1-\alpha^2)(1+\alpha^2)^{-1}$. This implies that $|\ell_1^*(\cos\phi, \sin\phi)^2|$ is locally constant and that this forces $\cos\theta = 1$. It means that $\ell_1 = I_2$, which completes the proof for $n = 2$.

Now consider the case where $n = 3$. Let ℓ be a linear automorphism of Ω_α^*. One can choose $\ell' \in S^1 \cdot O(3)$ such that $\ell_1 = \ell' \circ \ell$ is of the form

$$\ell_1(e_1 + i\alpha e_2) = (e_1 + i\alpha e_2), \quad \ell_1(e_2 + i\alpha e_1) = e^{i\phi}(x + i\alpha y), \quad \ell_1(e_3) = u + iv.$$

Since l_1 is a unitary map, one can easily check that $x_2 = y_1, y_1 = \pm 1$ or x_1, and $x_3, y_2, y_3 = 0$. If we put $y_1 = 1$ then, by an argument similar to that for the case $n = 2$, we show that $\phi = 0$. Then $\ell_1(e_3) = e^{i\psi} e_3$ and we easily check that $\psi = 0$. This means that $\ell_1 = I_3$, which completes the proof.

REMARK 4.13. To prove Proposition 4.12 we did not need an effective formula for N_α^*. However, such a formula will be found later (see Lemma 4.18).

PROPOSITION 4.14. *If $\alpha, \beta \in [0,1]$ and $\alpha \neq \beta$, then the unit balls Ω_α and Ω_β are not biholomorphically equivalent.*

PROOF. Suppose that Ω_α and Ω_β are (linearly) biholomorphically equivalent. Then the dual balls $\overline{\Omega_\alpha^*}$ and $\overline{\Omega_\beta^*}$ are linearly equivalent. Let $L \in \mathrm{GL}(\mathbb{C}^n)$ and $L(\overline{\Omega_\alpha^*}) = \overline{\Omega_\beta^*}$. Then

$$L(\exp N_\alpha^*) = \exp N_\beta^*.$$

Since $\Gamma(N_\alpha^*) = S^1 \cdot O(n)$, the mapping

$$\Phi(\ell) = L\ell L^{-1}$$

is an automorphism of the group $S^1 \cdot O(n)$. We also have $\det \Phi(\ell) = \det \ell$ and

$$\Phi(O(n)) = O(n).$$

Since we know the structure of automorphisms of the orthogonal group (see e.g. [DIE]), $\Phi_{|O(n)}$ must be of the form

$$\Phi(l) = \ell_0 \ell \ell_0^{-1} \quad \text{with } \ell_0 \in O(n).$$

Let $L = L_1 + iL_2$. Then, for all $\ell \in O(n)$, we have

$$\ell_0^{-1} L_1 \ell = \ell \ell_0^{-1} L_1, \quad \ell_0^{-1} L_2 \ell = \ell \ell_0^{-1} L_2$$

and since the Euclidean space $\mathbb{R}^n$ has enough symmetries,

$$\ell_0^{-1} L_1 = aI_n, \quad \ell_0^{-1} L_2 = bI_n, \quad L = c\ell_0, \quad c = a + ib.$$

If $z_0 \in \exp N_\alpha^*$, then $T_n(z_0) = 1$. This implies $|c| = 1$, $\exp N_\alpha^* = \exp N_\beta^*$ and $N_\alpha = N_\beta$, which is impossible if $\alpha \neq \beta$. The contradiction completes the proof.

Recently K. T. Kim [KI2] has proved that the Turowicz ball $\mathbb{T}_n$ has compact automorphism group and

$$\mathrm{Aut}(\mathbb{T}_n) = \mathrm{Aut}_0(\mathbb{T}_n) = S^1 \cdot O(n).$$

Motivated by Kim's result, at the end of this section we prove that the groups $\mathrm{Aut}(\Omega_\alpha)$, $\alpha \in (0,1)$, are compact for $n = 2$. Our proof will be also based on Kim's idea (see [KI1, 2]), which is closely related to his so-called *scaling lemma*. This method is very helpful in the situation where we cannot apply the Wong–Rosay theorem. We need the following version of the scaling lemma.

LEMMA 4.15 (see [KI1]). *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and $p \in \exp \overline{\Omega}$. Assume that there exist $\{g_j\} \subset \mathrm{Aut}(\Omega)$ and $p_0 \in \Omega$ such that $g_j(p_0) \to p$. Then there is a sequence $\{A_j\} \subset \mathrm{GL}(\mathbb{C}^n)$ such that $\|A_j\| \to 0$ as $j \to \infty$ and $\lim_{j\to\infty} A_j^{-1}(\Omega - p)$ =: $\widehat{\Omega}$ exists and is biholomorphic to Ω. Here the limit is taken in the sense of the local Hausdorff set convergence.*

REMARK 4.16. If $p \in \exp \overline{\Omega}$, then p is a local peak point for Ω. This property is essential for the proof of the above lemma (see [KI1]).

PROPOSITION 4.17. *The group* $\operatorname{Aut}(\Omega_\alpha)$, $\alpha \in (0,1)$, *is compact for* $n = 2$.

PROOF. We start with the observation that $\partial\Omega_\alpha$ is strictly pseudoconvex at all points which do not belong to the subsets $\partial Q_1 := \left\{\frac{1}{1+\alpha}e^{i\theta}(1,\pm i) : \theta \in \mathbb{R}\right\}$ or $\partial Q_2 := \{e^{i\theta}\omega : \theta \in \mathbb{R},\ \omega \in \mathbb{R}^2,\ |\omega| = 1\}$. At these points $\partial\Omega_\alpha$ is not of class $\mathcal{C}^2$. Observe also that

$$\partial Q_j = \{\phi(q_j) : \phi \in S^1 \cdot O(2)\}, \quad j = 1, 2,$$

where $q_1 = \frac{1}{1+\alpha}(1, i)$ and $q_2 = (1, 0)$. This means that it suffices to apply the scaling technique only at q_1 and q_2. (At points of strong pseudoconvexity we may apply the Wong–Rosay theorem.) However, we need to know first that $q_1, q_2 \in \exp \overline{\Omega}_\alpha$. To get this we prove

LEMMA 4.18. $N_\alpha^* = \max\left(T_n, \frac{1}{1+\alpha}L_n\right)$ *for* $n \geq 2$.

PROOF. Since $\max\left(T_n(z), \frac{1}{1+\alpha}L_n(z)\right) = 1$ for $z \in \exp N_\alpha^*$, it follows that $N_\alpha^* \geq \max\left(T_n, \frac{1}{1+\alpha}L_n\right)$. Now it suffices to check the opposite inequality for $z \in \mathbb{C}^n$ such that $x \cdot y = 0$. Put $x' = x/|x|$, $y' = y/|y|$ and take $w = u + iv$, $u \cdot v = 0$, $u' = u/|u|$, $v' = v/|v|$, $r = \max(|x' \cdot u'|, |y' \cdot v'|)$, $\varrho = \max(|x' \cdot v'|, |y' \cdot u'|)$. Then

$$\begin{aligned}|z \cdot w| &\leq [(|x||u|x' \cdot u' + |y||v|y' \cdot v')^2 + (|x||v|x' \cdot v' + |y||u|y' \cdot u')^2]^{1/2} \\ &\leq [(|x||u| + |y||v|)^2 r^2 + (|x||v| + |y||u|)^2 \varrho^2]^{1/2} \\ &\leq \max(|x||u| + |y||v|, |x||v| + |y||u|),\end{aligned}$$

since $r^2 + \varrho^2 \leq 1$. If $N_\alpha(w) \leq 1$, then $|u| + \alpha|v| \leq 1$, $|v| + \alpha|u| \leq 1$ and we easily deduce

$$|z \cdot w| \leq \max\left(|x|, |y|, \frac{|x| + |y|}{1+\alpha}\right) = \max\left(T_n(z), \frac{L_n(z)}{1+\alpha}\right).$$

Thus $N_\alpha^*(z) \leq \max\left(T_n(z), \frac{L_n(z)}{1+\alpha}\right)$, which completes the proof.

Now we have

$$T_2(q_2) > \frac{1}{1+\alpha}L_2(q_2), \quad \frac{1}{1+\alpha}L_2(q_1) > T_2(q_1)$$

and

$$\operatorname{grad} T_2(q_2) = q_2, \quad \operatorname{grad}\left(\frac{1}{1+\alpha}L_2\right)(q_1) = q_1.$$

Thus $q_1, q_2 \in \exp \overline{\Omega}_\alpha$. For simplicity of calculations we transform linearly Ω_α onto $D_\alpha := \ell(\Omega_\alpha)$, where $\ell(z_1, z_2) = (z_1 - iz_2, z_1 + iz_2)$. Then $\widehat{q}_1 = \frac{2}{1+\alpha}(1, 0) = \ell(q_1)$ and $\widehat{q}_2 = (1, 1) = \ell(q_2)$ are new points for scaling. The domain D_α has a simple description:

$$D_\alpha : \quad \frac{1-\alpha}{2}|z_1| + \frac{1+\alpha}{2}|z_2| < 1, \quad \frac{1+\alpha}{2}|z_1| + \frac{1-\alpha}{2}|z_2| < 1.$$

Assume that there exist $p_1 \in D_\alpha$ (resp. $p_2 \in D_\alpha$) and $\{g_j\} \subset \operatorname{Aut}(D_\alpha)$ such that

$$\lim_{j\to\infty} g_j(p_1) = \widehat{q}_1$$

(resp. $\lim_{j\to\infty} g_j(p_2) = \widehat{q}_2$). Our goal is to show that this gives a contradiction. First we consider the case of $\widehat{q}_1$. Let $A_j = (a_j^{lk})_{k,l=1,2} \in \mathrm{GL}(\mathbb{C}^2)$ be a sequence obtained in

Lemma 4.15. In particular, $a_j^{kl} \to 0$ as $j \to \infty$ and $\widehat{D}_\alpha = \lim_{j\to\infty} A_j^{-1}(D_\alpha - \widehat{q}_1)$ is biholomorphic to D_α and thus it is an unbounded hyperbolic domain (in the sense of Kobayashi, see e.g. [J-P]). The domain $A_j^{-1}(D_\alpha - \widehat{q}_1)$ is represented by

$$\left(\frac{1+\alpha}{2}\right)^2 |z_j^{(1)}|^2 - \left(\frac{1-\alpha}{2}\right)^2 |z_j^{(2)}|^2 + (1+\alpha)\Re(z_j^{(1)}) + (1-\alpha)|z_j^{(2)}| < 0,$$
$$\frac{1-\alpha}{2}\left|z_j^{(1)} + \frac{2}{1+\alpha}\right| + \frac{1+\alpha}{2}|z_j^{(2)}| < 1,$$

where $z_j^{(1)} = a_j^{11}z_1 + a_j^{12}z_2$, $z_j^{(2)} = a_j^{21}z_1 + a_j^{22}z_2$. We may assume that $a_j^{1k}/|a_j^{11}| \to a^{1k}$ and $a_j^{2k}/|a_j^{21}| \to a^{2k}$, $k = 1, 2$. Further, choosing a subsequence if necessary, we may also assume that either (I) $|a_j^{11}|/|a_j^{21}| \to b^{12}$ or (II) $|a_j^{21}|/|a_j^{11}| \to b^{21}$. In the first case $\widehat{D}_\alpha$ has the following description:

$$\widehat{D}_\alpha: \quad \Re(c^{11}z_1 + c^{12}z_2) + |c^{21}z_1 + c^{22}z_2| < 0,$$

where $c^{1k} = b^{12}a^{1k}$ and $c^{2k} = a^{2k}$ (details are analogous to those of Kim's proof of his Proposition 1 in [KI2]). Since $\widehat{D}_\alpha$ is a hyperbolic domain, $\det(c^{lk}) \neq 0$ (see also [KI2]). Thus, $\widehat{D}_\alpha \overset{\text{bih.}}{\sim} \{\Re(z_1) + |z_2| < 0\}$ and considering the mapping $\phi(z_1, z_2) = \left(-\frac{1+z_1}{1-z_1}, \frac{z_2}{(1-z_1)^2}\right)$ we easily see that $\widehat{D}_\alpha \overset{\text{bih.}}{\sim} \{|z_1|^2 + |z_2| < 1\} = D'_\alpha$. Hence, by 4.15, $D_\alpha \overset{\text{bih.}}{\sim} D'_\alpha$. But this is impossible. Indeed, since D_α and D'_α are circular, they are linearly equivalent. In particular, the sets of points at which ∂D_α and $\partial D'_\alpha$ are not smooth are also linearly equivalent. The set of points of nonsmoothness of $\partial D'_\alpha$ is the circle $S^1 \times \{0\}$, while the set of points of nonsmoothness of ∂D_α has three disjoint components. Thus we have a contradiction in this case. We analogously obtain the same contradiction in case (II).

Now we investigate the case of $\widehat{q}_2$, which is a little more laborious. Define

$$\begin{aligned}\varphi(z_1, z_2) := {}& \left[\left(\frac{1-\alpha}{2}\right)^2 (|z_2|^2 + 2\Re(z_2)) - \left(\frac{1+\alpha}{2}\right)^2 (|z_1|^2 + 2\Re(z_1))\right]^2 \\ & + 2(1-\alpha)\left[\left(\frac{1-\alpha}{2}\right)^2 (|z_2|^2 + 2\Re(z_2)) - \left(\frac{1+\alpha}{2}\right)^2 (|z_1|^2 + 2\Re(z_1))\right] \\ & - (1-\alpha)^2(|z_2|^2 + 2\Re(z_2)), \\ \delta(z_1, z_2) := {}& \frac{1+\alpha}{2}|z_1 + 1| + \frac{1-\alpha}{2}|z_2 + 1|.\end{aligned}$$

Then the domain $A_j^{-1}(D_\alpha - \widehat{q}_2)$ is represented by the conditions

$$\delta(z_j^{(1)}, z_j^{(2)}) < 1, \quad \varphi(z_j^{(1)}, z_j^{(2)}) > 0; \quad \delta(z_j^{(2)}, z_j^{(1)}) < 1, \quad \varphi(z_j^{(2)}, z_j^{(1)}) > 0,$$

where $z_j^{(1)}$, $z_j^{(2)}$ are given as above. By analogous considerations as in the case of $\widehat{q}_1$ we obtain

$$\widehat{D}_\alpha \overset{\text{bih.}}{\sim} \{\Re((1+\varepsilon\alpha)z_1 + (1-\varepsilon\alpha)z_2) < 0,\ \varepsilon = -1, 1\} \overset{\text{bih.}}{\sim} \{|\zeta| < 1\}^2,$$

and thus $D_\alpha \overset{\text{bih.}}{\sim} \{|\zeta| < 1\}^2$, which is impossible if $\alpha \in (0, 1)$. Here also the sets of points of nonsmoothness on ∂D_α and on the boundary of the polydisc are topologically different—this set is equal to the torus $S^1 \times S^1$ for the polydisc. We obtain a contradiction, which completes the proof of Proposition 4.17.

THEOREM 4.19. $\operatorname{Aut}(\Omega_\alpha) = S^1 \cdot O(2)$ *for* $\alpha \in (0,1)$.

PROOF. This is a consequence of Propositions 4.14, 4.17 and Kim's result [KI2, Prop. 2].

REMARK 4.20. It follows from Propositions 4.12 and 4.14 that

$$\Omega^*_\alpha \overset{\text{bih.}}{\sim} \Omega^*_\beta \Leftrightarrow \alpha = \beta$$

and

$$\operatorname{Aut}_0(\Omega^*_\alpha) = S^1 \cdot O(n) \quad \text{for all } \alpha \in [0,1].$$

Here the domains Ω^*_α are the unit balls with respect to the norms $\max\left(T_n, \frac{L_n}{1+\alpha}\right)$. We can also obtain these domains in the following way. Let $G = S^1 \cdot O(n)$. This group acts on S_{T_n} and S_{L_n} and acts transitively on $\exp T_n$ and $\exp L_n$. Introduce an equivalence relation on the sphere S_{T_n} by

$$z_1 \sim z_2 \text{ iff there exists } \ell \in G \text{ with } z_2 = \ell z_1.$$

Let $\Omega(z_0) = \operatorname{int}(\operatorname{conv}\{\ell z_0 : \ell \in G\})$ be a convex circular domain induced by the orbit of the point z_0. Thus, we have

$$\Omega^*_\alpha = \Omega(e_1 + i\alpha e_2), \quad \alpha \in [0,1], \quad \Omega(z_1) \overset{\text{bih.}}{\sim} \Omega(z_2) \quad \text{iff} \quad z_1 \sim z_2$$

and $\operatorname{Aut}_0(\Omega(z_0)) = G$ for each $z_0 \in S_{T_n}$.

5. The complex method of interpolation and conjugate norms in $\mathbb{C}^n$

In this section we present some results related to complex interpolation in $\mathbb{C}^n$. Our goal is to calculate the interpolation norms for the couples $(\mathbb{C}^n, L_n)$ and $(\mathbb{C}^n, T_n)$.

First, we recall some basic facts on the complex method of interpolation. This method was introduced by A. P. Calderón [CA] and J. L. Lions [LI] in the early sixties. For details we refer to the monograph [B-L].

Let $A = ((A_0, \|\cdot\|_{A_0}), ((A_1, \|\cdot\|_{A_1}))$ be a couple of complex Banach spaces such that A_0 and A_1 are subspaces of some complex Hausdorff linear topological space. Define the intersection $\Delta(A) = \Delta(A_0, A_1)$ and the sum $\Sigma(A) = \Sigma(A_0, A_1)$ of A_0 and A_1 by

$$A_0 \cap A_1 \quad \text{with the norm} \quad \|z\|_\Delta = \max\{\|z\|_{A_0}, \|z\|_{A_1}\}$$

and

$$A_0 + A_1 \quad \text{with the norm} \quad \|z\|_\Sigma = \inf\{\|z_0\|_{A_0} + \|z_1\|_{A_1} : z = z_0 + z_1,\ z_j \in A_j\},$$

respectively. Then $\Delta(A)$ and $\Sigma(A)$ are Banach spaces. Let

$$S = \{\zeta \in \mathbb{C} : 0 \le \Re\zeta \le 1\}, \quad S_0 = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < 1\}$$

and put

$$\mathcal{F}(A_0, A_1) = \{f : S \to \Sigma(A_0, A_1),\ f \in \mathcal{C}(S) \cap L^\infty(S) \cap \mathcal{O}(S_0), \\ f(j+it) \in A_j,\ \lim_{|t|\to\infty} \|f(j+it)\|_{A_j} = 0,\ j = 0, 1\}.$$

Then $\mathcal{F}(A_0, A_1)$ is a Banach space with the norm

$$\|f\|_{\mathcal{F}} = \max\{\sup_{\mathbb{R}} \|f(it)\|_{A_0}, \sup_{\mathbb{R}} \|f(1+it)\|_{A_1}\}.$$

Define also

$$A_{[\theta]} = \{a \in A_0 + A_1 : f(\theta) = a \text{ for some } f \in \mathcal{F}(A_0, A_1)\}$$

and for $a \in A_{[\theta]},\ 0 \le \theta \le 1$,

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} : f \in \mathcal{F}(A_0, A_1),\ f(\theta) = a\}.$$

The Banach space $(A_{[\theta]}, |\cdot\|_{[\theta]})$ is called the *interpolation space for the couple* A or the *interpolation space for the complex method*.

The basic properties of this method were obtained by Calderón [CA]. For the convenience of the reader, we present some important facts. The proofs can be found in [B-L].

THEOREM 5.1. *Let* $A = (A_0, A_1)$ *be a couple of complex Banach spaces. Then*

$$(A_0, A_0)_{[\theta]} = A_0, \quad (A_1, A_0)_{[\theta]} = (A_0, A_1)_{[1-\theta]}, \tag{5.1.1}$$

and $\Delta(A_0, A_1)$ *is dense in* $(A_0, A_1)_{[\theta]},\ 0 \le \theta \le 1$. *If* $z \in A_0 \cap A_1$, *then*

$$\|z\|_{[\theta]} \le \|z\|_{A_0}^{1-\theta} \|z\|_{A_1}^{\theta}. \tag{5.1.2}$$

($A_{[\theta]}$ *is an interpolation space of exact exponent* θ.) *If* A_0 *is reflexive, then so is* $A_{[\theta]},\ 0 < \theta < 1$. *Moreover, if* $\Delta(A_0, A_1)$ *is dense in* A_0 *and* A_1, *then we have the* duality theorem

$$(A_0, A_1)^*_{[\theta]} = (A_0^*, A_1^*)_{[\theta]}. \tag{5.1.3}$$

If $\Delta(A_0, A_1)$ *is dense in* $A_0, A_1, \Delta(A_{[\theta_1]}, A_{[\theta_2]})$, *then we have the* reiteration theorem

$$(A_{[\theta_1]}, A_{[\theta_2]})_{[\sigma]} = (A_0, A_1)_{[\theta]},\ 0 \le \sigma \le 1,\ \theta = (1-\sigma)\theta_1 + \sigma\theta_2. \tag{5.1.4}$$

(Interpolation theorem) *Let* (A_0, A_1) *and* (B_0, B_1) *be two couples of complex Banach spaces and let* $T_j : A_j \to B_j,\ j = 0, 1$, *be two continuous linear operators with norms* M_0 *and* M_1, *respectively. Assume* $T_0 = T_1$ *on* $A_0 \cap A_1$. *Then there exists a continuous linear operator* $T_\theta : (A_0, A_1)_{[\theta]} \to (B_0, B_1)_{[\theta]},\ T_\theta = T_j$ *on* $A_0 \cap A_1$, *with norm* M_θ, *which satisfies the inequality*

$$M_\theta \le M_0^{1-\theta} M_1^{\theta}. \tag{5.1.5}$$

All these assumptions are satisfied if A_0 and A_1 are equal to $\mathbb{C}^n$ equipped with any norms F and G. In this special case we also have the following properties, which easily follow from the definitions or from Theorem 5.1. In particular, (5.2.3) is a consequence of (5.1.3) and (5.2.2).

PROPOSITION 5.2. *Let* $F,\ G$ *be norms in* $\mathbb{C}^n$ *and let* L *be a linear automorphism of* $\mathbb{C}^n$. *If* $((\mathbb{C}^n, F), (\mathbb{C}^n, G))_{[\theta]} = (\mathbb{C}^n, \|\cdot\|_{[\theta]})$, *then*

$$((\mathbb{C}^n, F \circ L), (\mathbb{C}^n, G \circ L))_{[\theta]} = (\mathbb{C}^n, \|\cdot\|_{[\theta]} \circ L), \tag{5.2.1}$$

$$((\mathbb{C}^n, F(\overline{z}), (\mathbb{C}^n, G(\overline{z}))_{[\theta]} = (\mathbb{C}^n, \|\overline{z}\|_{[\theta]}), \tag{5.2.2}$$

$$((\mathbb{C}^n, F^*), (\mathbb{C}^n, G^*))_{[\theta]} = (\mathbb{C}^n, \|\cdot\|^*_{[\theta]}). \tag{5.2.3}$$

If $F \le G$, *then* $F(z) \le \|z\|_{[\theta]} \le G(z)$ *for all* $z \in \mathbb{C}^n$, $\theta \in [0,1]$. *If* $((\mathbb{C}^n, F), (\mathbb{C}^n, F^*))_{[\theta]} = (\mathbb{C}^n, \|\cdot\|^*_{[\theta]})$, *then*

$$\|z\|^*_{[\theta]} = \|z\|_{[1-\theta]}. \tag{5.2.4}$$

In particular, $\|z\|^*_{[1/2]} = \|z\|_{[1/2]} = |z|$.

As an application we prove part (5) of Proposition 2.7.

Proof of Proposition 2.7(5). Assume $L, L^* \in \Gamma(F)\}$. Then $L \in \Gamma(F^*)$ and, by 5.2.3 and 5.2.4, we have

$$(\mathbb{C}^n, |\cdot| \circ L) = ((\mathbb{C}^n, F \circ L), (\mathbb{C}^n, F^* \circ L))_{[1/2]} = ((\mathbb{C}^n, F), (\mathbb{C}^n, F^*))_{[1/2]} = (\mathbb{C}^n, |\cdot|).$$

This means that $L \in U(n)$, as claimed.

Now we show that there exists a norm F such that the Banach space $(\mathbb{C}^2, F)$ is isometrically isomorphic to $(\mathbb{C}^2, F^*)$ but $(\mathbb{C}^2, F)$ is not isometrically isomorphic to the Hilbert space $(\mathbb{C}^2, |\cdot|)$. The analogous problem in $\mathbb{R}^2$ is solved by the linear map Q represented by the matrix

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

and the norm $f(x_1, x_2) = 2^{-1/4}(|x_1| + |x_2|)$. Here $f^*(x) = f(Qx)$ and $(\mathbb{R}^2, f)$ is not isometrically isomorphic to the Euclidean space $\mathbb{R}^2$. We now show that Q is also "bad" in $\mathbb{C}^2$. In the construction of a counterexample we use the properties of the complex method of interpolation.

We denote by $\Gamma_1(n)$ the group of isometries of $\mathbb{C}^n$ equipped with the norm $\|z\|_1 = |z_1| + \ldots + |z_n|$ or $\|z\|_\infty = \max(|z_1|, \ldots, |z_n|)$. It is well known that $\Gamma_1(n)$ is a subgroup of $U(n)$ and each $L \in \Gamma_1(n)$ is of the form

$$L(z) = (e^{i\theta_1} z_{j_1}, \ldots, e^{i\theta_n} z_{j_n}),$$

where $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$.

As shown in Section 2, the condition $F^*(z) = F(Lz)$ for some $L \in U(n)$ (e.g. L has odd rank) implies $F(z) = |z|$. Now we show that some $L \in U(n)$ does not have this property.

Proposition 5.3. *Let* $L \in U(n) \setminus \Gamma_1(n)$ *and* $L^2 \in \Gamma_1(n)$. *Define a norm* F *in* $\mathbb{C}^n$ *by the condition*

$$((\mathbb{C}^n, \|\cdot\|_\infty), (\mathbb{C}^n, \|\cdot\|_1 \circ L))_{[1/2]} = (\mathbb{C}^n, F).$$

Then $F^*(z) = F(Lz)$ *and* $F(z)$ *is not equal to* $|z|$.

Proof. By the duality theorem,

$$\begin{aligned}(\mathbb{C}^n, F^*) &= ((\mathbb{C}^n, \|\cdot\|_1), (\mathbb{C}^n, \|\cdot\|_\infty \circ L))_{[1/2]} \\ &= ((\mathbb{C}^n, \|\cdot\|_\infty \circ L), (\mathbb{C}^n, \|\cdot\|_1))_{[1/2]}.\end{aligned}$$

Hence,

$$(\mathbb{C}^n, F^* \circ L^*) = ((\mathbb{C}^n, \|\cdot\|_\infty), (\mathbb{C}^n, \|\cdot\|_1 \circ L^*))_{[1/2]} = (\mathbb{C}^n, F),$$

because $\|L^* z\|_1 = \|L^2 L^* z\|_1 = \|Lz\|_1$.

Suppose $F(z) = |z|$. Then $|z|^2 \le \|z\|_\infty \|Lz\|_1$. Let

$$L = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The condition $L \notin \varGamma_1(n)$ implies that $\min(\|a_1\|_\infty, \ldots, \|a_n\|_\infty) < 1$. Let, for example, $\|a_1\|_\infty < 1$. Put $z_0 = \overline{a}_1$. Then $Lz_0 = e_1$ and thus

$$1 = |z_0|^2 \le \|a_1\|_\infty \cdot 1 < 1.$$

This contradiction completes the proof.

Now consider the special case of $n = 2$. If $L \in (U(2) \setminus \varGamma_1(2)) \cap O(2)$ and $L^2 \in \varGamma_1(2)$, then there are only the following possibilities:

$$L = \pm \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} \quad \text{or} \quad L = \pm \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad \text{with } \phi = \frac{\pi}{4}.$$

Let

$$Q = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \quad \phi = \frac{\pi}{4}.$$

Then $Q \notin \varGamma_1(2)$ and $Q^2 \in \varGamma_1(2)$. Let F be the norm in $\mathbb{C}^2$ given by

$$((\mathbb{C}^2, \|\cdot\|_\infty), (\mathbb{C}^2, \|\cdot\|_1 \circ Q))_{[1/2]} = (\mathbb{C}^2, F).$$

THEOREM 5.4. *Q is an isometric isomorphism between $(\mathbb{C}^2, F^*)$ and $(\mathbb{C}^2, F)$ but $(\mathbb{C}^2, F)$ is not isometrically isomorphic to $(\mathbb{C}^2, |\cdot|)$.*

PROOF. By Proposition 5.3, F is not the canonical Hilbertian norm in $\mathbb{C}^2$. For this reason it is enough to prove that if $F(z) = |Lz|$ for some L, then $F(z) = |z|$. If such an L exists, then $L = L_0 P$, where $L_0 = L_0^*$, $L \ge 0$ and $P \in U(2)$. Without loss of generality we may assume that

$$L_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \alpha, \beta > 0.$$

Then $F^*(z) = |L_0^{-1} P z|$, and $F^*(z) = F(Qz)$ iff

$$|L_0^{-1} P z| = |L_0 P Q z| \quad \text{or equivalently} \quad |L_0^{-1} z| = |L_0 P Q P^* z| = |L_0 B z|,$$

where $B = PQP^*$. This equality gives

$$\max(\alpha, \beta) = \|L_0\| = \|L_0^{-1}\| = \max(1/\alpha, 1/\beta)$$

and necessarily $\alpha \cdot \beta = 1$. Suppose $\alpha > 1$ and let $Bz = (B_1(z), B_2(z))$. Then

$$\alpha^{-2}|z_1|^2 + \alpha^2 |z_2|^2 = \alpha^2 |B_1(z)|^2 + \alpha^{-2} |B_2(z)|^2$$

and, for $|z| = 1$,

$$\alpha^{-2}(|z_1|^2 - |B_2(z)|^2) = \alpha^2(|z_1|^2 - |B_2(z)|^2),$$
$$\alpha^{-2}(|z_2|^2 - |B_1(z)|^2) = \alpha^2(|z_2|^2 - |B_1(z)|^2).$$

Therefore, $B_2(z) = e^{i\theta_1} z_1$, $B_1(z) = e^{i\theta_2} z_2$, so that

$$B = \begin{bmatrix} 0 & e^{i\theta_1} \\ e^{i\theta_2} & 0 \end{bmatrix}.$$

Since $\det Q = 1$, we obtain $\det B = -e^{i\theta_1}e^{i\theta_2} = 1$, and thus,

$$B = \begin{bmatrix} 0 & -e^{i\theta} \\ e^{-i\theta} & 0 \end{bmatrix} \quad \text{and} \quad PQ = \begin{bmatrix} 0 & -e^{i\theta} \\ e^{-i\theta} & 0 \end{bmatrix} P.$$

Let

$$P = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}.$$

Then we get two linear systems of equations

$$\begin{cases} \alpha_1 \cos\phi + \beta_1 \sin\phi = -\alpha_2 e^{i\theta}, \\ -\alpha_1 \sin\phi + \beta_1 \cos\phi = -\beta_2 e^{i\theta}, \end{cases} \quad \text{and} \quad \begin{cases} \alpha_2 \cos\phi + \beta_2 \sin\phi = \alpha_1 e^{-i\theta}, \\ -\alpha_2 \sin\phi + \beta_2 \cos\phi = \beta_1 e^{-i\theta}. \end{cases}$$

These equalities are equivalent to

$$(\alpha_2, \beta_2) = -e^{-i\theta} Q^*(\alpha_1, \beta_1), \quad (\alpha_1, \beta_1) = e^{i\theta} Q^*(\alpha_2, \beta_2).$$

Since $(\alpha_2, \beta_2) \cdot (\alpha_1, \beta_1) = 0$, we must have

$$(\alpha_1 \cos\phi + \beta_1 \sin\phi)\overline{\alpha}_1 + (-\alpha_1 \sin\phi + \beta_1 \cos\phi)\overline{\beta}_1 = 0,$$
$$\cos\phi + 2i \sin\phi \Im(\beta_1 \overline{\alpha}_1) = 0.$$

But this is possible only if $\cos\phi = 0$. This contradiction completes the proof.

Corollary 5.5. *Let F and Q be the norm and the unitary map from Theorem* 5.4, *respectively. Then*

$$((\mathbb{C}^2, F), (\mathbb{C}^2, F \circ Q))_{[1/2]} = (\mathbb{C}^2, |\cdot|) = (\mathbb{C}^2, \|\cdot\|_{[1/2]})$$

but $(\mathbb{C}^2, \|\cdot\|_{[1/2]})$ is not isometrically isomorphic to $(\mathbb{C}^2, F)$.

Corollary 5.6. *For each $n \geq 2$ there exists a Banach space $(\mathbb{C}^n, G)$ which is isometrically isomorphic to $(\mathbb{C}^n, G^*)$ and is not isometrically isomorphic to the Hilbert space $\mathbb{C}^n$.*

Proof. For $z \in \mathbb{C}^2,\ w \in \mathbb{C}^{n-2}$ define $G(z, w) = (F(z)^2 + |w|^2)^{1/2}$, where F is as in Theorem 5.4. Applying arguments similar to the proof of 2.11 we get

$$G^*(z, w) = (F^*(z)^2 + |w|^2)^{1/2} = (F(Qz)^2 + |w|^2)^{1/2} = G(Qz, w) = G(L(z, w)).$$

(Observe that $L \in U(n) \setminus \Gamma_1(n)$ and $L^2 \in \Gamma_1(n)$.) It is easily seen that $(\mathbb{C}^n, G)$ is not isometrically isomorphic to $(\mathbb{C}^n, |\cdot|)$. This completes the proof.

Now we introduce a certain class of norms in $\mathbb{C}^n$. The main goal of this section is to show that these norms can be obtained by applying the complex interpolation method to $\mathbb{C}^n$ equipped with the Turowicz and Lie norms. Fix $1 \leq p \leq \infty$. Define a function ℓ_p as follows:

$$\ell_p(z) = 2^{-1/p}\{(\lambda_1(z) + \lambda_2(z))^p + (\lambda_1(z) - \lambda_2(z))^p\}^{1/p}$$
$$= 2^{-1/p}[L_n(z)^p + (2T_n(z) - L_n(z))^p]^{1/p}$$

for $z \in \mathbb{C}^n$, where

$$\lambda_1(z) = \left(\frac{|z|^2 + |z^2|}{2}\right)^{1/2}, \quad \lambda_2(z) = \left(\frac{|z|^2 - |z^2|}{2}\right)^{1/2}.$$

Proposition 5.7. *ℓ_p is a norm in $\mathbb{C}^n$ such that*

(1) $\ell_p(x) = |x|$ *for* $x \in \mathbb{R}^n$, $\ell_1 = T_n$, $\ell_\infty = L_n$, $\ell_2 = |\cdot|$;

(2) $\ell_p(z) = 2^{-1/p}\{(|x| + |y|)^p + ||x| - |y||^p\}^{1/p}$ *if* $x \cdot y = 0$;

(3) $\ell_p(z) \geq (\leq)\ 2^{-1/p}\{(|x| + |y|)^p + ||x| - |y||^p\}^{1/p}$ *for* $z \in \mathbb{C}^n$ *and* $1 \leq p \leq 2$ *(resp.* $2 \leq p \leq \infty$);

(4) $\ell_p^* = \ell_q$, *where* $1/p + 1/q = 1$;

(5) $(\mathbb{C}^n, \ell_p)$ *is an interpolation space between* $(\mathbb{C}^n, \ell_\infty)$ *and* $(\mathbb{C}^n, \ell_1)$ *of exact exponent* $\theta = 1/p$.

Proof. The proof that ℓ_p is a norm which satisfies (1)–(4) is similar to the proof of Theorem 2.31. We may assume $1 < p < \infty$. It is easy to check that ℓ_p is a homogeneous function in $\mathbb{C}^n$ which has properties (1), (2). We have

$$(\lambda_1(z) \pm \lambda_2(z))^2 = |x|^2 + |y|^2 \pm 2(|x|^2|y|^2 - (x \cdot y)^2)^{1/2}.$$

Observe that the function $g(t) := (c+t)^r + (c-t)^r$, $0 \leq t \leq c$, where $c > 0$ is a constant, is decreasing if $0 < r < 1$ and increasing for $r > 1$. Hence we easily deduce the statement in (3). Note that the right-hand side of the inequalities in (3), which we denote by $f_p(x, y)$, is a norm in $\mathbb{R}^{2n}$. Here the subadditivity is a consequence of the fact that the function $(c + t)^p + |c - t|^p$, with fixed $c \geq 0$, is increasing for $t \geq 0$. Thus, applying Drużkowski's lemma, we easily verify that ℓ_p is a subadditive function if $1 < p \leq 2$ and we conclude that ℓ_p is a norm for such p. For the proof that ℓ_p is also a norm for $p > 2$, we need the following:

Claim 5.8. *$f_p^*(x, y) = f_q(x, y)$ in $\mathbb{R}^{2n}$, where $1/p + 1/q = 1$.*

Proof. It is easy to check that the norm f_p, $1 < p < \infty$, is differentiable on $\mathbb{R}^{2n} \setminus \{0\}$ and $f_q(\operatorname{grad} f_p(x, y)) = 1$. This implies $f_p^* \geq f_q$. On the other hand, by Proposition 1.1(4), we have

$$\begin{aligned} f_p^*(x, y) &\leq 2^{1/p-1} \sup\Big\{\sum_{j=0}^{1}(|x| + (-1)^j|y|)(|u| + (-1)^j|v|) : 2^{1/p} f_p(u, v) \leq 1\Big\} \\ &\leq 2^{-1/q} \sup\{(|x| + |y|)t + ||x| - |y||\tau : t^p + \tau^p \leq 1\} = f_q(x, y), \end{aligned}$$

which completes the proof.

Now let $p \in (1, 2]$. By 1.1(2) and 5.8 we have $\ell_p^*(z) \leq f_q(z)$. It suffices to prove the equality in (4) for $z \neq 0$ such that $x \cdot y = 0$. Then $\operatorname{grad} f_q(x + iy) = \alpha x + i\beta y = u + iv$, where $\alpha, \beta \in \mathbb{R}$. Thus, $u \cdot v = 0$. Moreover, $\ell_p(u + iv) = f_p(u, v) = 1$ and $x \cdot u + y \cdot v = f_q(x, y) = \ell_q(z)$. This gives $\ell_p^*(z) \geq f_q(z)$ and $\ell_p^*(z) = \ell_q(z)$, which completes the proof of (4). In particular, ℓ_q is a norm for $q \geq 2$. We can easily verify that property (5) holds.

Remark 5.9. In the special case $p = 4$ we have the following simple formula:

$$\begin{aligned} \ell_4(z) &= (|x|^4 + |y|^4 + 6|x|^2|y|^2 - 4(x \cdot y)^2)^{1/4} \\ &= 2^{-1/4}[(|x| + |y|)^4 + (|x| - |y|)^4 - 8(x \cdot y)^2]^{1/4}. \end{aligned}$$

If $n = 2$ we have the simplest formula for ℓ_p with arbitrary p:

$$\ell_p(z) = 2^{-1/p}(|z_1 - iz_2|^p + |z_1 + iz_2|^p)^{1/p}.$$

Hence, by a Braun–Kaup–Upmeier result [B-K], we obtain

$$\mathrm{Aut}(\{\ell_p < 1\}) = S^1 \cdot O(2) \quad \text{for } 1 \le p < \infty,\ p \ne 2.$$

THEOREM 5.10.

$$((\mathbb{C}^n, \ell_\infty), (\mathbb{C}^n, \ell_1))_{[1/p]} = (\mathbb{C}^n, \ell_p)$$

for $1 < p < \infty$, $n \ge 2$.

The basic role in the proof is played by the following proposition (we omit its purely technical proof).

PROPOSITION 5.11. *Let* $\mu_p(z) = 2^{1/p}\ell_p(z)$. *Then* $\ell_p \in \mathcal{C}^1(\mathbb{R}^{2n} \setminus \{0\})$ *for* $1 < p < \infty$ *and*

$$\mathrm{grad}\,\ell_p = 2^{-1/p}(\mu_p)^{1-p}[(L_n^{p-1}\,\mathrm{grad}\,L_n + (2T_n - L_n)^{p-1}(2\,\mathrm{grad}\,T_n - \mathrm{grad}\,L_n))].$$

Note that the singularities of $L_n(z)$ and $T_n(z)$ are cancelled in the above formula by the factor $L_n^{p-1} - (2T_n - L_n)^{p-1}$ and $(2T_n - L_n)^{p-1}$, respectively. As an immediate consequence of 5.11 we obtain, by the Shmul'yan theorem and by Drużkowski's lemma,

COROLLARY 5.12.

$$\exp \ell_p = \{e^{i\theta}\,\mathrm{grad}\,\ell_q(z) : \theta \in \mathbb{R},\ z = x + iy,\ x \cdot y = 0\}.$$

Here, *for* $z = x + iy$, $x \cdot y = 0$, *we have*

$$\text{(5.12.1)} \qquad \mathrm{grad}\,\ell_q(z) = \begin{cases} 2^{-1/q}(\mu_q)^{1-q}L_n^{q-1}\sqrt{2}\dfrac{z}{|z|}, \\ 2^{-1/q}(\mu_q)^{1-q}(2T_n - L_n)^{q-1}\dfrac{1}{\max(|x|,|y|)}(z + \mathrm{sign}(|x| - |y|)\overline{z}), \\ 2^{-1/q}(\mu_q)^{1-q}\left[L_n^{q-1}\left(\dfrac{x}{|x|} + i\dfrac{y}{|y|}\right) + (2T_n - L_n)^{q-1}\left(\dfrac{x}{|x|} - i\dfrac{y}{|y|}\right)\right], \end{cases}$$

for $z^2 = 0$, $|z|^2 = |z^2|$ *or* $|z|^2 > |z^2| > 0$, *respectively.*

It is also an important fact that $\mathrm{grad}\,\ell_q(\overline{z}) = \overline{\mathrm{grad}\,\ell_q(z)}$. Applying formulas for $\mathrm{grad}\,L_n$ and $\mathrm{grad}\,T_n$ one can easily check

PROPOSITION 5.13.

$$L_n(e^{i\tau}\,\mathrm{grad}\,L_n(z) + (2\,\mathrm{grad}\,T_n(z) - \mathrm{grad}\,L_n(z))) = 2,$$
$$T_n(e^{i\tau}L_n^q(z)\,\mathrm{grad}\,L_n(z) + (2T_n(z) - L_n(z))^q(2\,\mathrm{grad}\,T_n(z) - \mathrm{grad}\,L_n(z))) = (\mu_q)^q$$

for all $\tau \in \mathbb{R}$ *and* $|z|^2 > |z^2| > 0$. *If* $z^2 = 0$, *then* $L_n(\mathrm{grad}\,L_n(z)) = 2T_n(\mathrm{grad}\,L_n(z)) = 2$ *and* $T_n(\mathrm{grad}\,T_n(z)) = L(\mathrm{grad}\,T_n(z)) = 1$ *in the case* $|z|^2 = |z^2|$.

PROOF OF THEOREM 5.10. We apply the main ideas of Thorin's proof of the Riesz–Thorin theorem and Calderón's proof of the interpolation theorem for L_p (see [B-L]) with suitable modifications. Fix $p \in (1, \infty)$. Define $h_q : S \times (\mathbb{C}^n \setminus \{0\}) \to \mathbb{C}^n$ by

$$h_q(\zeta, z) = 2^{\zeta-1}(\mu_q(z))^{-q\zeta}[(L_n(z))^{q\zeta}\,\mathrm{grad}\,L_n(z) + (2T_n(z) - L_n(z))^{q\zeta}(2\,\mathrm{grad}\,T_n(z) - \mathrm{grad}\,L_n(z))]$$

for $|z|^2 > |z^2| > 0$ and

$$h_q(\zeta,z) = \begin{cases} 2^{\zeta-1}(\mu_q(z))^{-q\zeta}(L_n(z))^{q\zeta}\operatorname{grad} L_n(z) & \text{for } z^2=0,\\ 2^{\zeta}(\mu_q(z))^{-q\zeta}(2T_n(z)-L_n(z))^{q\zeta}\operatorname{grad} T_n(z) & \text{for } |z|^2=|z^2|. \end{cases}$$

We have, by 5.13,

$$(*) \qquad L_n(h_q(it,z)) = 1, \quad T_n(h_q(1+it,z)) = 1$$

for any $t\in\mathbb{R}$. Now, for fixed $z\in\mathbb{C}^n\setminus\{0\}$ and $\varepsilon>0$, define $f:S\to\mathbb{C}^n$ as

$$f(\zeta) = e^{\varepsilon\zeta^2-\varepsilon p^{-2}}h_q(\zeta,z).$$

Then $f\in\mathcal{F}((\mathbb{C}^n,\ell_\infty),(\mathbb{C}^n,\ell_1))$, $f(1/p)=\operatorname{grad}\ell_q(z)$ and, by $(*)$, $\|f\|_{\mathcal{F}}\le e^{\varepsilon}$, which implies

$$\|w\|_{[1/p]}\le \ell_p(w) \quad \text{for every } w\in\mathbb{C}^n.$$

Now assume $\|z\|_{[1/p]}=1$. Let $f\in\mathcal{F}$ be such that $f(1/p)=z$ and $\|f\|_{\mathcal{F}}\le 1+\varepsilon$. Define $F:S\to\mathbb{C}$ by $F(\zeta)=f(\zeta)\cdot\overline{h_p(1-\zeta,\overline{z})}$. It is obvious that $F\in\mathcal{O}(S_0)\cap\mathcal{C}(S)$, $\|F\|_S<\infty$ and, by $(*)$ and 2.5(7),

$$|F(it)|\le 1+\varepsilon, \quad |F(1+it)|\le 1+\varepsilon, \quad t\in\mathbb{R}.$$

Hence, by the Hadamard three lines theorem, $|F(\zeta)|\le 1+\varepsilon$ for all $\zeta\in S$. In particular,

$$\ell_p(z)\le |z\cdot\operatorname{grad}\ell_p(z)| = |F(1/p)|\le 1+\varepsilon,$$

which implies $\ell_p(w)\le\|w\|_{[1/p]}$ for $w\in\mathbb{C}^n$. The proof is complete.

Applying the reiteration theorem for the complex method we obtain the following generalization of Theorem 5.10.

Theorem 5.14. *Let $1\le p_1,p_2\le\infty$ and $\theta\in[0,1]$. Then*

$$((\mathbb{C}^n,\ell_{p_1}),(\mathbb{C}^n,\ell_{p_2}))_{[\theta]} = (\mathbb{C}^n,\ell_p),$$

where $\frac{1}{p}=(1-\theta)\frac{1}{p_1}+\theta\frac{1}{p_2}$.

Remark 5.15. Note the following interpretation of Theorem 5.10:

$$((\mathbb{R}^n,|\cdot|)\,\hat{\otimes}_{\mathbb{R}}\,\mathbb{C}),(\mathbb{R}^n,|\cdot|)\,\check{\otimes}_{\mathbb{R}}\,\mathbb{C}))_{[1/p]} = (\mathbb{C}^n,\ell_p).$$

The complex Banach space $(\mathbb{C}^n,\|\cdot\|_{[1/p]})=(\mathbb{C}^n,\ell_p)$ from Theorem 5.10 is uniformly smooth and uniformly convex for arbitrary $1<p<\infty$. Moreover, $\|a\|^*_{[1/p]}=\|a\|_{[1/q]}$ with $1/p+1/q=1$. We show below that this is a special case of a more general situation. As an application we obtain Clarkson's type inequalities for the norms ℓ_p:

$$(\ell_p(z+w)^q+\ell_p(z-w)^q)^{1/q}\le 2^{1/q}(\ell_p(z)^p+\ell_p(w)^p)^{1/p}, \quad 1\le p\le 2,$$
$$(\ell_p(z+w)^p+\ell_p(z-w)^p)^{1/p}\le 2^{1/q}(\ell_p(z)^p+\ell_p(w)^p)^{1/p}, \quad 2\le p\le\infty,$$

where $1/p+1/q=1$.

Let F be a norm in $X=\mathbb{C}^n$ and let $((X,F),(X,F^*))_{[\theta]}=(X,\|\cdot\|_{[\theta]})$. Define the following norms in X^k:

$$G_p(z_1,\dots,z_k) = (\|z_1\|^p_{[1/p]}+\dots+\|z_k\|^p_{[1/p]})^{1/p},$$
$$H_q(z_1,\dots,z_k) = (\|z_1\|^q_{[1/p]}+\dots+\|z_k\|^q_{[1/p]})^{1/q}$$

for $1 \le p \le \infty,\ 1/p + 1/q = 1$. In particular, we have

$$G_\infty(z_1, \ldots, z_k) = \max(F(z_1), \ldots, F(z_k)), \qquad G_1(z_1, \ldots, z_k) = F^*(z_1) + \ldots + F^*(z_k),$$
$$H_\infty(z_1, \ldots, z_k) = \max(F^*(z_1), \ldots, F^*(z_k)), \quad H_1(z_1, \ldots, z_k) = F(z_1) + \ldots + F(z_k).$$

By the duality theorem we have $\|\cdot\|^*_{[1/p]} = \|\cdot\|_{[1/q]}$, which gives, by the formula from Remark 2.17,

$$G_p^* = G_q, \quad H_q^* = H_p, \quad 1/p + 1/q = 1.$$

Now we formulate the following interpolation result which may be interpreted as a special case of Calderón's theorem on interpolation of vector-valued functions (see [B-L, Thm. 5.1.2]). We present here a different proof of this version of Calderón's theorem. Our method permits one also to prove other interpolation results which are not a special case of Calderón's theorem. (For instance, if F is a norm in $\mathbb{C}^n$, G is a norm in $\mathbb{C}^k$, $((\mathbb{C}^n, F),\ (\mathbb{C}^n, F^*))_{[1/p]} = (\mathbb{C}^n, F_p), ((\mathbb{C}^k, G), (\mathbb{C}^k, G^*))_{[1/p]} = (\mathbb{C}^k, G_p)$ and $H_p(z, w) = (F_p(z)^p + G_p(w)^p)^{1/p}$ for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^k$ then

$$((\mathbb{C}^n \times \mathbb{C}^k, H_\infty), (\mathbb{C}^n \times \mathbb{C}^k, H_1))_{[1/p]} = (\mathbb{C}^n \times \mathbb{C}^k, H_p).$$

Theorem 5.16 is (for $k = 2$) a special case of this result.)

THEOREM 5.16. *Let $X = \mathbb{C}^n$ and $k \ge 1$. Then for all $1 < p < \infty$ we have*

$$((X^k, G_\infty), (X^k, G_1))_{[1/p]} = (X^k, G_p),$$
$$((X^k, H_\infty), (X^k, H_1))_{[1/p]} = (X^k, H_p).$$

PROOF. It is enough to prove the first equality; the second follows by changing the roles of F and F^* and by the duality

$$((X, F^*), (X, F))_{[1/p]} = ((X, F), (X, F^*))_{[1/q]}.$$

We consider, for simplicity, the case $k = 2$. Let $A_0 = (X, F),\ A_1 = (X, F^*),\ B_0 = (X^k, G_\infty),\ B_1 = (X^k, G_1)$. First, observe that for $1 < p < \infty$,

$$\exp G_p = \{G_q(z, w)^{1-q}(\|z\|^{q-1}_{[1/q]} \operatorname{grad} \|\cdot\|_{[1/q]}(z), \|w\|^{q-1}_{[1/q]} \operatorname{grad} \|\cdot\|_{[1/q]}(w)) : (z, w) \in \mathcal{D}(G_q)\},$$

where $\mathcal{D}(G_q) = (\mathcal{D}(\|\cdot\|_{[1/q]}) \cup \{0\})^2 \setminus \{(0, 0)\}$. Fix $(z, w) \in \mathcal{D}(G_q)$ and let

$$z_0 = \operatorname{grad} \|\cdot\|_{[1/q]}(z) \quad \text{and} \quad w_0 = \operatorname{grad} \|\cdot\|_{[1/q]}(w).$$

Then $\|z_0\|_{[1/p]} = \|w_0\|_{[1/p]} = 1$. Let $f, g \in \mathcal{F}(A_0, A_1)$ be chosen such that $f(1/p) = z_0$, $g(1/p) = w_0$, $\|f\|_{\mathcal{F}} \le 1 + \varepsilon$, $\|g\|_{\mathcal{F}} \le 1 + \varepsilon$. Define

$$\phi_p(\zeta) = \phi_p(z, w, \zeta) = G_q(z, w)^{-q\zeta}(\|z\|^{q\zeta}_{[1/q]} f(\zeta), \|w\|^{q\zeta}_{[1/q]} g(\zeta)).$$

Then $\phi_p \in \mathcal{F}(B_0, B_1)$ and $\phi_p(1/p) = \operatorname{grad} G_q(z, w)$. We have

$$G_\infty(\phi_p(it)) = \max(F(f(it)), F(g(it))),$$
$$G_1(\phi_p(1 + it)) = G_q(z, w)^{-q}(\|z\|^q_{[1/q]} F^*(f(1 + it)) + \|w\|^q_{[1/q]} F^*(g(1 + it))),$$

which implies

$$\sup_{\mathbb{R}} \|\phi_p(it)\|_{B_0} = \max(\sup_{\mathbb{R}} \|f(it)\|_{A_0}, \sup_{\mathbb{R}} \|g(it)\|_{A_0})$$
$$\le \max(\|f\|_{\mathcal{F}}, \|g\|_{\mathcal{F}}) \le 1 + \varepsilon,$$

$$\sup_{\mathbb{R}} \|\phi_p(1+it)\|_{B_1}$$
$$\le G_q(z,w)^{-q}(\|z\|^q_{[1/q]} \sup_{\mathbb{R}} \|f(1+it)\|_{A_1} + \|w\|^q_{[1/q]} \sup_{\mathbb{R}} \|g(1+it)\|_{A_1})$$
$$\le G_q(z,w)^{-q}(\|z\|^q_{[1/q]} \|f\|_{\mathcal{F}} + \|w\|^q_{[1/q]} \|g\|_{\mathcal{F}} \le 1+\varepsilon.$$

This gives $\|\phi_p\|_{\mathcal{F}} \le 1+\varepsilon$ and therefore

$$\|\operatorname{grad} G_q(z,w)\|_{[1/p]} \le 1.$$

Hence, by Corollary 1.4, we get

$$\|(z,w)\|_{[1/p]} \le G_p(z,w), \quad (z,w) \in X^2.$$

Now let $(z,w) \in X^2$ and $\|(z,w)\|_{[1/p]} = 1$. Fix $\operatorname{grad} G_p(z_1,w_1) \in \exp G_q$. Define

$$\Phi(\zeta) = \phi_q(z_1,w_1,1-\zeta) \cdot h(\overline{\zeta}),$$

where $h \in \mathcal{F}(B_0,B_1)$, $h(1/p) = (z,w)$, $\|h\|_{\mathcal{F}} \le 1+\varepsilon$. Then $\Phi \in \mathcal{C}(S) \cap L^\infty(S) \cap \mathcal{O}(S_0)$. We have

$$|\Phi(it)| \le G_\infty(h(-it))G_1(\phi_q(z_1,w_1,1-it)) \le (1+\varepsilon)^2,$$
$$|\Phi(1+it)| \le G_1(h(1-it))G_\infty(\phi_q(z_1,w_1,-it)) \le (1+\varepsilon)^2,$$

which gives, by the Hadamard three lines theorem, $|\Phi(\zeta)| \le (1+\varepsilon)^2$. In particular,

$$|\Phi(1/p)| = |(z,w)\cdot \operatorname{grad} G_p(z_1,w_1)| \le (1+\varepsilon)^2,$$

and thus

$$G_p(z,w) = \sup\{|(z,w)\cdot \operatorname{grad} G_p(z_1,w_1)| : (z_1,w_1) \in \mathcal{D}(G_p)\} \le (1+\varepsilon)^2,$$

which implies $G_p(z,w) \le \|(z,w)\|_{[1/p]}$ for all $(z,w) \in X^2$. This completes the proof.

Now applying the reiteration theorem we obtain the following:

Corollary 5.17.

(1) $$((X^2,G_\infty),(X^2,G_2))_{[2/p]} = (X^2,G_p) \quad \textit{for } 2 \le p \le \infty,$$

(2) $$((X^2,H_1),(X^2,H_2))_{[2/p]} = (X^2,H_q) \quad \textit{for } 2 \le p \le \infty,\ 1/p+1/q=1,$$

(3) $$((X^2,G_2),(X^2,G_1))_{[2/p-1]} = (X^2,G_p) \quad \textit{for } 1 \le p \le 2,$$

(4) $$((X^2,H_2),(X^2,H_\infty))_{[2/p-1]} = (X^2,H_q) \quad \textit{for } 2 \le p \le \infty,\ 1/p+1/q=1.$$

Theorem 5.18 (Clarkson's type inequalities). *With the same assumptions on F we have the inequalities*

(1) $$(\|z+w\|^p_{[1/p]} + \|z-w\|^p_{[1/p]})^{1/p} \le 2^{1/q}(\|z\|^p_{[1/p]} + \|w\|^p_{[1/p]})^{1/p} \quad \textit{for } 2 \le p \le \infty,$$

(2) $$(\|z+w\|^p_{[1/p]} + \|z-w\|^p_{[1/p]})^{1/p} \le 2^{1/p}(\|z\|^q_{[1/p]} + \|w\|^q_{[1/p]})^{1/q} \quad \textit{for } 2 \le p \le \infty,$$

(3) $$(\|z+w\|^q_{[1/p]} + \|z-w\|^q_{[1/p]})^{1/q} \le 2^{1/q}(\|z\|^p_{[1/p]} + \|w\|^p_{[1/p]})^{1/p} \quad \textit{for } 1 \le p \le 2,$$

(4) $$(\|z+w\|^q_{[1/p]} + \|z-w\|^q_{[1/p]})^{1/q} \le 2^{1/p}(\|z\|^q_{[1/p]} + \|w\|^q_{[1/p]})^{1/q} \quad \textit{for } 1 \le p \le 2.$$

Proof. Since (2)⇒(1) and (3)⇒(4), it is enough to prove only (2) and (3). The idea of the proof is based on Maligranda–Persson's proof of the classical Clarkson–Hausdorff–

Young inequality (see [M-P1, 2]). Define a linear operator $T : X^2 \to X^2$ as $T(z,w) = (z+w, z-w)$. Then $T : (X^2, H_2) \to (X^2, G_2)$ has norm $M_0 = 2^{1/2}$ and $T : (X^2, H_1) \to (X^2, G_\infty)$ has norm ≤ 1. Hence, by the interpolation theorem, for $2 \le p \le \infty$,

$$T : ((X^2, H_1), (X^2, H_2))_{[2/p]} \to ((X^2, G_\infty), (X^2, G_2))_{[2/p]},$$

i.e. $T : (X^2, H_q) \to (X^2, G_p)$ has norm $M_{2/p} \le 2^{\frac{1}{2}\frac{2}{p}} = 2^{1/p}$, which gives the second inequality.

We also have

$$H_\infty(z+w, z-w) = \max(F^*(z+w), F^*(z-w)) \le F^*(z) + F^*(w) = G_1(z,w),$$

which means that $T : (X^2, G_1) \to (X^2, H_\infty)$ has norm ≤ 1. This implies that for $1 \le p \le 2$ the operator

$$T : ((X^2, G_2), (X^2, G_1))_{[2/p-1]} \to ((X^2, H_2), (X^2, H_\infty))_{[2/p-1]},$$

i.e. $T : (X^2, G_p) \to (X^2, H_q)$, has norm $M_{2/p-1} \le 2^{\frac{1}{2}(1-(2/p-1))} = 2^{1/q}$, which gives the third inequality. The proof is complete.

Corollary 5.19. *For every $1 < p < \infty$ the Banach space $(\mathbb{C}^n, \|\cdot\|_{[1/p]})$ is uniformly convex, uniformly smooth and the unit ball has strictly convex boundary.*

Remark 5.20. In particular, Theorem 5.18 and Corollary 5.19 hold for every norm $\mathcal{F}(f,z)$. Considering the norm $f(x_1,\dots,x_n) = \max(|x_1|,\dots,|x_n|)$ we obtain as a special case of Theorem 5.18 classical Clarkson's inequalities in $\mathbb{C}^n$. If we put $F(z) = |z|$ then we obtain the following inequality for the Euclidean norm:

$$(|z-w|^p + |z+w|^p)^{1/p} \le 2^{1/q}(|z|^q + |w|^q)^{1/q}$$

for $z, w \in \mathbb{C}^n$, $p \ge 2$. For $n = 1$ this inequality reduces to the Clarkson–Hausdorff–Young inequality.

We proved in the first section that for every Banach space $(\mathbb{R}^n, f)$ there exists a sequence of smooth spaces $(\mathbb{R}^n, f_k)$ such that $f_k \to f$ almost uniformly. Now we prove that for a complex Banach space $(\mathbb{C}^n, F)$ there exists a sequence of uniformly smooth and uniformly convex spaces $(\mathbb{C}^n, F_k)$ such that F_k is almost uniformly convergent to F. This is a corollary of

Proposition 5.21. *Let F be a norm in $\mathbb{C}^n$, put*

$$(\mathbb{C}^n, F_p) = ((\mathbb{C}^n, F), (\mathbb{C}^n, F^*))_{[1/p]}$$

for $1 < p < \infty$ and let

$$A = \sup_{|z|=1} \frac{F^*(z)}{F(z)}, \quad B = \inf_{|z|=1} \frac{F^*(z)}{F(z)}.$$

Then for all $1 < p < \infty$,

$$B^{1/p} F \le F_p \le A^{1/p} F.$$

Proof. Define $G_p = F_q$, $1/p + 1/q = 1$. Then

$$F_p \le F^{1-1/p}(F^*)^{1/p}, \quad G_p \le (F^*)^{1-1/p} F^{1/p},$$

which implies

$$F_p \leq A^{1/p} F, \quad G_p \leq B^{-1/p} F^*.$$

The second inequality gives $F_p \geq B^{1/p} F$, which completes the proof.

COROLLARY 5.22. *If F is a norm in $\mathbb{C}^n$ then, as $p \to \infty$, the interpolating norms F_p and G_p are almost uniformly convergent to F and F^*, respectively. The Banach spaces $(\mathbb{C}^n, F_p)$ and $(\mathbb{C}^n, G_p)$ are uniformly smooth and uniformly convex. Moreover, we have the following estimates for the modulus of uniform convexity and smoothness for $(\mathbb{C}^n, F_p)$ and $(\mathbb{C}^n, G_p)$, $p \geq 2$:*

$$\Delta_{F_p}(\varepsilon) \geq 1 - (1 - (\varepsilon/2)^p)^{1/p}, \quad \Delta_{G_p}(\varepsilon) \geq 1 - (1 - (\varepsilon/2)^p)^{1/p}, \quad 0 < \varepsilon \leq 2;$$
$$\varrho_{F_p}(\tau) \leq (1 + \tau^q)^{1/q} - 1, \quad \varrho_{G_p}(\tau) \leq (1 + \tau^q)^{1/q} - 1, \quad \tau > 0.$$

PROOF. The convergence is a consequence of Proposition 5.21. The first two inequalities follow by Clarkson's inequalities for F_p. The second couple of inequalities is easily obtained from the Lindenstrauss duality formulas.

6. On tensor products $\mathbb{R}^n \otimes \mathbb{R}^k$ and $\mathbb{C}^n \otimes \mathbb{C}^k$

Let $\mathbb{X} = \mathbb{K}^n$, $\mathbb{Y} = \mathbb{K}^k$ $(n \geq k,\ \mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C})$ be linear spaces with canonical Euclidean or Hermitian structure. We denote by $\mathbb{X} \check{\otimes} \mathbb{Y}$ and $\mathbb{X} \hat{\otimes} \mathbb{Y}$ the injective and projective tensor products, respectively, i.e. $\mathbb{X} \otimes \mathbb{Y}$ equipped with the smallest and greatest reasonable crossnorm (see [P1]). Let $L(\mathbb{X}, \mathbb{Y})$ be the Banach space of all linear operators from $\mathbb{X}$ into $\mathbb{Y}$ with the usual operator norm. The following is a special case of known facts (see e.g. [L-C], [P2]):

$$(\mathbb{X} \check{\otimes} \mathbb{Y})^* = \mathbb{X} \hat{\otimes} \mathbb{Y}, \quad \mathbb{X} \check{\otimes} \mathbb{Y} \simeq L(\mathbb{X}, \mathbb{Y}) \quad \text{(isometrical isomorphism).}$$

Now observe that $L(\mathbb{X}, \mathbb{Y}) \simeq (\mathbb{X}^k, \mathcal{T}_k)$, where

$$\mathcal{T}_k(x_1, \ldots, x_k) = \max_{|w|=1} (|w \cdot x_1|^2 + \ldots + |w \cdot x_k|^2)^{1/2}.$$

Let $\mathcal{L}_k = \mathcal{T}_k^*$. Note that $\mathcal{T}_1(x) = \mathcal{L}_1(x) = |x|$ and $\mathcal{T}_2 = T$, $\mathcal{L}_2 = L$ for $\mathbb{K} = \mathbb{R}$. Thus, we can write

$$\mathbb{X} \check{\otimes} \mathbb{Y} \simeq (\mathbb{X}^k, \mathcal{T}_k), \quad \mathbb{X} \hat{\otimes} \mathbb{Y} \simeq (\mathbb{X}^k, \mathcal{L}_k).$$

As mentioned earlier, the structure of exposed points of projective tensor products is known and thus, so is the structure of $\exp \mathcal{L}_k$. Namely, by the Heinrich–Ruess–Stegall theorem (see [R-S]) we have

$$\exp \mathcal{L}_k = \{(\alpha_1 x, \ldots, \alpha_k x) : x \in \mathbb{X},\ |x| = 1,\ \alpha \in \mathbb{Y},\ |\alpha| = 1\}.$$

In this section we show that

$$\exp \mathcal{T}_k = V_k(\mathbb{X})$$

where

$$V_k(\mathbb{X}) = \{(x_1, \ldots, x_k) \in \mathbb{X}^k : x_i \cdot x_j = \delta_{ij},\ i, j = 1, \ldots, k\}$$

is the kth Stiefel manifold in $\mathbb{X}$. To get this we calculate the norms $\mathcal{T}_k$ and $\mathcal{L}_k$ for $k = 1, \ldots, n$. The result is well known for $k = n$ (see [G-L], where it is given without proof) and in the general case the formulas are analogous. Applying these formulas we find the interpolating spaces $((\mathbb{X}^k, \mathcal{T}_k), (\mathbb{X}^k, \mathcal{L}_k))_{[\theta]}$ and thus $(\mathbb{C}^n \check{\otimes} \mathbb{C}^k, \mathbb{C}^n \hat{\otimes} \mathbb{C}^k)_{[\theta]}$, which is the main goal of this section.

First, we consider the case $\mathbb{K} = \mathbb{R}$.

For $a_1, \ldots, a_k \in \mathbb{R}^n$ we define the quadratic form

$$Q_{\mathbf{A}}(w) = (w \cdot a_1)^2 + \ldots + (w \cdot a_k)^2, \quad w \in \mathbb{R}^n,$$

where $\mathbf{A} = (a_1, \ldots, a_k)$ is the matrix with $a_1, \ldots, a_k$ as columns. Then

$$Q_{\mathbf{A}}(w) = w^* \mathbf{A}\mathbf{A}^* w, \quad \max_{|w|=1} Q_{\mathbf{A}}(w) = \max_{1 \le i \le k} \lambda_i(\mathbf{A}\mathbf{A}^*)$$

where $\lambda_i(\mathbf{A}\mathbf{A}^*)$ are the eigenvalues of $\mathbf{A}\mathbf{A}^*$. It is known that the nonzero eigenvalues of $\mathbf{A}\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{A}$ are the same (see e.g. [M-O]). Thus

$$\max_{|w|=1} Q_{\mathbf{A}}(w) = \max_{1 \le i \le k} \lambda_i(\mathbf{A}^*\mathbf{A}) = \max_{1 \le i \le k} \lambda_i(\mathbf{G}),$$

where $\mathbf{G} = \mathbf{G}(a_1, \ldots, a_k)$ is the Gram matrix. This implies

$$\mathcal{T}_k(a_1, \ldots, a_k) = \max_{1 \le i \le k} \lambda_i(\mathbf{G}(a_1, \ldots, a_k))^{1/2} = \max_{1 \le i \le k} \lambda_i(\mathbf{G}(a_1, \ldots, a_k)^{1/2}) = \max_{1 \le i \le k} \sigma_i(\mathbf{A}),$$

where $\sigma_i(\mathbf{A}) = \lambda_i((\mathbf{A}^*\mathbf{A})^{1/2})$ are called the *singular numbers* of $\mathbf{A}$ (see e.g. [M-O] or [G-L]). We have, by the Schur theorem (see [M-O]),

$$(|a_1|^2, \ldots, |a_k|^2) \prec (\lambda_1(\mathbf{G}(a_1, \ldots, a_k)), \ldots, \lambda_k(\mathbf{G}(a_1, \ldots, a_k)))$$

and therefore

$$\max_{1 \le i \le k} \lambda_i(\mathbf{G}(a_1, \ldots, a_k)^{1/2}) \ge \max(|a_1|, \ldots, |a_k|),$$
$$\operatorname{tr} \mathbf{G}(a_1, \ldots, a_k)^{1/2} \le |a_1| + \ldots + |a_k|.$$

Here $x \prec y$ denotes the relation of majorization:

$$\sum_{i=1}^{j} x_{[i]} \le \sum_{i=1}^{j} y_{[i]}, \quad j = 1, \ldots, k-1, \quad \sum_{i=1}^{k} x_{[i]} = \sum_{i=1}^{k} y_{[i]},$$

where the vectors $(x_{[1]}, \ldots, x_{[k]})$ and $(y_{[1]}, \ldots, y_{[k]})$ are permutations of x and y such that $x_{[1]} \ge \ldots \ge x_{[k]}$, $y_{[1]} \ge \ldots \ge y_{[k]}$.

Now we show that

$$\mathcal{L}_k(a_1, \ldots, a_k) = \operatorname{tr} \mathbf{G}(a_1, \ldots, a_k)^{1/2} = \sum_{i=1}^{k} \lambda_i(\mathbf{G}(a_1, \ldots, a_k))^{1/2}.$$

If $a_1, \ldots, a_k$ are orthogonal then it is easy to check that

$$\mathcal{L}_k(a_1, \ldots, a_k) = |a_1| + \ldots + |a_k| = \operatorname{tr} \mathbf{G}(a_1, \ldots, a_k)^{1/2}.$$

The group $O(k)$ acts on $\mathbb{X}^k$ in the following way. Let $\mathbf{M} = [\alpha_{ij}] \in O(k)$. Then

$$(a_1, \ldots, a_k)\mathbf{M} = \mathbf{A}\mathbf{M} = \Big(\sum_{j=1}^{k} \alpha_{j1} a_j, \ldots, \sum_{j=1}^{k} \alpha_{jk} a_j \Big) = (b_1, \ldots, b_k).$$

Observe the following facts:

$$Q_{\mathbf{AM}}(w) = Q_{\mathbf{A}}(w),$$
$$(a_1, \dots, a_k)\mathbf{M} \cdot (x_1, \dots, x_k) = (a_1, \dots, a_k) \cdot (x_1, \dots, x_k)\mathbf{M}^*,$$
$$\mathbf{G}(b_1, \dots, b_k) = \mathbf{M}^{-1}\mathbf{G}(a_1, \dots, a_k)\mathbf{M},$$

which implies (for all $\mathbf{M} \in O(k)$)

$$\mathcal{T}_k((a_1, \dots, a_k)\mathbf{M}) = \mathcal{T}_k(a_1, \dots, a_k),$$
$$\mathcal{L}_k((a_1, \dots, a_k)\mathbf{M}) = \mathcal{L}_k(a_1, \dots, a_k),$$
$$\operatorname{tr}\mathbf{G}((a_1, \dots, a_k)\mathbf{M})^{1/2} = \operatorname{tr}\mathbf{G}(a_1, \dots, a_k)^{1/2}.$$

This means that the above functions are orthogonally invariant. We need the following lemma:

Lemma 6.1. *If $a_1, \dots, a_k \in \mathbb{R}^n$, then there exists $\mathbf{M} \in O(k)$ such that $(a_1, \dots, a_k)\mathbf{M} = (b_1, \dots, b_k)$ with $b_i \cdot b_j = \delta_{ij}\lambda_i(\mathbf{G}(a_1, \dots, a_k))$.*

Proof. There exists $\mathbf{M} \in O(k)$ such that

$$\mathbf{M}^*\mathbf{G}(a_1, \dots, a_k)\mathbf{M} = \mathbf{D}(\lambda_1(\mathbf{G}(a_1, \dots, a_k)), \dots, \lambda_k(\mathbf{G}(a_1, \dots, a_k))).$$

(Here $\mathbf{D}(\alpha_1, \dots, \alpha_k)$ denotes the diagonal matrix with entries $\alpha_1, \dots, \alpha_n$.)

Since $\mathbf{M}^*\mathbf{G}(a_1, \dots, a_k)\mathbf{M} = \mathbf{M}^*\mathbf{A}^*\mathbf{A}\mathbf{M} = (\mathbf{AM})^*\mathbf{AM} = \mathbf{G}(b_1, \dots, b_k)$, we obtain $b_i \cdot b_j = \delta_{ij}\lambda_i(\mathbf{G}(a_1, \dots, a_k))$, which completes the proof.

As a corollary we obtain

$$\mathcal{L}_k(a_1, \dots, a_k) = \operatorname{tr}\mathbf{G}(a_1, \dots, a_k)^{1/2}.$$

Indeed, these functions are equal if $a_1, \dots, a_k$ are orthogonal. Applying Lemma 6.1 we conclude

$$\mathcal{L}_k(a_1, \dots, a_k) = \mathcal{L}_k(b_1, \dots, b_k) = \operatorname{tr}\mathbf{G}(b_1, \dots, b_k)^{1/2} = \operatorname{tr}\mathbf{G}(a_1, \dots, a_k)^{1/2}.$$

Summarizing, we have proved

Proposition 6.2. *For all $(a_1, \dots, a_k) \in \mathbb{X}^k$ we have*

$$\mathcal{T}_k(a_1, \dots, a_k) = \max_{1 \le i \le k} \lambda_i(\mathbf{G}(a_1, \dots, a_k)^{1/2}), \quad \mathcal{L}_k(a_1, \dots, a_k) = \operatorname{tr}\mathbf{G}(a_1, \dots, a_k)^{1/2}.$$

Now we consider the problem of differentiability of the norms $\mathcal{L}_k$. Since $\mathcal{L}_k(\mathbf{AM}) = \mathcal{L}_k(\mathbf{A})$ and $\operatorname{grad}\mathcal{L}_k(\mathbf{AM}) = \operatorname{grad}\mathcal{L}_k(\mathbf{A})\mathbf{M}$ for all $\mathbf{M} \in O(k)$, it suffices to consider the case where the vectors a_j are orthogonal. Then a necessary condition for differentiability of $\mathcal{L}_k$ at $\mathbf{A}$ is $a_j \neq 0$, $j = 1, \dots, k$. For instance, if $a_1 = 0$, then $\mathcal{L}_k(tv, a_2, \dots, a_k) = |t| + |a_2| + \dots + |a_k| = |t| + \mathcal{L}_k(\mathbf{A})$, where $v \in S^{n-1}$ is orthogonal to $a_2, \dots, a_k$. This implies that $\mathcal{L}_k$ is not differentiable at such a point.

Assume $a_j \neq 0$ for $j = 1, \dots, k$. Then $\mathcal{L}_k$ is differentiable at this point and

$$\operatorname{grad}\mathcal{L}_k(a_1, \dots, a_k) = (a_1/|a_1|, \dots, a_k/|a_k|).$$

To show that, it suffices to prove that

$$D_{v_i}\phi_j(0) = a_j \cdot v_i/|a_j|$$

for $j = 1, \ldots, k,\ i = 1, \ldots, n$, where $\phi_j(v) = \mathcal{L}_k(a_1, \ldots, a_j + v, \ldots, a_k)$, and $v_1, \ldots, v_n$ is an orthonormal basis in $\mathbb{R}^n$ such that $v_i = a_i/|a_i|$ for $1 \le j \le k$. This is a consequence of the known fact that a convex function $f : \mathbb{R}^m \supset \Omega \to \mathbb{R}$ is differentiable at a point if and only if there exist partial derivatives at this point or, equivalently, the directional derivatives for m linearly independent directions. We have

$$\phi_j(tv_i) = \begin{cases} \phi_j(0) + |a_j + tv_i| - |a_j|, & i = j, k+1, \ldots, n, \\ \phi_j(0) + [(|a_j| + |a_i|)^2 + t^2]^{1/2} - |a_j| - |a_i|, & i \neq j, k+1, \ldots, n, \end{cases}$$

which gives $D_{v_i}\phi_j(0) = a_j \cdot v_i/|a_j|$. We see that $\operatorname{grad} \mathcal{L}_k$ is a surjection of $\{(a_1, \ldots, a_k) \in \mathbb{X}^k : a_j \neq 0, a_i \cdot a_j = 0 \text{ for } i \neq j\}$ onto $V_k(\mathbb{X})$. It is easy to check that $V_k(\mathbb{X})\mathbf{M} = V_k(\mathbb{X})$ for all $\mathbf{M} \in O(k)$. So, we have proved

Proposition 6.3.

$$\exp \mathcal{T}_k = V_k(\mathbb{X}), \quad k = 1, \ldots, \dim \mathbb{X}.$$

In particular, since $V_n(\mathbb{X}) = O(n)$ is compact in X^n, by the Krein–Milman theorem we obtain the following known result:

Corollary 6.4. *If* $\mathbf{L}$ *is a linear endomorphism of* $\mathbb{R}^n$ *with* $\|\mathbf{L}\| \le 1$, *then there exist* $\mathbf{M}_1, \ldots, \mathbf{M}_j \in O(n)$ *and* $\alpha_1, \ldots, \alpha_j \ge 0$ *with* $\alpha_1 + \ldots + \alpha_j = 1$ *such that* $\mathbf{L} = \alpha_1\mathbf{M}_1 + \ldots + \alpha_j\mathbf{M}_j$.

Now we consider the case $\mathbb{K} = \mathbb{C},\ \mathbb{X} = \mathbb{C}^n$.

For $(a_1, \ldots, a_k) = \mathbf{A} \in \mathbb{X}^k$ define, analogously to the real case,

$$Q_{\mathbf{A}}(w) = |w \cdot a_1|^2 + \ldots + |w \cdot a_k|^2, \quad w \in \mathbb{C}^n,$$
$$\mathbf{G}(a_1, \ldots, a_k) = [a_i \cdot a_j]_{i,j=1,\ldots,k},$$
$$(a_1, \ldots, a_k)\mathbf{M} = \Big(\sum_{j=1}^{k} \alpha_{j1}a_j, \ldots, \sum_{j=1}^{k} \alpha_{jk}a_j\Big) = (b_1, \ldots, b_k) = \mathbf{AM},$$

for $\mathbf{M} = [\alpha_{ij}] \in U(k)$. Then we may write

$$Q_{\mathbf{A}}(w) = w^*\mathbf{AA}^*w, \quad Q_{\mathbf{AM}} = Q_{\mathbf{A}},$$
$$\mathbf{G}(\overline{a}_1, \ldots, \overline{a}_k) = \mathbf{A}^*\mathbf{A}, \quad \mathbf{G}(\overline{b}_1, \ldots, \overline{b}_k) = \mathbf{M}^{-1}\mathbf{G}(\overline{a}_1, \ldots, \overline{a}_k)\mathbf{M},$$
$$(a_1, \ldots, a_k)\mathbf{M} \cdot (x_1, \ldots, x_k) = (a_1, \ldots, a_k) \cdot (x_1, \ldots, x_k)\mathbf{M}^*,$$

which implies

$$\mathcal{T}_k(\mathbf{AM}) = \mathcal{T}_k(\mathbf{A}) \ge \max(|a_1|, \ldots, |a_k|),$$
$$\mathcal{L}_k(\mathbf{AM}) = \mathcal{L}_k(\mathbf{A}) \le |a_1| + \ldots + |a_k|,$$

with equalities if $a_1, \ldots, a_k$ are orthogonal. The first case of equality is a consequence of

$$\mathcal{T}_k(a_1, \ldots, a_k) = \mathcal{T}_{2k}((\Re a_1, \Im a_1), (-\Im a_1, \Re a_1), \ldots, (\Re a_k, \Im a_k), (-\Im a_k, \Re a_k)),$$

and $(\Re a_1, \Im a_1), (-\Im a_1, \Re a_1), \ldots, (\Re a_k, \Im a_k), (-\Im a_k, \Re a_k)$ are orthogonal in $\mathbb{R}^{2n}$ if $a_1, \ldots$ $\ldots, a_k$ are orthogonal in $\mathbb{C}^n$.

Similarly, we have

$$\max_{1\le j\le k}\lambda_j(\mathbf{G}(\mathbf{AM})^{1/2})=\max_{1\le j\le k}\lambda_j(\mathbf{G}(\mathbf{M})^{1/2})\ge \max(|a_1|,\dots,|a_k|),$$
$$\operatorname{tr}\mathbf{G}(\mathbf{AM})^{1/2}=\operatorname{tr}\mathbf{G}(\mathbf{A})^{1/2}\le |a_1|+\dots+|a_k|,$$

with equalities for orthogonal $a_1,\dots,a_k$.

Note that the group $T(k)=S^1\times\dots\times S^1$ also acts on $\mathbb{X}^k$:

$$(e^{i\theta_1},\dots,e^{i\theta_k})(a_1,\dots,a_k)=(e^{i\theta_1}a_1,\dots,e^{i\theta_k}a_k).$$

We need a counterpart of Lemma 6.1.

Lemma 6.5. *If $a_1,\dots,a_k\in\mathbb{C}^n$, then there exist $\mathbf{M}\in U(k)$ and $\mathbf{N}\in T(k)$ such that $\mathbf{N}(a_1,\dots,a_k)\mathbf{M}=(b_1,\dots,b_k)$ with $b_i\cdot b_j=\delta_{ij}\lambda_i(\mathbf{G}(a_1,\dots,a_k))$ and $\Re b_j\cdot\Im b_j=0$.*

Proof. There exists $\mathbf{M}\in U(k)$ such that

$$\mathbf{M}^*\mathbf{G}(\overline{a}_1,\dots,\overline{a}_k)\mathbf{M}=\mathbf{D}(\lambda_1(\mathbf{G}(\overline{a}_1,\dots,\overline{a}_k)),\dots,\lambda_k(\mathbf{G}(\overline{a}_1,\dots,\overline{a}_k))).$$

Let $(c_1,\dots,c_k)=\mathbf{AM}$. Since

$$\mathbf{M}^*\mathbf{G}(\overline{a}_1,\dots,\overline{a}_k)\mathbf{M}=\mathbf{M}^*\mathbf{A}^*\mathbf{AM}=(\mathbf{AM})^*\mathbf{AM}=\mathbf{G}(\overline{c}_1,\dots,\overline{c}_k),$$

we get $c_i\cdot c_j=\delta_{ij}\lambda_j(\mathbf{G}(\overline{a}_1,\dots,\overline{a}_k))$. Now using the Drużkowski lemma we complete the proof.

Applying Lemma 6.5 we easily prove

Proposition 6.6. *For all $(a_1,\dots,a_k)\in\mathbb{X}^k$ we have*

$$\mathcal{T}_k(a_1,\dots,a_k)=\max_{1\le i\le k}\lambda_i(\mathbf{G}(a_1,\dots,a_k)^{1/2}),\quad \mathcal{L}_k(a_1,\dots,a_k)=\operatorname{tr}\mathbf{G}(a_1,\dots,a_k)^{1/2}.$$

In the special case of $k=2$ we have

$$\lambda_1(\mathbf{G}(z,w))=\frac{|z|^2+|w|^2}{2}+\frac12((|z|^2+|w|^2)^2-4|\mathbf{G}(z,w)|)^{1/2},$$
$$\lambda_2(\mathbf{G}(z,w))=\frac{|z|^2+|w|^2}{2}-\frac12((|z|^2+|w|^2)^2-4|\mathbf{G}(z,w)|)^{1/2},$$

and thus,

$$\mathcal{T}_2(z,w)=\left[\frac{|z|^2+|w|^2}{2}+\frac12((|z|^2+|w|^2)^2-4|\mathbf{G}(z,w)|)^{1/2}\right]^{1/2},$$
$$\mathcal{L}_2(z,w)=[|z|^2+|w|^2+2|\mathbf{G}(z,w)|^{1/2}]^{1/2}.$$

Moreover,

$$\mathcal{T}_2(z,w)=T_n(z+iw),\quad \mathcal{L}_2(z,w)=L_2(z+iw)\quad \textit{for } z,w\in\mathbb{R}^n.$$

Now we consider the problem of differentiability of the norms $\mathcal{L}_k$. We have

$$\operatorname{grad}\mathcal{L}_k(\mathbf{AM})=\operatorname{grad}\mathcal{L}_k(\mathbf{A})\mathbf{M}\quad\text{for }\mathbf{M}\in U(k).$$

Therefore it is enough to consider the case where $a_1,\dots,a_k$ are orthogonal. The condition $a_j\ne0,\ j=1,\dots,k$, as in the real case, is necessary for differentiability of $\mathcal{L}_k$ at $\mathbf{A}$. We now show that it is also sufficient and then

$$\operatorname{grad}\mathcal{L}_k(\mathbf{A})=(a_1/|a_1|,\dots,a_k/|a_k|).$$

It suffices to prove that

$$D_{v_m}\phi_j(0) = \Re(a_j \cdot v_m), \quad D_{w_m}\phi_j(0) = \Im(a_j \cdot v_m),$$

for $m = 1, \ldots, n,\ j = 1, \ldots, k$, where $\phi_j(v) = \mathcal{L}_k(a_1, \ldots, a_j + v, \ldots, a_k)$ and $v_1, \ldots, v_n$ is an orthonormal basis in $\mathbb{C}^n$ such that $v_j = a_j/|a_j|,\ j = 1, \ldots, k$, and $w_j = iv_j$ — $v_1, \ldots, v_n, w_1, \ldots, w_n$ is a basis in $\mathbb{R}^{2n}$. We have

$$\phi_j(tv_i) = \begin{cases} \phi_j(0) + |a_j + tv_i| - |a_j|, & i = j, k+1, \ldots, n, \\ \phi_j(0) + [(|a_j| + |a_i|)^2 + t^2]^{1/2} - |a_j| - |a_i|, & i \neq j, k+1, \ldots, n, \end{cases}$$

and

$$\phi_j(tw_i) = \begin{cases} \phi_j(0) + |a_j + tw_i| - |a_j|, & i = j, k+1, \ldots, n, \\ \phi_j(0) + [(|a_j| + |a_i|)^2 + t^2]^{1/2} - |a_j| - |a_i|, & i \neq j, k+1, \ldots, n, \end{cases}$$

This easily implies $D_{v_m}\phi_j(0) = \Re(a_j \cdot v_m)$ and $D_{w_m}\phi_j(0) = \Im(a_j \cdot v_m)$. Since $V_k(\mathbb{X})\mathbf{M} = V_k(\mathbb{X})$ for all $\mathbf{M} \in U(k)$, we have proved

PROPOSITION 6.7. *If* $\mathbb{K} = \mathbb{C},\ \mathbb{X} = \mathbb{C}^n$, *then*

$$\exp \mathcal{T}_k = V_k(\mathbb{X}), \quad k = 1, \ldots, n.$$

In particular, $\exp \mathcal{T}_n = U(n)$ *and thus an analogue of Proposition* 6.4 *holds.*

It is known (see [H]) that $V_k(\mathbb{X})$ is the Shilov boundary of the unit ball with respect to the norm $\mathcal{T}_k$. That ball is a classical Cartan homogeneous domain of type $\Re_I$ (see [H] or [SG]). Note that the Lie ball is also a Cartan homogeneous domain of type $\Re_{IV}$ (cf. [H]).

Now we find the interpolation spaces between $(\mathbb{X}^k, \mathcal{T}_k)$ and $(\mathbb{X}^k, \mathcal{L}_k)$ for the complex method. For $1 \le p \le \infty$ define the following $U(k)$-invariant functions:

$$\Lambda_{k,p}(a_1, \ldots, a_k) = \Big(\sum_{j=1}^k \lambda_j(\mathbf{G}(a_1, \ldots, a_k)^{1/2})^p\Big)^{1/p} = \Big(\sum_{j=1}^k \lambda_j(\mathbf{G}(a_1, \ldots, a_k))^{p/2}\Big)^{1/p}.$$

In particular, $\Lambda_{k,1} = \mathcal{L}_k$ and $\Lambda_{k,\infty} = \mathcal{T}_k$. Since $\mathbf{G}(a_1, \ldots, a_k)$ is a positive semidefinite Hermitian matrix, we have the majorization

$$(|a_1|^2, \ldots, |a_k|^2) \prec (\lambda_1(\mathbf{G}(a_1, \ldots, a_k)), \ldots, \lambda_k(\mathbf{G}(a_1, \ldots, a_k))),$$

which implies

$$\Lambda_{k,p}(a_1, \ldots, a_k) \le (|a_1|^p + \ldots + |a_k|^p)^{1/p} \quad \text{for } 1 \le p \le 2,$$
$$\Lambda_{k,p}(a_1, \ldots, a_k) \ge (|a_1|^p + \ldots + |a_k|^p)^{1/p} \quad \text{for } 2 \le p \le \infty,$$

with equality if $a_1, \ldots, a_k$ are orthogonal. One can also prove

PROPOSITION 6.8. *If* $(a_1, \ldots, a_k) \in \mathbb{X}^k$ *and* $a_i \cdot a_l = 0$ *for* $1 \le i \le j,\ j+1 \le l \le k$, *then*

$$\Lambda_{k,p}(a_1, \ldots, a_k)^p = \Lambda_{j,p}(a_1, \ldots, a_j)^p + \Lambda_{k-j,p}(a_{j+1}, \ldots, a_k)^p.$$

Applying Lemma 6.5 we easily prove that $\Lambda_{k,p}$ is a norm in $\mathbb{X}^k$ for $p \ge 2$. If $a_1, \ldots, a_k$ are orthogonal and $p \ge 2$, then

$$\Lambda^*_{k,p}(a_1, \ldots, a_k) = (|a_1|^q + \ldots + |a_k|^q)^{1/q} = \Lambda_{k,q}(a_1, \ldots, a_k),$$

where $1/p + 1/q = 1$. Since $\Lambda^*_{k,p}$ and $\Lambda_{k,q}$ are both $U(k)$-invariant we easily obtain

$$\Lambda^*_{k,p} = \Lambda_{k,q}, \quad 1/p + 1/q = 1.$$

For $k = n$ the above norms were introduced by J. von Neumann [N]. The norms $\Lambda_{k,p}$ are also given by the following

PROPOSITION 6.9. *For every* $\mathbf{A} = (a_1, \ldots, a_k) \in \mathbb{X}^k$ *we have*

$$\Lambda_{k,p}(\mathbf{A}) = \begin{cases} \min\Big\{\Big(\sum_{i=1}^{k}\Big|\sum_{j=1}^{k}\alpha_{ji}a_j\Big|^p\Big)^{1/p} : \mathbf{M} = [\alpha_{ij}] \in U(k)\Big\}, & 1 \le p \le 2, \\ \max\Big\{\Big(\sum_{i=1}^{k}\Big|\sum_{j=1}^{k}\alpha_{ji}a_j\Big|^p\Big)^{1/p} : \mathbf{M} = [\alpha_{ij}] \in U(k)\Big\}, & 2 \le p \le \infty. \end{cases}$$

PROOF. Denote the right-hand side of the above equality by $\phi(\mathbf{A})$. In the space $\mathbb{X}^k$ consider the norms

$$\|\mathbf{A}\|_p = (|a_1|^p + \ldots + |a_k|^p)^{1/p}.$$

Then $\phi(\mathbf{A}) = \min\{\|\mathbf{AM}\|_p : \mathbf{M} \in U(k)\}$ for $1 \le p \le 2$ and $\phi(\mathbf{A}) = \max\{\|\mathbf{AM}\|_p : \mathbf{M} \in U(k)\}$ for $2 \le p \le \infty$. It follows from the definition of $\phi(\mathbf{A})$ that this function is $U(k)$-invariant: $\phi(\mathbf{AM}) = \phi(\mathbf{A})$. For this reason it is enough to prove $\phi(\mathbf{A}) = \|\mathbf{A}\|_p$ for $a_1, \ldots, a_k$ being orthogonal. Consider two cases: $1 \le p \le 2$ and $2 \le p \le \infty$. In the first case we have $\phi(\mathbf{A}) \le \|a\|_p$. Since the function $\mathbb{R}_+ \ni t \mapsto t^{p/2}$ is concave for $1 \le p \le 2$, we get the estimates

$$\begin{aligned} \|\mathbf{AM}\|_p^p &= \sum_{i=1}^{k}\Big(\sum_{j=1}^{k}|\alpha_{ji}|^2|a_j|^2\Big)^{p/2} \ge \sum_{i=1}^{k}\sum_{j=1}^{k}|\alpha_{ji}|^2|a_j|^p \\ &= \sum_{j=1}^{k}\Big(\sum_{i=1}^{k}|\alpha_{ji}|^2\Big)|a_j|^p = \sum_{j=1}^{k}|a_j|^p = \|\mathbf{A}\|_p^p, \end{aligned}$$

which imply $\phi(\mathbf{A}) \ge \|\mathbf{A}\|_p$ and thus $\phi(\mathbf{A}) = \|\mathbf{A}\|_p$.

Analogously, if $2 \le p < \infty$ then $\phi(\mathbf{A}) \ge \|\mathbf{A}\|_p$ and applying the convexity of the function $t \mapsto t^{p/2}$ we obtain $\phi(\mathbf{A}) \le \|\mathbf{A}\|_p$.

If $p = \infty$, then we also have $\phi(\mathbf{A}) \ge \|\mathbf{A}\|_\infty$ and

$$\|\mathbf{AM}\|^2 = \max_i\Big(\sum_{j=1}^{k}|\alpha_{ji}|^2|a_j|^2\Big) \le \max_i\sum_{j=1}^{k}|\alpha_{ji}|^2\max_j|a_j|^2 = \|\mathbf{A}\|_\infty^2,$$

which completes the proof.

Now we show that $\Lambda_{k,p}$ is differentiable on $\mathbb{X}^k \setminus \{0\}$ for $1 < p < \infty$. Since $\Lambda_{k,p}$ is $U(k)$-invariant it suffices to consider the case where $a_1, \ldots, a_k$ are orthogonal. Let $\phi_j(w) = \Lambda_{k,p}(a_1, \ldots, a_j + v, \ldots, a_k)$, $j = 1, \ldots, k$, and let $v_1, \ldots, v_n$ be an orthonormal basis in $\mathbb{C}^n$ such that $v_j = a_j/|a_j|$ if $a_j \ne 0$. Put $w_j = iv_j$, $j = 1, \ldots, n$. Applying Proposition 6.8 we can explicitly calculate $\phi_j(tv_i)$ and $\phi_j(tw_i)$ and check that

$$D_{v_i}\phi_j(0) = \delta_{ij}\Lambda_{k,p}(a_1, \ldots, a_k)^{1-p}|a_j|^{p-1}, \quad j = 1, \ldots, k,\ i = 1, \ldots, n,$$
$$D_{w_i}\phi_j(0) = 0 \quad \text{for all } i, j,$$

which implies differentiability of $\Lambda_{k,p}$ at each $\mathbf{A} \in X^k \setminus \{0\}$ and gives

$$\operatorname{grad}\Lambda_{k,p}(a_1, \ldots, a_k) = \Lambda_{k,p}(a_1, \ldots, a_k)^{1-p}(|a_1|^{p-2}a_1, \ldots, |a_k|^{p-2}a_k)$$

for $a_1, \ldots, a_k$ orthogonal (if $1 < p < 2$ and $a_j = 0$ we put $|a_j|^{p-2}a_j := 0$). Now observe that $(\mathbb{X}^k, \Lambda_{k,p})$ is an interpolating space between $(\mathbb{X}^k, \mathcal{T}_k)$ and $(\mathbb{X}^k, \mathcal{L}_k)$ of exact exponent $\theta = 1/p$:

$$\Lambda_{p,k} \le \mathcal{T}_k^{1-1/p} \mathcal{L}_k^{1/p}.$$

We are in a position to prove the main result of this section:

THEOREM 6.10. *For all $1 \le k \le n$ and $1 < p < \infty$ we have*

$$((\mathbb{X}^k, \mathcal{T}_k), (\mathbb{X}^k, \mathcal{L}_k))_{[1/p]} = (\mathbb{X}^k, \Lambda_{k,p}).$$

PROOF. Let $\Lambda_{k,p}(a_1, \ldots, a_k) = \Lambda_{k,p}(\mathbf{A}) = 1$. Then there exist $\mathbf{M} \in U(k)$ and $\mathbf{B} = (b_1, \ldots, b_k) \in \mathbb{X}^k$ such that $b_1, \ldots, b_k$ are orthogonal and

$$\mathbf{A} = \Lambda_{k,q}(\mathbf{B})^{1-q}(|b_1|^{q-2}b_1, \ldots, |b_k|^{q-2}b_k)\mathbf{M}, \quad 1/q + 1/p = 1.$$

Define $f, h_q : \{0 \le \Re\zeta \le 1\} = S \to X^k$ as

$$h_q(\zeta) = h_q(\zeta, \mathbf{B}, \mathbf{M}) = \Lambda_{k,q}(\mathbf{B})^{-q\zeta}(|b_1|^{q\zeta-1}b_1, \ldots, |b_k|^{q\zeta-1}b_k)\mathbf{M},$$
$$f(\zeta) = e^{\varepsilon\zeta^2 - \varepsilon p^{-2}} h_q(\zeta).$$

Then $f \in \mathcal{F}((X^k, \mathcal{T}_k), (\mathbb{X}^k, \mathcal{L}_k))$ and $f(1/p) = \mathbf{A}$. It is easily seen that for all $t \in \mathbb{R}$ we have

$$\mathcal{T}_k(h_q(it)) = 1, \quad \mathcal{L}_k(h_q(1 + it)) = 1,$$

which implies $\|f\|_{\mathcal{F}} \le e^{\varepsilon}$ and therefore

$$\|\mathbf{A}\|_{[1/p]} \le \Lambda_{k,p}(\mathbf{A}) \quad \text{for every } \mathbf{A} \in \mathbb{X}^k.$$

Now assume $\|\mathbf{A}\|_{[1/p]} = 1$. Let $f \in \mathcal{F}$ be such that $f(1/p) = \mathbf{A}$ and $\|f\|_{\mathcal{F}} \le 1+\varepsilon$. Let $\mathbf{A} = (b_1, \ldots, b_k)\mathbf{M}$, where $\mathbf{M} \in U(k)$ and $b_1, \ldots, b_k$ are orthogonal. Define $F : S \to \mathbb{C}$ by

$$F(\zeta) = h_p(1 - \zeta, \overline{\mathbf{B}}, \overline{\mathbf{M}}) \cdot \overline{f(\zeta)}.$$

Then $F \in \mathcal{O}(S_0) \cap \mathcal{C}(S) \cap L^{\infty}(S)$ and

$$|F(it)| \le \mathcal{L}_k(h_p(1 - it))\mathcal{T}_k(f(it)) \le 1 + \varepsilon,$$
$$|F(1 + it)| \le |\mathcal{T}_k(h_p(-it))\mathcal{L}_k(f(1 + it)) \le 1 + \varepsilon,$$

for $t \in \mathbb{R}$. Hence, by Hadamard's three lines theorem, $|F(\zeta)| \le 1 + \varepsilon$ for all $\zeta \in S$. In particular,

$$|F(1/p)| = \Lambda_{k,p}(\overline{\mathbf{B}}) = \Lambda_{k,p}(\overline{\mathbf{A}}) = \Lambda_{k,p}(\mathbf{A}) \le 1 + \varepsilon,$$

and thus $\Lambda_{k,p}(\mathbf{A}) \le \|\mathbf{A}\|_{[1/p]}$ for $\mathbf{A} \in \mathbb{X}^k$, which completes the proof.

COROLLARY 6.11.

$$(\mathbb{C}^n \check{\otimes} \mathbb{C}^k, \mathbb{C}^n \hat{\otimes} \mathbb{C}^k)_{[1/p]} \simeq (\mathbb{X}^k, \Lambda_{k,p}).$$

COROLLARY 6.12. *For all $\mathbf{A}, \mathbf{B} \in \mathbb{X}^k$ we have Clarkson's type inequalities*

$$(\Lambda_{k,p}(\mathbf{A} + \mathbf{B})^p + \Lambda_{k,p}(\mathbf{A} - \mathbf{B})^p)^{1/p} \le 2^{1/q}(\Lambda_{k,p}(\mathbf{A})^p + \Lambda_{k,p}(\mathbf{B})^p)^{1/p} \quad \text{for } 2 \le p \le \infty,$$
$$(\Lambda_{k,p}(\mathbf{A} + \mathbf{B})^p + \Lambda_{k,p}(\mathbf{A} - \mathbf{B})^p)^{1/p} \le 2^{1/p}(\Lambda_{k,p}(\mathbf{A})^q + \Lambda_{k,p}(\mathbf{B})^q)^{1/q} \quad \text{for } 2 \le p \le \infty,$$
$$(\Lambda_{k,p}(\mathbf{A} + \mathbf{B})^q + \Lambda_{k,p}(\mathbf{A} - \mathbf{B})^q)^{1/q} \le 2^{1/q}(\Lambda_{k,p}(\mathbf{A})^p + \Lambda_{k,p}(\mathbf{B})^p)^{1/p} \quad \text{for } 1 \le p \le 2,$$
$$(\Lambda_{k,p}(\mathbf{A} + \mathbf{B})^q + \Lambda_{k,p}(\mathbf{A} - \mathbf{B})^q)^{1/q} \le 2^{1/p}(\Lambda_{k,p}(\mathbf{A})^q + \Lambda_{k,p}(\mathbf{B})^q)^{1/q} \quad \text{for } 1 \le p \le 2.$$

Remark 6.13. Applying Corollary 6.12 we deduce that the norms Λ_p introduced in Section 2 satisfy Clarkson's type inequalities. These norms are also extremal in the following sense:

Proposition 6.14. *Fix* $1 < p < \infty$. *Let* $\mathcal{I}_p$ *be the class of norms* N *in* $\mathbb{C}^n$ *that satisfy*:

$$N(x) = |x| \quad \text{on } \mathbb{R}^n; \tag{1}$$

$$T_n \le N \le L_n, \quad N \le L_n^{1-1/p} T_n^{1/p}; \tag{2}$$

$$\left.\begin{aligned} (N(z+w)^p + N(z-w)^p)^{1/p} &\le 2^{1/q}(N(z)^p + N(w)^p)^{1/p} \\ (N(z+w)^p + N(z-w)^p)^{1/p} &\le 2^{1/p}(N(z)^q + N(w)^q)^{1/q} \end{aligned}\right\} \quad \text{for } 2 \le p < \infty, \tag{3'}$$

$$\left.\begin{aligned} (N(z+w)^q + N(z-w)^q)^{1/q} &\le 2^{1/q}(N(z)^p + N(w)^p)^{1/p} \\ (N(z+w)^q + N(z-w)^q)^{1/q} &\le 2^{1/p}(N(z)^q + N(w)^q)^{1/q} \end{aligned}\right\} \quad \text{for } 1 < p \le 2, \tag{3''}$$

where, as usual, $1/p + 1/q = 1$. *Then* $\Lambda_p, \Lambda_q \in \mathcal{I}_p$ *and for every* $N \in \mathcal{I}_p$,

$$\Lambda_p \le N \le \Lambda_q \quad \text{for } 2 \le p < \infty,$$
$$\Lambda_q \le N \le \Lambda_p \quad \text{for } 1 < p \le 2$$

and $(1) + (3')$ (*or* $(3'')$) $\Rightarrow (2)$.

Proof. We easily check that $\Lambda_p, \Lambda_q \in \mathcal{I}_p$. In particular, the second condition in (2) holds in view of the elementary inequalities

$$1 + \delta^q \le 1 + \delta, \quad 1 + \delta^p \le (1+\delta)^{p-1} \quad \text{for } 1 < p < \infty,\ \delta \in [0,1].$$

Now, let $N \in \mathcal{I}_p$ and let $p \ge 2$. By (1) and (3′) (applied to $z = x$, $w = iy$, $x, y \in \mathbb{R}^n$, $x \cdot y = 0$) we have, by the second Clarkson inequality,

$$N(z) \le (|x|^q + |y|^q)^{1/q}, \quad x \cdot y = 0.$$

Analogously, the first Clarkson inequality gives

$$N(z) \ge (|x|^p + |y|^p)^{1/p}, \quad x \cdot y = 0.$$

This implies $\Lambda_p \le N \le \Lambda_q$. Applying the second couple of Clarkson's inequalities we derive $\Lambda_q \le N \le \Lambda_p$ for $1 < p \le 2$. The last statement is a consequence of the fact that the norms Λ_p, Λ_q satisfy condition (2).

7. The complex interpolation of a complexification of a real Hilbert space

In this section we generalize the main result of Section 6 to the case of the complexification of a real Hilbert space. The proof of the general case is based on the result in $\mathbb{C}^n$.

Let H be a real Hilbert space with a scalar product $x \cdot y$ and norm $\|x\| = \sqrt{x^2}$. We consider the complexification $\widehat{H}$ of H obtained in the following way. Let $\widehat{H} = H \times H$. Then $\widehat{H}$ is a real Hilbert space with the scalar product $(x,y) \cdot (u,v) = x \cdot u + y \cdot v$ and norm $\|(x,y)\| = (x^2 + y^2)^{1/2}$. We introduce a complex structure on $\widehat{H}$ by putting

$$(x,y) + (u,v) = (x+u, y+v),$$

$$re^{i\theta}(x,y) = r(\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y).$$

We write $z = x + iy$ instead of $z = (x, y)$ and $\overline{z} = x - iy$ if we consider the complex structure of $\widehat{H}$, which also shows the meaning of the equality $\widehat{H} = H + iH$. Defining

$$(x + iy)\cdot(u + iv) := (x\cdot u + y\cdot v) + i(y\cdot u - x\cdot v)$$

we obtain the complex Hilbertian structure on $\widehat{H}$ with the norm $\|(x,y)\| = (x^2 + y^2)^{1/2}$ the same as in $H \times H$. Note also that

$$(x,y)\cdot(u,v) = \Re((x+iy)\cdot(u+iv)).$$

The group $O(2)$ acts on $H \times H$ by

$$(x,y)[\alpha_{ij}] = (\alpha_{11}x + \alpha_{21}y, \alpha_{12}x + \alpha_{22}y).$$

In particular,

$$(x,y)\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} = e^{i\theta}(x,y).$$

So the multiplication $e^{i\theta}(x,y)$ agrees with the action of $O^+(2)$ on $H\times H$. We denote by $\mathbf{G}(x,y)$ the Gram matrix of $x, y \in H$:

$$\mathbf{G}(x,y) = \begin{bmatrix} x^2 & x\cdot y\\ x\cdot y & y^2\end{bmatrix}.$$

Let $\lambda_1(x,y), \lambda_2(x,y)$ be the eigenvalues of $\mathbf{G}(x,y)^{1/2}$, explicitly given by

$$\lambda_1(x,y) = \left(\frac{\|z\|^2 + |z^2|}{2}\right)^{1/2}, \quad \lambda_2(x,y) = \left(\frac{\|z\|^2 - |z^2|}{2}\right)^{1/2},$$

where $z^2 = (x+iy)^2 = x^2 - y^2 + 2ix\cdot y$. As in the case of $\mathbb{C}^n$, for $x + iy \in \widehat{H}$ there exists $\theta \in \mathbb{R}$ such that $e^{i\theta}(x+iy) = x' + iy'$ and $x'\cdot y' = 0$. Equivalently, there exists $\mathbf{A} \in O^+(2)$ such that

$$\mathbf{A}^*\mathbf{G}(x,y)\mathbf{A} = \begin{bmatrix}\lambda_1(x,y)^2 & 0\\ 0 & \lambda_2(x,y)^2\end{bmatrix},$$

which gives

$$\min(\|x'\|, \|y'\|) = \min(\lambda_1(x,y), \lambda_2(x,y)),$$
$$\max(\|x'\|, \|y'\|) = \max(\lambda_1(x,y), \lambda_2(x,y)).$$

Define

$$\mathcal{T}(x+iy) = \max(\lambda_1(x,y), \lambda_2(x,y)), \quad \mathcal{L}(x+iy) = \lambda_1(x,y) + \lambda_2(x,y).$$

Since $\mathbf{G}((x,y)\mathbf{A}) = \mathbf{G}(x,y)$ for all $\mathbf{A} \in O(2)$, the functions $\mathcal{T}$ and $\mathcal{L}$ are $O(2)$-invariant. It is easily seen that this implies their homogeneity. The subadditivity of $\mathcal{L}$ and $\mathcal{T}$ is a consequence of the same result in $\mathbb{C}^n$. To prove this consider $\overline{x} + i\overline{y}$ and $\overline{u} + i\overline{v}$. Then $\overline{x} = x_1e_1 + \ldots + x_me_m$, $\overline{y} = y_1e_1 + \ldots + y_me_m$, $\overline{u} = u_1e_1 + \ldots + u_me_m$, $\overline{v} = v_1e_1 + \ldots + v_me_m$, where $e_1, \ldots, e_m$, $1 \le m \le 4$, is an orthonormal system in H. Then subadditivity of $\mathcal{T}$ and $\mathcal{L}$ for $\overline{x} + i\overline{y}$ and $\overline{u} + i\overline{v}$ is equivalent to the same condition for $x + iy$ and $u + iv$ in $\mathbb{C}^m$. For similar reasons the function

$$\begin{aligned}\ell_p(x+iy) &= 2^{-1/p}((\lambda_1(x,y) + \lambda_2(x,y))^p + |\lambda_1(x,y) - \lambda_2(x,y)|^p)^{1/p}\\ &= 2^{-1/p}((x^2 + y^2 + 2|\mathbf{G}(x,y)|^{1/2})^{p/2} + (x^2 + y^2 - 2|\mathbf{G}(x,y)|^{1/2})^{p/2})^{1/p}\end{aligned}$$

is a norm in $\widehat{H}$ for $1 < p < \infty$ which satisfies appropriate Clarkson inequalities. In particular,

$$\ell_p(x+iy) = 2^{-1/p}((\|x\|+\|y\|)^p + |\|x\|-\|y\||^p)^{1/p} \quad \text{if } x\cdot y = 0.$$

For other z the left-hand side and the right-hand side are related as in the case of $\mathbb{C}^n$.

Now we are in a position to formulate our interpolating result.

THEOREM 7.1. *For every* $1 < p < \infty$,

$$((\widehat{H},\mathcal{L}),(\widehat{H},\mathcal{T}))_{[1/p]} = (\widehat{H},\ell_p).$$

PROOF. First, observe that all norms ℓ_p are equivalent to the Hilbertian norm in $\widehat{H}$. In particular, $(\widehat{H},\ell_p)$ is reflexive. For every norm F in $\widehat{H}$ $(H\times H)$ which is equivalent to the Hilbertian (Euclidean) norm, we can define the conjugate norm F^* by

$$F^*(z) = \sup\{|z\cdot w| : F(w)\le 1\} \quad (= F^*(x,y) = \sup\{|x\cdot u + y\cdot v| : F(u,v)\le 1\})$$

(the second "=" sign means that for the complex norm, as in $\mathbb{C}^n$, both definitions agree). Then $(\widehat{H},F^*)$ $((H\times H,F^*))$ is also isomorphic to $(\widehat{H},\|\cdot\|)$ $((H\times H,\|\cdot\|))$ and, by the Riesz theorem, $F^{**} = F$. The operation * has some properties similar to the case of $\mathbb{C}^n$. The statements (1), (2) and (7) of 2.5 are also true in the general case. This follows immediately from the definition.

Now we show that

$$\ell_p^* = \ell_q, \quad 1/p + 1/q = 1, \quad 1 < p < \infty.$$

It is enough to prove this equality for $z = x+iy$ with $x\cdot y = 0$. Then we easily obtain (as in the case of $\mathbb{C}^n$), for $1 < p \le 2$,

$$\ell_p^*(z) \le \sup\{|x\cdot u + y\cdot v| : 2^{-1/p}((\|u\|+\|v\|)^p + |\|u\|-\|v\||)^p)^{1/p}) \le 1\} = \ell_q(z).$$

Define $\phi(z) = u+iv = \alpha x + \alpha y$ formally as the right-hand side in formula (5.12.1) for $\operatorname{grad}\ell_q(z)$ in $\mathbb{C}^n$. Then one can easily check that

$$\ell_p(\phi(z)) = 1, \quad \Re(\phi(z)\cdot z) = \ell_q(z).$$

This implies $\ell_p^*(z) = \ell_q(z)$ for $x\cdot y = 0$ and, therefore, for all $z\in\widehat{H}$. Since each ℓ_q, $1 < q < \infty$, satisfies Clarkson's inequalities, $(\widehat{H},\ell_q)$ is uniformly convex and therefore $(\widehat{H},\ell_p)$ is uniformly smooth, i.e. ℓ_p is Fréchet differentiable on $\widehat{H}\setminus\{0\}$ and by the Shmul'yan theorem,

$$\exp\ell_p = \{e^{i\theta}\operatorname{grad}\ell_q(z) : z = x+iy\in\widehat{H},\ x\cdot y = 0,\ \theta\in\mathbb{R}\}.$$

Here $\operatorname{grad}\ell_q(z) = \operatorname{grad}\ell_q(x,y)$ is a unique vector in $H\times H$ which is defined by the Riesz isometry for $d_{(x,y)}f$:

$$d_{(x,y)}f(u,v) = \operatorname{grad}\ell_q(x,y)\cdot(u,v) = \Re(\operatorname{grad}\ell_q(z)\cdot(u+iv)).$$

Since $\operatorname{grad}\ell_q(z)$ is determined by the conditions

$$\ell_p(\operatorname{grad}\ell_q(z)) = 1, \quad \Re(\operatorname{grad}\ell_q(z)\cdot z) = \ell_q(z),$$

we have $\operatorname{grad}\ell_q(z) = \phi(z)$ for $x\cdot y = 0$. If we define, as in $\mathbb{C}^n$, the function $h_p(z,\zeta)$, the next steps are analogous to those for $\mathbb{C}^n$. We complete the proof without any problems.

COROLLARY 7.2. $(H\mathbin{\hat{\otimes}_{\mathbb{R}}}\mathbb{C}, H\mathbin{\check{\otimes}_{\mathbb{R}}}\mathbb{C})_{[1/p]} \simeq (\widehat{H},\ell_p).$

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