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DIRECTED AND ANTIDIRECTED HAMILTONIAN CYCLES AND PATHS IN BIPARTITE GRAPHS

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We give conditions on the number of arcs sufficient for bipartite digraphs to have either directed or antidirected hamiltonian cycles and paths.

Let D be a balanced bipartite digraph on n vertices. First, in Theorem 2, we give a minimum function $f(n)$ such that if D has $f(n)$ or more arcs, then it is hamiltonian. Next, in Theorem 6, we obtain another minimum function $f(n)$ such that if D has a factor and at least $f(n)$ arcs, then it is, once more, hamiltonian. Finally, we give conditions, on the number of arcs or dealing with half-degrees, sufficient for bipartite digraphs to have antidirected cycles and paths of various lengths. For analogous results concerning digraphs the reader is encouraged to consult [4], [5].

Formally, throughout this paper, $D = (X, Y, E)$ denotes a bipartite digraph of order n with bipartition (X, Y), where we suppose that $|X| \leqslant |Y|$. Then $V(D)$ $(= X \cup Y)$ denotes the set of vertices and $E(D)$ denotes the set of arcs of D. If x and y are vertices of D, then we say that x *dominates* y if the arc (x, y) is present. For $A, B \subseteq V(D)$, we define $E(A \to B) = \{(x, y)\colon x \in A,\ y \in B,\ (x, y) \in E(D)\}$ and $E(A, B) = E(A \to B) \cup E(B \to A)$. The *outdegree*, *indegree* and *degree* of a vertex x are defined as $|E(x \to V(D))|$, $|E(V(D) \to x)|$ and $|E(x, V(D))|$ respectively and are denote by $d^+(x)$, $d^-(x)$ and $d(x)$ respectively. We say that D is *balanced* if $|X| = |Y|$ and *almost balanced* if $|Y| = |X| + 1$. We define an *antidirected cycle*, or shortly ADC, to be a cycle such that no two consecutive arcs form a directed path. Analogously we define *antidirected hamiltonian* *cycles* (ADHC), *antidirected paths* (ADP) and *antidirected hamiltonian paths* (ADHP). The *opposite* of D is the graph obtained from D by reversing the orientation of each arc of D.

We begin with the following easy proposition, which we need for the proof of Theorem 2.

PROPOSITION 1. *Let D be a bipartite digraph on 6 vertices. If for each vertex x of D we have $d(x) \geqslant 5$, then D is hamiltonian.*

Proof. Since D is bipartite and since any vertex x satisfies $d(x) \geqslant 5$, D is balanced. In addition, the condition $d(x) \geqslant 5$ implies that $d^+(x) \geqslant 2$ and $d^-(x) \geqslant 2$ and therefore D has a cycle C of length four [2]. Now, if x and y denote the vertices of $D-C$, then it suffices to consider arcs between $\{x, y\}$ and C in order to complete the proof.

This proposition is best possible because of the graph with bipartition (X, Y), where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, and arc-set $E(D) = \{(x_1, y_1), (y_1, x_2), (x_2, y_2), (y_2, x_1), (x_1, y_3), (y_3, x_1), (x_2, y_3), (y_3, x_2), (x_3, y_1), (y_1, x_3), (y_2, x_3), (x_3, y_2)\}$.

In Theorem 2, we shall consider the digraph $B_1(n)$ defined as follows (see also Fig. 1): Let A (resp. B) be an independent set on $k-1$ vertices (resp. on k vertices) such that $n = 2k$. $B_1(n)$ is the bipartite digraph obtained from the disjoint union of A, B by adding a new vertex x and all the arcs between A and

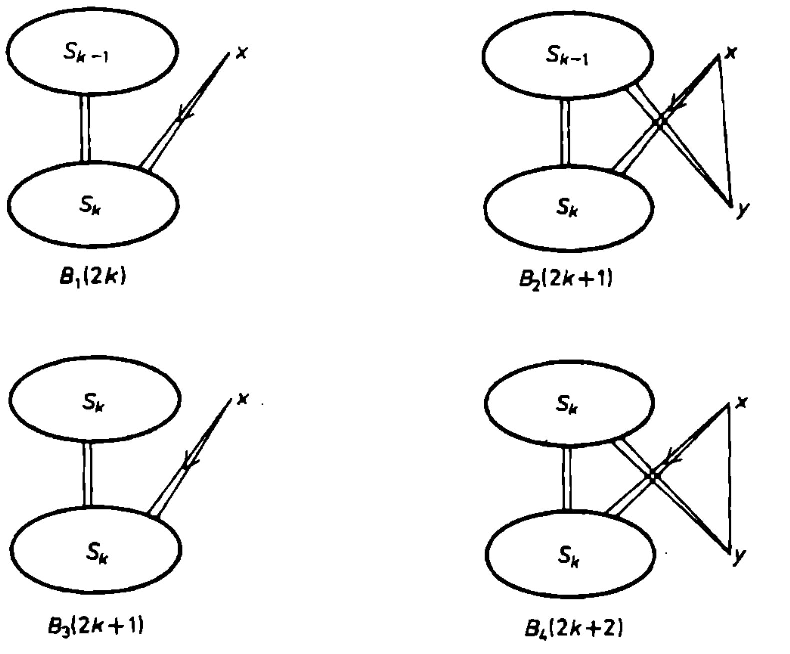

Fig. 1. The graph $B_1(2k)$ has no hamiltonian cycle; $B_2(2k+1)$ has no hamiltonian path with terminus y; $B_3(2k+1)$ has no path with terminus x; $B_4(2k+2)$ has no hamiltonian path with origin y and terminus x

B, and all the arcs from x to B. The resulting bipartite digraph has no cycle through x.

In Fig. 1, S_k denotes an independent set on k vertices.

THEOREM 2. *Let $D=(X, Y, E)$ be a balanced bipartite digraph with n vertices such that $|E(D)| \geqslant (n^2-n)/2$. Then D is hamiltonian unless $|E(D)| = (n^2-n)/2$ and moreover D is isomorphic either to $B_1(n)$ or to its opposite or else to R_1 or to R_2 of Fig. 2.*

Fig. 2

Proof. First, for any vertex x of D we have $d(x) \geqslant n/2$, since

$$d(x) = |E(D)| - |E(D-x)| \geqslant \frac{n^2-n}{2} - 2\left(\frac{n}{2}-1\right)\frac{n}{2} = \frac{n}{2}.$$

We now distinguish two cases depending upon the degree of x.

(a) *There exists a vertex x in D such that $d(x)=n/2$.* By a simple calculation we obtain

$$|E(D-x)| \geqslant \frac{n^2-n}{2} - \frac{n}{2} = \frac{(n-1)^2-1}{2}.$$

It follows that $D-x$ is the almost balanced complete bipartite digraph $K^*_{n/2,n/2-1}$ and therefore any two vertices of Y are connected by a hamiltonian path in $D-x$. Consequently, if D is not hamiltonian, then either $n=4$ and D is isomorphic to R_1 or to $B_1(4)$, or else $n \geqslant 6$ and then D is isomorphic to $B_1(n)$ or to its opposite.

(b) *For any vertex x of D we have $d(x) \geqslant n/2+1$.* The proof of this case is by induction on n. It is easy to see that any bipartite digraph on 4 vertices and no less than 6 arcs is hamiltonian unless it has exactly 6 arcs and moreover it is isomorphic to R_2 of Fig. 2. In what follows, assume that $n \geqslant 6$ and that the theorem is true for any bipartite digraph with no more than $n-2$ vertices.

Assume first that there exists a vertex x such that $d(x) \leqslant n-2$. Since $d(x) \geqslant n/2+1$, there exist two vertices y_1, y_2 such that both the arcs (y_1, x) and (x, y_2) are present. Now, let D' denote the bipartite digraph obtained from $D-\{x, y_1, y_2\}$ by adding a new vertex s and the set of arcs

$$\{(z, s)\colon z \in V(D'),\ (z, y_1) \in E(D)\} \cup \{(s, z)\colon z \in V(D'),\ (y_2, z) \in E(D)\}.$$

Clearly D' has $n-2$ vertices and moreover it satisfies

$$|E(D')| \geqslant \frac{n(n-1)}{2} - \left(n-2+2\left(\frac{n}{2}-1\right)\right) = \frac{(n-2)(n-3)}{2}+1.$$

It follows that D' is hamiltonian by induction, and therefore D is also hamiltonian, as required.

Assume next that for any vertex we have $d(x) \geqslant n-1$. This hypothesis implies that $d^+(x) \geqslant n/2-1$ and $d^-(x) \geqslant n/2-1$. Now, if $n = 6$, then we can easily verify that the graph is hamiltonian, using Proposition 1. If, on the other hand, $n > 6$, then, since in this case we have $d^+(x) \geqslant n/2-1 \geqslant (n+3)/4$ and $d^-(x) \geqslant n/2-1 \geqslant (n+3)/4$, it follows from a theorem of [1] that the graph is, once more, hamiltonian. This completes the proof of Theorem 2.

In order to formulate Theorem 3, we define the following bipartite digraphs $B_2(n)$, $B_3(n)$ and $B_4(n)$ (see also Fig. 1).

$B_2(n)$ is obtained as follows: Let A (resp. B) be an independent set on $k-1$ (resp. on k) vertices, $n = 2k+1$. $B_2(n)$ is the almost balanced bipartite digraph obtained from the complete bipartite graph with bipartition (A, B), by adding two vertices x and y and all the arcs between y and A, between x and y and from x to B. This graph has no hamiltonian path with terminus y.

$B_3(n)$ is obtained from $B_1(n)$ by putting $|A| = |B| = k$, $n = 2k+1$. The graph $B_3(n)$ has no path with terminus x.

$B_4(n)$ is obtained from $B_2(n)$ by putting $|A| = |B| = k$, $n = 2k+2$. Clearly, the graph $B_4(n)$ has no hamiltonian path with origin y and terminus x.

THEOREM 3. *Let $D = (X, Y, E)$ be a bipartite digraph on n vertices such that $|E(D)| \geqslant (n^2-n)/2+\varepsilon$, $\varepsilon = 0, 1$. Then:*

(i) *If D is almost balanced and if $\varepsilon = 0$, then any two vertices of Y are connected by a hamiltonian path unless $|E(D)| = (n^2-n)/2$ and D is isomorphic either to $B_2(n)$ or to $B_3(n)$ or to their opposites or else, for $n = 5$, to one of the graphs A_i, $i = 1, 2, 3$, presented in Fig. 3.*

(ii) *If D is balanced and if $\varepsilon = 1$, then any two vertices which are not in the same class of D are connected by a hamiltonian path unless D has exactly $(n^2-n)/2+1$ arcs and moreover D is isomorphic to $B_4(n)$ or to its opposite.*

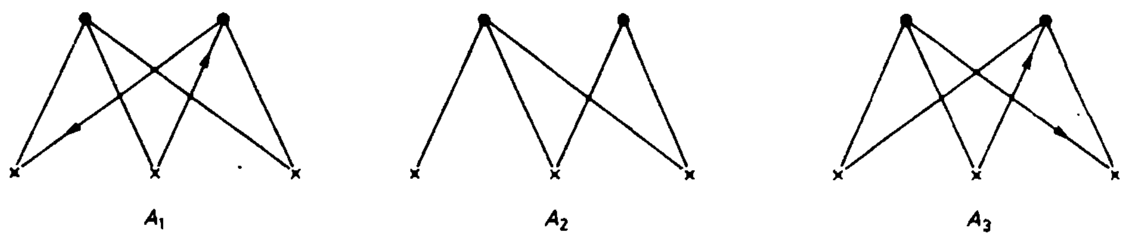

Fig. 3

Proof of (i). In what follows assume $n \geqslant 5$, since the only graph with 3 vertices and 3 arcs which does not satisfy the conclusion of (i) is the graph $B_2(3)$.

Let now y_1, y_2 be two vertices of Y and then let D' be the graph obtained from $D-\{y_1, y_2\}$ by adding a new vertex x and the arcs $\{(y_1, z): (x, z)$ in $E(D)\} \cup \{(z, y_2): (z, x)$ in $E(D)\}$. Clearly D' satisfies

$$|E(D')| \geqslant \frac{n^2-n}{2} - 2\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}.$$

Now, if D' is hamiltonian, then there exists a hamiltonian path from y_1 to y_2 in D, so, in what follows, assume that this is not the case. Consequently, it follows from Theorem 2 that $|E(D')| = (n-1)(n-2)/2$, i.e., $d^-(x_1) = d^+(x_2) = (n-1)/2$ and, moreover, D' is isomorphic to $B_1(n-1)$ or to its opposite for $n \geqslant 5$ or, for $n = 5$, to one of the graphs R_1 or R_2 of Fig. 2.

Now, if D' is isomorphic to $B_1(n-1)$, then there exists a vertex z in D' such that either $E(z \to D') = \emptyset$ or $E(D' \to z) = \emptyset$. Consequently, if z is in Y, then D is isomorphic to $B_2(n)$ or to its opposite, otherwise, if z is in X, then D is isomorphic to $B_3(n)$ or to its opposite. On the other hand, if D' is isomorphic to one of the graphs R_i $(i = 1, 2)$ of Fig. 2, then D is isomorphic to one of the graphs A_i, $i = 1, 2, 3$, of Fig. 3.

Proof of (ii). Let x, y be two vertices of D such that x is in X and y is in Y. We shall prove that there exists a hamiltonian path from x to y. The graph $D-y$ satisfies

$$|E(D-y)| \geqslant \frac{n^2-n}{2} + 1 - n \geqslant \frac{(n-1)(n-2)}{2}.$$

Observe now that if any two vertices of X are connected by a hamiltonian path in $D-y$, then there exists a hamiltonian path from y to x in D (in fact, it suffices to consider a vertex z of X which is dominated by y and then to take a hamiltonian path from z to x in $D-y$). Consequently, it follows from (i) that $|E(D-y)| = (n-1)(n-2)/2$, i.e., $d(y) = n$ and moreover $D-y$ is isomorphic to one of $B_2(n-1)$, $B_3(n-1)$, $B_4(n-1)$ or to one of the graphs A_i, $i = 1, 2, 3$, of Fig. 3. Now, if $D-y$ is isomorphic either to $B_2(n-1)$ or to $B_3(n-1)$, then D is isomorphic to $B_4(n)$. On the other hand, if $D-y$ is isomorphic to one of the graphs A_i, $i = 1, 2, 3$, then it is easy to find a hamiltonian path from y to x in D. This completes the proof.

The corollary below follows directly from Theorems 2 and 3.

COROLLARY 4. *Let D be a bipartite digraph. Then D has a hamiltonian path if*

(i) *it is balanced and has at least* $n^2/2-n+1$ *arcs, or*
(ii) *it is almost balanced and has at least* $(n^2-n)/2$ *arcs.*

Proof of (i). Assume first that there exists a vertex x in X (or in Y) such that either $E(x \to D) = \emptyset$ or $E(D \to x) = \emptyset$. Suppose without loss of generality that $E(x \to D) = \emptyset$. Now, let D' denote the bipartite digraph obtained from D by adding all the arcs from x to Y. Clearly D' has at least $(n^2 - n)/2 + 1$ arcs and therefore it is hamiltonian by Theorem 2. It follows that D has a hamiltonian path, as required.

Assume next that for any vertex x of D we have $E(x \to D) \neq \emptyset$ and $E(D \to x) \neq \emptyset$. Then we can complete the proof by using arguments similar to those of case (b) in the proof of Theorem 2.

Proof of (ii). This follows directly from part (i) of Theorem 3.

Note that Corollary 4 is best possible. To see that consider a) for (i), the bipartite digraph obtained from $B_1(n)$ by deleting all arcs incident to x, and b) for (ii), the graph obtained from $B_2(n)$ by deleting the arc (y, x).

For the proof of Theorem 6, we need the following lemma.

LEMMA 5. *Let D be a bipartite digraph. Moreover, assume that there are two pairwise vertex-disjoint cycles C_1 and C_2 in D, covering all the vertices of D. If $|E(C_1, C_2)| \geqslant \frac{1}{2}|V(C_1)||V(C_2)| + 1$, then D is hamiltonian.*

Proof. The proof is by contradiction. Put C_1: $c_1 \to c_2 \to \dots \to c_{2m}$ and C_2: $s_1 \to s_2 \to \dots \to s_{2k}$, where $2(m+k) = n$. Take a vertex c_i, $1 \leqslant i \leqslant 2m$, on C_1. Since D is not hamiltonian, we have

$$|E(c_i \to s_j)| + |E(s_{j-1} \to c_{i+1})| + |E(c_i \to s_{j+1})| + |E(s_j \to c_{i+1})| \leqslant 1, \quad 1 \leqslant j \leqslant 2k,$$

where i (resp. j) is taken modulo $2m$ (resp. modulo $2k$). Consequently, we obtain

$$\begin{aligned} |E(C_1, C_2)| &= \sum_{i=1}^{2m} \left(|E(c_i \to C_2)| + |E(C_2 \to c_{i+1})|\right) \\ &= \sum_{i=1}^{2m} \sum_{1 \leqslant j \leqslant 2k} \left(|E(c_i \to s_j)| + |E(s_{j-1} \to c_{i+1})| + |E(c_i \to s_{j+1})| + |E(s_j \to c_{i+1})|\right) \\ &\leqslant \sum_{i=1}^{2m} k = 2mk = \tfrac{1}{2}|V(C_1)|\,|V(C_2)|, \end{aligned}$$

a contradiction. This completes the proof.

THEOREM 6. *Let D be a bipartite digraph which contains a factor, in other words, a spanning regular subgraph with half-degrees one. If D has $n^2/2 - n + 2$ or more arcs, then it is hamiltonian unless it is isomorphic to the digraph obtained from $B_4(n)$ by deleting either all the arcs from x to B or all the arcs from A to y.*

Proof. Choose a factor of D, consisting of cycles $C_1, \dots, C_m$, such that m is the least possible. Clearly, if $m = 1$, then there is nothing to prove, so, in what follows, assume $m \geqslant 2$. Observe now that the bipartite digraph induced by $V(C_1) \cup V(C_i)$, $i = 2, \ldots, m$, is not hamiltonian, by the minimality property of m, and therefore we have $|E(C_1, C_i)| \leqslant \frac{1}{2}|V(C_1)|\,|V(C_i)|$, by Lemma 5. Consequently, we obtain

$$\sum_{i=2}^{m} |E(C_1, C_i)| \leqslant \tfrac{1}{2}|V(C_1)| \sum_{i=2}^{m} |V(C_i)| = \frac{p(n-p)}{2}, \quad \text{where } p = |V(C_1)|.$$

It follows that

$$|E(D)| \leqslant \frac{p^2}{2} + \frac{(n-p)^2}{2} + \frac{p(n-p)}{2} = \frac{n^2}{2} - \frac{p(n-p)}{2}.$$

Put $f(p) = p(n-p)/2$, where $2 \leqslant p \leqslant n-2$. Now, by studying the function $f(p)$ we can see that the minimum of $f(p)$ is attained either for $p = 2$ or for $p = n-2$. Then, by a simple calculation, we get $f(2) = f(n-2) = n-2$. It follows that all the above inequalities are equalities. In particular, the length of C_1 is two and moreover, the graph induced by $\bigcup_{i=2}^{m} V(C_i)$ is the complete bipartite digraph $K^*_{n/2-1,n/2-1}$. Now, it is very easy to verify the conclusion of the theorem and this completes the proof.

It was proved in [6] that any hamiltonian bipartite digraph with $n^2/4+n$ or more arcs has two distinct cycles of each even length m, $2 \leqslant m \leqslant n$. It is clear that, using this result, we can extend the conclusion of Theorem 2, for $n \geqslant 6$, and of Theorem 6, for $n \geqslant 8$. Note also that, by using arguments similar to those of the proof of Theorem 6, we may prove that any digraph which contains a factor and has n^2-3n+5 or more arcs is hamiltonian.

We shall conclude this paper with some results on antidirected cycles and paths. For that we need some additional definitions. Let D be any bipartite digraph. Then $G1$ (resp. $G2$) is the nondirected bipartite graph obtained from D by replacing any arc (x, y) (resp. any arc (y, x)), x in X and y in Y, by an edge.

THEOREM 7. *A bipartite digraph D has an ADC (resp. an ADP) of length k if and only if either $G1$ or $G2$ defined above has a cycle (resp. a path) of length k.*

Proof. Trivial.

From this theorem we obtain a series of corollaries.

COROLLARY 8. *Any balanced bipartite digraph with n vertices and $n^2/2-n+3$ or more arcs has an ADC of each even length m, $4 \leqslant m \leqslant n$.*

Proof. It follows from $|E(D)| \geqslant n^2/2-n+3$ that either $|E(X \to Y)| \geqslant n^2/4 - n/2+2$ or $|E(Y \to X)| \geqslant n^2/4-n/2+2$ holds. Now, in order to complete the proof, it suffices to use these inequalities together with Theorem 6 and a result given in [3], p. 207.

COROLLARY 9. *A bipartite digraph has an ADHP if*

(i) *it is balanced and has* $n^2/2-n+1$ *or more arcs, or*

(ii) *it is almost balanced and has* $(n^2+3)/2-n$ *or more arcs.*

Proof. It is similar to that of Corollary 8.

COROLLARY 10. *Any balanced bipartite digraph with n vertices and half-degrees at least* $n/4+1$ *has two pairwise arc-disjoint ADC of each even length m,* $4 \leqslant m \leqslant n$.

Proof. Observe that the minimum degree of $G1$ (resp. of $G2$) is $(n+2)/4$. It follows from a result of [1] that $G1$ (resp. $G2$) is bipancyclic, so it suffices to use Theorem 7 in order to complete the proof.

COROLLARY 11. *Let D be a bipartite digraph with half-degrees at least r. Let x and y be two vertices of D. Then:*

(i) *If D is balanced, if* $n \leqslant 4r-8$ *and if x and y are not in the same class of D, then x and y are connected by two pairwise arc-disjoint ADP of each odd length m,* $3 \leqslant m \leqslant n$.

(ii) *If D is almost balanced, if* $n \leqslant 4r-9$ *and if both the vertices x and y are in Y, then x and y are connected by two pairwise arc-disjoint ADP of each even length m,* $2 \leqslant m \leqslant n$.

Proof. It is very similar to that of Corollary 10.

Note that Corollaries 8–11 are best possible, since so are the corresponding results for $G1$ or $G2$.

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