

## ON SUBDIFFERENTIALS ON NON-CONVEX SETS

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Let  $(X, \tau)$  be a topological space. Let  $f(x)$  be a function defined on  $X$  with values in the extended real line  $\overline{R} = R \cup \{-\infty\} \cup \{+\infty\}$ .

Let  $\Phi$  be a class of functions defined on  $X$  and with values in  $\overline{R}$ . We assume that the class  $\Phi$  is invariant under addition of constants,  $\phi + c \in \Phi$  for  $\phi \in \Phi$  and  $c \in R$ . The function  $f(x)$  is said to be  $\Phi$ -convex if it can be represented as the supremum of a family of functions belonging to  $\Phi$ . We say that the function  $f(x)$  is *locally  $\Phi$ -convex* if for each  $x_0 \in X$  there is a neighbourhood  $U$  of  $x_0$  such that the function  $f|_U(x)$  is  $\Phi|_U$ -convex, where by  $f|_U(x)$  and  $\Phi|_U$  are denoted the restrictions of the function  $f(x)$  and the class  $\Phi$  to the set  $U$ , respectively. Of course, every  $\Phi$ -convex function is locally  $\Phi$ -convex. The converse is not true [1].

A function  $\phi \in \Phi$  is called a *local  $\Phi$ -subgradient* of the function  $f$  at a point  $x_0$  if there is a neighbourhood  $U$  of the point  $x_0$  such that

$$(1) \quad f(x) - f(x_0) \geq \phi(x) - \phi(x_0)$$

for all  $x \in U$ .

A function  $\phi \in \Phi$  is called a *global  $\Phi$ -subgradient* (briefly:  $\Phi$ -subgradient) of the function  $f$  at a point  $x_0$  if (1) holds for all  $x \in X$ .

It is easy to show that the existence of a local  $\Phi$ -subgradient at each point does not imply that the function  $f$  has a  $\Phi$ -subgradient at each point. Even more, the function  $f$  need not be  $\Phi$ -convex as follows from [1].

It is interesting, however, that there are classes  $\Phi$  such that the existence of a local  $\Phi$ -subgradient of a locally  $\Phi$ -convex function  $f(x)$  at each point  $x_0 \in X$

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implies the existence of a global  $\Phi$ -subgradient of  $f(x)$  at each point. If this is the case then we say that the family  $\Phi$  has the *globalization property*. If every local  $\Phi$ -subgradient can be extended to a global one, we say that the family  $\Phi$  has the *strong globalization property*.

If this property is enjoyed by all the functions  $f(x)$  satisfying the additional condition that there is  $\phi \in \Phi$  such that

$$(2) \quad \inf[f(x) - \phi(x)] > -\infty,$$

then we say that the family  $\Phi$  has the *bounded globalization property* (resp. *bounded strong globalization property*).

If this property holds for Lipschitz functions  $f(x)$  then we say that the family  $\Phi$  has the *Lipschitz globalization property* (resp. *Lipschitz strong globalization property*).

Let  $(X, \|\cdot\|)$  be a Banach space and let  $\Phi$  be the conjugate space,  $\Phi = X^*$ , i.e. the space of all linear continuous functionals. Let  $A \subset X$ . In this case locally  $\Phi$ -convex functions are called briefly *locally convex*. We say that the set  $A$  has

- the *linear globalization property* if the family  $X^*$  restricted to  $A$  has the globalization property;
- the *strong linear globalization property* if  $X^*$  restricted to  $A$  has the strong globalization property;
- the *bounded linear globalization property* if  $X^*$  restricted to  $A$  has the bounded globalization property;
- the *Lipschitz linear globalization property* if  $X^*$  restricted to  $A$  has the Lipschitz globalization property.

**PROPOSITION 1.** *A closed set  $A$  has the strong linear globalization property if and only if it is convex.*

**PROOF.** Suppose that the set  $A$  is not convex. Thus there are points  $x_1, x_2 \in A$  and  $0 < t < 1$  such that  $p_0 = tx_1 + (1-t)x_2 \notin A$ . We put  $v = x_2 - x_1$ . Let  $L = \{x \in X : x = p_0 + tv, t \in R\}$ . By  $d_L(x)$  we denote the distance of a point  $x$  from the line  $L$ ,  $d_L(x) = \inf\{\|x - p_0 - tv\| : t \in R\}$ . Since  $p_0 \notin A$  and the set  $A$  is closed, there is a  $d > 0$  such that  $d < \inf\{\|z - p_0\| : z \in A\}$ . Let  $C = \{x \in X : d_L(x) < d\}$ . Let  $B(p_0, d)$  be the open ball with center at  $p_0$  and radius  $d$ ,  $B(p_0, d) = \{p \in X : \|p - p_0\| < d\}$ . Now we shall define a function  $f(x)$  in the following way:

$$(3) \quad f(x) = \begin{cases} d_L(x) - d & \text{if } x \in C \text{ and } x = p + tv, t > 0, p \in B(p_0, d), \\ \max[0, d_L(x) - d] & \text{if } x \in C \text{ and } x = p + tv, t < 0, p \in B(p_0, d), \\ d_L(x) - d & \text{if } x \notin C. \end{cases}$$

It is easy to see that the function  $f(x)$  has a local linear subgradient at each point. Observe that  $l(x) \equiv 0$  is a local subgradient of the function  $f$  at the points  $x_1$  and  $x_2$ . On the other hand, since  $f(x_1) = -d$  and  $f(x_2) = 0$ ,  $l(x) \equiv 0$  is not a

global subgradient of the function  $f(x)$  at the point  $x_2$ . Thus the set  $A$  does not have the strong globalization property. ■

However, there are non-convex sets having the linear globalization property (bounded linear globalization property).

**PROPOSITION 2** (compare [1]). *Let  $A$  be the boundary of a convex bounded open set  $B$  in a Banach space  $(X, \|\cdot\|)$ ,  $A = \text{Fr } B$ . Then the set  $A$  has the bounded linear globalization property.*

**Proof.** Without loss of generality we may assume that  $0 \in \text{Int } B$ . Let  $\|x\|$  denote the Minkowski quasinorm of the point  $x$ :

$$(4) \quad \|x\| = \inf\{t > 0 : x/t \in B\}.$$

Let  $f_1(x) = f(x) - \inf f(x)$ . It is easy to see that  $f_1(x) \geq 0$ . Thus similarly to [1], we extend the function  $f_1$  to the whole space  $Y$  putting

$$f_2(x) = \|x\| f_1\left(\frac{x}{\|x\|}\right).$$

The function  $f_2(x)$  is well-defined since  $\frac{x}{\|x\|} \in A$ . It is easy to see that the function  $f_2$  possesses a local  $\Phi$ -subgradient 0 at 0. Observe that for any point  $x_0 \neq 0$  the function  $f_1(\frac{x}{\|x\|})$  has a local  $\Phi$ -subgradient  $\phi_1$  at the point  $\frac{x_0}{\|x_0\|}$  and  $f_1(\frac{x_0}{\|x_0\|}) - \phi_1(\frac{x_0}{\|x_0\|}) \geq 0$ .

Denote by  $\phi_2$  the functional supporting  $B$  at  $\frac{x_0}{\|x_0\|}$  such that  $\phi_2(\frac{x_0}{\|x_0\|}) = 1$ . Observe that the functional  $\phi(x) = \phi_1(x) - b\phi_2(x)$  is a local linear subgradient of  $f_1$  at the point  $\frac{x_0}{\|x_0\|}$  for all  $b \geq 0$ . If  $b = f_1(\frac{x_0}{\|x_0\|}) - \phi_1(\frac{x_0}{\|x_0\|})$  then  $\phi(\frac{x_0}{\|x_0\|}) = f_1(\frac{x_0}{\|x_0\|})$  and  $f_1(x) \geq \phi(x)$  holds in a neighbourhood  $V$  of the point  $\frac{x_0}{\|x_0\|}$  on  $B$ . Then by the homogeneity of the functions  $f_2(x)$  and  $\phi(x)$ ,  $f_2(x_0) \geq \phi(x_0)$  and  $f_2(x) \geq \phi(x)$  hold in a certain neighbourhood  $U$  of the point  $x_0$ . Thus  $\phi(x)$  is a local linear subgradient of  $f_2$  at the point  $x_0$ .

By Proposition 1, every local linear subgradient is also a global linear subgradient. Observe that its restriction to  $A$  gives a linear subgradient on  $A$ . ■

There is an open question: in Proposition 2, can we replace the bounded linear globalization property by the linear globalization property?

Using local compactness arguments we obtain

**COROLLARY 3** (compare [1]). *Let  $A$  be the boundary of a convex bounded open set  $B$  in a finite dimensional Banach space  $(X, \|\cdot\|)$ ,  $A = \text{Fr } B$ . Then the set  $A$  has the linear globalization property.*

Without the boundedness of the set  $B$  Proposition 2 does not hold, as follows from

**EXAMPLE 4.** Let  $X = R^2$  and let  $A = \{(x, y) : |y| = 1\}$ . It is easy to see that  $A$  is the boundary of an open convex set  $B = \{(x, y) : |y| < 1\}$  and that the set  $A$  does not have the linear globalization property.

The set  $A$  is not connected. As an example of a connected set we can take the set  $A_0 = \text{Fr } B_0$ , where  $B_0 = \{(x, y) : |y| < 1, x > 0\}$ .

Now we shall show that a large class of sets in Banach spaces do not have the linear globalization property.

Modifying the proof of Proposition 1, we obtain

**PROPOSITION 5.** *Let  $A$  be a closed set in a Banach space  $(X, \|\cdot\|)$ . Let  $\Phi$  be the restrictions of linear functionals to  $A$ . If there exist a point  $p_0 \notin A$  and a vector  $v$  such that there are  $t_1 < 0 < t_2 < t_3$  such that  $p_0 + t_i v \in A$ ,  $i = 1, 2, 3$ , then the set  $A$  does not have the bounded linear globalization property.*

**Proof.** Let  $L = \{x \in X : x = p_0 + tv, t \in \mathbb{R}\}$ . By  $d_L(x)$  we denote the distance of a point  $x$  from the line  $L$ ,  $d_L(x) = \inf\{\|x - p_0 - tv\| : t \in \mathbb{R}\}$ . Since  $p_0 \notin A$  and the set  $A$  is closed, there is a  $d > 0$  such that  $d < \inf\{\|z - p_0\| : z \in A\}$ . Let  $C = \{x \in X : d_L(x) < d\}$ . Let  $B(p_0, d)$  be the open ball with center at  $p_0$  and radius  $d$ ,  $B(p_0, d) = \{p \in X : \|p - p_0\| < d\}$ . Now we shall define a function  $f(x)$  in the following way:

$$(5) \quad f(x) = \begin{cases} d_L(x) - d & \text{if } x \in C \text{ and } x = p + tv, t > 0, p \in B(p_0, d), \\ \max[0, d_L(x) - d] & \text{if } x \in C \text{ and } x = p + tv, t < 0, p \in B(p_0, d), \\ d_L(x) - d & \text{if } x \notin C. \end{cases}$$

It is easy to see that the function  $f(x)$  has a local linear subgradient at each point. On the other hand, since  $f(p_0 + t_1 v) = -d$  and  $f(p_0 + t_2 v) = f(p_0 + t_3 v) = 0$ , the function  $f$  restricted to the line  $L$  is not convex. Thus  $f$  does not have a global linear subgradient.

Since the function  $f(x)$  is bounded from below, the set  $A$  does not have the bounded linear globalization property. ■

**COROLLARY 6.** *Let  $A$  be a closed set in a Banach space  $(X, \|\cdot\|)$ . If the set  $A$  has non-empty interior,  $\text{Int } A \neq \emptyset$ , then the set  $A$  has the linear globalization property if and only if it is convex.*

**Proof.** Let  $a \in \text{Int } A$ . Let  $b$  be an arbitrary point belonging to  $A$ . If the closed interval  $[a, b] = \{a + t(b - a) : 0 \leq t \leq 1\}$  is not contained in  $A$ , we can find  $p_0 \in [a, b]$ ,  $p_0 \notin A$ . Thus putting  $v = b - a$  we can easily find  $t_1 < 0 < t_2 < t_3$  such that  $p_0 + t_i v \in A$ ,  $i = 1, 2, 3$ , and by Proposition 5 the set  $A$  does not have the linear globalization property. Therefore if the set  $A$  has the linear globalization property, then  $[a, b] \subset A$ . This implies that  $\text{Int } A$  is a convex set and moreover  $A = \overline{\text{Int } A}$ . ■

**COROLLARY 7.** *Let  $A$  be a closed bounded set in a Banach space  $(X, \|\cdot\|)$ . Suppose that  $\text{Int } A \neq \emptyset$  and that there is a closed set  $B \subset A$  such that the set  $B$  is the boundary of an open set  $C$ ,  $B = \text{Fr } C$ . Then the set  $A$  has the bounded linear globalization property if and only if  $A = B$  and the set  $C$  is convex.*

**Proof.** Suppose that  $A \neq B$ . Let  $x_0 \in A$ ,  $x_0 \notin B$ . Let  $y_0$  be an arbitrary interior point of the set  $C$ . We put  $v = x_0 - y_0$  and we take the line  $L$  passing

through the points  $x_0, y_0$ . This line intersects the set  $B$  in at least two points, which we denote by  $y_1, y_2$ , respectively. The interval  $[x_0, y_0]$  is not contained in  $B$ . Thus there is a point  $p_0 \in [x_0, y_0]$ ,  $p_0 \notin B$ . Then we can easily find  $t_1 < 0 < t_2 < t_3$  such that  $p_0 + t_i v \in B$ ,  $i = 1, 2, 3$ . By Proposition 5, the set  $B$  does not have the linear globalization property. Therefore, if the set  $A$  has the linear globalization property, then  $A = B$ .

Suppose now that  $A = B$  and that the set  $C$  is not convex. Thus there are points  $x_1, x_2 \in \text{Int } C$  and a point  $p_0 \in [x_1, x_2]$ ,  $p_0 \notin A$ . Since  $B = \text{Fr } C$  there are  $t_1 < 0 < t_2 < t_3$  such that  $p_0 + t_i v \in A$ ,  $i = 1, 2, 3$ . By Proposition 5, the set  $A = B$  does not have the bounded linear globalization property. ■

It is interesting which sets located on the surface of a convex body have the linear globalization property. The situations may be different.

**PROPOSITION 8.** *Let  $(X, \|\cdot\|)$  be a Banach space. Let  $B$  be a convex bounded open set in  $X$  and let  $S$  be the boundary of  $B$ . Let  $A$  be a closed subset of  $S$ . Denote by  $\text{Fr}_S A$  the boundary of the set  $A$  in  $S$ . Suppose that for each  $x_0 \in \text{Fr}_S A$ , there is a linear functional  $a_{x_0}(x)$  such that for a  $c_{x_0} > 0$ ,*

$$(6) \quad a_{x_0}(x - x_0) \leq -c_{x_0} \|x - x_0\|$$

for  $x \in A$ . Then the set  $A$  has the Lipschitz linear globalization property.

**PROOF.** Let  $f(x)$  be an arbitrary Lipschitz function having a local subgradient  $\phi_0$  at each point  $x_0 \in A$ . Denote by  $L_f$  the Lipschitz constant of the function  $f$ . Without loss of generality, multiplying  $a_{x_0}$  by scalars we may assume that for all  $x_0 \in \text{Fr}_S A$ ,  $c_{x_0} > L_f$ . Let

$$(7) \quad f_{x_0}(x) = f(x_0) + a_{x_0}(x - x_0).$$

By (6) and the assumption that  $c_{x_0} > L_f$ , we obtain

$$(8) \quad f_{x_0}(x) \leq f(x)$$

for  $x \in A$ .

Thus the convex function  $\tilde{f}(x) = \max\{f_{x_0}(x) : x_0 \in \text{Fr}_S A\}$  also satisfies the condition

$$(9) \quad \tilde{f}(x) \leq f(x).$$

Now we define on  $S$  the following function:

$$(10) \quad F(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \tilde{f}(x) & \text{if } x \in S \setminus A. \end{cases}$$

It is easy to see that the function  $F(x)$  has a local subgradient at each point. Then by Proposition 2 it has a global subgradient at each point. ■

As a consequence we obtain

**EXAMPLE 9.** Let  $X = R^n$ . Let  $A = \{x = (x_1, \dots, x_n) \in R^n : x_1 \geq 0, x_1^2 + \dots + x_n^2 = 1\}$ . Then the set  $A$  has the linear globalization property.

As a limit case of Proposition 5 we obtain

PROPOSITION 10. *Let  $A$  be a closed set in a Banach space  $(X, \|\cdot\|)$ . Let  $\Phi$  be the restrictions of linear functionals to  $A$ . Let a vector  $v$  of norm one be cotangent to the set  $A$  at a point  $x_0$ , i.e. there is a sequence  $\{x_n\}$  of elements of  $A$  tending to  $x_0$  such that*

$$(11) \quad \frac{x_n - x_0}{\|x_n - x_0\|} \rightarrow v.$$

*If there are  $y \in A$ ,  $p_0 \notin A$ , such that  $y = x_0 - \alpha v$ ,  $p_0 = x_0 - \beta v$ ,  $0 < \beta < \alpha$ , then the set  $A$  does not have the bounded linear globalization property.*

PROOF. Let  $L = \{x \in X : x = p_0 + tv, t \in R\}$ . By  $d_L(x)$  we denote the distance of the point  $x$  from the line  $L$ ,  $d_L(x) = \inf\{\|x - p_0 - tv\| : t \in R\}$ . Since  $p_0 \notin A$  and the set  $A$  is closed, there is a  $d > 0$  such that  $d < \inf\{\|z - p_0\| : z \in A\}$ . Let  $C = \{x \in X : d_L(x) < d\}$ . Now we shall define a function  $f(x)$  in the following way:

$$(12) \quad f(x) = \begin{cases} d_L(x) - d & \text{if } x \in C \text{ and } x = p + tv, t < 0, p \in B(p_0, d), \\ \max[0, d_L(x) - d] & \text{if } x \in C \text{ and } x = p + tv, t > 0, p \in B(p_0, d), \\ d_L(x) - d & \text{if } x \notin C. \end{cases}$$

It is easy to see that the function  $f(x)$  has a local linear subgradient at each point. On the other hand,  $f(y) = -d$  and  $f(x_0) = f(x_n) = 0$ . Let  $\phi$  be a global subgradient of the function  $f$  at the point  $x_0$ . Then  $\phi(x_n) \leq \phi(x_0)$  and  $\phi(v) \leq 0$ . Thus by the linearity of  $\phi$ ,  $\phi(y) \geq 0 > -d = f(y)$  and we conclude that  $\phi$  is not a global subgradient of the function  $f$ . ■

Now we shall give an example of application of Proposition 10.

EXAMPLE 11. Let  $X = R^2$  and let

$$(13) \quad A = \{(x, y) : x^2 + y^2 = 1, x > 0\} \cup \{(x, y) : x + y = 1, -2 \leq x \leq 0\}.$$

Then the set  $A$  does not have the linear globalization property.

Propositions 5 and 10 practically give necessary and sufficient conditions for connected sets in  $R^2$  to have the linear globalization property.

In  $R^3$  the situation is more complicated. For example, we do not know whether the set

$$A = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1, \inf[x, y, z] \leq 0\}$$

has the linear globalization property.

Now we shall give other examples of sets with the linear globalization property which are not contained in boundaries of convex sets.

PROPOSITION 12. *Let  $(X, \|\cdot\|)$  be a Banach space. Let  $X_1, \dots, X_n$  be a decomposition of the space  $X$  into a direct sum*

$$(14) \quad X = X_1 \oplus X_2 \oplus \dots \oplus X_n.$$

Then the union  $A$  of  $X_1, \dots, X_n$ ,  $A = X_1 \cup \dots \cup X_n$  has the linear globalization property.

**Proof.** Let  $f(x)$  be a real valued locally convex function defined on  $A$  with a local subgradient at each point. Without loss of generality we may assume that  $f(0)=0$ . Let  $l_0(x)$  be a local subgradient of the function  $f(x)$  at the point  $x_0=0$ . Observe that for each  $k = 1, 2, \dots, n$ , the restriction  $l_0|_{X_k}$  is a local subgradient of the function  $f|_{X_k}$ . Since the spaces  $X_1, \dots, X_n$  are linear,  $l_0|_{X_k}$  is a global subgradient of the function  $f|_{X_k}$  on  $X_k$ ,  $k = 1, 2, \dots, n$ .

Take the function  $g(x) = f(x) - l_0(x)$ . It is easy to observe that  $g|_{X_k}$  are convex functions on  $X_k$ ,  $k = 1, \dots, n$ . Extend the function  $g(x)$  to the whole space putting for  $x = x_1 + \dots + x_n$ ,  $x_k \in X_k$ ,

$$\tilde{g}(x_1 + \dots + x_n) = \max[g(x_1, 0, \dots, 0), \dots, g(0, \dots, 0, x_n)].$$

Clearly,  $\tilde{g}(x)$  is a continuous convex function on the whole space which coincides with  $g(x)$  on  $A$ . Thus  $\tilde{g}(x)$  has a global subgradient at each point. Therefore the function  $\tilde{f}(x) = \tilde{g}(x) + l_0(x)$  is a convex function on the whole space which coincides with  $f(x)$  on  $A$ . This implies that the function  $f(x)$  has a global subgradient at each point. ■

Proposition 12 can be extended to an infinite number of spaces in the following way.

**PROPOSITION 13.** Let  $(X, \|\cdot\|)$  be a Banach space. Let  $X_1, \dots, X_k, \dots$  be a basic sequence of subspaces of the space  $X$  (i.e. every element  $x$  of the space  $X$  can be written in a unique way as a sum

$$(15) \quad x = x_1 + \dots + x_n + \dots,$$

where  $x_i \in X_i$ ,  $i = 1, 2, \dots$ ). Then the union  $A$  of  $X_1, \dots, X_n$ ,  $A = X_1 \cup \dots \cup X_n$  has the linear globalization property.

The proof runs along the same lines as the proof of Proposition 12.

**PROPOSITION 14.** Let  $(X, \|\cdot\|)$  be a Banach space. Let  $A_1, \dots, A_n$  be either closed intervals  $A_i = [0, x_i] = \{tx_i : 0 \leq t \leq 1\}$  or halflines  $A_i = [0, x_i] = \{tx_i : 0 \leq t\}$ . Suppose that for each  $i = 1, 2, \dots, n$  there is a continuous linear functional  $a_i(x)$  such that

$$(16) \quad a_i(x) \begin{cases} > 0 & \text{for } x \in A_i, x \neq 0, \\ < 0 & \text{for } x \in A_j, j = 1, 2, \dots, i-1, i+1, \dots, n, x \neq 0. \end{cases}$$

Then the union  $A = A_1 \cup \dots \cup A_n$  has the linear globalization property.

**Proof.** Let  $f(x)$  be a real valued locally convex function defined on  $A$  with a local subgradient at each point. Without loss of generality we may assume that  $f(0) = 0$  and that the local subgradient  $\partial f|_0(x) \equiv 0$  (cf. the proof of Proposition 12). Observe that for each  $i = 1, 2, \dots, n$ , zero is a local subgradient of the function

$f|_{A_i}$ . Since  $A_i$  is one dimensional, we can extend the convex function  $f_i(x) = f|_{A_i}(x)$  to the whole space in the following way:

$$(17) \quad \tilde{f}_i(x) = \begin{cases} f_i(y) & \text{if } a_i(x) = a_i(y), y \in A_i, a_i(x) \geq 0, \\ 0 & \text{if } a_i(x) < 0, \\ +\infty & \text{if } a_i(x) > 0, \text{ there is no } y \in A_i \text{ such that } a_i(x) = a_i(y). \end{cases}$$

By (16),  $\tilde{f}_i(x) = 0$  for  $x \in A_j, j \neq i$ . Thus

$$(18) \quad \tilde{f}(x) = \tilde{f}_1(x) + \dots + \tilde{f}_n(x)$$

is a convex extension of the function  $f(x)$  on the whole space and has a global subdifferential at each point of its domain. ■

### References

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