MAPPINGS AND INDUCTIVE INVARIANTS

BY

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1. Introduction. All spaces considered are presumed to be separable metrizable.

For any space \( X \), the inductive invariant \( \text{loc com} X \) is defined as follows:
\[ \text{loc com} X = -1 \text{ if and only if } X \text{ is locally compact and, for } n \geq 0, \]
\[ \text{loc com} X \leq n \text{ if each point of } X \text{ has arbitrarily small neighborhoods } U \]
in \( X \) with \( \text{loc com} \text{Fr} U \leq n - 1 \).

Moreover, the deficiency of \( X \), \( \text{def} X \), is defined to be the integer
\[ \min \{ \dim (\gamma X - X) | \gamma X \text{ is a compactification of } X \} \].

Lelek (see [3], Theorem 3.2) proved the following interesting result:

1.1. Theorem. If \( f \) is a continuous mapping of a space \( X \) such that
\( f^{-1}(y) \) is locally compact for each \( y \in f(X) \), then
\[ \dim X \leq \dim f(X) + \max \{ \dim f, \text{def} X \} \].

At the same time, Lelek (see [3], P 469) posed the following question:

1.2. Question. If \( f \) is a continuous mapping of \( X \), is it true that
\[ \dim X \leq \dim f(X) + \max \{ \dim f, \text{def} X \} + \text{loc com} f + 1 \]?

As usual, we write
\[ \dim f = \max \{ \dim f^{-1}(y) | y \in f(X) \} \]
and
\[ \text{loc com} f = \max \{ \text{loc com} f^{-1}(y) | y \in f(X) \} \].

This paper originated from an attempt to answer Question 1.2. In fact, we obtain an affirmative answer to this question in the case \( \text{Fr} f^{-1}(y) \) is locally compact for each \( y \) in \( f(X) \). It should be pointed out that an affirmative answer to this question has already been obtained by Nishiura in a paper [4]. However, in the case we considered here, we obtain a stronger result (see Main Theorem 3). The result will be a consequence of a generalization of Theorem 1.1.
2. Some properties of $\text{loc com and def}$. The following lemmas, whose easy proofs are omitted, are needed for the proof of our main result:

2.1. Lemma. If $A$ is an open subset of a space $X$, then

$$\text{loc com } A \leq \text{loc com } X.$$ 

2.2. Lemma. If $\{X_\alpha | \alpha \in \mathcal{A}\}$ is a covering of $X$ consisting of pairwise disjoint open subsets of $X$, then

$$\text{loc com } X \leq \max \{\text{loc com } X_\alpha | \alpha \in \mathcal{A}\}.$$ 

2.3. Lemma ([1], Theorem 4.2.1). If $A$ is a closed subset of a space $X$, then

$$\text{def } A \leq \text{def } X.$$ 

3. Main Theorem. If $f$ is a continuous mapping of a finite dimensional space $X$ such that $\text{Fr } f^{-1}(y)$ is locally compact for each $y \in f(X)$, then

$$\dim X \leq \max \{\text{loc com } f + \dim f + 1, \dim f(X) + \max \{\dim f, \text{def } X\}\}.$$ 

Proof. Let

$$X_0 = \bigcup_{y \in f(X)} \text{int } f^{-1}(y),$$

and let

$$X_1 = \bigcup_{y \in f(X)} \text{Fr } f^{-1}(y).$$

Then $X_0$ is open in $X$ and $X_1 = X - X_0$ is closed in $X$, and $X = X_0 \cup X_1$, so that

$$\dim X = \max \{\dim X_0, \dim X_1\}$$

by [2], Corollary 2b, p. 289.

The restriction

$$g = f|_{X_1}: X_1 \rightarrow f(X_1)$$

is continuous with $g^{-1}(y) = \text{Fr } f^{-1}(y)$ locally compact for each $y \in f(X_1)$, by assumption. By Theorem 1.1,

$$\dim X_1 \leq \dim f(X_1) + \max \{\dim g, \text{def } X_1\} \leq \dim f(X) + \max \{\dim f, \text{def } X\},$$

where the last inequality follows from Lemma 2.3.

On the other hand, for each compact subset $C$ of $X_0$, there exists a finite number of $y$'s, say $y_1, y_2, \ldots, y_k$, such that

$$C = \bigcup_{i=1}^k \text{int } f^{-1}(y_i),$$
so that, by [2], Corollary 2b, p. 289,
\[ \dim C \leq \dim \bigcup_{i=1}^{k} \operatorname{int} f^{-1}(y_i) \]
\[ \leq \max \{ \dim \operatorname{int} f^{-1}(y_i) \mid i = 1, 2, \ldots, k \} \]
\[ \leq \max \{ \dim f^{-1}(y_i) \mid i = 1, 2, \ldots, k \} \leq \dim f. \]

Hence, by [3], Section 2, and since \( \dim X_0 \leq \dim X < +\infty \),
\[ \dim X_0 \leq \operatorname{subcom} X_0 + \dim f + 1 \leq \operatorname{loccom} X_0 + \dim f + 1. \]

By Lemmas 2.1 and 2.2,
\[ \operatorname{loccom} X_0 \leq \max \{ \operatorname{loccom} \operatorname{int} f^{-1}(y) \mid y \in f(X) \} \]
\[ \leq \max \{ \operatorname{loccom} f^{-1}(y) \mid y \in f(X) \} = \operatorname{loccom} f, \]
so that
\[ \dim X_0 \leq \operatorname{loccom} f + \dim f + 1. \]

Putting things together, we have
\[ \dim X = \max \{ \dim X_0, \dim X_1 \} \]
\[ \leq \max \{ \operatorname{loccom} f + \dim f + 1, \dim f(X) + \max \{ \dim f, \operatorname{def} X \} \}. \]

This completes the proof of the theorem.

**COROLLARY.** If \( f \) is a continuous mapping of the finite dimensional space of \( X \) with \( \operatorname{Fr} f^{-1}(y) \) locally compact for each \( y \in f(X) \), then
\[ \dim X \leq \dim f(X) + \max \{ \dim f, \operatorname{def} X \} + \operatorname{loccom} f + 1. \]

**REFERENCES**


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