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Integrable solutions of functional equations

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Introduction

In this paper we are concerned with the integrable solutions of functional equations in a single variable. The functional equations which appear in this paper have been thoroughly investigated in many classes of functions, such as continuous, differentiable, analytic functions, etc. (see [12] and the bibliography therein). Concerning the integrable solutions of functional equations, the situation is different. There are two papers on this subject. In [16] a particular case of the linear functional equation was considered, and in [13] M. Kuczma investigated the uniqueness of integrable solutions of the homogeneous linear equation.

In the first chapter of the present paper we quote some fixed point theorems due to Browder [4], Kirk [11], and Boyd and Wong [3]. Moreover, we give a contribution to the theorem of Boyd and Wong, as well as proof of a result which has been published (without proof) in [17].

In Chapters 2–5 we consider the Lebesgue integrable solutions, in turn for linear equations of the 1-st order, non-linear equations of the 1-st order, systems of equations and for equations of higher orders.

In the last chapter we examine solutions integrable with respect to general measure for a linear equation of order 1.

Moreover, in Chapters 2 and 6 we apply some of the above results in order to obtain absolutely continuous solutions of a special linear equation. These results are then applied to a Goursat problem for a hyperbolic partial differential equation.

0. Explanatory notes, definitions and a lemma

Let R be the set of real numbers and write $\bar{R} = R \cup \{-\infty, +\infty\}$. Let (X, S, μ) be a measure space. For a $p > 0$ we denote by $L^p(X, S, \mu)$ the set of all S-measurable functions $\varphi\colon X \to R$ such that $\int\limits_X |\varphi|^p d\mu < \infty$.

The relation "$\sim$" in $L^p(X, S, \mu)$ defined as follows:

$$\varphi_1 \sim \varphi_2 \quad \text{iff} \quad \varphi_1 = \varphi_2 \quad \text{a.e. in } X$$

is an equivalence. We denote by $\boldsymbol{L}^p(X, S, \mu)$ the set $L^p(X, S, \mu)/\sim$ and by $[\varphi]$ the class of equivalence of a $\varphi \in L^p(X, S, \mu)$.

It is known that for every p, $0 < p < 1$, the space $L^p(X, S, \mu)$ with the metric

$$\varrho([\varphi_1], [\varphi_2]) = \int_X |\varphi_1 - \varphi_2|^p d\mu$$

is a complete metric space, and for $p \geqslant 1$, $L^p(X, S, \mu)$ with the norm

$$\|[\varphi]\| = \Big(\int_X |\varphi|^p d\mu\Big)^{1/p}$$

is a Banach space. Put

$$a(p) = \begin{cases} 1, & 0 < p < 1, \\ 1/p, & p \geqslant 1. \end{cases}$$

For every $p > 0$

$$\Big(\int_X |\varphi|^p d\mu\Big)^{a(p)}, \qquad [\varphi] \in L^p(X, S, \mu),$$

s a paranorm. In particular, we have the following Minkowski's inequality

$$\Big(\int_X |\varphi_1 + \varphi_2|^p d\mu\Big)^{a(p)} \leqslant \Big(\int_X |\varphi_1|^p d\mu\Big)^{a(p)} + \Big(\int_X |\varphi_2|^p d\mu\Big)^{a(p)}$$

for $\varphi_1, \varphi_2 \in L^p(X, S, \mu)$. The convergence of φ_n to φ in the sense of this paranorm denotes the convergence in measure. But we have the following:

LEMMA 0.1. *Let $\varphi_n \in L^p(X, S, \mu)$, $n = 1, 2, \ldots$ If*

$$\sum_{n=1}^{\infty} \Big(\int_X |\varphi_n|^p d\mu\Big)^{a(p)} < \infty,$$

then the series $\sum_{n=1}^{\infty} \varphi_n$ converges a.e. in X and its sum φ belongs to $L^p(X, S, \mu)$.

Proof. Suppose that $0 < p < 1$. By the integral test of convergence ([22], p. 277), the series $\sum_{n=1}^{\infty} |\varphi_n|^p$ converges a.e. in X. Since $0 < p < 1$, $\sum_{n=1}^{\infty} |\varphi_n|$ converges a.e. in X and, consequently, so does the series $\sum_{n=1}^{\infty} \varphi_n$. Now, we have by Fatou's lemma,

$$\int_X |\varphi|^p d\mu = \int_X \lim_{m\to\infty} \Big|\sum_{k=1}^{m} \varphi_k\Big| d\mu \leqslant \liminf_{m\to\infty} \int_X \Big|\sum_{k=1}^{m} \varphi_k\Big|^p d\mu$$

$$\leqslant \liminf_{m\to\infty} \int_X \sum_{k=1}^{m} |\varphi_k|^p d\mu = \sum_{k=1}^{\infty} \int_X |\varphi_k| d\mu < \infty,$$

which completes the proof for $p \in (0, 1)$.

The proof for $p \geqslant 1$ is similar, and can be obtained by a simple modification of the proof of Theorem 1.3, p. 214, in [1]. If $X = (a, b) \subset R$ and μ is the Lebesgue measure, we write $L^p(a, b)$ and $\mathbf{L}^p(a, b)$ instead of $L^p(X, S, \mu)$ and $\mathbf{L}^p(X, S, \mu)$.

To simplify the formulation of the results, in the sequel we assume the following convention. The expression "$\varphi \in L^p(X, S, \mu)$ is a solution of some functional equation" means, in particular, that after inserting φ into this equation both its sides are identically equal in X, whereas the statement "$[\varphi] \in \mathbf{L}^p(X, S, \mu)$ is a solution of some functional equation" denotes that for every $\psi \in [\varphi]$, ψ satisfies this equation a.e. in X. Besides these conventions, we treat the elements of $\mathbf{L}^p(X, S, \mu)$ as functions.

For $A \subset X$ put $S(A) = \{B \in S\colon B \subset A\}$. Evidently, $S(A)$ is a σ-ring and μ_A, defined by the formula $\mu_A(B) = \mu(B)$, $B \in S(A)$, is a measure. Suppose that $f_k\colon X \to X$, $k = 1, \ldots, n$, $F\colon X \times R^{n+1} \to R$ and $\mu(X) > 0$.

DEFINITION 0.1. We say that the solution $\varphi \in L^p(X, S, \mu)$ of a functional equation

$$F\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) = 0$$

depends on an arbitrary function if there exists a set $A \in S$ of positive measure such that for every function $\varphi_0 \in L^p\big(A, S(A), \mu_A\big)$ there exists a $\varphi \in L^p(X, S, \mu)$ satisfying this equation in X and such that $\varphi = \varphi_0$ in A.

If $h\colon X \times Y \to Z$, then for every $y \in Y$, the symbol $h(\cdot, y)$ denotes the function of one variable x defined as follows:

$$h(\cdot, y)(x) = h(x, y), \qquad x \in X.$$

The composition of functions f and g is denoted by $f \circ g$.

1. Some fixed point theorems

1. In our study of integrable solutions of functional equations we shall apply the well-known Banach's principle, its generalization given by Boyd and Wong [3], and some results of Browder [4] and Kirk [11] (cf. [9] for an elementary proof).

Let (X, ϱ) be a metric space and write $P = \{\varrho(x, y)\colon x, y \in X\}$. For $T\colon X \to X$ we denote by T^n the n-th iterate of T.

THEOREM 1.1 ([3]). *Let (X, ϱ) be a complete metric space and let $T\colon X \to X$ satisfy*

$$\varrho(Tx, Ty) \leqslant \gamma\big(\varrho(x, y)\big), \qquad x, y \in X, \tag{1.1}$$

where $\gamma\colon \bar{P} \to \langle 0, \infty)$ is upper semicontinuous from the right on $\bar{P}$ and satisfies $\gamma(t) < t$ for all $t \in \bar{P} \setminus \{0\}$ ($\bar{P}$ denotes the closure of P). Then T has a unique fixed point x_0 and $\varrho(T^n x, x_0)$ tends to 0 for every $x \in X$.

Now we shall prove a result which shows that sometimes in Theorem 1.1 the condition of the upper semicontinuity of γ may be omitted.

THEOREM 1.2. *Let (X, ϱ) be a complete metric space and let $T: X \to X$ satisfy* (1.1), *where $\gamma: \langle 0, \infty) \to \langle 0, \infty)$ fulfils the following conditions: γ is increasing*[1] *in $\langle 0, \infty)$, and $\lim_{n\to\infty} \gamma^n(t) = 0$ for every $t > 0$. Then T has a unique fixed point x_0 and $\lim_{n\to\infty} \varrho(T^n x, x_0) = 0$ for every $x \in X$*

Proof. For an $x \in X$ we put $x_n = T^n x$, $n = 1, 2, \ldots$ It follows from (1.1) by induction that

$$\varrho(x_{n+1}, x_n) \leqslant \gamma^n\big(\varrho(x_1, x)\big), \qquad n = 1, 2, \ldots,$$

and, consequently, $\lim_{n\to\infty} \varrho(x_{n+1}, x_n) = 0$. Thus, for an $\varepsilon > 0$ we can choose an n such that

$$\varrho(x_{n+1}, x_n) \leqslant \varepsilon - \gamma(\varepsilon).$$

Put $K(x_n, \varepsilon) = \{x \in X: \varrho(x, x_n) \leqslant \varepsilon\}$. By (1.1) and by the monotonicity of γ, we have for $z \in K(x_n, \varepsilon)$

$$\begin{aligned}\varrho(Tz, x_n) &\leqslant \varrho(Tz, Tx_n) + \varrho(Tx_n, x_n) \leqslant \gamma\big(\varrho(z, x_n)\big) + \varrho(x_{n+1}, x_n) \\ &\leqslant \gamma(\varepsilon) + \big(\varepsilon - \gamma(\varepsilon)\big) = \varepsilon.\end{aligned}$$

This means that T maps $K(x_n, \varepsilon)$ into itself, which implies that $\varrho(x_k, x_m) \leqslant 2\varepsilon$ for $k, m \geqslant n$. Consequently, $\{x_n\}$ is a Cauchy sequence and hence converges in view of the completeness of X. By the continuity of T (resulting from (1.1)), we have

$$x_0 = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx_0.$$

The uniqueness of the fixed point is obvious.

Remark. Let (X, ϱ) be a complete metric space and suppose that $T: X \to X$ satisfies (1.1) with the function

$$\gamma(t) = \begin{cases} 1, & t > 1, \\ \dfrac{1}{n+1}, & \dfrac{1}{n+1} < t \leqslant \dfrac{1}{n},\ n = 1, 2, \ldots, \\ 0, & t = 0. \end{cases}$$

Let us note that all the assumptions of Theorem 1.2 are fulfilled. Since γ is not upper semicontinuous from the right, we cannot apply Theorem 1.1. Meir and Keeler [18] proved a generalization of Theorem 1.1, but

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[1] Here and in the sequel increasing and decreasing denotes non-decreasing and non-increasing, respectively.

it does not contain our Theorem 1.2. Moreover, the argument given in the proof of Theorem 1.2 may be used to obtain a shorter proof of the theorem in [18], as well as of the result in [3].

Let B denote a Banach space with norm $\|\cdot\|$.

DEFINITION 1.1. A Banach space B is called *uniformly convex* if there exists a non-negative function $\delta\colon (0,2\rangle \to (0,1\rangle$ such that for some positive r the inequalities $\|x\| \leqslant r$, $\|y\| \leqslant r$ and $\|x-y\| \geqslant \varepsilon r$ imply that

$$\|\tfrac{1}{2}(x+y)\| \leqslant \big(1-\delta(\varepsilon)\big)r \quad \text{for} \quad x, y \in B.$$

Remark 1.1. Clarkson [6] proved that for $p > 1$ the space $L^p(X, S, \mu)$ is uniformly convex.

We quote the following theorem.

THEOREM 1.3 ([4], [11]). *Let B be a uniformly convex Banach space and let K be a non-empty bounded closed and convex subset of B. If a transformation $T\colon K \to K$ fulfils the condition*

$$\|Tx - Ty\| \leqslant \|x-y\|, \qquad x, y \in K,$$

then T has at least one fixed point in K.

2. Now we shall prove a theorem concerning the existence and uniqueness of solutions of some systems of equations in metric spaces.

Let $(c_{ik}^{(0)})$ be a square matrix with $c_{ik}^{(0)} \in R$, $i, k = 1, \ldots, n$. Define the sequence of matrices $(c_{ik}^{(l)})$ as follows:

$$c_{ik}^{(l+1)} = \begin{cases} c_{11}^{(l)} c_{i+1,k+1}^{(l)} - c_{i+1,1}^{(l)} c_{1,k+1}^{(l)} & i = k, \\ c_{11}^{(l)} c_{i+1,k+1}^{(l)} + c_{i+1,1}^{(l)} c_{1,k+1}^{(l)}, & i \neq k, \end{cases} \tag{1.2}$$

$i, k = 1, \ldots, n-l-1$, $l = 0, \ldots, n-2$. Evidently, $c_{ik}^{(l)}$ is an $(n-l)\times(n-l)$ square matrix.

We start with the following:

LEMMA 1.1. *Let $c_{ik}^{(0)} \geqslant 0$ for $i, k = 1, \ldots, n$, $n \geqslant 2$. The system of inequalities*

$$\sum_{\substack{k=1 \\ k\neq i}}^{n} c_{ik}^{(0)} r_k < c_{ii}^{(0)} r_i, \qquad i = 1, \ldots, n, \tag{1.3}$$

has a positive solution $r_1, \ldots, r_n$ if and only if

$$c_{ii}^{(l)} > 0, \quad i = 1, \ldots, n-l;\ l = 0, \ldots, n-1. \tag{1.4}$$

Proof. Suppose that $n = 2$. Since in the case $c_{12}^{(0)} = c_{21}^{(0)} = 0$ the proof of the lemma is trivial, we assume that one of these numbers, say $c_{12}^{(0)}$, is different from 0. Then (1.4) may be written in the form

$$\frac{c_{21}^{(0)}}{c_{22}^{(0)}} < \frac{c_{11}^{(0)}}{c_{12}^{(0)}}.$$

Note that the last inequality is fulfilled if and only if there exist positive numbers r_1 and r_2 such that

$$\frac{c_{21}^{(0)}}{c_{22}^{(0)}} < \frac{r_2}{r_1} < \frac{c_{11}^{(0)}}{c_{12}^{(0)}}.$$

Rewriting the above inequalities in the form

$$c_{12}^{(0)} r_2 < c_{11}^{(0)} r_1, \qquad c_{21}^{(0)} r_1 < c_{22}^{(0)} r_2,$$

we get (1.3) for $n = 2$. Thus the lemma is true for $n = 2$.

Now suppose that the lemma is valid for $n-1$, $n \geqslant 3$, and consider system (1.3) formed of n inequalities. The first of these inequalities can be written in the form

$$\frac{1}{c_{11}^{(0)}} \sum_{k=2}^{n} c_{1k}^{(0)} r_k < r_1. \tag{1.5}$$

If positive numbers $r_1, \ldots, r_n$ satisfy inequalities (1.3), then

$$\frac{c_{i1}^{(0)}}{c_{11}^{(0)}} \sum_{k=2}^{n} c_{1k}^{(0)} r_k + \sum_{\substack{k=2\\k\neq i}}^{n} c_{ik}^{(0)} r_k < c_{ii}^{(0)} r_i, \qquad i = 2, \ldots, n. \tag{1.6}$$

((1.6) is obtained through replacing r_1 in (1.3) by a smaller value standing on the left-hand side of (1.5)). Hence we get

$$\sum_{\substack{k=2\\k\neq i}}^{n} (c_{11}^{(0)} c_{ik}^{(0)} + c_{i1}^{(0)} c_{1k}^{(0)}) r_k < (c_{11}^{(0)} c_{ii}^{(0)} - c_{i1}^{(0)} c_{1i}^{(0)}) r_i, \qquad i = 2, \ldots, n.$$

After taking into account (1.2), the last system of inequalities can be written in the form

$$\sum_{\substack{k=1\\k\neq i}}^{n-1} c_{ik}^{(1)} r_{k+1} < c_{ii}^{(1)} r_{i+1}, \qquad i = 1, \ldots, n-1. \tag{1.7}$$

Thus positive numbers $r_2, \ldots, r_n$ fulfil the system of inequalities (1.7). It follows from the definition of $c_{ik}^{(1)}$ for $i \neq k$ and from (1.7) that $c_{ii}^{(1)} > 0$, $i = 1, \ldots, n-1$, which means that all the numbers $c_{ik}^{(1)}$, $i, k = 1, \ldots, n-1$, are positive. Now from the induction hypothesis we obtain $c_{ii}^{(l)} > 0$, $i = 1, \ldots, n-l$; $l = 2, \ldots, n-1$. So we have proved that if the system of inequalities (1.3) has a positive solution, then inequalities (1.4) are fulfilled.

Conversely, if inequalities (1.4) hold, then again by the induction hypothesis, there exist positive numbers $r_2, \ldots, r_n$ satisfying (1.7). Putting

$$r_1 = \frac{1}{c_{11}^{(0)}} \sum_{k=2}^{n} c_{1k}^{(0)} r_k + \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small, one can easily verify that the numbers $r_1, \ldots, r_n$ satisfy system (1.3). This completes the proof.

Remark. Lemma 1.1 and Theorem 1.4 have been published in [17] without proofs.

THEOREM 1.4. *Let (X_i, ϱ_i), $i = 1, \ldots, n$, be complete metric spaces and let $T_i\colon X_1 \times \ldots \times X_n \to X_i$, $i = 1, \ldots, n$, be mappings. If there exist numbers a_{ik}, $i, k = 1, \ldots, n$, such that*

$$\varrho_i\big(T_i(x_1, \ldots, x_n),\ T_i(\bar{x}_1, \ldots, \bar{x}_n)\big) \leqslant \sum_{k=1}^{n} a_{ik}\, \varrho_k(x_k, \bar{x}_k), \tag{1.8}$$

$$x_k, \bar{x}_k \in X_k, \qquad i, k = 1, \ldots, n,$$

and the numbers

$$c_{ik}^{(0)} = \begin{cases} 1 - a_{ik}, & i = k, \\ a_{ik}, & i \neq k, \end{cases} \qquad i, k = 1, \ldots, n, \tag{1.9}$$

fulfil the conditions

$$c_{ii}^{(l)} > 0, \quad i = 1, \ldots, n-l;\ l = 0, \ldots, n-1, \tag{1.10}$$

where $c_{ik}^{(l)}$ are defined by (1.2), *then the system of equations*

$$x_i = T_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{1.11}$$

has exactly one solution $x_1, \ldots, x_n$ such that $x_i \in X_i$, $i = 1, \ldots, n$. For any arbitrarily fixed $\overset{0}{x}_i \in X_i$, $i = 1, \ldots, n$, the sequence of successive approximations

$$\overset{m+1}{x}_i = T_i(\overset{m}{x}_1, \ldots, \overset{m}{x}_n), \quad m = 0, 1, \ldots,;\ i = 1, \ldots, n \tag{1.12}$$

converges and

$$x_i = \lim_{m \to \infty} \overset{m}{x}_i, \quad i = 1, \ldots, n. \tag{1.13}$$

Moreover, there exist numbers $r_i = r_i(\overset{0}{x}_1, \ldots, \overset{0}{x}_n) > 0$ and an s, $0 < s < 1$, such that

$$\varrho_i(\overset{m+1}{x}_i, \overset{m}{x}_i) \leqslant s^m r_i, \quad i = 1, \ldots, n. \tag{1.14}$$

Proof. In view of (1.9) and (1.10), the numbers $c_{ik}^{(0)}$ are non-negative. According to Lemma 1.1, there exist positive numbers $r_1, \ldots, r_n$ satisfying the system of inequalities (1.3). It follows from (1.3) and (1.9) that $r_1, \ldots, r_n$ satisfy

$$\sum_{k=1}^{n} a_{ik} r_k < r_i, \quad i = 1, \ldots, n. \tag{1.15}$$

Since these inequalities are sharp, there is an s, $0 < s < 1$, such that

$$\sum_{k=1}^{n} a_{ik} r_k \leqslant s r_i, \qquad i = 1, \dots, n. \tag{1.16}$$

Let us note that if $r_1, \dots, r_n$ fulfil (1.15), then so do $tr_1, \dots, tr_n$ for every $t > 0$. Fix arbitrary $\overset{0}{x}_i \in X_i$, $i = 1, \dots, n$, and consider the sequence $\{\overset{m}{x}_i\}$ defined by (1.12). Increasing, if necessary, the numbers r_i by a constant functor, we may assume that

$$\varrho_i(\overset{1}{x}_i, \overset{0}{x}_i) \leqslant r_i, \qquad i = 1, \dots, n. \tag{1.17}$$

Now we shall prove estimation (1.14). It follows from (1.17) that (1.14) holds for $m = 0$. Suppose for induction that (1.14) is fulfilled for some $m \geqslant 0$. Then, from (1.12), (1.8), (1.14) and (1.16), we obtain

$$\varrho_i(\overset{m+2}{x}_i, \overset{m+1}{x}_i) = \varrho_i\big(T_i(\overset{m+1}{x}_1, \dots, \overset{m+1}{x}_n), T_i(\overset{m}{x}_1, \dots, \overset{m}{x}_n)\big)$$
$$\leqslant \sum_{k=1}^{n} a_{ik} \varrho_k(\overset{m+1}{x}_k, \overset{m}{x}_k) \leqslant s^m \sum_{k=1}^{n} a_{ik} r_k \leqslant s^{m+1} r_i, \qquad i = 1, \dots, n,$$

and induction completes the proof of (1.14).

Now (1.14) yields that, for each $1 \leqslant i \leqslant n$, $\{\overset{m}{x}_i\}$ is a Cauchy sequence, and in view of the completeness of (X_i, ϱ_i), there exist x_i defined by (1.13). Since all the transformations T_i are continuous, from (1.13) it follows that x_i, $i = 1, \dots, n$, fulfil the system of equations (1.11).

We shall prove that the solution just obtained is unique. Suppose that $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are solutions of system (1.11). Without loss of generality we can assume that

$$\varrho_i(x_i, y_i) \leqslant r_i, \qquad i = 1, \dots, n.$$

Now, from the equations

$$x_i = T_i(x_1, \dots, x_n), \qquad y_i = T_i(y_1, \dots, y_n), \qquad i = 1, \dots, n,$$

and from (1.8) and (1.16), we obtain by induction

$$\varrho_i(x_i, y_i) \leqslant s^m r_i, \qquad m = 0, 1, \dots; \; i = 1, \dots, n.$$

Hence $\varrho_i(x_i, y_i) = 0$, $i = 1, \dots, n$, which completes the proof of the theorem.

Remark. In the case $m = 2$ Theorem 1.4 was proved by Păvăloiu [19] and Rus [20] by using a different method.

LEMMA 1.2. *Let (a_{ik}), $i, k = 1, \dots, n$, be a non-negative matrix with characteristic roots $\lambda_1, \dots, \lambda_n$, and let $c_{ik}^{(0)}$ be defined by* (1.9). *Then*

(i) *the systems of inequalities* (1.15) *and* (1.3) *are equivalent*;

(ii) *conditions* (1.10) *are equivalent to the following*:

$$s = \max\{|\lambda_i|: i = 1, \ldots, n\} < 1. \tag{1.18}$$

Proof. (i) follows immediately from the definition of $c_{ik}^{(0)}$. Note ([8], p. 365) that if a matrix (a_{ik}) is non-negative and its characteristic roots have the absolute values less than 1, then there exists an $\varepsilon > 0$ such that the matrix $(a_{ik}+\varepsilon)$ has the same property. Suppose that a non-negative matrix (a_{ik}) satisfies conditions (1.10). By Lemma 1.1, the system of inequalities (1.15) has a positive solution $r_1, \ldots, r_n$. Since these inequalities are sharp, the numbers $r_1, \ldots, r_n$ satisfy

$$\sum_{k=1}^{n} (a_{ik}+\varepsilon) r_k < r_i, \qquad i = 1, \ldots, n,$$

where $\varepsilon > 0$ is sufficiently small. Hence, by Lemma 1.1, the matrix $(a_{ik}+\varepsilon)$ satisfies (1.10). Therefore we can assume that (a_{ik}) is positive.

To prove (ii), we first suppose that (1.18) holds. By a theorem of Perron and Frobenius ([8], pp. 354–355) the number s is one of the characteristic roots of the matrix (a_{ik}) and the corresponding eigenvector $(r_1, \ldots, r_n)$ has all coordinates positive. Thus we have

$$\sum_{k=1}^{n} a_{ik} r_k = s r_i, \qquad i = 1, \ldots, n, \; r_i > 0. \tag{1.19}$$

Since $0 < s < 1$, we see that the positive numbers $r_1, \ldots, r_n$ satisfy system (1.15). Now, from (i) and Lemma 1.1, it follows that (1.10) must be fulfilled.

Conversely, let conditions (1.10) be fulfilled and suppose that $s \geqslant 1$. By the theorem of Perron and Frobenius, there exist positive numbers $r_1, \ldots, r_n$ such that (1.19) holds, and, consequently, we have

$$\sum_{k=1}^{n} a_{ik} r_k \geqslant r_i, \qquad i = 1, \ldots, n. \tag{1.20}$$

Using (1.9) we can write this system of inequalities in the equivalent form

$$\sum_{\substack{k=1\\k\neq i}}^{n} c_{ik}^{(0)} r_k \geqslant c_{ii}^{(0)} r_i, \qquad i = 1, \ldots, n. \tag{1.21}$$

Since $c_{11}^{(0)} > 0$, the first of these inequalities can be written in the form

$$(c_{11}^{(0)})^{-1} \sum_{k=2}^{n} c_{ik}^{(0)} r_k \geqslant r_1. \tag{1.22}$$

Replacing in (1.21) for $i = 2, \ldots, n$ the number r_1 by the left-hand side of (1.22), we obtain (by a simple computation)

$$\sum_{\substack{k=2\\k\neq i}}^{n} c_{11}^{(0)} c_{ik}^{(0)} + c_{i1}^{(0)} c_{1k}^{(0)}) r_k \geqslant (c_{11}^{(0)} c_{ii}^{(0)} - c_{i1}^{(0)} c_{1i}^{(0)}) r_i, \qquad i = 2, \ldots, n.$$

Using (1.2) we can write this system of inequalities in the form

$$\sum_{\substack{k=1\\k\neq i}}^{n-1} c_{ik}^{(1)} r_{k+1} \geqslant c_{ii}^{(1)} r_{i+1}, \qquad i = 1, \ldots, n-1.$$

Taking into account (1.10), we obtain by induction

$$\sum_{\substack{k=1\\k\neq i}}^{n-1} c_{ik}^{(l)} r_{k+l} \geqslant c_{ii}^{(l)} r_{i+1}, \qquad i = 1, \ldots, n-l;\ l = 1, \ldots, n-1. \tag{1.23}$$

Putting in (1.23), $l = n-1$, we get $0 \geqslant c_{11}^{(n-1)} r_n$, and consequently, $c_{11}^{(n-1)} \leqslant 0$. This contradiction proves that $0 \leqslant s < 1$ and completes the proof of the lemma.

Now, by Lemma 1.2 and Theorem 1.4, we get

THEOREM 1.5. *Let (X_i, ϱ_i), $i = 1, \ldots, n$, be complete metric spaces and let mappings $T_i\colon X_1 \times \ldots \times X_n \to X_i$ satisfy* (1.8). *If the absolute values of the characteristic roots of (a_{ik}) are less than* 1, *then system* (1.11) *has exactly one solution $x_1, \ldots, x_n$ such that $x_i \in X_i$, $i = 1, \ldots, n$. Moreover, for every $\overset{0}{x}_i \in X_i$, $i = 1, \ldots, n$, the sequence of successive approximations* (1.12) *converges and* (1.13) *and* (1.14) *hold.*

We shall need the following lemma.

LEMMA 1.3. *Let (a_{ik}) be a non-negative square matrix and let the numbers* (1.9) *fulfil conditions* (1.10). *If $r_1, \ldots, r_n$ satisfy the system of inequalities* (1.20), *then $r_i \leqslant 0$, $i = 1, \ldots, n$.*

Proof. By Lemma 1.1, there exist $R_i > 0$, $i = 1, \ldots, n$, and an s, $0 < s < 1$, such that

$$\sum_{k=1}^{n} a_{ik} R_k \leqslant s R_i, \qquad i = 1, \ldots, n. \tag{1.24}$$

We may assume that

$$r_i \leqslant R_i, \qquad i = 1, \ldots, n. \tag{1.25}$$

Consequently, the inequalities

$$r_i \leqslant s^m R_i, \qquad i = 1, \ldots, n, \tag{1.26}$$

hold for $m = 0$. Suppose that (1.26) hold for an $m \geqslant 0$. Then by (1.20) and (1.24), we have

$$r_i \leqslant \sum_{k=1}^{n} a_{ik} r_k \leqslant s^m \sum_{k=1}^{n} a_{ik} R_k \leqslant s^{m+1} R_i.$$

By induction, (1.26) hold for every non-negative integer m. This completes the proof of the lemma.

2. Integrable solutions of a linear functional equation of order 1

The general linear functional equation of order n has the form([2])

$$g_0 \varphi + g_1 \varphi \circ f_1 + \ldots + g_n \varphi \circ f_n = h,$$

where g_k, f_k and h are given and φ is an unknown function (cf. Kuczma [12], p. 27). In this chapter we study the linear equation of order 1. Here and in the sequel all functions are real-valued.

1. In this section we consider the homogeneous linear functional equation

(2.1) $$\varphi = g \varphi \circ f$$

and the following functional inequality

(2.2) $$|\varphi| \leqslant |g| \, |\varphi \circ f| .$$

We assume that

(2.i) *f is strictly increasing in an interval $I = (0, a)$, $0 < a \leqslant \infty$; f and f^{-1} are absolutely continuous in I and $f(I)$, respectively; and*

(2.3) $$0 < f(x) < x, \quad x \in I.$$

By f^n we denote the n-th iterate of f:

$$f^0(x) = x, \quad x \in I; \quad f^{n+1} = f \circ f^n, \quad f^{-n-1} = f^{-1} \circ f^{-n}, \quad n = 0, 1, \ldots$$

Remark 2.1. It follows from (2.i) that:

1. For every $x \in I$, the sequence $\{f^n(x)\}$ is strictly decreasing and $\lim_{n \to \infty} f^n(x) = 0$;

2. For every interval $J = (0, b)$, $b \leqslant a$, we have $f(J) \subset J$;

3. For every positive integer n, f^n is absolutely continuous in I, and f^{-n} is absolutely continuous in $f^n(I)$;

([2]) Here and in the sequel the symbol $g\varphi \circ f$ denotes the function $g(x)\varphi[f(x)]$.

4. $f' \neq 0$ a.e.(³) in I.

Statements 1–3 are obvious (cf. also [12], p. 21). To prove 4, denote by $m(A)$ the Lebesgue measure of $A \subset R$. It follows from (2.i) (cf. [15], p. 174) that for every measurable set $A \subset I$ we have

$$m(f(A)) = \int_A f'.$$

Put $A = \{x \in I: f'(x) = 0\}$ and suppose that $m(A) > 0$. Then $m(f(A)) = 0$, and consequently, f^{-1} is not absolutely continuous. This contradiction proves 4.

Remark 2.2. In the sequel we assume the following convention

$$\sum_{i=k}^{k-1} a_i = 0, \qquad \prod_{i=k}^{k-1} a_i = 1, \qquad k = 0, 1, \ldots$$

Let $x_0 \in I$. We put

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \ldots$$

Now we quote the following three theorems of M. Kuczma [13].

THEOREM 2.1. *Let* (2.i) *be fulfilled, let g be a measurable function in I and let u be a positive decreasing function on I such that for a certain $x_0 \in I$ the series*

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} u(x_i) \tag{2.4}$$

diverges. Suppose that for a certain $b > 0$ and $p > 0$ we have

$$f' \geqslant |g|^p u \quad \text{a.e. in } (0, b). \tag{2.5}$$

If a $[\varphi] \in L^p(I)$ satisfies equation (2.1), *then $[\varphi] = [0]$.*

THEOREM 2.2. *Let conditions* (2.i) *be fulfilled, let g be a measurable function in I, and let u be a positive decreasing function on I such that for a certain $x_0 \in I$ series* (2.4) *converges. Suppose that for a certain $b > 0$ and $p > 0$ we have*

$$0 < f' \leqslant |g|^p u \quad \text{a.e. in } (0, b). \tag{2.6}$$

Then every measurable function φ_0 defined on $\langle x_1, x_0)$ and belonging to $L^p(\langle x_1, x_0))$ admits a unique extension to a function φ satisfying equation (2.1) *a.e. in I. This extension is in $L^p((0, x_0))$ whenever $x_0 \in (0, b)$.*

THEOREM 2.3. *Under conditions of Theorem* 2.2, *if, moreover,*

$$\operatorname*{inf\,ess}_{(0,d)} f' |g|^{-p} > 0 \quad \textit{for every } d \in I,$$

then the extension φ of φ_0 belongs to $L^p(0, d)$ for every $d \in I$.

(³) In Chapters 2–5 measure refers to the Lebesgue measure.

Remark. Some conditions for the convergence or divergence of series (2.4) have been given by Kuczma [14].

We shall generalize Theorems 2.1–2.3 as follows.

THEOREM 2.4. *Let conditions* (2.i) *be fulfilled, let g be measurable in I and let $p > 0$. Suppose that for a certain $x_0 \in I$ the series*

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_i(x_0), \tag{2.7}$$

where

$$\varkappa_i(x_0) = \operatorname*{inf\,ess}_{\langle x_{i+1}, x_i)} f' |g|^{-p}, \tag{2.8}$$

diverges. If $[\varphi] \in L^p(I)$ satisfies inequality (2.2), *then* $[\varphi] = [0]$.

Proof. Since the sequence $\{x_n\}$ is strictly decreasing (cf. Remark 2.1), we have

$$\int_0^{x_0} |\varphi|^p = \sum_{n=0}^{\infty} \int_{x_{n+1}}^{x_n} |\varphi|^p \tag{2.9}$$

for $\varphi \in L^p(0, x_0)$. Suppose that $[\varphi] \in L^p(0, x_0)$ is a solution of inequality (2.2). By (2.2) and (2.8), we have

$$\int_{x_{n+1}}^{x_n} |\varphi|^p = \int_{x_n}^{x_{n-1}} |\varphi \circ f|^p f' \geqslant \int_{x_n}^{x_{n-1}} f' |g|^{-p} |\varphi|^p \geqslant \varkappa_{n-1}(x_0) \int_{x_n}^{x_{n-1}} |\varphi|^p.$$

After n steps we obtain

$$\int_{x_{n+1}}^{x_n} |\varphi|^p \geqslant \Big(\prod_{i=0}^{n-1} \varkappa_i(x_0)\Big) \int_{x_1}^{x_0} |\varphi|^p, \qquad n = 1, 2, \ldots \tag{2.10}$$

In virtue of (2.9) and (2.10), we have

$$\infty > \int_0^{x_0} |\varphi|^p \geqslant \Big(\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_i(x_0)\Big) \int_{x_1}^{x_0} |\varphi|^p.$$

Now it follows from the divergence of series (2.7) that $\varphi = 0$ a.e. in (x_1, x_0). Since

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_i(x_0) = \sum_{n=0}^{k-1} \prod_{i=0}^{n-1} \varkappa_i(x_0) + \prod_{i=0}^{k} \varkappa_i(x_0) \Big(\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_i(x_k)\Big)$$

(cf. Kuczma [13]), the series

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_i(x_k)$$

diverges. Replacing in the above argument the interval $\langle x_1, x)$ by $\langle x_{k+1}, x_k)$, $k = 1, 2, \ldots$, we obtain $\varphi = 0$ a.e. in $\langle x_{k+1}, x_k)$, $k = 1, 2, \ldots$, and, consequently, $\varphi = 0$ a.e. in $(0, x_0)$. Now it follows from the form of inequality (2.2) that $\varphi = 0$ a.e. in (x_0, a). This completes the proof.

By a simple modification of the proof of Theorem 2 in [13] we get the following:

THEOREM 2.5. *Let conditions* (2.i) *be fulfilled and let g be measurable in I. If for a certain $x_0 \in I$ series* (2.7) *with*

$$\varkappa_i(x_0) = \operatorname*{sup\,ess}_{\langle x_{i+1}, x_i)} f' |g|^{-p} > 0, \qquad i = 0, 1, \ldots, \tag{2.11}$$

converges, then the solution $\varphi \in L^p(0, x_0)$ of equation (2.1) *depends on an arbitrary function, viz., for every function $\varphi_0 \in L^p\langle x_1, x_0)$ there exists exactly one function φ satisfying equation* (2.1) *in I such that $\varphi = \varphi_0$ in $\langle x_1, x_0)$. Moreover, $\varphi \in L^p(0, x_0)$.*

Theorem 2.3 will be true if we replace "Under conditions of Theorem 2.2" by "Under conditions of Theorem 2.5".

Remark 2.3. Suppose that $f' |g|^{-p} \geqslant 1$ a.e. in an interval $(0, b)$, $b \in I$. If $[\varphi] \in L^p(I)$ is a solution of inequality (2.2), then by Theorem 2.4, $[\varphi] = [0]$ in I. If there exist an s, $0 < s < 1$, and a $b \in I$ such that $0 < f' |g|^{-p} \leqslant s$ a.e. in $(0, b)$, then, by Theorem 2.2, the solution $\varphi \in L^p(0, x_0)$, $x_0 \leqslant b$, of equation (2.1) depends on an arbitrary function.

2. Now we confine ourselves to the equation

$$\varphi = \varphi \circ f, \tag{2.12}$$

which is a particular case of equation (2.1).

LEMMA 2.1. *If conditions* (2.i) *are fulfilled, then for every function $u: I \to R$, positive, decreasing and such that*

$$f' \geqslant u \quad \text{a.e. in } I,$$

series (2.4) *converges.*

Proof. Suppose that series (2.4) diverges. Then it follows from Theorem 2.1 that $[\varphi] = [0]$ is the unique solution of equation (2.12) belonging to $L^p(I)$. But every function $\varphi = \text{const} \neq 0$ satisfies equation (2.12) and $[\varphi] \in L^p(0, a)$, $a < \infty$. This contradiction completes the proof of the lemma.

Setting $u = f'$ in Lemma 2.1, we obtain the following:

COROLLARY 2.1. *If f fulfils conditions* (2.i) *and f is concave in I, then the series $\sum_{n=1}^{\infty} (f^n)'$ converges a.e. in I.*

THEOREM 2.6. *If conditions* (2.i) *are fulfilled and f is convex or concave in I, then for every $d \in I$ equation* (2.12) *has a solution $\varphi \in L^p(0, d)$ depending on an arbitrary function.*

Proof. Suppose that f is convex. By (2.i), there exist an s, $0 < s < 1$, and an $\varepsilon > 0$ such that $0 \leqslant f' \leqslant s$ a.e. in $(0, \varepsilon)$. Hence (cf. Remark 2.1), $0 < f' \leqslant s$ a.e. in $(0, \varepsilon)$ and the theorem follows from Remark 2.3 and Theorem 2.3.

If f is concave, then by (2.i), we have $\operatorname{inf\,ess}_{(0,d)} f' > 0$ for every $d \in I$. Now the theorem follows from Corollary 2.1 and Theorem 2.3.

3. In this section we are going to study the non-homogeneous linear functional equation

$$\varphi = g\varphi\circ f + h. \tag{2.13}$$

We assume

(2.ii) *g and h are measurable in I.*

We shall prove a few theorems on the existence and uniqueness of solutions $[\varphi] \in L^p(0, x_0)$, $x_0 \in I$, of equation (2.13).

THEOREM 2.7. *Let conditions* (2.i) *and* (2.ii) *be fulfilled. If there exist an s, $0 < s < 1$, and an $x_0 \in I$ such that $h \in L^p(0, x_0)$ and*

$$|g|^p \leqslant sf', \quad \text{a.e. in } (0, x_0), \tag{2.14}$$

then equation (2.13) *has exactly one solution $[\varphi] \in L^p(0, x_0)$. This solution is given by the series*

$$\varphi = \sum_{k=0}^{\infty}\Big(\prod_{i=0}^{k-1} g\circ f^i\Big) h\circ f^k, \tag{2.15}$$

which converges a.e. in $(0, x_0)$.

Proof. The part on uniqueness follows immediately from Theorem 2.1. By (2.14), we have for $k = 0, 1, \ldots$

$$\Big[\int_0^{x_0}\Big(\prod_{i=0}^{k-1}|g\circ f^i|^p\Big)|h\circ f^k|^p\Big]^{\alpha(p)} \leqslant s^{\alpha(p)k}\Big[\int_0^{x_0}\Big(\prod_{i=0}^{k-1} f'\circ f^i\Big)|h\circ f^k|^p\Big]^{\alpha(p)}$$
$$= s^{\alpha(p)k}\Big[\int_0^{x_0}(f^k)'\,|h\circ f^k|^p\Big]^{\alpha(p)} = s^{\alpha(p)k}\Big[\int_0^{x_k}|h|^p\Big]^{\alpha(p)} \leqslant (s^{\alpha(p)})^k\Big[\int_0^{x_0}|h|^p\Big]^{\alpha(p)}.$$

By Lemma 0.1, series (2.15) converges a.e. in $(0, x_0)$ and its sum belongs to $L^p(0, x_0)$. Now the theorem results from the following:

LEMMA 2.2. *Let conditions* (2.i) *and* (2.ii) *be fulfilled. If series* (2.15) *converges a.e. in $(0, x_0)$, $x_0 \leqslant a$, then its sum φ satisfies equation* (2.13) *a.e. in $(0, x_0)$.*

Proof. Let A denote the set of all $x \in (0, x_0)$ such that series (2.15) diverges. Therefore A has measure zero. It follows from the absolute continuity of f and f^{-1} that the set $\bigcup\limits_{-\infty}^{+\infty} f^n(A)$ has measure zero. Put

$$B = (0, x_0)\setminus\bigcup_{-\infty}^{+\infty} f^n(A).$$

Since f is strictly increasing, $f\big((0, x_0)\big) \subset (0, x_0)$ and the set $\bigcup\limits_{-\infty}^{+\infty} f^n(A)$ is invariant under f, we have $f(B) \subset B$, and consequently, if $x \in B$, then series (2.15) converges for x and $f(x)$. Therefore for $x \in B$ we have

$$\begin{aligned}\varphi(x) &= \sum_{k=0}^{\infty}\prod_{i=0}^{k-1} g[f^i(x)]h[f^k(x)] = \sum_{k=1}^{\infty}\prod_{i=0}^{k-1} g[f^i(x)]h[f^k(x)] + h(x)\\ &= \sum_{k=0}^{\infty}\prod_{i=0}^{k} g[f^i(x)]h[f^{k+1}(x)] + h(x)\\ &= g(x)\sum_{k=0}^{\infty}\prod_{i=1}^{k} g[f^i(x)]h[f^{k+1}(x)] + h(x)\\ &= g(x)\sum_{k=0}^{\infty}\prod_{i=1}^{k-1} g[f^{i+1}(x)]h[f^{k+1}(x)] + h(x) = g(x)\varphi[f(x)] + h(x).\end{aligned}$$

This completes the proof of the lemma.

Remark. One can easily observe that if series (2.15) converges at the point x such that $g(x) \neq 0$, then it must converge also at the point $f(x)$.

THEOREM 2.8. *Let conditions* (2.i) *and* (2.ii) *be fulfilled. If there exist an s, $0 < s < 1$, and an $x_0 \in I$ such that $h \in L^p(0, x_0)$ and*

$$\frac{g(x)h[f(x)]}{h(x)} \leqslant s \quad \textit{a.e. in } (0, x_0), \tag{2.16}$$

then equation (2.13) *has at least one solution* $[\varphi] \in L^p(0, x_0)$. *This solution is given by formula* (2.15). *Moreover,*

a. *if series* (2.7) *defined by* (2.8) *diverges, then φ defined by* (2.15) *establishes the unique sulution of equation* (2.13) *in* $L^p(0, x_0)$;

b. *if series* (2.7) *defined by* (2.11) *converges, then the solution $\varphi \in L^p(0, x_0)$ of equation* (2.13) *depends an on arbitrary function.*

Proof. Since

$$\prod_{i=0}^{k-1} g[f^i(x)]h[f^k(x)] = \left(\prod_{i=0}^{k-1}\frac{g[f^i(x)]h[f^{i+1}(x)]}{h[f^i(x)]}\right)h(x),$$

we have, from (2.16),

$$\Big(\prod_{i=0}^{k-1} |g[f^i(x)]|\Big)\, |h[f^k(x)]| \leqslant s^k |h(x)| \quad \text{a.e. in } (0, x_0), \quad k = 0, 1, \ldots$$

Now using Lemma 0.1, we can easily verify that series (2.15) converges a.e. in $(0, x_0)$ and its sum φ belongs to $L^p(0, x_0)$. By Lemma 2.2, φ satisfies equation (2.13) a.e. in $(0, x_0)$. In order to prove a and b it suffices to apply Theorems 2.4 and 2.5, respectively.

THEOREM 2.9. *Let conditions* (2.i) and (2.ii) *be fulfilled. If there exists an* $x_0 \epsilon I$ *such that* $h \epsilon L^p(0, x_0)$ *and*

$$\sum_{k=0}^{\infty} \Big[\Big(\prod_{i=0}^{k-1} \eta_i\Big) \int_0^{x_k} |h|^p\Big]^{\alpha(p)} < \infty, \tag{2.17}$$

where

$$\eta_i = \operatorname*{supess}_{(0, x_i)} (f')^{-1} |g|^p, \quad i = 0, 1, \ldots, \tag{2.18}$$

then equation (2.13) *has at least one solution* $[\varphi] \epsilon L^p(0, x_0)$. *This solution is given by formula* (2.15). *Moreover,*

a. *if there exists an* $M > 0$ *such that*

$$\prod_{i=0}^{k-1} \eta_i \leqslant M, \quad k = 1, 2, \ldots, \tag{2.19}$$

or series (2.7) *defined by* (2.8) *diverges, then solution* (2.15) *is unique in* $L^p(0, x_0)$;

b. *if series* (2.7) *defined by* (2.11) *converges, then the solution* $\varphi \epsilon L^p(0, x_0)$ *of equation* (2.13) *depends on an arbitrary function.*

Proof. From (2.18) we have

$$\Big(\int_0^{x_0} \prod_{i=0}^{k-1} |g\circ f^i|^p\, |h\circ f^k|^p\Big)^{\alpha(p)} \leqslant \Big(\prod_{i=0}^{k-1} \eta_i \int_0^{x_0} |h\circ f^k|^p (f^k)'\Big)^{\alpha(p)}$$
$$= \Big(\prod_{i=0}^{k-1} \eta_i \int_0^{x_k} |h|^p\Big)^{\alpha(p)}.$$

Now, from the convergence of series (2.17) and from Lemma 0.1, follows the first part of the theorem.

Let us note that if φ is a solution of equation (2.13), then

$$\varphi = \Big(\prod_{i=0}^{n} g\circ f^i\Big)\, \varphi\circ f^{n+1} + \sum_{k=0}^{n} \Big(\prod_{i=0}^{k-1} g\circ f^i\Big)\, h\circ f^k, \quad n = 0, 1, \ldots \tag{2.20}$$

If $\varphi \in L^p(0, x_0)$, then, by (2.18) and (2.19), we have

$$\int_0^{x_0} \Big|\prod_{i=0}^{n} g\circ f^i\Big|^p |\varphi\circ f^{n+1}|^p \leqslant \Big(\prod_{i=0}^{n} \eta_i\Big)\int_0^{x_0} |\varphi\circ f^{n+1}|^p \prod_{i=0}^{n} f'\circ f^i$$
$$\leqslant M\int_0^{x_0} |\varphi\circ f^{n+1}|^p (f^{n+1})' = M\int_0^{x_{n+1}} |\varphi|^p.$$

Since $\lim_{n\to\infty} x_n = 0$, it follows that the first component of (2.20) tends to zero in measure. The remaining component of (2.20) is a partial sum of series (2.15) which, as we have proved, tends to φ given by (2.15) a.e. in $(0, x_0)$. Thus, when $n\to\infty$, the right-hand side of (2.20) tends to the sum of series (2.15) in measure, and, consequently, φ is uniquely determined. The remaining part of a and b follows from Theorems 2.4 and 2.5, respectively.

COROLLARY 2.2. *Let* (2.i), (2.ii) *and* (2.19) *be fulfilled. If* $\varphi \in L^p(0, x_0)$ *is a solution of equation* (2.13), *then it must be of the form* (2.15).

THEOREM 2.10. *Let* (2.i) *and* (2.ii) *be fulfilled. If there exist a number* $s > 1$ *and an* $x_0 \in I$ *such that*

$$(2.21) \qquad |g|^p \geqslant sf' \quad \text{a.e. in } (0, x_0)$$

and the series

$$(2.22) \qquad \sum_{k=1}^{\infty}\Big(\int_{x_{k+1}}^{x_k} \Big|\frac{h\circ f^{-1}}{g\circ f^{-1}}\Big|^p\Big)^{\alpha(p)}$$

converges, then equation (2.13) *has a solution* $\varphi \in L^p(0, x_0)$ *depending on an arbitrary function.*

This theorem can be deduced from Theorem 3.8.

4. Now we shall apply the results of the preceding section to a study of absolutely continuous solutions of the equation

$$(2.23) \qquad \Phi = s\Phi\circ f + H.$$

LEMMA 2.3. *Let* (2.i) *be fulfilled and let* $f(0) = 0$. *Suppose that* H: $\langle 0, a) \to R$. *Then*

a. *if* $-1 \leqslant s \leqslant 1$, *equation* (2.23) *has at most one absolutely continuous solution in* $\langle 0, a)$;

b. *if* $s = 1$, *equation* (2.23) *has at most a one-parameter family of absolutely continuous solutions in* $\langle 0, a)$.

Proof. If Φ_i: $\langle 0, a) \to R$, $i = 1, 2$, are abslutely continuous solutions of equation (2.23), then $\Phi = \Phi_1 - \Phi_2$ satisfies the equation

$$(2.24) \qquad \Phi = s\Phi\circ f,$$

and, consequently, $\varphi = \Phi'$ satisfies the equation

$$\varphi = sf'\Phi\circ f$$

and $\varphi \in L^1(0, x_0)$ for every $x_0 \in I$. Put $g = sf'$ and $p = 1$. It follows from Theorem 2.1 that for $|s| \leqslant 1$, $\varphi = \Phi' = 0$ a.e. in I. By the absolute continuity of Φ, there exists a constant c such that $\Phi = c$ in $\langle 0, a)$. Inserting $\Phi = c$ in (2.23), we obtain $\Phi = 0$ for $-1 \leqslant s < 1$. Consequently, $\Phi_1 = \Phi_2$ for $-1 \leqslant s < 1$, and $\Phi_1 - \Phi_2 = c$ for $s = 1$. This completes the proof of the lemma.

THEOREM 2.11. *Let* (2.i) *be fulfilled and let* $f(0) = 0$. *Suppose that* $H\colon \langle 0, a) \to R$ *is absolutely continuous.*

a. *If* $|s| < 1$, *then equation* (2.23) *has exactly one absolutely continuous solution* $\Phi\colon \langle 0, a) \to R$. *This solution has the form*

$$\Phi = \sum_{k=0}^{\infty} s^k H\circ f^k.$$

b. *If* $s = 1$, $H(0) = 0$ *and the series*

$$\sum_{k=0}^{\infty} \operatorname{Var} H \,|\, \langle 0, f^k(x)\rangle \tag{2.25}$$

converges for a certain $x_0 \in (0, a)$, *then* (2.25) *converges for every* $x \in (0, a)$ *and equation* (2.23) *has a unique one-parameter family of absolutely continuous solutions in* $(0, a)$. *These solutions are given by the formula*

$$\Phi = c + \sum_{k=0}^{\infty} H\circ f^k. \tag{2.26}$$

c. *If* $s = -1$ *and series* (2.25) *converges for a certain* $x_0 \in (0, a)$, *then equation* (2.23) *has exactly one absolutely continuous solution* $\Phi\colon \langle 0, a) \to R$. *This solution is given by the formula*

$$\Phi = \sum_{k=0}^{\infty} (-1)^k H\circ f^k.$$

d. *Let* $x_0 \in (0, a)$ *and let* $\Phi_0\colon \langle x_1, x_0\rangle \to R$ *be absolutely continuous and such that*

$$\Phi_0(x_0) = s\Phi_0[f(x_0)] + H(x_0) \quad (s \neq 0). \tag{2.27}$$

Then there exists exactly one function $\Phi\colon (0, a) \to R$ *satisfying equation* (2.23) *in* $(0, a)$ *and such that* $\Phi = \Phi_0$ *in* $\langle x_1, x_0\rangle$. *If, moreover,* $|s| > 1$ *and* $H'\circ f^{-1} \in L^1\big(f(0, a)\big)$, *then* Φ *is absolutely continuous in* $\langle 0, a)$, *where* $\Phi(0)$ *is assumed to be* $H(0)\,(1-s)^{-1}$ (*in other words, the absolutely continuous solution of equation* (2.23) *in* $\langle 0, a)$ *depends on an arbitrary function*).

Proof. Put $h = H'$ and consider the equation

$$\varphi = sf'\varphi\circ f + h. \tag{2.28}$$

a. Put $g = sf'$ and $p = 1$. Since $h \in L^1(0, x_0)$ for every $x_0 \in (0, a)$, in view of Theorem 2.7 there exists exactly one solution $[\varphi] \in L^1(0, x_0)$ of equation (2.28). Moreover, by (2.15), we have

$$\varphi = \sum_{k=0}^{\infty} s^k \Big(\prod_{i=0}^{k-1} f'\circ f^i\Big) h\circ f^k = \sum_{k=0}^{\infty} s^k (f^k)' h\circ f^k.$$

It is easy to verify that the function

$$\begin{aligned} \Phi(x) &= \int_0^x \varphi(t)\,dt + H(0)(1-s)^{-1} \\ &= \sum_{k=0}^{\infty} s^k \int_0^x [f^k(t)]' h[f^k(t)]\,dt + H(0)(1-s)^{-1} \\ &= \sum_{k=0}^{\infty} s^k \int_0^{f^k(x)} H'(t)\,dt + H(0)(1-s)^{-1} \\ &= \sum_{k=0}^{\infty} \big(H[f^k(x)] - H(0)\big) s^k + H(0)(1-s)^{-1} = \sum_{k=0}^{\infty} s^k H[f^k(x)] \end{aligned}$$

(the termwise integration is justified by Theorem 6.7, [22], p. 277), is an absolutely continuous solution of equation (2.23). The uniqueness of the solution Φ follows from Lemma 2.3.

b. Let us take an $x_0 \in I$ such that series (2.25) converges for $x = x_0$ and apply Theorem 2.9 with $p = 1$. We have $\eta_i = 1$, $i = 0, 1, \ldots$ Then, by the well-known property of absolutely continuous functions (cf. [21], p. 404, Theorem 5.6), series (2.17) takes the form

$$\sum_{k=0}^{\infty} \int_0^{x_k} |h| = \sum_{k=0}^{\infty} \int_0^{x_k} |H'| = \sum_{k=0}^{\infty} \operatorname{Var} H \,|\, \langle 0, f^k(x_0)\rangle$$

and, consequently, it is convergent. According to Theorem 2.9, there exists a solution φ of equation (2.28) such that $[\varphi] \in L^1(0, x_0)$ for every $x_0 \in (0, a)$ and, moreover,

$$\varphi = \sum_{k=0}^{\infty} \Big(\prod_{i=0}^{k-1} f'\circ f^i\Big) h\circ f^k = \sum_{k=0}^{\infty} H'\circ f^k (f^k)'.$$

The functions

$$\Phi(x) = c + \int_0^x \varphi = c + \sum_{k=0}^{\infty} H\circ f^k$$

(where c is a parameter) are absolutely continuous in $\langle 0, a)$ and satisfy equation (2.23).

Uniqueness is ensured by Lemma 2.3.

c. The proof is similar to that of b.

d. Let us put $\varphi_0 = \Phi_0'$ in $\langle x_1, x_0)$. Since $|s| > 1$ and the integrability of $H' \circ f^{-1}$ imply all the conditions of Theorem 2.10, there exists exactly one solution $\varphi \in L^1(0, x_0)$ such that $\varphi = \varphi_0$ in $\langle x_1, x_0)$. Now we can uniquely extend φ to a solution onto the whole interval $(0, a)$, in the obvious manner. Denote this extension again by φ. Integrating both sides of equation (2.28) in turn over the intervals $\big(x_0, f^{-1}(x_0)\big), \ldots, \big(f^{-n}(x_0), f^{-n-1}(x_0)\big), \ldots,$ we see that $\varphi \in L^1(0, b)$ for every $b \in (0, a)$. Consequently, the function

$$\Phi(x) = \int_{x_0}^{x} \varphi + \Phi_0(x_0), \qquad x \in \langle 0, a),$$

is absolutely continuous in $\langle 0, a)$ and $\Phi = \Phi_0$ in $\langle x_1, x_0)$. Moreover, by (2.28) and (2.27) for $x \in \langle 0, a)$, we have

$$\begin{aligned}
\Phi(x) &= \int_{x_0}^{x} \varphi + \Phi_0(x_0) = \int_{x_0}^{x} (sf'\varphi\circ f + H') + \Phi_0(x_0) \\
&= s \int_{f(x_0)}^{f(x)} \varphi + H(x) - H(x_0) + \Phi_0(x_0) \\
&= s\Big(\int_{f(x_0)}^{x_0} \varphi + \int_{x_0}^{f(x)} \varphi\Big) + H(x) - H(x_0) + \Phi_0(x_0) \\
&= s \int_{x_0}^{f(x)} \varphi + H(x) + \Big(\Phi_0(x_0) - s\int_{x_0}^{f(x_0)} \varphi - H(x_0)\Big) \\
&= s\Phi[f(x)] + H(x) + \big(\Phi_0(x_0) - s\Phi_0[f(x_0)] - H(x_0)\big) \\
&= s\Phi[f(x)] + H(x).
\end{aligned}$$

Remark 2.6. If $H' \in L^1(I)$ and there is a $K > 0$ such that $f' \leqslant K$ a.e. in I, then $H' \circ f^{-1} \in L^1\big(f(I)\big)$. Indeed, we have

$$\int_{f(I)} |H' \circ f^{-1}| \leqslant \int_{f(I)} |H' \circ f^{-1}|(f^{-1})' K = K \int_{I} |H'| < \infty.$$

5. In this section we give an application of Theorem 2.11 to the Goursat problem for a hyperbolic partial differential equation.

We assume the following conditions:

1. $D = \{(x, y)\colon 0 \leqslant x \leqslant a_1,\ 0 \leqslant y \leqslant a_2\}$, $a_1 > 0, a_2 > 0$, and $G\colon D \to R$ is an integrable function with respect to the two-dimensional Lebesgue measure;

2. f_1: $\langle 0, a_1\rangle \to \langle 0, a_2\rangle$, f_2: $\langle 0, a_2\rangle \to \langle 0, a_1\rangle$ are strictly increasing and absolutely continuous functions and f_1^{-1}, f_2^{-1} are absolutely continuous in $f_1(\langle 0, a_1\rangle)$ and $f_2(\langle 0, a_1\rangle)$, respectively;

3. $f_1(0) = f_2(0)$ and the curves $y = f_1(x)$, $x \in \langle 0, a_1\rangle$; $x = f_2(y)$, $y \in \langle 0, a_2\rangle$ have no point in common except the origin;

4. P: $\langle 0, a_1\rangle \to R$, Q: $\langle 0, a_2\rangle \to R$ are absolutely continuous in $\langle 0, a_1\rangle$ and $\langle 0, a_2\rangle$, respectively, and such that $P(0) = Q(0)$.

The Goursat problem: Find a function u: $D \to R$ possessing Lebesgue integrable derivatives u_{xy} and u_{yx}, $u_{xy} = u_{yx}$ a.e. in D, satisfying the equation

$$u_{xy} = G \quad \text{a.e. in } D, \tag{2.29}$$

and the conditions

$$u\big(x, f_1(x)\big) = P(x) \quad \text{for } x \in \langle 0, a_1\rangle, \tag{2.30}$$

$$u\big(f_2(y), y\big) = Q(y) \quad \text{for } y \in \langle 0, a_2\rangle. \tag{2.31}$$

Integrating both sides of equation (2.29), we obtain

$$u(x, y) = U(x, y) + \Phi(x) + \Psi(y), \tag{2.32}$$

where

$$U(x, y) = \int_0^x \int_0^y G(s, t)\, ds\, dt, \tag{2.33}$$

and Φ and Ψ are absolutely continuous functions to be determined. Put

$$V(x) = P(x) - U\big(x, f_1(x)\big), \qquad W(y) = Q(y) - U\big(f_2(y), y\big). \tag{2.34}$$

We shall prove the following:

LEMMA 2.4. *Let 1 be fulfilled and suppose that f_1: $\langle 0, a_1\rangle \to \langle 0, a_2\rangle$ (resp. f_2: $\langle 0, a_2\rangle \to \langle 0, a_1\rangle$) is absolutely continuous. If there exists an $M > 0$ such that*

$$\int_0^{a_2} |G(x, y)|\, dy \leqslant M \quad \textit{a.e. in } \langle 0, a_1\rangle \tag{2.35}$$

and

$$\int_0^{a_1} |G(x, y)|\, dx \leqslant M \quad \textit{a.e. in } \langle 0, a_2\rangle, \tag{2.36}$$

then the function $U\big(x, f_1(x)\big)$ (resp. $U\big(f_2(y), y\big)$) is absolutely continuous in $\langle 0, a_1\rangle$ (resp. $\langle 0, a_2\rangle$).

Proof. By (2.35) and (2.36), we have

$$|U(x,f_1(x)) - U(\bar{x}, f_1(\bar{x}))| = \left|\int_0^x ds \int_0^{f_1(x)} G(s,t)\,dt - \int_0^{\bar{x}} ds \int_0^{f_1(\bar{x})} G(s,t)\,dt\right|$$

$$= \left|\int_{\bar{x}}^{x}\left(\int_0^{f_1(x)} G(s,t)\,dt\right) ds - \int_0^{\bar{x}}\left(\int_{f_1(\bar{x})}^{f_1(x)} G(s,t)\,ds\right) dt\right|$$

$$\leqslant \left|\int_{\bar{x}}^{x}\left(\int_0^{a_2} |G(s,t)|\,dt\right) ds\right| + \left|\int_{f_1(\bar{x})}^{f_1(x)}\left(\int_0^{a_1} |G(s,t)|\,ds\right) dt\right|$$

$$\leqslant M|x-\bar{x}| + M|f_1(x) - f_1(\bar{x})|.$$

This inequality easily implies the absolute continuity of $U(x, f_1(x))$. (For $U(f_2(y), y)$ the proof is analogous.)

Suppose that inequalities (2.35) and (2.36) are fulfilled. Then, by 2, 4 and Lemma 2.4, the functions V and W defined by (2.34) are absolutely continuous in $\langle 0, a_1\rangle$ and $\langle 0, a_2\rangle$, respectively.

Now (2.32), (2.30) and (2.31) lead to the following system of equations for Φ and Ψ:

$$\Phi(x) + \Psi[f_1(x)] = V(x), \quad \Phi[f_2(y)] + \Psi(y) = W(y), \tag{2.37}$$

whence the elimination of Ψ yields the equation

$$\Phi(x) = \Phi\{f_2[f_1(x)]\} + V(x) - W[f_1(x)], \quad x \in \langle 0, a_1\rangle. \tag{2.38}$$

Putting in (2.38)

$$f = f_2 \circ f_1, \quad H = V - W \circ f_1, \tag{2.39}$$

we obtain the equation

$$\Phi = \Phi \circ f + H, \tag{2.40}$$

i.e. equation (2.23) with $s = 1$. By 3, $0 < f(x) < x$ for $x \in (0, a_1\rangle$. This together with 2 shows that f satisfies conditions (2.i). Moreover, it follows from (2.34), (2.33), 4 and from the definition of H that $H(0) = 0$. In view of Lemma 2.3, equation (2.40) has at most a one-parameter family of absolutely continuous solutions. They must have form (2.26). By (2.30)–(2.33), we have

$$\Phi(0) + \Psi(0) = u(0, 0) = P(0).$$

Hence, the constant c disappears in (2.32) and we may assume that $c = 0$. Taking into account Theorem 2.11 b, we get the following:

THEOREM 2.12. *If conditions* 1–4 *are fulfilled, then the Goursat problem* (2.29)–(2.31) *has at most one solution. If, besides* 1–4, *inequalities* (2.35) *and* (2.36) *are fulfilled, and series* (2.25), *where f and H are defined by* (2.39), *converges for a certain* $x_0 \in (0, a_1\rangle$, *then there exists exactly one solution* $u\colon D \to R$ *of the Goursat problem* (2.29)–(2.31). *This solution has the form*

$$u(x, y) = U(x, y) + W(y) + \sum_{k=0}^{\infty} \{H[f^k(x)] - H[f^k(f_2(y))]\}.$$

Remark 2.7. Bielecki and Kisyński [2], making use of equation (2.40), considered the Goursat problem (2.29)–(2.31) in the class of functions $C^1(D)$ with continuous derivatives u_{xy} and u_{yx} (cf. also [12], p. 102).

Similar functional equations in connection with a more general Goursat problem occur also in Deimling [7].

3. Integrable solutions of a non-linear functional equation of order 1

The general functional equation of order 1 has the form $(Fx, \varphi(x), \varphi[f(x)]) = 0$, where F and f are given and φ is unknown. We confine ourselves to the less general equations, namely, to the equation $\varphi(x) = h(x, \varphi[f(x)])$, when we are interested in the uniqueness of solutions, or $\varphi[f(x)] = g(x, \varphi(x))$, when the problem of the dependence of solutions on an arbitrary function is considered.

1. In this section we formulate the general assumptions on given functions for the equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{3.1}$$

and we prove a uniqueness theorem.

We assume:

(3.i) *f is strictly increasing in an interval* $I = (0, a)$, $0 < a \leqslant \infty$; *f and* f^{-1} *are absolutely continuous in I and f(I), respectively; and*

$$0 < f(x) < x, \qquad x \in I.$$

(3.ii) $h\colon I \times R \to R$ *fulfils the following conditions: for every* $y \in R$, $h(\cdot, y)$ *is measurable in I; for almost every x in I,* $h(x, \cdot)\colon R \to R$ *is continuous.*

Remark 3.1. Carathéodory [5] (cf. also Шрагин [23]) proved that if conditions (3.ii) are fulfilled, then for every measurable function $\varphi\colon I \to R$, the function $h(x, \varphi(x))$ is measurable in I.

(3.iii) *There exist an* $x_0 \in I$ *and a non-negative function* $\eta\colon (0, x_0) \to R$ *such that*

$$|h(x, y) - h(x, \bar{y})| \leqslant \eta(x)|y - \bar{y}| \quad \text{a.e. in } (0, x_0), \qquad y, \bar{y} \in R. \tag{3.2}$$

For the point x_0 we define the sequence $\{x_k\}$:

$$x_{k+1} = f(x_k), \quad k = 0, 1, \ldots$$

THEOREM 3.1. *Let conditions* (3.i)–(3.iii) *be fulfilled. Put*

$$\varkappa_i(x_0) = \inf_{\langle x_{i+1}, x_i)} \operatorname{ess} f' \eta^{-p}, \quad i = 0, 1, \ldots \tag{3.3}$$

If series (2.7) *diverges, then equation* (3.1) *has at most one solution* $[\varphi] \in L^p(I)$.

Proof. Suppose that $[\varphi_1]$, $[\varphi_2] \in L^p(I)$ satisfy equation (3.1). It follows from (3.2) that

$$|\varphi_1(x) - \varphi_2(x)| = \big|h\big(x, \varphi_1[f(x)]\big) - h\big(x, \varphi_2[f(x)]\big)\big|$$
$$\leqslant \eta(x)|\varphi_1[f(x)] - \varphi_2[f(x)]| \quad \text{a.e. in } (0, x_0).$$

Then $\varphi = \varphi_1 - \varphi_2$ satisfies the inequality

$$|\varphi(x)| \leqslant \eta(x)|\varphi[f(x)]|.$$

By Theorem 2.4, $\varphi = 0$ a.e. in $(0, a)$. This completes the proof.

COROLLARY 3.1. *Let* (3.i)–(3.iii) *be fulfilled. If there is an* $\varepsilon > 0$ *such that*

$$\eta^p \leqslant f' \quad \textit{a.e. in } (0, \varepsilon), \tag{3.4}$$

then equation (3.1) *has at most one solution* $[\varphi] \in L^p(I)$.

2. Now we shall prove a few theorems on the existence and uniqueness of integrable solutions of equation (3.1).

THEOREM 3.2. *Let* (3.i)–(3.iii) *be fulfilled. If* $h(\cdot, 0) \in L^p(0, x_0)$ *and there exists a number* s, $0 < s < 1$, *such that*

$$\eta^p \leqslant s f' \quad \textit{a.e. in } (0, x_0), \tag{3.5}$$

then equation (3.1) *has exactly one solution* $[\varphi] \in L^p(0, x_0)$.

Moreover, for every fixed $\varphi_0 \in L^p(0, x_0)$, *the sequence of successive approximations*

$$\varphi_{n+1}(x) = h\big(x, \varphi_n[f(x)]\big), \quad n = 0, 1, \ldots \tag{3.6}$$

converges a.e. in $(0, x_0)$ *and*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \quad \textit{a.e. in } (0, x_0). \tag{3.7}$$

Proof. $L^p(0, x_0)$ with the metric

$$\varrho([\varphi_1], [\varphi_2]) = \Big(\int_0^{x_0} |\varphi_1(x) - \varphi_2(x)|^p dx\Big)^{a(p)}$$

is a complete metric space. The mapping T, defined as

$$T([\varphi]) = \big[h\big(x, \varphi[f(x)]\big)\big], \tag{3.8}$$

maps $L^p(0, x_0)$ into itself. Indeed, take a $[\varphi] \in L^p(0, x_0)$. By (3.i), $\varphi \circ f$ is measurable and, consequently, $h\big(x, \varphi[f(x)]\big)$ is measurable (cf. Remark 3.1). From (3.2), (3.5) and from the inequality

$$(a+b)^p \leqslant \big(2\max(a, b)\big)^p \leqslant 2^p(a^p+b^p), \qquad a \geqslant 0, b \geqslant 0, p > 0,$$

we have

$$\begin{aligned}\big|h\big(x, \varphi[f(x)]\big)\big|^p &\leqslant \Big(\big|h\big(x, \varphi[f(x)]\big) - h(x, 0)\big| + |h(x, 0)|\Big)^p \\ &\leqslant \big(\eta(x)\,|\varphi[f(x)]| + |h(x, 0)|\big)^p \leqslant 2^p\big(\eta(x)^p\,|\varphi[f(x)]|^p + |h(x, 0)|^p\big) \\ &\leqslant 2^p\big(s\,|\varphi[f(x)]|^p f'(x) + |h(x, 0)|^p\big).\end{aligned}$$

Hence, by (3.8), we get

$$\int_0^{x_0} |T([\varphi])|^p \leqslant 2^p\Big(s\int_0^{x_1} |\varphi(x)|^p dx + \int_0^{x_0} |h(x, 0)|^p dx\Big),$$

which proves that T maps $L^p(0, x_0)$ into itself.

Now, in view of (3.2), (3.5) and (3.i), we have for $\varphi_1, \varphi_2 \in L^p(0, x_0)$

$$\begin{aligned}\varrho(T[\varphi_1], T[\varphi_2]) &= \Big(\int_0^{x_0} \big|h\big(x, \varphi_1[f(x)]\big) - h\big(x, \varphi_2[f(x)]\big)\big|^p dx\Big)^{a(p)} \\ &\leqslant \Big(s\int_0^{x_0} |\varphi_1[(x)] - \varphi_2[f(x)]|^p f'(x)\,dx\Big)^{a(p)} \leqslant s^{a(p)}\varrho([\varphi_1], [\varphi_2]).\end{aligned}$$

Since $s^{a(p)} < 1$, the first part of the theorem follows from Banach's principle. It follows also from this principle that for every $\varphi_0 \in L^p(0, x_0)$ the sequence $\{\varphi_n\}$ defined by (3.6) tends to φ in the sense of the metric, i.e., in measure.

To prove the second statement of the theorem, let us note that

$$|\varphi_{n+1}(x) - \varphi_n(x)|^p \leqslant s^n[f^n(x)]'\,|\varphi_1[f^n(x)] - \varphi_0[f^n(x)]|^p \qquad \text{a.e. in } (0, x_0)$$

for $n = 0, 1, \ldots$ This inequality easily follows from (3.2), (3.6) and (3.5) by induction. Hence, we obtain

$$\begin{aligned}\Big(\int_0^{x_0} |\varphi_{n+1}(x) - \varphi_n(x)|^p dx\Big)^{a(p)} &\leqslant \Big(s^n\int_0^{x_n} |\varphi_1(x) - \varphi_0(x)^p dx\Big)^{a(p)} \\ &\leqslant (s^{a(p)})^n\Big(\int_0^{x_0} |\varphi_1(x) - \varphi_2(x)|^p dx\Big)^{a(p)}.\end{aligned}$$

Now it follows from Lemma 0.1 that the series

$$\varphi_0 + \sum_{n=1}^{\infty} (\varphi_{n+1} - \varphi_n)$$

converges a.e. in $(0, x_0)$, and this completes the proof.

THEOREM 3.3. *Let* (3.i) *and* (3.ii) *be fulfilled. If there exist an* $x_0 \in I$ *and functions* $a_k\colon (0, x_0) \to R$, $k = 1, 2$, *such that*

$$(3.9) \qquad a_1 \leqslant a_2 \quad \textit{a.e. in } (0, x_0), \ a_k \in L^p(0, x_0), \ k = 1, 2;$$

$$(3.10) \qquad a_1(x) \leqslant h\big(x, a_1[f(x)]\big), \quad h\big(x, a_2[f(x)]\big) \leqslant a_2(x) \quad \textit{a.e. in } (0, x_0);$$

$$(3.11) \quad a_1[f(x)] \leqslant y_1 < y_2 \leqslant a_2[f(x)] \Rightarrow h(x, y_1) \leqslant h(x, y_2) \quad \textit{a.e. in } (0, x_0),$$

then there exist at least one solution $[\varphi] \in L^p(0, x_0)$ *of equation* (3.1). *If, moreover, conditions* (3.iii) *and* (3.4) *are fulfilled, then equation* (3.1) *has exactly one solution* $[\varphi] \in L^p(0, x_0)$.

Proof. It follows by induction from (3.9)–(3.11) that the sequence (3.6), where φ_0 is chosen to be equal to a_1, is an increasing sequence of measurable functions bounded by a_1 and a_2 a.e. in $(0, x_0)$. Then the function $\varphi = \lim_{n\to\infty} \varphi_n$ is defined a.e. in $(0, x_0)$, measurable and integrable with the power p. Since, in view of (3.ii), $h(x, \cdot)$ is continuous in R for all x in $(0, x_0)$ except a set of measure zero, we can pass to the limit in (3.6) a.e. in $(0, x_0)$. Thus, φ satisfies equation (3.1) a.e. in $(0, x_0)$. The part on uniqueness follows from Corollary 3.1.

Remark 3.2. Note that if $h(x, 0) \geqslant$ a.e. in $(0, x_0)$, $h(\cdot, 0)$ belongs to $L^p(0, x_0)$ and there is a $c > 0$ such that

$$h\big(x, ch(f(x), 0)\big) \leqslant ch(x, 0) \qquad \textit{a.e. in } (0, x_0),$$

then conditions (3.9) and (3.10) are fulfilled with $a_1 = 0$ and $a_2(x) = ch(x, 0)$.

EXAMPLE 3.1. Apply Theorem 3.3 to the equation

$$\varphi(x) = 2(x - x^2) \frac{\varphi(x^2)}{1 + |\varphi(x^2)|} + \frac{1}{\sqrt{x}}, \qquad 0 < x < 1,$$

assuming $p = 1$. Since $h(x, y) = 2(x - x^2) y (1 + |y|)^{-1} + x^{-1/2}$, (3.11) is fulfilled. Taking a $c \geqslant 3/2$ we can easily verify that h satisfies conditions given in Remark 3.2 and, consequently, (3.9) and (3.10) are fulfilled. Moreover, we have $\eta(x) = 2(x - x^2)$, and

$$\frac{\eta(x)}{f'(x)} = 1 - x < 1, \qquad 0 < x < 1.$$

By Theorem 3.3, there exists exactly one solution $[\varphi] \in L^p(0, 1)$.

Let us note that Theorem 3.2 cannot be applied here.

In Theorem 3.3 the monotonicity of h with respect to the second variable is assumed. It turns out that in the part of Theorem 3.3 concerning the uniqueness of the solution we can replace (3.10) and (3.11) by a weaker condition, whenever $p > 1$. Namely, we have the following:

THEOREM 3.4. *Let* (3.i)–(3.iii) *be fulfilled. Suppose that there exist* $a_k\colon (0, x_0) \to R$, $k = 1, 2$, *satisfying* (3.9) *and such that*

(3.12) $$a_1[f(x)] \leqslant y \leqslant a_2[f(x)] \Rightarrow a_1(x) \leqslant h(x, y) \leqslant a_2(x) \quad \textit{a.e. in } (0, x_0).$$

If $p > 1$ *and* $\eta^p \leqslant f'$ *a.e. in* $(0, x_0)$, *then there exists exactly one solution* $[\varphi] \in L^p(0, x_0)$ *of equation* (3.1).

Proof. Put $K = \{[\varphi] \in L^p(0, x_0)\colon a_1 \leqslant \varphi \leqslant a_2$ a.e. in $(0, x_0)\}$. K is a closed subset of the uniformly convex space $L^p(0, x_0)$ (cf. Remark 1.1). For $[\varphi] \in K$ we have $|\varphi| \leqslant |a_1| + |a_2|$ a.e. in $(0, x_0)$, $a_1, a_2 \in L^p(0, x_0)$. Hence, by Minkowski's inequality, K is bounded in $L^p(0, x_0)$. Convexity of K is trivial. We shall show that the mapping T defined by (3.8) maps K into itself. By the same argument as that applied in the proof of Theorem 3.2, we obtain that $T(L^p(0, x_0)) \subset L^p(0, x_0)$. Now take a $[\varphi] \in K$. We have $a_1 \leqslant \varphi \leqslant a_2$ a.e. in $(0, x_0)$. Hence, and from (3.12), we obtain

$$a_1(x) \leqslant h(x, \varphi[f(x)]) \leqslant a_2(x) \quad \text{a.e. in } (0, x_0)$$

and, consequently, $T([\varphi]) \in K$. Finally, by (3.iii) and the inequality $\eta^p \leqslant f'$ a.e. in $(0, x_0)$, we get for $[\varphi_1], [\varphi_2] \in K$

$$\begin{aligned}\|T([\varphi_1]) - T([\varphi_2])\| &= \Big(\int_0^{x_0} |h(x, \varphi_1[f(x)]) - h(x, \varphi_2[f(x)])|^p dx\Big)^{1/p} \\ &\leqslant \Big(\int_0^{x_0} \eta(x)^p |\varphi_1[f(x)] - \varphi_2[f(x)]|^p dx\Big)^{1/p} \\ &\leqslant \Big(\int_0^{x_0} f'(x) |\varphi_1[f(x)] - \varphi_2[f(x)]|^p dx\Big)^{1/p} \\ &= \Big(\int_0^{x_1} |\varphi_1(x) - \varphi_2(x)|^p\Big)^{1/p} \leqslant \|\varphi_1 - \varphi_2\|,\end{aligned}$$

so T is non-expansive. In view of Theorem 1.3, we obtain the existence of at least one solution $[\varphi] \in L^p(0, x_0)$ of equation (3.1). The uniqueness follows by Corollary 3.1. This completes the proof.

Now we shall prove a lemma which contains a condition for the uniqueness of integrable solution of equation (31) which is slightly different from the one occurring in Theorem 3.1.

LEMMA 3.1. *Let* (3.i)–(3.iii) *be fulfilled and let*

(3.13) $$\eta_i = \operatorname*{sup\,ess}_{(0, x_i)} \eta^p (f')^{-1}, \quad i = 0, 1, \ldots$$

If there exists a number $M > 0$ such that

$$\prod_{i=0}^{n} \eta_i \leqslant M, \quad n = 0, 1, \ldots, \tag{3.14}$$

then equation (3.1) *has at most one solution in $L^p(0, x_0)$.*

Proof. Suppose that $\varphi_1, \varphi_2 \in L^p(0, x_0)$ satisfy equation (3.1) a.e. in $(0, x_0)$. Hence, and from (3.2), we obtain by induction

$$|\varphi_1 - \varphi_2| \leqslant \Big(\prod_{i=0}^{n} \eta \circ f^i\Big) |\varphi_1 \circ f^{n+1} - \varphi_2 \circ f^{n+1}| \quad \text{a.e. in } (0, x_0),\ n = 0, 1, \ldots$$

Now from (3.13) and (3.14) we have

$$\int_0^{x_0} |\varphi_1 - \varphi_2|^p \leqslant \int_0^{x_0} \Big[\prod_{i=0}^{n} (\eta \circ f^i)^p (f' \circ f^i)^{-1}\Big] |\varphi_1 \circ f^{n+1} - \varphi_2 \circ f^{n+1}|^p (f^{n+1})'$$
$$\leqslant \Big(\prod_{i=0}^{n} \eta_i\Big) \int_0^{x_0} |\varphi_1 \circ f^{n+1} - \varphi_2 \circ f^{n+1}|^p (f^{n+1})' \leqslant M \int_0^{x_{n+1}} |\varphi_1 - \varphi_2|^p.$$

Since $\lim_{n\to\infty} x_n = 0$, we have $\varphi_1 = \varphi_2$ a.e. in $(0, x_0)$.

THEOREM 3.5. *Let* (3.i)–(3.iii) *be fulfilled. If $h(\cdot, 0) \in L^p(0, x_0)$ and the series* (*where η_i are defined by* (3.13))

$$\sum_{k=0}^{\infty} \Big(\prod_{i=0}^{k-1} \eta_i\Big)^{\alpha(p)} \Big(\int_0^{x_k} |h(\cdot, 0)|^p\Big)^{\alpha(p)} \tag{3.15}$$

converges, then there exists at least one solution $[\varphi] \in L^p(0, x_0)$ of equation (3.1). *The function φ can be obtained as the limit of the sequence of successive approximations* (3.6), *where $\varphi_0 = 0$. If, moreover,* (3.14) *is fulfilled, then this solution is unique.*

Proof. Consider sequence (3.6) with $\varphi_0 = 0$ and put $\bar{h} = h(\cdot, 0)$. From (3.2) we obtain by induction

$$|\varphi_{n+1} - \varphi_n| \leqslant \Big(\prod_{i=0}^{n-1} \eta \circ f^i\Big) |\bar{h} \circ f^n| \quad \text{a.e. in } (0, x_0),\ n = 0, 1, \ldots$$

Hence we get

$$\Big(\int_0^{x_0} |\varphi_{n+1} - \varphi_n|^p\Big)^{\alpha(p)} \leqslant \Big(\int_0^{x_0} \Big[\prod_{i=0}^{n-1} \frac{(\eta \circ f^i)^p}{f' \circ f^i}\Big] |\bar{h} \circ f^n|^p (f^n)'\Big)^{\alpha(p)}$$
$$\leqslant \Big(\prod_{i=0}^{n-1} \eta_i\Big)^{\alpha(p)} \Big(\int_0^{x_0} |\bar{h} \circ f^n|^p (f^n)'\Big)^{\alpha(p)} = \Big(\prod_{i=0}^{n-1} \eta_i\Big)^{\alpha(p)} \Big(\int_0^{x_n} |\bar{h}|^p\Big)^{\alpha(p)}.$$

Now it follows from the convergence of (3.15) that $\varphi_n \in L^p(0, x_0)$, $n = 0, 1, \ldots$, and hence, in view of Lemma 0.1, the sequence

$$\varphi_n = \sum_{k=1}^{n} (\varphi_k - \varphi_{k-1})$$

converges a.e. in $(0, x_0)$ to a function $\varphi \in L^p(0, x_0)$.

In order to prove that φ satisfies equation (3.1) a.e. in $(0, x_0)$, denote by A the set of those points $x \in (0, x_0)$ for which the sequence $\{\varphi_n(x)\}$ diverges, and by B the set of those points $x \in (0, x_0)$ for which the function $h(x, \cdot)$: $R \to R$ is not continuous. Assumptions (3.i) and (3.ii) imply that the set

$$C = \bigcup_{-\infty}^{+\infty} f^n(A \cup B)$$

has measure zero, and evidently $f(C) = C$. We have

$$f\big((0, x_0) \setminus C\big) = f\big((0, x_0)\big) \setminus f(C) = f\big((0, x_0)\big) \setminus C \subset (0, x_0) \setminus C.$$

Therefore, if $x \in (0, x_0) \setminus C$, then $f(x) \in (0, x_0) \setminus C$, $h(x, \cdot)$: $R \to R$ is continuous and there exist $\lim_{n\to\infty} \varphi_n(x)$ and $\lim_{n\to\infty} \varphi_n[f(x)]$. Consequently, we have for $x \in (0, x_0) \setminus C$

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} h\big(x, \varphi_{n-1}[f(x)]\big) = h\big(x, \lim_{n\to\infty} \varphi_{n-1}[f(x)]\big)$$
$$= h\big(x, \varphi[f(x)]\big).$$

The uniqueness of the solution is a consequence of Lemma 3.1.

At the end of this section we give one more result, whose proof is based on Boyd–Wong's fixed point theorem. We shall use the following lemma which contains Jensen's inequality for concave functions (cf. W. Feller, *An introduction to probability theory and its applications*, Vol. II, Chapter V, § 8b).

LEMMA 3.2. *If* γ: $(a, b) \to R$, $-\infty \leqslant a < b \leqslant +\infty$, *is concave, then for every function* $\varphi \in L^1(X, S, \mu)$, $\mu(X) = 1$, *such that* φ: $X \to (a, b)$, *we have*

$$\int_X \gamma \circ \varphi \, d\mu \leqslant \gamma\Big(\int_X \varphi \, d\mu\Big).$$

Now we replace assumption (3.iii) by

(3.iv) *There exist functions* η: $I \to \langle 0, \infty)$ *and* γ: $\langle 0, \infty) \to \langle 0, \infty)$ *such that for almost every* x *in* I

(3.16) $$|h(x, y) - h(x, \bar{y})| \leqslant \eta(x)\gamma(|y - \bar{y}|), \quad y, \bar{y} \in R,$$

where γ *is increasing, concave and fulfils the condition*

$$\gamma(t) < t, \quad t > 0. \tag{3.17}$$

THEOREM. 3.6. *Let* $I = (0, 1)$ *and let* (3.i), (3.ii) *and* (3.iv) *be fulfilled. If* $h(\cdot, 0) \in L^1(0.1)$ *and*

$$\eta \leqslant f' \quad a.e. \text{ in } I, \tag{3.18}$$

then equation (3.1) *has exactly one solution* $[\varphi] \in L^1(0, 1)$. *Moreover, for every* $\varphi_0 \in L^1(0, 1)$ *the sequence of successive approximations* (3.6) *converges to* φ *in measure.*

Proof. We shall prove that the transformation T defined by (3.8) maps $L^1(0, 1)$ into itself. Take a $\varphi \in L^1(0, 1)$. Then, by (3.16) and (3.17), we have

$$|h(x, \varphi[f(x)])| \leqslant \eta(x)|\varphi[f(x)]| + |h(x, 0)|$$

and, consequently, in view of (3.i) and (3.18), we get

$$\int_I |T([\varphi])| \leqslant \int_I f' |\varphi \circ f| + \int_I |h(\cdot, 0)| \leqslant \int_{f(I)} |\varphi| + \int_I |h(\cdot, 0)| < \infty.$$

Let $\varphi_1, \varphi_2 \in L^1(0, 1)$. It follows from (3.16), (3.18), and (3.i) that

$$\begin{aligned}
\varrho\big(T([\varphi_1]), T([\varphi_2])\big) &= \int_I \big|h(x, \varphi_1[f(x)]) - h(x, \varphi_2[f(x)])\big| dx \\
&\leqslant \int_I \gamma\big(|\varphi_1[f(x)] - \varphi_2[f(x)]|\big) f'(x) dx \\
&= \int_{f(I)} \gamma\big(|\varphi_1(x) - \varphi_2(x)|\big) dx \leqslant \int_I \gamma\big(|\varphi_1(x) - \varphi_2(x)|\big) dx.
\end{aligned}$$

Hence, by Lemma 3.2, we have

$$\varrho\big(T([\varphi_1]), T([\varphi_2])\big) \leqslant \gamma\Big(\int_I |\varphi_1(x) - \varphi_2(x)|\, dx\Big) = \gamma\big(\varrho(\varphi_1, \varphi_2)\big).$$

Now the result follows from Theorem 1.1 of Boyd and Wong or from Theorem 1.2.

3. Theorems 3.1–3.5 have a local character. Namely, they guarantee the existence and uniqueness of integrable solutions of equation (3.1) in a certain neighbourhood of the fixed point of f. It is known that, in general, every such solution may be uniquely extended onto the whole interval I (cf. [12], Chapter III, § 2). Now the question arises: Is this extension an integrable function?

LEMMA 3.3. *Let* (3.i) *and* (3.ii) *be fulfilled. If* $x_0 \in I$ *and* φ_0: $(0, x_0) \to R$ *satisfies equation* (3.1) *a.e. in* $(0, x_0)$, *then there exists exactly one function* φ: $I \to R$ *satisfying equation* (3.1) *a.e. in* I *and such that* $\varphi = \varphi_0$ *in* $(0, x_0)$. *If, moreover,* $\varphi_0 \in L^p(0, x_0)$ *and there exist a* $K > 0$ *and a measurable func-*

tion $\eta: (x_0, a)\to R$ *such that*

(3.19) $$h(\cdot, 0)\in L^p(x_0, a),$$

(3.20) $$|h(x, y)-h(x, \bar{y})| \leqslant \eta(x)|y-\bar{y}| \quad a.e. \text{ in } (x_0, a), \qquad y, \bar{y}\in R,$$

(3.21) $$\eta^p \leqslant Kf' \quad a.e. \text{ in } (x_0, a),$$

then $\varphi\in L^p(0, b_0)$ *for every* $b_0\in(x_0, a)$.

Proof. The first part of the lemma follows easily by (3.1), (3.i) and (3.ii). Take a $b_0\in(x_0, a)$. The sequence $b_n = f^n(b_0)$, $n = 1, 2, \ldots$, is strictly decreasing and $\lim_{n\to\infty} b_n = 0$ (cf. Remark 2.1). Consequently, there exists an N such that $\langle b_{N+1}, b_N) \subset (0, x_0)$ and we can put

(3.22) $$\varphi_k(x) = h\big(x, \varphi_{k-1}[f(x)]\big), \quad x\in\langle b_{N-k+1}, b_{N-k}), \quad k = 1, \ldots, N.$$

It follows from the uniqueness of φ that

(3.23) $$\varphi(x) = \begin{cases}\varphi_0(x), & 0 < x < b_N,\\ \varphi_k(x), & b_{N-k+1} \leqslant x < b_{N-k},\ k = 1, \ldots, N.\end{cases}$$

By (3.20), (3.22) and (3.21), we have

$$\begin{aligned}\int_{b_{N-k+1}}^{b_{N-k}} |\varphi_k(x)|^p dx &\leqslant \int_{b_{N-k+1}}^{b_{N-k}} \Big(\big|h\big(x, \varphi_{k-1}[f(x)]\big) - h(x, 0)\big| + |h(x, 0)|\Big)^p dx\\ &\leqslant \int_{b_{N-k+1}}^{b_{N-k}} \big(\eta(x)|\varphi_{k-1}[f(x)]| + |h(x, 0)|\big)^p dx\\ &\leqslant 2^p\Big(\int_{b_{N-k+1}}^{b_{N-k}} \eta(x)^p |\varphi_{k-1}[f(x)]|^p dx + \int_{b_{N-k+1}}^{b_{N-k}} |h(x, 0)|^p dx\Big)\\ &\leqslant 2^p\Big(K\int_{b_{N-k+1}}^{b_{N-k}} |\varphi_{k-1}[f(x)]|^p f'(x) dx + \int_{b_{N-k+1}}^{b_{N-k}} |h(x, 0)|^p dx\Big)\\ &= 2^p\Big(K\int_{b_{N-k+2}}^{b_{N-k+1}} |\varphi_{k-1}(x)|^p dx + \int_{b_{N-k+1}}^{b_{N-k}} |h(x, 0)|^p dx\Big)\end{aligned}$$

for $k = 1, \ldots, N$. Hence and by (3.19), we obtain $\varphi_k\in L^p(b_{N-k+1}, b_{N-k})$ for $k = 1, \ldots, N$. Now, in view of (3.23), we have $\varphi\in L^p(0, b_0)$. This completes the proof.

For a $b_0\in I$ we put

$$a_n = f^{-n}(b_0), \qquad n = 0, 1, \ldots$$

Note that the sequence $\{a_n\}$ is strictly increasing and infinite if $\lim_{x\to a-} f(x) = a$.

THEOREM 3.7. *Let all the assumptions of Lemma* 3.3 *be fulfilled.*

(i) *If*

$$b = \lim_{x\to a-} f(x) < a, \tag{3.24}$$

then the extension φ *belongs to* $L^p(I)$.

(ii) *Suppose that*

$$\lim_{x\to a-} f(x) = a. \tag{3.25}$$

If, moreover, there exist a $b_0 \in (x_0, a)$ *and a* c, $0 < c < 1$, *such that*

$$\eta^p \leqslant cf' \quad \text{a.e. in } (b_0, a), \tag{3.26}$$

and

$$\sum_{n=1}^{\infty}\Big(\int_{a_{n-1}}^{a_n} |h(x, 0)|^p dx\Big)^{a(p)} < \infty, \tag{3.27}$$

then the extension φ *belongs to* $L^p(I)$.

Proof. (i) Without any loss of generality we can assume that $x_0 \leqslant b$. By Lemma 3.3 and (3.24), there exists $\varphi_1 \in L^p(0, b)$ such that

$$\varphi_1(x) = h\big(x, \varphi_1[f(x)]\big) \quad \text{a.e. in } (0, b)$$

and

$$\varphi_1 = \varphi_0 \quad \text{in } (0, x_0).$$

By (3.i) and (3.24), we have $f(\langle b, a)) \subset (0, b)$. Therefore we can define φ_2 as

$$\varphi_2(x) = h\big(x, \varphi_1[f(x)]\big), \quad x \in \langle b, a). \tag{3.28}$$

Now it follows from the uniqueness of the extension φ that

$$\varphi(x) = \begin{cases} \varphi_1(x), & x \in (0, b), \\ \varphi_2(x), & x \in \langle b, a). \end{cases}$$

Hence, from (3.28), (3.20), (3.21) and (3.24), we have

$$\int_b^a |\varphi(x)|^p dx = \int_b^a \big|h\big(x, \varphi_1[f(x)]\big)\big|^p dx \leqslant \int_b^a \big(\eta(x)|\varphi_1[f(x)]| + |h(x, 0)|\big)^p dx$$

$$\leqslant 2^p\Big(K\int_{f(b)}^{b} |\varphi_1(x)|^p dx + \int_b^a |h(x, 0)|^p dx\Big).$$

In view of (3.19), we have

$$\int_b^a |\varphi(x)|^p dx < \infty.$$

This completes the proof of (i).

(ii) By Lemma 3.3, we have $\varphi \in L^p(0, a_0)$. Define the sequence of functions ψ_n as follows:

$$\psi_0 = \varphi \quad \text{in } (0, a_0), \qquad \psi_n(x) = h\big(x, \psi_{n-1}[f(x)]\big), \quad x \in \langle a_{n-1}, a_n), \; n = 1, 2, \ldots \tag{3.29}$$

Evidently, we have

$$\varphi = \psi_n \quad \text{in } \langle a_{n-1}, a_n). \tag{3.30}$$

By (3.20) and (3.29), we have

$$|\psi_n| \leqslant \eta |\psi_{n-1} \circ f| + |h(\cdot, 0)| \quad \text{a.e. in } \langle a_{n-1}, a_n).$$

Hence, by Minkowski's inequality and (3.26), we obtain

$$\Big(\int_{a_{n-1}}^{a_n} |\psi_n|^p\Big)^{\alpha(p)} \leqslant c^{\alpha(p)} \Big(\int_{a_{n-2}}^{a_n-1} |\psi_{n-1}|^p\Big)^{\alpha(p)} + \Big(\int_{a_{n-1}}^{a_n} |h(\cdot, 0)|^p\Big)^{\alpha(p)}.$$

Put

$$A_n = \Big(\int_{a_{n-1}}^{a_n} |\psi_n|^p\Big)^{\alpha(p)}, \qquad B_n = \Big(\int_{a_{n-1}}^{a_n} |h(\cdot, 0)|^p\Big)^{\alpha(p)}, \qquad C = c^{\alpha(p)}. \tag{3.31}$$

Thus we have

$$0 \leqslant A_n \leqslant C A_{n-1} + B_n, \quad n = 1, 2, \ldots, \; 0 < C < 1.$$

Using this inequality we obtain by induction

$$A_n \leqslant C^n A_0 + \sum_{k=0}^{n-1} C^k B_{n-k}, \quad n = 1, 2, \ldots \tag{3.32}$$

Taking into account (3.30), Minkowski's inequality, (3.31), (3.32) and (3.27), we have

$$\Big(\int_{a_0}^{a} |\varphi|^p\Big)^{\alpha(p)} \leqslant \sum_{n=1}^{\infty} A_n \leqslant \sum_{n=1}^{\infty} \Big(C^n A_0 + \sum_{k=0}^{n-1} C^k B_{n-k}\Big)$$

$$= \frac{CA_0}{1-C} + \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} C^k B_{n-k} = \frac{CA_0}{1-C} + \sum_{k=0}^{\infty} C^k \Big(\sum_{n=1}^{\infty} B_n\Big)$$

$$= \frac{CA_0}{1-C} + \frac{1}{1-C} \sum_{n=1}^{\infty} B_n < \infty.$$

This completes the proof.

4. We shall now consider the equation

$$\varphi[f(x)] = g\big(x, \varphi(x)\big). \tag{3.33}$$

Assume:

(3.v) $g\colon I\times R\to R$ *fulfils the following conditions: for every* $y\in R$, $g(\cdot, y)\colon I\to R$ *is measurable in* I; *there exists a measurable function* $\gamma\colon I\to\langle 0, \infty)$ *such that*

$$|g(x, y)-g(x, \bar{y})|\leqslant \gamma(x)\,|y-\bar{y}| \quad \textit{a.e. in } I,\ y, \bar{y}\in R. \tag{3.34}$$

From (3.34) there follows the continuity of the function $g(x, \cdot)\colon R\to R$ for almost every x in I. Consequently, for every measurable function $\varphi\colon I\to R$, the function $g\big(x, \varphi(x)\big)$ is measurable in I (cf. Remark 3.1).

THEOREM 3.8. *If* $g\colon I\times R\to R$, *conditions* (3.i) *are fulfilled and* $x_0\in I$, *then for every function* $\varphi_0\colon \big\langle f(x_0), x_0\big)\to R$ *there exists exactly one function* $\varphi\colon (0, x_0)\to R$ *satisfying equation* (3.33) *in* $(0, x_0)$ *and such that* $\varphi=\varphi_0$ *in* $\big\langle f(x_0), x_0\big)$.

If, moreover, conditions (3.v) *are fulfilled,* $\varphi_0\in L^p\big(f(x_0), x_0\big)$ *and there exists an* s, $0< s< 1$, *such that*

$$(\gamma\circ f^{-1})^p\leqslant s f^{-1} \quad \textit{a.e. in } \big(0, f(x_0)\big), \tag{3.35}$$

and

$$\sum_{k=1}^{\infty}\Big(\int_{x_{k+1}}^{x_k}|g(f^{-1}(x), 0)|^p\,dx\Big)^{\alpha(p)}< \infty, \tag{3.36}$$

then $\varphi\in L^p(0, x_0)$; *in other words, the solution* $\varphi\in L^p(0, x_0)$ *of equation* (3.33) *depends on an arbitrary function.*

Proof. Put

$$\varphi_k(x) = g\big(f^{-1}(x), \varphi_{k-1}[f^{-1}(x)]\big), \quad x\in\langle x_{k+1}, x_k), \ k=1, 2, \ldots, \tag{3.37}$$

and

$$\varphi = \varphi_k \quad \text{in } \langle x_{k+1}, x_k), \quad k=0, 1, \ldots \tag{3.38}$$

It follows from (3.37) and (3.38) that φ satisfies equation (3.33) in $(0, x_0)$ and $\varphi=\varphi_0$ in $\big\langle f(x_0), x_0\big)$. The uniqueness of such φ is obvious. This completes the proof of the first statement of the theorem.

By (3.34) and (3.37), we obtain

$$|\varphi_k|\leqslant(\gamma\circ f^{-1})|\varphi_{k-1}\circ f^{-1}|+\big|g\big(f^{-1}(\cdot), 0\big)\big| \quad \text{a.e. in } \langle x_{k+1}, x_k).$$

Hence, by Minkowski's inequality and (3.35), we get

$$\Big(\int_{x_{k+1}}^{x_k}|\varphi_k|^p\Big)^{\alpha(p)}\leqslant s^{\alpha(p)}\Big(\int_{x_k}^{x_{k-1}}|\varphi_{k-1}|^p\Big)^{\alpha(p)}+\Big(\int_{x_{k+1}}^{x_k}\big|g\big(f^{-1}(\cdot), 0\big)\big|^p\Big)^{\alpha(p)}.$$

Putting

$$A_k = \Big(\int_{x_{k+1}}^{x_k} |\varphi_k|^p\Big)^{\alpha(p)}, \qquad B_k = \Big(\int_{x_{k+1}}^{x_k} |g(f^{-1}(\cdot), 0)|^p\Big)^{\alpha(p)}, \qquad C = s^{\alpha(p)},$$

we have

$$0 \leqslant A_k \leqslant CA_{k-1} + B_k, \quad k = 1, 2, \ldots,$$

where $C < 1$. Further, the argument is the same as in the proof of Theorem 3.7; the convergence of the series $\sum_{k=1}^{\infty} B_k$ results now from (3.36).

Remark 3.3. If $0 < p \leqslant 1$, then the convergence of series (3.36) means that $g(f^{-1}(\cdot), 0) \in L^p(0, f(x_0))$. For $p > 1$, denote by $\tilde{L}_f^p(0, x_0)$ the class of all measurable functions φ: $(0, x_0) \to R$ such that

$$\sum_{k=0}^{\infty} \Big(\int_{x_{k+2}}^{x_{k+1}} |\varphi|^p\Big)^{1/p} < \infty.$$

Let us note that $\tilde{L}_f^p(0, x_0) \subset L^p(0, x_0)$ and $\tilde{L}_f^p(0, x_0) \neq L^p(0, x_0)$. Actually, for $p > 1$ we have proved that if $g(f^{-1}(\cdot), 0) \in \tilde{L}_f^p(0, x_0)$, then $\varphi \in \tilde{L}_f^p(0, x_0)$.

4. Integrable solutions of systems of functional equations

1. We shall consider the system of functional equations

$$(4.1) \qquad \varphi_i(x) = h_i\big(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)]\big), \quad i = 1, \ldots, n.$$

We begin with the formulation of the assumptions for given functions.

(4.i) *For every $i, k = 1, \ldots, n$, f_{ik} is strictly increasing in an interval $I = (0, a)$, $0 < a \leqslant \infty$, f_{ik} and f_{ik}^{-1} are absolutely continuous in I and $f_{ik}(I)$, respectively, and*

$$(4.2) \qquad 0 < f_{ik}(x) < x, \quad x \in I.$$

(4.ii) *For every $i = 1, \ldots, n$, h_i: $I \times R^n \to R$ fulfils the following conditions: for every $y_1, \ldots, y_n \in R$, $h_i(\cdot, y_1, \ldots, y_n)$ is measurable in I; there exist measurable functions η_{ik}: $I \to \langle 0, \infty)$, $k = 1, \ldots, n$, such that*

$$(4.3) \qquad |h_i(x, y_1, \ldots, y_n) - h_i(x, \bar{y}_1, \ldots, \bar{y}_n)| \leqslant \sum_{k=1}^{n} \eta_{ik}(x)|y_k - \bar{y}_k|$$

$$\textit{a.e. in } I, \quad y_m, \bar{y}_m \in R, \; m = 1, \ldots, n.$$

Remark 4.1. Inequality (4.3) ensures the continuity of the functions $h_i(x, \cdot, \ldots, \cdot)$: $R^n \to R$ for almost every x in I. This, together with the measurability of $h_i(\cdot, y_1, \ldots, y_n)$ for every $y_1, \ldots, y_n \in R$, guarantees the measurability of the composition $h_i(x, \varphi_1(x), \ldots, \varphi_n(x))$ for any measurable function φ_i: $I \to R$, $i = 1, \ldots, n$, ([5], [23]).

THEOREM 4.1. *Let conditions* (4.i) *and* (4.ii) *be fulfilled and let* $h_i(\cdot, 0, \ldots, 0) \in L^p(I)$, $i = 1, \ldots, n$. *If there exist* $s_{ik} \geqslant 0$ *such that*

$$(4.4) \qquad \left(\frac{\eta_{ik}^p}{f'_{ik}}\right)^{\alpha(p)} \leqslant s_{ik} \quad \text{a.e. in } I, \quad i, k = 1, \ldots, n,$$

and the numbers

$$c_{ik}^{(0)} = \begin{cases} 1 - s_{ik}, & i = k, \\ s_{ik}, & i \neq k, \end{cases}$$

fulfil inequalities (1.10), *where the numbers* $c_{ik}^{(l)}$ *are defined by* (1.2) (*or, what amounts to the same, the characteristic roots of the matrix* (s_{ik}) *have absolute values less than* 1), *then the system of equations* (4.1) *has exactly one solution* $[\varphi_i] \in L^p(I)$, $i = 1, \ldots, n$.

Moreover, for every fixed $\varphi_i \in L^p(I)$, $i = 1, \ldots, n$, *the sequence of successive approximations*

$$\overset{k+1}{\varphi_i}(x) = h_i\big(x, \overset{k}{\varphi_1}[f_{i1}(x)], \ldots, \overset{k}{\varphi_n}[f_{in}(x)]\big), \quad k = 0, 1, \ldots; \; i = 1, \ldots, n,$$

converges a.e. in I *and*

$$\varphi_i = \lim_{k \to \infty} \overset{k}{\varphi_i} \quad \text{a.e. in } I; \; i = 1, \ldots, n.$$

Proof. Put $X_i = L^p(I)$, $i = 1, \ldots, n$, and define the mapping T_i as follows:

$$(4.5) \quad T_i([\varphi_1], \ldots, [\varphi_n]) = [h_i(\cdot, \varphi_1 \circ f_{i1}, \ldots, \varphi_n \circ f_{in})], \quad i = 1, \ldots, n.$$

We shall verify that

$$(4.6) \qquad T_i(X_1 \times \ldots \times X_n) \subset X_i, \quad i = 1, \ldots, n.$$

Let $[\varphi_k] \in X_k$, $k = 1, \ldots, n$. It follows from the absolute continuity of f_{ik} and from (4.ii) (cf. Remark 4.1) that the function

$$h_i\big(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)]\big)$$

is measurable in I. Taking into account the inequality

$$(a_1 + \ldots + a_n)^p \leqslant n^p (a_1^p + \ldots + a_n^p),$$
$$a_i \geqslant 0, \quad i = 1, \ldots, n; \; p > 0,$$

we have by (4.3) and (4.4),

$$\begin{aligned}&\big|h_i(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)])\big|^p\\ &\quad\leqslant \Big(\sum_{k=1}^{n} \eta_{ik}(x)|\varphi_k[f_{ik}(x)]| + |h_i(x, 0, \ldots, 0)|\Big)^p\\ &\quad\leqslant (n+1)^p\Big(\sum_{k=1}^{n} \eta_{ik}(x)^p|\varphi_k[f_{ik}(x)]|^p + |h_i(x, 0, \ldots, 0)|^p\Big)\\ &\quad\leqslant (n+1)^p\Big(\sum_{k=n}^{n} (s_{ik})^{1/\alpha(p)}f'_{ik}(x)|\varphi_k[f_{ik}(x)]|^p + |h_i(x, 0, \ldots, 0)|^p\Big),\end{aligned}$$

and, consequently,

$$\begin{aligned}&\int_I |T_i([\varphi_1], \ldots, [\varphi_n])|^p\\ &\qquad\leqslant (n+1)^p\Big(\sum_{k=1}^{n} (s_{ik})^{1/\alpha(p)} \int_{f_{ik}(I)} |\varphi_k|^p + \int_I |h_i(\cdot, 0, \ldots, 0)|^p\Big) < \infty.\end{aligned}$$

This proves (4.6).

Let $[\overset{m}{\varphi}_i] \in X_i$, $i = 1, \ldots, n$; $m = 1, 2$. By (4.5), (4.3), (4.4) and Minkowski's inequality, we have

$$\begin{aligned}&\varrho_i\big(T_i([\overset{1}{\varphi}_1], \ldots, [\overset{1}{\varphi}_n]), T_i([\overset{2}{\varphi}_1], \ldots, [\overset{2}{\varphi}_n])\big)\\ &\qquad\leqslant \Big\{\int_I \Big(\sum_{k=1}^{n} \eta_{ik}|\overset{1}{\varphi}_k\circ f_{ik} - \overset{2}{\varphi}_k\circ f_{ik}|\Big)^p\Big\}^{\alpha(p)}\\ &\qquad\leqslant \sum_{k=1}^{n}\Big(\int_I (s_{ik})^{1/\alpha(p)}|\overset{1}{\varphi}_k\circ f_{ik} - \overset{2}{\varphi}_k\circ f_{ik}|^p f'_{ik}\Big)^{\alpha(p)}\\ &\qquad= \sum_{k=1}^{n} s_{ik}\Big(\int_{f_{ik}(I)} |\overset{1}{\varphi}_k - \overset{2}{\varphi}_k|^p\Big)^{\alpha(p)} \leqslant \sum_{k=1}^{n} s_{ik}\,\varrho_k([\overset{1}{\varphi}_k], [\overset{2}{\varphi}_k]).\end{aligned}$$

Thus, the first statement of the theorem results from Theorem 1.4.

To prove that $\overset{k}{\varphi}_i$ tend to φ_i a.e. in I, let us note that in view of Theorem 1.4 there exist numbers r_i and $0 < s < 1$ such that

$$\varrho_i([\overset{k+1}{\varphi}_i], [\overset{k}{\varphi}_i]) = \Big(\int_I |\overset{k+1}{\varphi}_i - \overset{k}{\varphi}_i|^p\Big)^{\alpha(p)} \leqslant s^k r_i, \qquad k = 0, 1, \ldots; i = 1, \ldots, n.$$

Now the desired convergence follows easily by Lemma 0.1. This completes the proof.

2. In this section we consider the system of equations

$$\varphi_i[f(x)] = g_i(x, \varphi_1(x), \dots, \varphi_n(x)), \quad i = 1, \dots, n. \tag{4.7}$$

We assume that

(4.iii) *The functions* g_i: $I \times R^n \to R$, $i = 1, \dots, n$, *fulfil the following conditions: for every* $y_1, \dots, y_n \in R$, $g_i(\cdot, y_1, \dots, y_n)$: $R \to R$ *is measurable in* I; *there exist functions* γ_{ij}: $I \to (0, \infty)$, $i, j = 1, \dots, n$, *such that*

$$|g_i(x, y_1, \dots, y_n) - g_i(x, \bar{y}_1, \dots, \bar{y}_n)| \leqslant \sum_{j=1}^{n} \gamma_{ij}(x) |y_j - \bar{y}_j| \tag{4.8}$$

a.e. in I, $y_j, \bar{y}_j \in R$.

THEOREM 4.2. a) *If conditions* (2.i) *are fulfilled and* g_i: $I \times R^n \to R$, $i = 1, \dots, n$, *then for every* $x_0 \in I$ *and for every system of functions* $\varphi_{i,0}$: $\langle f(x_0), x_0) \to R$, $i = 1, \dots, n$, *there exists exactly one system of functions* φ_i: $(0, x_0) \to R$ *satisfying* (4.7) *and such that* $\varphi_i = \varphi_{i,0}$ *in* $\langle f(x_0), x_0)$, $i = 1, \dots, n$.

b) *Moreover, suppose that* (4.iii) *is fulfilled,* $\varphi_{i,0} \in L^p(f(x_0), x_0)$, $i = 1, \dots, n$, *and there exist numbers* $s_{ij} \geqslant 0$ *such that*

$$\left(\frac{(\gamma_{ij} \circ f^{-1})^p}{(f^{-1})'} \right)^{\alpha(p)} \leqslant s_{ij} \quad \text{a. e. in } (0, f(x_0)), \; i, j = 1, \dots, n, \tag{4.9}$$

and the numbers

$$c_{ij}^{(0)} = \begin{cases} 1 - s_{ij}, & i = j, \\ s_{ij}, & i \neq j, \end{cases}$$

fulfil inequalities (1.10). *If the series*

$$\sum_{k=0}^{\infty} \Big(\int_{x_{k+2}}^{x_{k+1}} |g_i(f^{-1}(x), 0, \dots, 0)|^p \, dx \Big)^{\alpha(p)}, \quad i = 1, \dots, n, \tag{4.10}$$

where $x_k = f^k(x_0)$, $k = 1, 2, \dots$, *converge, then* $\varphi_i \in L^p(0, x_0)$, $i = 1, \dots, n$, *and so the integrable solution of system* (4.7) *depends on an arbitrary function.*

Proof. a) Put

$$\varphi_{i,k+1}(x) = g_i(f^{-1}(x), \varphi_{1,k}[f^{-1}(x)], \dots, \varphi_{n,k}[f^{-1}(x)]), \tag{4.11}$$

$$x \in \langle x_{k+2}, x_{k+1}), \; k = 0, 1, \dots; \; i = 1, \dots, n,$$

and

$$\varphi_i = \varphi_{i,k} \quad \text{in } \langle x_{k+1}, x_k), \quad k = 0, 1, \dots; \; i = 1, \dots, n. \tag{4.12}$$

It follows by (4.11) and (4.12) that φ_i, $i = 1, \dots, n$, is a solution of system (4.7). The uniqueness is trivial.

(b) In view of (4.11) and (4.8), we have

$$|\varphi_{i,k+1}| \leqslant \sum_{j=1}^{n} \gamma_{ij}\circ f^{-1} |\varphi_{j,k}\circ f^{-1}| + |g_i(f^{-1}(\cdot), 0, \ldots, 0)|$$

$$\text{a.e. in } \langle x_{k+2}, x_{k+1}), \quad k = 0, 1, \ldots; \ i = 1, \ldots, n.$$

Hence, by Minkowski's inequality and (4.9), we have

$$\Big(\int_{x_{k+2}}^{x_{k+1}} |\varphi_{i,k+1}|^p\Big)^{a(p)} \leqslant \sum_{j=1}^{n} s_{ij} \Big(\int_{x_{k+1}}^{x_k} |\varphi_{j,k}|^p\Big)^{a(p)} + \Big(\int_{x_{k+2}}^{x_{k+1}} |g_i(f^{-1}(\cdot), 0, \ldots, 0)|^p\Big)^{a(p)},$$

for $k = 0, 1, \ldots; \ i = 1, \ldots, n$. Putting

$$(4.13) \qquad a_{i,k} = \Big(\int_{x_{k+1}}^{x_k} |\varphi_{i,k}|^p\Big)^{a(p)}, \quad k = 0, 1, \ldots; \ i = 1, \ldots, n,$$

$$(4.14) \qquad b_{i,k} = \Big(\int_{x_{k+2}}^{x_{k+1}} |g_i(f^{-1}(\cdot), 0, \ldots, 0)|^p\Big)^{a(p)}, \quad k = 0, 1, \ldots; \ i = 1, \ldots, n,$$

we obtain the following system of recurrence inequalities

$$(4.15) \qquad a_{i,k+1} \leqslant \sum_{j=1}^{n} s_{ij} a_{j,k} + b_{i,k}, \quad k = 0, 1, \ldots; \ i = 1, \ldots, n.$$

To complete the proof it suffices to show that $\sum_{k=1}^{\infty} a_{i,k} < \infty$, $i = 1, \ldots, n$. This follows from the following:

LEMMA 4.1. *Let $s_{ij} \geqslant 0$, $i, j = 1, \ldots, n$, and let $b_{i,k} \geqslant 0$, $k = 0, 1, \ldots$; $i = 1, \ldots, n$. Suppose that the characteristic roots of the matrix (s_{ij}) have absolute values less than 1 and let $a_{i,k} \geqslant 0$, $k = 0, 1, \ldots$; $i = 1, \ldots, n$, be a solution of recurrence system* (4.15).

(i) *If* $\sum_{k=1}^{\infty} b_{i,k} < \infty$, *then* $\sum_{k=1}^{\infty} a_{i,k} < \infty$, $i = 1, \ldots, n$.

(ii) *If sequences* $\{b_{i,k}\}$, $i = 1, \ldots, n$, *are bounded, then so are* $\{a_{i,k}\}$.

(iii) *If* $\lim_{k\to\infty} b_{i,k} = 0$, $i = 1, \ldots, n$, *then* $\lim_{k\to\infty} a_{i,k} = 0$, $i = 1, \ldots, n$.

Proof. (i) By induction on k, we have

$$(4.16) \qquad a_{i,k+1} \leqslant \sum_{j_1, \ldots, j_{k+1}=1}^{n} s_{ij_1} s_{j_1 j_2} \ldots s_{j_k j_{k+1}} a_{j_{k+1},0} +$$

$$+ \sum_{m=1}^{k} \sum_{j_1, \ldots, j_m=1}^{n} s_{ij_1} s_{j_1 j_2} \ldots s_{j_{m-1} j_m} b_{j_m, k-m} + b_{i,k}, \quad i = 1, \ldots, n; \ k = 0, 1, \ldots$$

In view of Lemma 1.1, there exist $r_i > 0$, $i = 1, \ldots, n$, s, $0 < s < 1$, such that (cf. (1.16))

$$\sum_{j=1}^{n} s_{ij} r_j \leqslant s r_i, \quad i = 1, \ldots, n. \tag{4.17}$$

Without loss of generality we can assume that

$$a_{i,0} \leqslant r_i, \quad i = 1, \ldots, n, \tag{4.18}$$

$$\sum_{k=0}^{\infty} b_{i,k} \leqslant r_i, \quad i = 1, \ldots, n. \tag{4.19}$$

Estimate the first component on the right-hand side of (4.16). By (4.18) and (4.17), we have for $i = 1, \ldots, n$

$$\sum_{j_1,\ldots,j_{k+1}=1}^{n} s_{ij_1} s_{j_1 j_2} \ldots s_{j_k j_{k+1}} a_{j_{k+1},0} \leqslant \sum_{j_1=1}^{n} s_{ij_1} \sum_{j_2=1}^{n} s_{j_1 j_2} \ldots \sum_{j_{k+1}=1}^{n} s_{j_k j_{k+1}} r_{j_{k+1}}$$
$$\leqslant s \sum_{j_1=1}^{n} s_{ij_1} \ldots \sum_{j_k=1}^{n} s_{j_{k-1} j_k} r_{j_k} \leqslant \ldots \leqslant s^{k+1} r_i,$$

and, consequently, we have

$$a_{i,k+1} \leqslant s^{k+1} r_i + \sum_{m=1}^{k} \sum_{j_1,\ldots,j_m=1}^{n} s_{ij_1} \ldots s_{j_{m-1} j_m} b_{j_m, k-m} + b_{i,k}, \tag{4.20}$$
$$k = 0, 1, \ldots;\ i = 1, \ldots, n.$$

Since the series $r_i \sum_{k=0}^{\infty} s^{k+1}$ and $\sum_{k=0}^{\infty} b_{i,k}$ converge, it suffices to show that the series formed of the middle summands on the right-hand side of inequality (4.20) is convergent. By (4.19) and (4.17), we have

$$\sum_{k=1}^{\infty} \sum_{m=1}^{k} \sum_{j_1,\ldots,j_m=1}^{n} s_{ij_1} s_{j_1 j_2} \ldots s_{j_{m-1} j_m} b_{j_m, k-m}$$
$$= \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \sum_{j_1,\ldots,j_m=1}^{n} s_{ij_1} s_{j_1 j_2} \ldots s_{j_{m-1} j_m} b_{j_m, k-m}$$
$$= \sum_{m=1}^{\infty} \sum_{j_1,\ldots,j_m=1}^{n} s_{ij_1} s_{j_1 j_2} \ldots s_{j_{m-1} j_m} \sum_{k=m}^{\infty} b_{j_m, k-m}$$
$$\leqslant \sum_{m=1}^{\infty} \sum_{j_1=1}^{n} s_{ij_1} \sum_{j_2=1}^{n} s_{j_1 j_2} \ldots \sum_{j_m=1}^{n} s_{j_{m-1} j_m} r_{j_m} \leqslant r_i \sum_{m=1}^{\infty} s^m < \infty.$$

This completes the proof of (i).

(ii) Choose $r_i > 0$, $i = 1, \ldots, n$, satisfying (4.17) in such a manner that

$$b_{i,k} \leqslant r_i, \quad i = 1, \ldots, n;\ k = 0, 1, \ldots$$

By (4.20) and (4.17), we have

$$a_{i,k+1} \leqslant s^{k+1} r_i + \sum_{m=1}^{k} \sum_{j_1=1}^{n} s_{ij_1} \sum_{j_2=1}^{n} s_{j_1 j_2} \cdots \sum_{j_m=1}^{n} s_{j_{m-1} j_m} r_{j_m} + r_i$$
$$\leqslant r_i + \sum_{m=1}^{k} r_i s^m + r_i \leqslant r_i \left(2 + \frac{s}{1-s}\right), \quad k = 0, 1, \ldots;\ i = 1, \ldots, n,$$

which proves (ii).

(iii) In view of (ii), the numbers $p_i = \limsup_{k\to\infty} a_{i,k}$, $i = 1, \ldots, n$, ulfil the conditions $0 \leqslant p_i < \infty$, $i = 1, \ldots, n$. By (4.15), we have

$$p_i \leqslant \sum_{j=1}^{n} s_{ij} p_j, \quad i = 1, \ldots, n,$$

and, consequently, $p_i = 0$, $i = 1, \ldots, n$ (cf. Lemma 1.3). This completes the proof of the lemma.

Remark 4.2. The convergence of series (4.10) for $0 < p \leqslant 1$ denotes simply that $g_i(f^{-1}(\cdot), 0, \ldots, 0) \in L^p(0, x_1)$, $i = 1, \ldots, n$. For $p > 1$, Theorem 4.2 states that if $g_i(f^{-1}(\cdot), 0, \ldots, 0) \in \tilde{L}_f^p(0, x_0)$, $i = 1, \ldots, n$, then also $\varphi_i \in \tilde{L}_f^p(0, x_0)$ (cf. Remark 3.3).

Remark 4.3. Assume that f_i, $i = 1, \ldots, n$, fulfil (2.i) and there exists an $x_0 \in I$ such that $f_i^k(x_0) = f_j^k(x_0)$, $i, j = 1, \ldots, n$; $k = 1, 2, \ldots$ If in Theorem 4.2 we replace conditions (4.9) by

$$\left(\frac{(\gamma_{ij} \circ f_i^{-1})^p}{(f_i^{-1})'}\right)^{a(p)} \leqslant s_{ij} \quad \text{a.e. in } (0, x_1),\ i, j = 1, \ldots, n,$$

and if in (4.10) we put f_i^{-1} in place of f^{-1}, then we get an analogous result for the more general system of equations

$$\varphi_i[f_i(x)] = g_i\big(x, \varphi_1(x), \ldots, \varphi_n(x)\big), \quad i = 1, \ldots, n.$$

5. Integrable solutions of equations of higher orders

In this chapter we study functional equations of order $n > 1$. The general form of such an equation is

$$F\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) = 0.$$

Similarly as previously, investigations of the existence and uniqueness of solutions are carried out for the equation

$$\varphi(x) = h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big).$$

For investigations of the dependence of the solution on an arbitrary function, the form

$$\varphi[f_n(x)] = g\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_{n-1}(x)]\big)$$

is more convenient. But in the latter case f_n must fulfil some additional conditions. We shall confine ourselves only to the case where all the functions f_k are iterates of the same function f, i.e., $f_k = f^k$.

1. In this section we consider the linear equation

$$\varphi = \sum_{k=1}^{n} g_k \varphi \circ f_k + h. \tag{5.1}$$

Assume:

(5.i) *f_k is strictly increasing in an interval $I = (0, a)$, f_k and f_k^{-1} are absolutely continuous in I and $f_k(I)$, respectively, and*

$$0 < f_k(x) < x, \quad x \in I, \ k = 1, \ldots, n;$$

(5.ii) *g_k, $k = 1, \ldots, n$, and h are measurable in I.*

THEOREM 5.1. *If* (5.i) *and* (5.ii) *are fulfilled and there exist an $x_0 \in I$ and numbers $s_k \geqslant 0$, $k = 1, \ldots, n$, such that $h \in L^p(0, x_0)$ and*

$$\left(\frac{|g_k|^p}{f_k'}\right)^{a(p)} \leqslant s_k \quad \text{a.e. in } (0, x_0), \ k = 1, \ldots, n, \tag{5.2}$$

$$\sum_{k=1}^{n} s_k < 1, \tag{5.3}$$

then equation (5.1) *has exactly one solution $[\varphi] \in L^p(0, x_0)$. This solution is given by the series*

$$\varphi = \sum_{k=0}^{\infty} \sum_{i_1, \ldots, i_k = 1}^{n} \Big(\prod_{j=1}^{k} g_{i_j} \circ f_{i_{j-1}} \circ \ldots \circ f_{i_1}\Big) h \circ f_{i_k} \circ \ldots \circ f_{i_1}, \tag{5.4}$$

which converges a.e. in $(0, x_0)$.

Proof. Write

$$x_{i_1 \ldots i_k} = f_{i_k} \circ \ldots \circ f_{i_1}(x_0), \quad i_1, \ldots, i_k = 1, \ldots, n; \ k = 1, 2, \ldots \tag{5.5}$$

It follows from (5.i) that $x_{i_1\ldots i_k} < x_0$ and

$$\lim_{k\to\infty} x_{i_1\ldots i_k} = 0. \tag{5.6}$$

Using (5.2), (5.5) and the relation

$$(f_{i_k}\circ\ldots\circ f_{i_1})' = \prod_{j=1}^{k} f'_{i_j}\circ f_{i_{j-1}}\circ\ldots\circ f_{i_1},$$

we obtain the following estimation

$$\Big\{\int_0^{x_0}\Big(\prod_{j=1}^{k}|g_{i_j}\circ f_{i_{j-1}}\circ\ldots\circ f_{i_1}|^p\Big)|h\circ f_{i_k}\circ\ldots\circ f_{i_1}|^p\Big\}^{\alpha(p)}$$

$$\leqslant\Big\{\int_0^{x_0}\Big(\prod_{j=1}^{k} s_{i_j}^{1/\alpha(p)} f'_{i_j}\circ f_{i_{j-1}}\circ\ldots\circ f_{i_1}\Big)|h\circ f_{i_k}\circ\ldots\circ f_{i_1}|^p\Big\}^{\alpha(p)}$$

$$=\Big(\prod_{j=1}^{k} s_{i_j}\Big)\Big\{\int_0^{x_0}(f_{i_k}\circ\ldots\circ f_{i_1})'|h\circ f_{i_k}\circ\ldots\circ f_{i_1}|^p\Big\}^{\alpha(p)}$$

$$=\Big(\prod_{j=1}^{k} s_{i_j}\Big)\Big(\int_0^{x_{i_1\ldots i_k}}|h|^p\Big)^{\alpha(p)}\leqslant\Big(\int_0^{x_0}|h|^p\Big)^{\alpha(p)}\prod_{j=1}^{k} s_{i_j}.$$

It follows from (5.3) that

$$\sum_{k=0}^{\infty}\sum_{i_1,\ldots,i_k=1}^{n}\prod_{j=1}^{k} s_{i_j} = \sum_{k=0}^{\infty}(s_1+\ldots+s_n)^k < \infty.$$

Hence, in view of Lemma 0.1, series (5.4) converges a.e. in $(0, x_0)$ and its sum φ belongs to $L^p(0, x_0)$.

We shall verify that φ satisfies equation (5.1) a.e. in $(0, x_0)$. For this purpose define

$$f_{-k} = f_k^{-1}, \quad k = 1, \ldots, n,$$

and denote by A the set of those points $x \in (0, x_0)$ for which series (5.4) diverges. According to what we have already proved, A has measure zero. By (5.i), the set

$$B = \bigcup_{k=1}^{\infty}\ \bigcup_{i_1,\ldots,i_k=-n}^{n} f_{i_k}\circ\ldots\circ f_{i_1}(A)$$

has measure zero. Moreover, we have $f_k(B) = B$, $k = 1, \ldots, n$, and, consequently,

$$f_k\big((0, x_0)\setminus B\big) = f_k\big((0, x_0)\big)\setminus f_k(B) \subset (0, x_0)\setminus B, \quad k = 1, \ldots, n.$$

Therefore, if $x \in (0, x_0)\setminus B$, then series (5.4) converges at the points x, $f_k(x)$, $k = 1, \ldots, n$, and we have

$$\varphi(x) = \sum_{k=0}^{\infty} \sum_{i_1 \ldots i_k = 1}^{n} \Big(\prod_{j=1}^{k} g_{i_j} \circ f_{i_{j-1}} \circ \ldots \circ f_{i_1}(x)\Big) h \circ f_{i_k} \circ \ldots \circ f_{i_1}(x)$$

$$= \sum_{k=1}^{\infty} \sum_{i_1 \ldots i_k = 1}^{n} \Big(\prod_{j=1}^{k} g_{i_j} \circ f_{i_{j-1}} \circ \ldots \circ f_{i_1}(x)\Big) h \circ f_{i_k} \circ \ldots \circ f_{i_1}(x) + h(x)$$

$$= \sum_{i_1=1}^{n} g_{i_1}(x) \sum_{k=1}^{\infty} \sum_{i_2 \ldots i_k = 1}^{n} \Big(\prod_{j=2}^{k} g_{i_j} \circ f_{i_{j-1}} \circ \ldots \circ f_{i_2} \circ f_{i_1}(x)\Big) \times$$

$$\times h \circ f_{i_k} \circ \ldots \circ f_{i_2} \circ f_{i_1}(x) + h(x)$$

$$= \sum_{t=1}^{n} g_t(x) \sum_{k=0}^{\infty} \sum_{i_1 \ldots i_k = 1}^{n} \Big(\prod_{j=1}^{k} g_{i_j} \circ f_{i_{j-1}} \circ \ldots \circ f_{i_1}\big(f_t(x)\big)\Big) \times$$

$$\times h \circ f_{i_k} \circ \ldots \circ f_{i_1}\big(f_t(x)\big) + h(x)$$

$$= \sum_{t=1}^{n} g_t(x) \varphi[f_t(x)] + h(x).$$

The uniqueness follows simply by (5.2) and (5.3). This completes the proof.

Remark 5.1. Replacing formally n by ∞ in Theorem 5.1, one can obtain the corresponding result for a linear equation of order ∞.

2. Now we present some results concerning the non-linear equation

$$\varphi(x) = h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big). \tag{5.7}$$

Assume:

(5.iii) *The function* $h\colon I \times R^n \to R$ *fulfils the following conditions: for every* $y_1, \ldots, y_n \in R$, $h(\cdot, y_1, \ldots, y_n)\colon I \to R$ *is measurable; there exist* $x_0 \in I$ *and measurable functions* $\eta_i\colon (0, x_0) \to \langle 0, \infty)$, $i = 1, \ldots, n$, *such that*

$$|h(x, y_1, \ldots, y_n) - h(x, \bar{y}_1, \ldots, \bar{y}_n)| \leqslant \sum_{i=1}^{n} \eta_i(x)\, |y_i - \bar{y}_i| \quad \textit{a.e. in } (0, x_0).$$

Let us put

$$s_i = \operatorname*{supess}_{(0,x_0)} \left(\frac{\eta_i^p}{f_i'}\right)^{a(p)}, \qquad i = 1, \ldots, n.$$

THEOREM 5.2. *Let* (5.i) *and* (5.iii) *be fulfilled. If* $h(\cdot, 0, \ldots, 0) \in L^p(0, x_0)$ *and*

$$\sum_{i=1}^{n} s_i < 1,$$

then equation (5.7) *has exactly one solution* $[\varphi] \in L^p(0, x_0)$.

Moreover, for every fixed $\varphi_0 \in L^p(0, x_0)$, the sequence of successive approximations

(5.8) $$\varphi_{k+1}(x) = h(x, \varphi_k[f_1(x)], \ldots, \varphi_k[f_n(x)]), \quad k = 0, 1, \ldots,$$

converges a.e. in $(0, x_0)$ and $\varphi = \lim_{k\to\infty}\varphi_k$ a.e. in $(0, x_0)$.

This theorem follows from Banach's principle and Lemma 0.1.

Now we put

$$\eta_{ti_1\ldots i_k} = \operatorname*{sup\,ess}_{(0,x_{i_1\ldots i_k})}\left(\frac{\eta_t^p}{f_t'}\right)^{a(p)}, \quad t, i_1, \ldots, i_k = 1, \ldots, n;\ k = 0, 1, \ldots,$$

where $x_{i_1\ldots i_k}$ is defined by (5.5).

THEOREM 5.3. *Let* (5.i) *and* (5.iii) *be fulfilled. If $h(\cdot, 0, \ldots, 0) \in L^p(0, x_0)$ and the series*

$$\sum_{k=0}^{\infty}\sum_{i_1,\ldots,i_k=1}^{n}\Big(\prod_{j=1}^{k}\eta_{i_j i_1\ldots i_{j-1}}\Big)\Big(\int_0^{x_{i_1\ldots i_k}}|h(\cdot, 0, \ldots, 0)|^p\Big)^{a(p)}$$

converges, then there exists at least one solution $[\varphi] \in L^p(0, x_0)$ of equation (5.7). *This solution is given by the formula*

$$\varphi = \lim_{k\to\infty}\varphi_k \quad a.e.\ in\ (0, x_0),$$

where φ_k are given by (5.8) *with $\varphi_0 = 0$. If, moreover, there exists an $M > 0$ such that*

$$\sum_{i,\ldots,i_k=1}^{n}\prod_{j=1}^{k}\eta_{i_j i_1\ldots i_{j-1}} \leqslant M, \quad k = 0, 1, \ldots,$$

then this solution is unique.

The proof of this theorem is similar to that of Theorem 2.9.

Using Theorem 5.3, one can prove the following:

THEOREM 5.4. *Let* (5.i) *be fulfilled and put $f = \max(f_1, \ldots, f_n)$. Let $s_1 + \ldots + s_n = 1$, $s_i > 0$, $i = 1, \ldots, n$. Suppose that $H\colon \langle 0, a) \to R$ is absolutely continuous and $H(0) = 0$. If for a certain $x_0 \in I$*

$$\sum_{k=0}^{\infty}\operatorname{Var} H\,|\langle 0, f^k(x_0)\rangle < \infty,$$

then the equation

$$\Phi = s_1\Phi\circ f_1 + \ldots + s_n\Phi\circ f_n + H$$

has a unique one-parameter family of absolutely continuous solutions in $\langle 0, a)$. These solutions are given by the formula

$$\Phi = \sum_{k=0}^{\infty}\sum_{i_1,\ldots,i_k=1}^{n}\Big(\prod_{j=1}^{k}s_{i_j}\Big)H\circ f_{i_k}\circ\ldots\circ f_{i_1} + c, \quad c \in R.$$

3. We shall now investigate the equation

$$\varphi[f^n(x)] = g\big(x, \varphi[f^{n-1}(x)], \ldots, \varphi[f(x)], \varphi(x)\big). \tag{5.9}$$

We assume:

(5.iv) *The function $g: I\times R^n \to R$ fulfils the conditions: for every $y_1, \ldots, y_n \in R$, $g(\cdot, y_1, \ldots, y_n): I \to R$ is measurable; there exist an $x_0 \in I$ and functions γ_i: $(0, x_0) \to \langle 0, \infty)$, $i = 1, \ldots, n$, such that*

$$|g(x, y_1, \ldots, y_n) - g(x, \bar{y}_1, \ldots, \bar{y}_n)| \leqslant \sum_{i=1}^{n} \gamma_i(x)\, |y_i - \bar{y}_i| \tag{5.10}$$

a.e. in $(0, x_0)$, $y_i, \bar{y}_i \in R$, $i = 1, \ldots, n$.

LEMMA 5.1. *Let* $s_i \geqslant 0$, $i = 1, \ldots, n$, *and* $b_k \geqslant 0$, $k = 1, 2, \ldots$ *Suppose that all the roots of the polynomial*

$$p(z) = z^n - s_1 z^{n-1} - \ldots - s_n \tag{5.11}$$

have absolute values less than 1 *and the sequence* $a_k \geqslant 0$, $k = 1, 2, \ldots$, *satisfies the inequality*

$$a_{k+n} \leqslant s_1 a_{k+n-1} + \ldots + s_n a_k + b_k, \quad k = 1, 2, \ldots \tag{5.12}$$

1. *If the series* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*
2. *If the sequence* $\{b_k\}$ *is bounded, then so is* $\{a_k\}$.
3. *If* $\lim_{k\to\infty} b_k = 0$, *then* $\lim_{k\to\infty} a_k = 0$.

Proof. Inequality (5.12) is equivalent to the following system of recurrent inequalities

$$a_{i,k+1} \leqslant s_1 a_{1,k} + \ldots + s_n a_{n,k} + b_k, \quad a_{i+1,k+1} = a_{i,k}, \quad i = 1, \ldots, n-1,$$

where $a_{1,k} = a_{n+k-1}$. Let us put

$$s_{i1} = s_i,\ i = 1, \ldots, n; \quad s_{i+1,1} = 1,\ i = 1, \ldots, n-1;$$
$$s_{ij} = 0, \quad i \neq 1,\ j \neq i-1,$$

and

$$b_{1,k} = b_k, \quad b_{i,k} = 0, \quad i = 2, \ldots, n;\ k = 1, 2, \ldots$$

The characteristic polynomial of the matrix (s_{ik}) has the form

$$\begin{vmatrix} s_1 - z & s_2 & s_3 & \ldots & s_n \\ 1 & -z & 0 & \ldots & 0 \\ 0 & 1 & -z & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots 1 & -z \end{vmatrix} = (-1)^n p(z),$$

and, consequently, the lemma follows from Lemma 4.1.

Put $x_k = f^k(x_0)$, $k = 1, 2, \ldots$ Applying Lemma 5.1, we shall prove the following:

THEOREM 5.5. a) *If* (2.i) *and* (5.iv) *are fulfilled, then for every system of functions*

$$\varphi_k\colon \langle x_k, x_{k-1})\to R, \quad k = 1, \ldots, n,$$

there exists exactly one function $\varphi\colon (0, x_0)\to R$ *satisfying equation* (5.9) *in* $(0, x_0)$ *and such that* $\varphi = \varphi_k$ *in* $\langle x_k, x_{k-1})$, $k = 1, \ldots, n$.

b) *If, moreover,* $\varphi_k \in L^p(x_k, x_{k-1})$, $k = 1, \ldots, n$, *there exist numbers* s_i *such that*

$$\left(\frac{(\gamma_i\circ f^{-n})^p}{(f^{-i})'}\right)^{\alpha(p)} \leqslant s_i \quad \text{a.e. in } (0, x_n),\ i = 1, \ldots, n, \tag{5.13}$$

absolute values of the characteristic roots of polynomial (5.11) *are less than* 1, *and the series*

$$\sum_{k=1}^{\infty}\Big(\int\limits_{x_{k+n}}^{x_{k+n-1}} |g(f^{-n}(\cdot), 0, \ldots, 0)|^p\Big)^{\alpha(p)} \tag{5.14}$$

converges, then $\varphi \in L^p(0, x_0)$. *In other words, the solution of equation* (5.9) *in the class* $L^p(0, x_0)$ *depends on an arbitrary function.*

Proof. Put

$$\varphi_{k+n}(x) = g\big(f^{-n}(x), \varphi_{k+n-1}\circ f^{-1}(x), \ldots, \varphi_k\circ f^{-n}(x)\big), \quad x\in\langle x_{k+n}; x_{k+n-1}),\ k = 1, 2, \ldots, \tag{5.15}$$

and

$$\varphi = \varphi_k \quad \text{in } \langle x_k, x_{k-1}), \quad k = 1, 2, \ldots \tag{5.16}$$

By (5.15) and (5.16), φ is a solution of equation (5.9) in $(0, v_0)$. The uniqueness is trivial. This proves a).

Now, by (5.15), (5.10) and (5.13), we have for $x\in\langle x_{k+n}, x_{k+n-1})$

$$\begin{aligned}|\varphi_{k+n}| &\leqslant \sum_{i=1}^{n}\gamma_i\circ f^{-n}|\varphi_{k+n-i}\circ f^{-i}| + |g(f^{-n}(\cdot), 0, \ldots, 0)|\\ &\leqslant \sum_{i=1}^{n} s_i^{1/p\alpha(p)}[(f^{-i})']^{1/p}|\varphi_{k+n-i}\circ f^{-i}| + |g(f^{-n}(\cdot), 0, \ldots, 0)|.\end{aligned}$$

Hence, in view of Minkowski's inequality, we have

$$\begin{aligned}\Big(\int\limits_{x_{k+n}}^{x_{k+n-1}}|\varphi_{k+n}|^p\Big)^{\alpha(p)} &\leqslant \sum_{i=1}^{n} s_i\Big(\int\limits_{x_{k+n}}^{x_{k+n-1}}|\varphi_{k+n-i}\circ f^{-i}|^p (f^{-i})'\Big)^{\alpha(p)} + \Big(\int\limits_{x_{k+n}}^{x_{k+n-1}}|g(f^{-n}(\cdot), 0, \ldots, 0)|^p\Big)^{\alpha(p)}\\ &= \sum_{i=1}^{n} s_i\Big(\int\limits_{x_{k+n-i}}^{x_{k+n-i-1}}|\varphi_{k+n-i}|^p\Big)^{\alpha(p)} + \Big(\int\limits_{x_{k+n}}^{x_{k+n-1}}|g(f^{-n}(\cdot), 0, \ldots, 0)|^p\Big)^{\alpha(p)}.\end{aligned}$$

Putting

$$a_k = \Big(\int_{x_k}^{x_{k-1}} |\varphi_k|^p\Big)^{\alpha(p)}, \quad b_k = \Big(\int_{x_{k+n}}^{x_{k+n-1}} \big|g\big(f^{-n}(\cdot), 0, \ldots, 0\big)\big|^p\Big)^{\alpha(p)}, \quad k = 1, 2, \ldots,$$

we see that the last inequality takes form (5.12). By assumption, the series $\sum_{k=1}^{\infty} b_k$ converges. Consequently, in view of Lemma 5.1, we obtain

$$\Big(\int_0^{x_0} |\varphi|^p\Big)^{\alpha(p)} \leqslant \sum_{k=1}^{\infty}\Big(\int_{x_k}^{x_{k-1}} |\varphi_k|^p\Big)^{\alpha(p)} = \sum_{k=1}^{\infty} a_k < \infty.$$

This completes the proof.

Remark 5.2. The convergence of series (5.14) for $0 < p \leqslant 1$ denotes that $g\big(f^{-n}(\cdot), 0, \ldots, 0\big) \in L^p(0, x_n)$. For $p > 1$ Theorem 5.5 states that if $g\big(f^{-n}(\cdot), 0, \ldots, 0\big) \in \tilde{L}_f^p(0, x_0)$, then $\varphi \in \tilde{L}_f^p(0, x_0)$ (cf. Remark 3.3).

6. The case of general measures

In this section we shall show that, with suitable modifications, some results obtained earlier can be extended to the case of an arbitrary measure.

1. Let (X, S, μ) be a measure space. First we consider the functional nequality

$$|\varphi| \leqslant |g|\,|\varphi \circ f| \tag{6.1}$$

and the corresponding equation

$$\varphi = g\varphi \circ f. \tag{6.2}$$

If $f\colon X \to X$ is an S-measurable transformation, i.e., for every $A \in S$, $f^{-1}(A) \in S$, then the function $\mu f^{-1}\colon S \to \langle 0, \infty\rangle$, defined by the formula

$$\mu f^{-1}(A) = \mu\big(f^{-1}(A)\big),$$

is a measure on S (cf. [10], p. 163). Note that if f is invertible and $f^{-1}\colon f(X) \to X$ is S-measurable, i.e., for every $A \in S$ we have $(f^{-1})^{-1}(A) = f(A) \in S$, then also $\mu f\colon S \to \langle 0, \infty\rangle$, defined as

$$\mu f(A) = \mu\big(f(A)\big),$$

is a measure on S.

Assume:

(6.i) *(X, S, μ) is a σ-finite measure space, $\mu(X) > 0$; $f\colon X \to X$ is one-to-one; f and f^{-1} are S-measurable; the measures μf^{-1} and μf are ab-*

solutely continuous with respect to μ, *and*

$$\mu\Big(X\setminus\bigcup_{k=0}^{\infty}[f^k(X)\setminus f^{k+1}(X)]\Big) = 0. \tag{6.3}$$

Remark 6.1. Condition (6.3) replaces here inequalities (2.3). In particular, it follows from (6.3) that the set of the fixed points of f has measure zero.

Remark 6.2. Conditions (6.i) imply that every two measures in the sequence μf^n, $n = 0, \pm 1, \ldots$, are absolutely continuous with respect to each other, and for each of the integers k and l

$$\frac{d\mu f^k}{d\mu f^l} > 0 \quad \text{a.e. in } X.$$

In fact, if $\mu f^n(A) = 0$, then we have by (6.i),

$$\mu f^{n-1}(A) = \mu f^{-1}\big(f^n(A)\big) = 0, \qquad \mu f^{n+1}(A) = \mu f\big(f^n(A)\big) = 0,$$

which implies the first statement contained in Remark 6.2. Now fix integers k and l and put

$$A = \left\{x \in X:\ \frac{d\mu f^k}{d\mu f^l}(x) = 0\right\}.$$

Suppose that $\mu(A) > 0$. We have

$$\mu f^k(A) = \int_A \frac{d\mu f^k}{d\mu f^l}\, d\mu f^l = 0,$$

whence $\mu(A) = 0$ by the absolute continuity of $\mu = \mu f^0$ with respect to μf^k. This contradiction proves the second statement.

Put

$$A_n = f^n(X)\setminus f^{n+1}(X). \tag{6.4}$$

LEMMA 6.1. *If conditions* (6.i) *are fulfilled, then for every* $n \geqslant 0$ *we have* $\mu(A_n) > 0$.

Proof. If $\mu(A_0) = 0$, then $\mu(A_0) = \mu\big(f^n(A_0)\big) = \mu f^n(A_0) = 0$, $n = 1, 2, \ldots$ (cf. Remark 6.2). By (6.3) and (6.4), we have $\mu\big(X\setminus\bigcup_{n=0}^{\infty}A_n\big) = 0$, and, consequently,

$$\mu(X) = \mu\Big(\bigcup_{n=0}^{\infty} A_n\Big) + \mu\Big(X\setminus\bigcup_{n=0}^{\infty} A_n\Big) = 0.$$

This contradiction proves that $\mu(A_0) > 0$. Hence we have

$$\mu(A_n) = \mu\big(f^n(A_0)\big) = \mu f^n(A_0) > 0, \qquad n = 1, 2, \ldots$$

Remark 6.3. If $\mu(X) < \infty$, then $\lim_{n\to\infty}\mu(f^n(X)) = 0$.

Indeed, $f^{n+1}(X) \subset f^n(X)$, $n = 0, 1, \ldots$ Therefore

$$\bigcap_{n=0}^{\infty} f^n(X) \subset \left(X \setminus \bigcup_{n=0}^{\infty} A_n\right),$$

and, consequently, by (6.3), we have

$$\lim_{n\to\infty}\mu(f^n(X)) = \mu\left(\bigcap_{n=0}^{\infty} f^n(X)\right) = 0.$$

For the function $g\colon X\to R$ we write $N_g = \{x \in X\colon g(x) = 0\}$ and we put

$$\varkappa_n = \operatorname*{inf\,ess}_{A_n\setminus N_g} |g|^{-p}\frac{d\mu f}{d\mu}, \qquad n = 0, 1, \ldots \tag{6.5}$$

THEOREM 6.1. *Let* (6.i) *be fulfilled and let* $g\colon X\to R$ *be measurable. If the series*

$$\sum_{n=0}^{\infty}\prod_{i=0}^{n-1}\varkappa_i \tag{6.6}$$

diverges and $[\varphi] \in L^p(X, S, \mu)$ *satisfies inequality* (6.1), *then* $[\varphi] = [0]$.

Proof. By (6.i) and (6.4), we have

$$A_n \in S, \quad f(A_n) = A_{n+1}, \quad f^{-1}(A_{n+1}) = A_n, \quad A_n \cap A_k = \emptyset, \quad n \neq k;\ n, k = 0, 1, \ldots \tag{6.7}$$

Moreover, it follows from (6.3) that

$$\int_X |\varphi|^p d\mu = \sum_{n=0}^{\infty}\int_{A_n}|\varphi|^p d\mu, \qquad \varphi \in L^p(X, S, \mu). \tag{6.8}$$

Suppose that $\varphi \in L^p(X, S, \mu)$ satisfies (6.1) a.e. in X. Applying Theorem D in [10], p. 164, we obtain by (6.i), (6.7), (6.1) and (6.5) (note that by (6.1) we have $\varphi = 0$ a.e. in N_g),

$$\begin{aligned}\int_{A_{n+1}} |\varphi|^p d\mu &= \int_{f^{-1}(A_{n+1})} |\varphi\circ f|^p d\mu f = \int_{A_n} |\varphi\circ f|^p \frac{d\mu f}{d\mu} d\mu \\ &\geqslant \int_{A_n\setminus N_g} \frac{d\mu f}{d\mu} |g|^{-p} |\varphi|^p d\mu \geqslant \varkappa_n \int_{A_n} |\varphi|^p d\mu.\end{aligned}$$

Having repeated this procedure n times, we arrive at the inequality

$$\int_{A_{n+1}} |\varphi|^p d\mu \geqslant \left(\prod_{i=0}^{n}\varkappa_i\right)\int_{A_0}|\varphi|^p d\mu.$$

Hence, and from (6.8), we have

$$\infty > \int_X |\varphi|^p d\mu \geqslant \Big(\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_i\Big) \int_{A_0} |\varphi|^p d\mu .$$

Now the divergence of series (6.6) implies $\varphi = 0$ a.e. in A_0. Since (cf. [13])

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_i = \sum_{n=0}^{k} \prod_{i=0}^{n-1} \varkappa_i + \Big(\prod_{i=0}^{k} \varkappa_i\Big) \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_{k+i+1},$$

the series $\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \varkappa_{k+i+1}$ diverges for $k = 1, 2, \ldots$ Replacing in the above argumentation A_0 by A_k and X by $f^k(X)$, we obtain $\varphi = 0$ a.e. in A_k for $k = 1, 2, \ldots$ In view of (6.3) and (6.4), we have $\mu(X \setminus \bigcup_{n=0}^{\infty} A_n) = 0$ and, consequently, $\varphi = 0$ a.e. in X. This completes the proof.

Let us put

(6.9) $$\lambda_n = \sup_{A_n \setminus N_g} \operatorname{ess} |g|^{-p} \frac{d\mu f}{d\mu}.$$

THEOREM 6.2. *Let* (6.i) *be fulfilled and let* $g\colon X \to R$ *be a measurable unction such that* $g \neq 0$ *a.e. in* X. *If the series*

(6.10) $$\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \lambda_i$$

converges, then the solution $\varphi \in L^p(X, S, \mu)$ *of equation* (6.2) *depends on an arbitrary function* (*cf. Definition* 0.1).

Proof. By Lemma 6.1, $\mu(A_0) > 0$. Take a $\varphi_0 \in L^p(A_0, S(A_0), \mu_A)$ and put

(6.11) $$\varphi_{k+1}(x) = \frac{\varphi_k \circ f^{-1}(x)}{g \circ f^{-1}(x)}, \quad x \in A_{k+1},\ k = 0, 1, \ldots$$

Since A_n, $n = 1, 2, \ldots$, are disjoint, the function φ, defined as

(6.12) $$\varphi = \varphi_k \quad \text{in} \quad A_k,\ k = 0, 1, \ldots,$$

is correctly defined in $\bigcup_{n=0}^{\infty} A_n$ and satisfies equation (6.2) a.e. in X. Now, by Theorem D in [10], p. 164, (6.2) and (6.9), we have

$$\int_{A_n} |\varphi|^p d\mu = \int_{f^{-1}(A_n)} |\varphi \circ f|^p d\mu f = \int_{A_{n-1} \setminus N_g} |\varphi|^p |g|^{-p} \frac{d\mu f}{d\mu} d\mu \leqslant \lambda_{n-1} \int_{A_{n-1}} |\varphi_0|^p d\mu .$$

After n steps we obtain

$$\int_{A_n} |\varphi|^p d\mu \leqslant \Big(\prod_{i=0}^{n-1} \lambda_i\Big) \int_{A_0} |\varphi|^p d\mu, \qquad n = 0, 1, \ldots$$

Hence, and by the assumption, we have

$$\int_X |\varphi|^p d\mu \leqslant \Big(\sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \lambda_i\Big) \int_{A_0} |\varphi|^p d\mu < \infty,$$

which completes the proof.

By Theorems 6.1 and 6.2, we get the following:

COROLLARY 6.1. *Suppose that* (6.i) *is fulfilled and* $g\colon X \to R$ *is measurable. If*

$$|g|^p \leqslant \frac{d\mu f}{d\mu} \quad \textit{a.e. in } X$$

and $\varphi \in L^p(X, S, \mu)$ *satisfies* (6.1) *a.e. in* X, *then* $\varphi = 0$ *a.e. in* X. *If there exists an* $s > 1$ *such that*

$$|g|^p \geqslant s\frac{d\mu f}{d\mu} \quad \textit{a.e. in } X,$$

then the solution of equation (6.2) *in the class* $L^p(X, S, \mu)$ *depends on an arbitrary function.*

COROLLARY 6.2. *If* (6.i) *is fulfilled and* $\mu(X) < \infty$, *then the series*

$$\prod_{n=0}^{\infty} \prod_{i=0}^{n-1} \operatorname*{inf\,ess}_{A_n} \frac{d\mu f}{d\mu} \tag{6.13}$$

converges.

Proof. For an indirect proof, suppose that series (6.13) diverges. This implies, by Theorem 6.1 (with $g = 1$), that if $\varphi \in L^p(X, S, \mu)$ satisfies the inequality $|\varphi| \leqslant |\varphi\circ f|$ a.e. in X, then $\varphi = 0$ a.e. in X. But every constant function satisfies this inequality and belongs to $L^p(X, S, \mu)$. This contradiction completes the proof of Corollary 6.2.

2. We shall now consider the equation

$$\varphi = g\varphi\circ f + h. \tag{6.14}$$

We assume:

(6.ii) *The functions* $g, h\colon X \to R$ *are measurable.*

THEOREM 6.3. *Let* (6.i) *and* (6.ii) *be fulfilled and let* $h \in L^p(X, S, \mu)$. *If there exists an* s, $0 < s < 1$, *such that*

$$|g|^p \leqslant s\frac{d\mu f}{d\mu} \quad \textit{a.e. in } X, \tag{6.15}$$

then equation (6.14) *has exactly one solution* $[\varphi]\in L^p(X, S, \mu)$. *This solution is given by the formula*

$$\varphi = \sum_{k=0}^{\infty}\Big(\prod_{i=0}^{k-1} g\circ f^i\Big)\, h\circ f^k. \tag{6.16}$$

Proof. Since for each of the integers k and l the measure μf^k is absolutely continuous with respect to μf^l (cf. Remark 6.2), we have (see [10], p. 133, Theorem A)

$$\frac{d\mu f^k}{d\mu} = \prod_{i=0}^{k-1}\frac{d\mu f^{i+1}}{d\mu f^i} \quad \text{a.e. in } X. \tag{6.17}$$

Moreover,

$$\frac{d\mu f^{i+1}}{d\mu f^i} = \frac{d\mu f}{d\mu}\circ f^i \quad \text{a.e. in } X, \quad i = 0, 1, \ldots \tag{6.18}$$

In fact, by the formula on the change of measure under the integral, we get

$$\int_B \Big(\frac{d\mu f}{d\mu}\circ f^i\Big)\, d\mu f^i = \int_{f^i(B)} \frac{d\mu f}{d\mu}\, d\mu = \mu\big(f^{i+1}(B)\big), \qquad B\in S.$$

On the other hand,

$$\int_B \frac{d\mu f^{i+1}}{du f^i}\, d\mu f^i = \mu\big(f^{i+1}(B)\big), \qquad B\in S,$$

and, consequently, since B can be arbitrary, we obtain (6.18). By the theorem on change of measure and by (6.15), (6.18) and (6.17), we get

$$\begin{aligned}
\Big(\int_X \Big|\Big(\prod_{i=0}^{k-1} g\circ f^i\Big) h\circ f^k\Big|^p d\mu\Big)^{\alpha(p)} &\leqslant \Big(\int_X s^k\Big(\prod_{i=0}^{k-1}\frac{d\mu f^{i+1}}{d\mu f^i}\Big)|h\circ f^k|^p d\mu\Big)^{\alpha(p)} \\
&\leqslant (s^{\alpha(p)})^k\Big(\int_X |h\circ f^k|^p\frac{d\mu f^k}{d\mu}\, d\mu\Big)^{\alpha(p)} \\
&= (s^{\alpha(p)})\Big(\int_{f^k(X)} |h|^p d\mu\Big)^{\alpha(p)} \leqslant (s^{\alpha(p)})^k\Big(\int_X |h|^p d\mu\Big)^{\alpha(p)}.
\end{aligned}$$

Hence, by Minkowski's inequality and Lemma 0.1, we infer that series (6.16) converges a.e. in X and its sum belongs to $L^p(X, S, \mu)$.

Denote by A the set of all those $x\in X$ at which series (6.16) diverges. By (6.i), the set

$$B = \bigcup_{-\infty}^{+\infty} f^k(A)$$

has measure zero. Now we can easily verify that φ defined by (6.16) satisfies equation (6.14) everywhere in $X\setminus B$.

The uniqueness of the solution just obtained follows immediately from Theorem 6.1 (see Corollary 6.1).

Similarly we can prove the following:

THEOREM 6.4. *Let conditions* (6.i) *and* (6.ii) *be fulfilled and let* $h \in L^p(X, S, \mu)$. *If the series*

$$\sum_{k=0}^{\infty}\Big(\prod_{i=0}^{k-1}\eta_i\Big)^{\alpha(p)}\Big(\int_{f^k(X)}|h|^p d\mu\Big)^{\alpha(p)}, \tag{6.19}$$

where

$$\eta_i = \operatorname*{sup\,ess}_{f^i(X)} |g|^p\Big(\frac{d\mu f}{d\mu}\Big)^{-1}, \qquad i = 0, 1, \ldots,$$

converges, then equation (6.14) *has at least one solution* $[\varphi] \in L^p(X, S, \mu)$. *This solution is given by series* (6.16) *which converges a.e. in* X. *Moreover,*

a) *if there exists an* $M > 0$ *such that*

$$\prod_{i=0}^{k-1}\eta_i \leqslant M, \qquad k = 1, 2, \ldots$$

or series (6.6) *defined by* (6.5) *diverges, then this solution is unique*;

b) *if series* (6.10) *defined by* (6.9) *converges, then the solution* $\varphi \in L^p(X, S, \mu)$ *of equation* (6.14) *depends on an arbitrary function.*

By an obvious modification of the proof of Theorem 3.8, we obtain the following:

THEOREM 6.5. *Let conditions* (6.i) *and* (6.ii) *be fulfilled. If there exists an* $s > 1$ *such that*

$$|g|^p \geqslant s\frac{d\mu f}{d\mu} \qquad \text{a.e. in } X$$

and the series

$$\sum_{k=1}^{\infty}\Big(\int_{A_k}\Big|\frac{h\circ f^{-1}}{g\circ f^{-1}}\Big|^p d\mu\Big)^{\alpha(p)} \tag{6.20}$$

converges, then equation (6.14) *has in* X *the solution* $\varphi \in L^p(X, S, \mu)$ *depending on an arbitrary function.*

Remark 6.3. For $0 < p \leqslant 1$ the convergence of series (6.20) means that $h\circ f^{-1}/g\circ f^{-1} \in L^p\big(f(X), S(f(X)), \mu_{f(X)}\big)$.

3. Now we apply Theorems 6.3–6.5 to obtain solutions $\varphi\colon S \to R$ of the equation

$$\Phi(A) = s\Phi[f(A)] + H(A), \qquad A \in S, \tag{6.21}$$

where Φ is a signed measure, absolutely continuous with respect to μ, Let Φ: $S \to R$ be a signed measure. By the Hahn decomposition theorem, there exist two sets A and B such that $A \cap B = \emptyset$, $A \cup B = X$, and for every $C \in S$

$$\Phi^+(C) = \Phi(A \cap C) \geqslant 0, \qquad \Phi^-(C) = -\Phi(B \cap C) \geqslant 0.$$

Let us put

$$\mathrm{Var}\, \Phi | C = \Phi^+(C) + \Phi^-(C), \qquad C \in S.$$

It is well known that the functions Φ^+, Φ^- and $\mathrm{Var}\, \Phi$ are measures on S and

$$\Phi = \Phi^+ - \Phi^-.$$

THEOREM 6.6. *Let conditions* (6.i) *be fulfilled and let* H: $S \to R$ *be a signed measure absolutely continuous with respect to* μ.

a) *If* $|s| < 1$, *then there exists exactly one signed measure* Φ: $S \to R$ *absolutely continuous with respect to* μ *and satisfying equation* (6.21) *on* S. *This solution has the form*

$$\Phi = \sum_{k=0}^{\infty} s^k H \circ f^k. \tag{6.22}$$

b) *If* $|s| = 1$ *and the series*

$$\sum_{k=0}^{\infty} \mathrm{Var}\, H | f^k(X) \tag{6.23}$$

converges, then there exists exactly one signed measure φ: $S \to R$ *absolutely continuous with respect to* μ *and satisfying equation* (6.21) *on* S. *This solution is given by formula* (6.22).

c) *If* $|s| > 1$ *and* $dH/d\mu \circ f^{-1} \in L^1(f(X), f(S), \mu_{f(X)})$, *then there exists a set* $A_0 \in S$, $\mu(A_0) > 0$, *such that for every signed measure* Φ_0: $S(A_0) \to R$ *absolutely continuous with respect to* μ_{A_0} *there exists exactly one signed measure* Φ: $S \to R$ *absolutely continuous with respect to* μ, *satisfying equation* (6.21) *on* S *and fulfilling the condition*

$$\Phi(A) = \Phi_0(A), \qquad A \in S(A_0).$$

Proof. Note that the existence of a signed measure Φ: $S \to R$ absolutely continuous with respect to μ and satisfying equation (6.21) is equivalent to the existence of an integrable solution φ: $X \to R$ (i.e., $\varphi \in L^1(X, S, \mu)$) of the equation

$$\varphi = s \frac{d\mu f}{d\mu} \varphi \circ f + h, \tag{6.24}$$

where $h = \dfrac{dH}{d\mu}$.

In fact, let $\Phi: S \to R$ be a signed measure, absolutely continuous with respect to μ and satisfying equation (6.21). Differentiating both sides of equation (6.21) (cf. [10], p. 133, Theorem A), we see that the Radon–Nikodym derivative $\frac{d\Phi}{d\mu}$ satisfies the equation

$$\frac{d\Phi}{d\mu} = s\,\frac{d(\Phi f)}{d\mu f}\,\frac{d\mu f}{d\mu} + \frac{dH}{d\mu}.$$

Since $\frac{d(\Phi f)}{d\mu f} = \frac{d\Phi}{d\mu} \circ f$ (cf. (6.18)), the function $\varphi = \frac{d\Phi}{d\mu}$ satisfies equation (6.24) with $h = \frac{dH}{d\mu}$ a.e. in X. Moreover,

$$\int_X \varphi\, d\mu = \int_X \frac{d\Phi}{d\mu}\, d\mu = \Phi(X) \in R,$$

and, consequently, $\varphi \in L^1(X, S, \mu)$. Conversely, if $\varphi \in L^1(X, S, \mu)$ is a solution of equation (6.24), then the function $\Phi = \int \varphi\, d\mu$ satisfies equation (6.21) and is a signed measure, absolutely continuous with respect to μ.

Now a) follows from Theorem 6.3 and b) follows from Theorem 6.4 and from the remark that in the present case series (6.19) reduces to (6.23) (cf. [21], p. 401, Theorem 4.14).

To prove c), note that, by Theorem 6.5, in this case the solution $\varphi \in L^1(X, S, \mu)$ of equation (6.24) depends on an arbitrary function. Let A_0 denote this set on which an integrable solution of equation (6.24) may be arbitrarily prescribed. Take a signed measure $\Phi_0: S(A_0) \to R$ which is absolutely continuous with respect to μ_{A_0}, and put $\varphi_0 = d\Phi_0/d\mu$. Since $\Phi_0(A_0)$ is finite, φ_0 is integrable on A_0. According to Theorem 6.5, there exists exactly one function $\varphi \in L^1(X, S, \mu)$ satisfying equation (6.24) a.e. in X and such that $\varphi = \varphi_0$ in A_0. The function $\Phi = \int \varphi\, d\mu$ is a signed measure, absolutely continuous with respect to μ and, moreover, we have for $A \in S(A_0)$

$$\Phi(A) = \int_A \varphi\, d\mu = \int_A \varphi_0\, d\mu = \int_A \frac{d\Phi_0}{d\mu}\, d\mu = \Phi_0(A),$$

which completes the proof.

4. The considerations of this chapter show how, making suitable changes in the theorems concerning the case of the Lebesgue measure, we can obtain an analogous result for a general measure μ. The conditions for the measurability of the composition $h(x, \varphi_1(x), \ldots, \varphi_n(x))$, where φ_k, $k = 1, \ldots, n$, are μ-measurable, have been given by Шрагин [23].

Studying the Lebesgue integrable solutions of functional equations, we have assumed possibly the weakest conditions on the regularity of given functions. This, however, forced us to impose some additional conditions on the behaviour of these functions in a neighbourhood of the fixed point of the function f (resp. f_k, f_{ik}, for equations of higher orders or for systems of equations). Let us note that for analytic, differentiable or continuous solutions of the functional equations the suitable conditions are usually imposed exactly at the fixed point of f (cf. [12]).

If besides (2.i), we assume that $f \in C^1(\langle 0, x_0))$ and $g \in C(\langle 0, x_0))$, then the condition for the uniqueness of solutions $[\varphi] \in L^p(0, x_0)$ of equation (2.13) will be fulfilled whenever $|g(0)|^p < f'(0)$. If, on the contrary, $|g(0)|^p > f'(0)$, then, in general, the solution $\varphi \in L^p(0, x_0)$ of (2.13) depends on an arbitrary function. By (2.i), we have $0 \leqslant f'(0) \leqslant 1$. Suppose that $0 < f'(0) < 1$ and $0 < |g(0)| < 1$. Then for $0 < p < \dfrac{\ln f'(0)}{\ln |g(0)|}$, equation (2.13) has a solution $\varphi \in L^p(0, x_0)$ depending on an arbitrary function, and for $p > \dfrac{\ln f'(0)}{\ln |g(0)|}$, equation (2.13) has a unique solution in $L^p(0, x_0)$.

Throughout this paper we were assuming that the fixed point of the function f was equal to 0 and that 0 was the left end-point of the interval I; consequently, the fixed point was finite. This could be done without loss of generality.

If $I = (c, \infty)$ and ∞ is a fixed point of f (i.e., $\lim\limits_{x\to\infty} f(x) = \infty$); then nstead of (2.3) we must assume that

$$f(x) > x, \qquad x \in I.$$

Example. Consider the linear equation

$$\varphi(x) = \varphi(x+1) + \frac{1}{x^2}, \qquad x \in (1, \infty). \tag{6.25}$$

Here $I = (1, \infty)$; $f(x) = x+1$, $g(x) = 1$ and $\dfrac{|g(x)|}{f'(x)} = 1$ for $x \in I$. Therefore (cf. Corollary 2.2), if $\varphi \in L^1(1, \infty)$ is a solution of equation (6.25), then

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \qquad \text{a.e. in } (1, \infty).$$

Since we have

$$\int_1^\infty \varphi(x)\,dx = \infty,$$

equation (6.25) has no integrable solution in $(1, \infty)$.

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