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MAPPINGS AND INDUCTIVE INVARIANTS

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1. Introduction. All spaces considered are presumed to be separable metrizable.

For any space X, the *inductive invariant* $\operatorname{loccom} X$ is defined as follows: $\operatorname{loccom} X = -1$ if and only if X is locally compact and, for $n \geqslant 0$, $\operatorname{loccom} X \leqslant n$ if each point of X has arbitrarily small neighborhoods U in X with $\operatorname{loccom} \operatorname{Fr} U \leqslant n-1$.

Moreover, the *deficiency* of X, $\operatorname{def} X$, is defined to be the integer

$$\min\{\dim(\gamma X - X) \mid \gamma X \text{ is a compactification of } X\}.$$

Lelek (see [3], Theorem 3.2) proved the following interesting result:

1.1. THEOREM. *If f is a continuous mapping of a space X such that $f^{-1}(y)$ is locally compact for each $y \in f(X)$, then*

$$\dim X \leqslant \dim f(X) + \max\{\dim f, \operatorname{def} X\}.$$

At the same time, Lelek (see [3], P 469) posed the following question:

1.2. QUESTION. If f is a continuous mapping of X, is it true that

$$\dim X \leqslant \dim f(X) + \max\{\dim f, \operatorname{def} X\} + \operatorname{loccom} f + 1 ?$$

As usual, we write

$$\dim f = \max\{\dim f^{-1}(y) \mid y \in f(X)\}$$

and

$$\operatorname{loccom} f = \max\{\operatorname{loccom} f^{-1}(y) \mid y \in f(X)\}.$$

This paper originated from an attempt to answer Question 1.2. In fact, we obtain an affirmative answer to this question in the case $\operatorname{Fr} f^{-1}(y)$ is locally compact for each y in $f(X)$. It should be pointed out that an affirmative answer to this question has already been obtained by Nishiura in a paper [4]. However, in the case we considered here, we obtain a stronger result (see Main Theorem 3). The result will be a consequence of a generalization of Theorem 1.1.

2. Some properties of loc com **and** def. The following lemmas, whose easy proofs are omitted, are needed for the proof of our main result:

2.1. LEMMA. *If A is an open subset of a space X, then*

$$\operatorname{loc\,com} A \leqslant \operatorname{loc\,com} X.$$

2.2. LEMMA. *If $\{X_a \mid a \in \mathscr{A}\}$ is a covering of X consisting of pairwise disjoint open subsets of X, then*

$$\operatorname{loc\,com} X \leqslant \max\{\operatorname{loc\,com} X_a \mid a \in \mathscr{A}\}.$$

2.3. LEMMA ([1], Theorem 4.2.1). *If A is a closed subset of a space X, then*

$$\operatorname{def} A \leqslant \operatorname{def} X.$$

3. MAIN THEOREM. *If f is a continuous mapping of a finite dimensional space X such that* $\operatorname{Fr} f^{-1}(y)$ *is locally compact for each $y \in f(X)$, then*

$$\dim X \leqslant \max\big\{\operatorname{loc\,com} f + \dim f + 1,\ \dim f(X) + \max\{\dim f,\ \operatorname{def} X\}\big\}.$$

Proof. Let

$$X_0 = \bigcup_{y \in f(X)} \operatorname{int} f^{-1}(y),$$

and let

$$X_1 = \bigcup_{y \in f(X)} \operatorname{Fr} f^{-1}(y).$$

Then X_0 is open in X and $X_1 = X - X_0$ is closed in X, and $X = X_0 \cup X_1$, so that

$$\dim X = \max\{\dim X_0,\ \dim X_1\}$$

by [2], Corollary 2b, p. 289.

The restriction

$$g = f|_{X_1}\colon\ X_1 \to f(X_1)$$

is continuous with $g^{-1}(y) = \operatorname{Fr} f^{-1}(y)$ locally compact for each $y \in f(X_1)$, by assumption. By Theorem 1.1,

$$\begin{aligned}\dim X_1 &\leqslant \dim f(X_1) + \max\{\dim g,\ \operatorname{def} X_1\}\\ &\leqslant \dim f(X) + \max\{\dim f,\ \operatorname{def} X\},\end{aligned}$$

where the last inequality follows from Lemma 2.3.

On the other hand, for each compact subset C of X_0, there exists a finite number of y's, say $y_1, y_2, \ldots, y_k$, such that

$$C \subset \bigcup_{i=1}^{k} \operatorname{int} f^{-1}(y),$$

so that, by [2], Corollary 2b, p. 289,

$$\begin{aligned}\dim C &\leqslant \dim \bigcup_{i=1}^{k} \operatorname{int} f^{-1}(y_i) \\ &\leqslant \max\{\dim \operatorname{int} f^{-1}(y_i) \mid i = 1, 2, \ldots, k\} \\ &\leqslant \max\{\dim f^{-1}(y_i) \mid i = 1, 2, \ldots, k\} \leqslant \dim f.\end{aligned}$$

Hence, by [3], Section 2, and since $\dim X_0 \leqslant \dim X < +\infty$,

$$\dim X_0 \leqslant \operatorname{sub com} X_0 + \dim f + 1 \leqslant \operatorname{loc com} X_0 + \dim f + 1.$$

By Lemmas 2.1 and 2.2,

$$\begin{aligned}\operatorname{loc com} X_0 &\leqslant \max\{\operatorname{loc com} \operatorname{int} f^{-1}(y) \mid y \in f(X)\} \\ &\leqslant \max\{\operatorname{loc com} f^{-1}(y) \mid y \in f(X)\} = \operatorname{loc com} f,\end{aligned}$$

so that

$$\dim X_0 \leqslant \operatorname{loc com} f + \dim f + 1.$$

Putting things together, we have

$$\begin{aligned}\dim X &= \max\{\dim X_0, \dim X_1\} \\ &\leqslant \max\big\{\operatorname{loc com} f + \dim f + 1, \dim f(X) + \max\{\dim f, \operatorname{def} X\}\big\}.\end{aligned}$$

This completes the proof of the theorem.

COROLLARY. *If f is a continuous mapping of the finite dimensional space of X with* $\operatorname{Fr} f^{-1}(y)$ *locally compact for each $y \in f(X)$, then*

$$\dim X \leqslant \dim f(X) + \max\{\dim f, \operatorname{def} X\} + \operatorname{loc com} f + 1.$$

REFERENCES

[1] J. de Groot and T. Nishiura, *Inductive compactness as a generalization of semi-compactness*, Fundamenta Mathematicae 58 (1966), p. 201-218.

[2] K. Kuratowski, *Topology I*, New York - Warszawa 1966.

[3] A. Lelek, *Dimension and mappings of spaces with finite deficiency*, Colloquium Mathematicum 12 (1964), p. 221-227.

[4] T. Nishiura, *Inductive invariants of closed extensions of mappings*, ibidem 25 (1972), p. 73-78.

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